1. [5 pt] Using the Jacobi iteration method, we want to solve a linear system $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{bmatrix} 5 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & -2 & -6 \end{bmatrix}. \tag{1}$$

   (a) Calculate the iteration matrix B_{J}.

   (b) By calculating $\|B_{\mathrm{J}}\|_\infty$, show that the Jacobi method converges for any vector $\mathbf{b}$.

2. [5 pt] Suppose you are solving a linear system $A\mathbf{x} = \mathbf{b}$ using an iterative method $\mathbf{x}^{(k)} = B\mathbf{x}^{(k-1)} + \mathbf{c}$ and the spectral radius of B is found to be $\frac{1}{2}$. If the relative residual error was estimated to be 0.01 after 5 iterations, how many additional iterations will be needed for the relative residual error to be smaller than 10^{-8}? (Use $2^{10} \approx 10^3$)

3. [10 pt] We want to solve the following 2×2 system using the Jacobi and Gauss-Seidel iterations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{2}$$

   Here we assume that a, b, c, d, u, and v are real numbers and $a \neq 0$, $d \neq 0$.

   (a) Compute the iteration matrices B_{J} and B_{GS}.

   (b) Compute the spectral radii of B_{J} and B_{GS}.

   (c) Show that the Jacobi and Gauss-Seidel both converge or both diverge.

4. [10 pt] Consider the successive overrelaxation (SOR) method to solve $A\mathbf{x} = \mathbf{b}$. Whenever needed, use notations $A = (a_{ij})$, $\mathbf{b} = (b_i)$, and $A = L + D + U$.

   (a) Write down the equation that shows how each $x_i^{(k)}$ is updated.

   (b) Write down a pseudo code (or MATLAB code) for the method.

   (c) Derive the matrix representation $\mathbf{x}^{(k)} = B_{\mathrm{SOR},\omega}\mathbf{x}^{(k-1)} + \mathbf{c}_{\mathrm{SOR},\omega}$.

   (d) For the SOR method, state either a necessary condition (about ω) or a sufficient condition (about special matrices) for convergence.

5. [5 pt] For a nonsingular matrix A and a nonzero vector $\mathbf{b}$, let $\mathbf{x}$ be the solution to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{x} + \delta\mathbf{x}$ be the solution to the perturbed system $A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$. Show the following inequality:

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}. \tag{3}$$

6. [5 pt] Choose one of the following statements and prove it.
   - For an induce norm $\|\cdot\|$, $\|AB\| \leq \|A\|\|B\|$.

- If $\rho(B) < 1$, $I - B$ is invertible.
- For $\|B\| < 1$,

$$\|(I-B)^{-1}\| \leq \frac{1}{1-\|B\|}. \tag{4}$$