Lecture 3

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1 General Formulation of Explicit 1-Step Method

We now write 1-step methods in a general form,

$$y_{n+1} = y_n + h\Phi(t_n, y_n, h) \quad (\dagger)$$

Def. **Truncation Error**: We insert true values of the solution into the formulation above, and define the truncation error:

$$T_n = \frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n), h)$$

Def. **Consistency**: The numerical method ($\dagger$) is *consistent* with $y' = f(t, y)$ if the truncation error goes to 0 as $h \to 0$, i.e. $\lim\limits_{h\to 0} T_n = 0$.

Cor. Let $\Phi(t_n, y(t_n), \cdot)$ continuous on h, then the definition of consistency is equivalent to

$$\lim_{h\to 0} \frac{y(t_n + h) - y(t_n)}{h} = \lim_{h\to 0} \Phi(t_n, y(t_n), h)$$

i.e. $y'(t_n) = \Phi(t_n, y(t_n), 0)$. Hence we have consistency $\iff$

$$y'(t_n) = \Phi(t_n, y(t_n), 0) \iff f(t_n, y(t_n)) = \Phi(t_n, y(t_n), 0)$$

Thm. (**Convergence and Zero-Stability**) If the following conditions holds

1. $f(t, y)$ is Lipschitz in y, continuous in t.
2. $\Phi(t, y, h)$ is continuous in t, Lipschitz in y in the same region as for the Lipschitz condition of $f(t, y)$, such that

$$|\Phi(x, y, h) - \Phi(x, z, h)| \le L_\Phi |y - z|$$

3. $\Phi(\cdot, \cdot, h)$ is uniformly continuous in h, $\forall\ h \in [0, h_0]$.

Then the numerical solution is zero-stable (zero-stability), and it converges to the real solution. (convergence)

Proof. We still use the similar trick as in the euler method's proof. I.e. write the numerical method in discrete terms and also in the exact values

$$y_{n+1} = y_n + h\Phi(t_n, y_n, h) \quad (*)$$

$$y(t_{n+1}) = y(t_n) + h\Phi(t_n, y(t_n), h) + hT_n \quad (**)$$

the second equation minus the first one, and let $e_n := y_n - y(t_n)$, T_n be the truncation error

$$\begin{aligned} e_{n+1} &= e_n + h\left[\Phi(t_n, y_n, h) - \Phi(t_n, y(t_n), h)\right] - hT_n \\ |e_{n+1}| &\le |e_n| + hL_\Phi |e_n| + hT_n \\ &\le (1 + hL_\Phi)|e_n| + hT \quad (\text{with } T = \max_n T_n) \end{aligned} \tag{1}$$

Solve this recursive equation, and let $e_0 := y_0 - y(t_0)$, the (round off) error of the initial data, we get

$$\begin{aligned} |e_n| &\le (1 + hL_\Phi)^n |e_0| + [(1 + hL_\Phi)^n - 1]\frac{T}{L_\Phi} \\ &\le \exp(nhL_\Phi)|e_0| + (\exp(nhL_\Phi) - 1)\frac{T}{L_\Phi} \end{aligned} \tag{2}$$

We have convergence: since the first term only concerns the error of initial data, which we can, using some technique, make it close to zero. The second term is constant times T, which is at least $O(h) \to 0$ as $h \to 0$. So the error itself is always bounded.
The zero stability goes in a similar fashion. Recall the definition, we have two initial data y_0 and $\tilde{y}_0$, and using the same numerical method, we obtain sequence $\{y_0, y_1, ..., y_n\}$ and $\{\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_n\}$. Substitute $y(t_n)$ in $(**)$ to $\tilde{y}_n$, we obtain

$$|y_n - \tilde{y}_n| \le \exp(nhL_\Phi)|y_0 - \tilde{y}_0| + O(h)$$

which matches the definition of zero-stability ($|y_n - \tilde{y}_n| \le c \max_{0 \le j \le s-1} |y_j - \tilde{y}_j|$ as $h \searrow 0$.)

2 Construction of Φ

2.1 The Taylor Expansion

One way to construct a higher-ordered method is using Taylor expansion of $y(t_{n+1})$ at $y(t_n)$:

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + ... + \frac{h^p}{p!}y^{(p)}(t_n) + O(h^{p+1})$$

The complexity lies in calculating the derivatives. We do some low ordered terms:

$$\begin{aligned} &y'(t_n) = f(t_n, y(t_n)) \\ &y''(t_n) = f_t + f_y y'(t_n) = f_t + f_y f \\ &y'''(t_n) = f_{tt} + f_{ty}y' + y' f_{yt} + y'^2 f_{yy} + y'' f_y = f_{tt} + 2f_{ty}f + f^2 f_{yy} + f_y f_t + f_y^2 f \end{aligned} \tag{3}$$

The number of terms grows exponentially, very hard to calculate...

2.2 Runge-Kutta Construction

We've got another idea, we want to approximate

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))dt$$

with

$$y(t_{n+1}) = y(t_n) + h\sum_{j=1}^{N} b_j f(t_n + c_j h, y(t_n + c_j h))$$

where we make a finer partition of $[t_n, t_{n+1}]$ to N *stages*: $t_n + c_j h$, $c_1 \le c_2 \le ... \le c_N$. And b_j being the weights assigned to each value.

Def. **N-Stages Runge Kutta Method**: we partition the interval $[t_n, t_{n+1}]$ to N stages, and update y by

$$y_{n+1} = y_n + h\sum_{j=1}^{N} b_j k_j$$

where

$$k_1 = f(t_n, y_n); \quad k_i = f(t_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j) \;\; 2 \le i \le N$$

There are three sets of parameters, $\boldsymbol{b}^\top$: the weights, $\boldsymbol{c}$: the nodes of partition of $[t_n, t_{n+1}]$ and $\boldsymbol{A} = \{a_{ij}\}$: the *Runge-Kutta matrix.* We can put then together in a $n+1 \times n+1$ table, which is often referred to as *butcher table.*

Prop. By the corollary listed in the previous section, consistency of 1-step method $\iff$ $\Phi(t_n, y(t_n), 0) = f(t_n, y(t_n))$. When $h = 0$, all the ks degenerate to $f(t_n, y(t_n))$. And Runge-Kutta has $\Phi(t_n, y(t_n), h) = \sum_{j=1}^{N} b_j k_j \Rightarrow \sum_{j=1}^{N} b_j f(t_n, y(t_n)) = f(t_n, y(t_n)) \Rightarrow \sum_{j=1}^{N} b_j = 1$. Hence the consistency requires all the weights sum to 1.

Ex. **Runge (1895)**:

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} = \begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^\top \end{array} = \begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

Gives

$$y_{n+1} = y_n + hk_2; \quad k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}f(t_n, y_n)\right)$$

Ex.

$$\begin{array}{c|c} \boldsymbol{c} & \boldsymbol{A} \\ \hline & \boldsymbol{b}^\top \end{array} = \begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

Gives

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}k_1 + \frac{h}{2}k_2 \\ k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + h, y_n + f(t_n, y_n)\right) \end{aligned} \tag{4}$$

3 Truncation Error

We now consider how to solve the coefficients