Functional Analysis Assignment I

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Problem 1. If x, y, z are points in metric space (X, d), show

$$d(x, y) \geq |d(x, z) - d(y, z)|$$

Proof. By defining properties of metric:

1. $d(x, y) \geq 0$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

If $d(x, z) - d(y, z) \geq 0$, rearrange terms: $d(x, z) - d(y, z) \leq d(x, y) \Rightarrow |d(x, z) - d(y, z)| \leq d(x, y)$. Otherwise, rearrange terms: $d(y, z) - d(x, z) \geq -d(x, y) \Rightarrow |d(y, z) - d(x, z)| \leq d(x, y)$. ■

Problem 2. $(X, d_X), (Y, d_Y)$ are metric spaces. Show that the *Cartesian product* (For which $z = (x, y) \in Z$ for $x \in X, y \in Y$.) $(Z := X \times Y, d)$ is a metric space with

$$d(z_1, z_2) := d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Proof. It suffices to check the defining properties of d:

1. $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2) \geq 0$ is clear, since d_X, d_Y are metrics. Moreover, if $d(z_1, z_2) = 0$, then it must be $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$, hence $x_1 = x_2, y_1 = y_2$ $\Rightarrow z_1 = z_2$; the other direction is trivial.
2. $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2) = d_X(x_2, x_1) + d_Y(y_2, y_1) =: d(z_2, z_1)$. (*symmetry*)
3. (*triangle ineq.*)

$$\begin{aligned} d(z_1, z_3) + d(z_3, z_2) &= d_X(x_1, x_3) + d_Y(y_1, y_3) + d_X(x_3, x_2) + d_Y(y_3, y_2) \\ &\leq d_X(x_1, x_2) + d_Y(x_1, x_2) \\ &=: d(z_1, z_2) \quad ■ \end{aligned} \tag{1}$$

Problem 3. $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces. $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Show that

$$h := g \circ f : X \to Z$$

is continuous.

Proof. Take arbitrary $\{x_n : n \geq 1\} \subseteq X$ converging to x. By continuity of f: $f(x_n) \to f(x)$. Consider sequence $\{f(x_n) : n \geq 1\}$, by continuity of g: $g(f(x_n)) \to g(f(x))$.
i.e. we have $g(f(x_n)) \to g(f(x))$ whenever $x_n \to x$. $\iff$ $h = g \circ f$ is a continuous mapping by definition. ■

Problem 4. Show

- Every compact subset of a metric space is closed and bounded.
- A closed subset of a compact space is compact.

Proof. (1) Let $K \subseteq X$ be a compact subset.

(*Boundedness*) By compactness, pick open cover $K \subseteq \bigcup_{x\in F} B_r(x)$, $\exists$ a finite subcover $K \subseteq \bigcup_{i=1}^n B_r(x_i)$.

Denote $d_1 := \max\{r, d(x_1, x_2)\}$, then clearly $B_r(x_1) \cup B_r(x_2) \subseteq B_{r+d_1}(x_2)$; $d_2 := \max\{r, d(x_2, x_3)\}$, then $B_r(x_3) \cup B_{r+d_1}(x_2) \subseteq B_{r+d_1+d_2}(x_3)$. Do this repeatedly until n, we have

$$K \subseteq \bigcup_{i=1}^n B_r(x_i) \subseteq B_{r+\sum_{i=1}^n d_i}(x_n)$$

Where $r + \sum_{i=1}^n d_i < \infty$. K is bounded.

(*Closedness*) Forall $x_0 \in X \setminus K$, we have $K \subseteq \bigcup_{x\in K} B_{\frac{1}{2}d(x,x_0)}(x)$. By compactness, $\exists$ a finite subcover $K \subseteq \bigcup_{i=1}^n B_{\frac{1}{2}d(x_i,x_0)}(x_i)$. Let $\delta := \frac{1}{4} \min\limits_{i=1,\dots,n} \{d(x_i, x_0)\}$; then $\forall y \in B_\delta(x_0), \forall i = 1, \dots, n$: $d(x_i, y) \geq \frac{3}{4} d(x_i, x_0)$, which implies $y \notin B_{\frac{1}{2}d(x_i,x_0)}(x_i)$ for any i. Hence $B_\delta(x_0) \subseteq X \setminus K \Rightarrow X \setminus K$ is open.

(2) Let $K \subseteq X$ be a closed subset. X is a compact metric space $\iff$ X is sequentially compact $\iff$ X is complete and totally bounded.

Therefore it suffices to show K is totally bounded. This follows the total-boundedness of X. Because $\forall \epsilon > 0$, $\exists\, n_\epsilon \in \mathbb{N}$, s.t.

$$K \subseteq X \subseteq \bigcup_{i=1}^{n_\epsilon} B_\epsilon(x_i)$$

Hence (under the condition that $K \subseteq X$, X is complete) K is closed and totally bounded $\iff$ K is sequentially compact $\iff$ K is compact. ■

Problem 5. Let (X, d) be complete metric space, and $Y \subset X$. Show that

$$(Y, d) \text{ is complete} \iff Y \text{ is a closed subset of } X$$

Proof. ($\Leftarrow$) $\forall$ Cauchy sequence $\{y_n\} \subset Y \subset X$. Since (X, d) is complete, $\{y_n\}$ is convergent sequence and $y_n \to y \in X$.

Since Y is closed, $\{y_n\}$ is convergent, limit point $y \in X \Rightarrow y \in Y$ by definition. Hence Y is complete.

($\Rightarrow$) $\forall$ convergent sequence $\{y_n\} \subset Y$, $y_n \to y \in X$. But convergence of $y_n \to y$ implies that $\{y_n\}$ is Cauchy sequence.

Since (Y, d) is complete, $y_n \to y \in Y$. Therefore Y is closed. ■

Problem 6. Let (X, d) be metric space, $\{x_n\} \subset X$. Show if $\{x_n\}$ has Cauchy subsequence, then, for any decreasing sequence of positive $\epsilon_k \searrow 0$, there exists $\{x_{n_k}\} \subset \{x_n\}$, s.t.

$$d(x_{n_k}, x_{n_l}) \leq \epsilon_k \quad \text{for all } k \leq l$$

Proof. Denote $\{x_{n_j}\} \subset \{x_n\}$ be a Cauchy. By definition, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s.t. $d(x_{n_p}, x_{n_q}) \leq \epsilon$ for $n_q \geq n_p \geq N$. Given the sequence $\{\epsilon_k\} \searrow 0$, we have:
For ϵ_1, $\exists N_1 \in \mathbb{N}$, s.t. $d(x_{n_{p_1}}, x_{n_{q_1}}) \leq \epsilon_1$ for all $n_{q_1} \geq n_{p_1} \geq N_1$. We pick $x^*_{n_1} := x_{n_{p_1}}$.
For ϵ_2, $\exists N_2 \in \mathbb{N}$, $N_2 \geq N_1$, s.t. $d(x_{n_{p_2}}, x_{n_{q_2}}) \leq \epsilon_2$ for all $n_{q_2} \geq n_{p_2} \geq N_2$. We pick n_{p_2} such that $n_{p_2} \geq \max\{N_2, n_{p_1}\}$, let $x^*_{n_2} := x_{n_{p_2}}$. It is clear that $d(x^*_{n_1}, x^*_{n_2}) \leq \epsilon_1$, because $x^*_{n_1} \geq N_1, x^*_{n_2} \geq x^*_{n_1}$.
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Do this repeatedly, we obtain $\{x^*_{n_k}\} \subset \{x_n\}$ that satisfies desired property. ∎

Problem 7. $f : X \to \mathbb{R}$ is lower semicontinuous and $M \in \mathbb{R}$. Define

$$f_M(x) := \min\{f(x), M\}$$

Show that f_M is lower semicontinuous.

Proof. By definition, $f_M(x) \leq f(x)$.
Since f is lower semicontinuous,

$$f_M(x) \leq f(x) \leq \liminf_{n\to\infty} f(x_n)$$

whenever $x_n \to x$. Since $f_M(x) \leq M$,

$$\begin{aligned} f_M(x) &\leq \min\{M, \liminf_{n\to\infty} f(x_n)\} = \min\{M, \sup_{m\geq 0} \inf_{n\geq m} f(x_n)\} \\ &\leq \liminf_{n\to\infty}(\min\{M, f(x_n)\}) \qquad (2) \\ &= \liminf_{n\to\infty}(f_M(x_n)) \end{aligned}$$ ∎

Problem 8. $f : X \to \mathbb{R}$. The *Epigraph* epi$f$ is the subset of $X \times \mathbb{R}$ consisting of points that lie above the graph of f, i.e.

$$\text{epi}f := \{(x, t) \in X \times \mathbb{R} : t \geq f(x)\}$$

Show that f is lower semicontinous $\iff$ its epigraph is a closed set.

Proof. ($\Rightarrow$) Take any convergent sequence $\{(x_n, t_n)\} \subset \text{epi}f$, $(x_n, t_n) \to (x, t)$. We have pointwisely $t_n \geq f(x_n)$. Take limit inf on both sides:

$$t = \lim_{n\to\infty} t_n = \liminf_{n\to\infty} t_n \geq \liminf_{n\to\infty} f(x_n)$$

Since f is lower semicontinous, we have

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

Hence $t \geq f(x)$, $(x, t) \in \text{epi}f$, implies that epi$f$ is a closed set.
($\Leftarrow$) Let $x_n \to x$ be convergent sequence. Denote $t := \liminf_{n\to\infty} f(x_n)$, then there exists $\{x_{n_k}\} \subseteq \{x_n\}$, s.t. $f(x_{n_k}) \to t$, i.e. $(x_{n_k}, f(x_{n_k})) \to (x, t)$, which is also convergent.
Since epi$f$ is closed $\Rightarrow$ $(x, t) \in \text{epi}f$. So $t = \liminf_{n\to\infty} f(x_n) \geq f(x)$. ∎

Problem 9. $\{x_n\}$ is sequence in compact metric space such that all its convergent subsequence has same limit x. Show that

$$x_n \to x$$

Proof. Assume otherwise. I.e. x_n does not converge to x.

Then by definition, $\exists\delta > 0$, $\forall N \in \mathbb{N}$, $\exists n > N$, s.t. $x_n \notin B_\delta(x)$. Which is to say that infinitely many points lie out of $B_\delta(x)$. Denote these as $\{\bar{x}_n\} := \{x_n\} \setminus \{x_i \in \{x_n\} : x_i \in B_\delta(x)\}$

Since X is compact $\Rightarrow \exists \{x_{n_k}\} \subseteq \{\bar{x}_n\} \subseteq \{x_n\}$ that is convergent. But it can not converge to x, because $d(x, x_{n_k}) > \delta$ for all k. Contradicts the fact that any subsequence of $\{x_n\}$ converges to x. ■