Functional Analysis Assignment V

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Problem 1. Let $f \in C^\infty(0,1)$. Use the **Lax-Milgram thm** to show that BVP

$$\begin{cases} -v'' + \frac{1}{10}v' + v = f \\ v(0) = 0,\ v(1) = 0 \end{cases}$$

Has a unique solution $v \in \mathcal{L}^2(0,1)$

Proof. Assume v is a classical solution, then $\forall \phi \in H_0^1$, we have

$$\int v'\phi' + \int \frac{1}{10}v'\phi + \int v\phi = \int f\phi \tag{1}$$

We define

$$A(v,\phi) := \int v'\phi' + \int \frac{1}{10}v'\phi + \int v\phi \tag{2}$$

It is clear that A is a bilinear function, but not symmetric. We further set $\xi = e^{\frac{-x}{10}}$. Then the former equation can be written as

$$-(\xi v')' + \xi v = \xi f \tag{3}$$

Define on H_0^1 the symmetric continuous bilinear form

$$B(v,\phi) = \int \xi v'\phi' + \int \xi v\phi \tag{4}$$

This form is coercive, apply Lax-Milgram, there exists a unique $v \in H_0^1$ such that $B(v,\phi) = \int \xi f\phi$, for any $\phi \in H_0^1$. $\square$

Problem 2. Show that the closed linear span of a set is the closure of its linear span.

Proof. By their definition, linear span and closed linear span of a set S is defined as

$$\begin{aligned} \mathrm{cspan}\{S\} &= \bigcap_{\substack{S\subseteq C \\ C \text{ closed}}} C \\ \mathrm{span}\{S\} &= \bigcap_{S\subseteq Y} Y \\ \overline{\mathrm{span}\{S\}} &= \bigcap_{\substack{\mathrm{span}\{S\}\subseteq Z \\ Z \text{ closed}}} Z \end{aligned} \tag{5}$$

Where C, Y, Z are all linear subspaces. Denote $\{C\}, \{Y\}, \{Z\}$ as the families of sets forms the above intersection.
First we show $\overline{\mathrm{span}\{S\}} \subseteq \mathrm{cspan}\{S\}$.

- Take any $x \in \mathrm{cspan}\{x\}$, then $x \in C$ for all $C \in \{C\}$.
- Since $S \subseteq \mathrm{span}\{S\} \subseteq Z$. We have $\{Z\} \subseteq \{C\}$. Hence $x \in Z$ for all $Z \in \{Z\}$.

Next we show the another direction.

- Take any $x \in \overline{\mathrm{span}\{S\}}$, then $x \in Z$ for all $Z \in \{Z\}$.

- Since $\text{span}\{S\} \subseteq \overline{\text{span}\{S\}} \Rightarrow x \in \text{span}\{S\}$. Hence $x \in Y$ for all $Y \in \{Y\}$.
- Clearly $\{C\} \subseteq \{Y\}$, so $x \in C$ for all $C \in \{C\}$

$\square$

Problem 3. (*Lemma 8.*) Let H be a Hilbert space $\{x_j\}$ an orthonormal set in H. $\overline{\text{span}\{S\}} = \{\sum a_j x_j, \sum a_j^2 < \infty\}$ Show that $y \in \overline{\text{span}\{S\}}$ converges. That is,
$\sum a_j^2 < \infty \iff$ For Λ an index set, $\forall \epsilon > 0$, exists a finite index subset $I_\epsilon \subset \Lambda$, such that $\forall I \supset I_\epsilon$,

$$\left\| \sum_I a_j x_j - x \right\|^2 < \epsilon^2$$

For some $x \in H$.

Proof. $(\Rightarrow)$ Given $\sum a_j^2 < \infty$, we define $a := \sup \sum a_j^2 < \infty$. Then for all $\epsilon > 0$, there exists I_ϵ, such that $\forall I \supset I_\epsilon$

$$\left| \sum_I a_j^2 - a \right| < \epsilon^2$$

Therefore

$$\begin{aligned} \left\| \sum_{I_\epsilon} a_j x_j - \sum_I a_j x_j \right\|^2 &\le \left\| \sum_{I_\epsilon} a_j x_j \right\|^2 + \left\| \sum_I a_j x_j \right\|^2 \\ &= \sum_{I_\epsilon} |a_j|^2 + \sum_I |a_j|^2 \\ &\le \left| \sum_{I_\epsilon} |a_j|^2 - a \right| + \left| \sum_I |a_j|^2 - a \right| \\ &< 2\epsilon^2 \end{aligned} \tag{6}$$

$(\Leftarrow)$ $\forall \epsilon > 0$, exists a finite index subset $I_\epsilon \subset \Lambda$, such that $\forall I \supset I_\epsilon$, $\left\| \sum_I a_j x_j - x \right\|^2 < \epsilon^2$. Hence

$$\left\| \sum_{I \setminus I_\epsilon} a_j x_j \right\|^2 \le \left\| \sum_I a_j x_j - x \right\|^2 + \left\| x - \sum_{I_\epsilon} a_j x_j \right\|^2 < 2\epsilon^2 \tag{7}$$

It is clear that $\left\| \sum_{I_\epsilon} a_j x_j \right\|^2 < C$ is bounded, since there are finitely many terms.

$$\sum_I |a_j|^2 = \left\| \sum_I a_j x_j \right\|^2 \le \left\| \sum_{I_\epsilon} a_j x_j \right\|^2 + \left\| \sum_{I \setminus I_\epsilon} a_j x_j \right\|^2 < C + 2\epsilon^2 < \infty \tag{8}$$

Finished the proof. $\square$

Problem 4. (*Thm.9'*) Let $\{y_j\}$ be a sequence of vectors in Hilbert space whose closed linear span is all of H. Then there exists an orthonormal basis $\{x_j\}$ such that the linear span of $\{x_1, ..., x_n\}$ contains $y_1, ..., y_n$.
(*Ex.8*) Let H be Hilbert space; show that any two orthonormal bases in H have same cardinality.
(*Thm.10*) Let H be Hilbert space, $\{x_j\}$, $\{y_j\}$ two orthonormal bases. For all $x \in H$, has representation $x = \sum a_j x_j$, $a_j = \langle x, x_j \rangle$. Then the mapping

$$x \to y = \sum a_j y_j$$

is an isometry of H onto H, $0 \mapsto 0$. Furthermore every isometry of H onto H $0 \mapsto 0$ can be obtained in this fashion.

Proof. By hypothesis, we have

$$\overline{\operatorname{span}\{y_j\}} = H$$

Let $u_1 = y_1$, $x_1 = \frac{u_1}{\|u_1\|}$. Then let

$$\begin{aligned} u_2 &= y_2 - \langle y_2, u_1 \rangle \cdot \frac{u_1}{\|u_1\|^2} \\ x_2 &= \frac{u_2}{\|u_2\|} \end{aligned} \tag{9}$$

It is easy to check that $\langle u_1, u_2 \rangle = \langle u_1, y_2 \rangle - \langle y_2, u_1 \rangle \frac{\langle u_1, u_1 \rangle}{\|u_1\|^2} = 0$, hence $\langle x_1, x_2 \rangle = 0$. Then keep on doing this,

$$\begin{aligned} u_k &= y_k - \sum_{i=1}^{k-1} \langle y_k, u_i \rangle \frac{u_i}{\|u_i\|^2} \\ x_k &= \frac{u_k}{\|u_k\|} \end{aligned} \tag{10}$$

Stop at n until $u_n = 0$. This must happen at some $n < \infty$ since $\overline{\operatorname{span}\{y_j\}} = H$.
Claim. $\{x_k\}$ are orthonormal.
Proof of Claim. $\|x_k\| = 1$ is straightforward in construction. It suffices to show they are orthogonal. We prove by induction.
Assume $u_k \perp u_s$ for all $1 \leq s \leq k-1$. The basic case $u_2 \perp u_1$ is checked in the first step. Now at $k+1$, for any $1 \leq s \leq k$:

$$\begin{aligned} \langle u_s, u_{k+1} \rangle &= \left\langle u_s, y_{k+1} - \sum_{i=1}^{k} \langle y_{k+1}, u_i \rangle \frac{u_i}{\|u_i\|^2} \right\rangle \\ &= \langle u_s, y_{k+1} \rangle - \sum_{i=1}^{k} \langle y_{k+1}, u_i \rangle \frac{\langle u_s, u_i \rangle}{\|u_i\|^2} \\ &= \langle u_s, y_{k+1} \rangle - \langle y_{k+1}, u_s \rangle \frac{\langle u_s, u_s \rangle}{\|u_s\|^2} \quad \text{(By assumption, } \langle u_s, u_i \rangle = \delta_{si}) \\ &= 0 \end{aligned} \tag{11}$$

Which finished the proof. □

Proof. If H has finite dimension, the statement is obvious.
If H is infinite dimensional, let $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$ be two orthonormal bases. I, J are infinite index sets. By Parseval's Identity, for any $z \in H$

$$\|z\|^2 = \sum_{i \in I} \langle z, x_i \rangle^2 \tag{12}$$

So $\langle z, x_i \rangle \neq 0$ for countable number of i. We pick $z = y_j$, denote $I_j = \{i \in I, \langle y_j, x_i \neq 0 \rangle\}$, we have $|I_j| = |\mathbb{N}|$. Now for any $i \in I$, using orthonormal basis $\{y_j\}$, we can also write

$$\|x_i\|^2 = \sum_{j \in J} \langle x_i, y_j \rangle^2 = 1 \tag{13}$$

So for any $i \in I$, there exists $j \in J$ such that $\langle x_i, y_j \rangle \neq 0$. Therefore, $I = \bigcup_{j \in J} I_j$. We conclude that $|I| = |J \times \mathbb{N}| \leq |J|$.
Apply similar argument for the reverse direction, we obtain $|J| = |I \times \mathbb{N}| \leq |I|$, $\Rightarrow |I| = |J|$, finished the proof. □

Proof. (1) Define this mapping $T : x \mapsto y = \sum a_j y_j$, $a_j = \langle x, x_j \rangle$. We have

$$\|x\| = \left(\sum |a_j|^2\right)^{\frac{1}{2}} = \left\|\sum_j a_j y_j\right\| = \|Tx\| \tag{14}$$

Hence T is an isometry. By lemma.8, $\forall x \in H$, x has orthonormal expansion. Hence T is onto.
(2) There exists a orthonormal basis $\{e_j\}$ for H. For any onto isometry $T : H \to H$,and any $x, y \in H$, we have

$$\|Tx + Ty\|^2 = \|x + y\|^2 \Rightarrow \langle Tx, Ty \rangle = \langle x, y \rangle \quad (\dagger) \tag{15}$$

We define $\{f_j\} = \{Te_j\}$. Clearly $\|f_j\| = \|Te_j\| = \|e_j\| = 1$. And by (†), for $i \neq j$

$$\langle f_i, f_j \rangle = \langle Te_i, Te_j \rangle = \langle e_i, e_j \rangle = 0 \tag{16}$$

Moreover, since T is onto $\{f_j\}$ is an orthonormal basis. □

Problem 5. Show that every infinite dimensional separable Hilbert space is isomophic with ℓ^2.

$$\ell^2 = \left\{x = (a_1, a_2, ...), \sum |a_j|^2 < \infty\right\}$$

$\|x\| = \left(\sum |a_j|^2\right)^{\frac{1}{2}}$

Proof. Since H is separable, it has countably dense orthonormal basis $\{e_i\}_{i=1}^{\infty}$. By previous results in lecture, ℓ^2 is Hilbert space with inner product $\langle x, y \rangle = \sum x_j \bar{y}_j$. Hence we define

$$\begin{aligned} T :& H \to \ell^2 \\ & x \mapsto (\langle x, e_1 \rangle, ..., \langle x_j, e_j \rangle, ...) =: (z_1, ..., z_j, ...) \end{aligned} \tag{17}$$

We check that T is isometry:

$$\|x\| = \left(\sum \langle x_j, e_j \rangle^2\right)^{\frac{1}{2}} = \left(\sum |z_j|^2\right)^{\frac{1}{2}} = \|Tx\| \tag{18}$$

And T is onto due to lemma 8. Thus H is isomorphic with ℓ^2. □

Problem 6. Show that $C_0^\infty(D)$ is an inner product space under $\langle f, g \rangle_0$ and $\langle f, g \rangle_1$. Where

$$\langle f, g \rangle_0 = \int_D fg \quad \langle f, g \rangle_1 = \int_D \sum \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_j}$$

Proof. For both cases, symmetry is clear. For $\langle \cdot \rangle_0$:

$$\langle f, \alpha g + \beta h \rangle_0 = \int_D f(\alpha g + \beta h) = \alpha \int_D fg + \beta \int_D fh = \alpha \langle f, g \rangle_0 + \beta \langle f, h \rangle_0 \tag{19}$$

$$\langle f, f \rangle_0 = \int_D f^2 = 0 \Rightarrow f = 0 \text{ almost surely} \tag{20}$$

Since $f \in C_0^\infty \Rightarrow f \equiv 0$.
For $\langle \cdot \rangle_1$:

$$\langle f, \alpha g + \beta h \rangle_1 = \int_D \sum \frac{\partial f}{\partial x_j} \left(\alpha \frac{\partial g}{\partial x_j} + \beta \frac{\partial h}{\partial x_j}\right) = \alpha \langle f, g \rangle_1 + \beta \langle f, h \rangle_1 \tag{21}$$

$$\langle f, f \rangle_1 = 0 \Rightarrow \int_D \sum \left(\frac{\partial f}{\partial x_j}\right)^2 = 0 \Rightarrow \sum \left(\frac{\partial f}{\partial x_j}\right)^2 = 0 \text{ a.s.} \tag{22}$$

Since $f \in C_0^\infty$, $S = \sum \left(\frac{\partial f}{\partial x_j}\right)^2$ is continuous, hence $S \equiv 0$, which implies that $\frac{\partial f}{\partial x_j} \equiv 0$ for all x_j, $x \in D$. Hence $f \equiv C$, C is constant.
Moreover, since f have compact support and supp$f \subset D$, we have $f = 0$ on $\partial D \Rightarrow C = 0$. □

Problem 7. (*Ex.1*) Show $Y^\perp$ is a closed linear subspace of X'.
(*Ex.2*) Let Y be a closed linear subspace of a normed linear space X. Show that the dual of (X/Y) is isometrically isomorphic with $Y^\perp$.

Proof. By its definition, the annihilator of subset $Y \subseteq X$ is

$$Y^\perp := \{\ell \in X' : \ell(x) = 0 \text{ for all } x \in Y\}$$

We define linear functional $\kappa_x \in (X')'$, i.e. $\kappa_x : X' \to \mathbb{R}$ as $\kappa_x(\ell) = \ell(x)$. Then κ_x is continuous by definition of dual space. Hence the null space of κ_x

$$\mathcal{N}(\kappa_x) = \{\ell \in X' : \kappa_x(\ell) = \ell(x) = 0\}$$

is closed, because of the fact that $\mathcal{N}(\kappa_x) = \kappa_x^{-1}(\{0\})$, and continuity.
Therefore by definition

$$Y^\perp = \bigcap_{x \in Y} \mathcal{N}(\kappa_x)$$

is intersection of closed sets, hence closed, finished the proof. □

Proof. We define

$$\sigma : (X/Y)' \to X' \quad \sigma(f) = f \circ Q$$

For all $f \in (X/Y)'$, and $Q : X \to X/Y$ is quotient map.
First we show $\sigma(\cdot)$ is onto $Y^\perp$, i.e. its range $\mathcal{R}(\sigma) = Y^\perp$.

- Notice that $\sigma(f)(Y) = f(QY) = f(0) = 0$, so $\mathcal{R}(\sigma) \subseteq Y^\perp$.
- For any $g \in Y^\perp$, $\mathcal{N}(g) \supseteq Y$. Hence there exists $\hat{g} \in (X/Y)'$ such that $\hat{g} \circ Q = g$ and $\|\hat{g}\| = \|g\|$. Hence $\mathcal{R}(\sigma) \supseteq Y^\perp$

Next we show σ is an isometry. For all $f \in (X/Y)'$, there exists $\{x_n\} \subset X$, s.t. $\|Qx_n\| < 1$ and $\|f(Qx_n)\| \to \|f\|$. We pick $y_n \in Y$ s.t. $\|x_n + y_n\| < 1$, then

$$\|f \circ Q(x_n + y_n)\| = \|f(Q(x_n))\| \to \|f\| \tag{23}$$

So $\|f \circ Q\| \geq \|f\|$. We already have $\|f \circ Q\| \leq \|f\|$. So we finish with the proof. □

Problem 8. (*Ex.3*) Show that Y' is isometrically isomorphic with $X'/Y^\perp$.
(*Ex.4*) Show that the closed linear span of $\{y_j\}$ is the closure of linear span Y of $\{y_j\}$, consisting of all finite linear combinations of the y_j:

$$y = \sum_F a_j y_j$$

Proof. By definition

$$Y^\perp := \{\ell \in X', \ell(y) = 0 \ \forall y \in Y\}$$

We define

$$\rho : X' \to Y' \quad \rho(\ell) = \ell|_Y \tag{24}$$

Then the null space $\mathcal{N}(\rho) = Y^\perp$, because $\ell(Y) = 0 \ \forall \ell \in Y^\perp$. Hence we have $\rho(Y^\perp + \ell) = \rho(\ell)$. We can define

$$\hat{\rho} : X'/Y^\perp \to Y' \quad \rho(\ell + Y^\perp) = \rho(\ell) = \ell|_Y \tag{25}$$

First we show $\hat{\rho}$ is onto. For any $\phi \in Y'$, by **Hahn-Banach**, there exists extension $f \in X'$ such that $f|_Y = \phi$. Hence we have $f + Y^\perp \in X'/Y^\perp$

$$\hat{\rho}(f + Y^\perp) = \rho(f) = \phi \tag{26}$$

indicates that $\hat{\rho}$ is onto.
Next we show $\hat{\rho}$ is isometry. For any $f + Y^{\perp} \in X'/Y^{\perp}$,

$$\left\|\hat{\rho}(f + Y^{\perp})\right\| = \|f|_Y\| = \|f\| \geq \inf_{m \in Y^{\perp}} \|f - m\| = \left\|f + Y^{\perp}\right\| \tag{27}$$

Since for all $m \in Y^{\perp}$, $\|f - m\| \geq \|f|_Y\|$, it's clear that $\left\|f + Y^{\perp}\right\| \geq \|f|_Y\| = \left\|\hat{\rho}(f + Y^{\perp})\right\|$. So $\left\|\hat{\rho}(f + Y^{\perp})\right\| = \left\|f + Y^{\perp}\right\|$. Completed the proof that $\hat{\rho}$ gives an isometric isomorphism. □

Proof. closedspan$\{y_j\} = \overline{\text{span}\{y_j\}}$ is a duplicate of problem 2. We only show this definitions are equivalent to definition using finite linear combinations. That is, it suffices to show

$$U_1 := \bigcap_{C \in \mathscr{C}} C = \overline{\left\{\sum_{j \in F} a_j y_j, F \text{ is finite}\right\}} =: \overline{U_2}$$

Where $\mathscr{C} = \{C : \{y_j\} \subseteq C, C \text{ closed linear subspace}\}$
(**Step.1**) U_2 is a linear subspace, since

$$\alpha \sum_{j \in F_1} a_j y_j + \beta \sum_{k \in F_2} b_k y_k = \sum_{i \in F_1 \cup F_2} (\alpha a_i \mathbb{1}_{\{i \in F_1\}} + \beta b_i \mathbb{1}_{\{i \in F_2\}}) y_i$$

And the closure of a linear subspace is again a linear subspace.
$\overline{U_2} \supseteq \{y_j\}$, we just take $a_i = \delta_{ij}$. (i.e. $a_i = 1$ for $i = j$, otherwise 0). Moreover, $\overline{U_2}$ is closed since it's a closure. Hence $\overline{U_2} \in \mathscr{C} \Rightarrow U_1 \subseteq \overline{U_2}$.
(**Step.2**) Pick $z \in U_2$. Then for any $C \in \mathscr{C}$, since $\{y_j\} \subseteq C$, and C is linear subspace $\Rightarrow z \in C$. Since C is an aribitray one in $\mathscr{C}$, we conclude that $U_2 \subseteq U_1$.
Since U_1 is closed, any limit point of U_2 is also in U_1.
Hence $\overline{U_2} \subseteq U_1$, finished the proof. □

Problem 9. Show that if the total measure equals 1, then $\|x\|_p$ is an increasing function of p. I.e. for $s \geq p$

$$\|x\|_p \leq \|x\|_s$$

Proof. For $s = p$ clearly the equility holds. We assume $s > p$. If the full measure $\mu(\Omega) = 1$, by Holder's Inequlity: for $r > 1$:

$$\left|\int_\Omega fg\right| \leq \left(\int_\Omega |f|^r\right)^{\frac{1}{r}} \left(\int_\Omega |g|^{\frac{r}{r-1}}\right)^{1-\frac{1}{r}} \tag{28}$$

We take $g = \mathbb{1}_\Omega \equiv 1$, $f = |x|^p$, $r = \frac{s}{p} > 1$, $\Rightarrow$

$$\begin{aligned} \left|\int_\Omega |x|^p\right| &\leq \left(\int_\Omega |x|^s\right)^{\frac{p}{s}} \left(\int_\Omega |\mathbb{1}_\Omega|^{\frac{s}{s-p}}\right)^{1-\frac{p}{s}} \\ \Rightarrow \left(\int_\Omega |x|^p\right)^{\frac{1}{p}} &\leq \left(\int_\Omega |x|^s\right)^{\frac{1}{s}} \end{aligned} \tag{29}$$

Since $\int_\Omega |\mathbb{1}_\Omega|^{\frac{s}{s-p}} = \mu(\Omega) = 1$. The proof is finished. □