Stochastic Process Assignment III

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Problem 1.

Solution. (a) Define Y_t be the number of family that is in the hotel on $t-1$-th day and spend another day (i.e. still in the hotel on t-th day). Then by illustration, Y_t follows binomial distribution with probability $1-p$ (They check **out** with probability p!) and total counts X_{t-1}. On the another day, the total number of families is constituted by Y_t and N_t, where N_t is # of new-comers $\sim$ Pois(λ). Hence

$$\begin{aligned} P_{ij} &= \mathbb{P}\left(X_t = j | X_{t-1} = i\right) \\ &= \mathbb{P}\left(Y_t + N_t = j | X_{t-1} = i\right) \\ &= \sum_{k=0}^{i} \mathbb{P}\left(Y_t + N_t = j | X_{t-1} = i, Y_t = k\right) \mathbb{P}\left(Y_t = k | X_{t-1} = i\right) \\ &= \sum_{k=0}^{\min\{i,j\}} \mathbb{P}\left(N_t = j-k\right) \binom{i}{k} (1-p)^k p^{i-k} \\ &= \sum_{k=0}^{\min\{i,j\}} \frac{e^{-\lambda}\lambda^{j-k}}{(j-k)!} \binom{i}{k} (1-p)^k p^{i-k} \end{aligned} \tag{1}$$

(b)

$$\begin{aligned} \mathbb{E}\left[X_t\right] &= \mathbb{E}\left[\mathbb{E}\left[X_t | X_{t-1}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[Y_t + N_t | X_{t-1}\right]\right] \\ &= \mathbb{E}\left[(1-p)X_{t-1} + \lambda\right] = (1-p)\mathbb{E}\left[X_{t-1}\right] + \lambda \end{aligned} \tag{2}$$

Solve for $\mathbb{E}\left[X_t\right]$ recurrsively, we get

$$\begin{aligned} \mathbb{E}\left[X_t\right] &= \lambda(1 + (1-p) + ... + (1-p)^{n-1}) + (1-p)^n \mathbb{E}\left[X_0\right] \\ \Rightarrow \mathbb{E}\left[X_t | X_0 = i\right] &= \frac{\lambda(1-(1-p)^n)}{p} + (1-p)^n \cdot i \end{aligned} \tag{3}$$

(c) *Claim.* Stationary distribution of $\{X_t\}$ is a Poisson with rate $a = \lambda/p$.
Proof of Claim. It suffices to show X_t has same distribution regardless of t. It is clear that $X_t = N_t + Y_t$, N_t is independent of Y_t.

$$\begin{aligned} \mathbb{P}\left(Y_t = y\right) &= \sum_{k \geq y} \mathbb{P}\left(Y_t = y | X_{t-1} = k\right) \mathbb{P}\left(X_{t-1} = k\right) \\ &= \sum_{k \geq y} \frac{k!}{y!(k-y)!} (1-p)^k p^{k-y} \frac{e^{-a}a^k}{k!} \\ &= \sum_{k \geq y} \frac{e^{-a(1-p)}(a(1-p))^y}{y!} \cdot \frac{e^{-ap}(ap)^{k-y}}{(k-y)!} \\ &= \frac{e^{-a(1-p)}(a(1-p))^y}{y!} \end{aligned} \tag{4}$$

Hence $Y_t \sim$ Pois($a(1-p)$). We conclude that $X_t = Y_t + N_t$ is a Poisson with rate $\lambda + a(1-p)$, where $a = \lambda/p \Rightarrow \lambda + \frac{\lambda}{p}(1-p) = \lambda/p = a$. I.e. X_t is identically distributed as X_{t-1}. This is the sufficient condition for stationary state. We finish the proof.

Problem 2.

Solution. (a) Denote state $\{0,1\} := \{\text{Good Year, Bad Year}\}$. Then the transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad \boldsymbol{P}^2 = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix}, \quad \boldsymbol{P}^3 = \begin{pmatrix} \frac{29}{72} & \frac{43}{72} \\ \frac{43}{108} & \frac{65}{108} \end{pmatrix} \tag{5}$$

Define RV $X_i := \#$ of storms in year i, event $A_i := \{\text{Year } i \text{ is good year, given that year 0 is good year.}\}$. Then

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{2} X_i\right] &= \sum_{i=1}^{2} \mathbb{E}\left[X_i | A_i\right] \mathbb{P}\left(A_i\right) + \mathbb{E}[X_i | A_i^{\complement}] \mathbb{P}(A_i^{\complement}) \\ &= 1 \cdot (P_{00} + P_{00}^2) + 3 \cdot (P_{01} + P_{01}^2) = \frac{25}{6} \end{aligned} \tag{6}$$

(b) Using the elements in $\boldsymbol{P}^3$,

$$\begin{aligned} \mathbb{P}\left(X_3 = 0\right) &= \mathbb{P}\left(X_3 = 0 | A_3\right) \mathbb{P}\left(A_3\right) + \mathbb{P}(X_3 = 0 | A_3^{\complement}) \mathbb{P}(A_3^{\complement}) \\ &= \frac{29}{72} e^{-1} + \frac{43}{72} e^{-3} \end{aligned} \tag{7}$$

(c) Let the stationary probability be $\boldsymbol{\pi} = (\pi_0, \pi_1)^\top$, then we have

$$\begin{aligned} \begin{pmatrix} 1 - P_{00} & -P_{10} \\ 1 & 1 \end{pmatrix} \boldsymbol{\pi} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \Rightarrow \boldsymbol{\pi} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{3} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix} \end{aligned} \tag{8}$$

Problem 3.

Solution. Denote $\boldsymbol{P}$ the transition matrix.

$$\boldsymbol{P}^3 = \begin{pmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{pmatrix} \tag{9}$$

Then

$$\begin{aligned} \mathbb{E}\left[X_3\right] &= \sum_{x=0}^{2} \mathbb{E}\left[X_3 | X_0 = x\right] \mathbb{P}\left(X_0 = x\right) \\ &= \sum_{x=0}^{2} \left(\sum_{z=0}^{2} z P_{xz}^3\right) \mathbb{P}\left(X_0 = x\right) \\ &= \left(\frac{11}{54} \cdot 1 + \frac{47}{108} \cdot 2\right) \frac{1}{4} + \left(\frac{4}{27} \cdot 1 + \frac{11}{27} \cdot 2\right) \frac{1}{4} + \left(\frac{2}{9} \cdot 1 + \frac{13}{36} \cdot 2\right) \frac{1}{2} \\ &= \frac{53}{54} \end{aligned} \tag{10}$$

Problem 4. Show that the symmetric random walk is recurrent in two dimensions.

Solution. In d-dimension, we can always decompose a random walk on d orthogonal degrees of freedom. I.e. the composed random walk is regarded as d-vector, denote $\boldsymbol{X}_t := (X_t^{[1]}, X_t^{[2]}, ..., X_t^{[d]})^\top$; such that on any one degree of freedom ($1 \le i \le d$), $X_t^{[i]}$ is a 1-dimensional random walk.
It is clear that in any dimensional space, all states still communicate. Hence it suffices to check state $\mathbf{0}$. I.e. whether $P_{\mathbf{00}}^{2n}$ is summable.

$$X_{t+1}^{[i]} = \begin{cases} X_t^{[i]} + 1 & \text{W.p. } 1/2, \\ X_t^{[i]} - 1 & \text{W.p. } 1/2. \end{cases} \tag{11}$$

Then it is clear that $X_t^{[i]}$ are mutually independent for $1 \leq i \leq d$. Recall the result on 1-dimensional, we have

$$\mathbb{P}\left(X_{2n}^{[i]} = 0 \middle| X_0^{[i]} = 0\right) = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \sim \frac{1}{\sqrt{\pi n}} \tag{12}$$

So by independence,

$$\mathbb{P}\left(\boldsymbol{X}_{2n} = \boldsymbol{0} | \boldsymbol{X}_0 = \boldsymbol{0}\right) = \prod_{i=1}^{d} \mathbb{P}\left(X_{2n}^{[i]} = 0 \middle| X_0^{[i]} = 0\right) \sim \left(\frac{1}{\pi n}\right)^{\frac{d}{2}} \tag{13}$$

Therefore, we know that $P_{\boldsymbol{00}}^{2n}$ is **Not** summable if and only if $d \leq 2$. I.e. The symmetric random walk is recurrent in 1D or 2D, and is transient in higher dimensional spaces.

Problem 5.

Solution. Since the given markov chain is irreducible and aperiodic, it has a unique limiting distribution, denote $\boldsymbol{\pi} := (\pi_0, \pi_1, ..., \pi_M)^\top$, which satisfies

$$\boldsymbol{\pi} = \begin{pmatrix} 1 - P_{00} & -P_{10} & -P_{20} & \dots & -P_{M0} \\ -P_{01} & 1 - P_{11} & -P_{21} & \dots & -P_{M1} \\ -P_{02} & -P_{11} & 1 - P_{21} & \dots & -P_{M2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{0M} & -P_{1M} & -P_{2M} & \dots & -P_{MM} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} =: \boldsymbol{X}^{-1}\boldsymbol{e}_M \tag{14}$$

Where in the last column we have $-P_{Mj} = \sum_{i=0}^{M-1} P_{ij} - 1$. One can invert the matrix by Mathematica to verify that $\pi_i = \frac{1}{M+1}$ $\forall 0 \leq i \leq M$ indeed.
Alternatively, by the fact that the process is irreducible and aperiodic, $\boldsymbol{X}$ must be invertible. Hence it suffices to check that $\pi_i = \frac{1}{M+1} \Rightarrow \pi_j = \sum_{i=0}^{M} \pi_i P_{ij}$ and $\sum_{i=0}^{M} \pi_i = 1$. Then by uniqueness we know $\boldsymbol{\pi}$ is the solution. This is also indeed the case.

Problem 6.

Solution. (a) Denote $R := \{\text{It rains}\}$. Define $X_t := \#$ of umbrella at his current location. It is clear that $X_t \in \{0, 1, ..., r\}$, and at time t, there are $r - X_t$ umbrellas at the other location. The man brings an umbrella to time $t+1$ if it rains and $X_t > 0$. Hence, at his next move we have

$$X_{t+1} = \begin{cases} r - X_t & \text{If } R^{\complement} \cup \{X_t = 0\} \\ r - X_t + 1 & \text{If } R \cap \{X_t > 0\} \end{cases} \tag{15}$$

By definition we can see X_{t+1} only depend on present X_t. So $\{X_t\}$ is markov chain. Transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1-p & p \\ 0 & 0 & 0 & \dots & 1-p & p & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1-p & p & \dots & 0 & 0 & 0 \\ 1-p & p & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

(b) Calculate limiting probability via

$$\begin{cases} \pi_0 = (1-p)\pi_r \\ \pi_i = (1-p)\pi_{r-i} + p\pi_{r-i+1} & 0 < i < r \\ \pi_r = \pi_0 + p\pi_1 \\ \sum_{i=0}^{r} \pi_i = 1 \end{cases} \Rightarrow \begin{cases} \pi_0 = \frac{1-p}{1+r-p} \\ \pi_i = \frac{1}{1+r-p} & 0 < i \leq r \end{cases} \tag{17}$$

(c) It is clear that X_t is independent w.r.t. R (Rainy or not).

$$\mathbb{P}\left(\{\text{Get Wet}\}\right) = \mathbb{P}\left(X_t = 0 | R\right)\mathbb{P}\left(R\right) = \mathbb{P}\left(X_t = 0\right)\mathbb{P}\left(R\right) = \frac{p(1-p)}{1+r-p} \tag{18}$$

(d) When $r = 3$, employ first order condition

$$\frac{d}{dp}\frac{p(1-p)}{4-p} = \frac{p^2 - 8p + 4}{(4-p)^2} = 0 \Rightarrow p^* = \frac{8 - 4\sqrt{3}}{2} \tag{19}$$

Where $\frac{d^2}{dp^2}\mathbb{P}\left(\{\text{Get Wet}\}\right)(p) < 0$. So we conclude that p^* maximizes the chance by which he gets wet.

Problem 7.

Solution.

$$\mathbb{P}\left(Y_n = (i,j) | Y_k = (x_{k-1}, x_k), 0 \le k \le n-1\right) = \begin{cases} 0 & \text{If } x_{n-1} \ne i \\ \mathbb{P}\left(X_n = j | X_{n-1} = x_{n-1}\right) & \text{If } x_{n-1} = i \end{cases} \tag{20}$$

Only dependent on present state. So Y_n has markovian property. Transition probability is given by

$$P_{(i,j),(k,l)} = \begin{cases} 0 & \text{If } j \ne k \\ P_{kl} & j = k \end{cases} \tag{21}$$

Where P_{kl} is transition probability of X_n.

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}\left(Y_n = (i,j)\right) &= \lim_{n\to\infty} \mathbb{P}\left(X_{n-1} = i, X_n = j\right) \\ &= \lim_{n\to\infty} \mathbb{P}\left(X_{n-1} = i\right)\mathbb{P}\left(X_n = i | X_{n-1} = j\right) \\ &= \pi_i P_{ij} \end{aligned} \tag{22}$$

Problem 8.

Solution. (a) Define $A_n := \{\text{Picked molecule is in urn 1 at } n\text{-th switch.}\}$, then

$$\begin{aligned} \mathbb{E}\left[X_{n+1}\right] &= \mathbb{E}\left[\mathbb{E}\left[X_{n+1}|X_n\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[X_{n+1}|X_n; A_{n+1}\right] + \mathbb{E}[X_{n+1}|X_n; A^{\complement}_{n+1}]\right] \\ &= \mathbb{E}\left[(X_n - 1)\cdot\frac{X_n}{M} + (X_n + 1)\cdot\frac{M - X_n}{M}\right] \\ &= 1 + \mathbb{E}\left[X_n\right] - \frac{2\mathbb{E}\left[X_n\right]}{M} \end{aligned} \tag{23}$$

(b) By the recurrence formula that we obtain in (a), we can check for $n = 1$: $\mu_1 = 1 + (1 - 2/M)\mathbb{E}\left[X_0\right] = M/2 + (1 - 2/M)(\mathbb{E}\left[X_0\right] - M/2)$. We show by induction. Assume

$$\mu_{n-1} = \frac{M}{2} + \left(\frac{M-2}{M}\right)^{n-1}\left(\mathbb{E}\left[X_0\right] - \frac{M}{2}\right) \tag{24}$$

then by recurrence formula:

$$\begin{aligned} \mu_n &= 1 + \left(1 - \frac{2}{M}\right)\mu_{n-1} \\ &= 1 + \frac{M}{2}\left(1 - \frac{2}{M}\right) + \left(\frac{M-2}{M}\right)^n\left(\mathbb{E}\left[X_0\right] - \frac{M}{2}\right) \\ &= \frac{M}{2} + \left(\frac{M-2}{M}\right)^n\left(\mathbb{E}\left[X_0\right] - \frac{M}{2}\right) \end{aligned} \tag{25}$$

Finished the proof.

(c) $X_n \in \{0, 1, ..., M\}$ has $M + 1$ states. X_n is a markov process with transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & & & & \\ \frac{1}{M} & 0 & \frac{M-1}{M} & & & \\ & \frac{2}{M} & 0 & \frac{M-2}{M} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{M-1}{M} & 0 & \frac{1}{M} \\ & & & & 1 & 0 \end{pmatrix} \tag{26}$$

Denote limiting proability π_i, then from $\boldsymbol{P}$ and definition of π_i, we get

$$\begin{cases} \pi_0 = \frac{1}{M}\pi_1 \\ \pi_i = \left(1 - \frac{i-1}{M}\right)\pi_{i-1} + \frac{i+1}{M}\pi_{i+1} \quad \text{For } 0 < i < M \\ \pi_M = \frac{1}{M}\pi_{M-1} \end{cases} \tag{27}$$

Which implies the recurrence formula $\pi_k = \frac{M-k}{k+1} \cdot \pi_{k+1}$ for any $0 \le k \le M$. Hence

$$\begin{aligned} \pi_0 &= \frac{k!}{M(M-1)(M-2)\cdot \ldots \cdot (M-k)}\pi_k \\ &= \frac{k!(M-k)!}{M!}\pi_k = \frac{1}{\binom{M}{k}}\pi_k \end{aligned} \tag{28}$$

Therefore we solve π_0 from

$$1 = \sum_{k=1}^{M} \pi_k = \sum_{k=1}^{M} \binom{M}{k} \pi_0 \Rightarrow \pi_0 = \left(\frac{1}{2}\right)^M \tag{29}$$

And obtain that

$$\pi_k = \binom{M}{k}\left(\frac{1}{2}\right)^M \tag{30}$$

Problem 9.

Solution. It can be easily seen that state $\{1,2,3\}$ communicate, and state 4 is absorbing. Since state $\{1,2,3\}$ can go to state 4, we conclude that they are all *transient.*

$$\boldsymbol{P}_T = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{pmatrix} \Rightarrow \boldsymbol{S} = (\boldsymbol{I} - \boldsymbol{P}_T)^{-1} = \begin{pmatrix} \frac{64}{29} & \frac{40}{29} & \frac{18}{29} \\ \frac{28}{29} & \frac{90}{29} & \frac{26}{29} \\ \frac{38}{29} & \frac{60}{29} & \frac{56}{29} \end{pmatrix} \tag{31}$$

The third column gives s_{i3}. $s_{13} = 18/29, s_{23} = 26/29, s_{33} = 56/29$. It follows that

$$f_{13} = \frac{s_{13}}{s_{33}} = \frac{9}{28}; \quad f_{23} = \frac{s_{23}}{s_{33}} = \frac{13}{28}; \quad f_{33} = \frac{s_{33}-1}{s_{33}} = \frac{27}{56} \tag{32}$$

Problem 10.

Solution. Denote $\mu = \sum jP_j$, when $\mu > 1$, π_0 is the smallest positive number that solves

$$\pi_0 = \sum_{j \ge 0} \pi_0^j P_j \tag{33}$$

Else $\pi_0 = 1$. Hence we have

(a) $\pi_0 = 1$ since $\mu = 3/4 < 1$.
(b) $\pi_0 = 1$ since $\mu = 1/2 + 2 \cdot 1/4 = 1$.
(c) $\mu = 1/2 + 2/3 > 1$,

$$\pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^2 \Rightarrow \pi_0 = \frac{1}{2} \tag{34}$$

Problem 11.

Solution. Define this as event E. Fatorize the probability by conditioning recursively:

$$
\begin{aligned}
\mathbb{P}(E) &= \sum_{i\neq 0} \mathbb{P}(X_{m-k-1}=i)\,\mathbb{P}(X_{m-k}=...=X_{m-1}=0, X_m\neq 0|X_{m-k-1}=i) \\
&= \sum_{i\neq 0} \pi_i \mathbb{P}(X_{m-k}=0|X_{m-k-1}=i)\cdot \mathbb{P}\left(\begin{matrix} X_{m-k+1}=..=X_{m-1}=0, \\ X_m\neq 0\end{matrix}\middle| \begin{matrix} X_{m-k-1}=i, \\ X_{m-k}=0\end{matrix}\right) \\
&= \sum_{i\neq 0} \pi_i P_{i0}\cdot \mathbb{P}\left(X_{m-k+1}=0\middle| \begin{matrix} X_{m-k-1}=i, \\ X_{m-k}=0\end{matrix}\right)\mathbb{P}\left(\begin{matrix} X_{m-k+2}=..=X_{m-1}=0, \\ X_m\neq 0\end{matrix}\middle| \begin{matrix} X_{m-k-1}=i \\ X_{m-k}=X_{m-k+1}=0\end{matrix}\right) \\
&= \ldots\ldots \quad \text{(Apply Markovian Property)} \\
&= \sum_{i\neq 0} \pi_i P_{i0}\cdot P_{00}^{k-1}\cdot \mathbb{P}(X_m\neq 0|X_{m-1}=0) \\
&= \sum_{i\neq 0} \pi_i P_{i0}\cdot P_{00}^{k-1}(1-P_{00}) \\
&= P_{00}^{k-1}(1-P_{00})\left(\sum_{i\geq 0}\pi_i P_{i0}-\pi_0 P_{00}\right) \\
&= P_{00}^{k-1}(1-P_{00})^2\pi_0
\end{aligned}
\tag{35}
$$

Problem 12.

Proof.

$$
\begin{aligned}
P_{ij}^{(n)} &= \mathbb{P}(X_n=j|X_0=i) \\
&= \sum_{k=1}^{n} \mathbb{P}\left(X_n=j\middle| \begin{matrix} X_k=j, \\ X_{k-1},...,X_1\neq j, \\ X_0=i\end{matrix}\right)\mathbb{P}\left(\begin{matrix} X_k=j, \\ X_{k-1},...,X_1\neq j,\end{matrix}\middle| X_0=i\right) \\
&= \sum_{k=1}^{n} \mathbb{P}(X_n=j|X_k=j)\, f_{ij}^{(k)} \quad \text{(By Markovian Property)} \\
&= \sum_{k=1}^{n} P_{jj}^{(n-k)} f_{ij}^{(k)} \\
&= \sum_{k=0}^{n} P_{jj}^{(n-k)} f_{ij}^{(k)} \quad (\text{Since } f_{ij}^{(0)}=0.)
\end{aligned}
\tag{36}
$$

□

Problem 13.

Solution. (a) Define f_n be the probability that first return occurs at time n; and P_n be the probability of returning at n, both conditional on $X_0=0$ if express by conventional notations, $f_n := f_{00}^{(n)}, P_n := P_{00}^{(n)}$.

$$
\begin{aligned}
P_n &:= \mathbb{P}(X_n=0|X_0=0) \\
f_n &:= \mathbb{P}(X_n=0|X_0=0, X_1,...,X_{n-1}\neq 0)
\end{aligned}
\tag{37}
$$

Then for the first question, it suffices to calculate $\sum_{n\geq 0} nf_n$.
(*Step.1*) Consider transition probability, by the result of problem 12:

$$
P_{00}^{(n)} = \sum_{k=1}^{n} P_{00}^{(n-k)} f_{00}^{(k)} \quad \Rightarrow \quad P_n = \sum_{k=1}^{n} P_{n-k} f_k \quad (\dagger) \tag{38}
$$

P_n can be easily obtained, for n odd, there is no chance to return. For n even, it must spend half of the time moving forward, and half backward to remained unmoved. Hence for $n \geq 0$

$$P_{2n} = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}, \quad P_{2n+1} = 0 \tag{39}$$

(*Step.2*) Define $P_0 = 1$, then define **Generating Function**:

$$\Phi_P(t) = \sum_{n\geq 0} P_n t^n, \quad \Phi_f(t) = \sum_{n\geq 1} f_n t^n \tag{40}$$

We can already write down Φ_P explicitly, by *Taylor Expansion* of $(1-x)^{-1/2}$:[1]

$$\Phi_P = \sum_{n\geq 0}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} t^{2n} = \frac{1}{\sqrt{1-t^2}} \tag{41}$$

Apply (†),

$$\begin{aligned}\Phi_P(t) &= 1 + \sum_{n\geq 1}\left(\sum_{k=1}^{n} P_{n-k} f_k\right) t^n \\ &= 1 + \sum_{k\geq 1} f_k t^k \left(\sum_{n\geq k} P_{n-k} t^{n-k}\right) \\ &= 1 + \Phi_P(t)\Phi_f(t)\end{aligned} \tag{42}$$

Which implies that $\Phi_f(t) = 1 - 1/\Phi_P(t) = 1 - \sqrt{1-t^2}$.
(*Step.3*) It is easy to check that

$$\sum_{n\geq 0} n f_n = \frac{\partial}{\partial t}\Phi_f(t)\bigg|_{t=1} = \frac{t}{\sqrt{1-t^2}}\bigg|_{t=1} = \infty \tag{43}$$

We therefore conclude the the expected returning time is infinity, i.e. the symmetric random walk on 1-d is **Null-Recurrent**.
(b) Denote $A_{2t} = \{\text{Return to origin at time } 2t.\}$, clearly, $\mathbb{P}(A_{2t}) = P_{2t}$. Then, we have

$$N_{2n} = \sum_{t=1}^{n} \mathbb{1}_{A_{2t}} \tag{44}$$

Hence

$$\mathbb{E}[N_{2n}] = \sum_{t=1}^{n}\mathbb{E}[\mathbb{1}_{A_{2t}}] = \sum_{t=1}^{n} P_{2t} = \sum_{t=0}^{n} P_{2t} - 1 = \sum_{t=0}^{n}\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} - 1 \quad (\triangle) \tag{45}$$

Claim.

$$\mathbb{E}[N_{2n}] = (2n+1)\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} - 1 \quad (\dagger) \tag{46}$$

Proof of Claim. We prove this by **induction**. For the boundary case $\mathbb{E}[N_0] = 0$ is clear. Now assume (†) holds for n, we check $n+1$: By ($\triangle$):

$$\begin{aligned}\mathbb{E}[N_{2n+2}] &= \mathbb{E}[N_{2n}] + P_{2n+2} \\ &= (2n+1)\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} + \binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2} - 1 \\ &= (2n+1)\frac{2n!}{n!n!}\left(\frac{1}{2}\right)^{2n} + \frac{(2n+2)!}{(n+1)!(n+1)!}\left(\frac{1}{2}\right)^{2n+2} - 1 \\ &= \frac{(2n+1)\cdot 4\cdot (n+1)^2\cdot 2n!}{(n+1)^2\cdot n!n!}\left(\frac{1}{2}\right)^{2n+2} + \frac{(2n+2)!}{(n+1)!(n+1)!}\left(\frac{1}{2}\right)^{2n+2} - 1 \\ &= (2n+2)\frac{(2n+2)!}{(n+1)!(n+1)!}\left(\frac{1}{2}\right)^{2n+2} + \frac{(2n+2)!}{(n+1)!(n+1)!}\left(\frac{1}{2}\right)^{2n+2} - 1 \\ &= (2n+3)\binom{2n+2}{n+1}\left(\frac{1}{2}\right)^{2n+2} - 1\end{aligned} \tag{47}$$

[1] $(1-x)^{-1/2} = \sum_{n\geq 0}\binom{2n}{n}(x/4)^n$

Which finished the induction proof.
(c) By **Stirling's formula** in the textbook problem, $P_{2t} \sim \frac{1}{\sqrt{t}}$, hence the summation

$$\mathbb{E}\left[N_{2n}\right] = \sum_{t=1}^{n} P_{2t} \sim \sum_{t=1}^{n} \frac{1}{\sqrt{t}} \tag{48}$$

Claim. $\sum_{t=1}^{n} 1/\sqrt{t} = \Theta(\sqrt{n})$.
Proof of Claim. Firstly, notice that

$$\frac{1}{\sqrt{t}} \leq \frac{2}{\sqrt{t}+\sqrt{t-1}} = \frac{2(\sqrt{t}+\sqrt{t-1})(\sqrt{t}-\sqrt{t-1})}{\sqrt{t}+\sqrt{t-1}} = 2(\sqrt{t}-\sqrt{t-1}) \tag{49}$$

Then one can easily see that

$$\sum_{t=1}^{n} \frac{1}{\sqrt{t}} \leq 2(\sqrt{n}-1) \tag{50}$$

Secondly, notice that

$$\frac{1}{\sqrt{t}} \geq \frac{1}{\sqrt{t}+\sqrt{t-1}} = \sqrt{t}-\sqrt{t-1} \tag{51}$$

we assume $\sum_{t=1}^{n-1} 1/\sqrt{t} \geq \sqrt{n-1}$, then the inequality above implies:

$$\sum_{t=1}^{n} \frac{1}{\sqrt{n}} \geq \sqrt{n-1} + \frac{1}{\sqrt{n}} \geq \sqrt{n} \tag{52}$$

It is easy to check boundary case $n=1$, then by induction, we obtain $\sum_{t=1}^{n} 1/\sqrt{n} \geq \sqrt{n}$. Therefore

$$\sqrt{n} \leq \sum_{t=1}^{n} \frac{1}{\sqrt{t}} \leq 2(\sqrt{n}-1) \tag{53}$$

Which finished the proof.

Problem 14.

Solution. (a) The boundary condition $M_0 = M_N = 0$ is clear by game rules. For $1 \leq i \leq N-1$. Define $X_n :=$ # of the rounds till gameover starting at initial fortune n. $W = \{\text{Win the next round.}\}$

$$\begin{aligned} \mathbb{E}\left[X_n\right] &= \mathbb{E}\left[X_n|W\right]\mathbb{P}\left(W\right) + \mathbb{E}[X_n|W^{\complement}]\mathbb{P}(W^{\complement}) \\ &= (1+\mathbb{E}\left[X_{n+1}\right])p + (1+\mathbb{E}\left[X_{n-1}\right])q \\ &= 1 + pM_{n+1} + qM_{n-1} \end{aligned} \tag{54}$$

(b) The formula

$$M_n = 1 + pM_{n+1} + qM_{n-1} \tag{55}$$

is a *second order linear nonhomogeneous* recurrence relation with constant coefficients. By related theory, it has same general solution as homogeneous one $M_n = pM_{n+1} + qM_{n-1}$.

- For $p = q = 1/2$, the general solution is

$$M_n = -n^2 + C_1 + C_2 n \tag{56}$$

  Where C_1, C_2 are undetermined constants. Applying boundary condtions $M_0 = M_N = 0 \Rightarrow C_1 = 0, C_2 = N$. $M_n = n(N-n)$.
- For $p \neq q$, the general solution is

$$M_n = \frac{n}{q-p} + C_1 + C_2\left(\frac{q}{p}\right)^n \tag{57}$$

  Boundary conditions yields $C_1 + C_2 = 0$ and $C_1 + C_2(q/p)^N = -N/(q-p)$. $\Rightarrow$

$$C_2 = \frac{-N}{(q-p)((q/p)^N - 1)}, \quad C_1 = -C_2 \tag{58}$$

  So

$$M_n = \frac{n}{q-p} + \frac{N}{q-p} \cdot \frac{1-(q/p)^n}{(q/p)^N - 1} \tag{59}$$

Problem 15.

Solution. $X_n = X_{n-1} - S_n + O_n$. S_n, O_n stand for sales and order. $\{O_n\}$ is independent of $\{X_n\}$, and S_n only depend on X_{n-1}, independent of other history $\{X_t\}_{t<n-1}$. Therefore $\{X_n\}$ is a markov chain. The state space is $\{0, 1, ..., S\}$.

(*Case.1*) If $0 \leq X_{n-1} < s$, then at the beginning of n-th period, the inventory is S. To remain j items at the end of this period, it should sell $(S-j)$ items, $j \leq S$.

$$P_{ij} = \begin{cases} 0 & j > S,\ 0 \leq i < s \\ \alpha_{S-j} & 0 < j \leq S,\ 0 \leq i < s \\ 1 - \sum_{k=0}^{S-1} \alpha_k & j = 0,\ 0 \leq i < s \end{cases} \tag{60}$$

(*Case.2*) Else if $X_{n-1} \geq s$, the inventory will be $X_{n-1} = i$. To remain j items at the end of this period, it should sell $(i-j)$ items, $j \leq i$.

$$P_{ij} = \begin{cases} 0 & j > i,\ i \geq s \\ \alpha_{i-j} & 0 < j \leq i,\ i \geq s \\ 1 - \sum_{k=0}^{i-1} \alpha_k & j = 0,\ i \geq s \end{cases} \tag{61}$$

Which gives the full representation of transition matrix $\boldsymbol{P}$.