Pen and paper

1. You are given m data points $\{(x_1,y_1),(x_2,y_2),\cdots,(x_m,y_m)\}$ and asked to perform a least-squares fitting of those points to a polynomial of order p, $a_p x^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0$. Construct the normal equation $V^T V\mathbf{a} = V^T\mathbf{y}$ by specifying the Vandemonde matrix V and vectors $\mathbf{y}$ and $\mathbf{a}$. (You will use this in MATLAB Problem 1.)

2. For $A \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, consider a least-squares problem $A\mathbf{x} = \mathbf{b}$. The squared residual error is defined as $E(\mathbf{x}) = \|\mathbf{b} - A\mathbf{x}\|^2$. The least-squares solution minimizing $E(\mathbf{x})$ is denoted by $\mathbf{x}^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$. The column vectors of A are denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_n \in \mathbb{R}^m$, i.e., $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$.
   (a) Using the condition that $\partial E/\partial x_i = 0$ for $\mathbf{x} = \mathbf{x}^*$, show
   $$(\mathbf{a}_i, \mathbf{b}) = \sum_{j=1}^{n} (\mathbf{a}_i, \mathbf{a}_j) x_j^*. \tag{1}$$
   (b) Show that Eq. (1) is equivalent to $A^T\mathbf{b} = A^T A\mathbf{x}^*$
   (c) Show that $\mathbf{x}^*$ satisfying Eq. (1) indeed minimizes E, that is,
   $$E(\mathbf{x}) \geq \|\mathbf{b} - A\mathbf{x}^*\|^2 \qquad \text{for any } \mathbf{x} \in \mathbb{R}^n. \tag{2}$$

3. Consider the space of polynomial functions on $[-1,1]$ equipped with inner product
   $$(p,q) = \int_{-1}^{1} p(x)q(x)dx. \tag{3}$$
   (a) Show that the following three polynomials are orthogonal:
   $$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{4}$$
   (b) Argue that any quadratic polynomial can be expressed as a linear combination of $P_0(x)$, $P_1(x)$, and $P_2(x)$.
   (c) Find an orthonormal basis $\{\phi_0(x), \phi_1(x), \phi_2(x)\}$ for the linear space of polynomial functions of order 2 or less.
   (d) Find an orthonormal basis $\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x)\}$ for the linear space of polynomial functions of order 3 or less.

MATLAB

1. (a) Write a MATLAB function

    ```
    function a = lsq(x,y,p)
    ```

    that finds the least-squares fit for the data point $\{(x_1,y_1),(x_2,y_2),\cdots,(x_m,y_m)\}$. The return value is the polynomial in MATLAB format; in other words, if the polynomial is

    $$a_p x^p + a_{p-1}x^{p-1} + \cdots + a_2 x^2 + a_1 x + a_0, \tag{5}$$

    then $\texttt{a} = [a_p, a_{p-1}, \ldots, a_2, a_1, a_0]$.

   (b) Set up a simple polynomial fitting problem to demonstrate that the code works properly.

2. We want to observe the convergence of the following function in the limit $n \to \infty$:

    $$f_n(x) = \sum_{i=1}^{n} \frac{\sin((2i-1)x)}{(2i-1)} \tag{6}$$

   (a) Write a MATLAB function `sinesum(n,x)` that computes the value of $f_n(x)$.

   (b) Explain what the following command does.

    ```
    yvals=arrayfun(@(x) sinesum(n,x),xvals)
    ```

   (c) Plot $f_n(x)$ $(-2\pi \le x \le 2\pi)$ for $n = 5$, 10, 15.

   (d) Discuss the convergence behavior of $f_n(x)$.