Functional Analysis Assignment II

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Problem 1. Let (X, d) be metric space. Suppose h is a *homeomorphism* of X onto X, i.e. h is continous bijective map and its inverse is continuous. Given $A \subset X$, show that A and $h(A)$ have same category in X.

Proof. Since the sets of second category is defined to be those that are *not* of first category, it suffices to show that

$$A \text{ is of first category} \iff h(A) \text{ is of first category}$$

($\Rightarrow$) Suppose A is of first category, then we write $A = \bigcup_{i=1}^{n} A_i$, A_i are nowhere dense sets. Since $h(\cdot)$ is bijective, define $B_i := h(A_i)$, then $h(A) = \bigcup_{i=1}^{n} B_i$.
Claim: B_i is nowhere dense for all $i = 1, 2, ..., n$.
Proof of claim: Show by contradiction. Assume otherwise, i.e. B_i is not nowhere dense for some i, i.e. the interior of $\bar{B}_i$ is not empty, denote as O. It is clear that O and $h^{-1}(O)$ are open. We have

$$h^{-1}(O) \subseteq h^{-1}(\bar{B}_i) \subseteq \overline{h^{-1}(B_i)} \tag{1}$$

The second subseteq is due to continuity of $h^{-1}(\cdot)$: pick a point $b \in \bar{B}_i$, either $b \in B_i$ or $\lim\limits_{n\to\infty} b_n = b$, $\{b_n\} \subset B_i$. For the first case, clearly $h^{-1}(b) \in h^{-1}(B_i)$. For the second, since h^{-1} is continuous, we have $h^{-1}(\lim\limits_{n\to\infty} b_n) = \lim\limits_{n\to\infty} h^{-1}(b_n)$, and $\{h^{-1}(b_n)\} \subset h^{-1}(B_i)$. In both cases we can obtain $h^{-1}(b) \in \overline{h^{-1}(B_i)}$ $\forall b \in \bar{B}_i$, gives the proof.
Now that we have (1), note that $\overline{h^{-1}(B_i)} = \overline{h^{-1}(h(A_i))} = \bar{A}_i$; and exists open set $h^{-1}(O) \subseteq \bar{A}_i$ that is not empty. By definition

$$\operatorname{int}(\bar{A}_i) := \bigcup_{Q \subseteq \bar{A}_i, \text{open}} Q \supseteq h^{-1}(O) \tag{2}$$

is therefore not empty. Contradict the fact that A is of first category, i.e. A_i is nowhere dense.
($\Leftarrow$) is just a symmetric argument. Assume $h(A) = \bigcup_{k=1}^{n} B_i$, claim $A_i := h^{-1}(B_i)$ is nowhere dense. Argue by contradiction with using the continuity of $h(\cdot)$. ☐

Problem 2. Show that $\mathcal{C}([a, b])$ is separable.

Proof. It suffices to show there exists a contable dense set contained in $\mathcal{C}([a, b])$.
Firstly, we denote

$$\mathcal{P}(\mathbb{Q}) := \left\{ q \middle| q(x) = \sum_{k=0}^{n} a_k x^k; n \in \mathbb{N}, a_k \in \mathbb{Q}, a_n \neq 0 \right\}$$

$$\mathcal{P}(\mathbb{R}) := \left\{ p \middle| p(x) = \sum_{k=0}^{n} b_k x^k; n \in \mathbb{N}, b_k \in \mathbb{R}, b_n \neq 0 \right\}$$

(*Step.1*) We show that $\mathcal{P}(\mathbb{Q})$ is countable. Define $\mathcal{P}_n := \{q | q(x) = \sum_{k=0}^{n} a_k x^k; a_k \in \mathbb{Q}, a_n \neq 0\}$. Then $|\mathcal{P}_n| = |\mathbb{Q} \setminus \{0\} \times \mathbb{Q}^{n-1}|$, and $\mathcal{P}(\mathbb{Q}) = \bigcap_{k=0}^{\infty} \mathcal{P}_n$. Countable union of contable set, cartesian product of finite number of countable sets are both countable, which gives the proof of $\mathcal{P}(\mathbb{Q})$'s countability.
(*Step.2*) WLOG assume $x \in [0, 1]$. Due to (**Weietrass**), for all $f \in \mathcal{C}([0, 1])$, we can find $p \in \mathcal{P}(\mathbb{R})$ such that $|f - p_n| < \frac{1}{2n}$.
Then for this p_n with however large n, we can find $q_n \in \mathcal{P}(\mathbb{Q})$ with same n. Further more, since $\mathbb{Q}$ is dense in $\mathbb{R}$, for $b_k \in \mathbb{R}$, we can find $a_k \in \mathbb{Q}$ for every k, such that $|a_k - b_k| < \frac{1}{2n^2}$ uniformly. Therefore

$$|q_n - p_n| = \left| \sum_{k=0}^{n} a_k x^k - b_k x^k \right| \leq \sum_{k=0}^{n} |a_k - b_k||x^k| \leq \sum_{k=0}^{n} |a_k - b_k| < \frac{1}{2n} \tag{3}$$

Hence $|f - q_n| \leq |f - p_n| + |q_n - p_n| < \frac{1}{n} \to 0$, i.e. $q_n \to f$. Since $\{q_n\} \subset \mathcal{P}(\mathbb{Q})$, we can conclude that $\overline{\mathcal{P}(\mathbb{Q})} \supseteq \mathcal{C}([0,1])$. $\overline{\mathcal{P}(\mathbb{Q})} \subseteq \mathcal{C}([0,1])$ is trivial. So we have $\mathcal{P}(\mathbb{Q})$ is dense in $\mathcal{C}([0,1])$.
(*Step.3*) We extend this to $[a,b]$ by defining

$$h := \mathcal{C}([0,1]) \to \mathcal{C}([a,b])$$

with $(h \circ f)(x) := f(a+(b-a)x)$. Clearly h is isometry, and h is invertible. We conclude that $h^{-1}(\mathcal{P}(\mathbb{Q}))$ is dense in $\mathcal{C}([a,b])$, implies that the latter is separable. □

Problem 3. Show that every sequentially compact metric space K is separable.

Proof. K is sequentially compact $\Rightarrow$ K is totally bounded; i.e. for all $\epsilon > 0$, there exists a finite ϵ-net s.t. $K \subseteq \bigcup_{i=1}^{n_\epsilon} B_\epsilon(x_i)$.
Let $\epsilon = 1$, we find $U_1 := \bigcup_{i=1}^{n_1} B_1(x_i)$ is a union of n_1 balls. Denote

$$C_1 := \{x_i : B_1(x_i) \text{ Belongs to finite 1-net that covers } K\}$$

I.e. C_1 is the collection of all *center points* of balls in 1-net.
Do this for $\epsilon = \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...$, we obtain $\{C_1, C_2, ..., C_n, ...\}$ as collection of center points of the balls that constituting $\frac{1}{n}$-net. Then for any $z \in K$, there exists $x_1 \in C_1, x_2 \in C_2, ..., x_n \in C_n, ...$ such that $d(z, x_n) \leq \frac{1}{n}$. Hence we can obtain a sequence $x_n \to z$, with $\{x_n\} \subset \bigcup_{n=1}^{\infty} C_n$, which implies

$$\overline{\bigcup_{n=1}^{\infty} C_n} = K \tag{4}$$

I.e. $\bigcup_{n=1}^{\infty} C_n$ is dense in K. And $\bigcup_{n=1}^{\infty} C_n$ is also countable since it's countable union of sets that each has finite number of elements. We conclude that K is separable. □

Problem 4. Let K be a compact subset in the complete metric space X. Suppose $f \in \mathcal{C}(K, \mathbb{R})$. Show that f is uniformly continuous.

Proof. Firstly since $f : K \to \mathbb{R}$ is continuous, $\forall \epsilon > 0$, $\forall x, y \in K$, there exists δ_x relevant to x, such that $d(x,y) < \delta_x \Rightarrow d(f(x), f(y)) < \epsilon$, i.e.

$$f(B_{\delta_x}(x)) \subseteq B_\epsilon(f(x)) \tag{5}$$

For same ϵ, exhaust all $x \in K$. Then clearly $\bigcup_{x \in K} B_{\frac{\delta_x}{2}}(x)$ is an open cover of K. Since K is compact, there exists a finite subcover $U := \bigcup_{j=1}^{n} B_{\frac{\delta_j}{2}}(x_j)$.
Claim. $\forall \epsilon > 0$, $\forall x, y \in K$, there exists $\delta = \min\limits_{j=1,...,n} \frac{\delta_j}{2}$ uniformly, we have $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.
Proof of Claim. Suppose $x \in B_{\frac{\delta_j}{2}}(x_j)$ for some ball j in the finite subcover U, then the choice of δ ensures that y must be in the ball with same center and radius δ_j. Because

$$d(x_i, y) \leq d(x_i, x) + d(x, y) = \frac{\delta_j}{2} + \min_{j=1,...,n} \frac{\delta_j}{2} \leq \delta_j \tag{6}$$

Hence $x, y \in B_{\delta_j}(x_j)$, and by the initial choice of δ_j: $f(B_{\delta_j}(x_j)) \subseteq B_\epsilon(f(x_i))$, implies that $f(x), f(y) \in B_\epsilon(f(x_i))$. Hence $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$, proves uniform continuity. □

Problem 5. Let K be a compact subset in the complete metric space X. Suppose $f \in \mathcal{C}(K, \mathbb{R})$. Show that f is bounded and attains its maximum and minimum.

Proof. Step.1 We first show that compactness is continous-invariant, i.e. for $f : K \to W$ continuous, K compact, then $f(K)$ is also compact.
For arbitrary open cover $U = \bigcup_{i \in A} O_i$ of $f(K)$, $f^{-1}(U)$ is a cover of K. Since $f^{-1}(U) = \bigcup_{i \in A} f^{-1}(O_i)$, and f is continuous $\Rightarrow$ $f^{-1}(O_i)$ are open sets. Hence $f^{-1}(U)$ is an open cover of $K \Rightarrow \exists \bigcup_{i=1}^{n} f^{-1}(O_i) \subseteq f^{-1}(U)$ and is a finite cover of K. Therefore $\bigcup_{i=1}^{n} O_i$ is a finite cover of $f(K)$. Proves that $f(K)$ is

compact.
Step.2 Since $f(K) \subseteq \mathbb{R}$ is compact, it is bounded and closed. Since it's bounded, $a := \inf f(K)$ and $b := \sup f(K)$ exists and are limit points of $f(K)$. Moreover since $f(K)$ is closed $\Rightarrow a, b \in f(K)$.
Therefore, $\forall x \in K$, $a \leq f(x) \leq b$; and $\exists x_a, x_b \in K$, s.t. $f(x_a) = a, f(x_b) = b$. Which proves that f is bounded on K and attains its maximum and minimum. $\square$

Problem 6. Let $\{f_n \in \mathcal{C}([0,1]) | n \in \mathbb{N}\}$ be equicontinuous. If $f_n \to f$ pointwise, show that f is continuous.

Proof. $\mathcal{F} = \{f_n \in \mathcal{C}([0,1]) | n \in \mathbb{N}\}$ is equicontinuous, and $[0,1]$ is compact $\Rightarrow \mathcal{F}$ is uniformly equicontinuous. So $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$, $\forall \epsilon > 0$, there exists $\bar{\delta}$ *having nothing to do* with n, x, such that $f_n(B_{\bar{\delta}}(x)) \subseteq B_{\frac{\epsilon}{3}}(f_n(x))$.
Since $f_n \to f$ pointwise, $\forall \epsilon > 0$, $\forall x \in [0,1]$, $\exists N \in \mathbb{N}$, s.t $d(f(x), f_n(x)) < \frac{\epsilon}{3}$ as long as $n > N$.
Now we can show the continuity of f. Consider $\forall \epsilon > 0$, there exists $\delta = \bar{\delta}$. Then due to uniform equicontinuity of $\mathcal{F}$: $f_n(B_\delta(x)) \subseteq B_{\frac{\epsilon}{3}}(f_n(x))$ *regardless of* n, x. Then pick $n = N+1$, we have $d(f_n(x), f(x)) < \frac{\epsilon}{3}$.
Finally restrict $d(x,y) < \delta$, we get

$$
\begin{aligned}
d(f(x), f(y)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y) - f(y)) \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned} \tag{7}
$$

Implies that f is continuous. $\square$

Problem 7. Show that $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \tan^{-1} x$$

has no fixed point. And

$$|T(x) - T(y)| < |x - y| \text{ For all } x \neq y \in \mathbb{R}.$$

Illustrate the reason why this example does not contradict the contraction mapping thm.

Proof. Suppose T has fixed point $\bar{z}$, then $T\bar{z} = \bar{z} \Rightarrow \frac{\pi}{2} - \tan^{-1} \bar{z} = 0$, which has no solution. Hence T has no fixed point.
Then consider $\forall x \neq y \in \mathbb{R}$. Since T is continous on $\mathbb{R}$, by mean-value theorem, there exists $\xi \in [x, y]$

$$
\begin{aligned}
|Tx - Ty| &= |T'(\xi)||x - y| \\
&= \left|1 - \frac{1}{1+\xi^2}\right| |x - y| \\
&= \frac{\xi^2}{1+\xi^2}|x - y| < |x - y|
\end{aligned} \tag{8}
$$

This does not contradict the contraction mapping thm because T is *Not* a contraction map. By definition, $T : \mathbb{R} \to \mathbb{R}$ is contraction map if there exists $L \in [0,1)$ *regardless of* x, y, such that $d(Tx, Ty) \leq Ld(x,y)$ $(\triangle)$ for all $x, y \in \mathbb{R}$.
But for this T it is clear that RHS in equation (9) $\to |x-y|$ when $\xi \to \infty$. For example, we let $y = x+1$ and $x \to \infty$. Then we can't find L strictly less than 1 such that $d(Tx, Ty) \leq Ld(x,y)$. Clearly this implies that we can't find $L < 1$ for all $x \neq y \in \mathbb{R}$ to make $(\triangle)$ hold. Therefore T is not contraction map on $\mathbb{R}$. $\square$

Problem 8. The following integral equation for $f : [-a, a] \to \mathbb{R}$ arises in a model of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)dy \quad \text{for } -a \leq x \leq a.$$

Show that this equation has unique, bounded, continuous solution for $0 < a < \infty$. Further show that the solution is non-negative. Also discuss the circumstance when $a = \infty$.

Proof. (**Step.1**) Define functional $T : \mathcal{C}[-a, a] \to \mathcal{C}[-a, a]$, such that

$$Tf := 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)dy \tag{9}$$

It is clear that RHS is continuous for $-a \leq x \leq a$. Define $d(f, g) := \sup_{x\in[-a,a]} |f(x) - g(x)|$, then

$$\begin{aligned}
|Tf - Tg| &= \left|\frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2}(f(y) - g(y))dy\right| \\
&\leq \left|\frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2}dy\right| d(f, g) \\
&= \left|\frac{-1}{\pi}\int_{x-a}^{x+a} \frac{1}{1+(x-y)^2}d(x-y)\right| d(f, g) \\
&= \left|\frac{1}{\pi}\left(\tan^{-1}(x-a) - \tan^{-1}(x+a)\right)\right| d(f, g) \\
&\leq \frac{2}{\pi}\tan^{-1}(2a)\cdot d(f, g)
\end{aligned} \tag{10}$$

Denote $L := \frac{2}{\pi}\tan^{-1}(2a)$, we have $d(Tf, Tg) \leq Ld(f, g)$. When a is finite, $L < 1$. Hence $d(Tf, Tg) < d(f, g) \Rightarrow T$ is a contraction map on $\mathcal{C}[-a, a]$, which is also complete.
By *Contraction mapping Thm.* we know that $Tf = f$ has unique fixed point $\bar{f} \in \mathcal{C}[-a, a]$. Hence $\bar{f}(x)$ is unique solution of the equation, and is continuous. Since $[-a, a]$ is compact $\Rightarrow \bar{f}$ is also bounded.
(**Step.2**) Now we show $\bar{f}$ is non-negative. By the fact that T is contraction map, we can approach by newton's method. I.e. let $g_n := Tg_{n-1}$, then $g_n \to \bar{f}$. We pick $g_0 = 0$. Then $g_1 = Tg_0 = 1 \geq 0$. Now we prove by **Induction**. Assume $g_n \geq 0\ \forall x \in [-a, a]$, then

$$g_{n+1}(y) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} g_n(y)dy \geq 1 \geq 0 \tag{11}$$

So $g_n \geq 0$ for all $n \geq 0$. Since inequality is preserved in limit, we have $\bar{f} \geq 0$ as desired.
• When $a \to \infty$, we have $L = \lim_{a\to\infty} \frac{2}{\pi}\tan^{-1}(2a) = 1$, hence $d(Tf, Tg) \leq d(f, g)$. T is no longer a contraction map. In fact I have checked[1] that $Tf = f$ has no continous and bounded solution under this circumstance. □

Problem 9. Show there is a unique solution for following nonlinear BVP when constant λ has sufficiently small absolute value, where $f : [0, 1] \to \mathbb{R}$ is a given continuous function.

$$\begin{cases} -u_{xx} + \lambda \sin u = f(x) \\ u(0) = 0,\ u(1) = 0 \end{cases}$$

Proof. (**Step.1**) First we claim without proof (it's PDE class's business) that solving the given BVP is equivalent to solving $Tu = u$, where $T : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1]$,

$$Tu := \int_0^1 [f(y) - \lambda \sin(u(y))]G(x, y)dy \tag{12}$$

Where

$$G(x, y) = \begin{cases} x(1-y) & 0 \leq x \leq y \leq 1 \\ y(1-x) & 0 \leq y \leq x \leq 1 \end{cases} \tag{13}$$

is Green's function of $-\partial^2/\partial x^2$ in 1-D given boundary condition $u(0) = u(1) = 0$. We also define

[1]By Mathematica.

$d(u,v) := \sup_{x\in[0,1]} |u(x)-v(x)|$. Then we have:

$$
\begin{aligned}
|Tu - Tv| &= \left|\int_0^1 \lambda[\sin v(y) - \sin u(y)]G(x,y)dy\right| \\
&\leq |\lambda| \left|\int_0^1 G(x,y)dy\right| d(\sin v, \sin u) \\
&= |\lambda| d(\sin v, \sin u) \left|\int_0^x y(1-x)dy + \int_x^1 x(1-y)dy\right| \\
&= |\lambda| d(\sin v, \sin u) \left|\frac{x^2}{2}(1-x) + x(\frac{1}{2} - x + \frac{x^2}{2})\right| \\
&= |\lambda| d(\sin v, \sin u) \left|\frac{x-x^2}{2}\right| \\
&\leq |\lambda| d(\sin v, \sin u)
\end{aligned}
\tag{14}
$$

Hence $d(Tu,Tv) \leq |\lambda| d(\sin v, \sin u)$. We let $\lambda = \frac{1}{2}$, Then T is a contraction map. Since $\mathcal{C}[0,1]$ is complete, by contraction mapping theorem, $Tu = u$ has unique solution. □

Problem 10. Prove the following theorem. (*Thm.1*) Given linear space X.

1. The sets $\{0\}$ and X are linear subspaces of X.
2. The sum of any collection of subspaces is a subspace.
3. The intersection of any collection of subspaces is a subspace.
4. The union of a collection of subspaces totally ordered by inclusion is a subspace.

Proof. (*Thm.1*)

1. Really trivial.
2. $Y_\alpha \subset X$ is linear subspace for index $\alpha \in A$. Consider any $x, y \in \sum_\alpha Y_\alpha$, by definition we can write $x = \sum_\alpha x_\alpha, y = \sum_\alpha y_\alpha$ with $x_\alpha, y_\alpha \in Y_\alpha$. Since Y_α is linear subspace $\Rightarrow ax_\alpha + by_\alpha \in Y_\alpha$. So

$$
ax + by = a\sum_\alpha x_\alpha + b\sum_\alpha y_\alpha = \sum_\alpha ax_\alpha + by_\alpha \in \sum_\alpha Y_\alpha \tag{15}
$$

3. Y_α is linear subspace for index $\alpha \in A$. Then for $x, y \in \bigcap_\alpha Y_\alpha$, we have x, y in Y_α for all α. Hence $ax + by \in Y_\alpha$ for all $\alpha \Rightarrow ax + by \in \bigcap_\alpha Y_\alpha$, finished the proof.
4. $Y_n \subset X$ is linear subspace for all $n \in \mathbb{N}$; $Y_n \subseteq Y_{n+1}$. Consider $x, y \in \bigcup_{n\geq 1} Y_n$, there exists $p, q \geq 1$ such that $x \in Y_p, y \in Y_q$. WLOG assume $p \leq q$, then by inclusion $x \in Y_p \subseteq Y_q$. Therefore $ax + by \in Y_q \subseteq \bigcup_{n\geq 1} Y_n$, finished the proof.

□

Problem 11. X is linear space, Y is linear subspace of X. For $x_1, x_2 \in X$, denote $x_1 \equiv x_2 \bmod Y$ if $x_1 - x_2 \in Y$. Verify the followings

1. If $x_1 \equiv z_1, x_2 \equiv z_2$, then $x_1 + x_2 \equiv z_1 + z_2 \bmod Y$.
2. If $x_1 \equiv z_1$, then $kx_1 \equiv kz_1 \bmod Y$.

Proof. Both are clear by the fact that Y is linear subspace. Since $x_1 - z_1, x_2 - z_2 \in Y \Rightarrow (x_1 - z_1) + (x_2 - z_2) \in Y$, i.e. $(x_1 + x_2) - (z_1 + z_2) \in Y$.
Since $x_1 - z_1 \in Y \Rightarrow k(x_1 - z_1) = kx_1 - kz_1 \in Y$. □

Problem 12. Prove the following theorems.
(*Thm.3*)

1. The image of a linear subspace Y of X under a linear map $\boldsymbol{M} : X \to U$ is a linear subspace of U.

2. The inverse image under $\boldsymbol{M}$ of a linear subspace V of U is a linear subspace of X.

(*Thm.4*) Let K be a convex subset of a linear space X over the reals. Suppose that $x_1, ..., x_n \in K$; then so does every x of the form

$$x = \sum_{j=1}^{n} a_j x_j \quad \text{where } a_j \geq 0, \sum_{j=1}^{n} a_j = 1 \ (\dagger)$$

Proof. (*Thm.3*) $\forall u_1, u_2 \in \boldsymbol{M}Y$, we denote $\boldsymbol{M}y_1 = u_1, \boldsymbol{M}y_2 = u_2$ for $y_1, y_2 \in Y$.
Since $\boldsymbol{M}$ is a linear map:

$$\begin{aligned} u_1 + u_2 &= \boldsymbol{M}y_1 + \boldsymbol{M}y_2 = \boldsymbol{M}(y_1 + y_2) \in \boldsymbol{M}Y \\ ku_1 &= k\boldsymbol{M}y_1 = \boldsymbol{M}(ky_1) \in \boldsymbol{M}Y \end{aligned} \tag{16}$$

indicates that $\boldsymbol{M}Y$ is a linear subspace of U. Also since $\boldsymbol{M}^{-1}$ is a linear map. $\forall z_1, z_2 \in \boldsymbol{M}^{-1}V$, we denote $\boldsymbol{M}z_1 = v_1, \boldsymbol{M}z_2 = v_2$ for $v_1, v_2 \in V$.

$$\begin{aligned} z_1 + z_2 &= \boldsymbol{M}^{-1}v_1 + \boldsymbol{M}^{-1}v_2 = \boldsymbol{M}^{-1}(v_1 + v_2) \in \boldsymbol{M}^{-1}V \\ kz_1 &= k\boldsymbol{M}^{-1}v_1 = \boldsymbol{M}^{-1}(kv_1) \in \boldsymbol{M}^{-1}V \end{aligned} \tag{17}$$

Bespeaks that $\boldsymbol{M}^{-1}V$ is a linear subspace of X. □

Proof. (*Thm.4*) (**Induction Proof**) $n = 1$ is trivial, $n = 2$ is the definition of convexity.
Assume theorem is true when $n = k$, then when $n = k + 1$

$$\sum_{n=1}^{k+1} a_n x_n = (1 - a_{k+1}) \sum_{n=1}^{k} \frac{a_n}{1 - a_{k+1}} x_n + a_{k+1} x_{k+1} \tag{18}$$

Since we have $\sum_1^{k+1} a_n = 1$, therefore $\sum_1^k a_n = 1 - a_{k+1} \Rightarrow \sum_1^k \frac{a_n}{1-a_{k+1}} = 1$. So by $n = k$ assumption, $y := \sum_1^k \frac{a_n}{1-a_{k+1}} x_n \in K$, i.e. $RHS = (1 - a_{k-1})y + a_{k+1}x_{k+1}$. It belongs to K by defintion of convex set and the fact that $y, x_{k+1} \in K$. □

Problem 13. Prove the following theorems.
(*Thm.5*) Let X be a linear space of the reals.

1. The empty set is convex.
2. A singleton is convex.
3. Every linear subspace of X is convex.
4. The sum of two convex subsets is convex.
5. If K is convex, so is $-K$.
6. The intersection of an arbitrary collection of convex sets is convex.
7. Let $\{K_j\}$ be a collection of convex subsets that is totally ordered by inclusion. Then their union is convex.
8. The image of a convex set under a linear map is convex.
9. The preimage of a convex set under a linear map is convex.

(*Thm.6*) Define *Convex Hull* of S as the intersection of all convex sets containing S, denote S^{co}. Show that

1. S^{co} is the smallest convex set containing S.
2. S^{co} consists of all convex combinations (†) of points of S.

Proof. (*Thm.5*) • **(1)** trivial since there is no convex combinations. • **(2)** trivial since the only convex combination is just the singleton itself. • **(3)** trivial since convex combination is a special linear combination.
• **(4)** Denote $K := K_1 + K_2$, K_1, K_2 convex. Pick any $x, y \in K$, form any convex combination $(1 - \lambda)x + \lambda y = (1 - \lambda)(x_1 + x_2) + \lambda(y_1 + y_2) = [(1 - \lambda)x_1 + y_1] + [(1 - \lambda)x_2 + \lambda y_2]$ for which we have $(1 - \lambda)x_1 + y_1 \in K_1, (1 - \lambda)x_2 + y_2 \in K_2$. Therefore $x + y \in K$.

• **(5)** $x, y \in -K$, then $-x, -y \in K \Rightarrow -(1-\lambda)x - \lambda y \in K \Rightarrow (1-\lambda)x + \lambda y \in -K$
• **(6)** Denote $K := \bigcap_\alpha K_\alpha$. Pick any $x, y \in K$, then $x, y \in K_\alpha$ for all α. Hence convex combination $(1-\lambda)x + \lambda y \in K_\alpha$ for all α, so it is in K.
• **(7)** Consider $K_n \subseteq K_{n+1}$, $K := \bigcup_{k\geq 1} K_n$. Clearly $K_n \nearrow K$. For $x, y \in K$, $x \in K_p, y \in K_q$ for some p, q. So $x, y \in K_{\max\{p,q\}}$, which is convex $\Rightarrow (1-\lambda)x + \lambda y \in K_{\max\{p,q\}} \subseteq K$.
• **(8)** $\boldsymbol{M} : X \to Z$ is linear map. $\forall x, y \in \boldsymbol{M}K$ we have $\boldsymbol{M}^{-1}x, \boldsymbol{M}^{-1}y \in K$. By linearity of $\boldsymbol{M}$, Convex combination

$$(1-\lambda)\boldsymbol{M}^{-1}x + \lambda\boldsymbol{M}^{-1}y = \boldsymbol{M}^{-1}((1-\lambda)x + \lambda y) \in K \tag{19}$$

$\Rightarrow (1-\lambda)x + \lambda y \in \boldsymbol{M}K$.
• **(9)** $\boldsymbol{M} : V \to X$. $\forall x, y \in \boldsymbol{M}^{-1}K$ we have $\boldsymbol{M}x, \boldsymbol{M}y \in K$.

$$(1-\lambda)\boldsymbol{M}x + \lambda\boldsymbol{M}y = \boldsymbol{M}((1-\lambda)x + \lambda y) \in K \tag{20}$$

$\Rightarrow (1-\lambda)x + \lambda y \in \boldsymbol{M}^{-1}K$.

□

Proof. (*Thm.6*) By its definiton

$$S^{co} := \bigcap_{S_\alpha \text{convex}, S \subseteq S_\alpha} S_\alpha \tag{21}$$

So $\forall \alpha$, $S_\alpha \supseteq S^{co}$. Inplies that S^{co} is contained in all convex sets containing S.
$\forall x_1, ..., x_n \in S \subseteq S^{co}$, since S^{co} is convex, and combinations x of the form

$$x = \sum_{j=1}^{n} a_j x_j \quad \text{where } a_j \geq 0, \sum_{j=1}^{n} a_j = 1 \ (\dagger)$$

Should be $x \in S^{co}$, by (*Thm.4*) shown in problem 12. □

Problem 14. Prove the following theorems.
(*Thm.7*) Let K be a convex set, E an extreme subset of K and F an extreme subset of E. Then F is an extreme subset of K.
(*Thm.8*) Let $\boldsymbol{M}$ be linear map of linear space X into linear space U. Let K be a convex subset of U, E an extreme subset of K. Then the inverse image of E is either empty or an extreme subset of the inverse image of K.
Give an example to show that the image of an extreme subset under a linear map need not be an extreme subset of the image.

Proof. (*Thm.7*) Since E an extreme subset of $K \Rightarrow E$ convex and non-empty. F is extreme subset of E $\Rightarrow F$ is also convex and non-empty by definition.
Now it suffices to check second property. $\forall x \in F$ that can be written as $x = (y+z)/2$, $y, z \in K$; note that $F \subseteq E$, we have $x \in E$. So by the fact that E is extreme subset of $K \Rightarrow y, z \in E$.
Now that $x = (y+z)/2 \in F$, $y, z \in E$, F is extreme subset of $K \Rightarrow y, z \in F$. □

Proof. (*Thm.8*) $\boldsymbol{M} : X \to U$. E is extreme subset of convex $K \subset U$. Then if $\boldsymbol{M}^{-1}(E)$ is non-empty, it must be convex (due to *Thm.5-9*). Furthermore, $\boldsymbol{M}^{-1}K$ is also convex.
$\forall x \in \boldsymbol{M}^{-1}E$ that can be written as $x = (y+z)/2$, $y, z \in \boldsymbol{M}^{-1}K$; we have $\boldsymbol{M}y, \boldsymbol{M}z \in K$, $\boldsymbol{M}x \in E$. And since $\boldsymbol{M}$ is linear map,

$$\boldsymbol{M}x = \boldsymbol{M}\left(\frac{y+z}{2}\right) = \frac{\boldsymbol{M}y + \boldsymbol{M}z}{2} \tag{22}$$

Since E is extreme subset of K, by definition we have $\boldsymbol{M}y, \boldsymbol{M}z \in E$. Therefore $y, z \in \boldsymbol{M}^{-1}E$ as desired. We obtain: $\forall x \in \boldsymbol{M}^{-1}E$ that can be written as $x = (y+z)/2$, $y, z \in \boldsymbol{M}^{-1}K \Rightarrow y, z \in \boldsymbol{M}^{-1}E$. Therefore in this case $\boldsymbol{M}^{-1}E$ is extreme set of $\boldsymbol{M}^{-1}K$. □

(*Exercise 9*) Map $\boldsymbol{M} : [0,1]^2 \to [0,1]$, $(x, y) \mapsto x$. $\boldsymbol{M}$ is a linear map, because

$$a\boldsymbol{M}(x_1, y_1) + b\boldsymbol{M}(x_2, y_2) = ax_1 + bx_2 = \boldsymbol{M}(a(x_1, y_1) + b(x_2, y_2)) \tag{23}$$

It is clear that both $[0,1]^2$ and $[0,1]$ are convex. Furthermore, take $E := \{(x, y) | 0.3 \leq x \leq 0.4, y = 0\} \subset [0,1]^2$. E is a extreme subset of it. But $\boldsymbol{M}E = [0.3, 0.4] \subset [0,1]$ is not a extreme subset of $[0,1]$.