If you cannot join our Zoom session and share your camera for any reasons,
you must let the instructor know. If you haven't done so, you are not allowed to start until you send an email to ckim103@ucmerced.edu.

If you have any internet connection issue and are disconnected during the exam,
please continue to work on the exam and finish it. After uploading your exam to CatCourses, you will need to inform the instructor via email (ckim103@ucmerced.edu) that you have not acted outside the university academic integrity guidelines.

Exam timeline

- 11:55am – 12:05pm (10 minutes): Download the exam (this PDF file).
  - Read the instructions carefully.
  - Copy and sign the honor code.
  - You can read the exam but are NOT allowed to start it yet.
- 12:05pm – 1:05pm (60 minutes): Work on the exam.
- 1:05pm – 1:20pm (15 minutes): Upload your finished exam to CatCourses.
  - If you finish early, you can proceed to upload it before the time.

Read the following instructions carefully

- This exam lasts 60 minutes.
- The total exam score is 40 points = 3 pts $\times$ 5 problems (Part I) + 5 pts $\times$ 5 problems (Part II). There are ***2 extra points*** for tasks in framed boxes. Your score will be converted using the formula min(your score, 30 pts).
- You must write your solution on paper (i.e. not on an electronic device) and upload a PDF copy of your exam to CatCourse.

[extra 1 pt] On the first page, copy the following Academic Integrity Statement and **sign** it:

By completing this exam, I acknowledge and confirm that I will not give or receive any unauthorized assistance on this exam. I will conduct myself within the university academic integrity guidelines.

Part I

(5 problems $\times$ 3 pts)

Determine whether each statement is true or false. Justify your answer.

1. For a given induced norm $\|\cdot\|$, the geometric series of a matrix B (i.e. $I+B+B^2+B^3+\cdots$) converges if and only if $\|B\|<1$.

2. An ill-conditioned problem is caused by an unstable algorithm.

3. Both Gauss-Seidel and successive over-relaxation methods can be implemented with a single solution vector for update.

4. For $A\mathbf{x}=\mathbf{b}$ with A being a strictly diagonally dominant matrix, both Jacobi and Gauss-Seidel methods always converge.

5. When an iteration $\mathbf{x}^{(k+1)}=B\mathbf{x}^{(k)}+\mathbf{c}$ converges to the exact solution $\mathbf{x}^*$ for any choice of $\mathbf{x}^{(0)}$, the error vector $\mathbf{e}^{(k)}=\mathbf{x}^{(k)}-\mathbf{x}^*$ satisfies $\mathbf{e}^{(k)}=B^k\mathbf{e}^{(k-1)}$.

Part II

(5 problems $\times$ 5 pts)

1. [5 pts] Using the Jacobi iteration method, we want to solve a linear system $A\mathbf{x}=\mathbf{b}$:

$$A=\begin{bmatrix} 5 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & -2 & -6 \end{bmatrix}. \tag{1}$$

   (a) Calculate the iteration matrix B_{J}.

   (b) By calculating $\|B_{\mathrm{J}}\|_\infty$, show that the Jacobi method converges for any vector $\mathbf{b}$.

2. [5 pts] Suppose you are solving a linear system using an iterative method and the relative residual error decreases as described in the table. Determine the iteration number k in the table.

iteration number	relative error
5	10^{-2}
8	10^{-4}
k	10^{-8}

3. [5 pts] We want to solve the following 2×2 system using the Gauss-Seidel method:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{2}$$

Here we assume that a, b, c, d, u, and v are real numbers and $a \neq 0$, $d \neq 0$.

(a) Compute the iteration matrix B_{GS}.

(b) Compute the 1-norm $\|B_{\text{GS}}\|_1$.

(c) Compute the spectral radius $\rho(B_{\text{GS}})$.

(d) Using the expressions obtained in (b) and (c), find an example of A for which the Gauss-Seidel method converges but $\|B_{\text{GS}}\|_1 \geq 1$.

4. [5 pts] Write down a convergence theorem that can be implied from the following theorem. Briefly justify your convergence theorem.

[Kahan] *When the successive over-relaxation (SOR) method is applied to* $A\mathbf{x} = \mathbf{b}$ where A is a square matrix with nonzero diagonal elements, the spectral radius of its iteration matrix $B_{\text{SOR},\omega}$ satisfies

$$\rho(B_{\text{SOR},\omega}) \geq |\omega - 1|. \tag{3}$$

5. [5 pts] Choose one of the following statements for an induced norm $\|\cdot\|$ and the spectral radius $\rho(\cdot)$ of a matrix and prove it.

- $\rho(B) \leq \|B\|$.
- If $B^k \to 0$ as $k \to \infty$, $\rho(B) < 1$.
- For $\|B\| < 1$,

$$\|(I - B)^{-1}\| \leq \frac{1}{1 - \|B\|}. \tag{4}$$

[extra 1 pt] Write down the time that you finish your exam.

- Proceed to scan (or take clear pictures of) your exam and upload it to CatCourses.
- Include your cheat sheet.
- If, for any reason, you failed to share your camera during the full period of the exam (e.g. disconnection), you must inform the instructor via email that you have not acted outside the university academic integrity guidelines.