Functional Analysis Assignment VII

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Problem 1. Show that both Laplace equation

$$\Delta u = \sum_{i=0}^{n} \partial_{x_i x_i} u = 0$$

And wave equation

$$\Box u = \partial_{x_0 x_0} u - \sum_{i=1}^{n} \partial_{x_i x_i} u = 0$$

can be written into the positive symmetric system

$$L\boldsymbol{U} = \sum_{j=0}^{n} A_j \partial_{x_j} \boldsymbol{U} + B\boldsymbol{U} = f$$

Where $A_j (j = 0, ..., n)$ are symmetric matrix functions and B is a square matrix satisfying

$$B + B^\top - \sum_{j=0}^{n} \partial_{x_j} A_j > kI \quad \text{For a constant } k > 0.$$

Proof. □

Problem 2. Show that a weakly sequentially compact set is bounded.

Proof. Let $K \subset X$ be a subset of normed space. Let $\{x_n\} \subset K$ and $x_n \rightharpoonup x \in K$. Then by the third part of *Principle o f uniform boundedness*: there exists constant $c > 0$

$$\|x_n\| \leq c \quad (\dagger)$$

Assume K is unbounded, then we can pick $\{x_n\} \in K$ such that $\|x_n\| \geq n$. Since K is weakly sequentially compact, there exists a weakly convergent subsequence $\{x_{n_k}\} \subset \{x_n\}$, and

$$\|x_{n_k}\| \geq n_k$$

Contradicts (†). Hence K is bounded. □

Problem 3. Show that if the sequence $\{u_n\}$ is weak* convergent to u,

$$\|u\| \leq \liminf \|u_n\|$$

Proof. Since $u_n \rightharpoonup u$, we have $|u(x)| = \lim_{n\to\infty} |u_n(x)|$, for $\forall x \in X$.
By definiton, we can find $x_0 \in X$, such that $|u(x_0)| = \|u\|$ and $\|x_0\| = 1$. Therefore

$$\begin{aligned} \|u\| = |u(x_0)| &= \lim_{n\to\infty} |u_n(x_0)| \\ &\leq \liminf_{n\to\infty} \|u_n\| \, \|x_0\| \\ &= \liminf_{n\to\infty} \|u_n\| \end{aligned} \tag{1}$$

Which finished the proof. □

Problem 4. Show that the norm of bounded linear map is sub-additive, that is

$$\|\boldsymbol{M}+\boldsymbol{K}\| \leq \|\boldsymbol{M}\|+\|\boldsymbol{K}\|$$

Proof. By definition

$$\begin{aligned}\|\boldsymbol{M}+\boldsymbol{K}\| &= \sup_{\|x\|=1}\|(\boldsymbol{M}+\boldsymbol{K})x\| \\ &\leq \sup_{\|x\|=1}(\|\boldsymbol{M}x\|+\|\boldsymbol{K}x\|) \\ &\leq \sup_{\|x\|=1}\|\boldsymbol{M}x\|+\sup_{\|x\|=1}\|\boldsymbol{K}x\| = \|\boldsymbol{M}\|+\|\boldsymbol{K}\|\end{aligned} \tag{2}$$

□

Problem 5. Let X and U be Banach spaces, U reflexive. Let $\boldsymbol{M}$ be a bounded linear map: $X \to U$. Let x_n be a sequence in X, $x_n \rightharpoonup x$. Then, $\boldsymbol{M}x_n \rightharpoonup \boldsymbol{M}x$.

Proof. It suffice to show that $\lim_{n\to\infty}\xi(\boldsymbol{M}x_n)=\xi(\boldsymbol{M}x)$ for all $\xi \in U'$
For any $\xi \in U'$, we have

$$\begin{aligned}\lim_{n\to\infty}\xi(\boldsymbol{M}x_n)-\xi(\boldsymbol{M}x) &= \lim_{n\to\infty}\xi(\boldsymbol{M}x_n-\boldsymbol{M}x) \\ &= \lim_{n\to\infty}\xi(\boldsymbol{M}(x_n-x)) \\ &= \lim_{n\to\infty}(\xi\circ\boldsymbol{M})(x_n-x) \quad (\triangle)\end{aligned} \tag{3}$$

The quantity $(\triangle)$ is zero. Because $(\xi\circ\boldsymbol{M}) := \ell \in X'$, and due to the weak convergence of $\{x_n\}$, $\ell(x)=\lim_{n\to\infty}\ell(x_n)$. Hence, we have $\lim_{n\to\infty}\xi(\boldsymbol{M}x_n)=\xi(\boldsymbol{M}x)$ for all $\xi\in U'$. Finished the proof. □

Problem 6. If $\boldsymbol{I}$ is identity map: $X\to X$, show $\boldsymbol{I}'$ is identity map: $X'\to X'$.

Proof. Since $\boldsymbol{I}: X\to X$, $\boldsymbol{I}(x)=x$ for all $x\in X$. By definition of transpose, $\boldsymbol{I}': X'\to X'$ such that for any $\ell\in X'$

$$\boldsymbol{I}'\ell(x)=\ell(\boldsymbol{I}x)=\ell(x) \tag{4}$$

Therefore $\boldsymbol{I}'\ell=\ell$, which implies that $\boldsymbol{I}'$ is identity map on X'. □

Problem 7. $\boldsymbol{M}^*$ is adjoint operator on Hilbert space, show thm.5 is valid for it, that is

1. $\boldsymbol{M}^*$ is bounded, and $\|\boldsymbol{M}^*\|=\|\boldsymbol{X}\|$.
2. The nullspace of $\boldsymbol{M}^*$ is the annihilator of the range of $\boldsymbol{M}$. That is, $N_{\boldsymbol{M}^*}=R_{\boldsymbol{M}}^{\perp}$.
3. The nullspace of $\boldsymbol{M}$ is the annihilator of the range of $\boldsymbol{M}^*$. $N_{\boldsymbol{M}}=R_{\boldsymbol{M}^*}^{\perp}$.
4. $(\boldsymbol{M}+\boldsymbol{N})^*=\boldsymbol{M}^*+\boldsymbol{N}^*$.

Proof. (1) Consider linear functional with respect to $\ell_y(x)=\langle\boldsymbol{M}x,y\rangle$, for fixed $y\in H$. By (**Riesz**), there exists unique $z\in H$, such that

$$\ell_y(x)=\langle\boldsymbol{M}x,y\rangle=\langle x,z\rangle$$

Now define $M^*: H\to H$ as $\boldsymbol{M}^*y=z$ $(\triangle)$. This exactly is the defining property of adjoint of bounded linear map $\boldsymbol{M}$. So it suffices to verify the theorem on $\boldsymbol{M}^*$ defined by $(\triangle)$.
For any $x\in H$,

$$\begin{aligned}\|\boldsymbol{M}^*x\|^2 &= |\langle\boldsymbol{M}^*x,\boldsymbol{M}^*x\rangle| = |\langle\boldsymbol{M}(\boldsymbol{M}^*x),x\rangle| \\ &= |\ell_x(\boldsymbol{M}^*x)| \leq \|\ell_x\|\,\|\boldsymbol{M}^*x\| \\ &\Rightarrow \|\boldsymbol{M}^*x\| \leq \|\ell_x\| \leq \|\boldsymbol{M}\|\,\|x\|\end{aligned} \tag{5}$$

Hence by definition, we have $\|M^*\| \leq \|M\|$. Since $\boldsymbol{M}$ is bounded linear map, $\boldsymbol{M}^*$ too.
Now we show $\|\boldsymbol{M}\| = \|\boldsymbol{M}^*\|$. Consider

$$|\langle \boldsymbol{M}x, y\rangle| = |\langle x, \boldsymbol{M}^* y\rangle| \leq \|x\| \, \|M^*\| \, \|y\| \tag{6}$$

So

$$\boldsymbol{M}^* = \sup_{\|x\|=\|y\|=1} |\langle \boldsymbol{M}x, y\rangle| = \sup_{\|x\|=1} \|\boldsymbol{M}x\| = \|\boldsymbol{M}\|$$

Finished the proof.
(2) $\forall y \in N_{\boldsymbol{M}^*}$, $\langle x, \boldsymbol{M}^* y\rangle = 0$ for all $x \in H$. Hence $\langle \boldsymbol{M}x, y\rangle = 0 \Rightarrow y \in R_{\boldsymbol{M}}^{\perp}$.
On the other hand, $\forall y \in R_{\boldsymbol{M}}^{\perp}$, $0 = \langle \boldsymbol{M}x, y\rangle = \langle x, \boldsymbol{M}^* y\rangle$. $\Rightarrow y \in N_{\boldsymbol{M}^*}$. We conclude that $N_{\boldsymbol{M}^*} = R_{\boldsymbol{M}}^{\perp}$.
(3) $\forall x \in N_{\boldsymbol{M}}$, $\langle \boldsymbol{M}x, y\rangle = 0$ for all $y \in H$. $\Rightarrow \langle x, \boldsymbol{M}^* y\rangle = 0$. So $N_{\boldsymbol{M}} \subseteq R_{\boldsymbol{M}^*}^{\perp}$.
On the other hand, $\forall x \in R_{\boldsymbol{M}^*}^{\perp}$, $0 = \langle x, \boldsymbol{M}^* y\rangle = \langle \boldsymbol{M}x, y\rangle$. And $\|\boldsymbol{M}x\| = \sup_{\|y\|=1} |\langle \boldsymbol{M}x, y\rangle|$. $\Rightarrow \boldsymbol{M}x = 0$, $x \in N_{\boldsymbol{M}}$. So $N_{\boldsymbol{M}} \supseteq R_{\boldsymbol{M}^*}^{\perp}$. We conclude that $N_{\boldsymbol{M}} = R_{\boldsymbol{M}^*}^{\perp}$.
(4) By bilinearity of inner product,

$$\begin{aligned}\langle (\boldsymbol{M}+\boldsymbol{N})x, y\rangle &= \langle \boldsymbol{M}x, y\rangle + \langle \boldsymbol{N}x, y\rangle \\ &= \langle x, \boldsymbol{M}^* y\rangle + \langle x, \boldsymbol{N}^* y\rangle \\ &= \langle x, (\boldsymbol{M}^* + \boldsymbol{N}^*)y\rangle\end{aligned} \tag{7}$$

Therefore, we have $(\boldsymbol{M}+\boldsymbol{N})^* = \boldsymbol{M}^* + \boldsymbol{N}^*$. □

Problem 8. (*Ex.6*) Show that if $\text{w-lim}_{n\to\infty} \boldsymbol{M}_n = \boldsymbol{M}$, then $\text{w-lim}_{n\to\infty} \boldsymbol{M}_n' = \boldsymbol{M}'$, provided that X is reflexive.
(*Ex.7*) (*Thm.6*) Let X, U be Banach spaces, $\boldsymbol{M}_n$ a sequence of linear maps: $X \to U$, uniformly bounded in norm:

$$|\boldsymbol{M}_n| \leq c \quad \text{for all } n.$$

Suppose further that $\text{s-lim}_{n\to\infty} \boldsymbol{M}_n x$ exists for a *dense set* of x in X. Then $\{\boldsymbol{M}_n\}$ converges strongly. I.e. the $\text{s-lim}_{n\to\infty} \boldsymbol{M}_n x$ exists for all $x \in X$. Show the thm above and formulate analogous theorem for weak convergence.

Proof. $\boldsymbol{M} : X \to U$ weakly converges. By definition, $\forall x \in X$ and $\ell \in U'$, we have

$$\ell(\boldsymbol{M}x) = \lim_{n\to\infty} \ell(\boldsymbol{M}_n x)$$

Since X is reflexive, and by definition of transpose,

$$\ell(\boldsymbol{M}_n x) = (\boldsymbol{M}_n' \ell)x$$

Since X is reflexive, $\text{w-lim}_{n\to\infty} \boldsymbol{M}'$ exists. Denote $\boldsymbol{M}_n' \rightharpoonup \boldsymbol{N}'$. For any $x \in X$, $\ell \in U$. So we have

$$(\boldsymbol{N}'\ell)x = \lim_{n\to\infty} (\boldsymbol{M}_n'\ell)x = \lim_{n\to\infty} \ell(\boldsymbol{M}_n x) = \ell(\boldsymbol{M}x) = (\boldsymbol{M}'\ell)x \tag{8}$$

Hence $\boldsymbol{N}' = \boldsymbol{M}'$, i.e. $\boldsymbol{M}_n' \rightharpoonup \boldsymbol{M}'$. □

Proof. (1) Denote E the set in which $\text{s-lim}_{n\to\infty} \boldsymbol{M}_n x$ exists. E is dense in X.
$\forall \epsilon > 0$, for any $x \in X$, since E dense, $\exists \tilde{x} \in E$, s.t. $\|x - \tilde{x}\| < \epsilon/4c$.
Since $\text{s-lim}_{n\to\infty} \boldsymbol{M}_n x$ exists in E, it is Cauchy sequence. $\exists N > 0$, for all $n, m > N$ we have

$$\|(\boldsymbol{M}_n - \boldsymbol{M}_m)\tilde{x}\| < \frac{\epsilon}{2}$$

So

$$\begin{aligned}\|(\boldsymbol{M}_n - \boldsymbol{M}_m)x\| &\leq \|(\boldsymbol{M}_n - \boldsymbol{M}_m)\tilde{x}\| + \|(\boldsymbol{M}_n - \boldsymbol{M}_m)(x - \tilde{x})\| \\ &\leq \frac{\epsilon}{2} + \|\boldsymbol{M}_n - \boldsymbol{M}_m\| \, \|x - \tilde{x}\| \\ &\leq \frac{\epsilon}{2} + 2c \cdot \frac{\epsilon}{4c} \\ &= \epsilon\end{aligned} \tag{9}$$

So $\{\boldsymbol{M}_n x\}$ is Cauchy sequence for all $x \in X$, which stongly convergent. Finished the proof.

(b) Analogous theorem: X, U Banach spaces. $\boldsymbol{M}_n : X \to U$, are uniformly bounded in norm, i.e. $\|\boldsymbol{M}_n\| \leq c$ for all n. And w-lim $\boldsymbol{M}_n x$ exists for $x \in E$ which is dense in X. Then, $\boldsymbol{M}_n$ converges weakly, i.e. w-lim $\boldsymbol{M}_n x$ exists for all $x \in X$.

Proof. For any fixed $\ell \in U'$, $\forall \epsilon > 0$, any $x \in X$, there exists $\tilde{x} \in E$, such that

$$\|x - \tilde{x}\| < \frac{\epsilon}{4c\,\|\ell\|}$$

Since $\boldsymbol{M}_n x$ weakly convergent on E, $\exists N$, for any $n, m \geq N$:

$$\|\ell(\boldsymbol{M}_n \tilde{x}) - \ell(\boldsymbol{M}_m \tilde{x})\| < \frac{\epsilon}{2}$$

Hence

$$\begin{aligned}
\|\ell(\boldsymbol{M}_n x) - \ell(\boldsymbol{M}_m x)\| &\leq \|\ell((\boldsymbol{M}_n - \boldsymbol{M}_m)\tilde{x})\| + \|\ell((\boldsymbol{M}_n - \boldsymbol{M}_m)(x - \tilde{x}))\| \\
&\leq \frac{\epsilon}{2} + \|\ell\|\,\|\boldsymbol{M}_n - \boldsymbol{M}_m\|\,\|x - \tilde{x}\| \\
&\leq \frac{\epsilon}{2} + \|\ell\| \cdot 2c \cdot \frac{\epsilon}{4c\,\|\ell\|} \\
&= \epsilon
\end{aligned} \tag{10}$$

So $\{\ell(\boldsymbol{M}_n x)\}$ is Cauchy sequence for all $x \in X$ and $\ell \in U'$, implies that $\{\boldsymbol{M}_n\}$ converges weakly. □

Problem 9. Show that in a complex Hilbert space $(\boldsymbol{N}\boldsymbol{M})^* = \boldsymbol{M}^*\boldsymbol{N}^*$

Proof. It is clear by definition.

$$\langle \boldsymbol{N}\boldsymbol{M}x, y\rangle = \langle \boldsymbol{M}x, \boldsymbol{N}^*y\rangle = \langle x, \boldsymbol{M}^*\boldsymbol{N}^*y\rangle \tag{11}$$

Hence $(\boldsymbol{N}\boldsymbol{M})^* = \boldsymbol{M}^*\boldsymbol{N}^*$. □
