Pen and paper

1. Using the method of undetermined coefficients, find the central finite difference of order Δx^4:

$$f'(x) = \frac{A_2 f(x+2\Delta x) + A_1 f(x+\Delta x) + A_0 f(x) + A_{-1} f(x-\Delta x) + A_{-2} f(x-2\Delta x)}{\Delta x} + O(\Delta x^4). \quad (1)$$

   Compute the truncation error.

2. Apply Richardson extrapolation technique to the the backward finite difference of order Δx to obtain the backward finite difference of order Δx^2.

3. Apply Richardson extrapolation technique to the following finite difference to obtain a finite difference formula of order Δx^6.

$$\mathscr{D}_2(\Delta x) = \frac{-f(x+2\Delta x) + 8f(x+\Delta x) - 8f(x-\Delta x) + f(x-2\Delta x)}{12\Delta x} = f'(x) + C\Delta x^4 + O(\Delta x^6). \quad (2)$$

   Why is the resulting finite difference different from the standard central difference of order Δx^6?

4. Using the Lagrange polynomial $F(x)$ interpolating $f(x)$ at three points $x_0 \pm \Delta x$ and x_0, obtain the central finite difference of order Δx^2 for $f''(x)$.

5. From Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_3} f(x)dx = h\left[\frac{3}{8}f_0 + \frac{9}{8}f_1 + \frac{9}{8}f_2 + \frac{3}{8}f_3\right] - \frac{3}{80}h^5 f^{(4)}(\xi), \quad (3)$$

   write down the corresponding composite quadrature and estimate its error.

6. Determine A_1, A_2, C, p, and k in the following open Newton-Cotes formula by considering $f(x) = 1$, x, x^2:

$$\int_{x_0}^{x_3} f(x)dx = h\left[A_1 f_1 + A_2 f_2\right] + Ch^p f^{(k)}(\xi). \quad (4)$$

7. Using the idea of Richardson extrapolation, obtain Simpson's rule from trapezoid rule.

MATLAB

1. (a) Write a MATLAB function
```
function diffval = CentralDiff(f,x,dx,n)
```
that computes the first derivative of `f` at `x` using the `n`th-order central finite difference with `dx`. Assume that $n = 2$, 4, or 6.

   (b) Validate your code by computing the derivative of the following function at $x = 0$:

$$f_1(x) = 1 - e^{-x}. \tag{5}$$

In other words, observe the slope in the log-log plot of the error versus Δx for $n = 2$, 4, 6.

2. (a) Using `CentralDiff` with $n = 2$, 4, 6, observe the slope in the log-log plot of the error versus Δx for the derivative of the following function at $x = 0$:

$$f_2(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ e^x - 1 & \text{for } x \leq 0. \end{cases} \tag{6}$$

   (b) [Pen and paper] Explain the results.

3. For `CentralDiff` with $n = 2$, 4, 6, numerically construct central difference approximations of order $n+2$ using the Richardson extrapolation. For validation, use the same set-up as in Problem 1(b).

4. (a) Write a MATLAB function
```
function intgval = SimpsonIntg(f,a,b,n,which)
```
that computes $\int_a^b f(x)dx$ using `n+1` equally spaced points by the first Simpson's rule (`which=1`) or the second Simpson's rule (`which=2`).

   (b) Using $f(x) = e^{-x}\cos 3x$ with $a = 0$ and $b = 2$, validate your code for `which` = 1 and 2. In other words, observe the slope in the log-log plot of the error versus $n = 6, 6^2, \cdots, 6^5$.

   (c) Which Simpson's rule do you think better? Explain why.