CE394M: Introduction to the Finite Element Method

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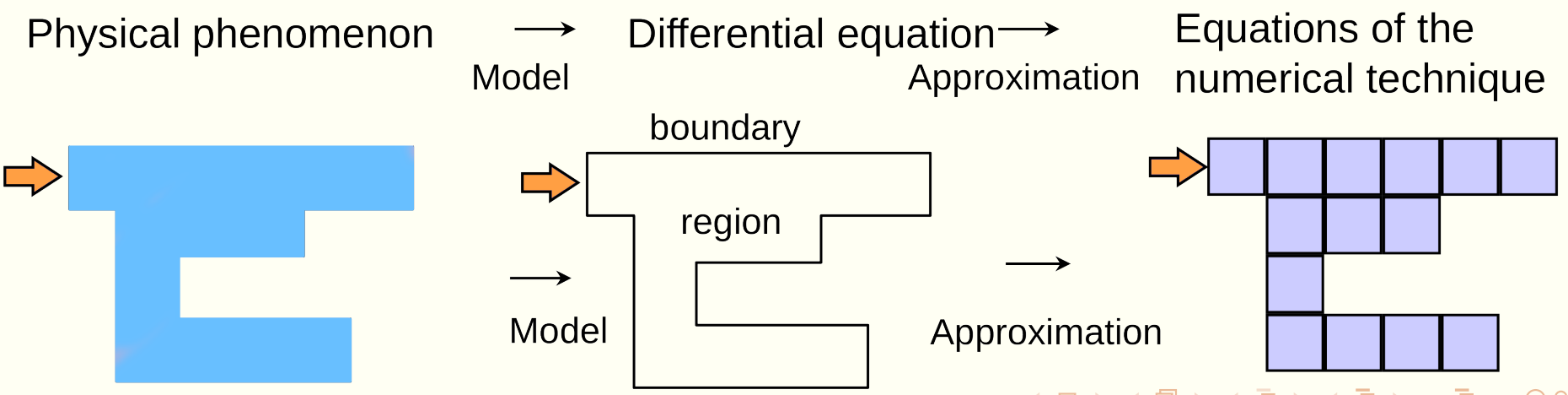
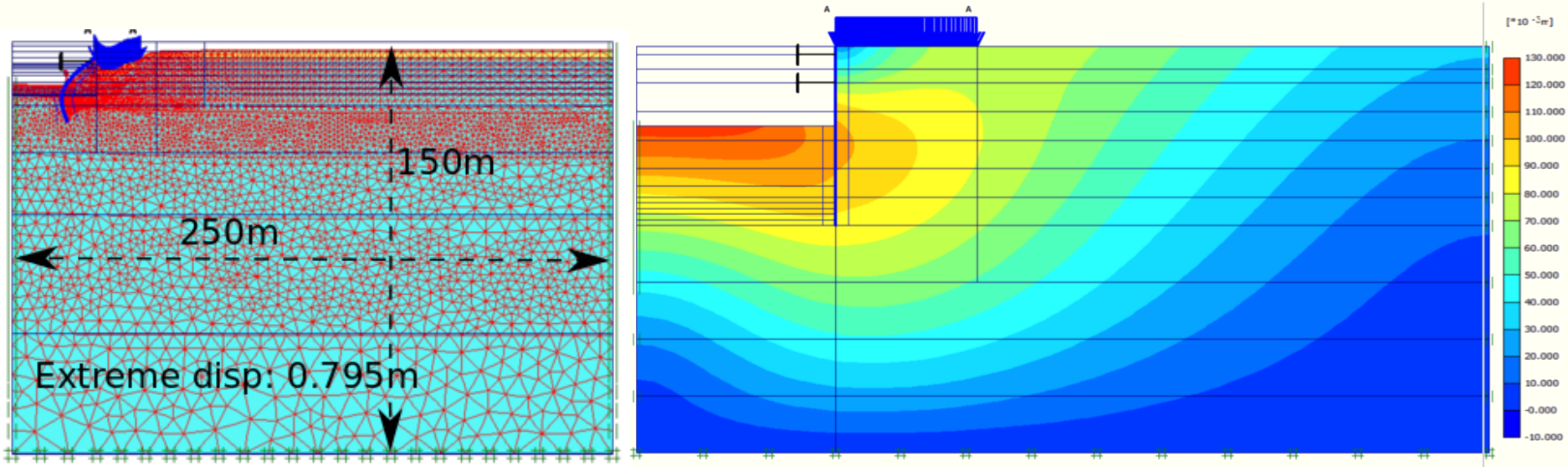
Overview

Numerical analysis of engineering problems

- Conceptualize the system
  - Geometry
  - Properties
  - Processes
- Describe it mathematically
  - Select the relevant differential equations
- Solve the equations (numerically)
  - Discretize the system
  - Settle for approximations (numerical techniques)
- Interpret the results



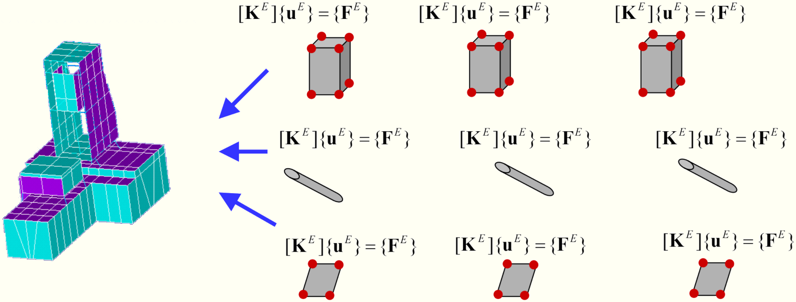
Finite Element Analysis



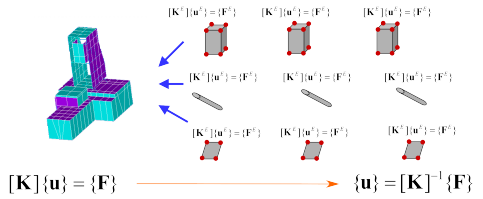
(a) FE Mesh (b) Displacement profile

Singapore Nicoll highway excavation FE analysis

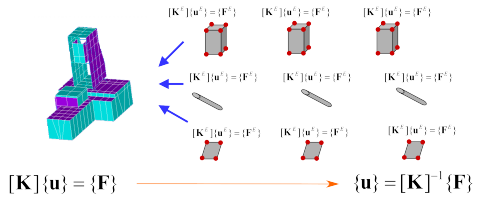
Finite Element Analysis



$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\} \longrightarrow \{\mathbf{u}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\}$$



Galerkin is a different one, a different starting point, he took some trial functions, functions that he hoped whose combinations could be close the right answer and the problem then is to find how much of each function goes into the good answer. We are talking about approximations, not exact solutions, geotechnical engineering involves approximations, and is quite OK. 100 years later because of the computers it was possible to work with a hundred thousand functions. Galerkin worked with 2 or 3 trial functions, so he had to make a very close guess of the solution, but if we have a hundred thousand functions they can be just maybe little hat functions, just up and down again simple functions and their combinations, if we have many many can give us close to the correct answer.



The whole idea of the FEM is a combination of the Galerkin idea of test functions with the idea of simple functions, where the physics is imple and the equations stay simple, but many many functions and that's what a computer is happy with. So instead of a differential equation, we have a big system to solve.

Our unknowns are how much of each hat function goes into the final solution, it's the coefficients we want, the number that multiplies each of the hat functions, when we add them together that we get a big system of equations for those numbers to multiply the hat functions and the system is very well organized for mathematics, so the question is how close is the approximation let's say if I approximate a function like e^x by a thousand hat functions and take a combination of the thousand hat functions to be close to e^x, if I take two-thousand hat functions how much closer do I get?

Galerkin:Ritz method

1. Define the functional u for which you wish to find stationary points.
2. Choose a combination of linearly independent functions that will be used to approximate the solution. These will be called 'basis functions'. The amplitudes of these functions will be the unknowns that you will determine. The basis functions must satisfy the Dirichlet ('fixed') boundary conditions.
3. Insert the approximate solution into the functional that is now denoted by u_h.
4. Take the directional derivative of u_h with respect to the unknown amplitudes of the basis functions.
5. Determine the amplitudes of the basis functions which yield a stationary point of u_h.

Galerkin:Ritz method

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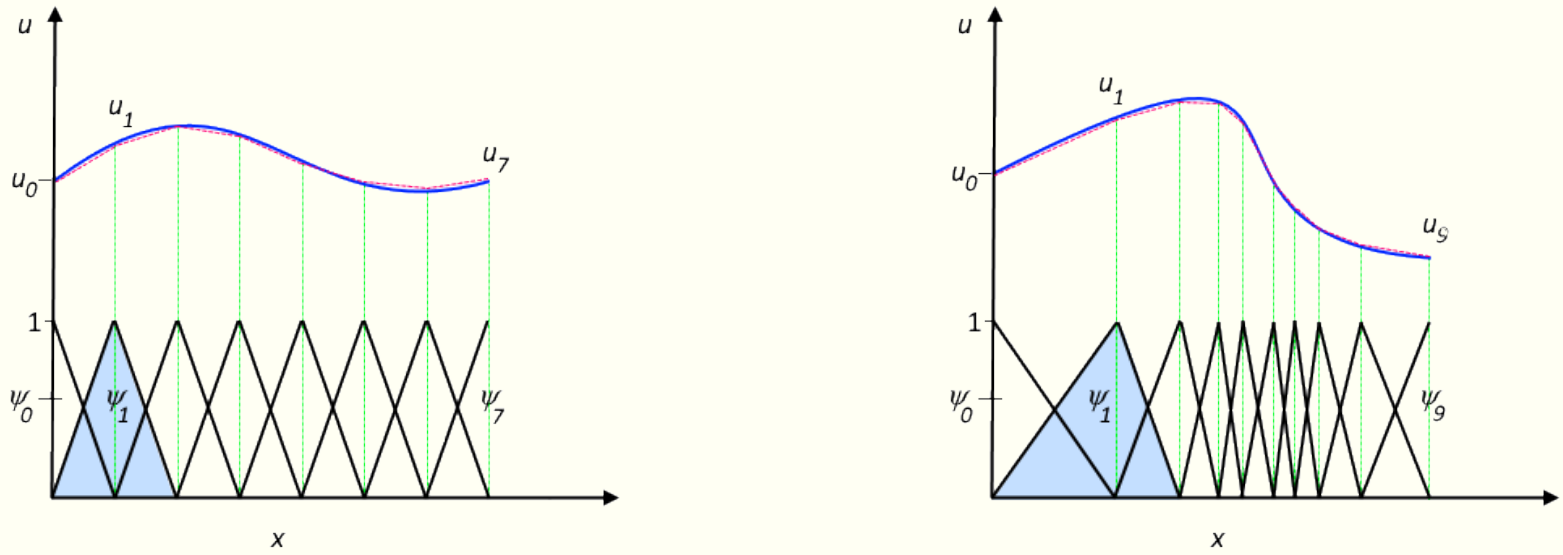
Numerical methods for partial differential equations are tools for finding approximate solutions and are normally used with the aid of a computer. A number of numerical methods are closely linked to a variational form of the differential equation. A group of such methods are known as Galerkin methods. The Ritz method is an example of a Galerkin method, and the finite element method is another. If a numerical method is properly formulated (and the equation is stable), the more effort (read computer time) that is expended, the closer one gets to the exact solution.

Finite Element Approximations

FE approximation of u, which is a dependent variable in a PDE.



FE basis functions

The function u can be approximated by a function u_h using linear combinations of basis functions according to the following expressions:

$$u \approx u_h \quad u_h = \sum_i u_i \psi_i$$

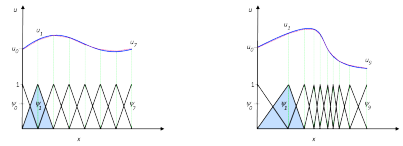
Finite Element Approximations

The reality of partial differential equations is that in most cases it is not possible to find an analytical solution. This is particularly so for equations on complicated geometries (as is common in engineering), nonlinear equations and equations with complicated source terms and boundary conditions.

Instead, an approximation of the equations needs to be constructed, typically based upon different types of discretizations. These discretization methods approximate the PDEs with numerical model equations, which can be solved using numerical methods. The solution to the numerical model equations are, in turn, an approximation of the real solution to the PDEs. The finite element method (FEM) is used to compute such approximations.

Finite Element Approximations

FE approximation of u, which is a dependent variable in a PDE.



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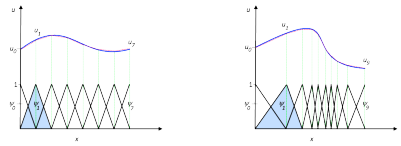
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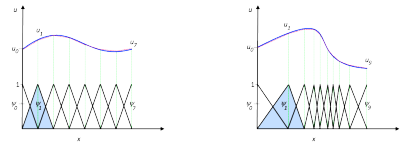
Take, for example, a function u that may be the dependent variable in a PDE (i.e., temperature, electric potential, pressure, etc.) The function u can be approximated by a function uh using linear combinations of basis functions according to the following expressions:

$$u \approx u_h \quad u_h = \sum_i u_i \psi_i$$

Here, ψ_i denotes the basis functions and u_i denotes the coefficients of the functions that approximate u with uh. The figure below illustrates this principle for a 1D problem. u could, for instance, represent the temperature along the length (x) of a rod that is nonuniformly heated. Here, the linear basis functions have a value of 1 at their respective nodes and 0 at other nodes. In this case, there are seven elements along the portion of the x-axis, where the function u is defined (i.e., the length of the rod).

Finite Element Approximations

FE approximation of u, which is a dependent variable in a PDE.



FE basis functions

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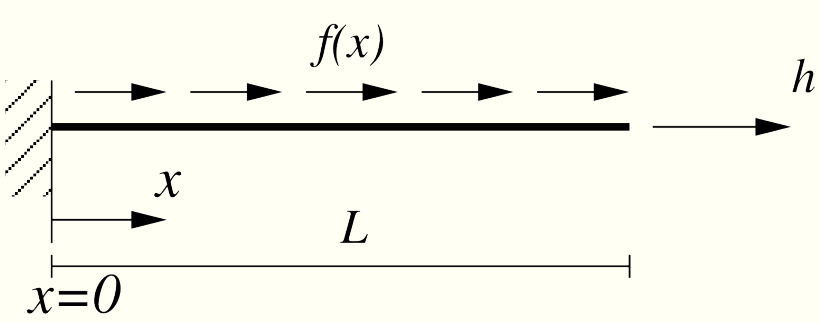
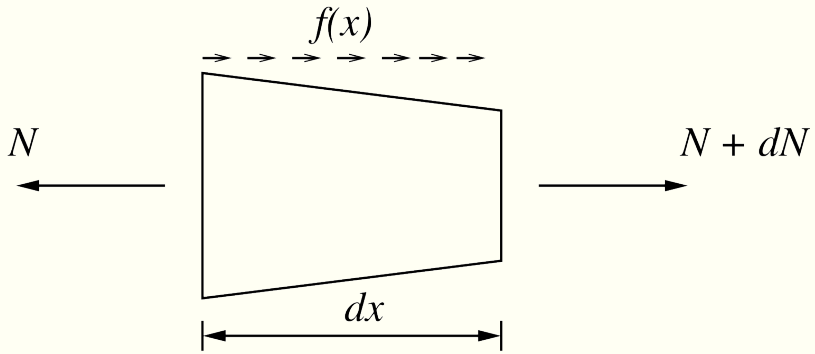
One of the benefits of using the finite element method is that it offers great freedom in the selection of discretization, both in the elements that may be used to discretize space and the basis functions. In the figure above, for example, the elements are uniformly distributed over the x-axis, although this does not have to be the case. Smaller elements in a region where the gradient of u is large could also have been applied.
Both of these figures show that the selected linear basis functions include very limited support (nonzero only over a narrow interval) and overlap along the x-axis. Depending on the problem at hand, other functions may be chosen instead of linear functions.

`https://www.comsol.com/multiphysics/finite-element-method`

Strong form of the equilibrium equation for a 1-D bar





where f is a distributed force and h as a force applied at the end of the bar

The equilibrium equation can be derived by considering an infinitesimal bar:

$$-\frac{dN}{dx} = f$$

where N is the normal force in the bar and f is the distributed force along the bar.

Boundary value problem of a 1-D bar

For linear elasticity

$$N = A\sigma = EA\frac{du}{dx} = EA\varepsilon$$

where $A(x)$ is the area of the bar, $E(x)$ is Young's modulus u is the displacement and $\varepsilon = du/dx$ is the strain.

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = f$$

which is a second-order differential equation. BCs:

1. $u = 0$ at $x = 0$ (displacement or 'Dirichlet' boundary condition),
2. $EA\varepsilon = h$ at $x = L$ (force or 'Neumann' boundary condition).

We now have a well-defined boundary value problem that can be solved.

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To formulate a boundary value problem, we need to assume a constitutive model which defines the relationship between stress and deformation, and we need to supply boundary conditions.

Dirichlet (or first-type) boundary condition is a type of boundary condition, when imposed on an ordinary or a partial differential equation, it specifies the values that a **solution needs to take on along the boundary of the domain**.

Neumann (or second-type) boundary condition is a type of boundary condition, when imposed on an ordinary or a partial differential equation, the condition specifies the values in which **the derivative of a solution is applied within the boundary of the domain**.

Weak form of the equilibrium equations of a 1D bar

The general derivation of the weak form of any equation from the strong form follows a standard procedure:

1. Multiply the strong equation by a weight function v which is equal to zero where Dirichlet (displacement) boundary conditions are applied, but is otherwise arbitrary (Another condition is that it must be sufficiently continuous. The degree of continuity required depends on the properties of the equation being considered.)
2. Use integration by parts to 'shift' derivatives to the weight function
3. Insert the Neumann (force) boundary conditions

We then want to find a solution u to the weak form that holds for all v . The weight function is also known as the '*test*' function.

Weak form of the equilibrium equations of a 1D bar

Multiplying equilibrium equation by an arbitrary weight function v and integrating along the bar:

$$-\int_0^L v\frac{dN}{dx}\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x.$$

we require that $v(0) = 0$ because of the displacement boundary condition at $x = 0$.

$$\int_0^L \frac{dv}{dx}N\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x + vN|_{x=0}^{x=L}.$$

Since $v(0) = 0$, inserting the constitutive relationship and taking into account the force boundary condition at $x = L$.

$$\int_0^L \frac{dv}{dx}EA\frac{du}{dx}\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x + v(L)h.$$

The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

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The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

Strong form is the conventional PDE. The strong form imposes continuity and differentiability requirements on the potential solutions to the equation. Weak form is an alternate representation of the differential equation. The weak form relaxes these requirements on solutions to a certain extent. This means that a larger set of functions are solutions of the weak form. By construction all solutions of the strong form satisfy the weak form but not vice-versa.

Weak formulations are often referred to as 'variational formulations'. In fact, the weak form is more general than the strong form. The weak form of an equation does not generally make an equation easier to solve analytically (it may make it harder), but is usually a more suitable form for mathematical analysis (allowing us to say things about the properties of the equation without knowing the solution) and for numerical solution methods.

Weak form of the equilibrium equations of a 1D bar

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The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

Weak form asks that the average value of $EA\frac{du}{dx}$ in the entire domain to be equal to the average value of force f. Indeed, it seems "too weak" as compared to the original differential equation, which asks that at every point $EA\frac{du}{dx}$ should be f.

Weak form of the equilibrium equations of a 1D bar

Multiplying equilibrium equation by an arbitrary weight function v and integrating along the bar:

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The task is to find u with $u(0) = 0$ such that this equation is satisfied for all v.

An important observation is that the weak form for the bar contains at most first-order derivatives, whereas the strong form for this problem contained second-order derivatives. We have transferred one derivative from the displacement field u to weight function v using integration by parts. This is a key feature of the weak form and is crucial for finite element methods. If the symbol v in the weak form was replaced by the symbol $\delta\varepsilon$, the weak form in the case of the bar resembles the virtual work equation. It is in fact equivalent. However, the process that we have followed is general and does not require any knowledge of virtual work, and can be applied to any differential equation.

FE formulation of a 1-D bar

Approximates the PDE by replacing the unknown function (the displacement u of the elastic bar) and the weight function (v) by approximate fields u_h and v_h. Inserting these fields into the weak equilibrium equation for a bar:

$$\int_0^L \frac{dv_h}{dx} EA \frac{du_h}{dx} \, \mathrm{d}x. = \int_0^L v_h f \, \mathrm{d}x + v_h(L)h.$$

We now allow only a limited number of possibilities for v_h and u_h. Therefore, it is now unlikely that u_h will be equal to the exact solution.

Basis functions

The approximate displacement field u_h is represented by a set of '*basis functions*' $N_i(x)$:

$$u_h(x) = \sum_i^n N_i(x) a_i$$

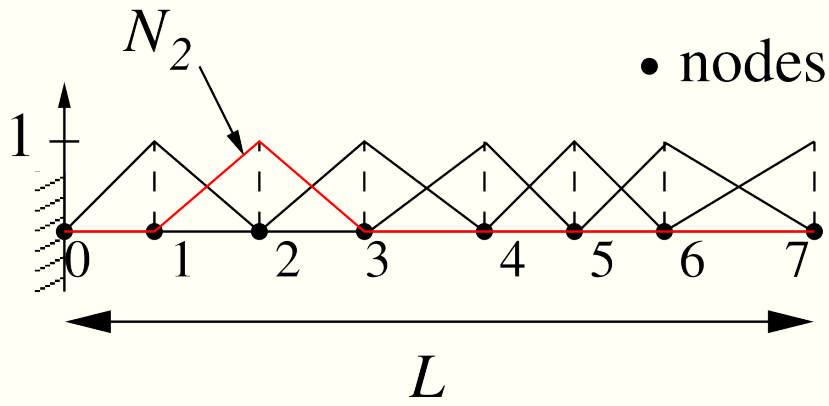
x_i: discrete number of points (known as nodes)
a_i: degrees of freedom
n: number of nodes
The approximate strain field:

$$\varepsilon_h = \frac{du_h}{dx} = \sum_i^n \frac{dN_i(x)}{dx} a_i$$

The task of the FE formulation will be to find the coefficients a_i, which will the approximate solution u_h and ε_h.

FE shape functions

The simplest finite element basis functions in 1D hat-like continuous, piece-wise linear polynomials.

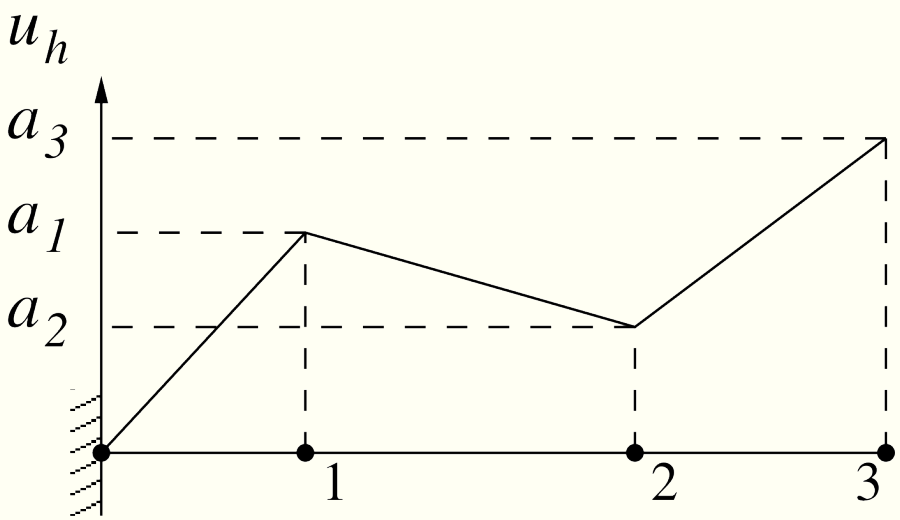


Each node i has its own shape function and its own degree of freedom. A shape function is equal to one at its own node, and zero at all others.

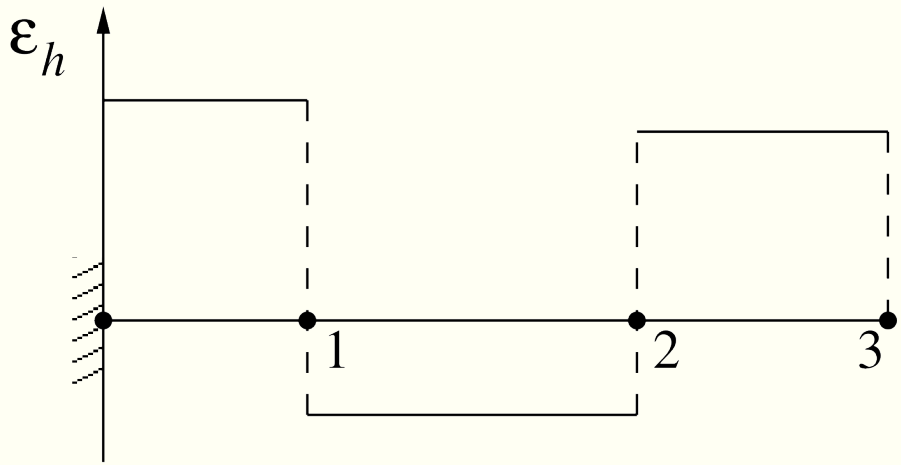
$$N_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} < x \leq x_i \,, \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x < x_{i+1} \,, \\ 0 & \text{otherwise.} \end{cases}$$

FE shape functions

For a bar divided into three elements, the displacement and strain fields could have the form



(a) displacement field



(b) strain field

FE formulation of a 1-D bar

Weak form:

$$\int_0^L EA \frac{dv_h}{dx} \frac{du_h}{dx} \, \mathrm{d}x. = \int_0^L v_h f \, \mathrm{d}x + v_h(L)h.$$

Using basis functions for u_h and v_h:

$$\int_0^L EA \left(\sum_i^n \frac{dN_i}{dx} a_i^* \right) \left(\sum_j^n \frac{dN_j}{dx} a_j \right) \, \mathrm{d}x = \int_0^L \left(\sum_i^n N_i a_i^* \right) f \, \mathrm{d}x + \left(\sum_i^n N_i(L) a_i^* \right) h$$

since a_i^* and a_j are not a function of x, we take it out.

$$\sum_i^n a_i^* \left(\sum_j^n a_j \int_0^L EA \frac{dN_i}{dx} \frac{dN_j}{dx} \, \mathrm{d}x \right) = \sum_i^n a_i^* \int_0^L N_i \, f \, \mathrm{d}x + \sum_i^n N_i(L) a_i^* \, h$$

FE formulation of a 1-D bar

Since a_i^* is arbitrary for each i we set $a^*_{k=i} = 1$ and $a^*_{k\neq i} = 0$. Then for each i we have an equation with n unknowns (the values of a_j):

$$i = 1: \quad \sum_j^n a_j \int_0^L EA \frac{dN_1}{dx}\frac{dN_j}{dx} dx = \int_0^L N_1 f dx + N_i(L)h,$$

$$i = 2: \quad \sum_j^n a_j \int_0^L EA \frac{dN_2}{dx}\frac{dN_j}{dx} dx = \int_0^L N_2 f dx + N_i(L)h,$$

$$\vdots$$

$$i = n: \quad \sum_j^n a_j \int_0^L EA \frac{dN_n}{dx}\frac{dN_j}{dx} dx = \int_0^L N_n f dx + N_i(L)h,$$

FE formulation of a 1-D bar

A system of linear equations is most conveniently expressed as a matrix:

$$\mathbf{Ka} = \mathbf{b}$$

Stiffness matrix:

$$K_{ij} = \int_0^L EA \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

right-hand side vector:

$$b_i = \int_0^L N_i f dx + N_i(L)h.$$

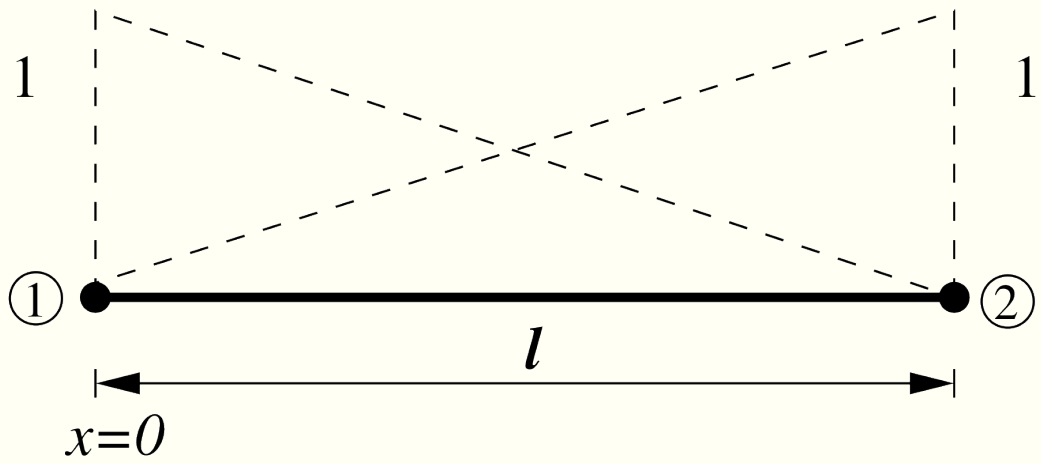
Finite Element Method: Formulation

$$[\mathbf{K}]\,\mathbf{u} = F$$

$$\mathbf{u} = [\mathbf{K}]^{-1}\,F$$

	Property [K]	**Behavior $\{\mathbf{u}\}$**	**Action $\{\mathbf{F}\}$**
Elastic	stiffness	displacement	force
Thermal	conductivity	temperature	heat source
Fluid	viscosity	velocity	body force

Linear shape functions



A polynomial shape function is equal to one at its own node, and zero at all other nodes of the element

The displacement field inside the element is given by

$$u_h(x) = N_1(x)a_1 + N_2(x)a_2$$
$$= \left(-\frac{x}{l} + 1\right) a_1 + \left(\frac{x}{l}\right) a_2$$

Linear shape functions: displacements

Writing displacement field using matrices and vectors:

$$u_h = \mathbf{N}\mathbf{a_e}$$

where the matrix $\mathbf{N}$ has the shape functions:

$$\mathbf{N} = [N_1(x) \quad N_2(x)]$$

Matrix $\mathbf{a_e}$ contains the degrees of freedom for an element:

$$\mathbf{a_e} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Linear shape functions: strains

The strain field is written as:

$$\begin{aligned}\varepsilon_h(x) &= \frac{dN_1(x)}{dx} a_1 + \frac{dN_2(x)}{dx} a_2 \\ &= \left(-\frac{1}{l}\right) a_1 + \left(\frac{1}{l}\right) a_2\end{aligned}$$

Strain inside an element:

$$\varepsilon_h = \mathbf{B}\mathbf{a_e}$$

where the matrix $\mathbf{B}$ is the derivatives of the shape functions:

$$\mathbf{B} = \begin{bmatrix} \frac{dN_1(x)}{dx} & \frac{dN_2(x)}{dx} \end{bmatrix}$$

Linear shape functions: strains

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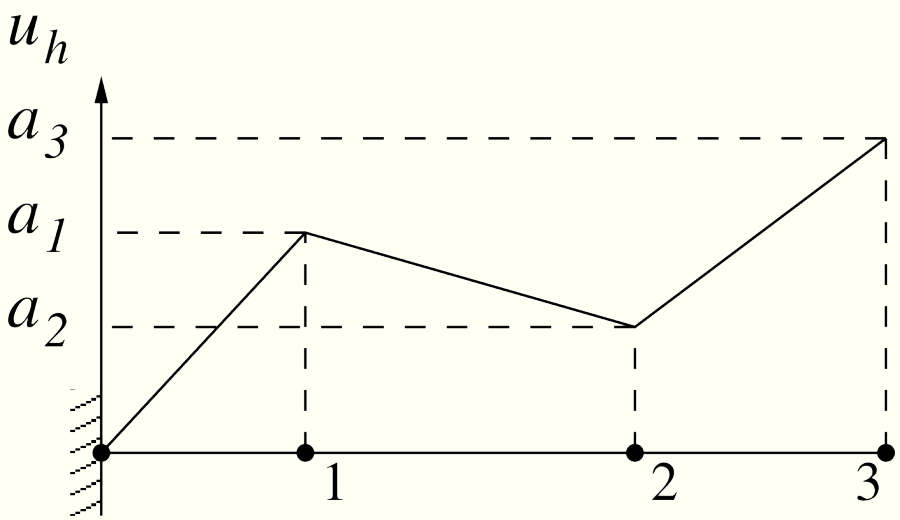
$$\mathbf{B} = \begin{bmatrix}\frac{dN_1(x)}{dx} & \frac{dN_2(x)}{dx}\end{bmatrix}$$

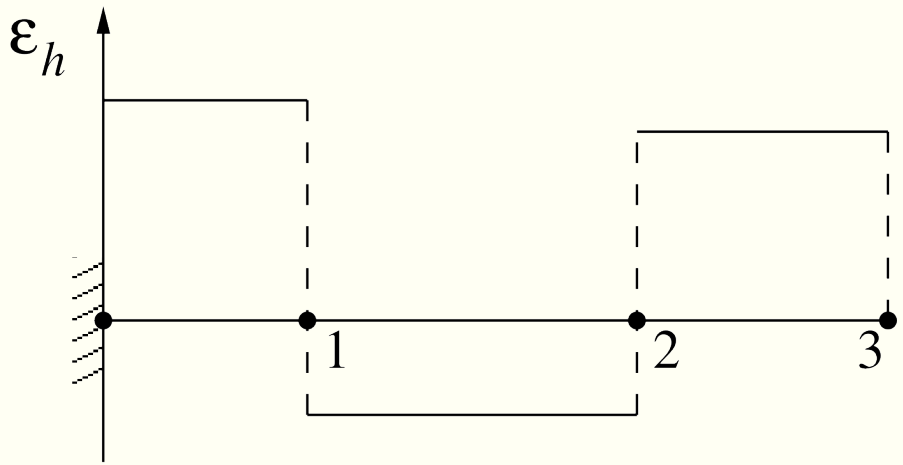
Using linear shape functions means that we are trying represent the exact solution using a collection of linear functions. Obviously, if the solution does not resemble a linear function and we use a limited number of elements, the result will not be very accurate. There are two ways to improve the accuracy; the obvious method is to use smaller elements, and the other is to use higher order polynomials within each element.

Continuity of finite element functions

For a bar divided into three elements, the displacement and strain fields could have the form:

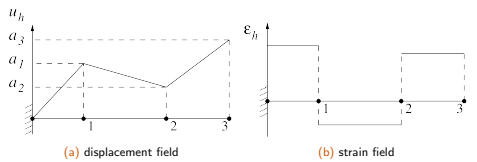


(a) displacement field



(b) strain field

For a bar divided into three elements, the displacement and strain fields could have the form:



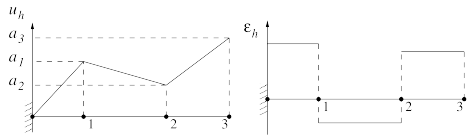
(a) displacement field (b) strain field

An important point is the continuity of the approximate displacement field u_h. For the shape functions that we have introduced, the field u_h is continuous, but its first derivative is discontinuous irrespective of the polynomial order of the basis functions. This is easy to see by considering two elements and computing the displacement and strain fields on either side of the node shared by the two elements.

It does not matter how high the polynomial order of the shape functions is, the strain will be discontinuous at element boundaries.

We can use these simple equations to construct shape functions that have discontinuous first derivatives in FEA of a bar because the weak form of the equation contains at most first-order derivatives. These shape functions are known as Lagrange polynomials. In the FEM, function with discontinuous derivatives are referred to as C^0 continuous functions. It turns out that these functions are not suitable for thin beam bending problems, and we will need to construct a different type of shape function for beams.

For a bar divided into three elements, the displacement and strain fields could have the form:



(a) displacement field (b) strain field

For the two elements: the displacement at node 2 from the left-hand side is given by:

$$u_h(x_2^-) = N_1(x_2)a_1 + N_2(x_2)a_2 = a_2$$

since $N_2 = 1$ at x_2 and all other shape functions are equal to zero. Similarly from the right-hand side of the node,

$$u_h(x_2^+) = N_2(x_2)a_2 + N_3(x_2)a_3 = a_2$$

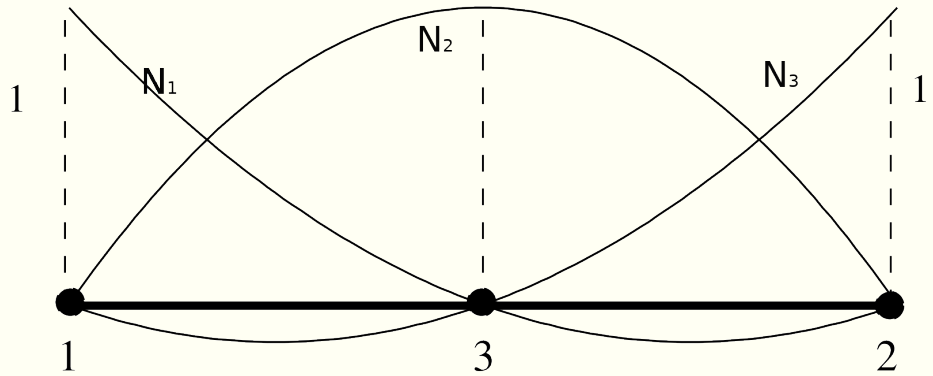
Now for the strain:

$$\epsilon_h(x_2^-) = \frac{dN_1(x_2)}{dx}a_1 + \frac{dN_2(x_2)}{dx}a_2; \quad \frac{dN_1(x_2)}{dx} \neq 0 \quad \& \frac{dN_2(x_2)}{dx} \neq 0$$

$$\epsilon_h(x_2^+) = \frac{dN_2(x_2)}{dx}a_1 + \frac{dN_3(x_2)}{dx}a_2; \quad \frac{dN_2(x_2)}{dx} \neq 0 \quad \& \frac{dN_3(x_2)}{dx} \neq 0$$

Clearly the strain is not continuous.

Quadratic element: shape functions



The shape functions:

$$N_1 = a_1 + b_1 x + c_1 x^2\,,$$
$$N_2 = a_2 + b_2 x + c_2 x^2\,,$$
$$N_3 = a_3 + b_3 x + c_3 x^2$$

$x_1 = -1$, $x_2 = 1$ and $x_3 = 0$:

$$N_1 = \frac{x^2}{2} - \frac{x}{2}\,,$$
$$N_2 = \frac{x^2}{2} + \frac{x}{2}\,,$$
$$N_3 = -x^2 + 1$$

Quadratic element: shape functions

The shape functions must satisfy three conditions and will be cubic polynomials of the form $N_i = a_i + b_i\, x + c_i\, x^2$. The SF must be equal to one at their node and zero at all others:

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} N_i(x_1) \\ N_i(x_2) \\ N_i(x_3) \end{bmatrix}$$

Inverting,

$$\begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} N_i(x_1) \\ N_i(x_2) \\ N_i(x_3) \end{bmatrix} = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$$

For node one, $[N_i(x_1) \quad N_i(x_2) \quad N_i(x_3)]^T = [1 \quad 0 \quad 0]^T$, and for node two $[0 \quad 0 \quad 1]^T$, for node three $[0 \quad 1 \quad 0]^T$. This leads to

$$N_1 = \frac{x^2}{2} - \frac{x}{2}, \quad N_2 = \frac{x^2}{2} + \frac{x}{2}, \quad N_3 = -x^2 + 1$$