Lecture 0: Quantumness and Qubits

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1 What are special in quantum world and missing in the classical realm?

In our quantum world, we have the following rules that are not applicable to our "mundane" classical Newtonian physics.

1. **Superposition:** Simultaneous existence of many states .
2. **Probabilistic measurement outcome:** A quantum measurement collapses the superposed state to one of the many possible states.
3. **Operator based representation:** Physical properties (observables) are described in terms of Hermitian matrices (operators).

2 Classical bits vs quantum bits

Classical bits:

Classical bits comprises of 2 digits, namely 0 and 1, depending on classical states such as *on/off*, *positive voltage/negative voltage*, *true/false*, *black/white*, *north pole/south pole*, *charged/uncharged* etc. Classical bits follow Boolean algebra.

Quantum bits a.k.a. qubits:

Instead of classical states, quantum bits or qubits are represented by quantum states which are often described quantum mechanical wavefunctions (eigenfunction ψ for the problem $\hat{H}\psi = E\psi$). Being wavefunction, the qubits are typically represented by kets or column matrices (vectors). Being vectors, the qubits follow linear algebra.
For instance,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

;

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

3 Two-level system (TLS): revisiting particle-in-a-box problem

We can revisit our familiar *particle-in-a-box* problem where a particle is confined between two infinite potentials ($V = \infty$ at positions $x = 0$ and $x = L$). The particle is free to move anywhere inside the box or well ($V = 0$ at $0 < x < L$) before it hits any of the barriers (see Fig. 1). The eigenenergies of the problem

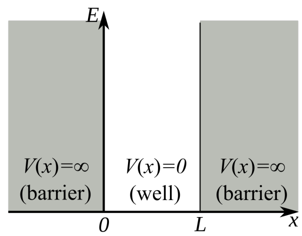

Figure 1: The one-dimensional particle-in-a-box problem.

can be found as

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \tag{1}$$

where E_n is the energy of the n-th levels, $n = \pm1, \pm2, \pm3, \ldots$, $\hbar$ is the reduced Planck's constant, m is the mass of the particle, L is the width of the potential well where the corresponding normalized eigenfunctions/wavefunctions become (see Fig. 2 for visualization)

$$\psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \,. \tag{2}$$

Now in this problem, we can easily any of the two quantum states and forget all other states. We can choose the lowest energy state or the ground state (ψ_1 with $E_1 = \pi^2\hbar^2/(2mL^2)$) as the state $|0\rangle$ and the first excites state (ψ_2 with $E_2 = 4\pi^2\hbar^2/(2mL^2)$) as the state $|1\rangle$.

4 Properties of a Hermitian Matrix

We mentioned in the begining that a quantum property is described by a Hermitian operator or matrix. A Hermitian operator is its own self-adjoint (adjoint = transpose + complex conjugation = complex conjugation + transpose). E.g., if A is a Hermitian operator, then $A^\dagger = A$. Now a Hermtian operator shows two distinct properties:

1. Its eigenvalues are always REAL.

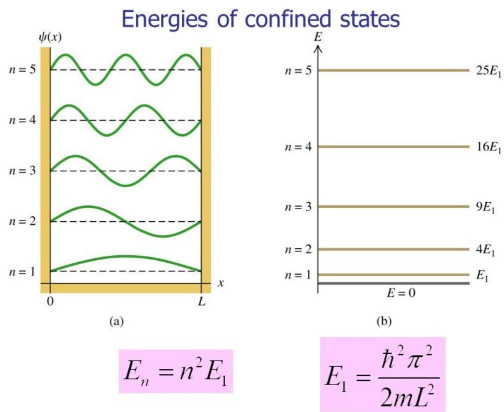

Figure 2: (a) Wavefunctions and (b) energy levels of the particle-in-a-box problem.

2. Its eigenfunctions are always ORTHOGONAL to each other, i.e. for two eigenfunction $|\psi_m\rangle$ and $|\psi_n\rangle$, we have the *inner product* $\langle\psi_m|\psi_n\rangle = K\delta_{mn}$.

Here K is a constant and δ_{mn} is a Kronecker delta. When we set $K = 1$, the eigenfunctions become ORTHONORMAL (we often do that to preserve the quantum probability).

5 Testing orthonormal properties for qubits

We can easily check that our qubits $|0\rangle$ and $|1\rangle$ are orthonormal to each other. When we represent these qubits by 2×2 column matrices (see Sec. 2), the inner product $\langle\alpha|\beta\rangle$ becomes just matrix multiplication of $\langle\alpha| = |\alpha\rangle^\dagger$ and $|\beta\rangle$. Hence

$$\langle 0|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}^T \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} = 0 + 0 = 0 \tag{3}$$

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Similarly,

$$\langle 1|1\rangle = \begin{bmatrix}0 & 1\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} = 0 + 1 = 1 \tag{5}$$

and

$$\langle 0|1\rangle = \langle 1|0\rangle^\dagger = 0\,. \tag{6}$$

6 Norm (or probability) conservation of a generic TLS

Looking at the orthonormal properties of $|0\rangle$ and $|1\rangle$, a generic two-level state (TLS) $|\psi\rangle$ can be expressed as a linear combination of the basis states $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \tag{7}$$

where a and b are arbitrary complex scalar coefficients.

Now in quantum physics, the probability is given by the *norm* or self inner product of the wave function, $||\psi|| \equiv |\psi|^2 \equiv \langle\psi|\psi\rangle$) and here we have

$$||\psi|| = (a\,|0\rangle + b\,|1\rangle)^\dagger (a\,|0\rangle + b\,|1\rangle) = |a|^2\,\langle 0|0\rangle + |b|^2\,\langle 1|1\rangle = |a|^2 + |b|^2\,. \tag{8}$$

Thus to preserve the total probabilty (which must be equal to 1), we must have

$$|a|^2 + |b|^2 = 1\,. \tag{9}$$

Eq. (9) is known as the norm or probability conservation condition.

The orthonormal bases can be considered as orthogonal unit vectors (say, $\hat{x}$ and $\hat{y}$) in vector algebra where the generic state $|\psi\rangle$ can be expressed as $\vec{\psi}$:

$$\vec{\psi} = a\hat{x} + b\hat{y}\,. \tag{10}$$

In this case, the inner product becomes the dot-product and we have

$$||\psi|| = (a\hat{x} + b\hat{y}) \cdot (a\hat{x} + b\hat{y}) = |a|^2\hat{x}\cdot\hat{x} + |b|^2\hat{y}\cdot\hat{y} = |a|^2 + |b|^2\,. \tag{11}$$