Math 223: Homework 11

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Problem 1

For the van der Pol oscillator, $y'' + y + \epsilon (y^2 - 1) y' = 0$, apply the method of multiple scales and show that it has a stable limit cycle that is nearly circular with radius $r = 2 + O(\epsilon)$.

Using the method of time averaging, we see that we can take $h = (y^2 - 1) y' \Longrightarrow (r \cos^2 \theta - 1)(-r \sin \theta)$. This means we compute the following integrals:

In[∘]:=
```
Integrate[-Sin[θ]^2 * Cos[θ]^2/(2π), {θ, 0, 2π}]
```

Out[∘]=

$$-\frac{1}{8}$$

In[∘]:=
```
Integrate[Sin[θ]^2/(2π), {θ, 0, 2π}]
```

Out[∘]=

$$\frac{1}{2}$$

This yields that the first average equation is given by $r'(\tau) = \frac{-1}{8} r^3 + \frac{1}{2} r$. To solve for $r(\tau)$, we take:

In[∘]:=
```
DSolve[r'[τ] + r[τ]^3/8 - r[τ]/2 == 0, r[τ], τ]
```

Out[∘]=

$$\left\{\left\{r[\tau] \to -\frac{2\, e^{\tau/2}}{\sqrt{e^{\tau} + e^{8 c_1}}}\right\}, \left\{r[\tau] \to \frac{2\, e^{\tau/2}}{\sqrt{e^{\tau} + e^{8 c_1}}}\right\}\right\}$$

Since we are discussing the radius, we take the positive solution. Taking the limit of this solution yields:

In[∘]:=
```
Limit[2 e^(τ/2)/Sqrt[e^τ + e^(8 c1)], τ → ∞]
```

Out[∘]=

$$2$$

This means that the amplitude of the oscillator has a stable limit cycle with a radius of $2 + O(\epsilon)$.

Problem 2

Use the method of multiple scales to study $\frac{d^2 y}{d t^2} + y + \epsilon y^2 \frac{d y}{d t} = 0$, $y(0) = 1$, $y'(0) = 0$. Describe the long time behavior of the solution based on your results.

Using the method of time averaging yields $h = y^2 y' = (r^2 \cos^2 \theta)(-r \sin \theta)$. This means we compute the following integral:

In[∘]:=
```
Integrate[-(Sin[θ]^2 * Cos[θ]^2)/(2 π), {θ, 0, 2 π}]
```

Out[∘]=

$$-\frac{1}{8}$$

This means that the first averaged equation is given by $r'(\tau) = \frac{-1}{8} r^3$. To solve for $r(\tau)$, we take:

In[∘]:=
```
DSolve[r'[τ] + r[τ]^3/8 == 0, r[τ], τ]
```

Out[∘]=

$$\left\{\left\{r[\tau] \to -\frac{2}{\sqrt{\tau - 8 c_1}}\right\}, \left\{r[\tau] \to \frac{2}{\sqrt{\tau - 8 c_1}}\right\}\right\}$$

Now we compute the second averaged equation given by:

In[∘]:=
```
Integrate[-(Sin[θ] * Cos[θ]^3)/(2 π), {θ, 0, 2 π}]
```

Out[∘]=

0

This means that the second averaged equation is given by $r(\tau)\, \phi'(\tau) = 0 \Longrightarrow \phi(\tau) = \phi_0 \in \mathrm{R}$. Taking both of theses together, we can solve for $A_0(\tau)$, $B_0(\tau)$. First we see that:

$$A_0(\tau) = r(\tau) \cos(\phi_0) = \frac{2}{\sqrt{\tau - 8 c_1}} \cos(\phi_0) = \frac{2}{\sqrt{\tau - 8 c_1}} D$$

We also see that:

$$B_0(\tau) = r(\tau) \sin(\phi_0) = \frac{2}{\sqrt{\tau - 8 c_1}} \sin(\phi_0) = \frac{2}{\sqrt{\tau - 8 c_1}} \sqrt{1 - D^2}\,.$$

From here, we see that to satisfy the conditions of $A_0(0) = 1$, $B_0(0) = 0$, we have: $\frac{2}{\sqrt{-8 c_1}} D = 1$ and $\frac{2}{\sqrt{-8 c_1}} \sqrt{1 - D^2} = 0$. In order to have a solution for c_1, we can satisfy the second condition by taking $D = 1 \Longrightarrow c_1 = \frac{-1}{2}$. This means the leading behavior is given by:

$Y_0(t, \tau) = \frac{2}{\sqrt{\tau + 4}} \cos(t).$

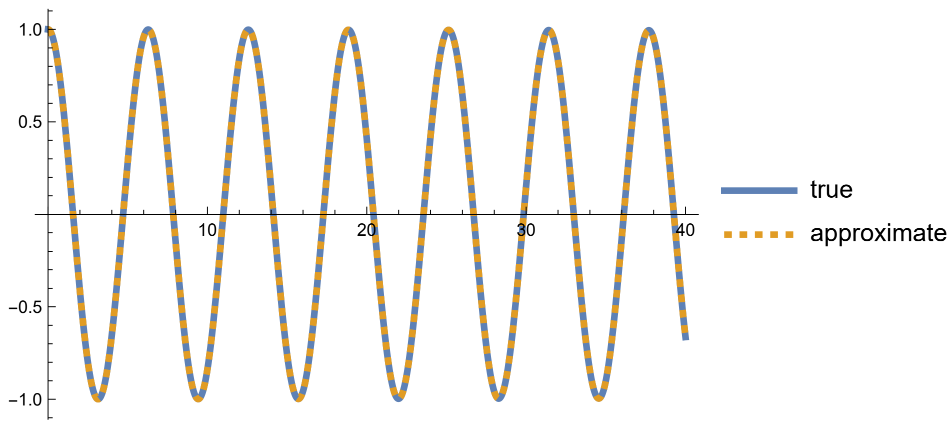
To validate, we see:

In[5]:=
```
truesol =
 NDSolveValue[{y''[t] + y[t] + 0.001 * y[t]^2 * y'[t] == 0, y[0] == 1, y'[0] == 0}, y, {t, 0, 40}]
```

Out[5]= InterpolatingFunction[Domain: {{0., 40.}} Output: scalar]

In[7]:=
```
Plot[{truesol[t], 2/Sqrt[0.001 * t + 4] Cos[t]}, {t, 0, 40},
 PlotStyle -> {Directive[Solid, Thickness[0.01]], Directive[Dashed, Thickness[0.01]]},
 PlotLegends -> {"true", "approximate"}, PlotRange -> All]
```

Out[7]=



The limiting behavior then shows that our approximation does in fact decay to 0:

In[8]:=
```
Limit[2 Cos[t]/Sqrt[4 + t * 0.001], t -> ∞]
```

Out[8]= 0.