Stochastic Process Assignment V

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May 4, 2016

Problem 1.

Solution. Define the followings

- $\{D_n\}$ be the demand of customers, i.i.d and has known distribution G.
- $\{T_n\}$ be the interarrival time of customers, i.i.d. and has common distribution F.
- X be the time between two occasions that the store make orders and bring the inventory up to S. Then X forms renewal process.
- Upon X, further define an alternating renewal process. The system is "On" if inventory is greater than or equal to y. Y is the time that system is in "On" in each cycle.

By theorem,

$$\lim_{t\to\infty} \mathbb{P}\left(\{\text{"On" at time } t\}\right) = \frac{\mathbb{E}[Y]}{\mathbb{E}[X]} \quad (\dagger) \tag{1}$$

is exactly the long-run proportion of time that system possesses inventory more than y. Further define

$$N(x) = \min\left\{n : \sum_{i=1}^{n} D_i > S - x\right\} \tag{2}$$

Be the number of customers to bring the inventory from full (S) to x. It is clear that $N(x) \perp T_n$ for all x, n, becasue $\{D_n\} \perp \{T_n\}$.
By this definition, we have Z is just the time it takes to bring the inventory from full to y, and X is just that from full to s, due to the (S, s) policy.

$$Z = \sum_{i=1}^{N(y)} T_i \qquad X = \sum_{i=1}^{N(s)} T_i; \tag{3}$$

Wald's Identity $\Rightarrow \mathbb{E}[Z] = \mathbb{E}[N(y)]\,\mathbb{E}[T]$, $\mathbb{E}[X] = \mathbb{E}[N(s)]\,\mathbb{E}[T]$.
The definition of $N(x)$ indicates that it means the same thing as the index of the first arrival that comes **After** time $S - x$, where the interarrival times have same distribution as D. Define $\tilde{N}(t)$ as renewal process associated with $\{D_n\}$, that is

$$\begin{aligned} &N(x) = \tilde{N}(S-x) + 1 \\ \Rightarrow &\mathbb{E}[N(x)] = m(S-x) + 1 = \sum_{n\geq 1} \mathbb{P}\left(\tilde{N}(S-x) \geq n\right) + 1 = \sum_{n\geq 1} G(S-x) + 1 \end{aligned} \tag{4}$$

So

$$(\dagger) = \frac{\mathbb{E}[N(y)]}{\mathbb{E}[N(s)]} = \frac{\sum_{n\geq 1} G(S-y) + 1}{\sum_{n\geq 1} G(S-s) + 1} \tag{5}$$

Problem 2.

Solution. (a) Consider a Poisson process with interarrival time $T \sim \text{Exp}(\lambda)$. And we *only* count the k^{th} event when k is a multiple of r. Then the counted events form our desired renewal process that has interarrival time $rT \sim \Gamma(r, \lambda)$.
Define $\{A(t)\}$ as the counting of initial Poisson; $N(t)$ as counting process associated with the new counted process. We have

$$\mathbb{P}(N(t) \geq n) = \mathbb{P}(A(t) \geq rn) = \sum_{k\geq rn} e^{\lambda t} \frac{(\lambda t)^k}{k!} \tag{6}$$

(b)

$$
\begin{aligned}
\mathbb{E}\left[N(t)\right] &= \sum_{n\geq 1} \mathbb{P}\left(N(t)\geq n\right) \\
&= \sum_{n\geq 1}\sum_{k\geq rn} e^{\lambda t}\frac{(\lambda t)^k}{k!} \\
&= \sum_{k\geq r}\sum_{n=1}^{\lfloor \frac{k}{r}\rfloor} e^{\lambda t}\frac{(\lambda t)^k}{k!} \\
&= \sum_{k\geq r}\left\lfloor \frac{k}{r}\right\rfloor e^{\lambda t}\frac{(\lambda t)^k}{k!}
\end{aligned} \tag{7}
$$

Problem 3.

Solution. We define following RVs:

- The completion of jobs forms a renewal process, denote X the interarrival time. We want to calculate the job completion rate in the long run, i.e. $\frac{1}{\mathbb{E}[X]}$
- Let Z be the time *required* to finish a job, Z has distribution F.
- Let T be the interarrival time of poisson shocks. $T \perp Z$, $T \sim \text{Exp}(\lambda)$.

Firstly condition on shock (T) and required time to complete current job (Z). If $T \geq Z$, the current job is not affected by the shock and will be finished upon Z. Otherwise, the job is restarted at T.

$$
\mathbb{E}\left[X|T,Z\right] = \begin{cases} Z & \text{If } T\geq Z, \\ \mathbb{E}\left[X\right]+T & \text{else if } T<Z. \end{cases} \tag{8}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}\left[X|Z\right] &= \int_0^\infty \mathbb{E}\left[X|Z,T=t\right] f_T(t)dt \\
&= \left(\int_0^Z + \int_Z^\infty\right)\mathbb{E}\left[X|Z,T=t\right]\lambda e^{-\lambda t}dt \\
&= \int_0^Z (\mathbb{E}\left[X\right]+t)\lambda e^{-\lambda t}dt + \int_Z^\infty Z\cdot \lambda e^{-\lambda t}dt \\
&= \mathbb{E}\left[X\right](1-e^{-\lambda Z}) - te^{-\lambda t}\Big|_0^Z + \frac{-1}{\lambda}e^{-\lambda x}\Big|_0^Z + (-e^{-\lambda t})\Big|_Z^\infty \\
&= \mathbb{E}\left[X\right](1-e^{-\lambda Z}) - Ze^{-\lambda Z} - \frac{e^{-\lambda Z}-1}{\lambda} + Ze^{-\lambda Z} \\
&= \left(\mathbb{E}\left[X\right]+\frac{1}{\lambda}\right)(1-e^{-\lambda Z})
\end{aligned} \tag{9}
$$

Finally, due to the fact that $T\perp Z$:

$$
\begin{aligned}
\mathbb{E}\left[X\right] = \mathbb{E}\left[\mathbb{E}\left[X|Z\right]\right] &= \left(\mathbb{E}\left[X\right]+\frac{1}{\lambda}\right)(1-\mathbb{E}\left[e^{-\lambda Z}\right]) \\
\Rightarrow \frac{1}{\mathbb{E}\left[X\right]} &= \frac{\lambda\mathbb{E}\left[e^{-\lambda Z}\right]}{1-\mathbb{E}\left[e^{-\lambda Z}\right]} = \frac{\lambda\int e^{-\lambda z}F'(z)dz}{1-\int e^{-\lambda z}F'(z)dz}
\end{aligned} \tag{10}
$$

Where F' is PDF of Z. F is known.

Problem 4.

Solution. (a) Define following RVs:

- $\cdot$ The machine replacements constitutes a renewal process. Denote X the time between replacements.
- $\cdot$ Define Z the lifespan of current machine. Z has distribution $F, (f)$.

Similar to the analysis in question 3, We have

$$\mathbb{E}[X|Z] = \begin{cases} Z & \text{If } Z \le T, \\ T & \text{else if } Z > T. \end{cases} \tag{11}$$

Therefore

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|Z]] \\ &= \mathbb{E}[\mathbb{E}[X|Z]; Z \le T] + \mathbb{E}[\mathbb{E}[X|Z]; Z > T] \\ &= \int_0^T zf(z)dz + \int_T^\infty Tf(z)dz \\ &= \int_0^T zf(z)dz + T(1 - F(T)) \quad (\dagger) \end{aligned} \tag{12}$$

Hence the rate is $1/\mathbb{E}[X] = (\dagger)^{-1}$.
(b) Further define

- $\cdot$ Y be the life between fails of machines. Forms another renewal process. If $Z \le T$, the current machine fails at Z. Otherwise, the current machine does not fail by T and it then starts up a new machine. Therefore

$$\mathbb{E}[Y|Z] = \begin{cases} Z & \text{If } Z \le T, \\ T + \mathbb{E}[Y] & \text{else if } Z > T. \end{cases} \tag{13}$$

Hence

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y|Z]] \\ &= \mathbb{E}[\mathbb{E}[Y|Z]; Z \le T] + \mathbb{E}[\mathbb{E}[Y|Z]; Z > T] \\ &= \int_0^T zf(z)dz + \int_T^\infty (T + \mathbb{E}[Y])f(z)dz \\ &= \int_0^T zf(z)dz + (T + \mathbb{E}[Y])(1 - F(T)) \quad (\dagger) \end{aligned} \tag{14}$$

$$\Rightarrow \mathbb{E}[Y] = \frac{\int_0^T zf(z)dz + T(1 - F(T))}{F(T)} = \frac{(\dagger)}{F(T)} \tag{15}$$

i.e. $1/\mathbb{E}[Y] = F(T) \times (\dagger)^{-1}$

Problem 5.

Solution. (1) We set two states as the two types of machine life distribution. i.e. rate μ_1 and μ_2. Define the CTMC associated with it. By regarding each machine failure as a transition, we obtain

$$q_{12} = \mu_1(1 - p), \quad q_{21} = \mu_2 p \tag{16}$$

Therefore

$$P_{11}(t) = \frac{\mu_1(1-p)}{\mu_1(1-p) + \mu_2 p} \exp\{-[\mu_1(1-p) + \mu_2 p]t\} + \frac{\mu_2 p}{\mu_1(1-p) + \mu_2 p} \tag{17}$$

And $P_{12}(t) = 1 - P_{11}(t)$. Similarly we have

$$P_{22}(t) = \frac{\mu_2 p}{\mu_1(1-p) + \mu_2 p} \exp\{-[\mu_1(1-p) + \mu_2 p]t\} + \frac{\mu_1(1-p)}{\mu_1(1-p) + \mu_2 p} \tag{18}$$

And $P_{21}(t) = 1 - P_{22}(t)$.
(2) Condition on initial machine type (Denote $X(t)$ as type, $Y(t)$ as operating time)

$$\begin{aligned} \mathbb{E}[Y(t)] &= p\mathbb{E}[Y(t)|X(0) = 1] + (1-p)\mathbb{E}[Y(t)|X(0) = 2] \\ &= p\left[\frac{P_{11}(t)}{\mu_1} + \frac{P_{12}(t)}{\mu_2}\right] + (1-p)\left[\frac{P_{21}(t)}{\mu_1} + \frac{P_{22}(t)}{\mu_2}\right] \end{aligned} \tag{19}$$

Where $P_{\cdot}(t)$s are specified in CTMC.
The renewal equation, i.e. $m(t)$ is given via

$$\mu[m(t)+1] = t + \mathbb{E}\left[Y(t)\right] \tag{20}$$

with $\mu = \frac{p}{\mu_1} + \frac{(1-p)}{\mu_2}$

Problem 6.

Solution. Define the followings

- T_1, T_2 be the lifespan of two components. $T_1 \sim \text{Exp}(\lambda_1)$, $T_2 \sim \text{Exp}(\lambda_2)$.
- The replacements of machines constitute a renewal process. Let X be the time between replacements.
- Further, $X = A + Y$, where A is the time during which 2 components are working together. Clearly, $A = \min\{T_1, T_2\}$. Therefore, due to memorylessness property, $Y \sim \text{Exp}(\lambda_1)$ if $A = T_2$; $Y \sim \text{Exp}(\lambda_2)$ if otherwise.

Hence

$$\begin{aligned} \mathbb{E}\left[X\right] &= \mathbb{E}\left[A\right] + \mathbb{E}\left[Y; A = T_1\right] + \mathbb{E}\left[Y; A = T_2\right] \\ &= \mathbb{E}\left[A\right] + \mathbb{E}\left[Y | A = T_1\right] \mathbb{P}\left(A = T_1\right) + \mathbb{E}\left[Y | A = T_2\right] \mathbb{P}\left(A = T_2\right) \\ &= \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1} \end{aligned} \tag{21}$$

Conduct similar analysis for the cost:

- Define $C = K + C_1 + C_2$ be the cost within X. where K is the fixed cost. C_2 is the cost incurred when 2 components are working together. Clearly, $C_2 = c_2 A$. C_1 is the cost incurred when 1 component is working. $C_1 = c_1 Y$.

$$\begin{aligned} \mathbb{E}\left[C\right] &= K + c_2\mathbb{E}\left[A\right] + c_2\left(\mathbb{E}\left[Y; A = T_1\right] + \mathbb{E}\left[Y; A = T_2\right]\right) \\ &= K + \frac{c_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_1} \end{aligned} \tag{22}$$

The long run average marginal cost of operation time is given by $r := \frac{\mathbb{E}[C]}{\mathbb{E}[X]}$.

$$r = \frac{\lambda_1\lambda_2(\lambda_1 + \lambda_2)K + \lambda_1\lambda_2 c_2 + (\lambda_1^2 + \lambda_2^2)c_1}{\lambda_1\lambda_2 + \lambda_1^2 + \lambda_2^2} \tag{23}$$

Problem 7.

Solution. (a) Settings:

- Denote $N(t)$ the arrival process.
- We regard an entering satellites becoming *Type I* satellites if it departs by time t, *Type II* satellites if otherwise. Denote # of type i satellites as $N_i(t)$.
- Consider the satellite enters at time s, $s \leq t$, then it will be a Type I satellite if its orbiting time is less than $t - s$. So the probability that he become Type I is $F(t - s)$.

It is clear that $X(t)$ tracks the number of Type II satellites by time t. Due to *Prop.5-3* In Ross, $X(t)$ i.e. $N_2(t)$ is a poisson variable with mean

$$m(t) = \lambda \int_0^t \overline{F}(s)ds = \lambda \int_0^t (1 - F(s))ds \tag{24}$$

Hence

$$\mathbb{P}\left(X(t) = k\right) = e^{-m(t)}\frac{m^k(t)}{k!} \tag{25}$$

(b) We view the system as an alternating renewal process,

- System is "On" if there is at least one satellite in the orbit. "Off" if otherwise. Denote X be the time between the ends of adjacent "Off" periods.
- $X = Y + Z$, Y is the time of "On" period. Z is that for "Off".

By theorem

$$\lim_{t\to\infty} \mathbb{P}\left(\{\text{Off at time } t\}\right) = \frac{\mathbb{E}[Z]}{\mathbb{E}[Y] + \mathbb{E}[Z]} \tag{26}$$

Once the system is off, it just waits for another arrival. So it's clear that Z is of same distribution as interarrival time of the Poisson arrival process. $\mathbb{E}[Z] = \frac{1}{\lambda}$.
Moreover, $\mathbb{P}\left(\{\text{Off at time } t\}\right) = \mathbb{P}\left(X(t) = 0\right) = e^{-m(t)}$.

$$\lim_{t\to\infty} \mathbb{P}\left(\{\text{Off at time } t\}\right) = \lim_{t\to\infty} e^{-m(t)} = e^{-\lambda \int_0^\infty \overline{F}(s)ds} \tag{27}$$

Therefore

$$e^{-\lambda \int_0^\infty \overline{F}(s)ds} = \frac{\frac{1}{\lambda}}{\mathbb{E}[Y] + \frac{1}{\lambda}} \quad \Rightarrow \quad \mathbb{E}[Y] = \frac{1 - e^{-\lambda \int_0^\infty \overline{F}(s)ds}}{\lambda e^{-\lambda \int_0^\infty \overline{F}(s)ds}} \tag{28}$$

And note that $\int_0^\infty \overline{F}(s)ds$ is just the expectation of service (orbiting) time. Y and Y_1 are identically distributed, hence the expected time remaining functional is given by $\mathbb{E}[Y]$.

Problem 8.

Solution. We must go for a stronger statement to obtain desired result.
Claim. $\{U_n\}$ i.i.d Uniform(0,1). For all $0 < x \le 1$, $N(x) := \min\{n : U_1 + ... + U_n > x\}$:

$$\mathbb{P}\left(N(x) > n\right) = \frac{x^n}{n!}$$

Proof of Claim. We proceed by induction. The $n = 1$ case is true. Since

$$\mathbb{P}\left(N(x) > 1\right) = \mathbb{P}\left(U_1 \le x\right) = x \tag{29}$$

Then

$$\begin{aligned}
\mathbb{P}\left(N(x) > n+1\right) &= \int_0^1 \mathbb{P}\left(N(x) > n+1 | U_1 = y\right) f_{U_1}(y)dy \\
&= \left(\int_0^x + \int_x^1\right) \mathbb{P}\left(N(x) > n+1 | U_1 = y\right) dy \\
&= \int_0^x \mathbb{P}\left(N(x) > n+1 | U_1 = y\right) dy \quad (\text{Since } N(x) = 1 \text{ if } U_1 \ge x.) \\
&= \int_0^x \mathbb{P}\left(N(x-y) > n\right) dy \\
&= \int_0^x \frac{(x-y)^n}{n!} dy \quad (\text{Let } z := x - y.) \\
&= \int_x^0 -\frac{z^n}{n!} dz = \frac{x^{n+1}}{(n+1)!}
\end{aligned} \tag{30}$$

Now take $x = 1 \Rightarrow \mathbb{P}\left(N > n\right) = \frac{1}{n!}$. Hence

$$\mathbb{E}[N] = \sum_{n\ge1} \mathbb{P}\left(N > n\right) = \sum_{n\ge1} \frac{1}{n!} = e \tag{31}$$

Problem 9.

Solution. Define the followings

- T_1, T_2 is the lifespan of two machine, $T_1, T_2 \sim \text{Exp}(\lambda)$.

- Busy and Idle period forms an alternating renewal process. Let $X = B + D$ be the time between two ends of idle periods. B for busy period, D for idle.
- Z be the repair time, which has known distribution G, g.

Then it suffices to calculate $\frac{\mathbb{E}[D]}{\mathbb{E}[B]+\mathbb{E}[D]}$. D equals the time until next machine failure, which equals $\min\{T_1, T_2\}$. Hence $\mathbb{E}[D] = \frac{1}{2\lambda}$.
Now consider busy period. Let T be the remaining life of the other machine when the repairman enters a busy period. Due to memorylessness, $T \sim \mathrm{Exp}(\lambda)$. Conditional on T, Z, if $T > Z$, i.e. the other machine does not break until the end of current busy period, the man will enter an idle. Otherwise if $T \leq Z$, the man restarts another busy period when finished with the current one.

$$\mathbb{E}[B|T,Z] = \begin{cases} Z & \text{If } T > Z, \\ \mathbb{E}[B] + Z & \text{else if } T \leq Z. \end{cases} \tag{32}$$

Hence

$$\begin{aligned} \mathbb{E}[B] &= \mathbb{E}[\mathbb{E}[B|Z]] \\ &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[B|Z, T=t]\, f_T(t)dt\right] \\ &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[B|Z, T=t]\, \lambda e^{-\lambda t}dt\right] \\ &= \mathbb{E}\left[\int_0^Z (\mathbb{E}[B] + Z)\lambda e^{-\lambda t}dt + \int_Z^\infty Z\lambda e^{-\lambda t}dt\right] \\ &= \mathbb{E}\left[Z + \mathbb{E}[B](1 - e^{-\lambda Z})\right] \\ &= \mathbb{E}[Z] + \mathbb{E}[B](1 - \mathbb{E}\left[e^{-\lambda Z}\right]) \end{aligned} \tag{33}$$

Implies that

$$\mathbb{E}[B] = \frac{\mathbb{E}[Z]}{\mathbb{E}[e^{-\lambda Z}]} = \frac{\int zG'(z)dz}{\int e^{-\lambda z}G'(z)dz} \tag{34}$$

So, proportion of idle time is:

$$\frac{\mathbb{E}[D]}{\mathbb{E}[B]} = \frac{1/2\lambda}{\mathbb{E}[B] + 1/2\lambda} \tag{35}$$

Problem 10.

Solution. The transition matrix is given by

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} \tag{36}$$

Denote limiting probability as $\boldsymbol{\pi}$, solve for $\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi} \Rightarrow \boldsymbol{\pi} = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8})$.
Denote mean time spent in state i as μ_i.

$$\begin{aligned} \mu_1 &= t_1 + P_{12}m_{12} = 11 \\ \mu_2 &= t_2 + P_{23}m_{23} = 22 \\ \mu_3 &= t_3 + P_{31}m_{31} + P_{32}m_{32} = \frac{67}{3} \end{aligned} \tag{37}$$

The limiting probabiliy is given by

$$\begin{aligned} P_j &= \frac{\pi_j \mu_j}{\sum_j \pi_j \mu_j} \Rightarrow \\ P_1 &= \frac{66}{465}, P_2 = \frac{198}{465}, P_3 = \frac{201}{465} \end{aligned} \tag{38}$$

Problem 11. (**Inspection Paradox**) For a renewal process with interarrival time X_n with distribution F, show

$$\mathbb{P}\left(X_{N(t)+1} > x\right) \geq \mathbb{P}\left(X_n > x\right)$$

Proof. Use the conventional notation $S_n = \sum_{i=1}^{n} X_i$ for waiting time. For $X_{N(t)+1}$. We condition on event $\{S_n = s, N(t) = n\}$. Note that we have following equivalence relationships:

- $\{X_{N(t)+1}|N(t) = n\} \iff \{X_{n+1}|N(t) = n\}$. $(*)$
- $\{N(t) = n\} \iff \{S_n \leq t, S_{n+1} > t\}$. Hence

$$\{S_n = s, N(t) = n\} \iff \{S_n = s, X_{n+1} > t - s\} \quad (**)$$

Apply these equivalent condition step by step, we have

$$\begin{aligned}
\mathbb{P}\left(X_{N(t)+1} > x|N(t) = n, S_n = s\right) &= \mathbb{P}\left(X_{n+1} > x|N(t) = n, S_n = s\right) \quad [\text{Apply } (*)] \\
&= \mathbb{P}\left(X_{n+1} > x|S_n = s, X_{n+1} > t - s\right) \quad [\text{Apply } (**)] \\
&= \mathbb{P}\left(X_{n+1} > x|X_{n+1} > t - s\right) \quad [\text{Since } X_{n+1} \perp S_n] \\
&= \frac{\mathbb{P}\left(X_{n+1} > x, X_{n+1} > t - s\right)}{\mathbb{P}\left(X_{n+1} > t - s\right)} \\
&= \frac{\overline{F}(\max\{x, t - s\})}{\overline{F}(t - s)}
\end{aligned} \tag{39}$$

Claim. $\frac{\overline{F}(\max\{x,t-s\})}{\overline{F}(t-s)} \geq \overline{F}(x)$. *Proof of Claim.*

- *Case.1* If $\max\{x, t - s\} = x$, then $LHS = \frac{\overline{F}(x)}{\overline{F}(t-s)} \geq \overline{F}(x)$. Because $\overline{F}(\cdot) \leq 1$.
- *Case.2* If $\max\{x, t - s\} = t - s$, then $LHS = \frac{\overline{F}(t-s)}{\overline{F}(t-s)} = 1 \geq \overline{F}(x)$ is clear. ■

Hence,

$$\mathbb{P}\left(X_{N(t)+1} > x|N(t) = n, S_n = s\right) = \mathbb{E}\left[\mathbb{1}_{\{X_{N(t)+1} > x\}}|N(t) = n, S_n = s\right] \geq \overline{F}(x)$$

Take expectation both sides:

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{X_{N(t)+1} > x\}}|N(t) = n, S_n = s\right]\right] &= \mathbb{E}\left[\mathbb{1}_{\{X_{N(t)+1} > x\}}\right] \\
&= \mathbb{P}\left(X_{N(t)+1} > x\right) \\
&\geq \overline{F}(x) = \mathbb{P}\left(X_n > x\right)
\end{aligned} \tag{40}$$

□

Problem 12. Define **Age** and **Residual** life: $A(t) := t - S_{N(t)}$, $Y(t) := S_{N(t)+1} - t$. Show that

a) If F is nonlattice and $\mu < \infty$, then

$$\lim_{t\to\infty} \mathbb{P}\left(A(t) \leq x\right) = \lim_{t\to\infty} \mathbb{P}\left(Y(t) \leq x\right) = \frac{\int_0^x \overline{F}(y)dy}{\mu}$$

b)

$$\lim_{t\to\infty} \mathbb{P}\left(X_{N(t)+1} > x\right) = \frac{\int_x^\infty dF(y)}{\mu}$$

c) If F is nonlattice and $X \in \mathcal{L}^2$, then the limiting mean excess life is

$$\lim_{t\to\infty} \mathbb{E}\left[Y(t)\right] = \frac{\mathbb{E}\left[X^2\right]}{2\mu}$$

Proof. a) We define an alternating renewal process as follows

- The full cycle corresponds to the initial renewal process, i.e. each cycle lasts for X.

· The system is "On" at t if the age $A(t) \leq x$. In another word, the **FIRST** x unit of time in the cycle is "On". Denote "On" time as Z. We have, by definition $Z = \min\{X, x\}$.

Apply *Thm.3.4.4* in the notes. I.e. if F is nonlattice, $\mathbb{P}(\{\text{On at } t\}) = \frac{\mathbb{E}[Z]}{\mathbb{E}[X]}$, i.e.

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{P}(A(t) \leq x) &= \frac{\mathbb{E}[Z]}{\mathbb{E}[X]} \\
&= \frac{\mathbb{E}[\min\{x, X\}]}{\mu} \\
&= \frac{1}{\mu}\int_0^\infty \mathbb{P}(\min\{x, X\} > y)\, dy \\
&= \frac{1}{\mu}\int_0^x \mathbb{P}(\min\{x, X\} > y)\, dy \quad (\text{Since } \min\{x, X\} \leq y \text{ when } y \geq x) \\
&= \frac{1}{\mu}\int_0^x \mathbb{P}(X > y)\, dy \quad (\text{Since } \{\min\{x, X\} > y\} \iff \{X > y\} \text{ when } y < x) \\
&= \frac{\int_0^x \overline{F}(y)dy}{\mu}
\end{aligned}
\tag{41}
$$

Proceed similarly for $Y(t)$. The system is "On (prime)" at t if the remaining life $Y(t) \leq x$. In another word, the **LAST** x unit of time in the cycle is "On (prime)". Denote "On (prime)" time as Z'. We have, by definition $Z' = \min\{X, x\}$. By exactly the same calculation;

$$
\lim_{t\to\infty} \mathbb{P}(Y(t) \leq x) = \frac{\mathbb{E}[Z']}{\mathbb{E}[X]} = \frac{\int_0^x \overline{F}(y)dy}{\mu} \tag{42}
$$

□

Proof. b) $X_{N(t)+1}$ is the current interval containing t. We define an alternating renewal process: The system is "On" for the **Entire** cycle if that cycle is longer than x, that is, for any cycle, it is either totally "On" (if longer than x) or totally "Off". So we have

$$
\mathbb{P}\left(X_{N(t)+1} > x\right) = \mathbb{P}(\{\text{Cycle containing } t \text{ is totally On}\}) = \mathbb{P}(\{\text{On at } t\}) \tag{43}
$$

Apply *Thm.3.4.4*,

$$
\mathbb{P}\left(X_{N(t)+1} > x\right) = \frac{\mathbb{E}[Z]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X; X > x]}{\mu} = \frac{\int_x^\infty yf(y)dy}{\mu} = \frac{\int_x^\infty ydF(y)}{\mu} \tag{44}
$$

□

Proof. c) We calculate $\mathbb{E}[Y(t)]$ by condition on $S_{N(t)}$:

$$
\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E}\left[Y(t)|S_{N(t)} = 0\right]\mathbb{P}\left(S_{N(t)} = 0\right) + \int_0^t \mathbb{E}\left[Y(t)|S_{N(t)} = y\right] dF_{S_{N(t)}}(y) \\
&= \mathbb{E}\left[Y(t)|S_{N(t)} = 0\right]\overline{F}(t) + \int_0^t \mathbb{E}\left[Y(t)|S_{N(t)} = y\right]\overline{F}(t-y)dm(y) \quad (\text{By lemma 3.4.3.})
\end{aligned}
\tag{45}
$$

Now we have,

$$
\begin{aligned}
\mathbb{E}\left[Y(t)|S_{N(t)} = 0\right] &= \mathbb{E}[X - t|X > t] \\
\mathbb{E}\left[Y(t)|S_{N(t)} = y\right] &= \mathbb{E}[X - (t-y)|X > t-y]
\end{aligned}
\tag{46}
$$

When given $S_{N(t)} = y$, $A(t) = t - y$, hence $Y(t) = X - A(t) = X - (t-y)$. And the condition implies $X > t - y$. So the second one above follows. We have

$$
\mathbb{E}[Y(t)] = \mathbb{E}[X - t|X > t]\overline{F}(t) + \int_0^t \mathbb{E}[X - (t-y)|X > t-y]\overline{F}(t-y)dm(y) \tag{47}
$$

Let $h(t) := \mathbb{E}[X-t|X>t]\overline{F}(t)$, we can check $h(t)$ is directly Riemann integrable since $X \in \mathcal{L}^2$. Apply *Key Renewal Thm*, the boundary term vanishes when $t \to \infty$. So

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}[Y(t)] &= \lim_{t\to\infty} \int_0^t h(t-y)dm(y) \\
&= \frac{1}{\mu}\int_0^t h(t)dt \\
&= \frac{1}{\mu}\int_0^t \mathbb{E}[X-t|X>t]\overline{F}(t)dt \\
&= \frac{1}{\mu}\int_0^t \left(\int_t^\infty (x-t)dF(x)\right)dt \\
&= \frac{1}{\mu}\int_0^\infty \left(\int_0^x (x-t)dt\right)dF(x) \\
&= \frac{1}{\mu}\int_0^\infty \frac{x^2}{2}dF(x) = \frac{\mathbb{E}\left[X^2\right]}{2\mu}
\end{aligned} \tag{48}$$

□

Problem 13. (**Elementary Renewal Thm**) Let $m(t) := \mathbb{E}[N(t)]$, $\mu := \mathbb{E}[X_1] < \infty$

$$\frac{m(t)}{t} \xrightarrow{t\to\infty} \frac{1}{\mu}$$

Proof. We have known in the lecture, $N(t)+1$ is a stopping time. And $S_{N(t)+1} \geq t$. Take expectation on both sides, by Wald's Identity:

$$\begin{aligned}
&\mathbb{E}\left[S_{N(t)+1}\right] = \mu\mathbb{E}[N(t)+1] = \mu(m(t)+1) \geq t \\
\Rightarrow &\frac{m(t)+1}{t} \geq \frac{1}{\mu} \quad \text{(Take liminf both sides)}: \\
\Rightarrow &\liminf_{t\to\infty} \frac{m(t)}{t} + \lim_{t\to\infty}\frac{1}{t} \geq \frac{1}{\mu} \\
\Rightarrow &\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu} \quad (*)
\end{aligned} \tag{49}$$

Define a **truncation** of X_n, for any fixed constant $M > 0$:

$$\overline{X}_n := \begin{cases} X_n & X_n \leq M \\ M & X_n > M \end{cases} \tag{50}$$

The truncation $\{\overline{X}_n\}$ forms another renewal process $\{\overline{N}(t)\}$ since they are i.i.d. Also define $\overline{S}_n$ associated with this process. We have $\mu_M := \mathbb{E}\left[\overline{X}_n\right] = \mathbb{E}[X_n; X_n \leq M] + M\mathbb{P}(X_n > M) \xrightarrow{M\to\infty} \mu$. And

$$\begin{aligned}
&S_{N(t)+1} = S_{N(t)} + \overline{X}_{N(t)+1} \leq t + M \\
\Rightarrow &\mu_M(\overline{m}(t)+1) \leq t + M \\
\Rightarrow &\frac{\overline{m}(t)+1}{t+M} \leq \frac{1}{\mu_M} \quad \text{For any fixed } M.
\end{aligned} \tag{51}$$

For every fixed M, we let $t \to \infty$ first, take limsup on both sides:

$$\limsup_{t\to\infty} \frac{\overline{m}(t)}{t} = \limsup_{t\to\infty} \frac{\overline{m}(t)}{t+M} \leq \frac{1}{\mu_M} \tag{52}$$

Since $\overline{X}_n$ is truncated X_n, we have $N(t) \leq \overline{N}(t)$, hence $m(t) \leq \overline{m}(t)$. So

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \limsup_{t\to\infty} \frac{\overline{m}(t)}{t} \leq \frac{1}{\mu_M} \tag{53}$$

LHS is a real number. Now let $M \to \infty$, the limit preserves inequility,

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \lim_{M\to\infty} \frac{1}{\mu_M} = \frac{1}{\mu} \quad (**) \tag{54}$$

$(*)$ and $(**)$ $\Rightarrow \lim\limits_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$, finished the proof. □

Problem 14. For the renewal reward process, show that if $R, X \in \mathcal{L}^1$, then

a) (Random variable)

$$\frac{R(t)}{t} \xrightarrow{a.s \quad t\to\infty} \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

b) (Quantity)

$$\frac{\mathbb{E}[R(t)]}{t} \xrightarrow{t\to\infty} \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$$

Proof. a) We write

$$\frac{R(t)}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \cdot \frac{N(t)}{t} \tag{55}$$

By Proposition in Notes (**SLLN**): $\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mathbb{E}[X]}$. And $\frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \xrightarrow{a.s.} \mathbb{E}[R]$. Hence their product as a whole:

$$\frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \cdot \frac{N(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}[R]}{\mathbb{E}[X]} \tag{56}$$

Finished the proof. □

Proof. b) Since $N(t)+1$ is a stopping time, we add an extra term to $R(t)$ and apply Wald's Identity:

$$\begin{aligned}\mathbb{E}[R(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)+1} R_i - R_{N(t)+1}\right] \\ &= \mathbb{E}[N(t)+1]\,\mathbb{E}[R] - \mathbb{E}\left[R_{N(t)+1}\right] \\ &= (m(t)+1)\mathbb{E}[R] - \mathbb{E}\left[R_{N(t)+1}\right]\end{aligned} \tag{57}$$

Hence

$$\frac{\mathbb{E}[R(t)]}{t} = \frac{(m(t)+1)\mathbb{E}[R]}{t} - \frac{\mathbb{E}\left[R_{N(t)+1}\right]}{t} \tag{58}$$

By elementary renewal theorem, the first part $\frac{(m(t)+1)\mathbb{E}[R]}{t} \to \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$. So now it suffices to show the second part has limit zero when $t \to \infty$.
We proceed by conditioning on $S_{N(t)}$, apply lemma 3.4.3 yields

$$\mathbb{E}\left[R_{N(t)+1}\right] = \mathbb{E}\left[R_{N(t)+1} | S_{N(t)} = 0\right] \overline{F}(t) + \int_0^t \mathbb{E}\left[R_{N(t)+1} | S_{N(t)} = y\right] \overline{F}(t-y) dm(y) \tag{59}$$

Where

$$\begin{aligned}\mathbb{E}\left[R_{N(t)+1} | S_{N(t)} = 0\right] &= \mathbb{E}[R_1 | R_1 > t] \\ \mathbb{E}\left[R_{N(t)+1} | S_{N(t)} = y\right] &= \mathbb{E}[R_n | R_n > t-y]\end{aligned} \tag{60}$$

So, let $h(t) := \mathbb{E}[R | R > t]\,\overline{F}(t)$, it is clear that $h(t) \searrow 0$ with $t \nearrow \infty$. Therefore, for all $\epsilon > 0$, exists T large, such that $h(t) < \epsilon$ whenever $t > T$. Moreover, $h(t) \leq \mathbb{E}[R]$ for all t.

$$\begin{aligned}\frac{\mathbb{E}\left[R_{N(t)+1}\right]}{t} &= \frac{1}{t}\left(\mathbb{E}[R_1 | R_1 > t]\,\overline{F}(t) + \int_0^t \mathbb{E}[R_n | R_n > t-y]\,\overline{F}(t-y) dm(y)\right) \\ &= \frac{h(t)}{t} + \frac{\int_0^t h(t-y) dm(y)}{t} \\ &= \frac{h(t)}{t} + \frac{\int_0^{t-T} h(t-y) dm(y)}{t} + \frac{\int_{t-T}^t h(t-y) dm(y)}{t} \\ &\leq \frac{\epsilon}{t} + \frac{\epsilon \cdot m(t-T)}{t} + \frac{\mathbb{E}[R] \cdot (m(t) - m(t-T))}{t}\end{aligned} \tag{61}$$

The first quantity $\frac{\epsilon}{t} \to 0$, the second one $\frac{\epsilon \cdot m(t-T)}{t} \to \frac{\epsilon}{\mu}$ for any fixed ϵ, T due to elementary renewal thm. The last quantity $\frac{\mathbb{E}[R]\cdot(m(t)-m(t-T))}{t} \to 0$ for any fixed T.
Hence, for any fixed $\epsilon, T(\epsilon)$, let t goes to infinity first

$$\frac{\mathbb{E}\left[R_{N(t)+1}\right]}{t} \xrightarrow{t\to\infty} \frac{\epsilon}{\mu} \tag{62}$$

Then let ϵ goes to zero, we have $\frac{\mathbb{E}\left[R_{N(t)+1}\right]}{t} \to 0$ as desired, finished the proof. ☐

Problem 15. For semi-Markov process, show that the long-run proportion of time that the process spends in state i is

$$P_i = \frac{\pi_i \mu_i}{\sum_{j=1}^{N} \pi_j \mu_j}$$

where π_j is the limiting probability of the embedded Markov chain.

Proof. We consider the *first n transitions* of the semi-Markov process. Define

- $P_i^{[n]}$ be the proportion of time in state i, during the first n transitions.
- $N_i^{[n]}$ # of visits to state i in the first n transitions.
- $Y_i^{[k]}$ be the amount of time stay in state i in the k^{th} visit to i. $\mu_i = \mathbb{E}\left[Y_i^{[k]}\right]$ for any $k \geq 1$.

By those definitons and logic reasoning,

$$P_i^{[n]} = \frac{\sum_{k=1}^{N_i^{[n]}} Y_i^{[k]}}{\sum_j \sum_{k=1}^{N_j^{[n]}} Y_j^{[k]}} \tag{63}$$

Where the numerator is the summation of all time spent in state i, and denominator is the summation of the time spent in all state. Rewrite it as

$$\begin{aligned} P_i^{[n]} &= \frac{\frac{1}{n}\sum_{k=1}^{N_i^{[n]}} Y_i^{[k]}}{\sum_j \frac{1}{n}\sum_{k=1}^{N_j^{[n]}} Y_j^{[k]}} \\ &= \frac{\frac{N_i^{[n]}}{n}\sum_{k=1}^{N_i^{[n]}} \frac{Y_i^{[k]}}{N_i^{[n]}}}{\sum_j \frac{N_j^{[n]}}{n}\sum_{k=1}^{N_j^{[n]}} \frac{Y_j^{[k]}}{N_j^{[n]}}} \end{aligned} \tag{64}$$

Since $N_i^{[n]} \to \infty$ as $n \to \infty$, and $Y_i^{[k]} \in \mathcal{L}^1$. By strong law:

$$\frac{\sum_{k=1}^{N_i^{[n]}} Y_i^{[k]}}{N_i^{[n]}} \xrightarrow{a.s \quad n\to\infty} \mathbb{E}\left[Y_i^{[k]}\right] = \mu_i \tag{65}$$

Moreover, $\frac{N_i^{[n]}}{n}$ is the proportion of visits to i during the first n visits. By definition of stationary probability,

$$\lim_{n\to\infty} \frac{N_i^{[n]}}{n} = \pi_i \tag{66}$$

Hence

$$P_i := \lim_{n\to\infty} P_i^{[n]} = \frac{\pi_i \mu_i}{\sum_j \pi_j \mu_j} \tag{67}$$

☐