

CE394M: Tresca and Mohr-Coulomb

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Overview

1 Constitutive modeling

2 Tresca model

3 Mohr-Coulomb model

Stress invariants

- The magnitudes of the component of the stress vector depend on the chosen direction of the coordinate axes (in 3D: 6 variables).
- Principal stresses always act on the same planes and have the same magnitude (invariant to the coordinate axes), but still need to define the corresponding orientations (in 3D: 6 variables).
- For isotropic materials, it is very convenient to work with alternative invariant quantities which are combinations of principal stresses.

Stress invariants

- Mean effective stress $p = \frac{1}{3}(\sigma_I + \sigma_{II} + \sigma_{III})$
- Deviatoric stress: $J = \frac{1}{\sqrt{6}}\sqrt{(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2}$
- Lode's angle $\theta = \tan^{-1} \left[\frac{1}{\sqrt{3}} \left(2 \frac{(\sigma_{II} - \sigma_{III})}{\sigma_I - \sigma_{III}} - 1 \right) \right]$

Principal stresses can be expressed in terms of invariants:

$$\begin{bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \end{bmatrix} = p \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{\sqrt{3}} J \begin{bmatrix} \sin \left(\theta + \frac{2\pi}{3} \right) \\ \sin \theta \\ \sin \left(\theta - \frac{2\pi}{3} \right) \end{bmatrix}$$

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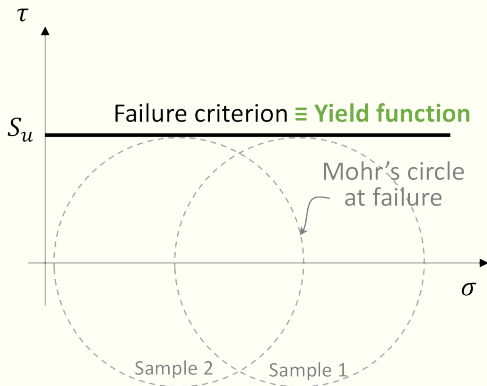
Tresca model

Simulation of undrained behavior of saturated clay

Failure criteria:

$$\tau_f = s_u$$

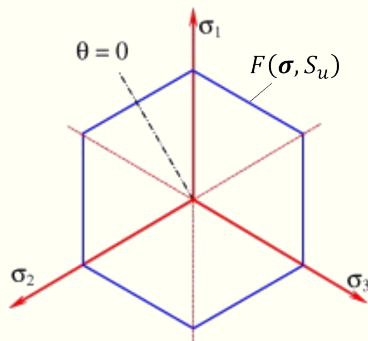
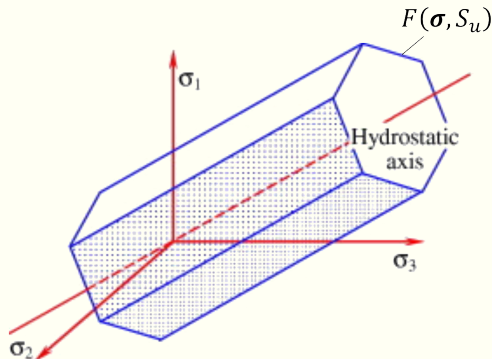
where τ_f is the shear stress at failure. s_u is the undrained strength.



Tresca model

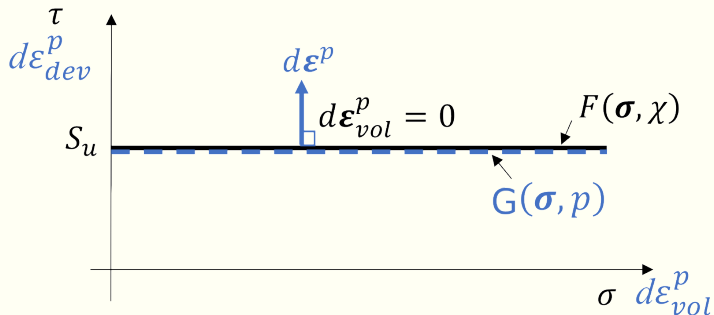
Yield function

$$\begin{aligned}F(\sigma, W_p) &= \sigma_I - \sigma_{III} - 2s_u = 0 \\&= J \cos \theta - s_u = 0\end{aligned}$$



Tresca model

- 1 The model is intended to predict the undrained behavior of saturated clay, hence it should predict zero volumetric strains.
- 2 This fact constrains the plastic potential, and an associated flow rule should be considered.
- 3 In general, this is a perfectly plastic model (no hard/soft law)
- 4 Parameters required (3): s_u, E_u, ν_u .



Overview

1 Constitutive modeling

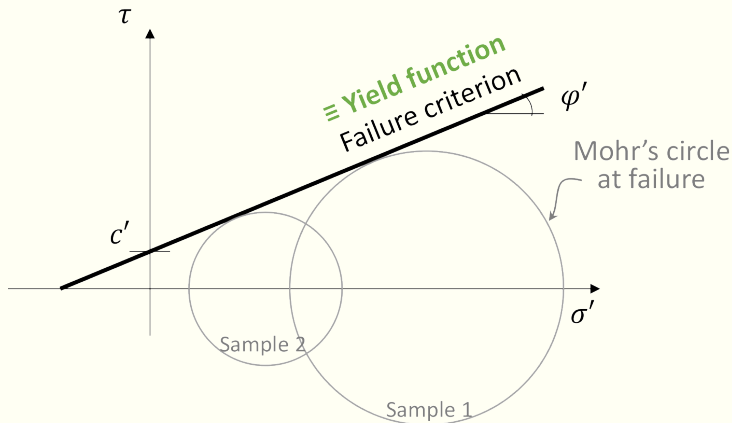
2 Tresca model

3 Mohr-Coulomb model

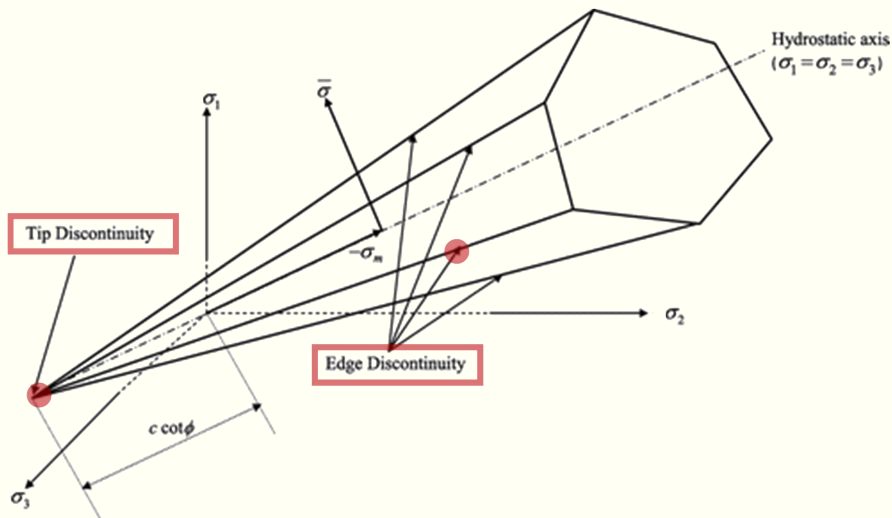
Mohr-Coulomb model

Failure criteria: $\tau_f = c' = \sigma'_f \tan \phi'$.

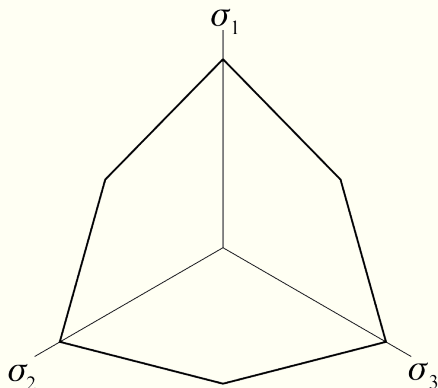
$$F(\sigma', c', \phi') = \frac{1}{2}(\sigma'_I - \sigma'_{III}) - \frac{1}{2}(\sigma'_I + \sigma'_{III}) \sin \phi' - c' \cos \phi' = 0.$$



Mohr-Coulomb model



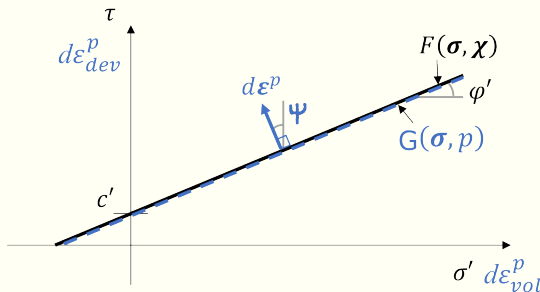
Mohr-Coulomb model



Mohr-Coulomb: flow rule?

Can we consider an associative flow rule? Failure criteria:

$$F(\sigma', c', \phi') = G(\sigma', c', \phi')$$



ϵ^p is inclined at an angle of ϕ' , and indicates negative (tensile) plastic strains. This results in a dilatant volumetric plastic strain ($d\epsilon_{vol}^p$), in which the angle of dilation ψ is equal to the effective friction angle (ϕ').

Mohr-Coulomb: Drawbacks (associative)

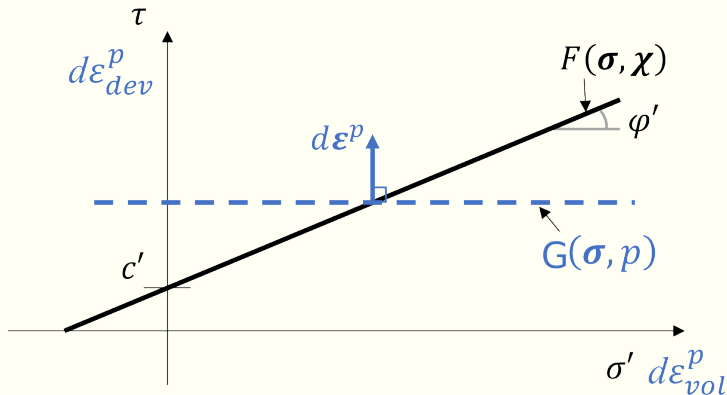
- 1 The magnitude of the plastic volumetric strains ($d\varepsilon$) is much larger than that observed in real soils.
- 2 Once the soil yields it will dilate for ever! (note: real soil, which may dilate initially on meeting the failure surface, will often reach a constant volume condition at large strains).

Solutions:

- 1 Adopt a non-associated flow rule with a more realistic angle of dilation ψ
- 2 Allow angle of dilation vary with plastic strain (hardening/softening law) $\psi = f(\varepsilon)$

Mohr-Coulomb: non-associative model

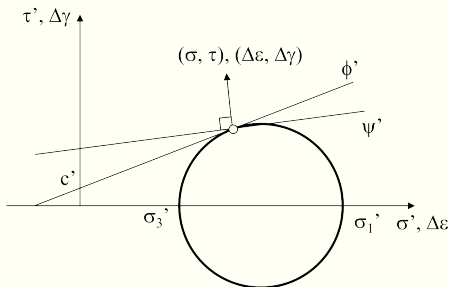
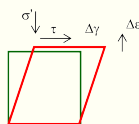
Consider a non-associated model with $\psi = 0$. In which case $d\varepsilon_{vol}^p = 0$.
Parameters (5): c', ϕ', ψ, E, ν .



Mohr-Coulomb: Yield and Potential fn

Yield fn: $F = \sigma'_1 - \sigma'_3 N_\phi + 2c\sqrt{N_\phi} = 0.$, where $N_\phi = \frac{1+\sin \phi'}{1-\sin \phi'}$

Potential fn: $G = \sigma'_1 - \sigma'_3 N_\psi + \text{constant} = 0.$, where $N_\psi = \frac{1+\sin \psi'}{1-\sin \psi'}$

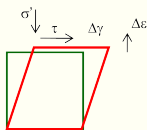


Mohr-Coulomb: plastic strains

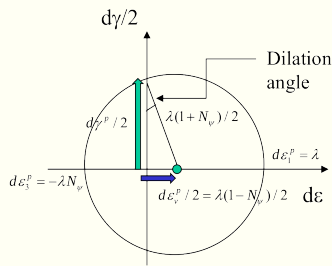
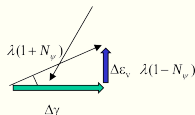
$$d\varepsilon_1^p = \lambda \frac{\partial G}{\partial \sigma_1} = \lambda \cdot 1 = \lambda$$

$$d\varepsilon_2^p = \lambda \frac{\partial G}{\partial \sigma_2} = \lambda \cdot 0 = 0 \quad \text{Zero strain increment in 2-dir}$$

$$d\varepsilon_3^p = \lambda \frac{\partial G}{\partial \sigma_3} = \lambda \cdot (-N_\psi) = -\lambda N_\psi$$



Dilation angle ψ



Interpretation of drained TXC

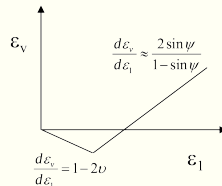
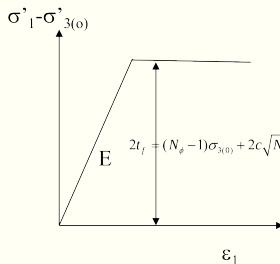
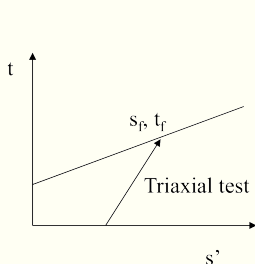
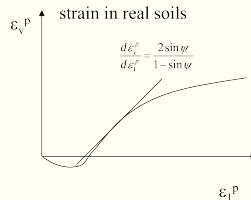
Plastic volumetric strain:

$$d\varepsilon_v^p = d\varepsilon_1^p + d\varepsilon_2^p + d\varepsilon_3^p = \lambda(1 - N_\psi)$$

$$= \lambda \frac{2 \sin \psi}{1 - \sin \psi}$$

$$\frac{d\varepsilon_v^p}{d\varepsilon_1^p} = \frac{\frac{2 \sin \psi}{1 - \sin \psi}}{\lambda} = \frac{2 \sin \psi}{1 - \sin \psi}$$

Note that ψ changes with plastic strain in real soils



Mohr-Coulomb: Tension cut-off

- 1 For $c = 0$, the standard Mohr-Coulomb criterion allows for tension. Allowable stress increase with cohesion.
- 2 In reality, soil can sustain none or only very small tensile stresses.
- 3 This behavior is usually included by specifying a tension cut-off, which is equivalent to include an additional yield surface:

$$F(\sigma, \sigma_t) = \sigma_i - \sigma_t = 0.$$

