Stochastic Process Assignment VII

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Problem 1.

Solution. Denote $D(s,t) = B(t) - B(s)$, then $\{D(s_i, t_i)\}$ are independent if intervals (s_i, t_i) are disjoint. Further, $D(s,t) \sim \mathcal{N}(0, \sigma^2(t-s))$. We have

$$\begin{aligned}\mathbb{E}\left[B(t_1)B(t_2)B(t_3)\right] &= \mathbb{E}\left[B(t_1)[B(t_1) + D(t_1,t_2)][B(t_1) + D(t_1,t_2) + D(t_2,t_3)]\right] \\ &\quad \text{(Denote } B(t_1), D(t_1,t_2), D(t_2,t_3) \text{ as } b_1, b_2, b_3.) \\ &= \mathbb{E}\left[b_1^3 + 2b_1^2 b_2 + b_1 b_2^2 + b_1^2 b_3 + b_1 b_2 b_3\right] \\ &= \mathbb{E}\left[B(t_1)^3\right] = 0\end{aligned} \tag{1}$$

Because for normal RVs with mean zero, the odd-order moments are all zero.

Problem 2.

Solution. (a) By relevant theories about passage time covered in lecture, we have

$$\mathbb{P}\left(T_a \le t\right) = \frac{2}{\sqrt{2\pi}} \int_{\frac{|a|}{\sqrt{t}}}^{\infty} e^{\frac{-y^2}{2}} dy \tag{2}$$

Hence

$$\mathbb{P}\left(T_a < \infty\right) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{\frac{-y^2}{2}} dy \tag{3}$$

To calculate expectation, we use

$$\begin{aligned}\mathbb{E}\left[T_a\right] &= \int_0^{\infty} (1 - \mathbb{P}\left(T_a \le t\right)) dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \int_0^{\frac{|a|}{\sqrt{t}}} e^{\frac{-y^2}{2}} dy dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \int_0^{\frac{a^2}{y^2}} e^{\frac{-y^2}{2}} dt dy \\ &\ge a^2 \int_0^1 \frac{1}{y^2} e^{\frac{-y^2}{2}} dy \ge a^2 e^{\frac{1}{2}} \int_0^1 \frac{1}{y^2} dy = \infty\end{aligned} \tag{4}$$

(b)

$$\begin{aligned}\mathbb{P}\left(T_1 < T_{-1} < T_2\right) &= \mathbb{P}\left(T_1 < T_{-1}\right) \mathbb{P}\left(T_{-1} < T_2 | T_1 < T_{-1}\right) \\ &= \mathbb{P}\left(\text{Up 1 before down 1 (at 0)}\right) \mathbb{P}\left(\text{Down 2 before up 1 (at 1)}\right) \\ &= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}\end{aligned} \tag{5}$$

Problem 3.

Solution. (a) Denote X_i the binary movements in each step. $X_i = 1$ *w.p.* $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t})$. $X_i = -1$ with $1-p$. Denote $X(t)$ be the position at time t. Then

$$X(t) = \sqrt{\Delta t} \sum_{i=1}^{\lfloor \frac{t}{\Delta t} \rfloor} X_i \tag{6}$$

And $\mathbb{E}[X_i] = 2p - 1 = \mu\sqrt{\Delta t}$, $X_i^2 \equiv 1$ hence $\mathbb{V}\mathrm{ar}[X_i] = 1 - \mu^2 \Delta t$. $\Rightarrow$

$$\begin{aligned} \mathbb{E}[X(t)] &= \sqrt{\Delta t} \left\lfloor \frac{t}{\Delta t} \right\rfloor \cdot \mu\sqrt{\Delta t} \xrightarrow{\Delta t \to 0} \mu t \\ \mathbb{V}\mathrm{ar}[X(t)] &= \Delta t \left\lfloor \frac{t}{\Delta t} \right\rfloor \cdot (1 - \mu^2) \xrightarrow{\Delta t \to 0} t \end{aligned} \tag{7}$$

(b) In gambler's ruin problem, the probability of up A before down B is

$$\mathbb{P}(\text{Up A before down B}) = \frac{1 - (q/p)^B}{1 - (q/p)^{A+B}} \tag{8}$$

A, B are the **counts**. In this problem, $\frac{q}{p} = \frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}$, $\frac{A}{\sqrt{\Delta t}}$, $\frac{B}{\sqrt{\Delta t}}$ are counts. Hence

$$\begin{aligned} \mathbb{P}(\text{Up A before down B}) &= \lim_{\Delta t \to 0 \to \infty} \frac{1 - \left(\frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right)^{\frac{B}{\sqrt{\Delta t}}}}{1 - \left(\frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right)^{\frac{A+B}{\sqrt{\Delta t}}}} \\ &= \frac{1 - e^{-2\mu B}}{1 - e^{-2\mu(A+B)}} \end{aligned} \tag{9}$$

Problem 4. $\{Y(t)\}$ is a continuous martingale if for $s < t$,

$$\mathbb{E}[Y(t) | Y(u), 0 \le u \le s] = Y(s)$$

Solution. (a) (Standard Brownian Motion)

$$\begin{aligned} \mathbb{E}[B(t)|B(u), \text{for } 0 \le u \le s] &= \mathbb{E}[B(s) + B(t) - B(s)|B(u), 0 \le u \le s] \\ &= \mathbb{E}[B(s)|B(u), 0 \le u \le s] + \mathbb{E}[B(t) - B(s)|B(u), 0 \le u \le s] \\ &= B(s) + \mathbb{E}[B(t) - B(s)|B(u), 0 \le u \le s] \\ &= B(s) + \mathbb{E}[B(t) - B(s)] \\ &= B(s) + 0 \end{aligned} \tag{10}$$

(b) $Y(t) = B(t)^2 - t$. Firstly we compute

$$\mathbb{E}\left[B^2(t)|B(u), 0 \le u \le s\right] = \mathbb{E}\left[B^2(t)|B(s)\right] = B^2(s) + t - s \tag{11}$$

Since $B(t)|B(s) \sim \mathcal{N}(B(s), t-s)$, $\Rightarrow \mathbb{E}\left[B^2(t) - t|B(u), 0 \le u \le s\right] = B^2(s) - s$. So

$$\begin{aligned} \mathbb{E}[Y(t)|Y(u), 0 \le u \le s] &= \mathbb{E}[\mathbb{E}[Y(t)|B(u), 0 \le u \le s] | Y(u), 0 \le u \le s] \\ &= \mathbb{E}\left[B(s)^2 - s|B(u)^2, 0 \le u \le s\right] \\ &= B^2(s) - s \end{aligned} \tag{12}$$

(c) $Y(t) = \exp\{cB(t) - \frac{ct^2}{2}\}$.

$$\begin{aligned} \mathbb{E}[Y(t)|Y(u), 0 \le u \le s] &= e^{\frac{-c^2 t}{2}} \mathbb{E}\left[e^{cB(t)}|B(u), 0 \le u \le s\right] \\ &= e^{\frac{-c^2 t}{2}} \mathbb{E}\left[e^{cB(t)}|B(s)\right] \quad (\dagger) \end{aligned} \tag{13}$$

We know that $B(t)|B(s) \sim \mathcal{N}(B(s), t-s)$. So

$$\begin{aligned} (\dagger) &= e^{\frac{-c^2 t}{2}} e^{cB(s) + \frac{(t-s)c^2}{2}} \\ &= e^{\frac{-c^2 s}{2} + cB(s)} = Y(s) \end{aligned} \tag{14}$$

Problem 5.

Solution. (1) By **Martingale Stopping Time Thm.**,

$$\mathbb{E}\left[B(T)\right] = \mathbb{E}\left[B(0)\right] = 0$$

Hence

$$\begin{aligned} 0 = \mathbb{E}\left[B(T)\right] &= \mathbb{E}\left[\frac{x-\mu T}{\sigma}\right] \\ \Rightarrow \mathbb{E}\left[T\right] &= \frac{x}{\mu} \end{aligned} \tag{15}$$

(2) $\{B^2(T) - T\}$ forms a martingale. By Martingale stopping time thm,

$$\begin{aligned} \mathbb{E}\left[B^2(T) - T\right] &= \mathbb{E}\left[B^2(0)\right] = 0 \\ \Rightarrow \mathbb{E}\left[\frac{(x-\mu T)^2}{\sigma^2} - T\right] &= 0 \\ \Rightarrow \mathbb{E}\left[(x-\mu T)^2\right] &= \sigma^2 \mathbb{E}\left[T\right] = \frac{\sigma^2 x}{\mu} \end{aligned} \tag{16}$$

That is, $\mathbb{E}\left[(\mu\mathbb{E}\left[T\right] - \mu T)^2\right] = \sigma^2 x/\mu$. $\Rightarrow \mathbb{V}\mathrm{ar}\left[\mu T\right] = \sigma^2 x/\mu$.
I.e. $\mathbb{V}\mathrm{ar}\left[T\right] = \sigma^2 x/\mu^3$

Problem 6.

Solution. (a) Clearly $\{Y(t)\}$ is Gaussian. And $\mathbb{E}\left[Y(t)\right] = t\mathbb{E}\left[B(\frac{1}{t})\right] = 0$. And that (assume $s \leq t$)

$$\begin{aligned} \mathbb{C}\mathrm{ov}\left[Y(s), Y(t)\right] &= \mathbb{C}\mathrm{ov}\left[sB\left(\frac{1}{s}\right), tB\left(\frac{1}{t}\right)\right] \\ &= st \cdot \mathbb{C}\mathrm{ov}\left[B\left(\frac{1}{s}\right), B\left(\frac{1}{t}\right)\right] = s \end{aligned} \tag{17}$$

Therefore, we conclude that $Y(t)$ is Standard Brownian motion.
(b) $Y(t) = \frac{B(a^2 t)}{a}$.

$$\mathbb{E}\left[Y(t)\right] = \frac{1}{a}\mathbb{E}\left[B(a^2 t)\right] = 0 \tag{18}$$

For $s \leq t$,

$$\mathbb{C}\mathrm{ov}\left[Y(s), Y(t)\right] = \frac{1}{a^2}\mathbb{C}\mathrm{ov}\left[B\left(a^2 s\right), B\left(a^2 t\right)\right] = \frac{1}{a^2} \cdot a^2 s = s \tag{19}$$

Which, add to the fact that $\{Y(t)\}$ is Gaussian, finished the proof.
(c) $\mathbb{E}\left[Y(t)\right] = (t+1)\mathbb{E}\left[Z\left(\frac{t}{t+1}\right)\right] = 0$.

$$\begin{aligned} \mathbb{C}\mathrm{ov}\left[Y(s), Y(t)\right] &= (s+1)(t+1)\mathbb{C}\mathrm{ov}\left[Z\left(\frac{s}{s+1}\right), Z\left(\frac{t}{t+1}\right)\right] \\ &= (s+1)(t+1)\frac{s}{s+1}\left[1 - \frac{t}{t+1}\right] \\ &= s \end{aligned} \tag{20}$$

Finished the proof.

Problem 7.

Proof. Denote N the number of iterations until getting X. Denote $P_0 = \mathbb{P}\left(U \le \frac{f(Y)}{cg(Y)}\right)$.

$$\begin{aligned}
\mathbb{P}(X \le x) &= \mathbb{P}(Y_N \le x) \\
&= \mathbb{P}\left(Y \le x \middle| U \le \frac{f(Y)}{cg(Y)}\right) \\
&= \frac{1}{P_0} \cdot \mathbb{P}\left(Y \le x, U \le \frac{f(Y)}{cg(Y)}\right) \\
&= \frac{1}{P_0} \int_{-\infty}^{x} \mathbb{P}\left(Y \le x, U \le \frac{f(Y)}{cg(Y)} \middle| Y = y\right) g(y)dy \\
&= \frac{1}{P_0} \int_{-\infty}^{x} \frac{f(y)}{cg(y)} g(y)dy \\
&= \frac{1}{P_0} \int_{-\infty}^{x} \frac{f(y)}{c} dy
\end{aligned} \tag{21}$$

On letting $x \to \infty$, $LHS = 1 = \frac{1}{cP_0}$, indicating that $P_0 = 1/c$. Hence

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f(y)dy \tag{22}$$

□

Problem 8.

Solution. (a) We hope to change the RVs in density function. $(X,Y) \to (R^2, \Theta)$. We have

$$\begin{aligned}
f_{R^2,\Theta}(d,\theta) &= f_{X,Y}(x,y)|\boldsymbol{J}|^{-1} \\
&= \frac{1}{2\pi} e^{-(x^2+y^2)/2} \begin{vmatrix} 2x & 2y \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}^{-1} \\
&= \frac{1}{2\pi} \cdot \frac{1}{2} e^{-2d}
\end{aligned} \tag{23}$$

It is clear that R^2, Θ are independent. Where $R^2 \sim \text{Exp}(\frac{1}{2})$. And $\Theta \sim \text{Unif}(0, 2\pi)$
(b) We can invert polar coordinates to rectangular. By previous results, $-c\log(U) \sim \text{Exp}(1/c)$. Hence X, Y can be obtained by

$$\begin{cases} X = \sqrt{R^2}\cos\Theta \sim (-2\log U)^{1/2}\cos(2\pi V) \\ Y = \sqrt{R^2}\sin\Theta \sim (-2\log U)^{1/2}\sin(2\pi V) \end{cases} \tag{24}$$

Where U, V are uniforms on $[0,1]$.

Problem 9.

Proof. (1) We proceed by induction. For $n = 1$, increasing function $f(\cdot), g(\cdot)$, we have

$$\begin{aligned}
&(f(X) - f(Y))(g(X) - g(Y)) \ge 0 \\
\Rightarrow &\mathbb{E}\left[(f(X) - f(Y))(g(X) - g(Y))\right] \ge 0 \\
\Rightarrow &\mathbb{E}\left[f(X)g(X)\right] + \mathbb{E}\left[f(Y)g(Y)\right] \ge \mathbb{E}\left[f(X)g(Y)\right] + \mathbb{E}\left[f(Y)g(X)\right]
\end{aligned} \tag{25}$$

Suppose X, Y are i.i.d, we conclude that

$$2\mathbb{E}\left[f(X)g(X)\right] \ge 2\mathbb{E}\left[f(X)\right]\mathbb{E}\left[g(X)\right] \tag{26}$$

as desired.
Then assume this holds for $\boldsymbol{X}^{[n-1]}$ with $n-1$ elements. For n:

$$\begin{aligned}
\mathbb{E}\left[f(\boldsymbol{X}^{[n]})g(\boldsymbol{X}^{[n]})|X_n = x\right] &= \mathbb{E}\left[f(\boldsymbol{X}^{[n-1]}, x)g(\boldsymbol{X}^{[n-1]}, x)\right] \\
&\ge \mathbb{E}\left[f(\boldsymbol{X}^{[n-1]}, x)\right]\mathbb{E}\left[g(\boldsymbol{X}^{[n-1]}, x)\right] \quad \text{(By hypothesis.)} \\
&= \mathbb{E}\left[f(\boldsymbol{X}^{[n]})|X_n = x\right]\mathbb{E}\left[g(\boldsymbol{X}^{[n]})|X_n = x\right]
\end{aligned} \tag{27}$$

So

$$\begin{aligned}\mathbb{E}\left[f(\boldsymbol{X}^{[n]})g(\boldsymbol{X}^{[n]})\right] &= \mathbb{E}\left[\mathbb{E}\left[f(\boldsymbol{X}^{[n]})g(\boldsymbol{X}^{[n]})\Big|X_n\right]\right] \\ &\geq \mathbb{E}\left[\mathbb{E}\left[f(\boldsymbol{X}^{[n]})\Big|X_n\right]\mathbb{E}\left[g(\boldsymbol{X}^{[n]})\Big|X_n\right]\right] \\ &\geq \mathbb{E}\left[f(\boldsymbol{X}^{[n]})\right]\mathbb{E}\left[g(\boldsymbol{X}^{[n]})\right]\end{aligned} \tag{28}$$

Finished the proof.
(b) WLOG suppose $k \nearrow$ with every dimension, then $-k(1-U_1,...,1-U_n) \nearrow$. By the result of (a) $\Rightarrow$

$$\begin{aligned}&\mathbb{C}\text{ov}\left[k(U_1,...,U_n), -k(1-U_1,...,1-U_n)\right] \geq 0 \\ \Rightarrow &\mathbb{C}\text{ov}\left[k(U_1,...,U_n), k(1-U_1,...,1-U_n)\right] \leq 0\end{aligned} \tag{29}$$

For $k \searrow$, exactly symmetric statement. □

Problem 10.

Solution. (a). Sampling F is as if sampling from F_i, respectively, with probability P_i. Hence we sample another indicator to determine which F_i to sample from.
In particular, draw $U \sim$ Unif$[0,1]$. Sample from F_i if

$$\sum_{j=1}^{i-1} P_j < U \leq \sum_{j=1}^{i} P_j$$

(b) Note that $F(\cdot)$ can be rewritten as

$$\begin{aligned}F(x) &= \frac{1}{3}F_1(x) + \frac{2}{3}F_2(x) \\ F_1(x) &= 1 - e^{-2x} \\ F_2(x) &= \min\{x, 1\}\end{aligned} \tag{30}$$

For $0 < x < \infty$. So $X = \frac{1}{3}X_1 + \frac{2}{3}X_2$ $X_1 \sim$Exp(2), and $X_2 \sim$Unif(0,1). The sample is therefore obtained by

$$X = \begin{cases}\frac{-\log U_2}{2} & U_1 < \frac{1}{3} \\ U_3 & U_1 \geq \frac{1}{3}\end{cases} \tag{31}$$

Where U_1, U_2, U_2 are unifrom(0,1).
