Objectives

Practice computing uniformly valid asymptotic approximations using boundary layer theory and the method of matched asymptotics.

Instructions

Use Mathematica to solve the following problems. Use the template introduced for the previous homework. Write up discussions of your results.

Problems

1. (Bender & Orszag, Problem 9.8) Use boundary-layer theory to find a uniform approximation with error of order ϵ^2 for the problem

$$\epsilon y'' + y' + y = 0, \quad 0 \le x \le 1, \quad 0 < \epsilon \ll 1$$
$$y(0) = e, \quad y(1) = 1.$$

Notice that there is no boundary layer in leading order, but one does appear in next order. Compare your solution with the exact solution to this problem.

2. (Bender & Orszag, Problem 9.8) Use boundary-layer methods to find an approximate solution to the *initial-value problem*

$$\epsilon y'' + a(x)y' + b(x)y = 0, \quad x > 0, \quad a(x) > 0, \quad 0 < \epsilon \ll 1$$
$$y(0) = 1, \quad y'(0) = 1,$$

Show that the leading-order uniform approximation satisfies $y(0) = 1$, but not $y'(0) = 1$ for arbitrary b. Compare the leading-order uniform approximation with the exact solution to the problem when $a(x)$ and $b(x)$ are constants.

3. (Hinch, Exercise 5.1) The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + (1+\epsilon)y' + y = 0, \quad 0 \le x \le 1,$$

and is subject to boundary conditions $y(0) = 0$ and $y(1) = e^{-1}$. Find two terms in the outer approximation, applying only the boundary condition at $x = 1$. Next find two terms in the inner approximation for the boundary layer near $x = 0$, which can be assumed to have width $O(\epsilon)$, and applying only the boundary condition at $x = 0$. Finally determine the constants of integration in the inner approximation by matching.