Eigenvalues and Eigenvectors

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1 Preliminaries

1.1 Subspaces

- $\mathbb{R}$, $\mathbb{C}$ field of real and complex numbers.
- Colspace of $\boldsymbol{A}^{m\times n}$:
$$\mathcal{C}(\boldsymbol{A}) := \operatorname{span}\{\operatorname{Cols}(\boldsymbol{A})\} \subseteq \mathbb{R}^n$$
And $\operatorname{Dim}(\mathcal{C}(\boldsymbol{A})) = r$. r is rank of $\boldsymbol{A}$.
- Rowspace of $\boldsymbol{A}^{m\times n}$:
$$\mathcal{R}(\boldsymbol{A}) := \operatorname{span}\{\operatorname{Rows}(\boldsymbol{A})\} = \mathcal{C}(\boldsymbol{A}^\top) \subseteq \mathbb{R}^m$$
And $\operatorname{Dim}(\mathcal{R}(\boldsymbol{A})) = r$.
- Nullspace of $\boldsymbol{A}^{m\times n}$:
$$\mathcal{N}(\boldsymbol{A}) := \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = 0\} \subset \mathbb{R}^n$$
$\operatorname{Dim}(\mathcal{N}(\boldsymbol{A})) = n - r$.
- $\mathcal{N}(\boldsymbol{A}^\top) := \{\boldsymbol{x} : \boldsymbol{A}^\top\boldsymbol{x} = 0\} \subset \mathbb{R}^m$. $\operatorname{Dim}(\mathcal{N}(\boldsymbol{A}^\top)) = m - r$.

2 Properties

2.1 Eigval

- Eigval λ, eigvec $\boldsymbol{v}$ such that: $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.
- $c\boldsymbol{v}$ is also an eigvec corresponding to λ, but only linearly indep. eigvecs are counted.

2.2 Characteristic Polynomial

- $\boldsymbol{A}$ is $n \times n$ square, then the characteristic polynomial of $\boldsymbol{A}$ is defined as.
$$P_{\boldsymbol{A}}(t) := \det(t\boldsymbol{I} - \boldsymbol{A})$$
- λ is eigval of $\boldsymbol{A}$ $\iff$ $P_{\boldsymbol{A}}(\lambda) = 0$ $\iff$ λ is root of $P_{\boldsymbol{A}}(t) = 0$.
Proof. ($\Rightarrow$) λ is eigval of $\boldsymbol{A}$: $(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = 0$.
$\boldsymbol{v}$ is an eigvec of $\boldsymbol{A}$, so $\boldsymbol{v} \neq 0 \Rightarrow \boldsymbol{v} \in \mathcal{N}(\boldsymbol{A})$. Hence $\operatorname{Dim}(\mathcal{N}(\boldsymbol{A})) > 0 \Rightarrow r < n$. □
- $P_{\boldsymbol{A}}(t)$ can be represented as, sence we know the roots,
$$P_{\boldsymbol{A}}(t) = \prod_{j=1}^{n}(t - \lambda_j)$$

- $\boldsymbol{D}$ being diagonal matrix, then $P_{\boldsymbol{D}} = \prod_{j=1}^{n}(t - d_j)$, which implies that $\lambda_j = d_j$, diagonal entries. Moreover, $\boldsymbol{De}_j = d_j\boldsymbol{e}_j$, so the j-th eigvec is $\boldsymbol{e}_j$ unit vec.
- $\boldsymbol{L}$ being lower triangular, then $P_{\boldsymbol{L}} = \prod_{j=1}^{n}(t - L_{jj})$, which implies that $\lambda_j = d_j$. The last col of lower tri is $L_{nn}\boldsymbol{e}_n$, therefore $\boldsymbol{Le}_n = L_{nn}\boldsymbol{e}_n$, i.e. the last eigvec is $\boldsymbol{e}_n$ unit vector. We can not tell about other eigvecs. Similar for $\boldsymbol{U}$. $\boldsymbol{Ue}_1 = U_{11}\boldsymbol{e}_1$, i.e. the first eigvec of $\boldsymbol{U}$ is $\boldsymbol{e}_1$ unit vector.

2.3 Multiplicity

- $\lambda(\boldsymbol{A})$ is the set of all eigvals of $\boldsymbol{A}$, it is acturally the spectrum of $\boldsymbol{A}$.
- If $\lambda \in \lambda(\boldsymbol{A})$ is a root of multiplicity m_λ of $P_{\boldsymbol{A}}(t) = 0$, define m_λ as multiplicity of eigval λ.
- Square matrix $\boldsymbol{A}$ has exactly n eigvals ($\lambda \in \mathbb{C}$), counted with multiplicity, i.e.

$$\sum_{\lambda \in \lambda(\boldsymbol{A})} m_\lambda = n$$

This is because, due to fundamental principal of algebra, $P_{\boldsymbol{A}}(t) = 0$ has exactly n roots, counted with multiplicity.

2.4 Eigspace

- Eigenspace (eigspace) of λ: $V_\lambda := \{\boldsymbol{v} : \boldsymbol{Av} = \lambda\boldsymbol{v}\}$, i.e. the set of all eigvecs corresponding to λ. $\mathrm{Dim}(V_\lambda)$ is the number of linearly indep. eigvecs corresponding to λ. We have

$$1 \leq \mathrm{Dim}(V_\lambda) \leq m_\lambda$$

Thm. eigvecs corresponding to *different* eigvals of $\boldsymbol{A}$ are linearly indep.

Proof. Let $\boldsymbol{v}_1, ..., \boldsymbol{v}_p$ correspond to different eigvals of $\boldsymbol{A}$: $\lambda_1, ..., \lambda_p$ ($p \leq n$). It suffices to show

$$c_1\boldsymbol{v}_1 + ... + c_p\boldsymbol{v}_p = \boldsymbol{0} \Rightarrow c_1 = ... = c_p = 0$$

Show by contradiction: suppose otherwise, i.e.

$$(\dagger) : c_1\boldsymbol{v}_1 + ... + c_{p-1}\boldsymbol{v}_{p-1} = \boldsymbol{v}_p$$

Apply $\boldsymbol{A}$ both sides:

$$\begin{aligned} \boldsymbol{A}(c_1\boldsymbol{v}_1 + ... + c_{p-1}\boldsymbol{v}_{p-1}) &= \boldsymbol{Av}_p \\ c_1\lambda_1\boldsymbol{v}_1 + ... + c_{p-1}\lambda_{p-1}\boldsymbol{v}_{p-1} &= \lambda_p\boldsymbol{v}_p \end{aligned} \tag{1}$$

$(\dagger) \times \lambda_p$, substracted from last equation:

$$\sum_{j=1}^{p-1} c_j(\lambda_1 - \lambda_p)\boldsymbol{v}_j = \lambda_p\boldsymbol{v}_p - \lambda_p\boldsymbol{v}_p = 0$$

Which is not possible since $\exists c_j \neq 0$, and $(\lambda_j - \lambda_p) \neq 0\ \forall j$. Contradiction. □

2.5 Miscellaneous

- $\boldsymbol{A}$ singular $\iff 0 \in \lambda(\boldsymbol{A})$.
  Proof. $\det(0\boldsymbol{I} - \boldsymbol{A}) = 0 \iff \boldsymbol{A}$ singular. □
- $\boldsymbol{A}$ invertible. $\lambda \in \lambda(\boldsymbol{A})$, $\boldsymbol{v} \in V_\lambda(\boldsymbol{A})$. Then $\frac{1}{\lambda} \in \lambda(\boldsymbol{A}^{-1})$, $\boldsymbol{v} \in V_{\frac{1}{\lambda}}(\boldsymbol{A}^{-1})$.
  Proof. $\boldsymbol{Av} = \lambda\boldsymbol{v} \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{Av} = \lambda\boldsymbol{A}^{-1}\boldsymbol{v} \Rightarrow \frac{1}{\lambda}\boldsymbol{v} = \boldsymbol{A}^{-1}\boldsymbol{v}$.

- $\boldsymbol{A}$ has eigtuple $(\lambda, \boldsymbol{v})$, then $\boldsymbol{A}^k$ with (λ^k, v). $P(\boldsymbol{A})$ is a polynomial of $\boldsymbol{A}$, has eigtuple $(P(\lambda), \boldsymbol{v})$.
- $\boldsymbol{A}$ and $\boldsymbol{A}^\top$ have same eigvals.
  Proof. $P_{\boldsymbol{A}}(t) = \det(t\boldsymbol{I} - \boldsymbol{A}) = \det((t\boldsymbol{I} - \boldsymbol{A})^\top) = \det(t\boldsymbol{I} - \boldsymbol{A}^\top) = P_{\boldsymbol{A}^\top}(t)$. Thus have same roots.
- First, second and the last term of $P_{\boldsymbol{A}}(t)$:

$$P_{\boldsymbol{A}}(t) = t^n - \operatorname{tr}(\boldsymbol{A})t^{n-1} + ... + (-1)^n \det(\boldsymbol{A})$$

  Moreover, $\sum_{j=1}^n \lambda_j = \operatorname{tr}(\boldsymbol{A})$ and $\prod_{j=1}^n \lambda_j = \det(\boldsymbol{A})$.
  Proof. $P_{\boldsymbol{A}}(t) = \det(t\boldsymbol{I} - \boldsymbol{A})$.
  $P_{\boldsymbol{A}}(0) = \det(-\boldsymbol{A}) = (-1)^n \det(\boldsymbol{A})$.
  t^n, t^{n-1} arises with prod term of diagonal entries:

$$\prod_{j=1}^n (t - A_{jj}) = t^n - t^{n-1} \sum_{j=1}^n A_{jj} + ...$$

  which have coefficients 1 and $-\operatorname{tr}(\boldsymbol{A})$. Moreover

$$P_{\boldsymbol{A}}(t) = \prod_{j=1}^n (t - \lambda_j) = t^n - t^{n-1} \sum_{j=1}^n \lambda_j + ... + (-1)^n \prod_{j=1}^n \lambda_j$$

  Finished the proof. □

3 Diagonal Form

- $\boldsymbol{A}$ square, $\boldsymbol{A}$ is diagonalizable iff $\exists$ diagonal $\boldsymbol{\Omega}$, invertible $\boldsymbol{V}$, s.t.

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$$

$$\boxed{1}$$