Functional Analysis Assignment VI

YANG, Ze (5131209043)

May 15, 2016

Problem 1. Show that a normed linear space X is finite dimensional iff its dual X' is finite dimensional

Proof. Show a stronger one: $\dim X = \dim X'$ for finite dimesional $X(\Rightarrow)$ or $X'(\Leftarrow)$.
$(\Rightarrow)$ Since X is of finite dimension, it has basis $\{x_j\}$. For aribitrary $x \in X$, $x = \sum_1^n a_j x_j$. Define $f_j \in X'$ as

$$f_j(x) = a_j$$

By this definition we also have $f_j(x_j) = \delta_{ij}$. And we obtain another vector $(f_1, ..., f_n)$.
Claim: It is a basis of X'.
Proof of Claim: First, for all $f \in X'$, $f(\cdot) = \left(\sum_{j=1}^n f(x_j) f_j\right)(\cdot)$, in the sense that $\forall x \in X$,

$$f(x) = f\left(\sum_{j=1}^n a_j x_j\right) = \sum_{j=1}^n f(x_j) a_j = \sum_{j=1}^n f(x_j) f_j(x) = \left(\sum_{j=1}^n f(x_j) f_j\right)(x)$$

Implies that $\text{span}\{f_j\} = X'$. Moreover 0 is a linear functional in X' with $0(x) = 0$ for all $x \in X$. So if given

$$\begin{aligned} \sum_{j=1}^n \lambda_j f_j &= 0 \\ \Rightarrow \left(\sum_{j=1}^n \lambda_j f_j\right)(x_j) &= 0(x_j) \\ \Rightarrow \lambda_j &= 0 \text{ for all } 1 \le j \le n \end{aligned} \tag{1}$$

We conclude that $\{f_j\}$ is the basis of X'.
For another direction $(\Leftarrow)$, just define a vector in X as $f(x_j) = (\sum_1^n \lambda_i f_i)(x_j) = \lambda_j$, for $\{f_j\}$ be the basis of X', then show $\{x_j\}$ is basis of X in the same fashion. □

Problem 2. Let $C[0,1]$ be the Banach space of all real-valued continuous functions $f : [0,1] \to \mathbb{R}$, with norm $\|f\| = \max_{x \in [0,1]} |f(x)|$.

- Show $X = \{f \in C[0,1]; f(0) = 0\}$ is a closed subspace of $C[0,1]$, hence a Banach space.
- Show that the map $f \mapsto \ell(f) = \int_0^1 f(x)dx$ is a continuous linear functional on X. Compute the norm

  $$\|\ell\| = \sup_{\|f\| \le 1, f \in X} |\ell(f)|$$

  Is this supremum over closed ball actually as maximum?

Proof. (a) Let $\{f_n\} \subset X$ be a convergent sequence. I.e $\|f_n - f\| \to 0$. Hence $\forall \epsilon > 0$, exists N, such that for $n > N$

$$\begin{aligned} \|f_n - f\| &= \max_{x \in [0,1]} |f_n(x) - f(x)| < \epsilon \\ \Rightarrow |(f_n - f)(0)| &= |f(0)| \le \|f_n - f\| < \epsilon \end{aligned} \tag{2}$$

Since ϵ is arbitrary, we let it goes to 0, and obtain $|f(0)| = 0$. Hence $f \in X \Rightarrow X$ is closed.
Since X is closed, $X \subset C[0,1]$, a Banach space. So X is also a Banach space.

(b) Since f is continuous function on compact set $[0,1]$, it is bounded and attains maximun/minimum. Which implies $|f(x)| \leq \|f\| < C$ for all $x \in [0,1]$. So $\ell(f) = \int_0^1 |f| \leq C$ is also bounded, hence continuous.

$$\|\ell\| = \sup_{\|f\| \leq 1, f \in X} |\ell(f)| = \sup_{\|f\| \leq 1, f \in X} \left| \int_0^1 f(x) \right| \tag{3}$$

The supremum is clearly 1, when $f(x)$ approaches 1 at every $x > 0$. The supremum is not attainable. Because $f(0) = 0$ and f is continuous. That is, $\forall \epsilon > 0$, exists δ, such that $|f(x)| < \epsilon$ whenever $0 \leq x \leq \delta$. Hence

$$\left| \int_0^1 f(x) \right| \leq (1 - \delta) + \delta\epsilon = 1 - \delta(1 - \epsilon) < 1 \tag{4}$$

□

Problem 3. In Banach space $X = L^\infty(\mathbb{R})$, consider the subspace V consisting of all bounded continuous functions.

- Show that there exists a bounded linear functional $\Lambda : L^\infty(\mathbb{R}) \to \mathbb{R}$ with $\|\Lambda\| = 1$ such that $\Lambda f = f(0)$ for every bounded continuous function f. However, show that there exists no function $g \in L^1(\mathbb{R})$ such that $\Lambda f = \int fg dx$ for every $f \in L^\infty(\mathbb{R})$.
- Conclude that the dual space of $L^\infty(\mathbb{R})$ cannot be identified with $L^1(\mathbb{R})$.

Proof. □

Problem 4. Given a sequence $\{x_n\}$ in Hilbert space H, show that the strong convergence $\|x_n - x\| \to 0$ holds if and only if

$$\|x_n\| \to \|x\| \quad \text{and} \quad x_n \rightharpoonup x$$

Proof. $(\Rightarrow)$ is clear, since strong convergence implies weak convergence and the convergence of norm.
$(\Leftarrow)$ Consider

$$\begin{aligned} \|x_n - x\|^2 &= \|x_n\|^2 + \|x\|^2 - 2\langle x_n, x\rangle \\ \Rightarrow \lim_{n\to\infty} \|x_n - x\|^2 &= \lim_{n\to\infty} \|x_n\|^2 + \|x\|^2 - 2 \lim_{n\to\infty} \ell(x_n) \end{aligned} \tag{5}$$

Where we denote $\ell(\cdot) = \langle \cdot, x\rangle$, clearly $\ell \in H'$. By weak convergence: $\lim_{n\to\infty} \ell(x_n) = \ell(x) = \langle x, x\rangle = \|x\|^2$. By another condition $\lim_{n\to\infty} \|x_n\| = \|x\|$. So RHS $= 2\|x\|^2 - 2\|x\|^2 = 0$.
$\Rightarrow x_n \to x$ strongly, finished the proof. □

Problem 5. Consider a bounded sequence of functions $f_n \in L^2[0,T]$. As $n \to \infty$, show that the weak convergence $f_n \rightharpoonup f$ holds iff

$$\lim_{n\to\infty} \int_0^b f_n(x)dx = \int_0^b f(x)dx \quad \text{For every } b \in [0,T] \quad (\dagger)$$

Proof. $(\Rightarrow)$ if $f_n \rightharpoonup f$, since L^2 is hilbert space, there is linear functional $\ell \in (L^2)'$, where

$$\ell(f_n) = \langle \mathbb{1}_{[0,b]}, f_n\rangle = \int_0^b f_n \tag{6}$$

For all $b \in [0,T]$. So due to weak convergence we have $\lim_{n\to\infty} \int_0^b f_n = \langle \mathbb{1}_{[0,b]}, f\rangle = \int_0^b f$.
$(\Leftarrow)$ Since b is arbitrary, $(\dagger)$ actually implies that $\int \mathbb{1}_D f_n \to \int \mathbb{1}_D f$ for any compact $D = [a,b] \subseteq [0,T]$, since $\int \mathbb{1}_{[a,b]} f = \int (\mathbb{1}_{[0,b]} - \mathbb{1}_{[0,a]}) f$, and $\int |\mathbb{1}_{[a,b]} f| \leq C(b-a)$ by boundedness of f.
Then we follow the real-analysis type construction.

- By linearity, $\int \phi f_n \to \int \phi f$, ϕ is simple function.
- By monotone convergence thm, this $\int g^\pm f_n \to \int g^\pm f$, $g^\pm$ are positive.

· For arbitrary $g \in L^2$, let $g = g^+ - g^-$, since g is bounded: $\int gf_n \to \int gf$.

All linear functionals on L^2 have such form, so we finish the proof. □

Problem 6. Suppose Ω is Lebesgue measurable set and $p \in (1, \infty)$. If $f_n \rightharpoonup f$ in $L^p(\Omega)$ and

$$\|f_n\|_{L^p} \to \|f\|_{L^p}$$

Then show that $f_n \to f$ strongly in $L^p(\Omega)$. How about $p = 1$?

Proof. (a) (**Radon-Riesz**) We first state a lemma
lemma. Assume X is a uniformly comvex Banach space, $x_n \rightharpoonup x$ and

$$\limsup_{n\to\infty} \|x_n\| \le \|x\|$$

Then $x_n \to x$ strongly.
Proof of lemma. If $x = 0$ we are done. Assume $x \neq 0$. Define

$$\lambda_n := \max\{\|x_n\|, \|x\|\} \quad y_n := \frac{x_n}{\lambda_n}, \quad y := \frac{x}{\|x\|}$$

So we get $\lambda_n \to \|x\|$ by the limit sup condition. And for linear functional $\ell \in X'$, we have

$$\lim_{n\to\infty} \ell(y_n) = \lim_{n\to\infty} \ell\left(\frac{x_n}{\lambda_n}\right) = \lim_{n\to\infty} \frac{1}{\lambda_n}\ell(x_n) = \frac{\ell(x)}{\|x\|} = \ell\left(\frac{x}{\|x\|}\right) \tag{7}$$

That is, $y_n \rightharpoonup y$. In fact we use $\frac{y_n+y}{2} \rightharpoonup y$ and by theorem

$$\|y\| \le \liminf_{n\to\infty} \left\|\frac{y_n+y}{2}\right\| \le \left\|\frac{y_n+1}{2}\right\| \tag{8}$$

By definition, $\|y_n\| \le 1$ and $\|y\| = 1$. So actually $\lim\limits_{n\to\infty} \left\|\frac{y_n+y}{2}\right\| = 1$. By uniform convexity $\Rightarrow \|y_n - y\| \to 0$, that is $\|x_n - x\| \to 0$, finished the proof. ■
In our previous result (HW4 problem 3), we have already shown that L^p is uniformly convex for $p \ge 2$. And now that $\|f_n\|_{L^p} \to \|f\|_{L^p}$, we have

$$\limsup_{n\to\infty} \|f_n\|_{L^p} = \|f\|_{L^p} \le \|f\|_{L^p}$$

Apply the lemma, we obtain the desired result.
(b) It is not the case for $p = 1$. We let $\Omega = [0, 2\pi]$, $f_n := \sin(nx) + 1$. Then clearly $f_n \rightharpoonup 1$;

$$\|f_n\|_{L^1} = \int_0^{2\pi} |\sin(nx) + 1| = 2\pi$$

but

$$\|f_n - 1\|_{L^1} = \int_0^{2\pi} |\sin(nx)| = 4$$

. □

Problem 7. Exercise 1. Show

$$y_K = \sum_{k=1}^{K} x_{n_k}(t) < 4$$

Exercise 2. If a sequence $\{x_n\} \subset \ell^1$ converges weakly, then it converges strongly.
Exercise 3. If a sequence of points $\{x_n\}$ in normed linear space satisfies

1. $\{x_n\}$ are uniformly bounded, i.e. $|x_n| \le c$.
2. $\lim\limits_{n\to\infty} \ell(x_n) = \ell(x)$ for a set of ℓ dense in X'.

Then $x_n \rightharpoonup x$.

Proof. (Ex.1) Draw a plot of $x_n(t)$, since $n_{k+1} > 2n_k$, for any $t \in [0,1]$, there exists an $M > 0$ such that $\frac{1}{n_M} < t < \frac{2}{n_M}$, hence $t > \frac{1}{n_M} > \frac{2}{n_{M+1}}$. So

$$\begin{aligned}\sum_{k=1}^{K} x_{n_k}(t) &\leq \sum_{k=1}^{\max\{M,K\}} x_{n_k}(t) \\ &= \left(\sum_{k=1}^{M} + \sum_{k=M}^{\max\{M,K\}}\right) x_{n_k}(t) \\ &= 2 - n_M t + \sum_{k=1}^{M-1} n_k t \\ &< 2 - n_M \frac{1}{n_M} + \sum_{k=1}^{M-1} \frac{n_M}{2^{M-1-k}} \frac{2}{n_M} \\ &= 1 + \frac{4}{2^M} \sum_{k=1}^{M-1} 2^k = 5\left(1 - \frac{1}{2^M}\right) < 5\end{aligned} \tag{9}$$

(Well...I didn't work out 4, but the purpose of this is just deducing an upper bound of y_K, so I think 5 is just fine.) ☐

Proof. (Ex.2) Let $\{\boldsymbol{y}^{[n]}\} \subset \ell^1$ be a sequence that converges weakly. WLOG $\boldsymbol{y}^{[n]} \rightharpoonup 0$. We argue by contradiction.
Assume $\boldsymbol{y}^{[n]}$ does not converge to 0 in norm, i.e. $\exists \epsilon > 0$, such that

$$\left\|\boldsymbol{y}^{[n]} - 0\right\| \geq 5\epsilon$$

By previous result we have known $(\ell^1)' = \ell^\infty$.
We consider $\boldsymbol{y}^{[0]} = (y_1^{[0]}, y_2^{[0]}...) \in \ell^1$, there exists n_0 s.t. $\sum_{k \geq n_0+1} |y_k^{[0]}| < \epsilon$; which implies that $\sum_{k=0}^{n_0} |y_k^{[0]}| > 3\epsilon - \epsilon = 4\epsilon$.
Now for this fixed n_0, pick $\boldsymbol{y}^{[1]} = (y_1^{[1]}, y_2^{[1]}...) \in \ell^1$, s.t. $\sum_{k=0}^{n_0} |y_k^{[1]}| < \epsilon$. Moreover, there exists $n_1 > n_0$ such that $\sum_{k \geq n_1+1} |y_k^{[1]}| < \epsilon$. Hence

$$\sum_{k=n_0+1}^{n_1} |y_k^{[1]}| = \left\|\boldsymbol{y}^{[1]}\right\| - \sum_{k=0}^{n_0} |y_k^{[1]}| - \sum_{k \geq n_1} |y_k^{[1]}| \geq 5\epsilon - \epsilon - \epsilon = 3\epsilon$$

We keep doing this and obtain $\{\boldsymbol{y}^{[j]}\}$. Extract n_{j-1} to n_j elements from each $\boldsymbol{y}^{[j]}$, normalize to 1 and concatenate toghther: That is, we take

$$\boldsymbol{x} := \left(0, ..., 0; \frac{y_{n_0+1}^{[1]}}{|y_{n_0+1}^{[1]}|}, ..., \frac{y_{n_1}^{[1]}}{|y_{n_1}^{[1]}|}; \frac{y_{n_1+1}^{[2]}}{|y_{n_1+1}^{[2]}|}, ..., \frac{y_{n_2}^{[2]}}{|y_{n_2}^{[2]}|}; ......\right)$$

$\boldsymbol{x} \in \ell^\infty$ and clearly $\|\boldsymbol{x}\|_\infty = 1$.

$$\begin{aligned}|\langle \boldsymbol{x}, \boldsymbol{y}^{[j]} \rangle| &= \left|\sum_{k \geq 0} x_k y_k^{[j]}\right| \\ &\geq \left|\sum_{k=n_{j-1}+1}^{n_j} x_k y_k^{[j]}\right| - \left|\sum_{k \geq n_j+1} x_k y_k^{[j]}\right| - \left|\sum_{k=0}^{n_{j-1}} x_k y_k^{[j]}\right| \\ &\geq \sum_{k=n_{j-1}+1}^{n_j} |y_k^{[j]}| - \|\boldsymbol{x}\|_\infty \sum_{k \notin \{n_{j-1}+1, ..., n_j\}} |y_k^{[j]}| \\ &\geq 3\epsilon - 1 \cdot (\epsilon + \epsilon) = \epsilon\end{aligned} \tag{10}$$

Define $\ell(\cdot) := \langle \boldsymbol{x}, \cdot \rangle$. It is clear that $\ell(\boldsymbol{y}^{[j]})$ does not converge to 0. But since $\boldsymbol{y}^{[j]} \rightharpoonup \boldsymbol{0}$, we should have $\lim_{j\to\infty} \ell(\boldsymbol{y}^{[j]}) = \ell(\boldsymbol{0}) = 0$, contradiction. □

Proof. (Ex.3) Suppose $\|x_n\| < c$. For any $\epsilon > 0$, for any $f \in X'$, we can choose $\{\phi_j\} \in D$, D is dense in X' and such that for j large

$$\|f_j - f\| \leq \frac{\epsilon}{3c}$$

Due to weak convergece in D, for this ϵ, exists N, for $n > N$ we have $|f_j(x_n) - f_j(x)| < \epsilon/3$ for any $f_j \in D$.

$$\begin{aligned} |f(x_n) - f(x)| &\leq |f(x_n) - f_j(x_n)| + |f_j(x_n) - f_j(x)| + |f_j(x) - f(x)| \\ &\leq |f_j(x_n) - f_j(x)| + 2\,\|f - f_j\| \cdot |x - x_n| \\ &\leq \frac{\epsilon}{3} + 2\frac{\epsilon}{3c} \cdot c = \epsilon \end{aligned} \tag{11}$$

Which implies that $x_n \rightharpoonup x$ in X', finished the proof. □

Problem 8. Deduce thm 10.5 from 10.6 applied to balls centered at origin $K = B_r : \{x : |x| \leq r\}$

Proof. The target is to show that if $x_n \rightharpoonup x$, then

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|$$

Denote $a := \liminf_{n\to\infty} \|x_n\|$. Now given $x_n \rightharpoonup x$, the norm is bounded: $\|x_n\| \leq c$ for some c. Further, we can pick $x_{n_1} \in \{x_n\}$, such that $\|x_{n_1}\| \leq a$. If $\{x_n\} \in B_a(0)$ then we are done, just apply theorm 6 on $B_a(0)$ yield the desired result.
Otherwise, $\exists\, x_{n_2} \in \{x_n\}$, $x_{n_2} \neq x_{n_1}$, we have $\{x_{n_1}, x_{n_2}\} \in B_{a_2}(0)$. Where $a_2 = \max\{a, \|x_{n_2}\|\}$.
...
Continue doing this we obtain a subsequence $\{x_{n_k}\}$, $x_{n_k} \rightharpoonup x$, and $\{x_{n_k}\} \subset B_{a_k}(0)$.
So apply theorem 5 yields $x \in B_{a_k}(0)$, $\Rightarrow$

$$\begin{aligned} \|x\| &\leq a_k \\ \Rightarrow \liminf_{n\to\infty} \|x\| &\leq \liminf_{k\to\infty} \max\{a, \|x_{n_k}\|\} \\ \Rightarrow \|x\| &\leq \max\{a, \liminf_{k\to\infty} \|x_{n_k}\|\} \\ \Rightarrow \|x\| &\leq \max\{a, a\} = a \end{aligned} \tag{12}$$

Finished the proof. □