Math 223: Homework 4

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Problem 1

$x\,y''' = y'$

Let $y = e^{S(x)}$. We can perform the substitution and solve the ODE in Mathematica as:

In[∘]:=
```
x * y'''[x] == y'[x] /. y → Function[x, E^S[x]] // FullSimplify
```

Out[∘]=

$$e^{S[x]}\left(x\,S'[x]^3 + S'[x]\,(-1 + 3\,x\,S''[x]) + x\,S^{(3)}[x]\right) == 0$$

Let $S''(x) << (S'(x))^2 \Longrightarrow x\,S''(x)\,S'(x) << x\,(S'(x))^3$, and $x\,S'''(x) << x\,(S'(x))\,(S''(x))$. We then obtain the asymptotic relation that $x\,(S'(x))^3 \sim S'(x)$. We can now find the control factors with:

In[∘]:=
```
DSolve[x * (S'[x])^3 == S'[x], S[x], x]
```

Out[∘]=

$$\{\{S[x] \to c_1\}, \{S[x] \to -2\sqrt{x} + c_1\}, \{S[x] \to 2\sqrt{x} + c_1\}\}$$

To verify our assumptions, we see that:

In[∘]:=

$$\left\{\text{Limit}\left[\frac{(D[-2\sqrt{x}, \{x, 2\}])}{D[-2\sqrt{x}, \{x, 1\}]^2}, x \to \text{Infinity}\right], \text{Limit}\left[\frac{(D[2\sqrt{x}, \{x, 2\}])}{D[2\sqrt{x}, \{x, 1\}]^2}, x \to \text{Infinity}\right]\right\}$$

Out[∘]=

$$\{0, 0\}$$

Using these control factors, we can add a correction term $c(x)$ to each $S(x)$ in order to solve for the leading behavior, for $c(x) << S(x)$ as $x \to \infty$. For the first control factor, this yields:

In[∘]:=
```
Solve[x * y'''[x] == y'[x] /. y → Function[x, E^(2 √x + c[x])], c'''[x]] //
  FullSimplify // Expand
```

Out[∘]=

$$\left\{\left\{c^{(3)}[x] \to -\frac{3}{4\,x^{5/2}} + \frac{3}{2\,x^2} + \frac{3\,c'[x]}{2\,x^{3/2}} - \frac{2\,c'[x]}{x} - \frac{3\,c'[x]^2}{\sqrt{x}} - c'[x]^3 - \frac{3\,c''[x]}{\sqrt{x}} - 3\,c'[x]\,c''[x]\right\}\right\}$$

By taking $c(x) << 2\sqrt{x}$, $c'(x) << x^{\frac{-1}{2}}$, $c''(x) << \frac{-1}{2}x^{\frac{-3}{2}}$, $c'''(x) << \frac{3}{4}x^{\frac{-5}{2}}$, we see the equation above asymptotically is equivalent to:

$-2x^{\frac{3}{2}} c'(x) \sim \frac{3}{2}\sqrt{x} \Longrightarrow c(x) \sim \frac{3}{4} ln(x).$

We verify the assumptions imposed on c(x) by

In[ ]:=
```
Limit[(3 Log[x]/4)/(2 Sqrt[x]), x → Infinity]
```

Out[ ]=
```
0
```

This means we have that $y_1(x) \sim x^{\frac{3}{4}} e^{2\sqrt{x}}$ as $x \to \infty$.

For the second control factor, we repeat to obtain:

In[ ]:=
```
Solve[x * y'''[x] == y'[x] /. y → Function[x, E^(-2 Sqrt[x] + c[x])], c'''[x]] //
  FullSimplify // Expand
```

Out[ ]=

$$\left\{\left\{c^{(3)}[x] \to \frac{3}{4x^{5/2}} + \frac{3}{2x^2} - \frac{3c'[x]}{2x^{3/2}} - \frac{2c'[x]}{x} + \frac{3c'[x]^2}{\sqrt{x}} - c'[x]^3 + \frac{3c''[x]}{\sqrt{x}} - 3c'[x]\,c''[x]\right\}\right\}$$

By taking $c(x) << -2\sqrt{x}$, $c'(x) << -x^{\frac{-1}{2}}$, $c''(x) << \frac{+1}{2} x^{\frac{-3}{2}}$, $c'''(x) << \frac{-3}{4} x^{\frac{-5}{2}}$, we see the equation above asymptotically is equivalent to:

$2x^{\frac{3}{2}} c'(x) \sim \frac{3}{2}\sqrt{x} \Longrightarrow c(x) \sim \frac{-3}{4} ln(x)$

We verify the assumptions imposed on c(x) by:

In[ ]:=
```
Limit[(3 Log[x]/4)/(2 Sqrt[x]), x → Infinity]
```

Out[ ]=
```
0
```

This means we have that $y_2(x) \sim x^{\frac{-3}{4}} e^{-2\sqrt{x}}$ as $x \to \infty$.

Problem 2

$y'' = \sqrt{x}\, y$

Let $y = e^{S(x)}$. We can perform the substitution and solve the ODE in Mathematica as:

In[ ]:=
```
y''[x] == Sqrt[x] * y[x] /. y → Function[x, E^S[x]] // FullSimplify
```

Out[ ]=

$$e^{S[x]}\left(-\sqrt{x} + S'[x]^2 + S''[x]\right) == 0$$

Let $S''(x) << (S'(x))^2$. We then obtain the asymptotic relation $S'(x)^2 \sim \sqrt{x}$. We can now find the control factors:

In[∘]:=
```
DSolve[S'[x]^2 == Sqrt[x], S[x], x]
```
Out[∘]=

$$\left\{\left\{S[x] \to -\frac{4\,x^{5/4}}{5} + \mathbb{c}_1\right\}, \left\{S[x] \to \frac{4\,x^{5/4}}{5} + \mathbb{c}_1\right\}\right\}$$

To verify our assumptions, we see that:

In[∘]:=
$$\left\{\text{Limit}\left[\frac{\left(D\left[-\frac{4\,x^{5/4}}{5}, \{x, 2\}\right]\right)}{D\left[-\frac{4\,x^{5/4}}{5}, \{x, 1\}\right]^2}, x \to \text{Infinity}\right], \text{Limit}\left[\frac{\left(D\left[\frac{4\,x^{5/4}}{5}, \{x, 2\}\right]\right)}{D\left[\frac{4\,x^{5/4}}{5}, \{x, 1\}\right]^2}, x \to \text{Infinity}\right]\right\}$$

Out[∘]=

{0, 0}

Using these control factors, we can add a correction term $c(x)$ to each $S(x)$ in order to solve for the leading behavior, for $c(x) << S(x)$ as $x \to \infty$. For the first control factor, this yields:

In[∘]:=
$$\text{Solve}\left[y''[x] == \text{Sqrt}[x] * y[x] \text{ /. } y \to \text{Function}\left[x, E^{\wedge}\left(\frac{4\,x^{5/4}}{5} + c[x]\right)\right], c''[x]\right] \text{ //}$$
```
  FullSimplify // Expand
```
Out[∘]=

$$\left\{\left\{c''[x] \to -\frac{1}{4\,x^{3/4}} - 2\,x^{1/4}\,c'[x] - c'[x]^2\right\}\right\}$$

By taking $c(x) << \frac{4}{5}x^{\frac{5}{4}}$, $c'(x) << x^{\frac{1}{4}}$, $c'''(x) << \frac{1}{4}x^{\frac{-3}{4}}$, we see the equation above is asymptotically equivalent to:

$2\,x\,c'(x) \sim \frac{-1}{4} \Longrightarrow c(x) \sim \frac{-1}{8}\,l\,n(x)$.

We verify the assumptions imposed on $c(x)$ by:

In[∘]:=
$$\text{Limit}\left[\frac{\left(\frac{-\text{Log}[x]}{8}\right)}{4/5 * x^{\wedge}(5/4)}, x \to \text{Infinity}\right]$$

Out[∘]=

0

This means we have that $y_1(x) \sim x^{\frac{-1}{8}}\,e^{\frac{4}{5}x^{\frac{5}{4}}}$ as $x \to \infty$.

For the second control factor, we repeat as:

In[∘]:=
$$\text{Solve}\left[y''[x] == \text{Sqrt}[x] * y[x] \text{ /. } y \to \text{Function}\left[x, E^{\wedge}\left(\frac{-4\,x^{5/4}}{5} + c[x]\right)\right], c''[x]\right] \text{ //}$$
```
  FullSimplify // Expand
```
Out[∘]=

$$\left\{\left\{c''[x] \to \frac{1}{4\,x^{3/4}} + 2\,x^{1/4}\,c'[x] - c'[x]^2\right\}\right\}$$

By taking $c(x) << \frac{-4}{5}x^{\frac{5}{4}}$, $c'(x) << -x^{\frac{1}{4}}$, $c'''(x) << \frac{-1}{4}x^{\frac{-3}{4}}$, we see the equation above is asymptotically equivalent to:

$$2\,x\,c'(x) \sim \frac{-1}{4} \Longrightarrow c(x) \sim \frac{-1}{8}\, l\,n(x).$$

We verify the assumptions imposed on $c(x)$ by:

```
Limit[(Log[x]/8)/(- 4 / 5 * x^(5 / 4)), x → Infinity] S
```

Out[∘]=

```
0
```

This means we have that $y_2(x) \sim x^{\frac{-1}{8}}\, e^{-\frac{4}{5}x^{\frac{5}{4}}}$ as $x \to \infty$.

Problem 3

$$y'' = e^{\frac{-3}{x}}\, y$$

Let $y = e^{S(x)}$. We can perform the substitution and solve the ODE in Mathematica as:

In[∘]:=
```
y''[x] == E^(-3 / x) * y[x] /. y → Function[x, E^S[x]] // FullSimplify
```

Out[∘]=

$$e^{S[x]}\,\left(S'[x]^2 + S''[x]\right) == e^{-\frac{3}{x}+S[x]}$$

Let $S''(x) < < (S'(x))^2$. We then obtain the asymptotic relation $S'(x)^2 \sim e^{\frac{-3}{x}}$. We can now find the control factors:

In[5]:=
```
DSolve[S'[x]^2 == E^(-3 / x), S[x], x]
```

Out[5]=

$$\left\{\left\{S[x] \to -e^{-\frac{3}{2}/x}\,x + c_1 - \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right]\right\}, \left\{S[x] \to e^{-\frac{3}{2}/x}\,x + c_1 + \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right]\right\}\right\}$$

To verify our assumptions, we see that:

In[∘]:=

$$\left\{\text{Limit}\left[\frac{\left(D\left[-e^{-\frac{3}{2}/x}\,x + c_1 - \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right], \{x, 2\}\right]\right)}{D\left[-e^{-\frac{3}{2}/x}\,x + c_1 - \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right], \{x, 1\}\right]^2}, x \to \text{Infinity}\right],\right.$$

$$\left.\text{Limit}\left[\frac{\left(D\left[e^{-\frac{3}{2}/x}\,x + c_1 + \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right], \{x, 2\}\right]\right)}{D\left[e^{-\frac{3}{2}/x}\,x + c_1 + \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right], \{x, 1\}\right]^2}, x \to \text{Infinity}\right]\right\}$$

Out[∘]=

```
{0, 0}
```

Using these control factors, we can add a correction term $c(x)$ to each $S(x)$ in order to solve for the leading behavior, for $c(x) < < S(x)$ as $x \to \infty$. For the first control factor, this yields:

In[]:=
```
Solve[y''[x] == E^(-3/x) * y[x] /.
   y → Function[x, E^(e^(-3/2/x) x + 3/2 ExpIntegralEi[-3/(2 x)] + c[x])],
  c''[x]] // FullSimplify // Expand
```

Out[]=

$$\left\{\left\{c''[x] \to -\frac{3\,e^{-\frac{3}{2}/x}}{2\,x^2} - 2\,e^{-\frac{3}{2}/x}\,c'[x] - c'[x]^2\right\}\right\}$$

In[]:=
```
D[e^(-3/2/x) x + 3/2 ExpIntegralEi[-3/(2 x)], {x, 1}] // FullSimplify
```

Out[]=

$$e^{-\frac{3}{2}/x}$$

In[]:=
```
D[e^(-3/2/x) x + 3/2 ExpIntegralEi[-3/(2 x)], {x, 2}] // FullSimplify
```

Out[]=

$$\frac{3\,e^{-\frac{3}{2}/x}}{2\,x^2}$$

By taking $c(x) << e^{-\frac{3}{2}/x}\,x + \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right]$ and using the two derivative expressions above to yield $c'(x) << e^{-\frac{3}{2}/x}$ and $c''(x) << \frac{3\,e^{-\frac{3}{2}/x}}{2\,x^2}$, we see the equation above is asymptotically equivalent to: $-4\,x^2\,c'(x) \sim 3 \Longrightarrow c(x) = \frac{3}{4\,x}$.

We verify the assumptions imposed on $c(x)$ by:

In[]:=
```
Limit[(3/(4*x)) / (e^(-3/2/x) x + 3/2 ExpIntegralEi[-3/(2 x)]), x → Infinity]
```

Out[]=
```
0
```

This means we have that $y_1(x) \sim e^\wedge\left(e^{-\frac{3}{2}/x}\,x + \frac{3}{2}\,\text{ExpIntegralEi}\left[-\frac{3}{2\,x}\right] + 3/4\,x\right)$

We repeat the same for the second control factor:

In[]:=
```
Solve[y''[x] == E^(-3/x) * y[x] /.
   y → Function[x, E^(-e^(-3/2/x) x + 3/2 ExpIntegralEi[-3/(2 x)] + c[x])],
  c''[x]] // FullSimplify // Expand
```

Out[]=

$$\left\{\left\{c''[x] \to \frac{9\,e^{-\frac{3}{2}/x}}{2\,x^3} - \frac{9\,e^{-3/x}}{x^2} - \frac{3\,e^{-\frac{3}{2}/x}}{2\,x^2} - \frac{6\,e^{-3/x}}{x} + 2\,e^{-\frac{3}{2}/x}\,c'[x] + \frac{6\,e^{-\frac{3}{2}/x}\,c'[x]}{x} - c'[x]^2\right\}\right\}$$

We now solve for $S''(x)$ using Solve.

In[6]:=
```
Solve[S'[x]^2 == E^(-3/x), S'[x]]
```
Out[6]= $\left\{\left\{S'[x] \to -e^{-\frac{3}{2}/x}\right\}, \left\{S'[x] \to e^{-\frac{3}{2}/x}\right\}\right\}$

From here, we can take the series expansion of our control factors :

In[8]:= Series$\left[-e^{-\frac{3}{2}/x}, \{x, \text{Infinity}, 3\}\right]$

Out[8]= $-1 + \frac{3}{2x} - \frac{9}{8x^2} + \frac{9}{16x^3} + O\left[\frac{1}{x}\right]^4$

In[9]:= Series$\left[e^{-\frac{3}{2}/x}, \{x, \text{Infinity}, 3\}\right]$

Out[9]= $1 - \frac{3}{2x} + \frac{9}{8x^2} - \frac{9}{16x^3} + O\left[\frac{1}{x}\right]^4$

We can now integrate the dominant term to yield our two control factors of $S(x) = x, -x$ as $x \to \infty$. Using these control factors, we can add a correction term $c(x)$ to each $S(x)$ in order to solve for the leading behavior, for $c(x) << S(x)$ as $x \to \infty$. For the first control factor, this yields:

In[14]:=
```
y''[x] == E^(-3/x) * y[x] /. y → Function[x, E^(x + c[x])] // FullSimplify
```
Out[14]=

$e^{x+c[x]} \left(1 - e^{-3/x} + c'[x] (2 + c'[x]) + c''[x]\right) == 0$

Taking derivatives, we see that we have $c'(x) << 1$, $c''(x) << 0$. To deal with the exponential term, we once again take the power series to see that:

In[15]:= Series$\left[1 - e^{-3/x}, \{x, \text{Infinity}, 3\}\right]$

Out[15]=

$\frac{3}{x} - \frac{9}{2x^2} + \frac{9}{2x^3} + O\left[\frac{1}{x}\right]^4$

Using these and the asymptotic relations above, we have that:
$c'(x) \sim \frac{-3}{2x} \Longrightarrow c(x) \sim \frac{-3}{2} \, ln(x)$. This means we have $y_1(x) \sim x^{\frac{-3}{2}} e^x$.
Repeating the same for the other control factor, we see that:

In[18]:=
```
y''[x] == E^(-3/x) * y[x] /. y → Function[x, E^(-x + c[x])] // FullSimplify
```
Out[18]=

$e^{-x+c[x]} \left(1 - e^{-3/x} + (-2 + c'[x]) c'[x] + c''[x]\right) == 0$

Taking derivatives, we see that we have $c'(x) << -1$, $c''(x) << 0$. To deal with the exponential term, we once again take the power series to see that:

In[19]:= Series$\left[1 - e^{-3/x}, \{x, \text{Infinity}, 3\}\right]$

Out[19]=

$\frac{3}{x} - \frac{9}{2x^2} + \frac{9}{2x^3} + O\left[\frac{1}{x}\right]^4$

Using these and the asymptotic relations above, we have that:

$c'(x) \sim \frac{3}{2x} \Longrightarrow c(x) \sim \frac{3}{2} \, ln(x)$. This means that we have $y_2(x) \sim x^{\frac{3}{2}} e^{-x}$.

To verify the assumptions we made we see that:

In[17]:=
```
{Limit[(D[x, {x, 2}])/D[x, {x, 1}]^2, x → Infinity], Limit[(D[x, {x, 2}])/D[x, {x, 1}]^2, x → Infinity]}
```

Out[17]=
```
{0, 0}
```