# Pen and paper

1. Prove the following sufficient and necessary condition for convergence:

   *For iteration $\mathbf{x}^{(k)} = B\mathbf{x} + \mathbf{c}$ with $\mathbf{x}^{(0)} = \mathbf{x}_0$, $\mathbf{x}^{(k)}$ converges for any choice of $\mathbf{x}_0$ if and only if the spectral radius $\rho(B)$ of the iteration matrix $B$ is smaller than 1.*

   (a) (if statement)
   - Show $\mathbf{x}^{(k)} = B^k\mathbf{x}_0 + (I + B + B^2 + \cdots + B^{k-1})\mathbf{c}$.
   - Show $I - B$ is invertible. (Hint: $I - B$ does not have a zero eigenvalue.)
   - Show $I + B + B^2 + \cdots + B^{k-1} = (I - B)^{-1}(I - B^k)$.
   - Using $\rho(B) < 1 \Rightarrow B^k \to 0$ (you do not need to prove this), conclude the proof.

   (b) (only if statement)
   - Let $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^*$ with $\mathbf{x}^*$ being the convergent solution. Show $\mathbf{e}^{(k)} = B^k\mathbf{e}^{(0)} = B^k(\mathbf{x}_0 - \mathbf{x}^*)$.
   - Suppose $\rho(B) \geq 1$ and derive a contradiction. (Hint: find $\mathbf{x}_0$ such that $\mathbf{e}^{(k)}$ does not converge to $\mathbf{0}$.)

2. Assume that iteration $\mathbf{x}^{(k)} = B\mathbf{x}^{(k-1)} + \mathbf{c}$ with $\mathbf{x}^{(0)} = \mathbf{x}_0$ converges to $\mathbf{x}^*$ and $\|B\| < 1$ for an induced matrix norm. Prove the following inequality:

$$\|\mathbf{x}^{(k)} - \mathbf{x}^*\| \leq \frac{\|B\|^k}{1 - \|B\|}\|\mathbf{x}^{(1)} - \mathbf{x}_0\| \tag{1}$$

   (a) Show

$$\mathbf{x}^{(k)} - \mathbf{x}^* = -B^k(I - B)^{-1}\left(\mathbf{x}^{(1)} - \mathbf{x}_0\right). \tag{2}$$

   (b) By showing

$$\|(I - B)^{-1}\| \leq \frac{1}{1 - \|B\|}, \tag{3}$$

   conclude the proof.

3. [Extra Credit Problem] Using Stein–Rosenberg theorem below, argue that if $A = (a_{ij})$ is symmetric positive definite with non-positive off-diagonal elements, then both Jacobi and Gauss–Seidel iterations converge and Gauss–Seidel converges faster. Assume $\rho(B_{\mathrm{GS}}) > 0$.

   Theorem (Stein–Rosenberg) If $a_{ij} \leq 0$ for $i \neq j$ and if $a_{ii} > 0$ for each $i$, then one and only one of the following holds:

   (a) $0 \leq \rho(B_{\mathrm{GS}}) < \rho(B_{\mathrm{J}}) < 1$,

   (b) $1 < \rho(B_{\mathrm{J}}) < \rho(B_{\mathrm{GS}})$,

   (c) $\rho(B_{\mathrm{J}}) = \rho(B_{\mathrm{GS}}) = 0$,

   (d) $\rho(B_{\mathrm{J}}) = \rho(B_{\mathrm{GS}}) = 1$.

# MATLAB

1. Write the following MATLAB function for the SOR iteration method to solve a general $n \times n$ system `A*x=b`:

   ```
   function [final_sol,sols] = SOR(A,b,x0,niter,omega)
   ```

   Here, `omega` is the relaxation parameter $\omega$, `final_sol` is the iterative solution after `niter` iterations starting with initial guess `x0`, and `sols` is the sequence of iterative solutions $\{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(\texttt{niter})}\}$ ($n \times (\texttt{niter}+1)$ matrix).

2. Choose a $6 \times 6$ symmetric positive-definite matrix. For several values of $0 < \omega < 2$, compare convergence rate by plotting the semi-log plot of relative residual errors versus iteration number.

3. For the same matrix, write a script to plot the spectral radius $\rho(B_{\mathrm{SOR}})$ versus $0 < \omega < 2$. Locate the optimal value of $\omega$.