Functional Analysis Assignment IV

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Problem 1. Let X be a linear space over $\mathbb{R}$ with metric d satisfying

$$d(x+z, y+z) = d(x,y) \quad \text{and} \quad d(ax, ay) = |a|d(x,y) \quad \text{for all } x, y, z \in X, a \in \mathbb{R}$$

Define

$$\|x\| = d(x, 0)$$

Show that $\|\cdot\|$ is a norm on X.

Proof. It suffices to check defining properties of norm.

Positivity is clear. Since $d(x,0) \geq 0$, $d(x,0) = 0 \iff x = 0$.
Homogeneity: $\|ax\| = d(ax, 0) = |a|d(x,0) = |a|\|x\|$.
Triangle Ineq.:

$$\begin{aligned} \|x+y\| &= d(x+y, 0) = d(x+y, -y+y) \\ &= d(x, -y) \leq d(x,0) + d(0,-y) \\ &= d(x,0) + |-1| \cdot d(y,0) = \|x\| + \|y\| \end{aligned} \tag{1}$$

□

Problem 2. Given $f \in \mathcal{C}(a,b)$, define its support as

$$\text{supp} f := \overline{\{x | x \in (a,b), f(x) \neq 0\}}$$

i.e. the closure of the set where f is non-zero. Let X be the space of all real-valued, continous functions f with compact support and the norm is defined as

$$\|f\| := \max_{x \in (a,b)} |f(x)|$$

Show that X is not a complete normed linear space.

Proof. Let ϕ be continuous function, $\text{supp}\phi = [0,1]$. ϕ_k be the horizontal translation of ϕ, i.e.

$$\phi_k(x) = \phi(x-k)$$

It is easy to find that $\text{supp}\phi_k = [k, k+1]$.
Now define $f_n := \sum_{k=1}^{n} \frac{1}{k}\phi_k$, for any fixed k, clearly f_k is continuous. $\text{supp} f_k = [1, k+1]$ is compact. Moreover for $n \leq m$

$$\begin{aligned} \|f_n - f_m\| &= \max_{x \in [0,m]} \left| \sum_{k=n}^{m} \frac{1}{k}\phi_k \right| \\ &\leq \frac{m-n}{n} \max_k \{\max_x \phi_k(x)\} \\ &\xrightarrow{m,n\to\infty} 0 \end{aligned} \tag{2}$$

Hence $\{f_n\}$ is a Cauchy sequence. $f_n \to f = \sum_{n\geq 1} \frac{1}{n}\phi_n$. However $\text{supp} f = [1, \infty)$, is not compact. So X is not complete. □

Problem 3. Let $\Omega \in \mathbb{R}^n$ be a Lebesgue measurable set. Show that $\mathcal{L}^p(\Omega)$ $(1 < p < \infty)$ is a uniformly convex space.

Proof. We first state a lemma.
Lemma.(**Clarkson's First Ineq.**) For $p \geq 2$, there exists a constant $c > 0$, such that for all $a, b \in \mathbb{R}$:

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{|a|^p}{2} + \frac{|b|^p}{2} \tag{3}$$

Proof of Lemma. Define

$$\phi(x) = (x^2+1)^{\frac{p}{2}} - x^p - 1 \quad x \geq 0 \tag{4}$$

It can be seen easily $\phi(0) = 0$ and calculate first derivative yields $\phi \nearrow$ on $[0, \infty)$. Hence letting $x = \alpha/\beta$, $\alpha, \beta \in \mathbb{R}^+ \cup \{0\}$, $\phi(x) \geq 0$ implies

$$x^p + 1 \leq (x^2+1)^{\frac{p}{2}} \Rightarrow \alpha^p + \beta^p \leq (\alpha^2 + \beta^2)^{\frac{p}{2}} \tag{5}$$

Now let $\alpha := \left|\frac{a+b}{2}\right|$, $\beta := \left|\frac{a-b}{2}\right|$ for $a, b \in \mathbb{R}$. equation (5) implies

$$\begin{aligned}\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p &\leq \left[\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right]^{\frac{p}{2}} \\ &= \left(\frac{a^2+b^2}{2}\right)^{\frac{p}{2}} \leq \frac{|a|^p}{2} + \frac{|b|^p}{2}\end{aligned} \tag{6}$$

The last leq follows that $x \mapsto x^{\frac{p}{2}}$ is a convex function when $p \geq 2$. Which proves the lemma.
Now for any fixed $x \in \Omega$, let $a := f(x), b := g(x),\ f, g \in \mathcal{L}^p$; $\|f\| = \|g\| = 1$. Integrate for both sides on $\Omega \Rightarrow$

$$\begin{aligned}\left\|\frac{f+g}{2}\right\|_p^p &\leq \left(\frac{\|f\|_p + \|g\|_p}{2}\right)^p - \left\|\frac{f-g}{2}\right\|_p^p \\ \left\|\frac{f+g}{2}\right\|_p &\leq \left(1 - \left\|\frac{f-g}{2}\right\|_p^p\right)^{\frac{1}{p}} \\ &= 1 - \left(1 - \left(1 - \left\|\frac{f-g}{2}\right\|_p^p\right)^{\frac{1}{p}}\right) \\ &= 1 - \epsilon(\|f-g\|_p)\end{aligned} \tag{7}$$

Where $\epsilon(r) = 1 - (1 - r^p)^{1/p}$ is increasing with $\epsilon(0) = 0$. ☐

Problem 4. Show that a uniformly convex normed linear space must be strictly convex.

Proof. Consider $\|x\| = \|y\| = 1$, $0 < \lambda < 1$,

$$\begin{aligned}\|\lambda x + (1-\lambda)y\| &= \|\lambda(x+y) + (1-2\lambda)y\| \\ &\leq 2\lambda\left\|\frac{x+y}{2}\right\| + 1 - 2\lambda \\ &\leq 2\lambda(1 - \epsilon(\|x-y\|)) + 1 - 2\lambda \\ &= 1 - 2\lambda\epsilon(\|x-y\|) \\ &< 1 \quad \text{for } x \neq y\end{aligned} \tag{8}$$

Which finished the proof.

☐

Problem 5. Let H be a Hilbert space and A, B are linear maps $H \to H$. Suppose that A and B satisfy

$$\langle x, Ay \rangle = \langle x, By \rangle \quad \text{for all } x, y \in H$$

Show that $A = B$. If H is a convex Hilbert space and A, B satisfy

$$\langle x, Ax \rangle = \langle x, Bx \rangle \quad \text{for all } x \in H$$

Show that $A = B$. What can we say about A, B for real Hilbert spaces?

Proof. (a.)

$$\langle x, Ay - By \rangle = 0$$

(b.)

$$\langle x, Ax - Bx \rangle = 0$$

□

Problem 6. *Exercise.4* Show that every finite-dimensional subspaces of a normed linear space is closed. (Hint: Use the fact that all norms are equivalent on finite-dimensional spaces to show that every finite dimensional subspaces is complete).
Exercise.5 Show the norms of (a), (c), (d), (e) are not strictly subadditive. Show (b), (f) are not subadditive for $p = 1$. The normed space below are all complete.

a. The space of all vectors with infinite components,

$$\ell^\infty := \{x : x = (a_1, a_2, ...) \;\; a_j \text{ complex; } |a_j| \text{ bounded.}\}$$

$$\|x\|_\infty := \sup_j |a_j|$$

b. The space of all vectors with infinite components,

$$\ell^p := \left\{x : x = (a_1, a_2, ...) \;\; p \geq 1; \sum |a_j|^p < \infty.\right\}$$

$$\|x\|_p := \left(\sum |a_j|^p\right)^{\frac{1}{p}}$$

c. S is an abstract set, X the space of all complex-valued functions f that are bounded. The norm is

$$\|f\|_\infty := \sup_{s \in S} |f(s)|$$

d. Q a topological space, X the space of all complex valued, continuous, bounded functions f on Q. The norm is

$$\|f\| := \sup_{q \in Q} |f(q)|$$

e. Q topological space, X the space of all complex-valued, continuous functions f with compact support. The norm is

$$\|f\|_{max} := \max_{q \in Q} |f(q)|$$

f. D some domain in $\mathbb{R}^n$, X the space of continuous functions f with compact support. The norm is

$$\|f\|_p := \left(\int_D |f(x)|^p dx\right)^{\frac{1}{p}}$$

Proof. Exercise 4. Suppose $(X, \|\cdot\|)$ is a normed inear space. $Y \subset X$ is a subspace of X, $\dim Y = n < \infty$. By problem (2) in last homework, any two norms on finite dimensional linear space, in particular, Y, are equivalent.
Claim. Y is complete with respect to $\|\cdot\|$ associated with X.
Proof of Claim. Since all norms are equivalent on Y, it suffices to show that Y is complete with respect to

$$\|x\|_1 = \sum_{j=1}^n |x^{[j]}|, \quad \text{where } x = \sum_{j=1}^n x^{[j]} e_j$$

For Cauchy sequence $\{x_n\}$, $\forall\epsilon$, exist N, whenever $n,m>n \Rightarrow$

$$\|x_n - x_m\|_1 = \sum_{j=1}^{n} |x_n^{[j]} - x_m^{[j]}| < \epsilon \tag{9}$$

Hence $|x_n^{[j]} - x_m^{[j]}| < \epsilon$ for $j = 1,2,...,n$. So $\{x_n^{[j]}\}$ is a Cauchy sequence of real numbers. By completeness of $\mathbb{R}$, $x_n^{[j]} \to x_j$. Therefore we conclude that $x_n \to x = (x_1, ..., x_n) \in Y$, finished the proof.
Exercise 5.

a. Let $x = (1,1,0,0,...), y = (0,1,1,0,...)$, x and y are not multiples of each other, $x+y = (1,2,1,0,...)$. And we have
$$\|x\|_\infty + \|y\|_\infty = 2 = \|x+y\|_\infty$$

b. For $p = 1$, $\|x\| = \sum |a_j|$. Let $x = (1,0,0,...), y = (0,1,0,...), x+y = (1,1,0,...)$. Clearly $\|x+y\| = \|x\| + \|y\|$.

c, d, e. are similar. Let f be a complex-valued, continuous function supported on $[0,1]$, imaginary part is zero, real part is positive, attains maximum at $f(\frac{3}{4}) = 1$. g be another function of same flavour, but supported on $[\frac{1}{2}, \frac{3}{2}]$, attains maximum at $g(\frac{3}{4}) = 1$. Obviously, for the sup norm defined in c,d,e, we all have
$$\|f+g\|_\infty = (f+g)\left(\frac{3}{4}\right) = 2 = \|f\|_\infty + \|g\|_\infty$$

f. For $p = 1$ we have $\|f\|_{\mathcal{L}^1} = \int |f|$. Let f be continuous, positive function supported on $[0,1]$, integrates to 1, g be horizontal translation of f supported on $[2,3]$. We have $\|f+g\|_{\mathcal{L}^1} = 2 = \|f\|_{\mathcal{L}^1} + \|g\|_{\mathcal{L}^1}$.

□

Problem 7. *Exercise.1* Show that a norm that satisfies *parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

is induced by a scalar product.
Exercise.2 Show that the scalar product depends continuously on its factors; that is, if $x_n \to x, y_n \to y$ in the sense of $\|x_n - x\| \to 0, \|y_n - y\| \to 0$, then $\langle x_n, y_n\rangle \to \langle x, y\rangle$. (Use Schwarz ineq.)

Proof. We first do exercise 2 for the preparation of exercise 1.
By symmetry, we only prove continuity wrt one factor, say y.

$$\begin{aligned} |\langle x,y\rangle - \langle x,z\rangle| &= |\langle x, y-z\rangle| \\ &\leq \|x\|\,\|y-z\| \end{aligned} \tag{10}$$

Due to (Cauchy-Schwartz). Hence for fixed x, whenever $\|y-z\| < \epsilon$, we have $|\langle x,y\rangle - \langle x,z\rangle| \leq c_x\epsilon$, c_x is constant. Which proves the continuity. ■
Claim. The norm $\|\cdot\|$ that supports parallelogram identity is induced by

$$\langle x,y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

Now it suffices to use the parallelogram identity to show that $\langle\cdot,\cdot\rangle$ is an inner product.

Symmetry is clear, $\langle x,y\rangle = \langle y,x\rangle$ by its definition.

Positivity: $\langle x,x\rangle = \|2x\|^2/4 \geq 0$. And $\langle x,x\rangle = 0 \iff \|x\|^2 = 0 \iff x = 0$.

We now work on Bilinearly.
(*Step.1*) Show $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$. By the identity,

$$\begin{aligned} &\|(x+z)+y\|^2 + \|(x+z)-y\|^2 = 2\|x+z\|^2 + 2\|y\|^2 \\ &\|x+(y+z)\|^2 + \|x-(y+z)\|^2 = 2\|x\|^2 + 2\|y+z\|^2 \\ \Rightarrow &\|x+y+z\|^2 = \|x\|^2 + \|y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \frac{1}{2}\|x+z-y\|^2 - \frac{1}{2}\|x-y-z\| \end{aligned} \tag{11}$$

Similarly

$$\begin{aligned}
&\|(x-z)+y\|^2 + \|(x-z)-y\|^2 = 2\|x-z\|^2 + 2\|y\|^2 \\
&\|x+(y-z)\|^2 + \|x-(y-z)\|^2 = 2\|x\|^2 + 2\|y-z\|^2 \\
\Rightarrow &\|x+y-z\| = \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 - \frac{1}{2}\|x-z-y\|^2 - \frac{1}{2}\|x+z-y\|
\end{aligned} \tag{12}$$

Hence

$$\begin{aligned}
\langle x+y, z\rangle &= \frac{1}{4}\left(\|x+y+z\| - \|x+y-z\|\right) \\
&= \frac{1}{4}\left(\|x+z\|^2 + \|y+z\|^2 - \|x-z\|^2 - \|y-z\|^2\right) \\
&= \langle x, z\rangle + \langle y, z\rangle
\end{aligned} \tag{13}$$

(*Step.2*) We show *Homogeneity* for integers. Take $y = x$, we obtain $\langle 2x, z\rangle = 2\langle x, z\rangle$. By definition it is clear that $\langle -x, z\rangle = -\langle x, z\rangle$. So by induction we conclude that $\langle kx, z\rangle = k\langle x, z\rangle$ for all $k \in \mathbb{N}$.
Now consider $\forall \frac{p}{q} \in \mathbb{Q}$,

$$q\langle \frac{p}{q}x, y\rangle = \langle px, y\rangle = p\langle x, y\rangle \Rightarrow \langle \frac{p}{q}x, y\rangle = \frac{p}{q}\langle x, y\rangle \tag{14}$$

(*Step.3*) For $\lambda \in \mathbb{R}$, we can find a series of rational $r_n \to \lambda$, for each r_n we have $\langle r_n x, y\rangle = r_n\langle x, y\rangle$. Take limit on both sides, and by result of exercise 2: $\langle \cdot, y\rangle$ is continuous. Hence $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.
(*Bilinearity*) Combining step 1 and 2 together we get the linearity w.r.t. x. Argue in the same way it is easy to obtain linearity w.r.t. y. Which finished the proof. □

Problem 8. Show ℓ^2 is complete. Where

$$\ell^2 := \left\{x = (a^{[1]}, a^{[2]}, ...), \sum |a^{[j]}|^2 < \infty\right\}$$

$$\|x\|_2 = \left(\sum |a^{[j]}|^2\right)^{\frac{1}{2}}$$

Proof. Suppose $\{x_n\} = \{(a_n^{[1]}, a_n^{[2]}, ...)\}$ is a Cauchy, that is, for fixed ϵ, exists N, whenever $m, n > N$

$$\|x_n - x_m\|_2 = \left(\sum_j |a_n^{[j]} - a_m^{[j]}|^2\right)^{1/2} < \epsilon \tag{15}$$

The elements in summation are positive, hence $|a_n^{[j]} - a_m^{[j]}|^2 < \epsilon^2$, so $\{a_n^{[j]}\}$ are also Cauchy for any $j \geq 1$. Denote $a_n^{[j]} \to a_j$, we have $x_n \to x = (a_1, a_2, ...)$ by letting $m \to \infty$ in equation (14). Moreover,

$$\begin{aligned}
\|x\|_2 &= \|x - x_n + x_n\|_2 \\
&\leq \|x - x_n\|_2 + \|x_n\|_2 \\
&\leq \epsilon + \|x_n\|_2
\end{aligned} \tag{16}$$

So $\|x\|_2^2 \leq (\epsilon + \|x_n\|)^2 < \infty$, since $\|x_n\|_2^2 < \infty$, implies that $x \in \ell^2$. □

Problem 9. (*Lemma.5*)

i. The nullspace of a linear functional that is not $\equiv 0$ is a linear subspace of codimension 1.

ii. If two linear functionals ℓ and m have the same nullspace, then they are constant multiples of each other.

$$\ell = cm$$

iii. The nullspace of a linear functional that is bounded in the sense of

$$|\ell(x)| \leq c\|x\|$$

is a closed subspace.

Proof. (*i.*) Denote $\ell(x) : X \to \mathbb{R}$ the linear functional. N_ℓ the null space of it. ℓ is not $\equiv 0 \Rightarrow \exists x_0 \in X$, such that $\ell(x_0) \neq 0$.
Consider $[y] = y + N_\ell \in X/N_\ell$, $y_1 \sim y_2$ if $\ell(y_1) = \ell(y_2)$. Note that

$$\ell\left(\frac{\ell(y)}{\ell(x_0)}x_0 - y\right) = \frac{\ell(y)}{\ell(x_0)}\ell(x_0) - \ell(y) = 0 \tag{17}$$

We have

$$[y] = \frac{\ell(y)}{\ell(x_0)}\left(x_0 + N_\ell\right) \tag{18}$$

So $\mathrm{codim} N_\ell = \dim X/N_\ell = \dim\{x_0 + N_\ell\} = 1$.
(*ii*). Denote these two functionals ℓ and m. If $\ell \equiv m \equiv 0$, the conclusion is trivial. Otherwise it is a direct result of (*i.*).
To argue, suppose $\ell(x_0) \neq 0$, then for all $x \in X$,

$$\ell\left(\frac{\ell(x)}{\ell(x_0)}x_0 - x\right) = 0 = m\left(\frac{\ell(x)}{\ell(x_0)}x_0 - x\right) \tag{19}$$

Since they have same null space. $\Rightarrow$

$$\frac{\ell(x)}{\ell(x_0)}m(x_0) - m(x) = 0 \Rightarrow \ell(x) = \frac{\ell(x_0)}{m(x_0)} \cdot m(x) \tag{20}$$

(*iii*). Let $\{x_n\}$ be a convergent sequence in $N_\ell \subset X$, $x_n \to x \in X$ in the sense that $\|x_n - x\| \to 0$ as $n \to \infty$.

$$|\ell(x) - \ell(x_n)| = |\ell(x - x_n)| \leq c\,\|x - x_n\| \to 0 \tag{21}$$

Hence $\ell(x) = 0$, implies that $x \in N_\ell$. Hence N_ℓ is closed. $\square$