Non-Hermitian Skin Effect: A Primer

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Hermiticity is a crucial condition for the observables to be real (hence measuarble) in our conventional quantum physics. However, the conventional quantum physics deals with the ideal scenario where the system is isolated from its environment. As soon as we take account of the interaction with the environment, we enter the generic world of non-equilibrium quantum physics where Hermiticity is not a required condition. The recent hunt of topological insulators for fault-tolerant quantum computation has opened the gate for studying the physics of non-Hermiticity into those materials. It has been recently discovered that non-Hermiticity induces an exotic phenomenon dubbed the *skin effect* which entails asymmetric localization of quantum states. In our article, we provide an introduction to this phenomenon and physics behind this with a flavor of students-friendly pedagogy.

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I. HATANO-NELSON MODEL: A FLAVOR OF NON-HERMITICITY IN A LATTICE

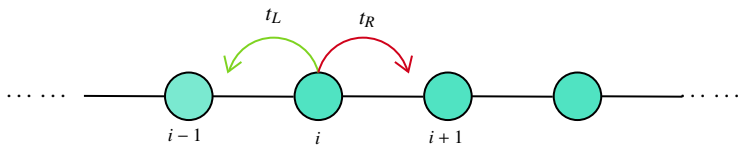

FIG. 1: Diagram showing the hopping in Hatano-Nelson Model

The Hatano-Nelson model (non-disordered) is the simplest tight-binding lattice model where the left and right-moving hopping amplitudes are unequal (*non-reciprocal*). The Hamiltonian for spinless particles in one dimension reads,

$$\hat{H}_{\mathrm{HN}} = \sum_{i=0}^{L-1} \left[t_R c_i^\dagger c_{i+1} + t_L c_{i+1}^\dagger c_i \right], \tag{1}$$

where L is the number of sites in the lattice and t_R and t_L are the right and left hopping amplitudes. The Hamiltonian is written using the second quantisation formalism, where c_i and $c_i^\dagger$ are the particle anihilation and creation operators at lattice site i, respectively. The Hamiltonian becomes non-Hermitian when $t_L \neq t_R$, that is, *asymmetric hopping*.

We can express the anihilation operator in the momentum space as:

$$c_i = \frac{1}{\sqrt{L}} \sum_{k=0}^{L-1} e^{ikx_i} c_k. \tag{2}$$

where x_i is the coordinate or position of site i. We then substitute this in the Hamiltonian:

$$\begin{aligned}
\hat{H}_{\mathrm{HN}} &= \sum_L \sum_k \sum_{k'} t_R e^{-ikx_i} e^{ik'(x_i+a)} c_k^\dagger c_{k'} + t_L e^{-ik(x_i+a)} e^{ik'x_i} c_k^\dagger c_{k'} && (3)\\
&= \sum_L \sum_k \sum_{k'} \left(t_R e^{ix_i(k'-k)} e^{ik'a} + t_L e^{ix_i(k'-k)} e^{-ika} \right) c_k^\dagger c_{k'} && (4)\\
&= \sum_k \left(t_R e^{ika} + t_L e^{-ika} \right) c_k^\dagger c_k \qquad \left(\because \sum_i e^{i(k-k')x_i} = \delta_{kk'} \right) && (5)\\
&= \sum_k E(k) c_k^\dagger c_k && (6)
\end{aligned}$$

where we utilize $x_{i+1} - x_i = a$, a being the lattice constant (spacing between two consecutive lattice sites). Hence we derive the energy spectrum in terms of the wave-number as:

$$E(k) = t_R e^{ika} + t_L e^{-ika} = (t_R + t_L)\cos(ka) + i(t_L - t_R)\sin(ka), \qquad k = \frac{2\pi i}{L} \quad (i = 0, \ldots, L-1)$$

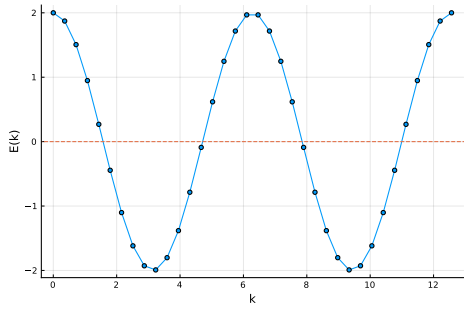

FIG. 2: Energy varying with k for the Hermitian Hatano-Nelson Model. The parameters taken are $a = 1, t = 1$

Note that, if $t_L = t_R = t$, then $E(k) = 2t\cos(ka) \in \mathbb{R}$ and the model becomes Hermitian, with real eigenvalues. However, if $t_L \neq t_R$, we will have a finite value for the imaginary part of the energy too.

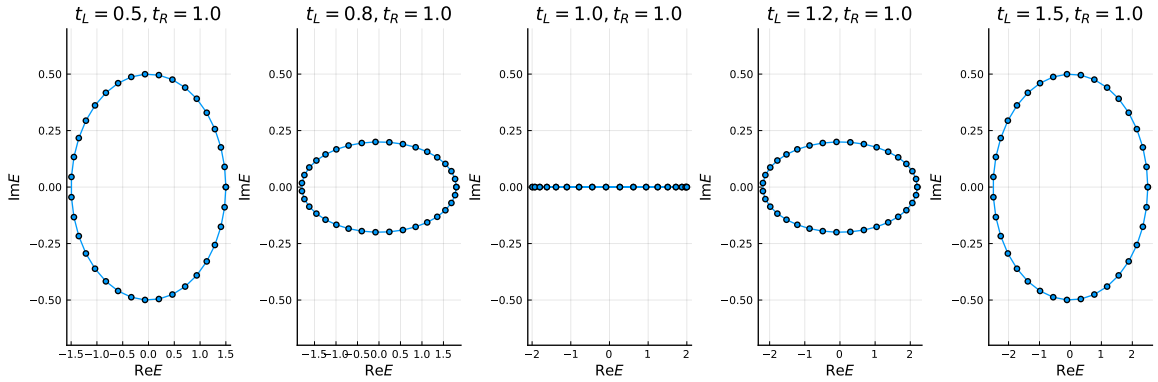

FIG. 3: For $t_L \neq t_R$, the imaginary and the real part as a function of k defines an ellipse and winds around the origin.

To this extent, we can define the spectral winding number for the system:

$$\omega = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \mathrm{d}k\partial_k \ln E(k) = \begin{cases} +1, & |t_L| > |t_R| \\ -1, & |t_L| < |t_R| \end{cases}$$

From Figure. I, we cannot observe the orientation of the winding, however, the transition between the two windings is clear, since at $t_L = t_R$, the ellipses collapse and we have a flat line (since $\mathrm{Im}\, E(k) = 0$) In the equivalent matrix representation under the periodic boundary and open chain case, we have the following:

$$[\hat{H}_{\mathrm{HN}}]_{OBC} = \begin{pmatrix} 0 & t_L & 0 & 0 & \cdots \\ t_R & 0 & t_L & 0 & \cdots \\ 0 & t_R & 0 & t_L & \cdots \\ 0 & 0 & t_R & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad [\hat{H}_{\mathrm{HN}}]_{PBC} = \begin{pmatrix} 0 & t_L & 0 & 0 & \cdots & t_R \\ t_R & 0 & t_L & 0 & \cdots & 0 \\ 0 & t_R & 0 & t_L & \cdots & 0 \\ 0 & 0 & t_R & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ t_L & 0 & 0 & 0 & & \end{pmatrix}$$

Using exact diagonalisation, we can hence find the energy eigenvalues of the Hatano-Nelson Hamiltonian. Under open boundary condition, all the eigenstates of the Hatano-Nelson Model are exponentially localised on the boundary. The direction of localisation depends on the hopping term, that is, the states are localised on the left if $t_L > t_R$ and otherwise,on the right. This phenomenon is termed as *non-Hermitian skin effect*.

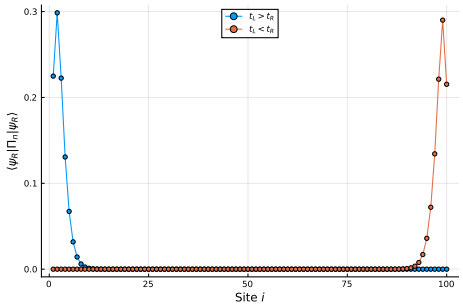

FIG. 4: Figure showing the localisation of states along the boundary. Parameters taken, for blue plot $t_L = 1.5, t_R = 0.5$ and for red plot $t_L = 0.5, t_R = 1.5$, $N = 100$.

For a non-Hermitian system, since $H^\dagger \neq H$, it is not guaranted that a state $|\psi\rangle$ satisfying $H|\psi\rangle = E|\psi_R\rangle$ will also satisfy the conjugate equation. Thus, we define two vectors for the Hamiltonian:

$$\text{Right Eigenvector:} \qquad H|\psi_R\rangle = E|\psi_R\rangle$$
$$\text{Left Eigenvector:} \qquad \langle\psi_L|H = E\langle\psi_L|$$

Studying the left and right eigenvectors separately, we can observe the skin-effect. We also define the *biorthogonal localisation* by:

$$C = \langle\psi_L|\Pi_n|\psi_R\rangle, \qquad \text{where } \Pi_n = c_n^\dagger|0\rangle\langle 0|c_n \text{ is the projector onto site } n$$

If the right eigenstate is localised to the right, the left eigenstate is localised to the left and hence the biorthogonal localisation nullifies this localisation, giving a 'bulk-state ' picture in the biorthogonal formalism.

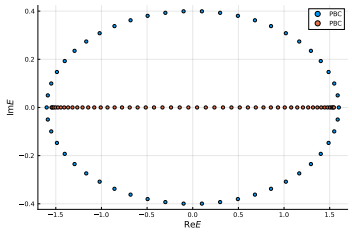

FIG. 5: Re E vs. ImE plot for both OBC and PBC case with parameters $N = 50, t_L = 0.6, t_R = 1.0$. We can see that while the PBC eigenvalues trace an ellipse, the OBC eigenvalues are purely real.

REFERENCES