MATH 233: Scientific Computing
Assignment 3 – Level set method
Due October 28th, 2022

1. **Semi-Lagrangian method for the advection equation**

   Consider the square domain $\Omega = [-1,1]^2$ and the linear advection equation

   $$\frac{\partial \phi}{\partial t} + \mathbf{V}\cdot\nabla\phi = 0 \qquad \forall x \in \Omega, \tag{1}$$

   with the following velocity field, initial condition and final time

   $$\begin{aligned} \mathbf{V} &= (-y, x), \\ \phi(x,y,0) &= \sqrt{(x-0.25)^2 + y^2} - 0.2 \\ t_f &= 2\pi. \end{aligned}$$

   Solve the above problem numerically using a second-order (in time and space)Semi-Lagrangian method. At any point $(\mathbf{x^{n+1}}, t^{n+1})$, the departure point $\mathbf{x_d}$ is constructed using a mid-point rule (RK2):

   $$\begin{aligned} \mathbf{x}^* &= \mathbf{x^{n+1}} - \frac{\Delta t}{2}\mathbf{V}(\mathbf{x^{n+1}}, t^{n+1}) \\ \mathbf{x^d} &= \mathbf{x^{n+1}} - \Delta t\mathbf{V}(\mathbf{x}^*, t^{n+\frac{1}{2}}). \end{aligned}$$

   To compute the interpolation of the solution at $(\mathbf{x^d}, t^n)$, $\tilde{\phi}(\mathbf{x^d}, t^n)$, find to which grid cell C the point $\mathbf{x^d}$ belongs to and use a the (ENO inspired) quadratic interpolation

   $$\begin{aligned} \tilde{\phi}(\mathbf{x^d}, t^n) &= \phi^n_{i,j}\frac{x_{i+1}-x_d}{\Delta x}\frac{y_{j+1}-y_d}{\Delta y} + \phi^n_{i,j+1}\frac{x_{i+1}-x_d}{\Delta x}\frac{y_d-y_i}{\Delta y} \\ &+ \phi^n_{i+1,j}\frac{x_d-x_i}{\Delta x}\frac{y_{j+1}-y_d}{\Delta y} + \phi^n_{i+1,j+1}\frac{x_d-x_i}{\Delta x}\frac{y_d-y_i}{\Delta y} \\ &- \frac{(x_d-x_i)(x_{i+1}-x_d)}{2}\mathrm{minmod}_C(\phi_{xx}) \\ &- \frac{(y_d-y_j)(y_{j+1}-y_d)}{2}\mathrm{minmod}_C(\phi_{yy}) \end{aligned}$$

   where $(i,j), (i+1,j), (i,j+1), (i+1,j+1)$ are the corners of the cell C, and $\Delta x, \Delta y$ are tits dimensions. $\mathrm{minmod}(\phi_{xx})$ denotes the minmod of the second order derivatives in x at the corners of C (calculated using central finite differences formulas). If $\mathbf{x^d}$ is outside of the computational, define C as the closest cell in Ω and use the exact same interpolation formula.

   (a) Test your method for increasing resolution and various $\frac{\Delta x}{\Delta t}$ ratios (for example $0.5, 1, 5$ and 10). Compute the error and study the convergence of the method.

   (b) What can you say about the stability of the method ?

   (c) Does the solution remain reinitialized ?

2. **Reinitialization Equation**

   Consider now the reinitialization equation, written in the form

   $$\frac{\partial \phi}{\partial t} + \frac{S(\phi_0)}{|\nabla\phi|}\nabla\phi\cdot\nabla\phi = S(\phi_0).$$

   Using a Godunov Scheme, write a function that reinitialize any arbitrary level set function. The term ϕ_x should be discretize as:

(a) $S(\phi_0)\phi_x^- \leq 0$ and $S(\phi_0)\phi_x^+ \leq 0 \quad \Rightarrow \quad \phi_x^+$

(b) $S(\phi_0)\phi_x^- \geq 0$ and $S(\phi_0)\phi_x^+ \geq 0 \quad \Rightarrow \quad \phi_x^-$

(c) $S(\phi_0)\phi_x^- \leq 0$ and $S(\phi_0)\phi_x^+ \geq 0 \quad \Rightarrow \quad 0$

(d) $S(\phi_0)\phi_x^- \geq 0$ and $S(\phi_0)\phi_x^+ \leq 0$

   i. $|\phi_x^-| \geq |\phi_x^+| \quad \Rightarrow \quad \phi_x^-$

   ii. $|\phi_x^-| \leq |\phi_x^+| \quad \Rightarrow \quad \phi_x^+$

(a) To test your method, consider a contour for which you know the reinitialized level set function (like the initial solution of the above problem) and construct a non reinitialized level set function by perturbing the reinitialized one (make sure that the contour is preserved). Use your code to reinitialize that level set function.

(b) Show the solution after various number of iterations.

(c) Approximately, how many does it take to obtained a "good" reinitialized function ?

3. **Level set method** Use the method you developped in part 1 and 2 to simulate the advection of a circular contour of radius 0.2 and center (0.25,0) under the deforming velocity field defined previously. What should you look at to verify that your method is working correctly ?

4. **Extra credit** Simulate the advection of a contour (of your choice) under a velocity field (of your choice). No credit will be given for example involving circles or rotations. Be creative !