Math 231 Homework #11

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(1) (a) Using the derivations from lecture, we can compute the terms:

- $a_0 = \frac{1}{\pi}(1,f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x\,dx\right)$ by definition of $f(x)$.

$= \frac{1}{\pi}\left(0 + \frac{x^2}{2}\Big]_0^{\pi}\right)$

$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \boxed{\frac{\pi}{2}}$

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot\cancel{\cos(nx)}\,dx + \int_0^{\pi} x\cos(nx)\,dx\right)$

$= \frac{1}{\pi}\left(\int_0^{\pi} x\cos(nx)\,dx\right) \Rightarrow u = x \quad du = dx \qquad dv = \cos(nx)\,dx \quad v = \frac{\sin(nx)}{n}$

$\Rightarrow \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} - \int_0^{\pi}\frac{\sin(nx)}{n}\,dx\right] \qquad w = nx \Rightarrow dw = n\,dx$

$= \frac{1}{\pi}\left[\frac{x\sin(nx)}{n}\Big]_0^{\pi} - \frac{1}{n^2}\int_0^{\pi}\sin(w)\,dw\right)$

$= \frac{1}{\pi}\left[\frac{\pi n\sin(\pi n) + \cos(\pi n) - 1}{n^2}\right]$

$= \boxed{\begin{cases} \frac{-2}{\pi n^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}}$

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cancel{\sin(nx)}\,dx + \int_0^{\pi} x\sin(nx)\,dx$

$= \frac{1}{\pi}\left(\int_0^{\pi} x\sin(nx)\,dx\right) \Rightarrow u = x \quad du = dx \qquad dv = \sin(nx)\,dx \quad v = -\frac{\cos(nx)}{n}$

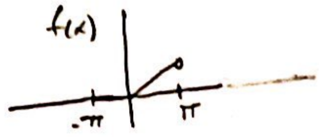
$\Rightarrow \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\Big]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right] \qquad w = nx \Rightarrow dw = n\,dx$

$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n}\Big]_0^{\pi} + \frac{1}{n^2}\int_0^{\pi}\cos(nx)\,dx\right]$

$= \frac{1}{\pi}\left[\frac{\sin(\pi n) - \pi n\cos(\pi n)}{n^2}\right]$

$= \boxed{\begin{cases} \frac{1}{n} & \text{for } n \text{ odd} \\ -\frac{1}{n} & \text{for } n \text{ even} \end{cases}}$

Hence $f(x) \simeq \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$ for the a_0, a_n, & b_n solved for above.



① (b) By definition, we know $\tilde{f}(x) = \frac{f(x^-) + f(x^+)}{2}$. We see our function f has a finite number of jump discontinuities since $f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ x & 0 \le x < \pi \end{cases}$ has a jump discontinuity at π as shown in the above diagram.

Hence, since $f(x^-)$ is defined to be 0, we see that:

$\hookrightarrow \tilde{f}(\pi) = \frac{f(x^-) + f(x^+)}{2}$ @ $x = \pi$ $= \frac{0 + \pi}{2} = \boxed{\frac{\pi}{2}}$

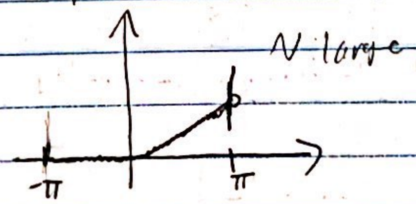
$$\boxed{\text{Hence, } \tilde{f}(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \pi/2 & x = \pi \\ x & 0 < x < \pi \end{cases} \text{ and } S_N(x) \to \tilde{f}(x)}$$

(c) (i) Since $\tilde{f}(x) \neq f(x)$, we say that $S_N(x) \not\to f(x)$ pointwise

$\boxed{S_N(x) \text{ does not converge to } f \text{ pointwise due to } S_N(x) \to \tilde{f}(x) \text{ pointwise and } \tilde{f}(x) \neq f(x)}$

(ii) $\boxed{S_N(x) \text{ does converge to } f \text{ in the sense of the } L^2 \text{ norm}}$, since we see that as $N \to \infty$, the area between f and S_N tends to 0, except at the end points. But since there are finitely many jump discontinuities present at the endpoints, they constitute a set of measure zero, hence do not impact the integral of $\|S_N - f\|_2^2 = \int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 dx \to 0$ as $N \to \infty$

A pictoral representation of this is shown below: (region of oscillation shrinks)

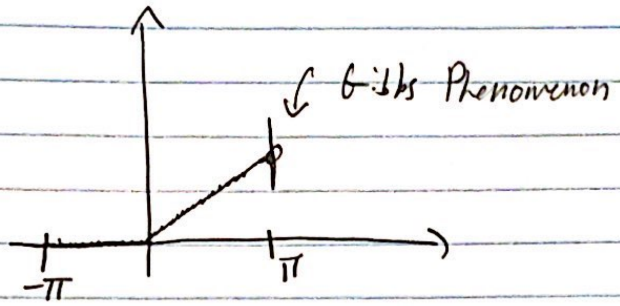


We see the inner area tends to 0, with only finitely many jump discontinuities at $x = \pi$ where this is not the case.

(iii) Since uniform convergence ⇒ pointwise convergence, we can say that not pointwise convergence ⇒ not uniform convergence, hence $\boxed{S_N \text{ does not converge to } f(x) \text{ uniformly}}$. Also since we have jump discontinuities, $\sup_{-\pi \le x \le \pi} |S_N(x) - f| \not\to 0$ as Gibbs phenomenon occurs at the endpoints, hence the maximum jump will never be 0.

1(a)

① d) For N-large and for f defined as in problem #1, we see:



Gibbs phenomenon occurs when observing the convergence behavior of the Fourier Series of a periodic that has a finite number of discontinuities. We see that at the jump discontinuity, there is a large jump in the oscillation and it does not smooth to 0 as the rest of the oscillations do at the continuous parts of f.

(e) By lecture, Parseval's identity says:

$\hookrightarrow \|f\|^2 = \frac{\pi}{2}a_0^2 + \pi\sum_{n=1}^{\infty} a_n^2 + \pi\sum_{n=1}^{\infty} b_n^2$ for a_0, a_n, b_n in 1(a).

We can compute $\|f\|^2$ using our inner product:

$$\Rightarrow \int_{-\pi}^{\pi} f^2(x)\,dx = \cancel{\int_{-\pi}^{0} 0^2\,dx} + \int_0^{\pi} x^2\,dX$$

$$= 0 + \frac{x^3}{3}\Big]_0^{\pi} = \frac{\pi^3}{3}$$

$$\Rightarrow \frac{\pi^3}{3} = \frac{\pi}{2}\left(\frac{\pi}{2}\right)^2 + \pi\sum_{n=1}^{\infty}\left(\frac{2}{\pi n^2}\right)^2 (n \text{ odd}) + \pi\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{1}{n}\right)^2$$

$$= \frac{\pi^3}{3} - \frac{\pi^3}{8} = \pi\left(\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^4}(n \text{ odd})\right) + \pi\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \frac{5\pi^3}{24} = \pi\left(\sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^4} + \sum_{n=1}^{\infty}\frac{1}{n^2}\right) \quad (*)$$

$$\Rightarrow \frac{5\pi^2}{24} = \sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^4} + \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \boxed{\frac{5\pi^2}{24} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} + \sum_{n=1}^{\infty}\frac{1}{n^2}} \checkmark$$

(*) (We use the fact that $a_n = 0$ for n even to reformulate $\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^4}(n \text{ odd}) + \sum_{n=1}^{\infty} 0\,(n \text{ even})$ to $\sum_{n=1}^{\infty}\frac{4}{\pi^2(2n-1)^4}$.)

(2) (a) From lecture, we know x_i are the roots of $P_3(x)$..

$\Rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x) = 0$

$\Rightarrow \frac{1}{2}x(5x^2 - 3) = 0 \Rightarrow \boxed{x = 0, \pm\sqrt{\frac{3}{5}}}$

Since we know x_i, we can compute w_i using the formula given:

$\hookrightarrow P_3'(x) = \frac{1}{2}(15x^2 - 3) \therefore P_3'(\pm\sqrt{\frac{3}{5}}) = 6, \; P_3'(0) = -3$

$\Rightarrow$ w_1: $\frac{2}{(1-\frac{3}{5})(\frac{1}{2}(15(\frac{3}{5}) - 3))^2} = \frac{2}{\frac{2}{5}(3)^2} = \boxed{\frac{5}{9}}$

w_2: $\frac{2}{(1-0^2)(\frac{1}{2}(-3))^2} = \frac{2}{\frac{9}{4}} = \boxed{\frac{8}{9}}$

w_3: $\frac{2}{(1-\frac{3}{5})(\frac{1}{2}(15(\frac{3}{5}) - 3))^2} = \frac{2}{\frac{2}{5}(3)^2} = \boxed{\frac{5}{9}}$

(b) $f(x) = x^k(x^2+2) + 1$ for $k = 1, 2, 3, 4$

$k=1$: $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x(x^2+2) + 1\,dx \approx \sum_{i=1}^{3} w_i f(x_i)$

$= \frac{5}{9}(-\sqrt{\frac{3}{5}}(\frac{3}{5}+2)+1) + \frac{8}{9}(1) + \frac{5}{9}(\sqrt{\frac{3}{5}}(\frac{3}{5}+2)+1)$

$= \boxed{2}$

$k=2$: $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2(x^2+2) + 1\,dx \approx \sum_{i=1}^{3} w_i f(x_i)$

$= \frac{5}{9}(\frac{3}{5}(\frac{3}{5}+2)+1) + \frac{8}{9}(1) + \frac{5}{9}(\frac{3}{5}(\frac{3}{5}+2)+1)$

$= \boxed{3.733}$

$k=3$: $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^3(x^2+2) + 1\,dx \approx \sum_{i=1}^{3} w_i f(x_i)$

$= \frac{5}{9}((-\sqrt{\frac{3}{5}})^3(\frac{3}{5}+2)+1) + \frac{8}{9}(1) + \frac{5}{9}((\sqrt{\frac{3}{5}})^3(\frac{3}{5}+2)+1)$

$= \boxed{2}$

$k=4$: $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^4(x^2+2) + 1\,dx \approx \sum_{i=1}^{3} w_i f(x_i)$

$= \frac{5}{9}((-\sqrt{\frac{3}{5}})^4(\frac{3}{5}+2)+1) + \frac{8}{9}(1) + \frac{5}{9}((\sqrt{\frac{3}{5}})^4(\frac{3}{5}+2)+1)$

$= \boxed{3.0394}$

(c) By lecture, we know this quadrature will be precise up to polynomials of degree $2n-1 \Rightarrow 2(3)-1 = \boxed{5}$

Hence we see, $\boxed{k = 1, 2, 3 \text{ are exact}}$ by $\deg(p) = 3, 4, 5$ respectively.

For $k=4$, the actual value is 3.0857.

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③ (a) Since we are given the weights and the values of x, we can directly use $\int_a^b f(x)\, w(x) \approx \sum_{i=1}^{3} w_i f(x_i)$ with the w_i : x_i given in the question.

$$\Rightarrow \int_{-\infty}^{\infty} (x^2+3)e^{-x^2}dx \approx \sum_{i=1}^{3} w_i f(x_i)$$

$$= \frac{\sqrt{\pi}}{6} f(-\sqrt{\tfrac{3}{2}}) + \frac{2\sqrt{\pi}}{3} f(0) + \frac{\sqrt{\pi}}{6} f(\sqrt{\tfrac{3}{2}})$$

$$= \boxed{6.2036}$$

(b) In order to use the weights and values for x in the previous equation, we must have the weight function as e^{-x^2} instead of $e^{-x^2/2}$. To do this, we can perform a substitution to get:

$$\begin{cases} u = \frac{x}{\sqrt{2}} & \Rightarrow \sqrt{2}u = x \\ du = \frac{1}{\sqrt{2}}dx & \Rightarrow \sqrt{2}\,du = dx \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} (x^2+3)e^{-x^2/2}dx = \int_{-\infty}^{\infty} \sqrt{2}(2u^2+3)e^{-u^2}du$$

$$= \sqrt{2}\int_{-\infty}^{\infty} (2u^2+3)e^{-u^2}du$$

Now, since $w(u) = e^{-u^2}$, we can use the same weights : values of x as before with $f(u) = 2u^2+3$

$$\Rightarrow \sqrt{2}\int_{-\infty}^{\infty} (2u^2+3)e^{-u^2}du \approx \sum_{i=1}^{3} w_i f(x_i)$$

$$= \frac{\sqrt{\pi}}{6} f(-\sqrt{\tfrac{3}{2}}) + \frac{2\sqrt{\pi}}{3} f(0) + \frac{\sqrt{\pi}}{6} f(\sqrt{\tfrac{3}{2}})$$

$$= \boxed{10.0265}$$

(c) $H_3(x) = 8x^3 - 12x$: To obtain x_i, we must find the roots of $H_3(x)$ $\Rightarrow$ $H_3(x) = 0$

$$\Rightarrow 8x^3 - 12x = 0 \Rightarrow x(8x^2-12) = 0 \Rightarrow \boxed{x = 0, \pm\sqrt{\tfrac{3}{2}}}$$ ✓ ✓

To find w_i, we use the formula in lecture of:

$$w_i = \int_a^b \frac{w(x)\, H_3(x)}{(x-x_i)\, H_3'(x_i)} dx$$

Now we can solve with $H_3'(x) = 24x^2 - 12$

$$\Rightarrow w_1 = \int_{-\infty}^{\infty} \frac{e^{-x^2}(8x^3-12x)}{(x-x_1)(24x_1^2-12)}dx = \boxed{\frac{\sqrt{\pi}}{6}}$$ ✓

$$\Rightarrow w_2 = \int_{-\infty}^{\infty} \frac{e^{-x^2}(8x^3-12x)}{(x-x_2)(24x_2^2-12)}dx = \boxed{\frac{2\sqrt{\pi}}{3}}$$ ✓

$$\Rightarrow w_3 = \int_{-\infty}^{\infty} \frac{e^{-x^2}(8x^3-12x)}{(x-x_3)(24x_3^2-12)}dx = \boxed{\frac{\sqrt{\pi}}{6}}$$ ✓

(3) (d) As with the other Gaussian Quadrature methods, the method will be exact for polynomials until degree $2n-1$. For this case, we use $H_3(x)$, hence we will be exact until degree $2(3)-1 = \boxed{5}$ ✓

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MATH 231 Homework 4 MATLAB

1. (a) Write a MATLAB function

```
function intg = trapezoid(f,a,b,n
```

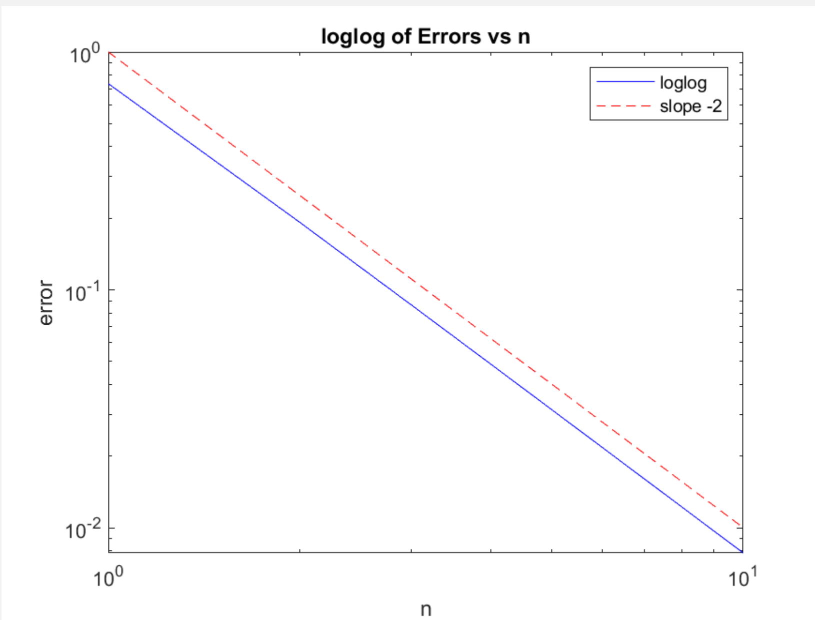
that computes $\int_a^b f(x)dx$ using the trapezoid method with $x_i = a + \frac{i}{n}(b-a)(i = 0, ..., n)$.
(b) Using $f(x) = e^{-x}$ with $a = -1$ and $b = 1$, show that the order of convergence is n^{-2}. In other words, in the log-log plot of the errors versus n, the slope should be -2.

SOLUTION:

```
TRAPEZOID METHOD
function intg = trapezoid(f,a,b,n)
dx = (b-a)/n;
xvals = linspace(a,b,n+1);
fvals = arrayfun(f,xvals);
wvals = dx*ones(size(xvals));
wvals(1) = 0.5*wvals(1);
wvals(end) = 0.5*wvals(end);
intg = sum(fvals.*wvals);
```

Using the psuedocode provided in discussion, we implement the trapezoid method with the following MATLAB code.

Using the above code, we can compute the errors and plot the log-log plot using the script loglog1b.m found in the appendix. We see here in the plot below that the slope of the line in the log-log plot is parallel to -2, hence validating that our method is correct and that the order of convergence is n^{-2}. (Actual slope computed to be -1.97 by computing the slope manually using $m = \frac{log(y_1)-log(y_0)}{log(x_1)-log(x_0)}$ for $x_0 = 1$, $x_1 = 10$ and y_0, y_1 being their respective errors).



2. (a)Write a `MATLAB` function

```
function intg = LegendreGauss(f,a,b,n)
```

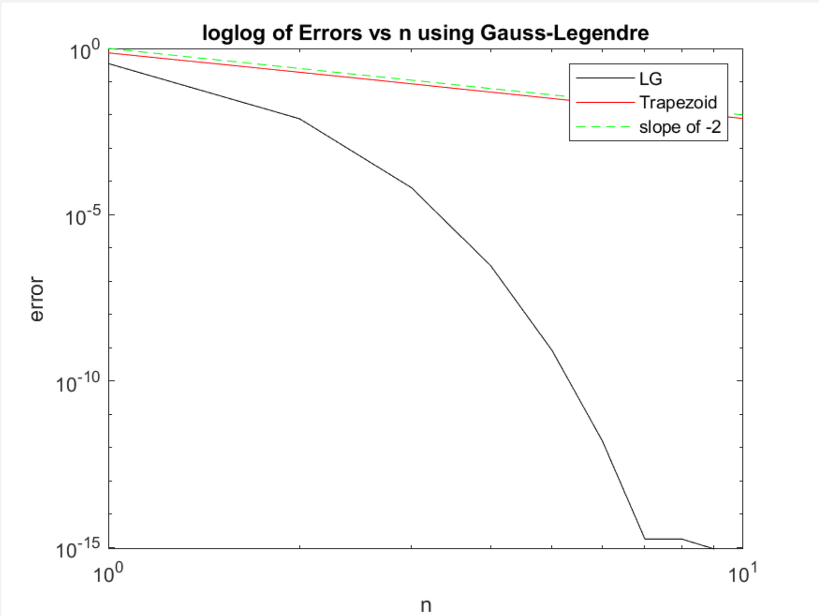
that computes $\int_a^b f(x)dx$ using the Legendre-Gauss quadrature with the nth-order polynomial. Use `MATLAB` function `lgwt` to obtain the quadrature points and weights.
(b) Using $f(x) = e^{-x}$ with $a = -1$ and $b = 1$, observe the spectral convergence.

SOLUTION:

```
LEGENDRE-GAUSS
function intg = LegendreGauss(f,a,b,n)
[xvals,wvals] = lgwt(n,a,b);
fvals = arrayfun(f,xvals);
intg = sum(fvals.*wvals);
```

The above script implements the Legendre-Gauss method using the x and w values obtained from `lgwt`.

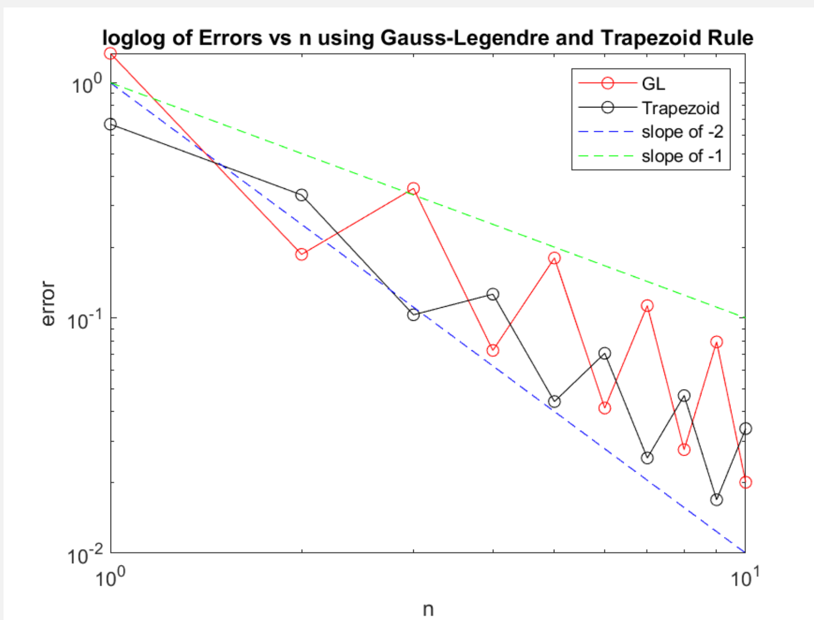
Using the script `plot2b.m` found in the appendix, we plot the log-log plot of the errors vs n. For reference, we include also the trapezoid method as well as a line of slope -2. We see the that Legendre-Gauss method converges extremely quickly, and it reaches the minimal computable error due to machine arithmetic. Since the function $f(x) = e^{-x}$ has no irregularities and behaves well at all points (differentiable) in its domain, this behavior is to be expected.



3. For $f(x) = \sqrt{|x|}$ $(-1 \leq x \leq 1)$, show and discuss the convergence behavior of both methods.

SOLUTION:

Below is the convergence behaviors of both the Legendre-Gauss method and the trapezoid method using the script `plot3b.m` found in the appendix. Upon inspection, we see that our function $f(x)$ is not differentiable at $x = 0$ due to the sharp kink at that point, hence it is not regularly behaved on the domain $[-1, 1]$. Since this is the case, we see that the convergence behavior of both methods is not exactly optimal. Both methods bounce around for varying values of n, and although the trend is still downwards, we do not see the convergence of decay in error in the log-log plot as we did in 1 and 2. We see the slope -1 and slope -2 mostly contain both lines as n gets large. For the trapezoid method, since it is not parallel to a line slope of -2, we see that the order of convergence for this one is not exactly n^{-2}. The spectral convergence of Legendre-Gauss is also sporadically behaved. We can empirically estimate the the slope of the line that best fits the endpoints of both method is around -1.5 since both rest between the dashed lines as n increases. Hence for the non-regularly behaved function $f(x)$, both methods still decay in error, but are not as optimal in convergence if we were to use an infinitely differentiable function on the domain.



```
loglog1b.m
f = @(x) exp(-x);
a = -1;
b = 1;
syms x
exactval = double(int(f(x),a,b));
n = [1,2,3,4,5,6,7,8,9,10];
errs = [];
for i=1:10
    app = trapezoid(f,a,b,n(i));
    err = abs(app - exactval);
    errs = [errs err];
end
sl = @(x) x.^(-2);
ys = sl(n);
loglog(n, errs, 'b')
hold on
loglog(n, ys, 'r--')
hold off
title('loglog of Errors vs n')
xlabel('n')
ylabel('error')
legend('loglog', 'slope -2')
%saveas(gcf, '1b.png')

plot2b.m
f = @(x) exp(-x);
a = -1;
b = 1;
syms x
exactval = double(int(f(x),a,b));
n = [1,2,3,4,5,6,7,8,9,10];
errs = [];
for i=1:10
    app = LegendreGauss(f,a,b,n(i));
    err = abs(app - exactval);
    errs = [errs err];
end
errs2 = [];
for i=1:10
    app = trapezoid(f,a,b,n(i));
    err = abs(app - exactval);
    errs2 = [errs2 err];
end
sl = @(x) x.^(-2);
ys = sl(n);
loglog(n, errs, 'k')
hold on
loglog(n, errs2, 'r')
hold off
hold on
loglog(n,ys, 'g--')
hold off
title('loglog of Errors vs n using Gauss-Legendre')
xlabel('n')
ylabel('error')
legend('LG', 'Trapezoid','slope of -2')
saveas(gcf, '2b.png')
```

```
plotq3.m
f = @(x) sqrt(abs(x));
a = -1;
b = 1;
syms x
exactval = double(int(f(x),a,b));
n = [1,2,3,4,5,6,7,8,9,10];
errs = [];
for i=1:10
    app = LegendreGauss(f,a,b,n(i));
    err = abs(app - exactval);
    errs = [errs err];
end
errs2 = [];
for j = 1:10
    app2 = trapezoid(f,a,b,n(j));
    err2 = abs(app2-exactval);
    errs2 = [errs2 err2];
end
sl = @(x) x.^(-2);
ys = sl(n);
sl2 = @(x) x.^(-1);
ys2= sl2(n);
loglog(n, errs,'ro-')
hold on
loglog(n,errs2, 'ko-')
hold off
hold on
loglog(n,ys,'b--')
hold off
hold on
loglog(n,ys2, 'g--')
hold off
title('loglog of Errors vs n using Gauss-Legendre and Trapezoid Rule')
xlabel('n')
ylabel('error')
legend('GL', 'Trapezoid', 'slope of -2', 'slope of -1')
saveas(gcf, '3.png')
```