1. [8pt] You are given m data points $\{(x_1,y_1),(x_2,y_2),\cdots,(x_m,y_m)\}$. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of order n, the coefficients of which are to be determined.
   (a) Construct a linear system that determines the coefficients of $p(x)$ so that $p(x)$ interpolates the data points. (Do not worry about whether the system is underdetermined or overdetermined.)
   (b) Construct a linear system that determines the least-squares fit $p(x)$. When is the least-squares fit uniquely determined?

2. [8 pts] Consider a 2π-periodic function and its Fourier series:

$$f(x) = \begin{cases} 0 & \text{for } -\pi \le x \le 0, \\ x & \text{for } 0 \le x < \pi, \end{cases} \tag{1}$$

$$f(x) \sim \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \tag{2}$$

   (a) Write down the function $\tilde{f}(x)$ to which the partial sum $S_N(x)$ of the Fourier series converges pointwise.
   (b) Does $S_N(x)$ converge to $f(x)$ (i) pointwise, (ii) in L^2 norm, and/or (iii) uniformly?
   (c) Sketch a typical plot of $S_N(x)$ for a large value of N. What is the name of this phenomenon?

3. [8 pts] Consider the following Hermite–Gauss quadrature:

$$\int_{-\infty}^{\infty} f(x)e^{-x^2} \approx \sum_{i=1}^{3} w_i f(x_i), \tag{3}$$

   where $x_1 = -\sqrt{\frac{3}{2}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{2}}$, $w_1 = \frac{\sqrt{\pi}}{6}$, $w_2 = \frac{2\sqrt{\pi}}{3}$, $w_3 = \frac{\sqrt{\pi}}{6}$.
   (a) Estimate the integral

$$\int_{-\infty}^{\infty}(x^2+x)e^{-x^2}dx. \tag{4}$$

   (b) State when the quadrature (3) becomes exact.

4. [8 pts]
   (a) Write the Lagrange polynomial $F(x)$ that interpolates $f(x)$ at three points x_0 and $x_0 \pm \Delta x$.
   (b) Using $F(x)$, obtain the second-order central finite difference for $f''(x_0)$.

5. [8 pts] Using Simpson's rule

$$\int_{x_0}^{x_2} f(x)dx = \Delta x\left(\frac{1}{3}f_0 + \frac{4}{3}f_1 + \frac{1}{3}f_2\right) - \frac{1}{90}\Delta x^5 f^{(4)}(\xi), \tag{5}$$

   (a) construct its composite rule for $\int_a^b f(x)dx$.
   (b) Obtain an error bound for the composite rule.

6. [8 pts] Apply Richardson extrapolation technique to the backward finite difference of order Δx to obtain the backward finite difference of order Δx^2.

7. [8 pts] Write a MATLAB function

```
function [tvals,uvals] = Euler(F,t0,T,u0,n)
```

that computes the solution $u(t)$ ($t_0 \le t \le T$) to the initial value problem

$$u'(t) = F(t, u(t)), \quad u(t_0) = u_0. \tag{6}$$

using the forward Euler method. Note that `tvals` contains points $t_k = t_0 + k\Delta t$ ($k = 0, 1, \cdots, n$) with $\Delta t = (T - t_0)/n$ and `uvals` contains the values of the approximate solution at these points.

8. Consider the initial value problem (6).
   (a) For a one-step scheme $u_{k+1} = u_k + \Delta t A(t_k, u_k, \Delta t)$, define the local truncation error τ_{k+1} at timestep t_{k+1}.
   (b) For the forward Euler scheme, compute the local truncation error τ_{k+1}.

9. [8 pts] For the initial value problem (6),
   (a) write down the backward Euler scheme.
   (b) Write down the leapfrog scheme.
   (c) Write down the explicit trapezoid scheme.

10. [8 pts] Fill out the blanks.
    (a) For $f(x) \sim \dfrac{a_0}{2} + \displaystyle\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$, $b_n = \dfrac{\int_{-\pi}^{\pi} f(x) \sin nx\, dx}{\boxed{}}$

    (write an expression not a number)
    (b) $f''(x) = \dfrac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\boxed{}} + O\left(\boxed{}\right)$
    (c) $\displaystyle\int_{x_0}^{x_4} f(x)dx \approx \Delta x \left[\frac{8}{3} f(x_1) + \boxed{}\, f(x_2) + \frac{8}{3} f(x_3)\right]$ where $x_i = x_0 + i\Delta x$, $i = 1, 2, 3, 4$
    (d) The implicit midpoint scheme is a $\boxed{}$-order method. That is, the local truncation error is $O(\boxed{})$ and the global truncation error is $O(\boxed{})$.