CE394M: Linear Elasticity

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Isotropic linear elastic stress-strain relations

The relationship between the stress and strain tensor is a linear one. The stress component is a linear combination of the strain tensor. The most general form for *linear* stress-strain relations for a *Cauchy elastic* material is given by:

Where σ_{ij}^0 is the components of initial stress tensor corresponding to the initial strain free (when all strain components $\varepsilon_{kl} = 0$). D_{ijkl} is the tensor of material *elastic constants*.

If it is assumed that the initial strain free state corresponds to an *initial stress free state*, that is $\sigma_{ij}^0 = 0$, the equations reduces to:

Observation on linear elasticity

1. $\sigma_{ij} = D_{ijkl}\varepsilon_{kl}$ is a general expression relating stress to strains for a linear solid.
2. D_{ijkl} is a 4th order tensor containing 81 terms (we trick using symmetry and reduce order).
3. D_{ijkl} material response functions having dimensions F/L^2.
4. Homogeneous: D_{ijkl} independent of position
5. Isotropic: D_{ijkl} independent of frame of reference.
6. Because the stress is symmetric: $\sigma_{ij} = \sigma_{ji}$, $D_{ijkl} = D_{jikl}$. Strain is symmetric $\varepsilon_{kl} = \varepsilon_{lk}$ and $D_{ijkl} = D_{ijlk}$. Hence the number of independent variables drop from 81 to 36.
7. Both the stress and the strain tensor have only 6 independent values, therefor write them as vectors, then the stiffness tensor can be written as a matrix (compromise I can not rotate tensor).

Stress-strain relationship

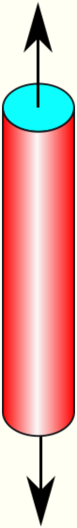
$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16}\\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26}\\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36}\\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46}\\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56}\\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{bmatrix}$$

D_{ijkl} is a tensor of material *elastic constants*. However, the above [**D**] is not a tensor anymore. So we can not rotate the matrix to another frame of reference. This relationship is useful for isotropic materials, where **D** is independent of the frame of reference.

The inverse of the relationship (Compliance matrix):

Hooke's law

Empirical observation:



Where E is defined as the *Young's modulus*.
The lateral strains are defined as:

Using superposition for principal stresses:

$$\varepsilon_{11} = (1/E)\left[\sigma_{11} - \nu\sigma_{22} - \nu\sigma_{33}\right]$$
$$\varepsilon_{22} = (1/E)\left[-\nu\sigma_{11} + \sigma_{22} - \nu\sigma_{33}\right]$$
$$\varepsilon_{33} = (1/E)\left[-\nu\sigma_{11} - \nu\sigma_{22} + \sigma_{33}\right]$$

Hooke's law

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix}$$

It is possible to invert the matrix to obtain the generalized Hooke's law:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \alpha \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ & (1-\nu) & \nu & 0 & 0 & 0 \\ & & (1-\nu) & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

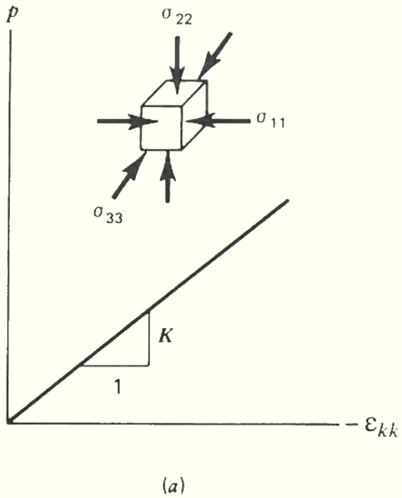
Where $\alpha = E/((1+\nu)(1-2\nu))$. Similarly, we can obtain the inverse matrix.

Hooke's law

The matrices [**C**] and [**D**] contains two indepdent variables E and μ, where $E > 0$ and $-1 \leq \nu \leq 0.5$. The matrix can also be defined in terms of Lame's constants.

$$\begin{cases} \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} & \text{Lame's modulus (wave propagation)} \\ \mu = G = \frac{E}{2(1+\nu)} & \text{Shear modulus (shear behavior)} \\ K = \frac{E}{3(1-2\nu)} & \text{Bulk modulus (volumetric behavior)} \end{cases}$$

Isotropic linear elastic

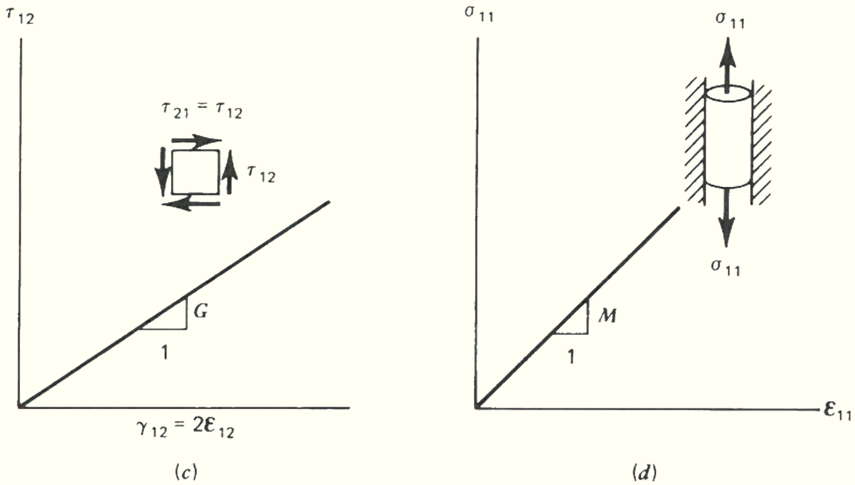


Behavior of isotropic linear elastic material in simple tests: (a) hydrostatic compression test ($\sigma_{11} = \sigma_{22} = \sigma_{33} = p$) and (b) simple tension test (Chen 1994)

Isotropic linear elastic

Hydrostatic compression test The non-zero components of stress:
The *Bulk modulus, K,* is defined as the ratio between the *hydrostatic pressure p* nd the corresponding volume change $\delta\varepsilon_v = \varepsilon_{kk}$.

Simple tension test The only non-zero components of stress:
The *Young's modulus, E,* and *Poisson's ratio,* ν as.



Behavior of isotropic linear elastic material in simple tests: (c) pure shear test, and (d) uniaxial strain test (Chen 1994)

Isotropic linear elastic

Simple shear test
The non-zero components of stress:
The *Shear modulus, G or μ*, is defined as:

Uniaxial strain test The test is carried out by applying a uniaxial stress component σ_{11} in the axial direction of a cylindrical sample, whose lateral surface is *restrained* against lateral movement (Oedometer test). Axial strain ε_{11} is the only nonvanishing component. The *constrained modulus* M or as PLAXIS calls it E_{oed} is defined as the ratio between σ_{11} and $\varepsilon_1 1$.

Plane stress v Plane strain

For a frame with an axis perpendicular to the plane of interest, x_3 or z:

Plane stress

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The strain in z is written as:

Plane strain

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The stress in z is written as:

Elastic solution for settlement under foundations

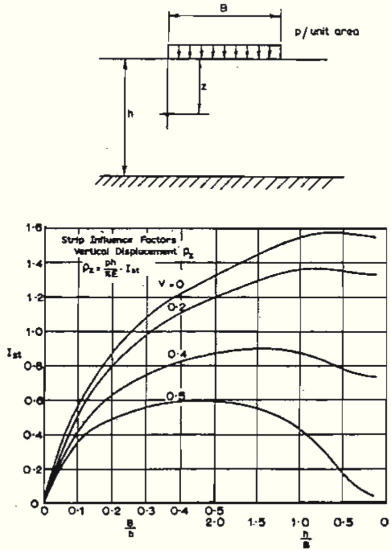


FIG.5.2 Strip curves for ρ_z.

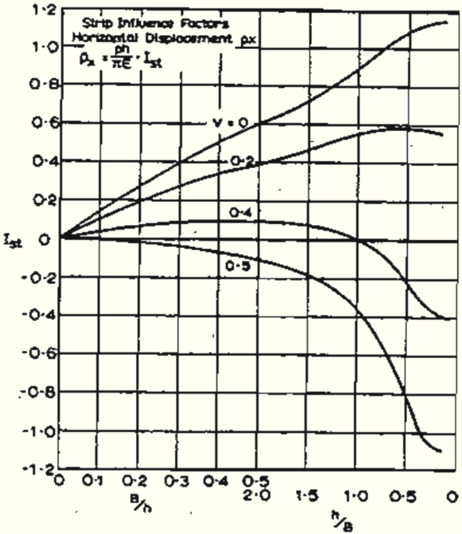
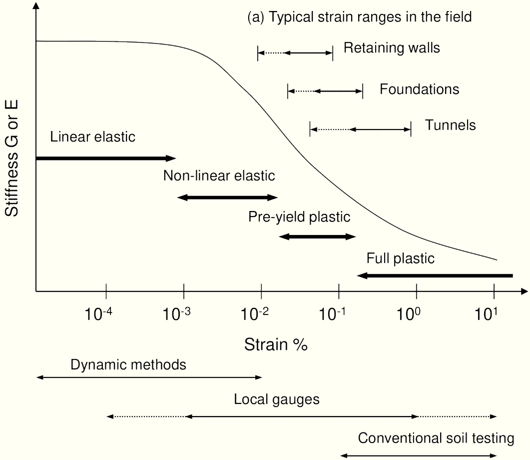


FIG.5.3 Strip curves for ρ_x.

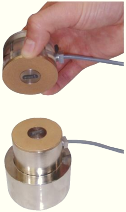
Elastic solutions

1. *Ease of use* Only two parameters, choose equivalent values representative of stran/stress levels.
2. *Disadvantage*: No failure criteria.
3. Validate code with chart solutions ("*Exact solutions*")., e.g., Poulos and Davis (1974).
4. Useful to get feeling of problem (low stress levels) not wide distribution of plastic zones.

Stiffness: small to large strains

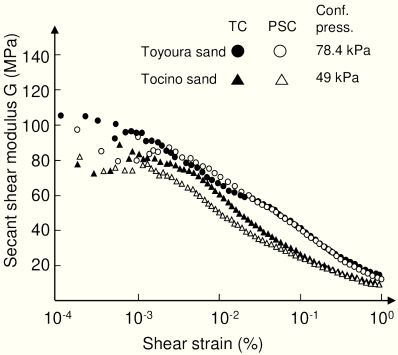


Local gauges

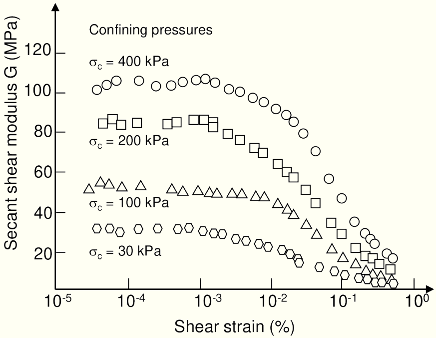


Bender element (GDS)

Stiffness: small to large strains

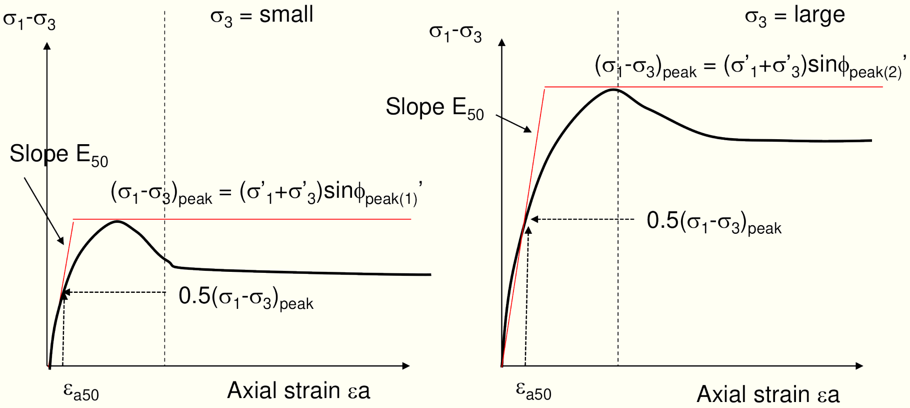


(a) Toyoura and Tocino sands

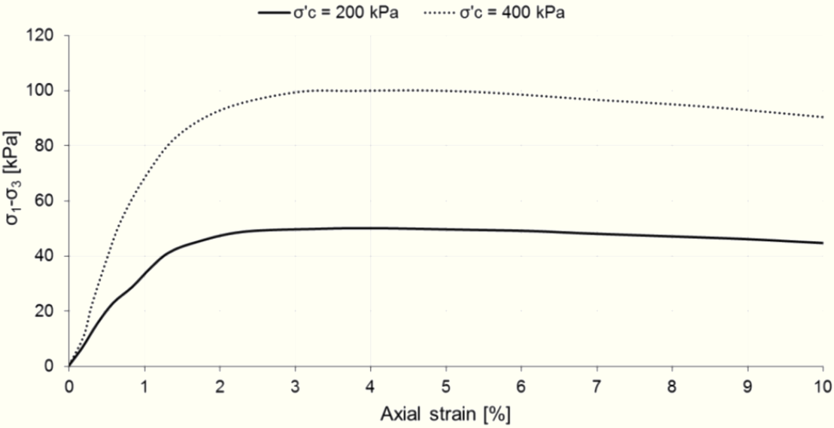


(b) Kaolin clay

Stiffness at intermediate strains

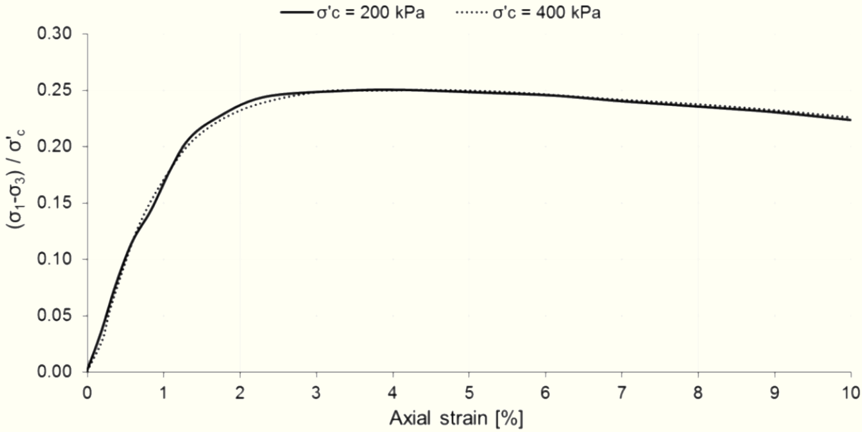


Undrained strength



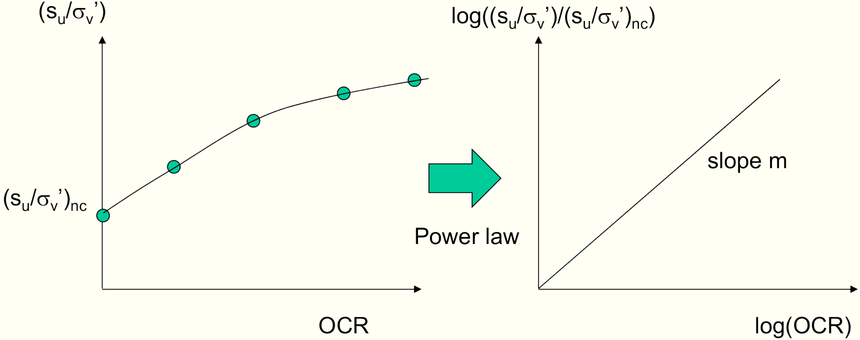
Triaxial compression test data of homogeneous clay (Ladd & Foott, 1974)

Normalized undrained strength



Normalised triaxial compression test data of homogeneous clay (Ladd & Foott, 1974)

Stress History and Normalized Soil Engineering Properties (SHANSEP) (Ladd and Foote, 1974)



Isotropic Linear Elastic Stress-strain relationship

The *isotropic tensor* D_{ijkl}:

$$D_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

Where $\lambda, \mu,$ and $, \alpha$ are scalar constants. Since D_{ijkl} must satisfy symmetry, $\alpha = 0$.

So the stress:

Hence for an isotropic linear elastic material, there are only two independent material constants, λ and μ, which are called *Lame's constants*.

SHANSEP procedure