CE394M: An introduction to plasticity

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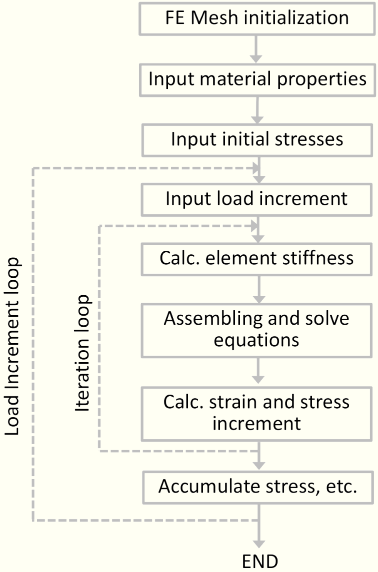
FE workflow



$\boldsymbol{\sigma}$

$\Delta \boldsymbol{f}$

$$K_{ij} \approx \sum_{p}^{n_p} B_i^p D^p B_j^p W_p$$

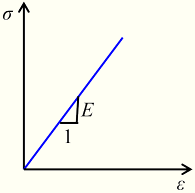
$\Delta \boldsymbol{u} = \boldsymbol{K}^{-1} \cdot \Delta \boldsymbol{f}$ **Equilibrium eq.**

$\Delta \boldsymbol{\varepsilon} = \boldsymbol{B} \cdot \Delta \boldsymbol{u}$ **Compatibility eq.**

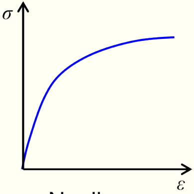
$\Delta \boldsymbol{\sigma} = \boldsymbol{D} \cdot \Delta \boldsymbol{\varepsilon}$ **Constitutive eq.**

$$\boldsymbol{u} = \boldsymbol{u} + \Delta \boldsymbol{u}$$
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + \Delta \boldsymbol{\varepsilon}$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma} + \Delta \boldsymbol{\sigma}$$

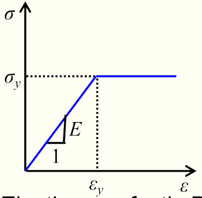
Constitutive law



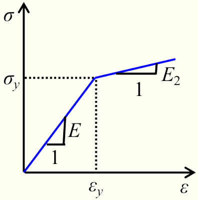
Linearly Elastic



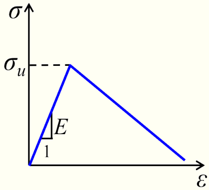
Nonlinear



Elastic – perfectly Plastic

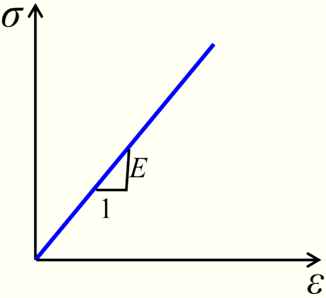


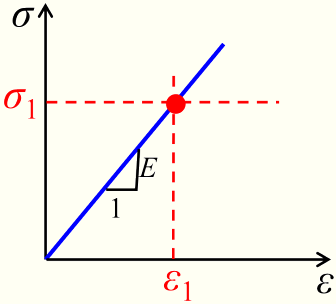
Elastic – Linear Hardening



Elastic – Linear Softening

Isotropic linear elasticity

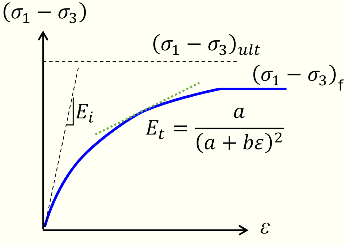




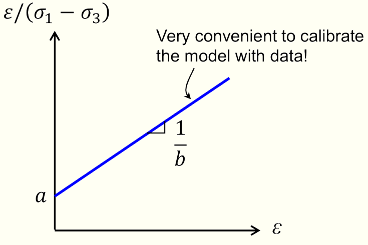
The knowledge of strain alone allows us to obtain the stress value.

Real soil behavior

Hyperbolic model (Duncan and Chang., 1970)



Hyperbolic stress-strain relationship



Transformation of hyperbolic relations

There is no physical meaning to $(\sigma_1 - \sigma_3)_{ult}$. The $(\sigma_1 - \sigma_3)_f$ is determined from the strength criteria $\tau = c + \sigma' \tan \phi'$ (drained) or $\tau = s_u$ (total stress undrained).

Drawbacks of hyperbolic model

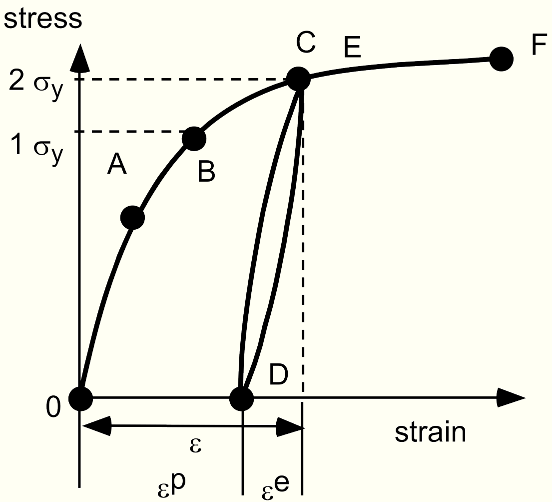
Positive aspects

- Relatively simple and easy to use
- Well validated & calibrated
- Based solely on TX testing
- Good for loading conditions (dams)

Original short comings

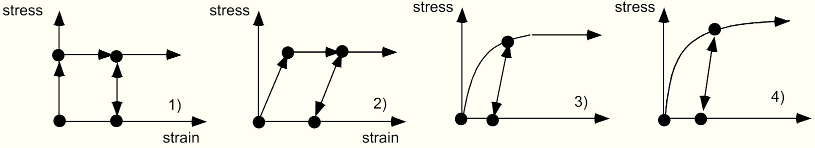
- Not an effective stress model (no treatment of Δu) - Plaxis can do
- No strain softening (still a problem)
- No dilation - important for dense sand / OC clay (Plaxis can do)
- Empirical model
- Poor performance in excavation problems (where it is linear elastic completely)

Classical plasticity

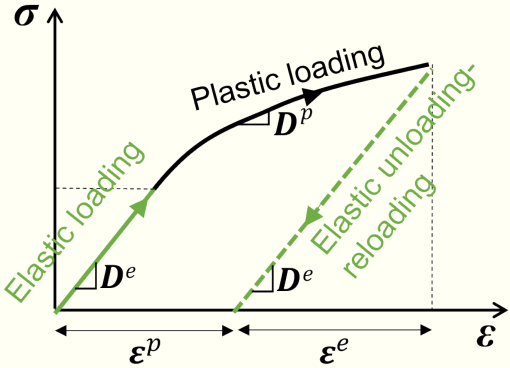


Specimen 1 at point 0 same as specimen 2 at point D, but yield stress of specimen 2 is greater than the yield stress of specimen 1 due to **plastic hardening**.

Classical plasticity



Elasto-plastic materials

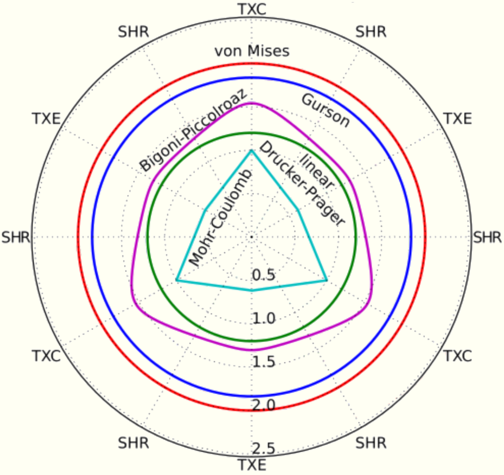


Basic concepts of classical plasticity

To formulate an elasto-plastic constitutive model we need:

Yield functions

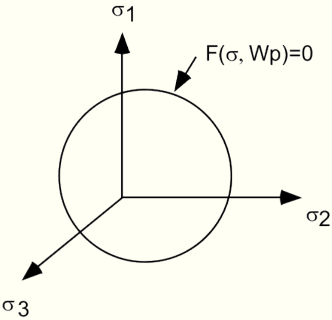
defines when plastic strains occur. If the material is isotropic, we can use the principal stresses to define the stress state.



Wikipedia

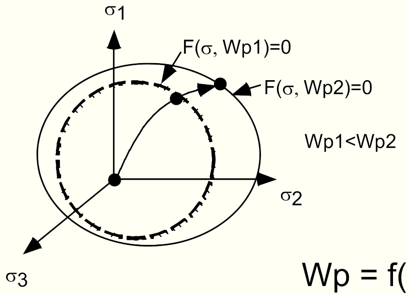
Yield functions

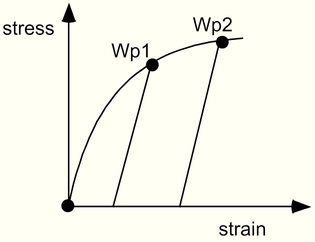
Yield function $F(\sigma, Wp) = 0$.



Hardening law

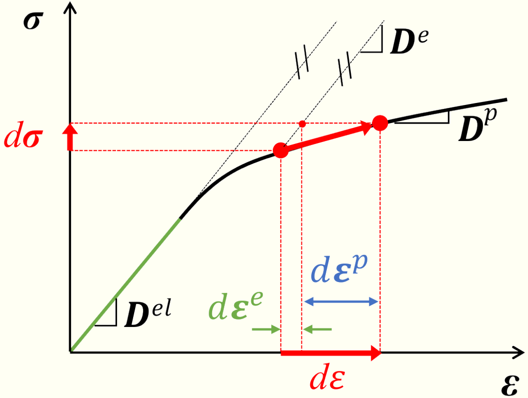
How the threshold of yielding changes with plastic strain or how the yield function changes with plastic strain.





$$Wp = f(\varepsilon^p)$$

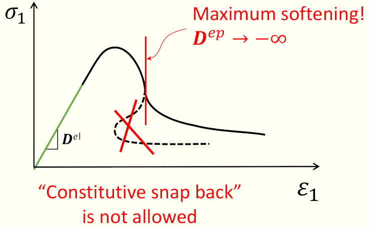
Equations of elasto-plasticity: 1. Stress-strain relation



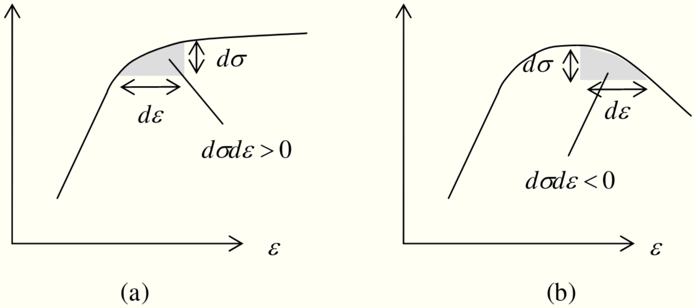
Equations of elasto-plasticity: 2. Flow rule

Drucker's postulate

Drucker (1952) established that for a stable inelastic material in a closed stress cycle, a positive work must be done.

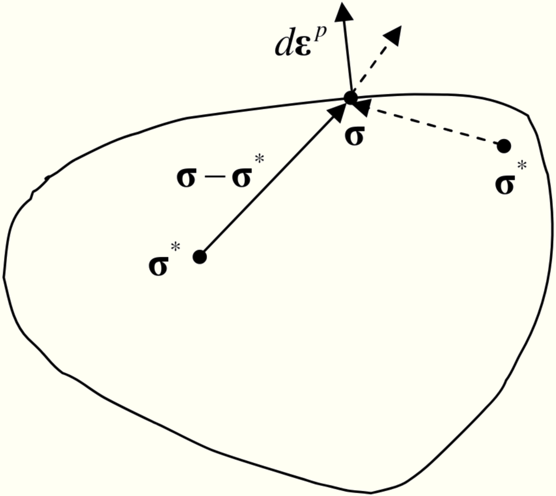


Drucker's postulate of a stable material

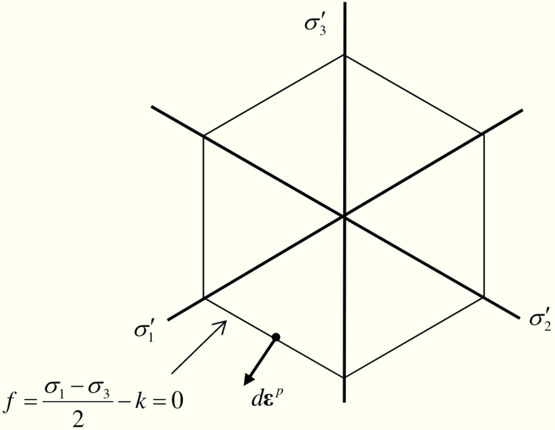


Normality

In terms of vectors in principal stress (plastic strain increment) space: $d\sigma \cdot d\varepsilon^p \geq 0$.

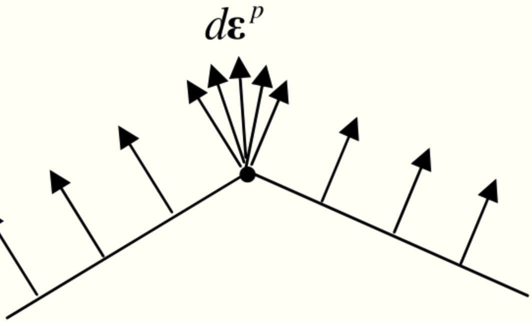


Normality in Tresca condition



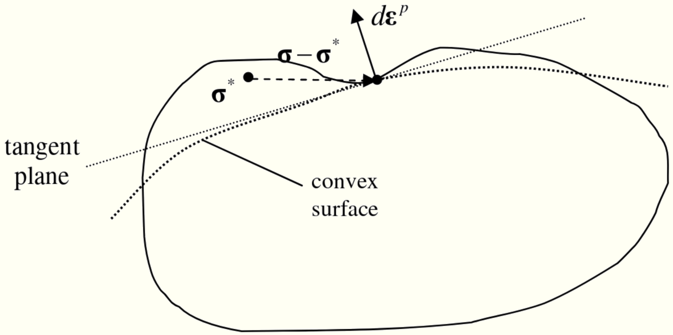
The plastic strain increment vector and the Tresca criterion in the pi-plane (for the associated flow-rule).

Normality in Tresca condition



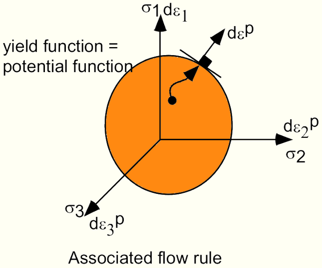
When the yield surface has sharp corners, as with the Tresca criterion, it can be shown that the plastic strain increment vector must lie within the cone bounded by the normals on either side of the corner

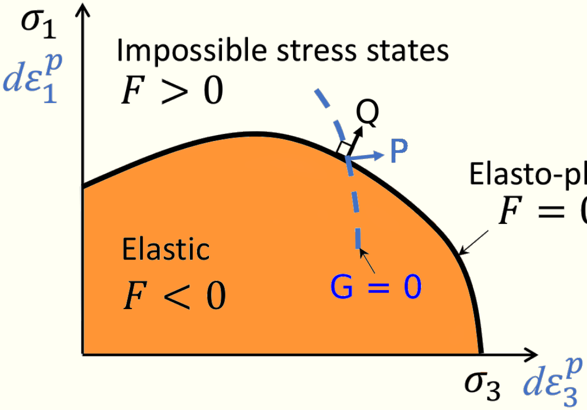
Normality in non-convex surfaces



Using the same arguments, one cannot have a yield surface like the one shown in Fig. In other words, the yield surface is convex: the entire elastic region lies to one side of the tangent plane to the yield surface

Associated and non-associated plasticity




σ_1
$d\varepsilon_1^p$
Impossible stress states
$F > 0$
Q
P
Elasto-plastic
$F = 0$
Elastic
$F < 0$
G = 0
σ_3
$d\varepsilon_3^p$

Equations of elasto-plasticity: 3. Consistency condition

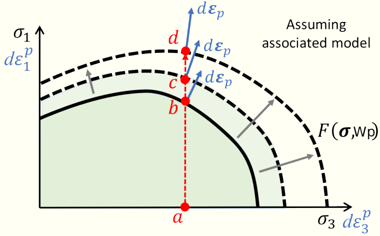
The consistency condition can be written as:

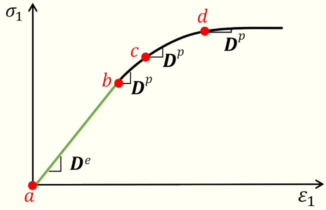
$$\begin{aligned}
dF &= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \frac{\partial F}{\partial Wp} \cdot dWp && = 0 \\
&= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \left(\frac{\partial F}{\partial Wp}\right) \cdot \left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot d\varepsilon^p && = 0 \\
&= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma + \left(\frac{\partial F}{\partial Wp}\right) \left(\frac{\partial Wp}{\partial \varepsilon^p}\right)^T \cdot \frac{\partial G}{\partial \sigma} d\lambda && = 0 \\
dF &= \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot d\sigma - H d\lambda && = 0.
\end{aligned}$$

if $H > 0$: Hardening, if $H = 0$: perfect plasticity, if $H < 0$: softening.

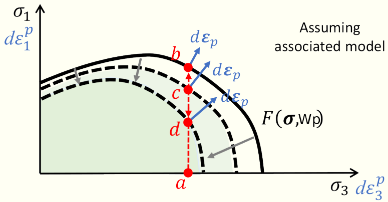
Hardening v Softening

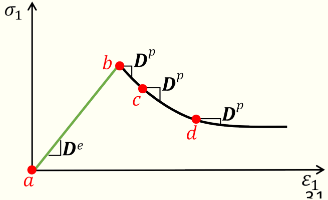
Linear elastic – hardening plastic material $H > 0$





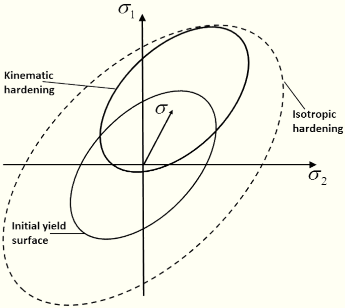
Linear elastic – softening plastic material $H < 0$





Isotropic v kinematic hardening

Two primary types of hardening laws: density and kinematic hardening.



1. **Density hardening**: also referred to as isotropic hardening. $(\sigma_1 - \sigma_3) - 2c(h) = 0$
2. **Kinematic hardening**: typically describing fabric anisotropy. No change in size of yield surface but translating in stress space. $F = (\sigma_1 - \sigma_3) - \alpha(h) - 2c = 0$.

Isotropic v kinematic hardening

Basic concepts of elasto-plasticity

The response of the material is:

- **elastic**:

- **elasto-plastic**:

Note: When the response is elastic, we have no problem! Knowing the strain, we directly can calculate the stress ($d\sigma = D^e \cdot d\varepsilon$).
The difficulty arises when the material response is elasto-plastic, we need to determine D^p!

Equations of elasto-plasticity

For an elasto-plastic material:

- Input material:
- Results from FE analysis:
- unknowns:
- What we are interested:

1. Stress-strain: $d\sigma = D^e \cdot (d\varepsilon - d\varepsilon^p)$
2. Consistency-condition: $dF = \frac{\partial F}{\partial \sigma}^T \cdot d\sigma - H d\lambda$
3. Flow rule: $d\varepsilon^p = \frac{\partial G}{\partial \sigma} d\lambda$.

Solution procedure:

Eq 1 and 3 (stress-strain & flow rule) in Eq 2 (consistency condition) for $d\lambda$

$$dF = \left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot \left(d\varepsilon - \frac{\partial G}{\partial \sigma} d\lambda\right) - H d\lambda = 0$$

$$d\lambda = \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H}$$

$$\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \frac{\partial G}{\partial \sigma} + H > 0$$

$d\lambda$ in Eq 3 $\rightarrow d\varepsilon^p$

$$d\varepsilon^p = \frac{\partial G}{\partial \sigma} d\lambda = \frac{\partial G}{\partial \sigma} \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H}$$

Equations of elasto-plasticity

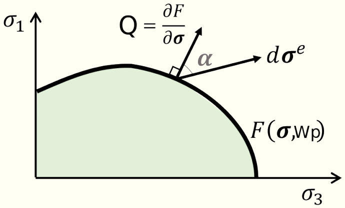
$d\varepsilon^p$ in Eq 1 $\rightarrow$ $d\sigma$

$$
\begin{aligned}
d\sigma &= D^e \cdot (d\varepsilon - d\varepsilon^p) \\
&= D^e \cdot \left(d\varepsilon - \frac{\partial G}{\partial \sigma} \frac{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \cdot d\varepsilon}{\left(\frac{\partial F}{\partial \sigma}\right)^T \cdot D^e \left(\frac{\partial G}{\partial \sigma}\right) + H} \right)
\end{aligned}
$$

$$
\boxed{d\sigma' = \left[D^e - \frac{D^e \left(\frac{\partial G}{\partial \sigma'}\right) \left(\frac{\partial F}{\partial \sigma'}\right)^T D^e}{-\left(\frac{\partial F}{\partial W_p}\right) \left(\frac{\partial W_p}{\partial \varepsilon^p}\right)^T \left(\frac{\partial G}{\partial \sigma'}\right) + \left(\frac{\partial F}{\partial \sigma'}\right)^T D^e \left(\frac{\partial G}{\partial \sigma'}\right)} \right] d\varepsilon}
$$

Plastic loading v elastic unloading

Depending on α, we might have three different scenarios:



If $\alpha < 90° \Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} > 0$

$\Rightarrow d\lambda > 0 \Rightarrow d\boldsymbol{\varepsilon}^p > 0$

$\Rightarrow$ **Elastoplastic loading**

$\Rightarrow dF = 0$

If $\alpha = 90° \Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} = 0$

$\Rightarrow d\lambda = 0 \Rightarrow d\boldsymbol{\varepsilon}^p = 0$

$\Rightarrow$ **Neutral loading**

$\Rightarrow dF = 0$

If $\alpha > 90° \Rightarrow \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}\right)^T \cdot \boldsymbol{D}^e \cdot d\boldsymbol{\varepsilon} < 0$

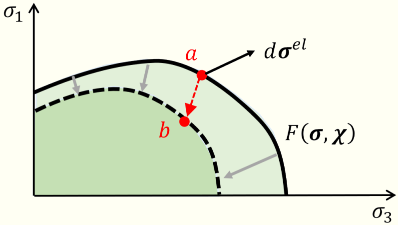
$\Rightarrow \cancel{d\Lambda < 0} \Rightarrow \cancel{d\boldsymbol{\varepsilon}^p = 0}$

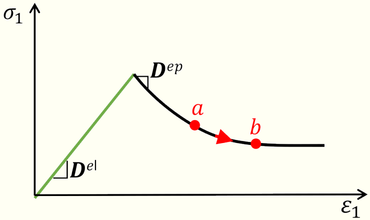
$\Rightarrow$ **Elastic unloading**

$\Rightarrow dF < 0$

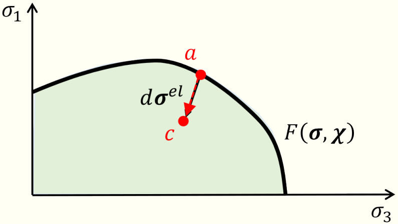
Plastic loading v elastic unloading

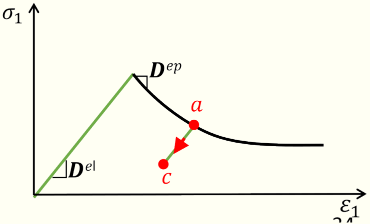
Don't confuse **elastoplastic loading with softenting**...





...with **elastic unloading**



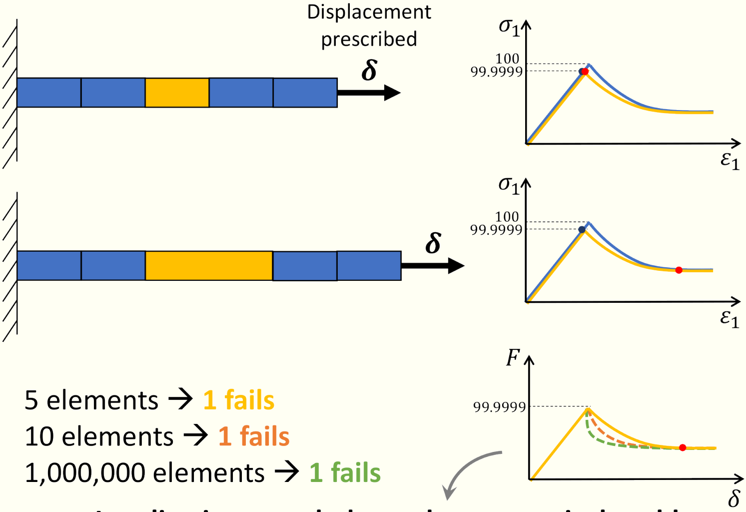


Problems associated to softening

1. The modeling of **strain softening** (i.e. strength decreases from peak to residual conditions) leads to the localization phenomenon.
2. Localization shows up when all deformation is absorbed by one element, while the neighboring elements remain under elastic conditions.
3. The solution depends on the element size, hence the localization is a **mesh dependent numerical problem**.
4. **Solution:** regularization techniques: non-local integration, strain softening models dependent on the element size,...
5. Be careful: Most of the commercial software do not include regularization techniques.

Problems associated to softening

Don't confuse **elastoplastic loading with softenting**...



5 elements → 1 fails
10 elements → 1 fails
1,000,000 elements → 1 fails

Localization =mesh dependent numerical problem