Pen and paper

1. By approximating $f(t,y(t))$ in

$$y(t_{n+3}) = y(t_{n+2}) + \int_{t_{n+2}}^{t_{n+3}} f(t,y(t))dt \tag{1}$$

using a Lagrange polynomial $F(t)$ that interpolates $f(t_{n+2},y_{n+2})$, $f(t_{n+1},y_{n+1})$, $f(t_n,y_n)$, obtain the 3-step Adams–Bashforth method.

2. Consider the following multi-step method:

$$y_{n+2} - 3y_{n+1} + 2y_n = h\left[\frac{13}{12}f(t_{n+2},y_{n+2}) - \frac{5}{3}f(t_{n+1},y_{n+1}) - \frac{5}{12}f(t_n,y_n)\right]. \tag{2}$$

   (a) Show that the local truncation error is $O(h^3)$.

   (b) Suppose you solve a trivial initial value problem

$$y'(t) = 0, \quad y(0) = 1, \tag{3}$$

   using this scheme with $y_0 = 1$ and $y_1 = 1 + \varepsilon$. Express y_n in terms of n and ε.

   (c) Would you use this scheme?

3. (a) Find the 6th-order and 7th-order backward finite difference formulas from a book or the internet (e.g. `http://web.media.mit.edu/~crtaylor/calculator.html`).

   (b) Obtain the 6th-order and 7th-order backward differentiation formula methods.

   (c) Show that BDF6 is a stable method but BDF7 is not.

MATLAB

1. (a) Write a MATLAB function

   ```
   function [tvals,yvals] = AB2(F,t0,T,y0,y1,n)
   ```

   that computes the solution $y(t)$ ($t_0 \le t \le T$) to the initial value problem

   $$y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \tag{4}$$

   using AB2 with provided y_1. Note that `tvals` contains points $t_k = t_0 + k\Delta t$ ($k = 0, 1, \cdots, n$) with $\Delta t = (T - t_0)/n$ and `yvals` contains the values of the approximate solution at these points.

   (b) Validate your code by using the following initial value problem:

   $$y' = \left(1 - \frac{4}{3}t\right) y, \quad y(0) = 1. \tag{5}$$

   For the value of y_1, use the exact solution $y(t) = \exp(t - \frac{2}{3}t^2)$.

2. (a) Write a *wrapper* MATLAB function

   ```
   function [tvals,yvals] = AB2_y1(F,t0,T,y0,n,opt)
   ```

   that computes the solution $y(t)$ to (4) by *internally calling* `AB2` with a value of y_1 computed by the forward Euler update (`opt=1`) or the explicit midpoint update (`opt=2`).

   (b) For the three cases (i.e., `AB2` with exact y_1 and `AB2_y1` with `opt=1,2`), draw a log-log plot of the error at the final time $t = 3$ versus the number of subintervals n and compare the results.