Pen and paper

1. Consider the following 2π-periodic function

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0, \\ x & \text{for } 0 \le x < \pi. \end{cases} \tag{1}$$

   (a) Compute the Fourier series of $f(x)$. You may use MATLAB (or a symbolic computation package such as Mathematica and Maple) to compute integrals.

   (b) Write down the function $\tilde{f}(x)$ to which the partial sum $S_N(x)$ converges pointwise.

   (c) Does $S_N(x)$ converge to $f(x)$ (i) pointwise, (ii) in L^2 norm, and/or (iii) uniformly?

   (d) Sketch a typical plot of $S_N(x)$ for a large value of N and explain Gibbs phenomenon.

   (e) Derive Parseval's identity for $f(x)$ to obtain

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} + \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{5}{24}\pi^2. \tag{2}$$

2. (a) Using the 3rd-order Legendre polynomial $P_3(x) = \frac{1}{2}(5x^3 - 3x)$, construct a Gaussian quadrature

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i) \quad \text{where } w_i = \frac{2}{(1-x_i^2)[P_n'(x_i)]^2}. \tag{3}$$

   In other words, find x_i and w_i.

   (b) Let $f(x) = x^k(x^2+2)+1$. For $k = 1,2,3,4$, compute the approximate value of the integral $\int_{-1}^{1} f(x)dx$ using the quadrature rule derived.

   (c) For which values of k does the quadrature give the exact value?

3. Consider the following Hermite–Gaussian quadrature expression

$$\int_{-\infty}^{\infty} f(x)e^{-x^2} \approx \sum_{i=1}^{3} w_i f(x_i), \tag{4}$$

   where

$$x_1 = -\sqrt{\frac{3}{2}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{2}}, \quad w_1 = \frac{\sqrt{\pi}}{6}, \quad w_2 = \frac{2\sqrt{\pi}}{3}, \quad w_3 = \frac{\sqrt{\pi}}{6}. \tag{5}$$

   (a) Estimate the following integral using the quadrature

$$\int_{-\infty}^{\infty} (x^2+3)e^{-x^2}dx. \tag{6}$$

   (b) Estimate the following integral using the quadrature

$$\int_{-\infty}^{\infty} (x^2+3)e^{-x^2/2}dx. \tag{7}$$

   (c) Explain how to obtain x_i and w_i from $H_3(x) = 8x^3 - 12x$.

   (d) State when (4) becomes exact.

MATLAB

1. (a) Write a MATLAB function
   `function intg = trapezoid(f,a,b,n)`
   that computes $\int_a^b f(x)dx$ using the trapezoid method with $x_i = a + \frac{i}{n}(b-a)$ $(i = 0, 1, \cdots, n)$.

   (b) Using $f(x) = e^{-x}$ with $a = -1$ and $b = 1$, show that the order of convergence is n^{-2}. In other words, in the log-log plot of the errors versus n, the slope should be 2.

2. (a) Write a MATLAB function
   `function intg = LegendreGauss(f,a,b,n)`
   that computes $\int_a^b f(x)dx$ using the Legendre–Gauss quadrature with the nth-order polynomial. Use MATLAB function `lgwt` to obtain the quadrature points and weights.

   (b) Using $f(x) = e^{-x}$ with $a = -1$ and $b = 1$, observe the spectral convergence.

3. For $f(x) = \sqrt{|x|}$ $(-1 \le x \le 1)$, show and discuss the convergence behavior of both method.