Objectives

Practice computing the asymptotic behavior of integrals by (i) expanding the function to be integrated in an asymptotic series, and (ii) applying integration-by-parts.

Instructions

Use Mathematica to solve the following problems. Use the template introduced for the previous homework. Write up discussions of your results.

Problems

1. The Fresnel sine integral is given by $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) \mathrm{d}t$. The command to evaluate $S(x)$ in Mathematica is `FresnelS[x]`.

   (a) Use integration-by-parts to find the leading behavior of $S(x)$ as $x \to 0^+$ and compare your result with what Mathematica computes, which might be slightly different from what you compute. Comment on what you find and which approximation you believe yields a better approximation overall.

   (b) Use $\int_0^\infty \sin\left(\frac{\pi t^2}{2}\right) \mathrm{d}t = \frac{1}{2}$, and integration-by-parts to find the leading behavior of $S(x)$ as $x \to +\infty$ (Hint: Consider a substitution prior to performing integration-by-parts). Compare your result with what Mathematica computes and discuss what you find.

2. (Bender & Orszag, 6.9) Show that

$$\int_0^1 \frac{e^x - e^{xt}}{1-t} \mathrm{d}t \sim e^x \log(x) + e^x \gamma + \cdots, \quad x \to +\infty,$$

   where γ is the Euler-Mascheroni constant, defined accoring to

$$\gamma = \int_0^\infty \left(\frac{1}{u+1} - e^{-u}\right) \frac{\mathrm{d}u}{u}.$$

   (Hint: make a substitution that yields x as a limit of integration, and consider the trick we used in Lecture 11 to manufacture the definition of γ given above.)