CE394M: Linear Elasticity

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Overview

Isotropic linear elastic stress-strain relations

The linear relationship between the stress and strain tensor is a linear one. The stress component is a linear combination of the strain tensor:

$$\sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13} + C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23} + C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}$$

The most general form for *linear* stress-strain relations for a *Cauchy elastic* material is given by:

Where B_{ij} is the components of initial stress tensor corresponding to the initial strain free (when all strain components $\varepsilon_{kl} = 0$). C_{ijkl} is the tensor of material *elastic constants*.
If it is assumed that the initial strain free state corresponds to an *initial stress free state*, that is $B_{ij} = 0$, the equations reduces to:

Observation on linear elasticity

Stress-strain relationship

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

C_{ijkl} is a tensor of material *elastic constants*. However, the above $[\mathbf{C}]$ is not a tensor anymore. So we can not rotate the matrix to another frame of reference. This relationship is useful for isotropic materials, where $\mathbf{C}$ is independent of the frame of reference.

The inverse of the relationship (Compliance matrix):

Isotropic Linear Elastic Stress-strain relationship

The *isotropic tensor* C_{ijkl}:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

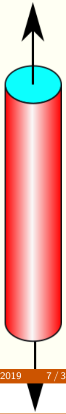
Where $\lambda, \mu,$ and $, \alpha$ are scalar constants. Since C_{ijkl} must satisfy symmetry, $\alpha = 0$.

So the stress:

Hence for an isotropic linear elastic material, there are only two independent material constants, λ and μ, which are called *Lame's constants*.

Hooke's law

Empirical observation:



Where E is defined as the *Young's modulus*.
The lateral strains are defined as:

Using superposition for principal stresses:

$$\begin{aligned}
\varepsilon_{11} &= (1/E)\left[\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right] \\
\varepsilon_{22} &= (1/E)\left[-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right] \\
\varepsilon_{33} &= (1/E)\left[-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right]
\end{aligned}$$

Hooke's law

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix}$$

It is possible to invert the matrix to obtain the generalized Hooke's law:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \\ \sigma_{11} \end{Bmatrix} = \alpha \begin{bmatrix} (1-\mu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

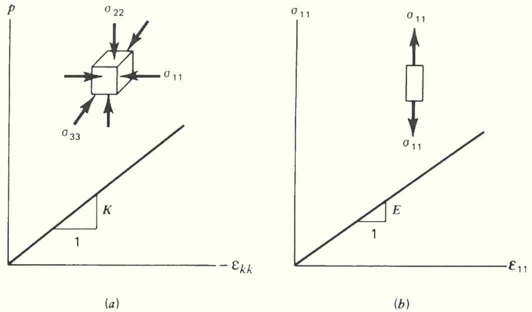
Where $\alpha = E/((1+\mu)(1-2\mu))$. Similarly, we can obtain the inverse matrix.

Hooke's law

The matrices [**C**] and [**D**] contains two indepdent variables E and μ, where $E > 0$ and $-1 \leq \mu \leq 0.5$. The matrix can also be defined in terms of Lame's constants.

$$\begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} & \text{Lame's modulus (wave propagation)} \\ \mu = G = \frac{E}{2(1+\nu)} & \text{Shear modulus (shear behavior)} \\ K = \frac{E}{3(1-2\nu)} & \text{Bulk modulus (volumetric behavior)} \end{cases}$$

Isotropic linear elastic



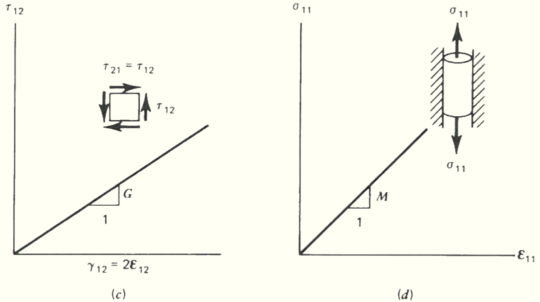
Behavior of isotropic linear elastic material in simple tests: (a) hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$) and (b) simple tension test (Chen 1994)

Isotropic linear elastic

Hydrostatic compression test The non-zero components of stress:
The *Bulk modulus, K*, is defined as the ratio between the *hydrostatic pressure p* nd the corresponding volume change $\delta\varepsilon_v = \varepsilon_{kk}$.

Simple tension test The only non-zero components of stress:
The *Young's modulus, K*, is defined as the ratio between the *hydrostatic pressure p* nd the corresponding volume change $\Delta\varepsilon_v = \varepsilon_{kk}$.

Isotropic linear elastic



Behavior of isotropic linear elastic material in simple tests: (c) pure shear test, and (d) uniaxial strain test (Chen 1994)

Isotropic linear elastic

Simple shear test
The non-zero components of stress:
The *Shear modulus, G or μ*, is defined as:

Uniaxial strain test The test is carried out by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical sample, whose lateral surface is *restrained* against lateral movement (Oedometer test). Axial strain $\varepsilon_1 1$ is the only nonvanishing component. The *constrained modulus* M or as PLAXIS calls it E_{oed} is defined as the ratio between σ_{11} and $\varepsilon_1 1$.

Plane stress v Plane strain

For a frame with an axis perpendicular to the plane of interest, x_3 or z:
Plane stress

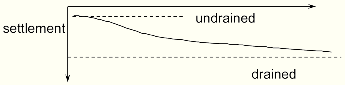
.
The strain in z is written as:

Plane strain

.
The stress in z is written as:

Elastic solutions

Pore-pressure analysis in geotechnical engineering



- Time independent
  - Total stress analysis
    - Undrained
  - Effective stress analysis
    - Drained
    - Undrained
- Consolidation
  - Coupled analysis in Effective stress

Drained analysis - Effective stress

1. Need to assign initial effective stresses before the analysis.
2. Can use any effective stress model: Elastic, Mohr-Coulomb/Drucker Prager and Cam-Clay models.
3. If plasticity models are used, need to update the effective stresses at each increment:
4. Very common.

Undrained analysis - Total stress

- Excess pore pressure cannot be calculated.
- Effective stress state of the soil cannot be examined.
- Elastic model is commonly used for deformation
- Von-Mises model is used for modeling undrained shear strength of clays. (C_u or s_u and undrained friction $\phi_u = 0$)
- Can assign different stiffness and strength at different depths explicity by assigning different model parameters at different depths.

Undrained analysis - Effective vs Total stress

Undrained analysis - Effective stress

- Need to assign initial effective stresses before the analysis.
- Can use any effective stress model, so the stiffness and strength variation with depth can be modeled implicitly with the one set of model parameters.
- The applied load is carried by the soil skeleton and pore water.
- The contribution of the bulk modulus of water needs to be added:
- Effective stress increment can be computed by:
- Need to update the effective stresses at each time step.

Effective stress approach or undrained (A)

- Effective stiffness and effective strength parameters are used.
- *Pore pressures are generated*, but may be **inaccurate** depending on the model.
- Undrained shear strength is *not* an input parameter but an outcome of the constitutive model. The resulting shear strength must be checked against known data!
- Consolidation analysis can be performed after the undrained calculation, which *affects the shear strength*!

Equivalent effective stress approach or undrained (B)

- Effective stiffness parameters and *undrained strength parameters* are used.
- *Pore pressures are generated*, but may be hightly **inaccurate**.
- Undrained shear strength is an input parameter.
- Consolidation analysis should not be performed after the undrained calculation, s_u must be updated, if consolidation is performed anyway!

Methods of undrained analysis for Mohr-Coulomb clay

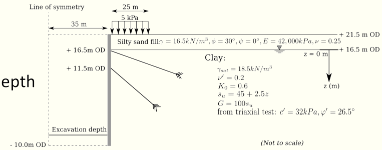
undrained analysis	**material type**	**deformation parameters**	**strength parameters**	**initial conditions**
Total stress	Non-porous / drained	E_u, ν_u	$c_u, \phi_u = 0$	$K_{0,u}$
Effective stress	Undrained (triaxial parameters)	E', ν'	c', ϕ'	K_0
Equivalent Effective stress	Undrained (strength profile)	E', ν'	c', ϕ'	K_0

Consolidation analysis - Effective stress

- Use Biot's 3D consolidation theory
- Pore pressure and displacement are computed at each time step.
- Need to use effective stress model
- Need permeability
- Lots of computational time
- More realistic. Undrained, partially drained, drained depending on the loading condition, drainage condition, permeability of soil.
- Stress path followed is correct, which should provide a good strain estimate when plasticity models are used.

Total stress evaluating varying K_0

- K_0 Varies with depth

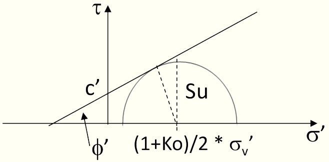


Depth	σ_v	u	σ_v'	$k_0\sigma_v'=\sigma_h'$	σ_h	K_0	K_0 Layer
0	82.5	0	82.5	49.5	49.5	0.6	
6.5	202.75	65	137.75	82.65	147.65	0.728	0.664
11.5	295.25	115	180.25	108.15	223.15	0.756	0.742
16.5	387.75	165	222.75	133.65	298.65	0.77	0.763

Undrained analysis using effective stress method

Effective stress Method A

- Define c' and ϕ' in terms of the real effective stress parameters, assuming zero dilation.
- ν' is the effective Poisson ratio
- E and K_0 should be '*effective stress*' based.



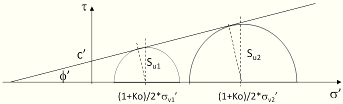
Mohr-Coulomb failure criteria: $\tau = c' + \sigma' \tan \phi'$

$$s_u = c' \cos \phi' + (1 + K_0)/2 \cdot \sin \phi' \cdot \sigma'_{v0}.$$

Undrained analysis using equivalent effective stress method

Effective stress Method B

- Define c' and ϕ' in terms of the "*equivalent*" effective stress parameters, with zero dilation. Parameters are defined based on the strength profile with depth.
- ν' is the effective Poisson ratio
- E and K_0 should be '*effective stress*' based.

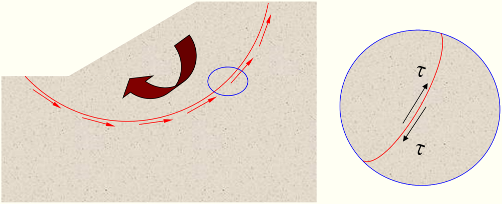


Mohr-Coulomb failure criteria: $\tau = c' + \sigma' \tan \phi'$

$$s_u = c' \cos \phi' + (1 + K_0)/2 \cdot \sin \phi' \cdot \sigma'_{v0}.$$

2014 Oso landslide

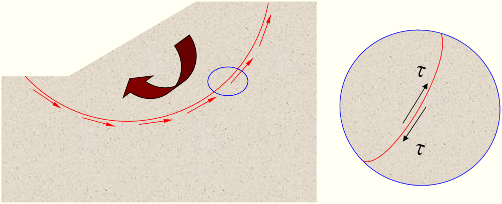
Shear failure plane



At failure, shear stress along the failure surface (τ) reaches the shear strength (τ_f).

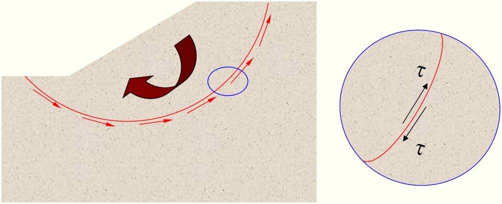
Factor of Safety =

Dry slope (total stress = effective stress)



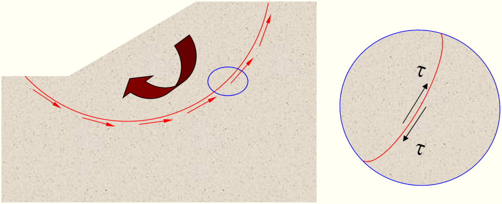
Saturated slope (total stress = effective stress + pwp)

Drained conditions - need to compute the steady state pore pressure field and then evaluate "effective stress-based" shear strength to find the overall stability (based on total stress equilibrium).



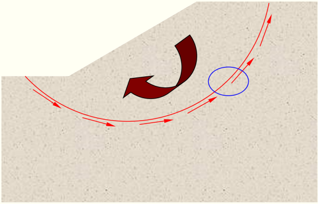
Saturated slope (total stress = effective stress + PWP)

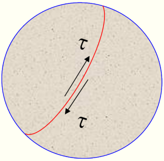
Undrained conditions - (total stress approach) – Use "*total-stress based*" shear strength (s_u) to find the overall stability (based on total stress equilibrium).



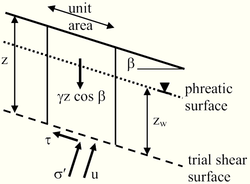
Saturated slope (total stress = effective stress + PWP)

Undrained conditions - (Effective stress approach-not common) - need to compute the pore pressure (including excess pore pressure) field and then evaluate "*effective stress -based*" shear strength to find the overall stability (based on total stress equilibrium).





Infinite slopes

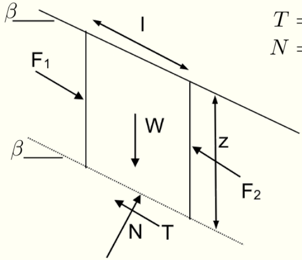


$$
\begin{aligned}
u &= \gamma_w z_w \cos^2\beta \\
\sigma &= \gamma z \cos^2\beta \\
\sigma' &= (\gamma z - \gamma_w z_w) \cos^2\beta \\
\tau &= \gamma z \cos\beta \sin\beta
\end{aligned}
$$

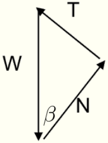
$$
\tan\phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan\beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}
$$

Soil fails when (dry):
Soil fails when (submerged):

Undraned infinite slope (total stress approach)

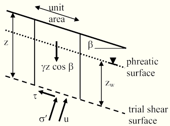


$$W = \gamma z l \cos(\beta)$$
$$T = W \sin(\beta)$$
$$N = W \cos(\beta)$$



But also the shear stress:

Infinite slope: Summary



$u = \gamma_w z_w \cos^2\beta$
$\sigma = \gamma z \cos^2\beta$
$\sigma' = (\gamma z - \gamma_w z_w) \cos^2\beta$
$\tau = \gamma z \cos\beta \sin\beta$

$$\tan\phi_{mob} = \frac{\tau}{\sigma'} = \frac{\tan\beta}{\left(1 - \frac{\gamma_w z_w}{\gamma z}\right)}$$

1. Factor of Safety = resistance / driving
2. Dry FoS = $\tan(\phi_{\mathrm{mob}})/\tan(\beta)$
3. Submerged FoS = $\tan(\phi_{\mathrm{mob}})/\tan(\beta)$
4. Undrained FoS = $2s_u/\gamma z \sin(2\beta)$
5. Steady state seepage FoS = $(1 - \gamma_w/\gamma)\tan(\phi_{\mathrm{mob}})/\tan(\beta)$ where the water table is located at the slope surface