Numerical Solutions for DEs HW1

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Problem 1. (1.1) Apply the method to Theorems 1.1 and 1.2 to show the convergence of the implicit midpoint rule (1.12) and of the theta method (1.13)

Proof. (**a.** *The implicit midpoint rule*) we let $\bar{t} = t_n + \frac{h}{2} = \frac{t_n + t_{n+1}}{2}$, $\boldsymbol{r}_n$ be the truncation error. And firstly we show that this method is of order 2. By definition

$$\begin{aligned} \boldsymbol{y}(t_{n+1}) &= \boldsymbol{y}(t_n) + h\boldsymbol{f}(\bar{t}, \tfrac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2}) + h\boldsymbol{r}_n \\ &= \boldsymbol{y}(t_n) + h\boldsymbol{f}(\bar{t}, \boldsymbol{y}(\bar{t})) + h\boldsymbol{u}_n + h\boldsymbol{r}_n \end{aligned} \tag{1}$$

in which we let $\|\boldsymbol{u}_n\| = \left\|\boldsymbol{f}(\bar{t}, \frac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2}) - \boldsymbol{f}(\bar{t}, \boldsymbol{y}(\bar{t}))\right\|$. Since $\boldsymbol{f}$ is Lipschitz in $\boldsymbol{y}$:

$$\|\boldsymbol{u}_n\| \le L\left\|\tfrac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2} - \boldsymbol{y}(\bar{t})\right\| = \tfrac{L}{2}\left\|\boldsymbol{y}(t_n) + \boldsymbol{y}(t_{n+1}) - 2\boldsymbol{y}(\bar{t})\right\| \tag{2}$$

With Taylor expansion at $\bar{t}$,

$$\|\boldsymbol{u}_n\| \le \tfrac{L}{2}\left\|(\boldsymbol{y}(\bar{t}) - \tfrac{h}{2}\boldsymbol{y}'(\bar{t})) + (\boldsymbol{y}(\bar{t}) + \tfrac{h}{2}\boldsymbol{y}'(\bar{t})) - 2\boldsymbol{y}(\bar{t}) + O(h^2)\right\| = O(h^2) \tag{3}$$

Hence use Taylor expansion at $\bar{t}$ for equation (1) $\Rightarrow$ (write $\boldsymbol{y} = \boldsymbol{y}(\bar{t})$ for simplicity)

$$\begin{aligned} h\boldsymbol{r}_n &= \boldsymbol{y}(t_{n+1}) - \boldsymbol{y}(t_n) - h\boldsymbol{y}'(\bar{t}) + O(h^3) \\ &= \boldsymbol{y} + \tfrac{h}{2}\boldsymbol{y}' + \tfrac{h^2}{8}\boldsymbol{y}'' - (\boldsymbol{y} - \tfrac{h}{2}\boldsymbol{y}' + \tfrac{h^2}{8}\boldsymbol{y}'') - h\boldsymbol{y}' + O(h^3) \\ &= O(h^3) \end{aligned} \tag{4}$$

So the method is of order 2. Now we let $\boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{y}(t_n)$, we substract equation (1) from the method, and using Lipschitz condition:

$$\begin{aligned} \boldsymbol{e}_{n+1} &= \boldsymbol{e}_n + h\left[\boldsymbol{f}(\bar{t}, \tfrac{\boldsymbol{y}_n+\boldsymbol{y}_{n+1}}{2}) - \boldsymbol{f}(\bar{t}, \tfrac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2})\right] + O(h^3) \\ \|\boldsymbol{e}_{n+1}\| &\le \|\boldsymbol{e}_n\| + \tfrac{hL}{2}\left\|\boldsymbol{y}_n + \boldsymbol{y}_{n+1} - \boldsymbol{y}(t_n) - \boldsymbol{y}(t_{n+1})\right\| + O(h^3) \\ &\le \|\boldsymbol{e}_n\| + \tfrac{hL}{2}(\|\boldsymbol{e}_n\| + \|\boldsymbol{e}_{n+1}\|) + O(h^3) \\ \Rightarrow \|\boldsymbol{e}_{n+1}\| &\le \frac{1+\frac{hL}{2}}{1-\frac{hL}{2}}\|e_n\| + \frac{c}{1-\frac{hL}{2}}h^3 \quad \text{(same as the bound for trapezoid method)} \\ &\le \frac{ch^2}{L}\exp\left(\frac{nhL}{1-\frac{hL}{2}}\right) \end{aligned} \tag{5}$$

We have $\lim\limits_{h\to 0}\|\boldsymbol{e}_n\| = 0$. By definition the method is convergent in $[0, nh]$. $\square$

(**b.** *the θ method*) We have known that the theta method is at least order 1. So

$$\boldsymbol{y}(t_{n+1}) = \boldsymbol{y}(t_n) + h\left[\theta\boldsymbol{f}(t_n, \boldsymbol{y}(t_n) + (1-\theta)\boldsymbol{f}(t_{n+1}, \boldsymbol{y}(t_{n+1})\right] + O(h^2) \tag{6}$$

Substract it from the numerical formula, and employing Lipschitz condition:

$$\begin{aligned} \|\boldsymbol{e}_{n+1}\| &\le \|\boldsymbol{e}_n\| + hL\theta\|\boldsymbol{e}_n\| + hL(1-\theta)\|\boldsymbol{e}_{n+1}\| + O(h^2) \\ \Rightarrow \|\boldsymbol{e}_{n+1}\| &\le \frac{1+hL\theta}{1-hL(1-\theta)}\|\boldsymbol{e}_n\| + \frac{c}{1-hL(1-\theta)}h^2 \\ &\le \frac{c}{L}\left[\left(\frac{1+hL\theta}{1-hL(1-\theta)}\right)^n - 1\right]h \\ &\le \frac{ch}{L}\exp\left(\frac{nhL}{1-hL(1-\theta)}\right) \end{aligned} \tag{7}$$

We have $\lim_{h\to 0} \|\boldsymbol{e}_n\| = 0$. By definition the method is convergent in $[0, nh]$. ☐ ☐

Problem 2. (1.2) The linear system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$, where $\boldsymbol{A}$ is a symmetric matrix, is solved by Euler's method.

a. Letting $\boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{y}(nh)$, $n = 0, 1, ...$, show that

$$\|\boldsymbol{e}_n\|_2 \leq \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} |(1+h\lambda)^n - e^{nh\lambda}|$$

where $\sigma(\boldsymbol{A})$ is the set of eigenvalues of $\boldsymbol{A}$ and $\|\cdot\|_2$ the Euclidean matrix norm.

b. Demonstrate that for every $-1 \ll x \leq 0$ and $n = 0, 1, ...$ it is true that

$$e^{nx} - \frac{1}{2}nx^2 e^{(n-1)x} \leq (1+x)^n \leq e^{nx}$$

c. Suppose that the maximal eigenvalue of $\boldsymbol{A}$ is $\lambda_{max} < 0$. Prove that, as $h \to 0$, and $nh \to t \in [0, t^*]$,

$$\|\boldsymbol{e}_n\|_2 \leq \frac{1}{2} t\lambda_{max}^2 e^{\lambda_{max}t} \|\boldsymbol{y}_0\|_2 h \leq \frac{1}{2} t^* \lambda_{max}^2 \|\boldsymbol{y}_0\|_2 h$$

d. Compare the order of magnitude of this bound with the upper bound from theorem 1.1 in the case

$$\boldsymbol{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad t^* = 10$$

Proof. (a.) The exact solution of the system is $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0$. The euler method gives $\boldsymbol{y}_n = (\boldsymbol{I}+h\boldsymbol{A})\boldsymbol{y}_{n-1}$, so $\boldsymbol{y}_n = (\boldsymbol{I}+h\boldsymbol{A})^n\boldsymbol{y}_0$. By its definition $\boldsymbol{e}_n = [(\boldsymbol{I}+h\boldsymbol{A})^n - e^{nh\boldsymbol{A}}]\boldsymbol{y}_0$. Take the spectral norm we have

$$\|\boldsymbol{e}_n\|_2 \leq \|\boldsymbol{y}_0\|_2 \left\|(\boldsymbol{I}+h\boldsymbol{A})^n - e^{nh\boldsymbol{A}}\right\|_2 = \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} \left|(1+h\lambda)^n - e^{nh\lambda}\right| \tag{8}$$

(b) For $x \leq 0$: $f'(x) = (e^x - x - 1)' = e^x - 1 \leq 0$, hence $f(x) \searrow$, $f(x) \geq f(0) = 0 \Rightarrow e^x \geq 1 + x$ *(we call it (1))*. At the same time $g'(x) = (e^x - \frac{x^2}{2} - x - 1)' = e^x - x - 1 \geq 0$, hence $g(x) \nearrow$, $g(x) \leq g(0) = 0 \Rightarrow e^x \leq 1 + x + \frac{x^2}{2}$, or $e^x - \frac{x^2}{2} \leq 1 + x$ *(we call it (2))*.

Let $h(\epsilon) = ((1-\epsilon)-\epsilon)^n + n(1-\epsilon)^{n-1}\epsilon - (1-\epsilon)^n$ for $1 \gg \epsilon \geq 0$, $h'(\epsilon) = n - n(1-\epsilon)^{n-1} \geq 0$, hence $h(\epsilon) \geq h(0) = 0 \Rightarrow ((1-\epsilon)-\epsilon)^n \geq (1-\epsilon)^n - n(1-\epsilon)^{n-1}\epsilon$. Now suppose $a \to 1^-$, $b \to 0^+$, consider $(a-b)^n \sim ((1-\epsilon)-\epsilon)^n$ with $a \sim 1-\epsilon, b \sim \epsilon$, we obtain $(a-b)^n \geq a^n - na^{n-1}b$. *(we call it (3))*.

Now let $a = e^x$, $b = \frac{x^2}{2}$, since $1 \ll x \leq 0$, $x \to 0^-$, $|b|, |a-1|$ are small. We can employ *(3)*:

$$e^{nx} - \frac{nx^2}{2}e^{(n-1)x} \leq (e^x - \tfrac{x^2}{2})^n \leq (1+x)^n \leq e^{nx}$$

where the first leq is due to *(3)*, the second is due to *(2)*, and the third one is due to *(1)*.

(c) Since $\lambda_{max} < 0, t \in [0, t^*]$, we have $te^{\lambda_{max}t} < t \leq t^*$. Hence $\frac{1}{2}te^{\lambda_{max}t}(\lambda_{max}^2 \|\boldsymbol{y}_0\|_2 h) \leq \frac{1}{2}t^*(\lambda_{max}^2 \|\boldsymbol{y}_0\|_2 h)$. And by (b): $-\frac{1}{2}nx^2e^{(n-1)x} \leq (1+x)^n - e^{nx} \leq 0 \Rightarrow \frac{1}{2}nx^2e^{(n-1)x} \geq |(1+x)^n - e^{nx}| \geq 0$. So

$$\|\boldsymbol{e}_n\|_2 \leq \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} \left|(1+h\lambda)^n - e^{nh\lambda}\right| \leq \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} \frac{1}{2}nh^2\lambda^2e^{(n-1)h\lambda} \quad (*) \tag{9}$$

with $h \to 0$, $nh = t = O(1)$, $(n-1)h \to t$. Hence

$$\|\boldsymbol{e}_n\|_2 \leq (*) = \frac{1}{2}t\|\boldsymbol{y}_0\|_2 h \max_{\lambda\in\sigma(\boldsymbol{A})} \lambda^2 e^{\lambda t} = \frac{1}{2}t\|\boldsymbol{y}_0\|_2 h\lambda_{max}^2 e^{t\lambda_{max}} \tag{10}$$

Becasue $\lambda^2 e^{\lambda t}$ is an increasing function with respect to λ. Finished the proof.

(d) In this case $\lambda_{max} = -1$, $t^* = 10$. This bound:

$$\|\boldsymbol{e}_n\|_2 \leq \frac{1}{2}t^*\lambda_{max}^2 \|\boldsymbol{y}_0\|_2 h = 5\|\boldsymbol{y}_0\|_2 h$$

Is linear in t^* and quadratic in λ. The bound in (1.1)

$$\|\boldsymbol{e}_n\| \le \frac{c}{\lambda}(e^{t^*\lambda}-1)h \approx \frac{1}{1}\|\boldsymbol{y}_0\|(e^{10}-1)h \approx 22026\,\|\boldsymbol{y}_0\|\,h$$

grows exponentially with λt^*.

□

Problem 3. (1.3) We solve the scalar linear system $y' = ay, y(0) = 1$.

a. Show that the 'continuous output' method

$$u(t) = \frac{1+\frac{1}{2}a(t-nh)}{1-\frac{1}{2}a(t-nh)}y_n, \qquad nh \le t \le (n+1)h, \quad n = 0, 1, ...$$

is consistent with the values of y_n and y_{n+1} which are obtained by the trapezoidal rule.

b. Demonstrate that u obeys the perturbed ODE

$$u'(t) = au(t) + \frac{\frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2}y_n \qquad t \in [nh, (n+1)h]$$

with initial condition $u(nh) = y_n$. Thus prove that

$$u((n+1)h) = e^{ha}\left[1+\frac{1}{4}a^3\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right]y_n$$

c. Let $e_n = y_n - y(nh)$, $n = 0, 1, ...$ show that

$$e_{n+1} = e^{ha}\left[1+\frac{1}{4}a^3\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}\right]e_n + \frac{1}{4}a^3 e^{(n+1)ha}\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1-\frac{1}{2}a\tau)^2}$$

In particular, deduce that $a < 0$ implies that the error propagetes subject to the inequality

$$|e_{n+1}| \le e^{ha}\left[1+\frac{1}{4}|a|^3\int_0^h e^{-\tau a}\tau^2 d\tau\right]|e_n| + \frac{1}{4}|a|^3 e^{(n+1)ha}\int_0^h e^{-\tau a}\tau^2 d\tau$$

Proof. (a) the formula for Trapezoid rule to solve $y' = ay$ is

$$y_{n+1} = y_n + \tfrac{h}{2}(ay_n + ay_{n+1}) \Rightarrow y_{n+1} = \frac{1+\frac{ah}{2}}{1-\frac{ah}{2}}y_n$$

And insert $t = nh, t = (n+1)h$ to $u(t)$: $u(nh) = y_n$, $u(nh+h) = \frac{1+\frac{1}{2}a(nh+h-nh)}{1-\frac{1}{2}a(nh+h-nh)} = y_{n+1}$. Hence it is consistent to trapezoid rule in terms of y_n and y_{n+1}.

(b)

$$\begin{aligned}
u'(t) &= \frac{\frac{1}{2}a[1-\frac{1}{2}a(t-nh)]+\frac{1}{2}a[1+\frac{1}{2}a(t-nh)]}{[1-\frac{1}{2}a(t-nh)]^2}y_n = \frac{a}{[1-\frac{1}{2}a(t-nh)]^2}y_n \\
&= \frac{a-\frac{1}{4}a^3(t-nh)^2+\frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2}y_n = \frac{a(1+\frac{1}{2}a(t-nh))(1-\frac{1}{2}a(t-nh))}{[1-\frac{1}{2}a(t-nh)]^2}y_n + \frac{\frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2}y_n \\
&= au(t) + \frac{\frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2}y_n
\end{aligned} \tag{11}$$

So the general solution of the ode is:

$$\begin{aligned}
u(t) &= e^{\int_{nh}^t -(-a)dz}\left(\int_{nh}^t e^{\int_{nh}^s -adz}\frac{\frac{1}{4}a^3(s-nh)^2}{[1-\frac{1}{2}a(s-nh)]^2}y_n ds + C\right) \\
&= e^{a(t-nh)}\left(\frac{1}{4}a^3\int_{nh}^t \frac{e^{-a(s-nh)}(s-nh)^2}{[1-\frac{1}{2}a(s-nh)]^2}y_n ds + C\right)
\end{aligned} \tag{12}$$

Clearly when $t = nh$, the integral vanishes, so $C = y_n$, we let $\tau := s - nh$ inside the integral:

$$u(t) = e^{a(t-nh)} \left(\frac{1}{4}a^3 \int_0^{t-nh} \frac{e^{-a\tau}\tau^2}{(1-\frac{1}{2}a\tau)^2} d\tau + 1 \right) y_n \tag{13}$$

Therefore, at $t = (n+1)h$,

$$u((n+1)h) = e^{ah} \left(\frac{1}{4}a^3 \int_0^{h} \frac{e^{-a\tau}\tau^2}{(1-\frac{1}{2}a\tau)^2} d\tau + 1 \right) y_n$$

(c) We can easily solve the scalar linear system analytically: $y(t) = e^{at}$. Hence $y((n+1)h) = e^{a(n+1)h} = e^{ah}y(nh)$. And we let the constant $J = \frac{1}{4}a^3 \int_0^h \frac{e^{-a\tau}\tau^2}{(1-\frac{1}{2}a\tau)^2} d\tau$. We have:

$$\begin{aligned} e_{n+1} &= y_{n+1} - y(t_{n+1}) = u((n+1)h) - y((n+1)h) \\ &= e^{ah}\left(J+1\right) y_n - e^{ah}y(nh) \\ &= e^{ah}\left(J+1\right) y_n - e^{ah}y(nh) - e^{ah}Jy(nh) + e^{ah}Jy(nh) \\ &= e^{ah}\left(J+1\right) e_n + Je^{a(n+1)h} \end{aligned} \tag{14}$$

If $a < 0$, $0 < 1 \le (1-\frac{1}{2}a\tau)^2$ i.e. $\frac{1}{(1-\frac{1}{2}a\tau)^2} \le 1 \ \forall \tau \in [0, h]$. So $|J| \le \frac{1}{4}|a|^3 \int_0^h e^{-\tau a}\tau^2 d\tau$. We conclude that

$$|e_{n+1}| \le e^{ah}\left(|J|+1\right)|e_n| + |J|e^{a(n+1)h} \le e^{ah}\left(\frac{1}{4}|a|^3 \int_0^h e^{-\tau a}\tau^2 d\tau + 1 \right) + \frac{1}{4}e^{a(n+1)h}|a|^3 \int_0^h e^{-\tau a}\tau^2 d\tau \tag{15}$$

□

Problem 4. (2.2) Let $\eta(z, w) = \rho(w) - z\sigma(w)$.

a. Demonstrate that the multistep method (2.8) is of order p iff

$$\eta(z, e^z) = cz^{p+1} + O(z^{p+2}), \qquad z \to 0$$

for some $c \in \mathbb{R} \setminus \{0\}$.

b. Show that, subject to $\partial\eta(0,1)/\partial w \neq 0$, there exists in a neighbourhood of the origin an analytic function $w_1(z)$ such that $\eta(z, w_1(z)) = 0$ and

$$w_1(z) = e^z - c\left(\frac{\partial\eta(0,1)}{\partial w}\right)^{-1} z^{p+1} + O(z^{p+2}), \qquad z \to 0 \quad (*)$$

c. Show that $(*)$ is true if the underlying method is convergent.

Proof. (a) Define sequence $\{c_m\}$ as

$$c_m = \begin{cases} \sum_{k=0}^s a_k, & m = 0 \\ \frac{1}{m!}\sum_{k=0}^s (a_k k^m - m b_k k^{m-1}), & m \ge 1 \end{cases} \tag{16}$$

The method (2.8) is of order p if and only if $c_m = 0$ for $m = 0, 1, ..., p$, $c_{p+1} \neq 0$; we call this condition (†). We further examine the generating polynomial of $\{c_m\}$: $P(z) = \sum_{m=0}^{\infty} c_m z^m$. And then we have $P(0) = c_0$, $P'(0) = c_1$, ..., $P^{(m)}(0) = c_m$. Hence (†) $\iff$ $P^{(p+1)}(0) \neq 0$, $P^{(m)}(0) = 0$ for $m = 1, 2, ..., p$. Since P is a polynomial of z, this is true if and only if

$$P(z) = cz^{p+1} + h.o.t. \ \ c \neq 0, z \to 0 \quad (‡)$$

And then we rewrite $P(z)$ as we have done in the course, and finally we can rewrite $P(z)$ as $P(z) = \sum_{k=0}^s a_k(e^z)^k - z\sum_{k=0}^s b_k(e^z)^k = \rho(e^z) - z\sigma(e^z) = \eta(z, e^z)$. Hence (‡) $\iff$ $\eta(z, e^z) = cz^{p+1} + O(z^{p+2})$, $c \neq 0$. □

(b) (c) *No Idea...* □

Problem 5. (2.3) Instead of (2.3), consider the identity

$$\boldsymbol{y}(t_{n+s}) = \boldsymbol{y}(t_{n+s-2}) + \int_{t_{n+s-2}}^{t_{n+s}} \boldsymbol{f}(\tau, \boldsymbol{y}(\tau))d\tau$$

a. Replace $\boldsymbol{f}(\tau, \boldsymbol{y}(\tau))$ by the interpolating polynomial $\boldsymbol{p}$ from section 2.1 and substitute $\boldsymbol{y}_{n+s-2}$ in place of $\boldsymbol{y}(t_{n+s-2})$. Show that the resultant explicit *Nystrom* method is of order $p = s$.
b. Derive the two-step Nystrom method in a closed form by using the above appraoch.
c. Find the coefficients of the two-step and three-step Nystrom methods by noticing that $\rho(w) = w^{s-2}(w^2-1)$ and evaluating σ from (2.13).
d. Derive the two-step third-order implicit *Milne* method. Again letting $\rho(w) = w^{s-2}(w^2-1)$ but allowing σ to be of degree s.

Proof. (a)

$$\begin{aligned}
\boldsymbol{y}(t_{n+s}) &= \boldsymbol{y}(t_{n+s-2}) + \int_{t_{n+s-2}}^{t_{n+s}} \boldsymbol{p}(\tau)d\tau + O(h^{s+1}) \\
&= \boldsymbol{y}(t_{n+s-2}) + \int_{t_{n+s-2}}^{t_{n+s}} \sum_{k=n}^{n+s-1} \boldsymbol{y}'(t_k) L_k^{[s]}(\tau)d\tau + O(h^{s+1}) \\
&= \boldsymbol{y}(t_{n+s-2}) + \sum_{k=n}^{n+s-1} \boldsymbol{y}'(t_k) \int_{t_{n+s-2}}^{t_{n+s}} L_k^{[s]}(\tau)d\tau + O(h^{s+1})
\end{aligned} \tag{17}$$

As before we translate the mesh to the left by nh, and find that $\int_{t_{n+s-2}}^{t_{n+s}} L_k^{[s]}(\tau)d\tau = \int_{t_{s-2}}^{t_s} L_{\tilde{k}}^{[s]}(\tau)d\tau$, where $\tilde{k} = 0, 1, ..., s-1$, with $L_k^{[s]}(t) = \prod_{j=0, j\neq k}^{s-1} \frac{t_j - t}{t_j - t_k}$. And this integral is a constant quantity times h, we denote $\int_{t_{s-2}}^{t_s} L_{\tilde{k}}^{[s]}(\tau)d\tau = h\left(\frac{1}{h}\int_{t_{s-2}}^{t_s} L_{\tilde{k}}^{[s]}(\tau)d\tau\right) =: hc_k$ (we change k to $k = 0, 1, ..., s-1$ in the following text). Therefore

$$\boldsymbol{y}(t_{n+s}) = \boldsymbol{y}(t_{n+s-2}) + h\sum_{k=0}^{s-1} c_k \boldsymbol{f}(t_{n+k}, \boldsymbol{y}(t_{n+k})) + O(h^{s+1}) \tag{18}$$

And the method is given by

$$\boldsymbol{y}_{n+s} = \boldsymbol{y}_{n+s-2} + h\sum_{k=0}^{s-1} c_k \boldsymbol{f}(t_{n+k}, \boldsymbol{y}_{n+k}) \tag{19}$$

h times the truncation error $h\boldsymbol{r}_n \sim O(h^{s+1})$, so the method is of order s.

(b) Two-step Nystrom: $s = 2$. $\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + h(c_0\boldsymbol{f}(t_n, \boldsymbol{y}_n) + c_1\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}))$

$$c_0 = \int_0^2 L_0^{[2]}dt = \int_0^2 \frac{1-t}{1-0}dt = 0; \qquad c_1 = \int_0^2 L_1^{[2]}dt = \int_0^2 \frac{0-t}{0-1}dt = 2 \tag{20}$$

So the 2-step Nystrom is given by

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})$$

(c) $\rho(w) = w^{s-2}(w^2-1)$.

· $s = 2$ case: $\rho(w) = w^2 - 1 = (w-1)(w+1) = v(v+2)$, with $v = w - 1$,

$$\sigma(v) = \frac{v+2}{1 - (\frac{1}{2}v - \frac{1}{3}v^2)} = (v+2)(1+\tfrac{1}{2}v) + O(v^2) = 2 + 2v + O(v^2)$$

$\sigma(w) = 2 + 2(w-1) = 2$, which matches our result in (b).

· $s=3$ case: $\rho(w) = w(w^2-1) = w(w-1)(w+1) = v(v+1)(v+2)$, with $v = w-1$,

$$\sigma(v) = \frac{(v+1)(v+2)}{1-(\frac{1}{2}v-\frac{1}{3}v^2)} = (v^2+3v+2)(1+\tfrac{1}{2}v-\tfrac{1}{12}v^2)+O(v^3) = 2+4v+\frac{7v^2}{3}+O(v^3)$$

$\sigma(w) = \frac{1}{3}-\frac{2}{3}w+\frac{7}{3}w^2$, which gives the 3-step *Nystrom* method:

$$\boldsymbol{y}_{n+3} = \boldsymbol{y}_{n+1} + h\left[\frac{1}{3}\boldsymbol{f}(t_n,\boldsymbol{y}_n) - \frac{2}{3}\boldsymbol{f}(t_{n+1},\boldsymbol{y}_{n+1}) + \frac{7}{3}\boldsymbol{f}(t_{n+2},\boldsymbol{y}_{n+2})\right] \tag{21}$$

(d) 2-step implicit: $\rho(w) = w^2-1 = (w-1)(w+1) = v(v+2)$

$$\sigma(v) = \frac{v+2}{1-(\frac{1}{2}v-\frac{1}{3}v^2)} = (v+2)(1+\tfrac{1}{2}v-\tfrac{1}{12}v^2)+O(v^3) = 1+2v+\frac{v^2}{3}+O(v^3)$$

$\sigma(w) = \frac{1}{3}+\frac{4}{3}w+\frac{1}{3}w^2$, which gives the 2-step *Mline* method:

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + h\left[\frac{1}{3}\boldsymbol{f}(t_n,\boldsymbol{y}_n) + \frac{4}{3}\boldsymbol{f}(t_{n+1},\boldsymbol{y}_{n+1}) + \frac{1}{3}\boldsymbol{f}(t_{n+2},\boldsymbol{y}_{n+2})\right] \tag{22}$$

□

Problem 6. Show that the explicit multistep method

$$\boldsymbol{y}_{n+3} + a_2\boldsymbol{y}_{n+2} + a_1\boldsymbol{y}_{n+1} + a_0\boldsymbol{y}_n = h[b_2\boldsymbol{f}(t_{n+2},\boldsymbol{y}_{n+2}) + b_1\boldsymbol{f}(t_{n+1},\boldsymbol{y}_{n+1}) + b_0\boldsymbol{f}(t_n,\boldsymbol{y}_n)]$$

is fourth order only if $a_0+a_2=8$ and $a_1=-9$. Hence deduce that this method cannot be both fourth order and convergent.

Proof. The method is of order 4 $\iff$

$$\sum_{k=0}^{3} a_k = 0; \qquad \sum_{k=0}^{3}(a_k k^m - m b_k k^{m-1}) = 0, \quad m = 1,2,3,4 \tag{23}$$

And we have already known $a_3 = 1$, $b_3 = 0$. Hence it suffices to solve

$$\begin{cases} a_0+a_1+a_2 = -1 \\ b_0+a_1-b_1+2a_2-b_2+3a_3 = 0 \\ a_1-2b_1+4a_2-4b_2+9 = 0 \\ a_1-3b_1+8a_2-12b_2+27 = 0 \\ a_1-4b_1+16a_2-32b_2+81 = 0 \end{cases} \tag{24}$$

Label the equations as a to e: $(d)-\frac{3}{4}(c)-\frac{1}{4}(e) \Rightarrow$

$$-3b_1+8a_2-12b_2-\frac{3(-2b_1+4a_2-4b_2)}{4}-\frac{-4b_1+16a_2-32b_2}{4} = 0$$

$$\Rightarrow -\frac{1}{2}b_1+a_2-b_2 = 0$$

Insert into (c) $\Rightarrow a_1 = -9$, hence $a_0+a_2 = 8$.
Claim This method, with $a_1=-9$ and $a_0+a_2=8$ is not stable.
Proof of claim: let $a_2 = c$, then characteristic polynomial $\rho(z) = z^3+cz^2-9z+8-c$. Use `Mathematica`, we find its zeros:

$$z_1 = 1, \quad z_2 = \frac{-(c+1)-\sqrt{32+(c-1)^2}}{2}, \quad z_3 = \frac{-(c+1)+\sqrt{32+(c-1)^2}}{2} \tag{25}$$

Which are neither (1) all inside the unit circle, nor (2) with norm 1 while having multiplicity of 1. Hence by theorem, the numerical method is not stable. $\Rightarrow$ the numerical method does not converge. □
