Eigenvalues and Eigenvectors

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1 Preliminaries

1.1 Subspaces

- $\mathbb{R}$, $\mathbb{C}$ field of real and complex numbers.
- Colspace of $\boldsymbol{A}^{m\times n}$:
$$\mathcal{C}(\boldsymbol{A}) := \text{span}\left\{\text{Cols}(\boldsymbol{A})\right\} \subseteq \mathbb{R}^n$$
And $\text{Dim}(\mathcal{C}(\boldsymbol{A})) = r$. r is rank of $\boldsymbol{A}$.
- Rowspace of $\boldsymbol{A}^{m\times n}$:
$$\mathcal{R}(\boldsymbol{A}) := \text{span}\left\{\text{Rows}(\boldsymbol{A})\right\} = \mathcal{C}(\boldsymbol{A}^\top) \subseteq \mathbb{R}^m$$
And $\text{Dim}(\mathcal{R}(\boldsymbol{A})) = r$.
- Nullspace of $\boldsymbol{A}^{m\times n}$:
$$\mathcal{N}(\boldsymbol{A}) := \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = 0\} \subset \mathbb{R}^n$$
$\text{Dim}(\mathcal{N}(\boldsymbol{A})) = n - r$.
- $\mathcal{N}(\boldsymbol{A}^\top) := \{\boldsymbol{x} : \boldsymbol{A}^\top\boldsymbol{x} = 0\} \subset \mathbb{R}^m$. $\text{Dim}(\mathcal{N}(\boldsymbol{A}^\top)) = m - r$.

2 Properties

2.1 Eigval

- Eigval λ, eigvec $\boldsymbol{v}$ such that: $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.
- $c\boldsymbol{v}$ is also an eigvec corresponding to λ, but only linearly indep. eigvecs are counted.

2.2 Characteristic Polynomial

- $\boldsymbol{A}$ is $n \times n$ square, then the characteristic polynomial of $\boldsymbol{A}$ is defined as.
$$P_{\boldsymbol{A}}(t) := \det(t\boldsymbol{I} - \boldsymbol{A})$$
- λ is eigval of $\boldsymbol{A}$ $\iff$ $P_{\boldsymbol{A}}(\lambda) = 0$ $\iff$ λ is root of $P_{\boldsymbol{A}}(t) = 0$.
Proof. ($\Rightarrow$) λ is eigval of $\boldsymbol{A}$: $(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = 0$.
$\boldsymbol{v}$ is an eigvec of $\boldsymbol{A}$, so $\boldsymbol{v} \neq 0 \Rightarrow \boldsymbol{v} \in \mathcal{N}(\boldsymbol{A})$. Hence $\text{Dim}(\mathcal{N}(\boldsymbol{A})) > 0 \Rightarrow r < n$. $\square$
- $P_{\boldsymbol{A}}(t)$ can be represented as, sence we know the roots,
$$P_{\boldsymbol{A}}(t) = \prod_{j=1}^{n}(t - \lambda_j)$$

- $\boldsymbol{D}$ being diagonal matrix, then $P_{\boldsymbol{D}} = \prod_{j=1}^{n}(t - d_j)$, which implies that $\lambda_j = d_j$, diagonal entries. Moreover, $\boldsymbol{D}\boldsymbol{e}_j = d_j\boldsymbol{e}_j$, so the j-th eigvec is $\boldsymbol{e}_j$ unit vec.
- $\boldsymbol{L}$ being lower triangular, then $P_{\boldsymbol{L}} = \prod_{j=1}^{n}(t - L_{jj})$, which implies that $\lambda_j = L_{jj}$. The last col of lower tri is $L_{nn}\boldsymbol{e}_n$, therefore $\boldsymbol{L}\boldsymbol{e}_n = L_{nn}\boldsymbol{e}_n$, i.e. the last eigvec is $\boldsymbol{e}_n$ unit vector. We can not tell about other eigvecs. Similar for $\boldsymbol{U}$. $\boldsymbol{U}\boldsymbol{e}_1 = U_{11}\boldsymbol{e}_1$, i.e. the first eigvec of $\boldsymbol{U}$ is $\boldsymbol{e}_1$ unit vector.

2.3 Multiplicity

- $\lambda(\boldsymbol{A})$ is the set of all eigvals of $\boldsymbol{A}$, it is acturally the spectrum of $\boldsymbol{A}$.
- If $\lambda \in \lambda(\boldsymbol{A})$ is a root of multiplicity m_λ of $P_{\boldsymbol{A}}(t) = 0$, define m_λ as algebraic multiplicity of eigval λ.
- Square matrix $\boldsymbol{A}$ has exactly n eigvals ($\lambda \in \mathbb{C}$), counted with (algebraic) multiplicity, i.e.

$$\sum_{\lambda \in \lambda(\boldsymbol{A})} m_\lambda = n$$

  This is because, due to fundamental principal of algebra, $P_{\boldsymbol{A}}(t) = 0$ has exactly n roots, counted with multiplicity.

2.4 Eigspace

- Eigenspace (eigspace) of λ: $V_\lambda := \{\boldsymbol{v} : \boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}\}$, i.e. the set of all eigvecs corresponding to λ. $\mathrm{Dim}(V_\lambda)$ is the number of linearly indep. eigvecs corresponding to λ. We have

$$1 \leq \mathrm{Dim}(V_\lambda) \leq m_\lambda$$

  Where $\mathrm{Dim}(V_\lambda)$ is the geometric multiplicity of λ.

Thm. eigvecs corresponding to *different* eigvals of $\boldsymbol{A}$ are linearly indep.

Proof. Let $\boldsymbol{v}_1, ..., \boldsymbol{v}_p$ correspond to different eigvals of $\boldsymbol{A}$: $\lambda_1, ..., \lambda_p$ ($p \leq n$).
It suffices to show

$$c_1\boldsymbol{v}_1 + ... + c_p\boldsymbol{v}_p = \boldsymbol{0} \Rightarrow c_1 = ... = c_p = 0$$

Show by contradiction: suppose otherwise, i.e.

$$(\dagger) : c_1\boldsymbol{v}_1 + ... + c_{p-1}\boldsymbol{v}_{p-1} = \boldsymbol{v}_p$$

Apply $\boldsymbol{A}$ both sides:

$$\boldsymbol{A}(c_1\boldsymbol{v}_1 + ... + c_{p-1}\boldsymbol{v}_{p-1}) = \boldsymbol{A}\boldsymbol{v}_p$$
$$c_1\lambda_1\boldsymbol{v}_1 + ... + c_{p-1}\lambda_{p-1}\boldsymbol{v}_{p-1} = \lambda_p\boldsymbol{v}_p$$

$(\dagger) \times \lambda_p$, substracted from last equation:

$$\sum_{j=1}^{p-1} c_j(\lambda_1 - \lambda_p)\boldsymbol{v}_j = \lambda_p\boldsymbol{v}_p - \lambda_p\boldsymbol{v}_p = 0$$

Which is not possible since $\exists c_j \neq 0$, and $(\lambda_j - \lambda_p) \neq 0 \ \forall j$. Contradiction. $\square$

2.5 Miscellaneous

- $\boldsymbol{A}$ singular $\iff$ $0 \in \lambda(\boldsymbol{A})$.
  Proof. $\det(0\boldsymbol{I} - \boldsymbol{A}) = 0 \iff \boldsymbol{A}$ singular. □
- $\boldsymbol{A}$ invertible. $\lambda \in \lambda(\boldsymbol{A})$, $\boldsymbol{v} \in V_\lambda(\boldsymbol{A})$. Then $\frac{1}{\lambda} \in \lambda(\boldsymbol{A}^{-1})$, $\boldsymbol{v} \in V_{\frac{1}{\lambda}}(\boldsymbol{A}^{-1})$.
  Proof. $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{A}^{-1}\boldsymbol{v} \Rightarrow \frac{1}{\lambda}\boldsymbol{v} = \boldsymbol{A}^{-1}\boldsymbol{v}$.
- $\boldsymbol{A}$ has eigtuple $(\lambda, \boldsymbol{v})$, then $\boldsymbol{A}^k$ with (λ^k, v). $P(\boldsymbol{A})$ is a polynomial of $\boldsymbol{A}$, has eigtuple $(P(\lambda), \boldsymbol{v})$.
- $\boldsymbol{A}$ and $\boldsymbol{A}^\top$ have same eigvals.
  Proof. $P_{\boldsymbol{A}}(t) = \det(t\boldsymbol{I} - \boldsymbol{A}) = \det((t\boldsymbol{I} - \boldsymbol{A})^\top) = \det(t\boldsymbol{I} - \boldsymbol{A}^\top) = P_{\boldsymbol{A}^\top}(t)$. Thus have same roots.
- First, second and the last term of $P_{\boldsymbol{A}}(t)$:

$$P_{\boldsymbol{A}}(t) = t^n - \mathrm{tr}(\boldsymbol{A})t^{n-1} + ... + (-1)^n \det(\boldsymbol{A})$$

  Moreover, $\sum_{j=1}^n \lambda_j = \mathrm{tr}(\boldsymbol{A})$ and $\prod_{j=1}^n \lambda_j = \det(\boldsymbol{A})$.
  Proof. $P_{\boldsymbol{A}}(t) = \det(t\boldsymbol{I} - \boldsymbol{A})$.
  $P_{\boldsymbol{A}}(0) = \det(-\boldsymbol{A}) = (-1)^n \det(\boldsymbol{A})$.
  t^n, t^{n-1} arises with prod term of diagonal entries:

$$\prod_{j=1}^n (t - A_{jj}) = t^n - t^{n-1}\sum_{j=1}^n A_{jj} + ...$$

  which have coefficients 1 and $-\mathrm{tr}(\boldsymbol{A})$. Moreover

$$P_{\boldsymbol{A}}(t) = \prod_{j=1}^n (t - \lambda_j) = t^n - t^{n-1}\sum_{j=1}^n \lambda_j + ... + (-1)^n \prod_{j=1}^n \lambda_j$$

  Finished the proof. □

3 Diagonal Form

- $\boldsymbol{A}$ square, $\boldsymbol{A}$ is diagonalizable iff $\exists$ diagonal $\boldsymbol{\Omega}$, invertible $\boldsymbol{V}$, s.t.

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$$

  Entries of $\boldsymbol{\Lambda}$ are eigvals of $\boldsymbol{A}$, and cols of $\boldsymbol{V}$ are corresponding eigvecs.
  Proof. $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Lambda}$.
  $\boldsymbol{A}\boldsymbol{V} = (\boldsymbol{A}\boldsymbol{v_1}|...|\boldsymbol{A}\boldsymbol{v}_n)$; $\boldsymbol{V}\boldsymbol{\Lambda} = (\boldsymbol{V}\lambda_1\boldsymbol{e}_1|...|\boldsymbol{V}\lambda_n\boldsymbol{e}_n) = (\boldsymbol{v}_1\lambda_1|...|\boldsymbol{v}_n\lambda_n)$. □
  Since $\boldsymbol{V}$ is nonsingular, $\{\boldsymbol{v}_k\}$ must be linearly indep.
- $\boldsymbol{A}$ is diagonalizable iff it has n linearly indep. eigvecs.
- If $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$, then $\boldsymbol{A}^p = \boldsymbol{V}\boldsymbol{\Lambda}^p\boldsymbol{V}^{-1}$.
  Proof. $\boldsymbol{A}^p = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}...\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1} = \boldsymbol{V}\boldsymbol{\Lambda}^p\boldsymbol{V}^{-1}$.□
  Similarly, $\boldsymbol{A}^{-1} = \boldsymbol{V}\boldsymbol{\Lambda}^{-1}\boldsymbol{V}^{-1}$, and $\boldsymbol{A}^{-p} = \boldsymbol{V}\boldsymbol{\Lambda}^{-p}\boldsymbol{V}^{-1}$.

4 Diagonally Dominant Matrices

- R_j defined as sum of absolute value of entries on j-th row except for the main diagonal one A_{jj}.

$$R_j := \sum_{k=1, k\neq j}^n |A_{jk}|$$

- $\boldsymbol{A}$ is a weakly diagonal dominant matrix iff $|A_{jj}| \geq R_j$ for all $j = 1, ..., n$.
- $\boldsymbol{A}$ is a strictly diagonal dominant matrix iff $|A_{jj}| > R_j$ for all $j = 1, ..., n$.

Thm. (**Gershgorin**) $\boldsymbol{A}^{n\times n}$, for any eigval λ, there exists index j, s.t.

$$|\lambda - A_{jj}| \leq R_j$$

Alternative statement:

$$\lambda(\boldsymbol{A}) \subseteq \bigcup_{j=1}^{n} D(A_{jj}, R_j)$$

Where $D(A_{jj}, R_j) := \{z \in \mathbb{C} : |z - A_{jj}| \leq R_j\}$ being disc in complex field, centerred at A_{jj}, with radius R_j. Let $\boldsymbol{v}$ be eigval corresponding to λ, index j is acturally the index of entry in $\boldsymbol{v}$ who has biggest absolute value.

Proof. $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, $\boldsymbol{v} = (v_1, ..., v_n)^\top$. $j = \text{argmax } |v_j|$.
$\langle \boldsymbol{a}_j, \boldsymbol{v}\rangle = \lambda v_j$, angle stands for inner product, $\boldsymbol{a}_j$ is j-th row of $\boldsymbol{A}$. I.e.

$$\begin{aligned}
\lambda v_j &= \sum_{i=1}^{n} A_{ji} v_i = A_{jj} v_j + \sum_{i=1, i\neq j}^{n} A_{ji} v_i \\
\lambda - A_{jj} &= \frac{1}{v_j} \sum_{i=1, i\neq j}^{n} A_{ji} v_i \\
|\lambda - A_{jj}| &\leq \frac{1}{|v_j|} \sum_{i=1, i\neq j}^{n} |A_{ji}||v_i| = \sum_{i=1, i\neq j}^{n} |A_{ji}| \frac{|v_i|}{|v_j|} \\
&\leq \sum_{i=1, i\neq j}^{n} |A_{ji}| = R_j
\end{aligned}$$

Thm. $\boldsymbol{A}$ is strictly diagonally dominant $\Rightarrow$ $\boldsymbol{A}$ is nonsingular.
Proof. By (Gershgorin): if λ is eigval, then exists j: $|\lambda - A_{jj}| \leq R_j$
$\Rightarrow |A_{jj}| - |\lambda| \leq R_j \Rightarrow |\lambda| \geq |A_{jj}| - R_j > 0$.
$\boldsymbol{A}$ is singular iff $\lambda = 0$, but any eigval has positive absolute value. □

- We can also examine sum of absolute values of entries on every column. $\boldsymbol{A}$ is strictly column diagonally dominant iff

$$|A_{jj}| > \sum_{k=1, k\neq j}^{n} |A_{kj}|$$

- $\boldsymbol{A}$ is (strictly) column diag dominant $\iff$ $\boldsymbol{A}^\top$ is (strictly) diag dominant. $\boldsymbol{A}$ being strictly column diag dominant $\Rightarrow$ nonsingular.

5 Eigvals of Tridiagonal Matrix

- Symmetric $N \times N$ tridiagonal matrix:

$$\boldsymbol{B}_N := \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}$$

Has eigval

$$\mu_j = 2 - 2\cos\left(\frac{j\pi}{N+1}\right)$$

Eigvec $\boldsymbol{v}_j$ with i-th entry:

$$\boldsymbol{v}_j(i) = \sin\left(\frac{ij\pi}{N+1}\right)$$

Proof. By showing $\boldsymbol{B}_N\boldsymbol{v}_j = \mu_j\boldsymbol{v}_j$.

· Any tridiagonal matrix

$$\boldsymbol{T} = \begin{pmatrix} d & -a & \cdots & 0 \\ -a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a \\ 0 & \cdots & -a & d \end{pmatrix}$$

Has eigval $\lambda_j = d - 2a + a\mu_j$, and same eigvec as $\boldsymbol{B}_N$.
Proof. $\boldsymbol{T} = (d-2a)\boldsymbol{I} + a\boldsymbol{B}_N$, $\Rightarrow \boldsymbol{T}\boldsymbol{v}_j = (d-2a)\boldsymbol{v}_j + a\boldsymbol{B}_N\boldsymbol{v}_j = (d-2a+\mu_j)\boldsymbol{v}_j$. □

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· $\boldsymbol{v}$ is $n \times 1$, $\boldsymbol{A} = \boldsymbol{v}\boldsymbol{v}^\top$ has rank 1. $\lambda_1 = \boldsymbol{v}^\top\boldsymbol{v}$, $m_{\lambda_1} = 1$. And $\lambda_2 = 0$ with $m_{\lambda_2} = n-1$.
Proof. Let λ be a nonzero eigval of $\boldsymbol{A}$, with eigvec $\boldsymbol{u}$, then by $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$:

$$\boldsymbol{v}(\boldsymbol{v}^\top\boldsymbol{u}) = \lambda\boldsymbol{u} \Rightarrow \boldsymbol{u} = \frac{\boldsymbol{v}^\top\boldsymbol{u}}{\lambda}\boldsymbol{v} = c\boldsymbol{v}$$

Hence $\boldsymbol{u}$ is a scaler multiple of $\boldsymbol{v}$. If $c \neq 0$:

$$\boldsymbol{v}\boldsymbol{v}^\top c\boldsymbol{v} = \lambda c\boldsymbol{v} \Rightarrow c\boldsymbol{v}^\top\boldsymbol{v} = \lambda c$$

So $\lambda_1 = \boldsymbol{v}^\top\boldsymbol{v}$ is the only nonzero case. Otherwise $\boldsymbol{u} = \boldsymbol{0}$, $\lambda_2 = 0$. □

· If $\boldsymbol{A}$ is idempotent, i.e. $\boldsymbol{A}^2 = \boldsymbol{A} \Rightarrow \lambda = 0$ or 1.

· If $\boldsymbol{A}$ is nilpotent, i.e. $\exists p$, s.t. $\boldsymbol{A}^p = \boldsymbol{O} \Rightarrow \lambda = 0$

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