MATH 232 Homework 2

1. **Problem 1**

SOLUTION:

(a) The boundary value problem for the Green's function corresponding to the BVP $u'' = f$ with a Neumann boundary condition is given as follows:

$$G''(x, x') = \delta(x - x')$$

for $x, x' \in (-1, 1)$ and boundary conditions $G'(0, x') = 0$ and $G(1, x') = 0$. To solve this, we solve for the coefficients of:

$$G(x, x') = \begin{cases} ax + b, & 0 < x < x' \\ cx + d, & x' < x < 1 \end{cases}$$

Applying the boundary conditions of $G'(0, x') = 0$ and $G(1, x') = 0$ yields:

$$G'(0, x') = a = 0 \tag{1}$$
$$G(1, x') = c + d = 0 \tag{2}$$

We must also have the continuity of the Green's function as well as the jump condition, which will assist in solving for these coefficients. Both of these yield:

$$G(x' - 0; x') = G(x' + 0; x') \longrightarrow b = cx' + d \tag{3}$$
$$G'(x' + 0; x') - G'(x' - 0'; x') = 1 \longrightarrow c - a = 1 \tag{4}$$

respectively. But from the boundary conditions, we know $a = 0$ and $c = -d$, so this directly shows that $c = 1$, $d = -1$, and $b = x' - 1$. Hence, the solution to our BVP comes from the Green's function given by:

$$G(x, x') = \begin{cases} x' - 1, & 0 < x < x' \\ x - 1, & x' < x < 1 \end{cases}$$

(b) From lecture, we derived that the solution to the boundary value problem $u'' = f$ is given by:

$$u(x) = c_0 + c_1 x + \int_0^1 G(x, x').f(x')dx'$$

But now, if we want to solve for the coefficients c_0, c_1, then we must apply the Neumann and Dirichlet boundary conditions. This yields the system:

$$\begin{cases} u'(0) = \alpha \longrightarrow c_1 + \frac{d}{dx}(\int_0^1 G(0, x').f(x')dx') = \alpha \implies c_1 = \alpha \\ u(1) = \beta \longrightarrow c_0 + \alpha + \int_0^1 G(1, x').f(x')dx' = \beta \implies c_0 = \beta - \alpha - f(1) \end{cases}$$

Hence, the solution is given by:

$$u(x) = \beta - \alpha - f(1) + \alpha x + \int_0^x G(x, x').f(x')dx' + \int_x^1 G(x, x').f(x')dx'$$
$$\implies u(x) = \beta - \alpha - f(1) + \alpha x + (x - 1)\int_0^x f(x')dx' + \int_x^1 (x' - 1).f(x')dx'$$

(c) To find the general formulas for the elements of the inverse of the matrix given in 2.54 of the text, see that the formulas can be given by (where $AB = I$, namely B being the inverse of A):

- $B_{i,0} = x_i - 1, i = 0, ..., m+1$
- $B_{i,j} = h \begin{cases} x_j - 1, & i = 1, ..., j \\ x_i - 1, & i = j, ..., m \end{cases}$
- $B_{i,m+1} = x_j, i = 0, ..., m+1$

using our computation in (b). Using the formulation from 2.54, we see:

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & h^2 \end{bmatrix}$$

With $h = 0.25$, we can use what we computed above to formulate A^{-1} and the above to formulate A yielding:

$$A = \begin{bmatrix} -4 & 4 & 0 & 0 & 0 \\ 16 & -32 & 16 & 0 & 0 \\ 0 & 16 & -32 & 16 & 0 \\ 0 & 0 & 16 & -32 & 16 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} -1 & -\frac{3}{16} & -\frac{1}{8} & -\frac{1}{16} & 1 \\ -\frac{3}{4} & -\frac{3}{16} & -\frac{1}{8} & -\frac{1}{16} & 1 \\ -\frac{1}{2} & -\frac{1}{8} & -\frac{1}{8} & -\frac{1}{16} & 1 \\ -\frac{1}{4} & -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. **Problem 2**

SOLUTION:
We start with A and F defined in (2.58),

$$A = \frac{1}{h^2}\begin{bmatrix} -h & h & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & 0 & 1 & -2 & 1 \\ & & & h & -h \end{bmatrix}, \quad F = \begin{bmatrix} \sigma_0 + \frac{h}{2}f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \\ -\sigma_1 + \frac{h}{2}f(x_{m+1}) \end{bmatrix}$$

Next, we compute A^T:

$$A^T = \frac{1}{h^2}\begin{bmatrix} -h & 1 & & & \\ h & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & 0 & 1 & -2 & h \\ & & & 1 & -h \end{bmatrix}$$

Next, recognize that the sum of the diagonals (excluding the first two and last two rows) is 0. If we use $\frac{1}{h}$ as the first and last entry of our vector, the first two and last two rows of our matrix will also sum to 0. Therefore, we construct $Null(A^T)$ as:

$$Null(A^T) = \begin{bmatrix} \frac{1}{h} \\ 1 \\ \vdots \\ 1 \\ \frac{1}{h} \end{bmatrix}$$

To check our solution, we take the product of A^T and $Null(A^T)$:

$$\frac{1}{h^2}\begin{bmatrix} -h & 1 & & & \\ h & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & 0 & 1 & -2 & h \\ & & & 1 & -h \end{bmatrix}\begin{bmatrix} \frac{1}{h} \\ 1 \\ \vdots \\ 1 \\ \frac{1}{h} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

See next page.

We can generate the solvability condition for (2.58) by taking the product of $Null(A^T)$ and F:

$$\begin{bmatrix} \frac{1}{h} & 1 & \dots & 1 & \frac{1}{h} \end{bmatrix} \begin{bmatrix} \sigma_0 + \frac{h}{2} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \\ -\sigma_1 + \frac{h}{2} f(x_{m+1}) \end{bmatrix} = \sigma_0 + \frac{h}{2} f(x_0) + h \sum_{i=1}^{m} f(x_i) + \frac{h}{2} f(x_{m+1}) - \sigma_1$$

Since a solution to (2.58) requires that F be orthogonal to $Null(A^T)$, we obtain our solvability condition, corresponding to (2.62):

$$\frac{h}{2} f(x_0) + h \sum_{i=1}^{m} f(x_i) + \frac{h}{2} f(x_{m+1}) = \sigma_1 - \sigma_0.$$

3. **Problem 3**

SOLUTION: (a) See the attached script `p3_01.m` in the appendix below. We essentially follow the text's approach of using Newton's method to solve the nonlinear problem.

(b) Using the script `p3_01.m`, we find a solution to the nonlinear pendulum problem as discussed in the text. We plot the solution for the initial conditions $\alpha = 0.7 = \beta$ for various choices of a time interval (parameter T). Below are some of the solution curves and error plots produced by the Newton iterative method.

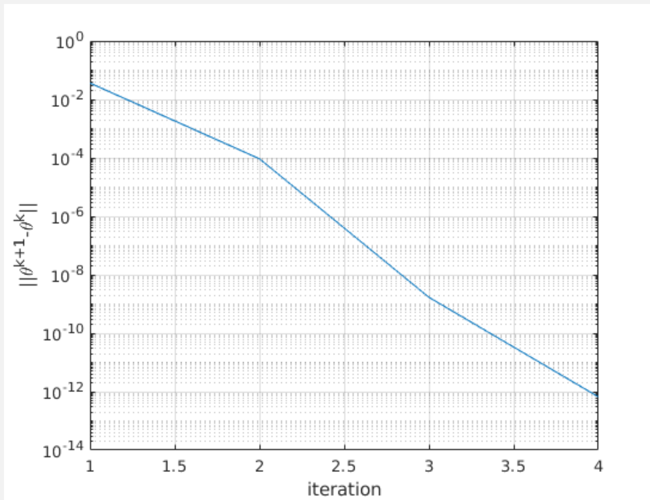

Figure 1: semilog plot of the error per iteration of the solution on the domain $[0, 2\pi]$.

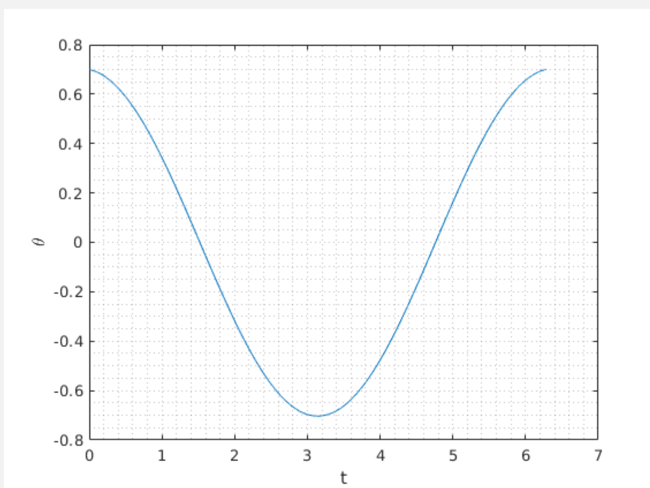

Figure 2: Solution curve of $\theta(t)$ in $[0, 2\pi]$.

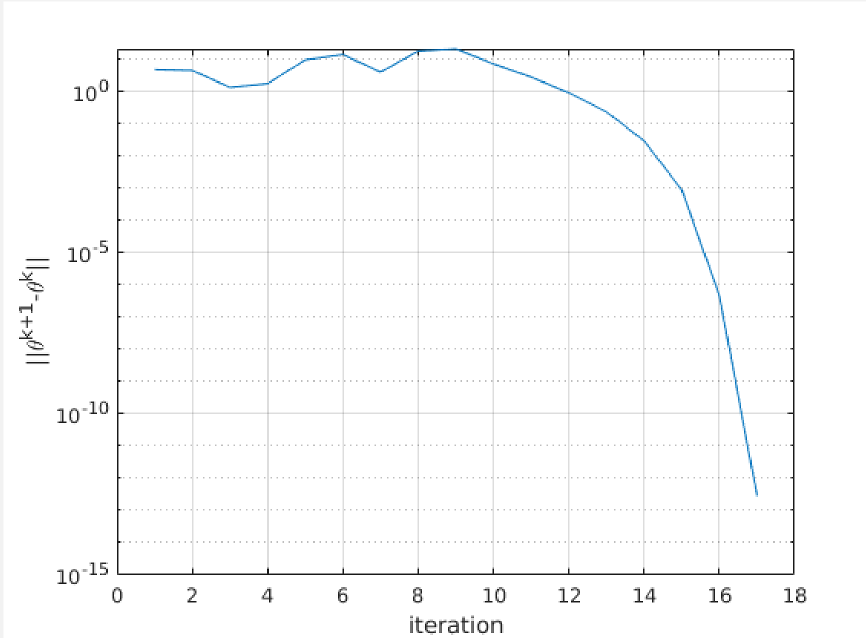

Figure 3: semilog plot of the error per iteration of the solution on the domain $[0, 3\pi]$.

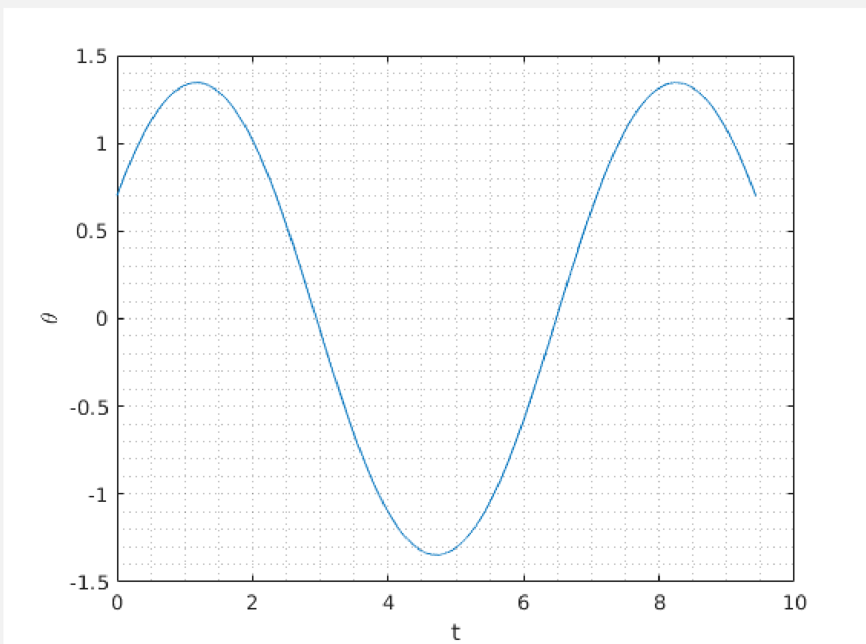

Figure 4: Solution curve of $\theta(t)$ in $[0, 3\pi]$.

As we increase the size of the domain to $[0, 10\pi]$, we see that our iteration method no longer converges. Furthermore, we see that θ_{max} in larger domains tends to increase. We can speculate that $\theta_{max} \to \infty$ as our domain extends to ∞.

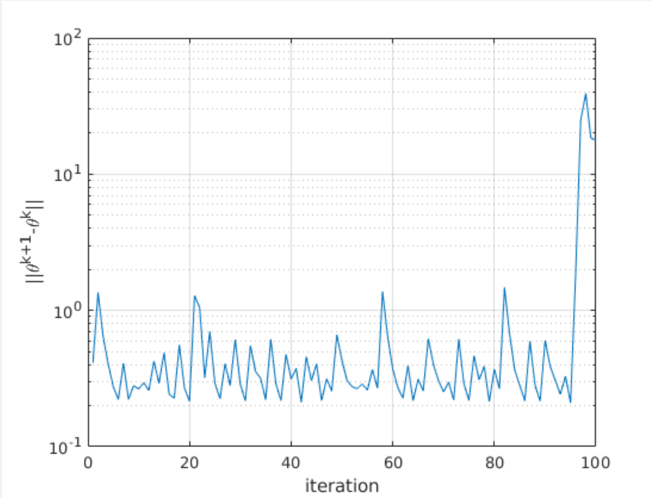

Figure 5: semilog plot of the error per iteration of the solution on the domain $[0, 10\pi]$.

We can also see that we start generating internal boundary layers in the solution at our domain increases, as shown below.

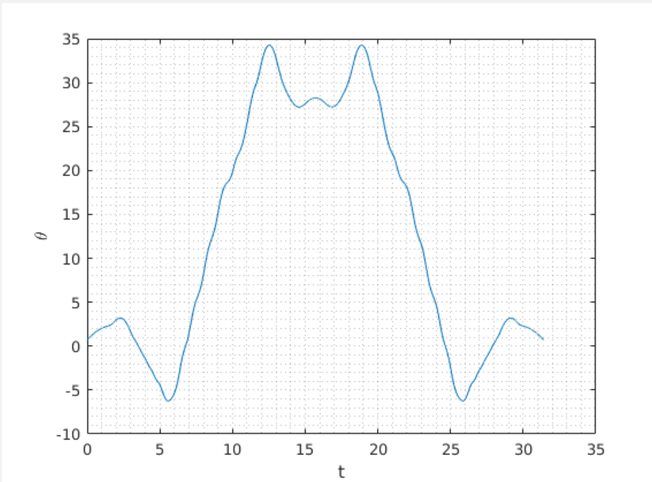

Figure 6: Solution curve of $\theta(t)$ in $[0, 10\pi]$.

Contributions:

Each group member completed the assignment individually, then the team we met together to discuss the answers complete the write up in Latex.

APPENDIX: p3 01.m

```
close all
clear
clc

% define grid
t0 = 0;
tf = 10*pi;
m = 1e4;
h = (tf-t0)/(m+1);
t = (1:1:m)*h;
t=t';

% define init conditions
alpha = 0.7;
beta = 0.7;

% initialize G, J, theta, theta_0
G = zeros(m,1);
delta = zeros(m,1);


% generate initial guess
theta = 0.7*cos(t);

% start iterations
k_lim = 100;
err = zeros(k_lim,1);
err_tol = 1e-12;

for k = 1:k_lim

    % update G
    for i = 1:m
        if i == 1
            G(1) = (alpha-2*theta(i)+theta(i+1))/h^2+sin(theta(i));
        elseif i == m
            G(m) = (beta-2*theta(i)+theta(i-1))/h^2+sin(theta(i));
        else
            G(i) = (theta(i+1)-2*theta(i)+theta(i-1))/h^2+sin(theta(i));
        end
    end

    % update J
    J_diag = -2+h^2*cos(theta);
    J = spdiags([ones(m,1) J_diag ones(m,1)], -1:1, m, m)/h^2;
```

```
    % solve linear system
    delta = -J\G;
    err(k) = norm(delta,inf);
    theta = theta + delta;

    if err(k) < err_tol
        k_lim = k;
        break
    end
end

figure
semilogy(1:k_lim,err(1:k_lim))
xlabel 'iteration'
ylabel '||\theta^{k+1}-\theta^k||'
grid

figure
plot(t,theta)
xlabel 't'
ylabel '\theta'
grid minor
```