Pen and paper

1.

$$A_1 = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \tag{1}$$

(a) You solve $A_1\mathbf{x} = \mathbf{b}$ using the Jacobi method.
   i. Write down the component-wise recursion formula.
   ii. Write down the matrix-form recursion formula.
   iii. Show that the Jacobi method converges.

(b) You solve $A_1\mathbf{x} = \mathbf{b}$ using the Gauss-Seidel method.
   i. Write down the component-wise recursion formula.
   ii. Write down the matrix-form recursion formula.
   iii. Show that the Gauss-Seidel method does not converge.

(c) You solve $A_2\mathbf{x} = \mathbf{b}$ using the Jacobi method.
   i. Write down the component-wise recursion formula.
   ii. Write down the matrix-form recursion formula.
   iii. Show that the Jacobi method does not converge.

(d) You solve $A_2\mathbf{x} = \mathbf{b}$ using the Gauss-Seidel method.
   i. Write down the component-wise recursion formula.
   ii. Write down the matrix-form recursion formula.
   iii. Show that the Gauss-Seidel method converges.

†† For ii. and iii., you can use MATLAB.

2. We want to solve the following 2×2 system using the Jacobi method:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad (a \neq 0,\ d \neq 0). \tag{2}$$

(a) The iteration scheme can be cast into the following form:

$$\mathbf{x}^{(k)} = B_{\mathrm{J}}\mathbf{x}^{(k-1)} + \mathbf{c}_{\mathrm{J}}. \tag{3}$$

Express B_{J} and $\mathbf{c}_{\mathrm{J}}$ in terms of a, b, c, d, u, and v.

(b) Express the following quantities in terms of a, b, c, and d:
   - $\|B_{\mathrm{J}}\|_1$,
   - $\|B_{\mathrm{J}}\|_\infty$,
   - $\|B_{\mathrm{J}}\|_2$,
   - Spectral radius $\rho(B_{\mathrm{J}})$.

(c) Sort those quantities by magnitude.

(d) State a necessary and sufficient condition for the Jacobi method to converge in terms of a, b, c, and d.

(e) Show that this condition holds for a 2×2 diagonally dominant matrix.

3. [Extra Credit Problem] We now want to solve the same system using the Gauss-Seidel method:

   (a) The iteration scheme can be cast into the following form:

   $$\mathbf{x}^{(k)} = B_{\text{GS}}\mathbf{x}^{(k-1)} + \mathbf{c}_{\text{GS}}. \tag{4}$$

   Express B_{GS} and $\mathbf{c}_{\text{GS}}$ in terms of a, b, c, d, u, and v.

   (b) Express the following quantities in terms of a, b, c, and d:
   - $\|B_{\text{GS}}\|_1$,
   - $\|B_{\text{GS}}\|_\infty$,
   - $\|B_{\text{GS}}\|_2$,
   - Spectral radius $\rho(B_{\text{GS}})$.

   (c) Sort those quantities by magnitude.

   (d) State a necessary and sufficient condition for the Gauss-Seidel method to converge in terms of a, b, c, and d.

   (e) Show that this condition holds for a 2×2 symmetric positive definite matrix.

   (f) Is it possible for one of Jacobi and Gauss-Seidel to converge and the other diverge for a 2×2 matrix?

MATLAB

1. Write the following MATLAB functions for the Jacobi and Gauss-Seidel methods to solve a general $n \times n$ system `A*x=b`:
   - `function [final_sol,sols] = Jacobi(A,b,x0,niter)`
   - `function [final_sol,sols] = GaussSeidel(A,b,x0,niter)`

   Here, `final_sol` is the iterative solution after `niter` iterations starting with initial guess `x0` and `sols` is the sequence of iterative solutions $\{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(\texttt{niter})}\}$ ($n \times (\texttt{niter}+1)$ matrix).

2. For each method, validate your implementation by using a 6×6 system (convergent case).
   - Plot relative residual errors versus iteration number k (using semi-log plot).
   - Compute the spectral radius ρ of the iteration matrix and plot ρ^k versus k (compare the slopes).

3. For each method, observe the divergent case.
   - Plot relative residual errors versus iteration number k (using semi-log plot).
   - Compute the spectral radius ρ of the iteration matrix and plot ρ^k versus k (compare the slopes).