Numerical Solutions for DEs HW1

YANG, Ze (5131209043)

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A Note to TA:
Hi, this is the senior student from Antai College who did not register for this course. I would like to do all the assignments for practice, but feel free to just skip my homework if you don't have time. Thank you again for allowing me to access the assignments and other class material! :)
- Ze

Problem 1. (1.1) Apply the method to Theorems 1.1 and 1.2 to show the convergence of the implicit midpoint rule (1.12) and of the theta method (1.13)

Proof. (**a.** *The implicit midpoint rule*) we let $\bar{t} = t_n + \frac{h}{2} = \frac{t_n + t_{n+1}}{2}$, $\boldsymbol{r}_n$ be the truncation error. And firstly we show that this method is of order 2. By definition

$$\begin{aligned} \boldsymbol{y}(t_{n+1}) &= \boldsymbol{y}(t_n) + h\boldsymbol{f}(\bar{t}, \tfrac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2}) + h\boldsymbol{r}_n \\ &= \boldsymbol{y}(t_n) + h\boldsymbol{f}(\bar{t}, \boldsymbol{y}(\bar{t})) + h\boldsymbol{u}_n + h\boldsymbol{r}_n \end{aligned} \tag{1}$$

in which we let $\|\boldsymbol{u}_n\| = \left\| \boldsymbol{f}(\bar{t}, \frac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2}) - \boldsymbol{f}(\bar{t}, \boldsymbol{y}(\bar{t})) \right\|$. Since $\boldsymbol{f}$ is Lipschitz in $\boldsymbol{y}$:

$$\|\boldsymbol{u}_n\| \le L \left\| \tfrac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2} - \boldsymbol{y}(\bar{t}) \right\| = \tfrac{L}{2} \left\| \boldsymbol{y}(t_n) + \boldsymbol{y}(t_{n+1}) - 2\boldsymbol{y}(\bar{t}) \right\| \tag{2}$$

With Taylor expansion at $\bar{t}$,

$$\|\boldsymbol{u}_n\| \le \tfrac{L}{2} \left\| (\boldsymbol{y}(\bar{t}) - \tfrac{h}{2}\boldsymbol{y}'(\bar{t})) + (\boldsymbol{y}(\bar{t}) + \tfrac{h}{2}\boldsymbol{y}'(\bar{t})) - 2\boldsymbol{y}(\bar{t}) + O(h^2) \right\| = O(h^2) \tag{3}$$

Hence use Taylor expansion at $\bar{t}$ for equation (1) $\Rightarrow$ (write $\boldsymbol{y} = \boldsymbol{y}(\bar{t})$ for simplicity)

$$\begin{aligned} h\boldsymbol{r}_n &= \boldsymbol{y}(t_{n+1}) - \boldsymbol{y}(t_n) - h\boldsymbol{y}'(\bar{t}) + O(h^3) \\ &= \boldsymbol{y} + \tfrac{h}{2}\boldsymbol{y}' + \tfrac{h^2}{8}\boldsymbol{y}'' - (\boldsymbol{y} - \tfrac{h}{2}\boldsymbol{y}' + \tfrac{h^2}{8}\boldsymbol{y}'') - h\boldsymbol{y}' + O(h^3) \\ &= O(h^3) \end{aligned} \tag{4}$$

So the method is of order 2. Now we let $\boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{y}(t_n)$, we substract equation (1) from the method, and using Lipschitz condition:

$$\begin{aligned} \boldsymbol{e}_{n+1} &= \boldsymbol{e}_n + h\left[\boldsymbol{f}(\bar{t}, \tfrac{\boldsymbol{y}_n + \boldsymbol{y}_{n+1}}{2}) - \boldsymbol{f}(\bar{t}, \tfrac{\boldsymbol{y}(t_n)+\boldsymbol{y}(t_{n+1})}{2}) \right] + O(h^3) \\ \|\boldsymbol{e}_{n+1}\| &\le \|\boldsymbol{e}_n\| + \tfrac{hL}{2} \|\boldsymbol{y}_n + \boldsymbol{y}_{n+1} - \boldsymbol{y}(t_n) - \boldsymbol{y}(t_{n+1})\| + O(h^3) \\ &\le \|\boldsymbol{e}_n\| + \tfrac{hL}{2}(\|\boldsymbol{e}_n\| + \|\boldsymbol{e}_{n+1}\|) + O(h^3) \\ \Rightarrow \|\boldsymbol{e}_{n+1}\| &\le \frac{1 + \frac{hL}{2}}{1 - \frac{hL}{2}} \|\boldsymbol{e}_n\| + \frac{c}{1 - \frac{hL}{2}} h^3 \quad \text{(same as the bound for trapezoid method)} \\ &\le \frac{ch^2}{L} \exp\left(\frac{nhL}{1 - \frac{hL}{2}} \right) \end{aligned} \tag{5}$$

We have $\lim_{h \to 0} \|\boldsymbol{e}_n\| = 0$. By definition the method is convergent in $[0, nh]$. $\square$

(**b.** *the θ method*) We have known that the theta method is at least order 1. So

$$\boldsymbol{y}(t_{n+1}) = \boldsymbol{y}(t_n) + h\left[\theta \boldsymbol{f}(t_n, \boldsymbol{y}(t_n)) + (1-\theta)\boldsymbol{f}(t_{n+1}, \boldsymbol{y}(t_{n+1}))\right] + O(h^2) \tag{6}$$

Substract it from the numerical formula, and employing Lipschitz condition:

$$\begin{aligned} \|\boldsymbol{e}_{n+1}\| &\leq \|\boldsymbol{e}_n\| + hL\theta \|\boldsymbol{e}_n\| + hL(1-\theta)\|\boldsymbol{e}_{n+1}\| + O(h^2) \\ \Rightarrow \|\boldsymbol{e}_{n+1}\| &\leq \frac{1+hL\theta}{1-hL(1-\theta)}\|\boldsymbol{e}_n\| + \frac{c}{1-hL(1-\theta)}h^2 \\ &\leq \frac{c}{L}\left[\left(\frac{1+hL\theta}{1-hL(1-\theta)}\right)^n - 1\right]h \\ &\leq \frac{ch}{L}\exp\left(\frac{nhL}{1-hL(1-\theta)}\right) \end{aligned} \tag{7}$$

We have $\lim_{h\to 0}\|\boldsymbol{e}_n\| = 0$. By definition the method is convergent in $[0, nh]$. □ □

Problem 2. (1.2) The linear system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$, where $\boldsymbol{A}$ is a symmetric matrix, is solved by Euler's method.

a. Letting $\boldsymbol{e}_n = \boldsymbol{y}_n - \boldsymbol{y}(nh)$, $n = 0, 1, ...$, show that

$$\|\boldsymbol{e}_n\|_2 \leq \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} |(1+h\lambda)^n - e^{nh\lambda}|$$

where $\sigma(\boldsymbol{A})$ is the set of eigenvalues of $\boldsymbol{A}$ and $\|\cdot\|_2$ the Euclidean matrix norm.

b. Demonstrate that for every $-1 \ll x \leq 0$ and $n = 0, 1, ...$ it is true that

$$e^{nx} - \frac{1}{2}nx^2 e^{(n-1)x} \leq (1+x)^n \leq e^{nx}$$

c. Suppose that the maximal eigenvalue of $\boldsymbol{A}$ is $\lambda_{max} < 0$. Prove that, as $h \to 0$, and $nh \to t \in [0, t^*]$,

$$\|\boldsymbol{e}_n\|_2 \leq \frac{1}{2}t\lambda_{max}^2 e^{\lambda_{max}t}\|\boldsymbol{y}_0\|_2 h \leq \frac{1}{2}t^*\lambda_{max}^2\|\boldsymbol{y}_0\|_2 h$$

d. Compare the order of magnitude of this bound with the upper bound from theorem 1.1 in the case

$$\boldsymbol{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \quad t^* = 10$$

Proof. (a.) The exact solution of the system is $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0$. The euler method gives $\boldsymbol{y}_n = (\boldsymbol{I}+h\boldsymbol{A})\boldsymbol{y}_{n-1}$, so $\boldsymbol{y}_n = (\boldsymbol{I}+h\boldsymbol{A})^n\boldsymbol{y}_0$. By its definition $\boldsymbol{e}_n = [(\boldsymbol{I}+h\boldsymbol{A})^n - e^{nh\boldsymbol{A}}]\boldsymbol{y}_0$. Take the spectral norm we have

$$\|\boldsymbol{e}_n\|_2 \leq \|\boldsymbol{y}_0\|_2 \left\|(\boldsymbol{I}+h\boldsymbol{A})^n - e^{nh\boldsymbol{A}}\right\|_2 = \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})}\left|(1+h\lambda)^n - e^{nh\lambda}\right| \tag{8}$$

(b) For $x \leq 0$: $f'(x) = (e^x - x - 1)' = e^x - 1 \leq 0$, hence $f(x) \searrow$, $f(x) \geq f(0) = 0 \Rightarrow e^x \geq 1 + x$ *(we call it (1))*. At the same time $g'(x) = (e^x - \frac{x^2}{2} - x - 1)' = e^x - x - 1 \geq 0$, hence $g(x) \nearrow$, $g(x) \leq g(0) = 0 \Rightarrow e^x \leq 1 + x + \frac{x^2}{2}$, or $e^x - \frac{x^2}{2} \leq 1 + x$ *(we call it (2))*.

Let $h(\epsilon) = ((1-\epsilon)-\epsilon)^n + n(1-\epsilon)^{n-1}\epsilon - (1-\epsilon)^n$ for $1 \gg \epsilon \geq 0$, $h'(\epsilon) = n - n(1-\epsilon)^{n-1} \geq 0$, hence $h(\epsilon) \geq h(0) = 0 \Rightarrow ((1-\epsilon)-\epsilon)^n \geq (1-\epsilon)^n - n(1-\epsilon)^{n-1}\epsilon$. Now suppose $a \to 1^-$, $b \to 0^+$, consider $(a-b)^n \sim ((1-\epsilon)-\epsilon)^n$ with $a \sim 1-\epsilon, b \sim \epsilon$, we obtain $(a-b)^n \geq a^n - na^{n-1}b$. *(we call it (3))*.

Now let $a = e^x$, $b = \frac{x^2}{2}$, since $1 \ll x \leq 0$, $x \to 0^-$, $|b|, |a-1|$ are small. We can employ *(3)*:

$$e^{nx} - \frac{nx^2}{2}e^{(n-1)x} \leq (e^x - \tfrac{x^2}{2})^n \leq (1+x)^n \leq e^{nx}$$

where the first leq is due to *(3)*, the second is due to *(2)*, and the third one is due to *(1)*.

(c) Since $\lambda_{max} < 0, t \in [0, t^*]$, we have $te^{\lambda_{max}t} < t \leq t^*$. Hence $\frac{1}{2}te^{\lambda_{max}t}(\lambda_{max}^2 \|\boldsymbol{y}_0\|_2 h) \leq \frac{1}{2}t^*(\lambda_{max}^2 \|\boldsymbol{y}_0\|_2 h)$. And by (b): $-\frac{1}{2}nx^2e^{(n-1)x} \leq (1+x)^n - e^{nx} \leq 0 \Rightarrow \frac{1}{2}nx^2e^{(n-1)x} \geq |(1+x)^n - e^{nx}| \geq 0$. So

$$\|\boldsymbol{e}_n\|_2 \leq \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} \left|(1+h\lambda)^n - e^{nh\lambda}\right| \leq \|\boldsymbol{y}_0\|_2 \max_{\lambda\in\sigma(\boldsymbol{A})} \frac{1}{2}nh^2\lambda^2e^{(n-1)h\lambda} \quad (*) \tag{9}$$

with $h \to 0$, $nh = t = O(1)$, $(n-1)h \to t$. Hence

$$\|\boldsymbol{e}_n\|_2 \leq (*) = \frac{1}{2}t\|\boldsymbol{y}_0\|_2 h \max_{\lambda\in\sigma(\boldsymbol{A})} \lambda^2e^{\lambda t} = \frac{1}{2}t\|\boldsymbol{y}_0\|_2 h\lambda_{max}^2e^{t\lambda_{max}} \tag{10}$$

Becasue $\lambda^2e^{\lambda t}$ is an increasing function with respect to λ. Finished the proof.

(d) In this case $\lambda_{max} = -1$, $t^* = 10$. This bound:

$$\|\boldsymbol{e}_n\|_2 \leq \frac{1}{2}t^*\lambda_{max}^2\|\boldsymbol{y}_0\|_2 h = 5\|\boldsymbol{y}_0\|_2 h$$

Is linear in t^* and quadratic in λ. The bound in (1.1)

$$\|\boldsymbol{e}_n\| \leq \frac{c}{\lambda}(e^{t^*\lambda} - 1)h \approx \frac{1}{1}\|\boldsymbol{y}_0\|(e^{10} - 1)h \approx 22026\|\boldsymbol{y}_0\|h$$

grows exponentially with λt^*.

□

Problem 3. (1.3) We solve the scalar linear system $y' = ay, y(0) = 1$.

a. Show that the 'continuous output' method

$$u(t) = \frac{1 + \frac{1}{2}a(t-nh)}{1 - \frac{1}{2}a(t-nh)}y_n, \qquad nh \leq t \leq (n+1)h, \quad n = 0, 1, ...$$

is consistent with the values of y_n and y_{n+1} which are obtained by the trapezoidal rule.

b. Demonstrate that u obeys the perturbed ODE

$$u'(t) = au(t) + \frac{\frac{1}{4}a^3(t-nh)^2}{[1 - \frac{1}{2}a(t-nh)]^2}y_n \qquad t \in [nh, (n+1)h]$$

with initial condition $u(nh) = y_n$. Thus prove that

$$u((n+1)h) = e^{ha}\left[1 + \frac{1}{4}a^3\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1 - \frac{1}{2}a\tau)^2}\right]y_n$$

c. Let $e_n = y_n - y(nh)$, $n = 0, 1, ...$ show that

$$e_{n+1} = e^{ha}\left[1 + \frac{1}{4}a^3\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1 - \frac{1}{2}a\tau)^2}\right]e_n + \frac{1}{4}a^3e^{(n+1)ha}\int_0^h \frac{e^{-\tau a}\tau^2 d\tau}{(1 - \frac{1}{2}a\tau)^2}$$

In particular, deduce that $a < 0$ implies that the error propagetes subject to the inequality

$$|e_{n+1}| \leq e^{ha}\left[1 + \frac{1}{4}|a|^3\int_0^h e^{-\tau a}\tau^2 d\tau\right]|e_n| + \frac{1}{4}|a|^3e^{(n+1)ha}\int_0^h e^{-\tau a}\tau^2 d\tau$$

Proof. (a) the formula for Trapezoid rule to solve $y' = ay$ is

$$y_{n+1} = y_n + \tfrac{h}{2}(ay_n + ay_{n+1}) \Rightarrow y_{n+1} = \frac{1 + \frac{ah}{2}}{1 - \frac{ah}{2}}y_n$$

And insert $t = nh, t = (n+1)h$ to $u(t)$: $u(nh) = y_n$, $u(nh+h) = \frac{1+\frac{1}{2}a(nh+h-nh)}{1-\frac{1}{2}a(nh+h-nh)} = y_{n+1}$. Hence it is consistent to trapezoid rule in terms of y_n and y_{n+1}.

(b)

$$\begin{aligned}
u'(t) &= \frac{\frac{1}{2}a[1-\frac{1}{2}a(t-nh)] + \frac{1}{2}a[1+\frac{1}{2}a(t-nh)]}{[1-\frac{1}{2}a(t-nh)]^2} y_n = \frac{a}{[1-\frac{1}{2}a(t-nh)]^2} y_n \\
&= \frac{a - \frac{1}{4}a^3(t-nh)^2 + \frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2} y_n = \frac{a(1+\frac{1}{2}a(t-nh))(1-\frac{1}{2}a(t-nh))}{[1-\frac{1}{2}a(t-nh)]^2} y_n + \frac{\frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2} y_n \\
&= au(t) + \frac{\frac{1}{4}a^3(t-nh)^2}{[1-\frac{1}{2}a(t-nh)]^2} y_n
\end{aligned} \tag{11}$$

So the general solution of the ode is:

$$\begin{aligned}
u(t) &= e^{\int_{nh}^{t} -(-a)dz} \left(\int_{nh}^{t} e^{\int_{nh}^{s} -adz} \frac{\frac{1}{4}a^3(s-nh)^2}{[1-\frac{1}{2}a(s-nh)]^2} y_n ds + C \right) \\
&= e^{a(t-nh)} \left(\frac{1}{4}a^3 \int_{nh}^{t} \frac{e^{-a(s-nh)}(s-nh)^2}{[1-\frac{1}{2}a(s-nh)]^2} y_n ds + C \right)
\end{aligned} \tag{12}$$

Clearly when $t = nh$, the integral vanishes, so $C = y_n$, we let $\tau := s - nh$ inside the integral:

$$u(t) = e^{a(t-nh)} \left(\frac{1}{4}a^3 \int_{0}^{t-nh} \frac{e^{-a\tau}\tau^2}{(1-\frac{1}{2}a\tau)^2} d\tau + 1 \right) y_n \tag{13}$$

Therefore, at $t = (n+1)h$,

$$u((n+1)h) = e^{ah} \left(\frac{1}{4}a^3 \int_{0}^{h} \frac{e^{-a\tau}\tau^2}{(1-\frac{1}{2}a\tau)^2} d\tau + 1 \right) y_n$$

(c) We can easily solve the scalar linear system analytically: $y(t) = e^{at}$. Hence $y((n+1)h) = e^{a(n+1)h} = e^{ah}y(nh)$. And we let the constant $J = \frac{1}{4}a^3 \int_0^h \frac{e^{-a\tau}\tau^2}{(1-\frac{1}{2}a\tau)^2} d\tau$. We have:

$$\begin{aligned}
e_{n+1} = y_{n+1} - y(t_{n+1}) &= u((n+1)h) - y((n+1)h) \\
&= e^{ah}(J+1)y_n - e^{ah}y(nh) \\
&= e^{ah}(J+1)y_n - e^{ah}y(nh) - e^{ah}Jy(nh) + e^{ah}Jy(nh) \\
&= e^{ah}(J+1)e_n + Je^{a(n+1)h}
\end{aligned} \tag{14}$$

If $a < 0$, $0 < 1 \le (1-\frac{1}{2}a\tau)^2$ i.e. $\frac{1}{(1-\frac{1}{2}a\tau)^2} \le 1 \ \forall \tau \in [0,h]$. So $|J| \le \frac{1}{4}|a|^3 \int_0^h e^{-\tau a}\tau^2 d\tau$. We conclude that

$$|e_{n+1}| \le e^{ah}(|J|+1)|e_n| + |J|e^{a(n+1)h} \le e^{ah}\left(\frac{1}{4}|a|^3 \int_0^h e^{-\tau a}\tau^2 d\tau + 1 \right) + \frac{1}{4}e^{a(n+1)h}|a|^3 \int_0^h e^{-\tau a}\tau^2 d\tau \tag{15}$$

□

Problem 4. (2.2) Let $\eta(z,w) = \rho(w) - z\sigma(w)$.

a. Demonstrate that the multistep method (2.8) is of order p iff

$$\eta(z, e^z) = cz^{p+1} + O(z^{p+2}), \qquad z \to 0$$

for some $c \in \mathbb{R} \setminus \{0\}$.

b. Show that, subject to $\partial\eta(0,1)/\partial w \neq 0$, there exists in a neighbourhood of the origin an analytic function $w_1(z)$ such that $\eta(z, w_1(z)) = 0$ and

$$w_1(z) = e^z - c\left(\frac{\partial\eta(0,1)}{\partial w}\right)^{-1} z^{p+1} + O(z^{p+2}), \qquad z \to 0 \quad (*)$$

c. Show that $(*)$ is true if the underlying method is convergent.

Proof. (a) Define sequence $\{c_m\}$ as

$$c_m = \begin{cases} \sum_{k=0}^{s} a_k, & m = 0 \\ \frac{1}{m!}\sum_{k=0}^{s}(a_k k^m - m b_k k^{m-1}), & m \geq 1 \end{cases} \tag{16}$$

The method (2.8) is of order p if and only if $c_m = 0$ for $m = 0, 1, ..., p$, $c_{p+1} \neq 0$; we call this condition (†). We further examine the generating polynomial of $\{c_m\}$: $P(z) = \sum_{m=0}^{\infty} c_m z^m$. And then we have $P(0) = c_0$, $P'(0) = c_1$, ..., $P^{(m)}(0) = c_m$. Hence (†) $\iff$ $P^{(p+1)}(0) \neq 0$, $P^{(m)}(0) = 0$ for $m = 1, 2, ..., p$. Since P is a polynomial of z, this is true if and only if

$$P(z) = cz^{p+1} + h.o.t. \quad c \neq 0, z \to 0 \quad (\ddagger)$$

And then we rewrite $P(z)$ as we have done in the course, and finally we can rewrite $P(z)$ as $P(z) = \sum_{k=0}^{s} a_k (e^z)^k - z\sum_{k=0}^{s} b_k (e^z)^k = \rho(e^z) - z\sigma(e^z) = \eta(z, e^z)$. Hence (‡) $\iff$ $\eta(z, e^z) = cz^{p+1} + O(z^{p+2})$, $c \neq 0$. □

(b) (c) *No Idea...* □

Problem 5. (2.3) Instead of (2.3), consider the identity

$$\boldsymbol{y}(t_{n+s}) = \boldsymbol{y}(t_{n+s-2}) + \int_{t_{n+s-2}}^{t_{n+s}} \boldsymbol{f}(\tau, \boldsymbol{y}(\tau))d\tau$$

a. Replace $\boldsymbol{f}(\tau, \boldsymbol{y}(\tau))$ by the interpolating polynomial $\boldsymbol{p}$ from section 2.1 and substitute $\boldsymbol{y}_{n+s-2}$ in place of $\boldsymbol{y}(t_{n+s-2})$. Show that the resultant explicit *Nystrom* method is of order $p = s$.
b. Derive the two-step Nystrom method in a closed form by using the above appraoch.
c. Find the coefficients of the two-step and three-step Nystrom methods by noticing that $\rho(w) = w^{s-2}(w^2 - 1)$ and evaluating σ from (2.13).
d. Derive the two-step third-order implicit *Milne* method. Again letting $\rho(w) = w^{s-2}(w^2 - 1)$ but allowing σ to be of degree s.

Proof. (a)

$$\begin{aligned} \boldsymbol{y}(t_{n+s}) &= \boldsymbol{y}(t_{n+s-2}) + \int_{t_{n+s-2}}^{t_{n+s}} \boldsymbol{p}(\tau)d\tau + O(h^{s+1}) \\ &= \boldsymbol{y}(t_{n+s-2}) + \int_{t_{n+s-2}}^{t_{n+s}} \sum_{k=n}^{n+s-1} \boldsymbol{y}'(t_k)L_k^{[s]}(\tau)d\tau + O(h^{s+1}) \\ &= \boldsymbol{y}(t_{n+s-2}) + \sum_{k=n}^{n+s-1} \boldsymbol{y}'(t_k)\int_{t_{n+s-2}}^{t_{n+s}} L_k^{[s]}(\tau)d\tau + O(h^{s+1}) \end{aligned} \tag{17}$$

As before we translate the mesh to the left by nh, and find that $\int_{t_{n+s-2}}^{t_{n+s}} L_k^{[s]}(\tau)d\tau = \int_{t_{s-2}}^{t_s} L_{\tilde{k}}^{[s]}(\tau)d\tau$, where $\tilde{k} = 0, 1, ..., s-1$, with $L_k^{[s]}(t) = \prod_{j=0, j\neq k}^{s-1} \frac{t_j - t}{t_j - t_k}$. And this integral is a constant quantity times h, we denote $\int_{t_{s-2}}^{t_s} L_{\tilde{k}}^{[s]}(\tau)d\tau = h\left(\frac{1}{h}\int_{t_{s-2}}^{t_s} L_{\tilde{k}}^{[s]}(\tau)d\tau\right) =: hc_k$ (we change k to $k = 0, 1, ..., s-1$ in the following text). Therefore

$$\boldsymbol{y}(t_{n+s}) = \boldsymbol{y}(t_{n+s-2}) + h\sum_{k=0}^{s-1} c_k \boldsymbol{f}(t_{n+k}, \boldsymbol{y}(t_{n+k})) + O(h^{s+1}) \tag{18}$$

And the method is given by

$$\boldsymbol{y}_{n+s} = \boldsymbol{y}_{n+s-2} + h \sum_{k=0}^{s-1} c_k \boldsymbol{f}(t_{n+k}, \boldsymbol{y}_{n+k}) \tag{19}$$

h times the truncation error $h\boldsymbol{r}_n \sim O(h^{s+1})$, so the method is of order s.

(b) Two-step Nystrom: $s = 2$. $\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + h(c_0 \boldsymbol{f}(t_n, \boldsymbol{y}_n) + c_1 \boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}))$

$$c_0 = \int_0^2 L_0^{[2]} dt = \int_0^2 \frac{1-t}{1-0} dt = 0; \qquad c_1 = \int_0^2 L_1^{[2]} dt = \int_0^2 \frac{0-t}{0-1} dt = 2 \tag{20}$$

So the 2-step Nystrom is given by

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + 2h\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1})$$

(c) $\rho(w) = w^{s-2}(w^2 - 1)$.

- $s = 2$ case: $\rho(w) = w^2 - 1 = (w-1)(w+1) = v(v+2)$, with $v = w - 1$,

$$\sigma(v) = \frac{v+2}{1 - (\frac{1}{2}v - \frac{1}{3}v^2)} = (v+2)(1 + \tfrac{1}{2}v) + O(v^2) = 2 + 2v + O(v^2)$$

  $\sigma(w) = 2 + 2(w-1) = 2$, which matches our result in (b).
- $s = 3$ case: $\rho(w) = w(w^2 - 1) = w(w-1)(w+1) = v(v+1)(v+2)$, with $v = w - 1$,

$$\sigma(v) = \frac{(v+1)(v+2)}{1 - (\frac{1}{2}v - \frac{1}{3}v^2)} = (v^2 + 3v + 2)(1 + \tfrac{1}{2}v - \tfrac{1}{12}v^2) + O(v^3) = 2 + 4v + \frac{7v^2}{3} + O(v^3)$$

  $\sigma(w) = \frac{1}{3} - \frac{2}{3}w + \frac{7}{3}w^2$, which gives the 3-step *Nystrom* method:

$$\boldsymbol{y}_{n+3} = \boldsymbol{y}_{n+1} + h\left[\frac{1}{3}\boldsymbol{f}(t_n, \boldsymbol{y}_n) - \frac{2}{3}\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) + \frac{7}{3}\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2})\right] \tag{21}$$

(d) 2-step implicit: $\rho(w) = w^2 - 1 = (w-1)(w+1) = v(v+2)$

$$\sigma(v) = \frac{v+2}{1 - (\frac{1}{2}v - \frac{1}{3}v^2)} = (v+2)(1 + \tfrac{1}{2}v - \tfrac{1}{12}v^2) + O(v^3) = 1 + 2v + \frac{v^2}{3} + O(v^3)$$

$\sigma(w) = \frac{1}{3} + \frac{4}{3}w + \frac{1}{3}w^2$, which gives the 2-step *Mline* method:

$$\boldsymbol{y}_{n+2} = \boldsymbol{y}_n + h\left[\frac{1}{3}\boldsymbol{f}(t_n, \boldsymbol{y}_n) + \frac{4}{3}\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) + \frac{1}{3}\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2})\right] \tag{22}$$

□

Problem 6. Show that the explicit multistep method

$$\boldsymbol{y}_{n+3} + a_2\boldsymbol{y}_{n+2} + a_1\boldsymbol{y}_{n+1} + a_0\boldsymbol{y}_n = h[b_2\boldsymbol{f}(t_{n+2}, \boldsymbol{y}_{n+2}) + b_1\boldsymbol{f}(t_{n+1}, \boldsymbol{y}_{n+1}) + b_0\boldsymbol{f}(t_n, \boldsymbol{y}_n)]$$

is fourth order only if $a_0 + a_2 = 8$ and $a_1 = -9$. Hence deduce that this method cannot be both fourth order and convergent.

Proof. The method is of order 4 $\iff$

$$\sum_{k=0}^{3} a_k = 0; \qquad \sum_{k=0}^{3}(a_k k^m - m b_k k^{m-1}) = 0, \quad m = 1, 2, 3, 4 \tag{23}$$

And we have already known $a_3 = 1$, $b_3 = 0$. Hence it suffices to solve

$$\begin{cases} a_0 + a_1 + a_2 = -1 \\ b_0 + a_1 - b_1 + 2a_2 - b_2 + 3a_3 = 0 \\ a_1 - 2b_1 + 4a_2 - 4b_2 + 9 = 0 \\ a_1 - 3b_1 + 8a_2 - 12b_2 + 27 = 0 \\ a_1 - 4b_1 + 16a_2 - 32b_2 + 81 = 0 \end{cases} \tag{24}$$

Label the equations as a to e: $(d) - \frac{3}{4}(c) - \frac{1}{4}(e) \Rightarrow$

$$-3b_1 + 8a_2 - 12b_2 - \frac{3(-2b_1 + 4a_2 - 4b_2)}{4} - \frac{-4b_1 + 16a_2 - 32b_2}{4} = 0$$

$$\Rightarrow -\frac{1}{2}b_1 + a_2 - b_2 = 0$$

Insert into (c) $\Rightarrow a_1 = -9$, hence $a_0 + a_2 = 8$.
Claim This method, with $a_1 = -9$ and $a_0 + a_2 = 8$ is not stable.
Proof of claim: let $a_2 = c$, then characteristic polynomial $\rho(z) = z^3 + cz^2 - 9z + 8 - c$. Use `Mathematica`, we find its zeros:

$$z_1 = 1, \quad z_2 = \frac{-(c+1) - \sqrt{32 + (c-1)^2}}{2}, \quad z_3 = \frac{-(c+1) + \sqrt{32 + (c-1)^2}}{2} \tag{25}$$

Which are neither (1) all inside the unit circle, nor (2) with norm 1 while having multiplicity of 1. Hence by theorem, the numerical method is not stable. $\Rightarrow$ the numerical method does not converge. $\square$
