Stochastic Process Assignment VI

Zed

May 17, 2016

Problem 1.

Solution. · Let X be # men in system. Note that N_a is the number found by **Next** arrival, not the one applied in PASTA.

· Clearly, if $X < n$, the next arrival won't see more than n men. In another word, N_a is at most X.

· For any $n, j > 0$, $\{N_a = n | X = n + j\}$ is equivalent to j services completed before next arrival, and that arrival before the $j+1$st service completion. So

$$\mathbb{P}(N_a = n | X = n + j) = \left(\frac{\mu}{\mu+\lambda}\right)^j \left(\frac{\lambda}{\mu+\lambda}\right)$$

$$\begin{aligned}
\mathbb{P}(N_a = n) &= \sum_{k\geq 0} \mathbb{P}(N_a = n | X = k)\, \mathbb{P}(X = k) \\
&= \sum_{k\geq n} \mathbb{P}(N_a = n | X = k) \left(\frac{\lambda}{\mu}\right)^k \left(1 - \frac{\lambda}{\mu}\right) \\
&= \sum_{j\geq 0} \mathbb{P}(N_a = n | X = n + j) \left(\frac{\lambda}{\mu}\right)^{n+j} \left(1 - \frac{\lambda}{\mu}\right) \\
&= \sum_{j\geq 0} \left(\frac{\mu}{\mu+\lambda}\right)^j \left(\frac{\lambda}{\mu+\lambda}\right) \left(\frac{\lambda}{\mu}\right)^{n+j} \left(1 - \frac{\lambda}{\mu}\right) \\
&= \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \frac{\lambda}{\mu+\lambda} \sum_{j\geq 0} \left(\frac{\lambda}{\mu+\lambda}\right)^i \\
&= \left(\frac{\lambda}{\mu}\right)^{n+1} \left(1 - \frac{\lambda}{\mu}\right)
\end{aligned} \tag{1}$$

Therefore we see $\mathbb{P}(N_a = n) = \frac{\lambda}{\mu} P_n$. Hence

$$\mathbb{E}[N_a] = \sum_{n\geq 0} n\mathbb{P}(N_a = n) = \frac{\lambda}{\mu} \sum_{n\geq 0} n P_n = \frac{\lambda}{\mu} L \tag{2}$$

Problem 2.

Proof. (1)

· It suffices to show that $N - 1$ is a Poisson, where N is the # men in the queue perceived by an arrival.

· Denote W^* the waiting time of this customer **in the queue**, then W^* contains N service times, which implies that $W^* | N = n$ is distributed as $\Gamma(n, \mu)$. (Note that W^* is different from W).

Then

$$\begin{aligned}
\mathbb{P}(N = n | W^* = t) &= \frac{\mathbb{P}(N = n)\, \mathbb{P}(W^* = t | N = n)}{\mathbb{P}(W^* = t)} \\
&= \frac{1}{f_{W^*}(t)} \left(\frac{\lambda}{\mu}\right)^n (1 - \frac{\lambda}{\mu}) \mu e^{-\mu t} \frac{(\mu t)^{n-1}}{(n-1)!} \\
&= K \frac{(\lambda t)^{n-1}}{(n-1)!}
\end{aligned} \tag{3}$$

By same method in which we solve for $N|W$, using

$$1 = \sum_{n\geq 1} \mathbb{P}\left(N = n | W^* = t\right) = K \sum_{n\geq 1} \frac{(\lambda t)^{n-1}}{(n-1)!} \tag{4}$$

gives $K = e^{-\lambda t}$. Hence $\mathbb{P}\left(N = n+1 | W^* = t\right) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \Rightarrow N - 1 \sim \text{Pois}(\lambda t)$.

(2) A byproduct of the proof above is the value of K:

$$\begin{aligned} e^{-\lambda t} &= K = \frac{1}{f_{W^*}(t)}(1 - \frac{\lambda}{\mu}) \cdot \frac{\lambda}{\mu} \\ \Rightarrow f_{W^*} &= e^{-\lambda t}(1 - \frac{\lambda}{\mu}) \cdot \frac{\lambda}{\mu} \end{aligned} \tag{5}$$

It is clear that $\mathbb{P}\left(W^* = 0\right) = \mathbb{P}\left(N = 0\right) = 1 - \frac{\lambda}{\mu}$. So for $x > 0$, we have

$$\mathbb{P}\left(W^* \leq t\right) = \mathbb{P}\left(W^* = 0\right) + \int_0^t f_{W^*}(t)dt = 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}(1 - e^{-(\mu-\lambda)x}) \tag{6}$$

(3) Conditional on $N \geq 1$, $W*$ is at least one service time. Hence $W^*|N > 1$ is identically distributed as W, which (shown in lecture) is $\text{Exp}(\mu - \lambda)$. And $W^*|N = 0$ is clearly zero. Hence

$$\begin{aligned} \mathbb{E}\left[W^*\right] &= \mathbb{E}\left[W^*|N = 0\right]\mathbb{P}\left(N = 0\right) + \mathbb{E}\left[W^*|N \geq 1\right]\mathbb{P}\left(N \geq 1\right) \\ &= 0 + \frac{1}{\mu - \lambda}\frac{\lambda}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} \end{aligned} \tag{7}$$

And $\mathbb{V}\text{ar}\left[W^*|N \geq 1\right] = \frac{1}{(\mu-\lambda)^2}$. Second moment is $\mathbb{E}\left[(W^*)^2|N \geq 1\right] = \frac{2}{(\mu-\lambda)^2} \Rightarrow \mathbb{E}\left[(W^*)^2\right] = \frac{2}{(\mu-\lambda)^2} \cdot \frac{\lambda}{\mu}$. So

$$\begin{aligned} \mathbb{V}\text{ar}\left[W^*\right] &= \mathbb{E}\left[(W^*)^2\right] - \mathbb{E}\left[W^*\right]^2 \\ &= \frac{2\lambda}{\mu(\mu-\lambda)^2} - \frac{\lambda^2}{\mu^2(\lambda-\mu)^2} \\ &= \frac{\lambda}{\mu(\mu-\lambda)^2} + \frac{\lambda}{\mu^2(\mu-\lambda)} \end{aligned} \tag{8}$$

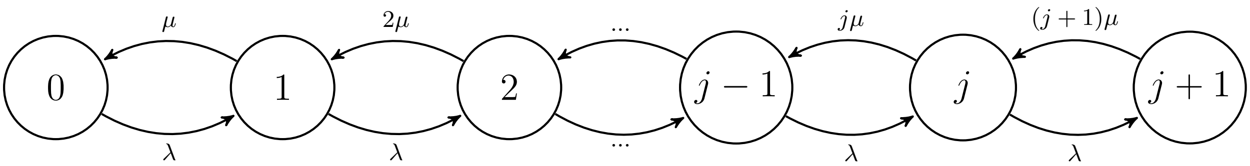
□

Problem 3.

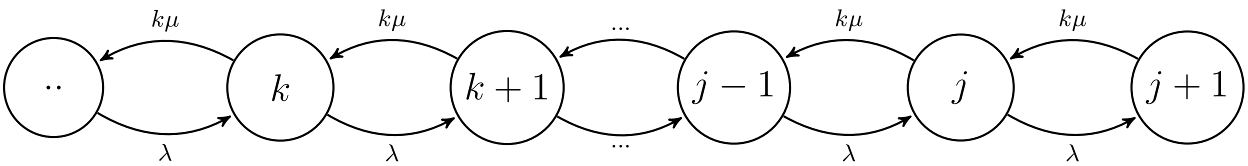
Solution.

Problem 4.

Solution. For $j < k$ case:



And when $j \geq k$:



The balance equation is given by

$$\begin{cases} (j\mu+\lambda)P_j = \lambda P_{j-1} + (j+1)\mu P_{j+1} & 0<j<k \\ (k\mu+\lambda)P_j = \lambda P_{j-1} + k\mu P_{j+1} & j \geq k \\ \lambda P_0 = \mu P_1 \end{cases} \tag{9}$$

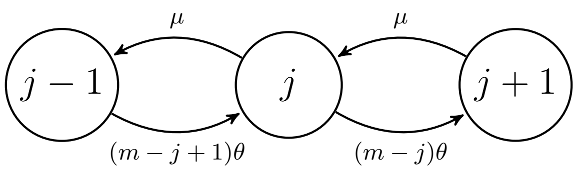
Dente $\rho = \frac{\lambda}{\mu}$. Solve with *Mathematica*, $\Rightarrow$

$$\begin{cases} P_0 = \left(\sum_{i=1}^{k-1} \frac{(k\rho)^i}{i!} + \frac{(k\rho)^k}{k!}\frac{1}{1-\rho}\right)^{-1} \\ P_j = P_0 \cdot \frac{(k\rho)^j}{j!}, \quad 0<j<k \\ P_j = P_0 \cdot \frac{k^k\rho^j}{k!}, \quad j \geq k \end{cases} \tag{10}$$

Problem 5.

Solution.

- Denote $P_j = \mathbb{P}(\{\text{j men in system.}\})$, $j \leq m$. Then there should be $m-j$ men finished before this time, and is about to rejoin the system.
- For these $m-j$ men, time to next rejoining is the minimum of $m-j$ $\text{Exp}(\theta)$s, hence the transition rate is $(m-j)\theta$. This will make system goes $j \to j+1$. Similar analysis for $j-1 \to j$ transition, this rate is $(m-j+1)\theta$
- The boundary condition is at node P_0 and P_m, which can be obtained by taking $j=1$ or $j=m-1$ in the graph below.



Hence we have BVP

$$\begin{cases} (m-j+1)\theta P_{j-1} + \mu P_{j+1} = \mu P_j + (m-j)\theta P_j & 0<j<m \\ m\theta P_0 = \mu P_1 \\ \mu P_m = \theta P_{m-1} \end{cases} \tag{11}$$

We postulate that P_j is proportional to a weight function w_j, i.e. $P_j = \frac{w_j}{Z}$, Z is normalizing factor. Solve via *Mathematica*:

$$w_j = \frac{m!}{(m-j)!}\mu^{m-j}\theta^j; \quad P_j = \frac{w_j}{\sum_{i=0}^m w_i} \tag{12}$$

Check:

$$\begin{aligned} Z \times LHS &= (m-j+1)\theta w_{j-1} + \mu w_{j+1} = \frac{m!(m-j+1)}{(m-j+1)!}\mu^{m-j+1}\theta^j + \frac{m!}{(m-j-1)!}\mu^{m-j}\theta^{j+1} \\ Z \times RHS &= \mu w_j + (m-j)\theta w_j = \frac{m!}{(m-j)!}\mu^{m-j+1}\theta^j + \frac{m!(m-j)}{(m-j)!}\mu^{m-j}\theta^{j+1} \end{aligned} \tag{13}$$

Check. Boundary also checks. Since stationary distribution is unique, this is exactly the solution.

(a)

$$\lambda_a = \sum_{j=0}^m \lambda_j P_j = \sum_{j=0}^m (m-j)\theta P_j \tag{14}$$

(b)

$$W = \frac{L}{\lambda_a} = \frac{\sum_{j=0}^m jP_j}{\sum_{j=0}^m (m-j)\theta P_j} \tag{15}$$

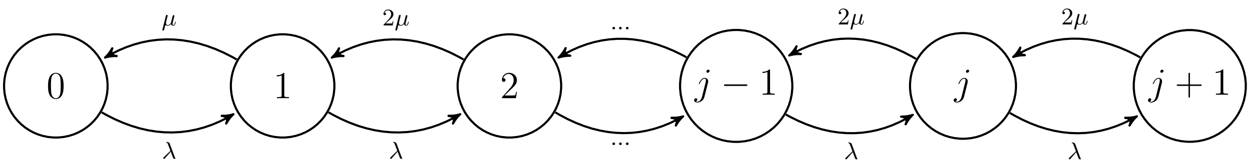
Where

$$P_j = \frac{\frac{m!}{(m-j)!}\mu^{m-j}\theta^j}{\sum_{i=1}^m \frac{m!}{(m-i)!}\mu^{m-i}\theta^i}$$

Problem 6.

Solution.

- The arrival rate is always λ, departure is 2μ at every $j \geq 2$. Graph is as below.



$$\begin{cases} \lambda P_{j-1} + 2\mu P_{j+1} = (2\mu + \lambda) P_j \quad j \geq 2 \\ \lambda P_0 = \mu P_1 \\ \lambda P_0 + 2\mu P_2 = (\mu + \lambda) P_1 \end{cases} \tag{16}$$

(a) It turns out that

$$P_0 = \frac{2\mu - \lambda}{2\mu + \lambda}, \quad P_j = \frac{\lambda^j}{2^{j-1}\mu^j} P_0 \text{ for } j \geq 1 \tag{17}$$

(b)

$$\begin{aligned} q(0,1) &= \lambda P_0 = \frac{\lambda(2\mu - \lambda)}{2\mu + \lambda} \\ q(2,1) &= 2\mu P_2 = \frac{\lambda^2}{\mu} \cdot \frac{2\mu - \lambda}{2\mu + \lambda} \end{aligned} \tag{18}$$

(c) It is clear that when $j \geq 2$ the stock clerk will be checking all the time. When $j = 1$, the stock clerk can be also checking. This happens when the other clerk finishes before arrival at $j = 2$, that is $S \leq T$ (S: service time, T: interarrival time). Hence

$$\mathbb{P}\left(\{\text{stock clerk is checking, } j = 1\}\right) = \mathbb{P}\left(S \leq T\right) P_2 = \frac{\lambda^2(2\mu - \lambda)}{2\mu(\mu + \lambda)(2\mu + \lambda)} \tag{19}$$

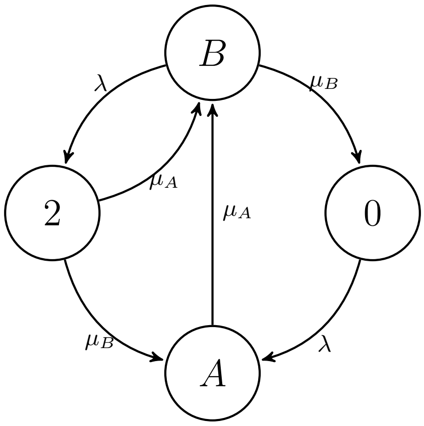
Total proportion of time that he is checking is:

$$\begin{aligned} p &= \mathbb{P}\left(\{\text{checking, } j = 1\}\right) + \sum_{j \geq 2} P_j \\ &= \frac{\lambda^2(2\mu - \lambda)}{2\mu(\mu + \lambda)(2\mu + \lambda)} + \frac{2\lambda^2}{\mu(2\mu + \lambda)} \end{aligned} \tag{20}$$

Problem 7.

Solution.

- There are only 4 states: system is full, only A or B is occupied, and 2 servers are both free. Denote these by $\{0, A, B, 2\}$.
- New arrivals make the transition $B \to 2$ and $0 \to A$ at state B or 0 when A is free, with rate $\lambda = 2$.
- Transition $B \to 0$ is made if B finishes early then next arrival, w.p. $\frac{\lambda}{\mu_2 + \lambda}$
- Transition $2 \to A$ and $B \to 0$ is made with completion at B, rate is $\mu_B = 2$. $2 \to B$, and $A \to B$ is made with completion at A, rate is $\mu_A = 4$.



$$\begin{cases} \lambda P_0 = \mu_B P_B \\ \mu_A P_A = \lambda P_0 + \mu_B P_2 \\ (\mu_B + \lambda) P_B = \mu_A P_A + \mu_A P_2 \\ (\mu_B + \mu_A) P_2 = \lambda P_B \end{cases} \tag{21}$$

(1) With $\lambda = 2$, $\mu_A = 4$, $\mu_B = 2$. It turns out that

$$P_0 = \frac{1}{3}, \quad P_A = \frac{2}{9}, \quad P_B = \frac{1}{3}, \quad P_2 = \frac{1}{9} \tag{22}$$

People can only enter at 0 or B. So define the events

$$\begin{aligned} &E_0 := \{\text{enters the system at state [0]}\} \\ &E := \{\text{enters the system.}\} = E_0 \cup E_B \\ &S_B := \{\text{receives service at B.}\} \end{aligned} \tag{23}$$

$$\mathbb{P}(E) = P_B + P_0 = \frac{2}{3} \tag{24}$$

(b) When enter at state 0, there is nobody ahead of him that potentially prevent him from entering B. But if entering at state B,

$$\mathbb{P}(S_B|E_B) = \mathbb{P}(S_B \le S_A) = \frac{\mu_B}{\mu_A + \mu_B} = \frac{1}{3}$$

So

$$\begin{aligned} \mathbb{P}(S_B|E) &= \mathbb{P}(S_B|E_B)\,\mathbb{P}(E_B|E) + \mathbb{P}(S_B|E_0)\,\mathbb{P}(E_0|E) \\ &= \frac{\mu_B}{\mu_A + \mu_B} \times \frac{P_B}{P_0 + P_B} + 1 \times \frac{P_0}{P_0 + P_B} = \frac{2}{3} \end{aligned} \tag{25}$$

(c)

$$\mathbb{E}[N] = 0 \cdot P_0 + 1 \cdot (P_A + P_B) + 2P_2 = \frac{7}{9}$$

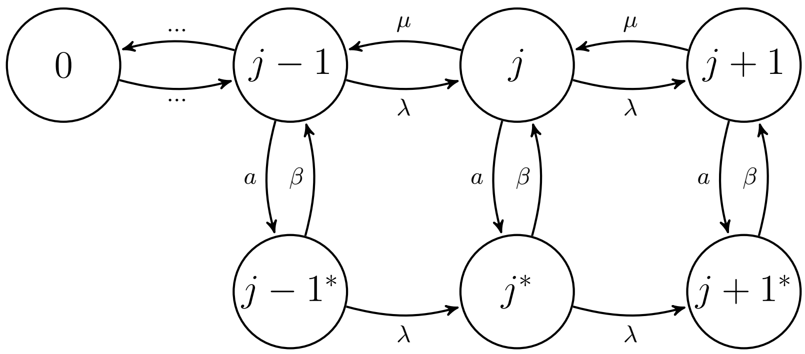
(d)

$$\begin{aligned} \mathbb{E}[W|E] &= \mathbb{E}[W|E_B]\,\mathbb{P}(E_B|E) + \mathbb{E}[W|E_0]\,\mathbb{P}(E_0|E) \\ &= (\mathbb{E}[S_A] + \mathbb{E}[S_B]\,\mathbb{P}(S_B|E_B)) \cdot \mathbb{P}(E_B|E) + \mathbb{E}[S_A + S_B] \cdot \mathbb{P}(E_0|E) \\ &= \left(\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3}\right) \cdot \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{2}\right) \cdot \frac{1}{2} \\ &= \frac{7}{12} \end{aligned} \tag{26}$$

Problem 8.

Solution.

- Define the states as $\{0, 1, ...\}$ and $\{0^*, 1^*, ...\}$. j represents there are j men in the system, and j^* means j men and breaking down.
- For any state* that is not 0^*, the rate of fixing up is β. And for any state that is not 0, the rate of breaking down is a.



So the balance equation is given by

$$\begin{cases} (\mu+\lambda+a)P_j = \lambda P_{j-1} + \mu P_{j+1} + \beta P_{j^*} & j \geq 1 \\ (\lambda+\beta)P_{j^*} = \lambda P_{(j-1)^*} + aP_j & j^* \geq 2 \\ \lambda P_0 = \mu P_1 \\ (\lambda+\beta)P_{1^*} = aP_1 \end{cases} \tag{27}$$

So $\lambda_a = \lambda$.

$$W = \frac{L}{\lambda_a} = \frac{\sum_{j\geq 0} j(P_j + P_{j^*})}{\lambda} \tag{28}$$

Problem 9.

Solution. · If we regard time waiting in breakdown period as part of the "service", then the problem becomes a M/G/1.

- Apply **Pollaczek-Khintchine**, we have

$$W = \mathbb{E}[S] + \frac{\lambda\mathbb{E}[S^2]}{2(1-\lambda\mathbb{E}[S])} \quad (\dagger)$$

Then it suffices to calculate $\mathbb{E}[S]$ and $\mathbb{E}[S^2]$.

Define the followings

- S be the new "service" time. $\tilde{S}$ be the initial service time, $\tilde{S} \sim \text{Exp}(\mu)$. A be the interarrivals of breakings, $A \sim \text{Exp}(a)$. B be the time to fix up, $B \sim \text{Exp}(\beta)$.
- N the # of breakdowns before finished. When a customer is serviced, $\mathbb{P}(\text{break down dose not occur}) = \mathbb{P}\left(\tilde{S} < A\right) = \frac{\mu}{\mu+a}$. We regard this as success probability, $N+1$ is therefore # of trials until a success arises. Hence $N+1 \sim$Geometric$(\frac{\mu}{\mu+a})$. We have

$$\mathbb{E}[N] = \frac{\mu+a}{\mu}, \quad \mathbb{V}\text{ar}[N] = \frac{a(\mu+a)}{\mu}$$

- $\{T_i\}$ be the sequence of time in service before another breakdown arises or the success. Then $T_i = \min\{\tilde{S}, A\}$. $T_i \sim \text{Exp}(\mu+a)$

The total "service" time is therefore

$$\begin{aligned} S &= \sum_{i=1}^{N}(T_i + B_i) + T_{N+1} \\ \mathbb{E}[S|N] &= \frac{N+1}{a+\mu} + \frac{N}{\beta} \\ \mathbb{V}\text{ar}[S|N] &= \frac{N+1}{(a+\mu)^2} + \frac{N}{\beta^2} \end{aligned} \tag{29}$$

Wald Identity $\Rightarrow$

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S|N]] = \frac{1}{\mu} + \frac{\mu+a}{\mu\beta} - \frac{1}{\beta} = \frac{1}{\mu} + \frac{a}{\mu\beta} \tag{30}$$

And

$$\mathbb{V}\text{ar}[S] = \left(\frac{1}{a+\mu} + \frac{1}{\beta}\right)^2 \frac{a(a+\mu)}{\mu^2} + \frac{1}{\mu(\mu+a)} + \frac{a}{\mu\beta^2} \tag{31}$$

With these and (†) we can calculate W, which is too lengthy to write.

Problem 10.

Solution. We have

- External arrivals: $r_1 = 5$, $r_2 = 10$, $r_3 = 15$.
- Service time $\mu_1 = 10$, $\mu_2 = 50$, $\mu_3 = 100$.

$$\boldsymbol{P} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{32}$$

And

$$\lambda_j = r_j + \sum_{i=1}^{3} \lambda_i P_{ij} \tag{33}$$

(a) We solve out λ_j, and $\rho_j = \frac{\lambda_j}{\mu_j}$, $L_j = \frac{\rho_j}{1-\rho_j} = \frac{\lambda_j}{\mu_j - \lambda_j}$

$$\begin{cases} \lambda_1 = 5 \\ \lambda_2 = 40 \\ \lambda_3 = \frac{170}{3} \end{cases} \quad \Rightarrow L = \sum_{j=1}^{3} \frac{\lambda_j}{\mu_j - \lambda_j} = 1 + 4 + \frac{17}{13} \approx 6.3 \tag{34}$$

(b)

$$W = \frac{L}{\lambda_a} = \frac{L}{r_1 + r_2 + r_3} \approx 0.21 \tag{35}$$

Problem 11.

Solution. Denote S the service time and V the value of customer. The system is a M/G/1 queue. $\mathbb{E}[V] = \frac{1}{2}$, $\mathbb{V}\text{ar}[V] = \frac{1}{12}$, $\mathbb{E}\left[V^2\right] = \frac{1}{3}$.

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[\mathbb{E}[S|V]] = \mathbb{E}[3+4V] = 5 \\ \mathbb{E}\left[S^2|V\right] &= \mathbb{E}^2[S|V] + \mathbb{V}\text{ar}[S|V] = 5 + (3+4V)^2 \\ \mathbb{E}\left[S^2\right] &= \mathbb{E}\left[\mathbb{E}\left[S^2|V\right]\right] = 5 + 9 + 24 \cdot \frac{1}{2} + \frac{16}{3} = \frac{94}{3} \end{aligned} \tag{36}$$

(a) Apply **Pollaczek-Khintchine**, we have

$$\begin{aligned} W &= \mathbb{E}[S] + \frac{\lambda \mathbb{E}\left[S^2\right]}{2(1-\lambda\mathbb{E}[S])} \\ &= \frac{94\lambda/3}{2(1-5\lambda)} + 5 \end{aligned} \tag{37}$$

(b)

$$W|\{V=x\} = \mathbb{E}[S|x] + W_Q = \frac{94\lambda/3}{2(1-5\lambda)} + 3 + 4x \tag{38}$$

Problem 12.

Solution. (a) Denote I the idle period, clearly $\mathbb{E}[I] = 1/\lambda$. P_0 be long run proportion of time there is 0 man in system. By results in the lecture, $1 - P_0 = \lambda\mathbb{E}[S]$ and

$$\begin{aligned} P_0 &= \frac{\mathbb{E}[I]}{\mathbb{E}[I] + \mathbb{E}[B]} = \frac{1/\lambda}{1/\lambda + \mathbb{E}[B]} \\ \Rightarrow\ \mathbb{E}[B] &= \frac{1-P_0}{\lambda P_0} = \frac{\mathbb{E}[S]}{1-\lambda\mathbb{E}[S]} \end{aligned} \tag{39}$$

By PASTA: $a_0 := \mathbb{P}(\{\text{Arrival observes 0 man in the queue.}\}) = P_0$. And hence

$$\mathbb{E}[S] = a_0 \int_0^\infty \bar{G}_1(t)dt + (1-a_0)\int_0^\infty \bar{G}_2(t)dt \tag{40}$$

Substitute a_0 with $P_0 = 1 - \lambda\mathbb{E}[S]$, yields

$$\mathbb{E}[S] = \frac{\int_0^\infty \bar{G}_1(t)dt}{1 + \lambda\int_0^\infty \bar{G}_1(t)dt + \lambda\int_0^\infty \bar{G}_2(t)dt} \tag{41}$$

Therefore

$$\mathbb{E}[B] = \frac{\int_0^\infty \bar{G}_1(t)dt}{1 - \lambda\int_0^\infty \bar{G}_2(t)dt} \tag{42}$$

(b) We have $a_0 = P_0 = 1/\mathbb{E}[C] \Rightarrow$

$$\mathbb{E}[C] = \frac{1}{1-\lambda\mathbb{E}[S]} = \frac{1 + \int_0^\infty \bar{G}_1(t)dt - \int_0^\infty \bar{G}_2(t)dt}{1 - \int_0^\infty \bar{G}_2(t)dt} \tag{43}$$

Problem 13.

Solution. Define the followings

- T be the actual time in service, which has distribution G.
- N be the number of breakdowns. Since the service is continued after fixing ups, the time spent to fix server has noting to do with arrivals of breakdowns. Since the interarrival time of berakdowns is Exp(α), the number of breakdowns occurred up to time t is a Poisson variable, i.e. $N \sim$ Pois(αt).
- R be the time spent fixing the server, has distribution H.
- S be the total "Service" time including $\{R_i\}$s, we have

$$S = \sum_{i=1}^N R_i + T$$

(a) Apply *Pollaczek-Khintchine*, we have

$$W = \mathbb{E}[S] + \frac{\lambda\mathbb{E}[S^2]}{2(1-\lambda\mathbb{E}[S])} \quad (\dagger)$$

Now it suffices to calculate $\mathbb{E}[S^2]$ and $\mathbb{E}[S]$.

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[\mathbb{E}[S|T]] \\ &= \mathbb{E}[\alpha T\mathbb{E}[R] + \mathbb{E}[T]] \\ &= \mathbb{E}[T](\alpha\mathbb{E}[R] + 1) \end{aligned} \tag{44}$$

$$\begin{aligned} \mathbb{E}\left[\mathbb{E}\left[S^2|T\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[T^2 + 2T\sum_{i=1}^N R_i + \left(\sum_{i=i}^N R\right)^2 \middle| T\right]\right] \\ &= \mathbb{E}\left[T^2 + 2T\cdot\alpha T\mathbb{E}[R] + \alpha T\mathbb{E}\left[R^2\right] + \alpha^2T^2\mathbb{E}^2[R]\right] \\ &= \alpha\mathbb{E}[T]\mathbb{E}\left[R^2\right] + \mathbb{E}\left[T^2\right]\cdot 2\alpha\mathbb{E}[R] + \mathbb{E}\left[T^2\right] + \mathbb{E}\left[T^2\right]\cdot\alpha^2\mathbb{E}^2[R] \\ &= \alpha\mathbb{E}[T]\mathbb{E}\left[R^2\right] + \mathbb{E}\left[T^2\right](1+\alpha\mathbb{E}[R])^2 \end{aligned} \tag{45}$$

Therefore

$$W_Q = \frac{\alpha \mathbb{E}\left[T\right] \mathbb{E}\left[R^2\right] + \mathbb{E}\left[T^2\right] \left(1 + \alpha \mathbb{E}\left[R\right]\right)^2}{2\left(1 - \lambda \mathbb{E}\left[T\right]\left(\alpha \mathbb{E}\left[R\right] + 1\right)\right)} \quad (\dagger) \tag{46}$$

Where T has known distribution G and R has known distribution H.
(b)

$$L_Q = \lambda W_Q = \lambda \cdot (\dagger) \tag{47}$$

(c)

$$W = W_Q + \mathbb{E}\left[S\right] = (\dagger) + \mathbb{E}\left[T\right]\left(\alpha \mathbb{E}\left[R\right] + 1\right) \tag{48}$$

(d)

$$L = \lambda W = \lambda\left((\dagger) + \mathbb{E}\left[T\right]\left(\alpha \mathbb{E}\left[R\right] + 1\right)\right) \tag{49}$$

Problem 14.

Solution. (a)

- Assume service time S has distribution G, WLOG assume G has density $g(\cdot)$.
- X_n be # men in the queue at the departure of nth man. Y_n be the # of arrivals during S_n. Conditioning on S_n, $Y_n|\{S_n = t\}$ is Poisson with mean λt. Hence we can compute

$$\begin{aligned} \mathbb{P}\left(Y_n = j\right) &= \int_0^\infty \mathbb{P}\left(Y_n = j | S_n = t\right) g(t) dt \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} g(t) dt \quad (\dagger) \end{aligned} \tag{50}$$

If $X_n > 0$, Y_{n+1} arrives in the $(n+1)$st service time, and when that guy departs, there is $X_{n+1} = X_n + Y_{n+1} - 1$ (not counting himself) men in the system.
Otherwise if $X_n = 0$, i.e. after n-th man departs, there is nobody in the queue. Then, 1 man arrives and S_{n+1} *starts* on his arrrival, then Y_{n+1} man arrives in S_{n+1}. Hence when he leaves, there are Y_{n+1} men in the queue. We conclude that

$$X_{n+1} = \begin{cases} X_n + Y_{n+1} - 1 & \text{if } X_n > 0 \\ Y_{n+1} & \text{if } X_n = 0 \end{cases} \tag{51}$$

It is clear that $\{X_n\}$ is a markov chain, since it is determined by X_n and Y_{n+1}, thus independent to any history before n-th departure.
(b) By the relation above and $(\dagger)$

$$\begin{aligned} P_{ij} &= \mathbb{P}\left(X_{n+1} = j | X_n = i\right) \\ &= \begin{cases} \mathbb{P}\left(i + Y_{n+1} - 1 = j\right) & \text{if } i > 0 \\ \mathbb{P}\left(Y_{n+1} = j\right) & \text{if } i = 0 \end{cases} \\ &= \begin{cases} \mathbb{P}\left(Y_{n+1} = j - i + 1\right) & \text{if } i > 0 \\ \mathbb{P}\left(Y_{n+1} = j\right) & \text{if } i = 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } i > 0 \text{ and } j - i + 1 < 0 \\ \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j-i+1}}{(j-i+1)!} g(t) dt & \text{if } i > 0 \text{ and } j - i + 1 \geq 0 \\ \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} g(t) dt & \text{if } i = 0 \end{cases} \end{aligned} \tag{52}$$

We denote $p_j = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} g(t) dt$, then transition matrix is

$$\boldsymbol{P} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ p_0 & p_1 & p_2 & p_3 & \cdots \\ 0 & p_0 & p_1 & p_2 & \cdots \\ 0 & 0 & p_0 & p_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{53}$$

Problem 15.

Solution.

- X_n now is # men in the system perceived by n-th arrival. Further define Y_n is a counting of expoential events with rate μ durting the n-th interarrival time T_n, i.e. the **potential** # of completion if there is sufficient customers in the queue.
- The queue is G/M/1, assume T_n has known distribution G, and assume that G has density $g(\cdot)$.
- Since the completion of service is Exp(μ), by same argument as Problem 14, conditioning on T_n, $Y_n\{T_n = t\}$ is Poisson variable with mean μt. So

$$\begin{aligned}\mathbb{P}\left(Y_n = j\right) &= \int_0^\infty \mathbb{P}\left(Y_n = j | T_n = t\right) g(t)dt \\ &= \int_0^\infty e^{-\mu t} \frac{(\mu t)^j}{j!} g(t)dt \quad (\dagger)\end{aligned} \tag{54}$$

  Denote this as q_j.

$X_n + 1$ is # of men in system after the observer joins the queue, Y_{n+1} is the # of completion according to rate μ, which is **not the actural** completion. When $X_n + 1 - Y_{n+1} < 0$, implies the customer in system does not meet the server's capacity. Hence $n+1$-st arrival see nobody remains. $X_{n+1} = 0$. Otherwise, $X_{n+1} = X_n + 1 - Y_{n+1}$. Therefore

$$X_{n+1} = \begin{cases} X_n + 1 - Y_{n+1} & \text{if } X_n + 1 - Y_{n+1} \leq 0 \\ 0 & \text{if } X_n + 1 - Y_{n+1} > 0 \end{cases} \tag{55}$$

It is clear that $\{X_n\}$ is a markov chain, since it is determined by X_n and Y_{n+1}, thus independent to any history before n-th arrival.
(b)

$$\begin{aligned}P_{ij} = \mathbb{P}\left(X_{n+1} = j | X_n = i\right) &= \\ &= \begin{cases} 0 & \text{if } j > 0 \text{ and } i - j + 1 < 0 \\ \mathbb{P}\left(Y_{n+1} = i - j + 1\right) & \text{if } j > 0 \text{ and } i - j + 1 \geq 0 \\ \mathbb{P}\left(Y_{n+1} \geq i + 1\right) & \text{if } j = 0 \end{cases}\end{aligned} \tag{56}$$

Hence the transition matrix is

$$\boldsymbol{P} = \begin{pmatrix} \sum_1^\infty q_k & q_0 & 0 & 0 & \cdots \\ \sum_2^\infty q_k & q_1 & q_0 & 0 & \cdots \\ \sum_3^\infty q_k & q_2 & q_1 & q_0 & \cdots \\ \sum_4^\infty q_k & q_3 & q_2 & q_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{57}$$