# CE394M: Introduction to the Finite Element Method

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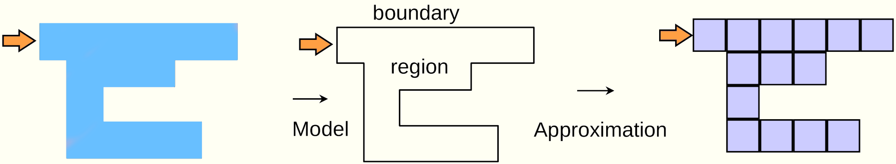
# Overview

1. Numerical analysis of engineering problems
2. Galerkin methods
3. Strong form
4. Weak form
5. Finite Element formulation
6. Shape functions

# Numerical analysis of engineering problems

Physical phenomenon $\xrightarrow[\text{Model}]{}$ Differential equation $\xrightarrow[\text{Approximation}]{}$ Equations of the numerical technique



# Boundary value problems

Differential equations coupled with boundary conditions

- Steady state (time-independent)

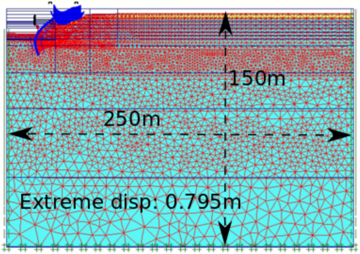
- Transient (time-dependent)

# Numerical solutions to differential equations

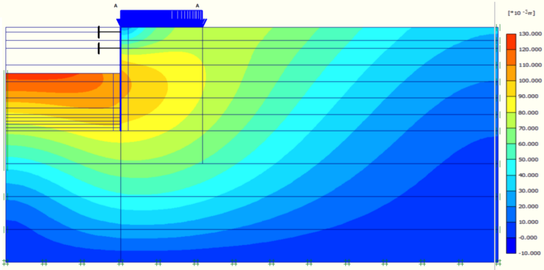
# Finite Difference Method

# Finite Difference Methods

# Finite Element Analysis




(a) FE Mesh



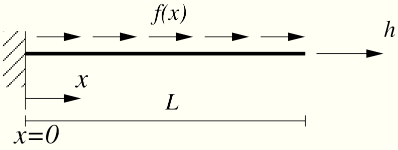

(b) Displacement profile

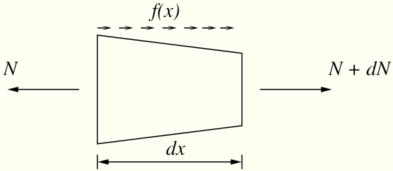
Singapore Nicoll highway excavation FE analysis

# Galerkin:Ritz method

# Finite Element Approximations

# Strong form of the equilibrium equation for a 1-D bar




where $f$ is a distributed force and $h$ as a force applied at the end of the bar

The equilibrium equation can be derived by considering an infinitesimal bar:

where $N$ is the normal force in the bar and $f$ is the distributed force along the bar.

# Boundary value problem of a 1-D bar

For linear elasticity

where $A(x)$ is the area of the bar, $E(x)$ is Young's modulus $u$ is the displacement and $\varepsilon = du/dx$ is the strain.

which is a second-order differential equation. BCs:

# Weak form of the equilibrium equations of a 1D bar

The general derivation of the weak form of any equation from the strong form follows a standard procedure:

1. Multiply the strong equation by a weight function $v$ which is equal to zero where Dirichlet (displacement) boundary conditions are applied, but is otherwise arbitrary (Another condition is that it must be sufficiently continuous. The degree of continuity required depends on the properties of the equation being considered.)
2. Use integration by parts to 'shift' derivatives to the weight function
3. Insert the Neumann (force) boundary conditions

We then want to find a solution $u$ to the weak form that holds for all $v$ . The weight function is also known as the '*test*' function.

# Weak form of the equilibrium equations of a 1D bar

Multiplying equilibrium equation by an arbitrary weight function v and integrating along the bar:

$$-\int_0^L v\frac{dN}{dx}\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x.$$

we require that $v(0) = 0$ because of the displacement boundary condition at $x = 0$.

$$\int_0^L \frac{dv}{dx}N\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x + vN|_{x=0}^{x=L}.$$

Since $v(0) = 0$, inserting the constitutive relationship and taking into account the force boundary condition at $x = L$.

$$\int_0^L \frac{dv}{dx}EA\frac{du}{dx}\,\mathrm{d}x. = \int_0^L vf\,\mathrm{d}x + v(L)h.$$

The task is to find $u$ with $u(0) = 0$ such that this equation is satisfied for all $v$.

# FE formulation of a 1-D bar

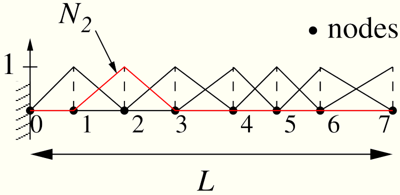
# Basis functions

The approximate displacement field $u_h$ is represented by a set of '*basis functions*' $N_i(x)$:

The approximate strain field:

# FE shape functions

The simplest finite element basis functions in 1D hat-like continuous, piece-wise linear polynomials.



# FE shape functions

For a bar divided into three elements, the displacement and strain fields could have the form

# FE formulation of a 1-D bar

Weak form:

$$\int_0^L EA \frac{dv_h}{dx}\frac{du_h}{dx}\,\mathrm{d}x. = \int_0^L v_h f\,\mathrm{d}x + v_h(L)h.$$

Using basis functions for $u_h$ and $v_h$:

since $a_i^*$ and $a_j$ are not a function of $x$, we take it out.

$$\sum_i^n a_i^* \left(\sum_j^n a_j \int_0^L EA \frac{dN_i}{dx}\frac{dN_j}{dx}\,\mathrm{d}x\right) = \sum_i^n a_i^* \int_0^L N_i\, f\,\mathrm{d}x + \sum_i^n N_i(L) a_i^*\, h$$

# FE formulation of a 1-D bar

Since $a_i^*$ is arbitrary for each $i$ we set $a_{k=i}^* = 1$ and $a_{k\neq i}^* = 1$. Then for each $i$ we have an equation with $n$ unknowns (the values of $a_j$):

$$i = 1: \quad \sum_j^n a_j \int_0^L EA \frac{dN_1}{dx}\frac{dN_j}{dx} dx = \int_0^L N_1 f dx + N_i(L)h,$$

$$i = 2: \quad \sum_j^n a_j \int_0^L EA \frac{dN_2}{dx}\frac{dN_j}{dx} dx = \int_0^L N_2 f dx + N_i(L)h,$$

$$\vdots$$

$$i = n: \quad \sum_j^n a_j \int_0^L EA \frac{dN_n}{dx}\frac{dN_j}{dx} dx = \int_0^L N_n f dx + N_i(L)h,$$

# FE formulation of a 1-D bar

A system of linear equations is most conveniently expressed as a matrix:

Stiffness matrix:

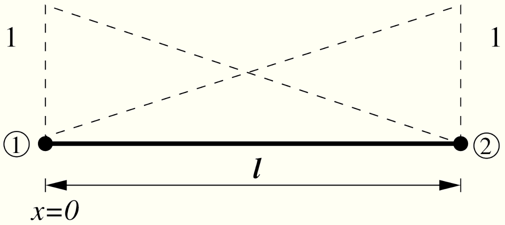
right-hand side vector:

# Finite Element Method: Formulation

$$[\mathbf{K}]\,\mathbf{u} = F$$
$$\mathbf{u} = [\mathbf{K}]^{-1}\,F$$

| | **Property [K]** | **Behavior $\{\mathbf{u}\}$** | **Action $\{\mathbf{F}\}$** |
|---|---|---|---|
| **Elastic** | stiffness | displacement | force |
| **Thermal** | conductivity | temperature | heat source |
| **Fluid** | viscosity | velocity | body force |

# Linear shape functions




The displacement field inside the element is given by

# Linear shape functions: displacements

Writing displacement field using matrices and vectors:

where the matrix **N** has the shape functions:

Matrix $\mathbf{a_e}$ contains the degrees of freedom for an element:

# Linear shape functions: strains

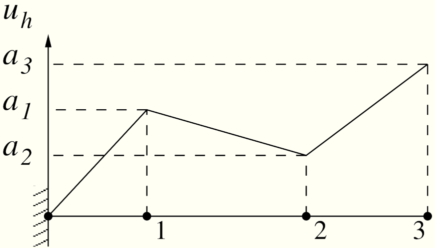
The strain field is written as:

Strain inside an element:

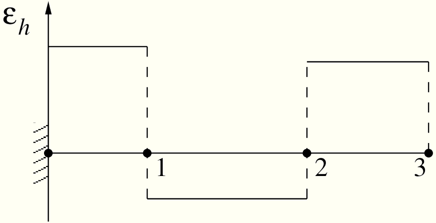
where the matrix **B** is the derivatives of the shape functions:

# Continuity of finite element functions

For a bar divided into three elements, the displacement and strain fields could have the form:




(a) displacement field




(b) strain field

# Quadratic element: shape functions

The shape functions: $x_1 = -1$, $x_2 = 1$ and $x_3 = 0$:

# Quadratic element: shape functions

The shape functions must satisfy three conditions and will be cubic polynomials of the form $N_i = a_i + b_i\, x + c_i\, x^2$. The SF must be equal to one at their node and zero at all others:

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} N_i(x_1) \\ N_i(x_2) \\ N_i(x_3) \end{bmatrix}$$

Inverting,

$$\begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \\ 0.5 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} N_i(x_1) \\ N_i(x_2) \\ N_i(x_3) \end{bmatrix} = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$$

For node one, $[N_i(x_1) \quad N_i(x_2) \quad N_i(x_3)]^T = [1 \quad 0 \quad 0]^T$, and for node two $[0 \quad 0 \quad 1]^T$, for node three $[0 \quad 1 \quad 0]^T$. This leads to

$$N_1 = \frac{x^2}{2} - \frac{x}{2}, \quad N_2 = \frac{x^2}{2} + \frac{x}{2}, \quad N_3 = -x^2 + 1$$