Lecture 5

Zed

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1 Newton's Method for Solving Nonlinear Equations

We want to find roots for nonlinear equation $g(x) = 0$.

Algo. **Newton's Method**

- Initialize a starting point x_0.
- Update x by $x_{n+1} \leftarrow x_n - \frac{g(x_n)}{g'(x_n)}$, suppose $g'(x_n) \neq 0$.
- Stop when $\|g(x_n)\| \leq thres$ (small), $\|x_{n+1} - x_n\| \leq thres$ (small) or $n \geq K$ (fail to converge).

The Newton's method only converges in a local sense. Suppose the real zero is α, s.t. $g(\alpha) = 0$, then using taylor expansion at x_n:

$$\begin{aligned} 0 = g(\alpha) &= g(x_n) + g'(x_n)(\alpha - x_n) + \frac{1}{2}g''(\xi_n)(\alpha - x_n)^2 \\ \Rightarrow 0 &= \left(\frac{g(x_n)}{g'(x_n)} - x_n\right) + \alpha + \frac{g''(\xi_n)}{2g'(x_n)}(\alpha - x_n)^2 \end{aligned} \tag{1}$$

Hence exist bound M such that

$$|\alpha - x_{n+1}| = \frac{g''(\xi_n)}{2g'(x_n)}|\alpha - x_n|^2 \leq M|\alpha - x_n|^2$$

That is, if the error at n-th iteration $|\alpha - x_n|$ is already small, the next error at $(n+1)$-th iteration will be the square of it. Which implies a (locally) quadratic convergence rate.

2 Implicit Methods

2.1 Implicit Euler

2.2 Implicit Runge-Kutta Method