Stochastic Process Assignment II

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March 20, 2016

Problem 1.

Solution. Define RV $N_s^{[k]} :=$ # of trials before obtaining k-consecutive successes, *given* that we have already had s-consecutive successes in the stack. We want $\mathbb{E}\left[N_0^{[k]}\right]$, and we have

$$N_0^{[k]} = N_0^{[k-1]} + N_{k-1}^{[k]} \tag{1}$$

Define $A :=$ {The next trial right after we have $k-1$ consecutive successes is again a success}, we can write

$$\begin{aligned}\mathbb{E}\left[N_{k-1}^{[k]}\right] &= \mathbb{E}\left[N_{k-1}^{[k]}; A\right] + \mathbb{E}\left[N_{k-1}^{[k]}; A^{\complement}\right] \\ &= 1 \cdot p + N_0^{[k]} \cdot (1-p)\end{aligned} \tag{2}$$

Insert back into equation (1) yields

$$\mathbb{E}\left[N_0^{[k]}\right] = \frac{1}{p}\left(1 + \mathbb{E}\left[N_0^{[k-1]}\right]\right) \tag{3}$$

Which is a recursive formula for sequence $\left\{\mathbb{E}\left[N_0^{[k]}\right] : k \geq 1\right\}$. Note $N_0^{[1]} \sim \text{Geometric}(p)$, we solve from recursion that $\mathbb{E}\left[N_0^{[k]}\right] = \sum_{i=1}^{k} 1/p^i$.

Problem 2.

Solution. By the definition given in the problem, it suffices to show $f_{Y|X}(y,i) = C' e^{-(\alpha+1)y} y^{s+i-1}$, where C' is irrelevant to y.

$$\begin{aligned}f_{Y|X}(y|i) &:= \frac{f_{X,Y}(i,y)}{p_X(i)} \\ &= \frac{p_{X|Y}(i|y) f_Y(y)}{p_X(i)} \\ &= \frac{1}{p_X(i)} \cdot \frac{e^{-y} y^i}{i!} \cdot C e^{-\alpha y} y^{s-1} \\ &= \frac{C}{p_X(i) i!} \cdot e^{-(\alpha+1)y} y^{s+i-1}\end{aligned} \tag{4}$$

Since $\{X = i\}$ is a known condition, $C' := C/p_X(i)i!$ is a constant. By the given definition in the problem, $Y|X$ is Gamma-distributed.

Problem 3.

Solution. Since $T(\boldsymbol{X}) = \sum_1^n X_i$, deterministically we have $t = \sum_1^n x_i$.

$$\begin{aligned}f_{\boldsymbol{X},T(\boldsymbol{X})}(\boldsymbol{x}, t) &= \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{x}, T(\boldsymbol{X}) = t\right) \\ &= \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{x}\right) \mathbb{P}\left(T(\boldsymbol{X}) = t | \boldsymbol{X} = \boldsymbol{x}\right) \\ &= \mathbb{P}\left(\boldsymbol{X} = \boldsymbol{x}\right) \cdot 1 \\ &= f_{\boldsymbol{X}}(\boldsymbol{x})\end{aligned} \tag{5}$$

(a). When $X \sim \mathcal{N}(\theta, 1)$, $T(\boldsymbol{X}) \sim \mathcal{N}(n\theta, n)$. And the gaussian vector $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, where $\boldsymbol{\theta} = [\theta, ..., \theta]$, $\boldsymbol{\Sigma} = \boldsymbol{I}$ is identity matrix.

$$
\begin{aligned}
f_{\boldsymbol{X}|T(\boldsymbol{X})}(\boldsymbol{x}|t) &= \frac{f_{\boldsymbol{X},T}(\boldsymbol{x}, t)}{f_T(t)} = \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{f_T(t)} \\
&= \frac{\exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\theta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\theta})^\top\right)/\sqrt{(2\pi)^n \det(\boldsymbol{\Sigma})}}{\exp(-\frac{1}{2n}(t-n\theta)^2)/\sqrt{2\pi n}} \\
&= C \exp\left(\frac{t^2}{2n} - \theta t + \frac{n\theta^2}{2} - \frac{\boldsymbol{x}\boldsymbol{x}^\top}{2} + \theta t - \frac{\boldsymbol{\theta}\boldsymbol{\theta}^\top}{2}\right) \\
&= C \exp\left(\frac{t^2}{2n} - \frac{\boldsymbol{x}\boldsymbol{x}^\top}{2}\right)
\end{aligned}
\tag{6}
$$

In which $\boldsymbol{x} = [x_1, x_2, ..., x_n]$, $C := \sqrt{1/(2\pi)^{n-1}}$. Since $f_{\boldsymbol{X}|T}$ is not a function of θ, by definition, T is a sufficient statistic.
(b). Given $X \sim \text{Exp}(\theta)$, we have $T(\boldsymbol{X}) \sim \Gamma(n, \theta)$.

$$
f_{\boldsymbol{X}|T(\boldsymbol{X})}(\boldsymbol{x}|t) = \frac{f_{\boldsymbol{X}}(\boldsymbol{x})}{f_T(t)} = \frac{\theta^n \exp(-\theta \sum_1^n x_i)}{\theta \exp(-\theta t)(\theta t)^{n-1}/\Gamma(n)} = \Gamma(n)/t^{n-1} \tag{7}
$$

(c) Given $X \sim \text{Bernoulli}(\theta)$, we have $T(\boldsymbol{X}) \sim \text{Binom}(n, \theta)$.

$$
p_{\boldsymbol{X}|T(\boldsymbol{X})}(\boldsymbol{x}|t) = \frac{p_{\boldsymbol{X}}(\boldsymbol{x})}{p_T(t)} = \frac{\theta^{\sum_1^n x_i}(1-\theta)^{n-\sum_1^n x_i}}{\binom{n}{t}\theta^t(1-\theta)^{(n-t)}} = \frac{1}{\binom{n}{t}} \tag{8}
$$

(d) Given $X \sim \text{Poi}(\theta)$, we have $T(\boldsymbol{X}) \sim \text{Poi}(n\theta)$.

$$
p_{\boldsymbol{X}|T(\boldsymbol{X})}(\boldsymbol{x}|t) = \frac{p_{\boldsymbol{X}}(\boldsymbol{x})}{p_T(t)} = \frac{e^{-n\theta}\theta^{\sum_1^n x_i}/\prod_1^n x_i!}{e^{-n\theta}(n\theta)^t/t!} = \frac{t!}{n^t \prod_1^n x_i!} \tag{9}
$$

Problem 4.

Solution. (a). Denote $D := \{\text{The observed person has disease.}\}$, then we are able to interpret the quantities in the illustration as: $\mathbb{P}(D|\{X = x\}) = P(x)$; $\mathbb{P}(X = x) = f(x)$. Hence $\mathbb{P}(D \cap \{X = x\}) = P(x)f(x)$.

$$
\begin{aligned}
\mathbb{P}(\{X = x\}|D) &= \frac{\mathbb{P}(D \cap \{X = x\})}{\mathbb{P}(D)} \\
&= \frac{\mathbb{P}(D \cap \{X = x\})}{\int_x \mathbb{P}(D \cap \{X = x\})\, dx} \\
&= \frac{P(x)f(x)}{\int_x P(x)f(x)dx}
\end{aligned}
\tag{10}
$$

(b). Just replace D with $D^\complement$, in which $\mathbb{P}\left(D^\complement \middle| \{X = x\}\right) = 1 - P(x)$, yields

$$
\mathbb{P}\left(\{X = x\} \middle| D^\complement\right) = \frac{(1-P(x))f(x)}{\int_x (1-P(x))f(x)dx} \tag{11}
$$

(c).

$$
\frac{\mathbb{P}(\{X = x\}|D)}{\mathbb{P}\left(\{X = x\}|D^\complement\right)} = \frac{\int_x (1-P(x))f(x)dx}{\int_x P(x)f(x)dx} \cdot \frac{1}{\frac{1}{P(x)} - 1} \tag{12}
$$

Note that in the first quantity we integrate x out, so it's just a constant. And the second quantity $\nearrow$ whenever $1 \geq P(x) \nearrow$, which finishes the proof.

Problem 5.

Solution. (a). Define RV $N^{[i]} := \#$ of rounds befone 2-consecutive hits when player i shoots first; $i = 1, 2$. $A_k := \{\text{The target is hitted in the } k^{th} \text{ round.}\}$. Then

$$\begin{aligned}
\mu_1 := \mathbb{E}\left[N^{[1]}\right] &= \mathbb{E}\left[N^{[1]}; A_1\right] + \mathbb{E}\left[N^{[1]}; A_1^{\complement}\right] \\
&= \left(\mathbb{E}\left[N^{[1]}; A_1 \cap A_2\right] + \mathbb{E}\left[N^{[1]}; A_1 \cap A_2^{\complement}\right]\right) + \mathbb{E}\left[N^{[1]}; A_1^{\complement}\right] \\
&= \mathbb{E}\left[N^{[1]}\middle| A_1 \cap A_2\right] \mathbb{P}\left(A_1 \cap A_2\right) + \mathbb{E}\left[N^{[1]}\middle| A_1 \cap A_2^{\complement}\right] \mathbb{P}\left(A_1 \cap A_2^{\complement}\right) + \mathbb{E}\left[N^{[1]}\middle| A_1^{\complement}\right] \mathbb{P}\left(A_1^{\complement}\right) \\
&= 2p_1p_2 + (\mu_1 + 2)p_1(1 - p_2) + (\mu_2 + 1)(1 - p_1)
\end{aligned} \tag{13}$$

By similar split of $N^{[2]}$, we have

$$\mu_2 = 2p_2p_1 + (\mu_2 + 2)p_1(1 - p_2) + (\mu_1 + 1)(1 - p_2) \tag{14}$$

Solving the equation system, yields

$$\begin{cases} \mu_1 = (2 + p_1^2p_2 - p_1p_2) \;/\; (p_1p_2(2 - p_1 - p_2 + p_1p_2)) \\ \mu_2 = (2 + p_2^2p_1 - p_1p_2) \;/\; (p_1p_2(2 - p_1 - p_2 + p_1p_2)) \end{cases} \tag{15}$$

(b). Define RV $X^{[i]} := \#$ of hits befone 2-consecutive hits when player i shoots first; A_i is same event as in (a).

$$\begin{aligned}
h_1 := \mathbb{E}\left[X^{[1]}\right] &= \mathbb{E}\left[X^{[1]}; A_1\right] + \mathbb{E}\left[X^{[1]}; A_1^{\complement}\right] \\
&= \left(\mathbb{E}\left[X^{[1]}; A_1 \cap A_2\right] + \mathbb{E}\left[X^{[1]}; A_1 \cap A_2^{\complement}\right]\right) + \mathbb{E}\left[X^{[1]}; A_1^{\complement}\right] \\
&= 2p_1p_2 + (h_1 + 1)p_1(1 - p_2) + h_2(1 - p_1)
\end{aligned} \tag{16}$$

By similar split of $X^{[2]}$, we have

$$h_2 = 2p_2p_1 + (h_2 + 1)p_1(1 - p_2) + h_1(1 - p_2) \tag{17}$$

Solving the equation system, yields

$$\begin{cases} h_1 = (p_1 + p_2 + p_1^2p_2^2 - p_1p_2^2) \;/\; (p_1p_2(2 - p_1 - p_2 + p_1p_2)) \\ h_2 = (p_1 + p_2 + p_1^2p_2^2 - p_1^2p_2) \;/\; (p_1p_2(2 - p_1 - p_2 + p_1p_2)) \end{cases} \tag{18}$$

Problem 6. Verify that following definitions for Poisson process are equivalent. Counting process $\{N(t) : t \geq 0\}$ is a poisson process if 1. $N(0) = 0$, 2. independent increments and

3. $\mathbb{P}\left(N(t + s) - N(s) = n\right) = e^{-\lambda t}(\lambda t)^n/n!$.
3' $\mathbb{P}\left(N(h + s) - N(s) = 1\right) = \lambda h + o(h)$; $\mathbb{P}\left(N(h + s) - N(s) \geq 2\right) = o(h)$ for all s and $h \to 0$.

Proof. (3) $\Rightarrow$ (3') is straightforward

$$\begin{aligned}
&\mathbb{P}\left(N(h + s) - N(s) = 0\right) = e^{-\lambda h} = 1 - \lambda h + o(h) \\
&\mathbb{P}\left(N(h + s) - N(s) = 1\right) = e^{-\lambda h}\lambda h = (1 - \lambda h + o(h))\lambda h = \lambda h + o(h)
\end{aligned} \tag{19}$$

Hence,

$$\mathbb{P}\left(N(h + s) - N(s) \geq 2\right) = 1 - \mathbb{P}\left(N(h + s) - N(s) \in \{0, 1\}\right) = o(h) \tag{20}$$

Finishes the proof.
(3') $\Rightarrow$ (3) (**Step.1**) We check MGF $\phi_{N(t)}(x) = \mathbb{E}\left[e^{xN(t)}\right]$ equal to that of Poisson(λt). For clearity of notations, we write $u(x, t) := \phi_{N(t)}(x)$. In particular for fixed $\bar{t}$, $u(x, \bar{t})$ is MGF of RV $N(\bar{t})$, and a univariate function of u. We further define increment $\Delta_{s,s+t} := N(s + t) - N(s)$, then $N(s) = \Delta_{0,s}$. By independent increment property, $\Delta_{a,b}, \Delta_{c,d}$ are independent if $(a, b) \cap (c, d) = \emptyset$.

$$\begin{aligned}
u(x, t + h) &= \mathbb{E}\left[e^{x(N(t+h) - N(t))}e^{xN(t)}\right] \\
&= \mathbb{E}\left[e^{x\Delta_{t,t+h}}e^{x\Delta_{0,t}}\right] \\
&= u(x, t)\mathbb{E}\left[e^{x\Delta_{t,t+h}}\right] \\
&= u(x, t)\left[1 - \lambda h + o(h) + e^x(\lambda h + o(h)) + o(h)\right] \\
&= u(x, t)\left[1 - \lambda h + e^x\lambda h + o(h)\right]
\end{aligned} \tag{21}$$

$$\Rightarrow \frac{u(x,t+h)-u(x,t)}{h} = u(x,t)\lambda(e^x-1)+\frac{o(h)}{h} \tag{22}$$

Let $h \to 0$ and note that $N(0)=0$, it suffices to solve following Boundary Value Problem

$$\begin{cases} u_t(x,t) = u(x,t)\lambda(e^x-1) \\ u(x,0)=1 \end{cases} \tag{23}$$

It turns out that $u(x,t)=\exp(\lambda t(e^x-1))$, implies that for every fixed $t \geq 0$, $N(t) \sim \mathrm{Poi}(\lambda t)$.
(**Step.2**) Now consider for any $s \geq 0$, $\Delta_{s,s+t} = N(s+t)-N(s) \Rightarrow \Delta_{s,s+t}+\Delta_{0,s} = \Delta_{0,s+t}$, and $\Delta_{s,s+t}, \Delta_{0,s}$ are independent increments; furthermore MGF of $\Delta_{0,s}$ is known to us, which is $u(x;s)$. Hence

$$\begin{aligned} &\phi_{\Delta_{0,s}} \cdot \phi_{\Delta_{s,s+t}} = \phi_{\Delta_{0,s+t}} \\ \Rightarrow\ &\phi_{\Delta_{s,s+t}} = \frac{g(x,s+t)}{g(x,s)} = \exp(\lambda t(e^x-1)) \end{aligned} \tag{24}$$

Which implies that $\Delta_{s,s+t} \sim \mathrm{Poi}(\lambda t)$. □

Problem 7. $\{T_n : n \geq 1\}$ are i.i.d exponential with mean $\frac{1}{\lambda}$. Define $N(t) := \max\{n : S_n \leq t\}$ where $S_0 = 0$ and $S_n = \sum_{i=1}^n T_i$. Show $\{N(t)\}$ is Poisson process with rate λ.

Proof. (**Step.1**) We check $S_n \sim \Gamma(n,\lambda)$. Since $\{T_n : n \geq 1\}$ are i.i.d exponential, we consider the MGF of S_n,

$$\phi_{S_n}(t) = \prod_{i=1}^n \phi_{T_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n \tag{25}$$

Which is exactly the MGF of a $\Gamma(n,\lambda)$ RV. Therefore we can write the CDF of S_n as $\sim \Gamma(n,\lambda)$

$$F_{S_n}(t) = \mathbb{P}(S_n \leq t) = 1 - \sum_{i=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \tag{26}$$

(**Step.2**) Then we derive the distribution of $N(t)$. By its definition, $\mathbb{P}(N(t)=n) = \mathbb{P}(S_n \leq t < S_{n+1}) = \mathbb{P}(\{S_n \leq t\} \setminus \{S_{n+1} \leq t\})$. It is clear that $\{S_{n+1} \leq t\} \subseteq \{S_n \leq t\}$ because $S_n \leq S_{n+1}$. Hence

$$\begin{aligned} \mathbb{P}(N(t)=n) &= \mathbb{P}(\{S_n \leq t\} \setminus \{S_{n+1} \leq t\}) \\ &= \mathbb{P}(S_n \leq t) - \mathbb{P}(S_{n+1} \leq t) \\ &= F_{S_n}(t) - F_{S_{n+1}}(t) \\ &= \frac{e^{-\lambda t}(\lambda t)^n}{n!} \end{aligned} \tag{27}$$

Which implies that $N(t) \sim \mathrm{Poi}(\lambda t)$.
(**Step.3**) We show that $\{N(t) : t \geq 0\}$ is of **stationary increments**, and further show that it is of **independent increments**. Define $\Delta_{t_1,t_2} := N(t_2) - N(t_1)$, then in particular we have $\Delta_{0,t} = N(t)$. Still employ same notations for interarrival time and waiting time (i.e. T_n, S_n).
$\forall$ starting point $s > 0$, Define $S_n^{[s]} := (S_{n+N(s)} - s)$ i.e. the waiting time of n^{th} event happening **after** s. We have

$$S_n^{[s]} = (S_{N(s)+1} - s) + \sum_{i=2}^n T_{N(s)+i} \tag{28}$$

Where $S_{N(s)+1}$ is the waiting time of the first event happening after s, we have $S_{N(s)+1} = S_{N(s)} + T_{N(s)+1}$; and $T_{N(s)+i}$ are i.i.d Exponential(λ). We notice that event $\{S_{N(s)+1} > s\}$ i.e. $\{S_1^{[s]} > 0\}$ is *surely true*[1], since $N(s)+1^{st}$ event has not yet happened at time s. So for all $t \geq 0$, by **memoryless** property of

[1]By saying event E surely true, we mean that $E = \Omega$ (which differs from almost surely true where we only require $\mathbb{P}(E)=1$). And of course any event with probability 0 or 1 must be independent of anything else.

$T_{N(s)+1}$:

$$\begin{aligned}
\mathbb{P}\left(T_{N(s)+1} > t\right) &= \mathbb{P}\left(T_{N(s)+1} > t + (s - S_{N(s)}) \middle| T_{N(s)+1} > (s - S_{N(s)})\right) \\
&= \mathbb{P}\left(S_{N(s)} + T_{N(s)+1} - s > t \middle| S_{N(s)} + T_{N(s)+1} - s > 0\right) \\
&= \mathbb{P}\left(S_1^{[s]} > t \middle| S_1^{[s]} > 0\right) \\
&= \frac{\mathbb{P}(\{S_1^{[s]} > t\} \cap \{S_1^{[s]} > 0\})}{\mathbb{P}(S_1^{[s]} > 0)} \\
&= \mathbb{P}(S_1^{[s]} > t)
\end{aligned} \tag{29}$$

Which implies that $S_1^{[s]}$ has identical distribution as $T_{N(s)+1}$, which is Exponential(λ) and is independent w.r.t. T_j, for all $j \neq N(s)+1$. Therefore $S_n^{[s]} = S_1^{[s]} + \sum_{i=2}^n T_{N(s)+i}$ is a summation of n copies of i.i.d Exponential(λ). Hence, $S_n^{[s]} \sim \Gamma(n, \lambda)$ is of identical distribution as S_n (†).
Since $\Delta_{s,s+t} = \max\{n : S_n^{[s]} < t\}$. Note that $\Delta_{0,t} = N(t) = \max\{n : S_n < t\}$ and fact (†), we finish the proof that $\Delta_{0,t}$ and $\Delta_{s,s+t}$ are identically distributed for all $s \geq 0$. (**Stationary Increments**)
Now for any s, t, we have

$$\begin{aligned}
\phi_{\Delta_{0,s}+\Delta_{s,s+t}}(x) &= \phi_{\Delta_{0,s+t}}(x) \\
&= \exp(\lambda(s+t)(e^x - 1)) \\
&= \exp(\lambda s(e^x - 1)) \cdot \exp(\lambda t(e^x - 1)) \\
&= \phi_{\Delta_{0,s}}(x) \cdot \phi_{\Delta_{0,t}}(x) \\
&= \phi_{\Delta_{0,s}}(x) \cdot \phi_{\Delta_{s,s+t}}(x) \quad (\triangle)(\text{By stationary increments})
\end{aligned} \tag{30}$$

Which implies that $\Delta_{s,s+t}$ and $\Delta_{0,s}$ are independent for all $t, s \geq 0$.
Now for any $a, b, c, d \geq 0$, $(a,b) \cap (c,d) = \emptyset$ and WLOG $a \leq b \leq c \leq d$. $(\triangle) \Rightarrow \Delta_{a,b}, \Delta_{c,d}$ are independent. (**Independent Increments**)
(**Step.4**) By stationary increments in step3 and distribution of $N(t)$ in step2, we conclude that

$$\mathbb{P}\left(\Delta_{s,s+t} = n\right) = \mathbb{P}\left(N(t) = n\right) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \tag{31}$$

Which finishes the proof of defining properties of Poisson process.

□

Problem 8.

Solution. (a) Denote RV J the type of battery that is drawn, $j = 1, 2, ..., n$.

$$\begin{aligned}
\mathbb{P}\left(X \leq t\right) &= \sum_{j=1}^n \mathbb{P}\left(X \leq t | J = j\right) \mathbb{P}\left(J = j\right)] \\
&= \sum_{j=1}^n (1 - e^{-\lambda_j t}) P_j
\end{aligned} \tag{32}$$

So $\bar{F}_X = \sum_{j=1}^n e^{-\lambda_j t} P_j$ and $f_X(t) = \sum_{j=1}^n \lambda_j e^{-\lambda_j t} P_j$.
(b) We want to consider $\mathbb{P}\left(J = 1 | X > t\right)$.

$$\begin{aligned}
\mathbb{P}\left(J = 1 | X > t\right) &= \frac{\mathbb{P}\left(X > t | J = 1\right) \mathbb{P}\left(J = 1\right)}{\mathbb{P}\left(X > t\right)} \\
&= \frac{e^{-\lambda_1 t} \cdot P_1}{e^{-\lambda_1 t} \cdot P_1 + \sum_{j=2}^n e^{-\lambda_j t} P_j} \\
&= \frac{P_1}{P_1 + \sum_{j=2}^n e^{(\lambda_1 - \lambda_j)t} P_j}
\end{aligned} \tag{33}$$

Since $\lambda_1 \geq \lambda_j$ for all j, $\mathbb{P}\left(J = 1 | X > t\right) \nearrow$ with t. And we also observe that $\mathbb{P}\left(J = 1 | X > t\right) \to 1$ when $t \to \infty$.

Problem 9. Issurance claims are made at times as Poisson process with λ, u.e. time of n^{th} claim is waiting time S_n. Amount C_n associated with each claim has known i.i.d dist with mean μ. So the PV of total insurance payment up to t is

$$D(t) = \sum_{i=1}^{N(t)} e^{-\alpha S_i} C_i$$

Solution.

$$\begin{aligned}
\mathbb{E}\left[D(t)\right] &= \sum_{n\geq 0} \mathbb{E}\left[D(t); \{N(t) = n\}\right] \\
&= \sum_{n\geq 0} \mathbb{E}\left[D(t)|N(t) = n\right] \cdot \mathbb{P}\left(N(t) = n\right) \\
&= \sum_{n\geq 0} \left(\sum_{i=1}^{n} \mathbb{E}\left[e^{-\alpha S_i} C_i | N(t) = n\right] \right) \mathbb{P}\left(N(t) = n\right) \\
&= \sum_{n\geq 0} \left(\sum_{i=1}^{n} \mu \int_0^\infty e^{-\alpha s} f_{S_i|N(t)}(s|n) ds \right) \mathbb{P}\left(N(t) = n\right) \quad \text{(We have } S_i|N(t) \sim \mathcal{U}(0,t)\text{)} \\
&= \sum_{n\geq 0} \left(\sum_{i=1}^{n} \frac{\mu}{\alpha t}(1 - e^{-\alpha t}) \right) \mathbb{P}\left(N(t) = n\right) \\
&= \sum_{n\geq 0} \frac{n\mu}{\alpha t}(1 - e^{-\alpha t}) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= 0 + \sum_{n\geq 1} \frac{n\mu}{\alpha t}(1 - e^{-\alpha t}) \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t}) \sum_{n\geq 1} \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t})
\end{aligned} \tag{34}$$

Problem 10. $\{N(t) : t \geq 0\}$ be Poisson process, indep. of $\{X_i\}$ i.i.d with mean μ variance σ^2. Find $\mathbb{C}\text{ov}\left[N(t), \sum_{i=1}^{N(t)} X_i\right]$

Solution. $N(t) \sim \text{Pois}(\lambda t)$, hence $\mathbb{E}\left[N(t)\right] = \lambda t$. Denote $S_N := \sum_{i=1}^{N(t)} X_i$. By **Wald's Identity**, $\mathbb{E}\left[S_N\right] = \mathbb{E}\left[N(t)\right] \mathbb{E}\left[X_1\right] = \lambda t \mu$. Then we calculate $\mathbb{E}\left[N(t) S_N\right]$:

$$\begin{aligned}
\mathbb{E}\left[N(t) S_N\right] &= \mathbb{E}\left[\mathbb{E}\left[N(t) \sum_{i=1}^{N(t)} X_i \middle| N(t) \right] \right] \\
&= \mathbb{E}\left[N(t) \sum_{i=1}^{N(t)} \mathbb{E}\left[X | N(t)\right] \right] \\
&= \mathbb{E}\left[N^2(t)\mu\right] = ((\lambda t)^2 + \lambda t)\mu
\end{aligned} \tag{35}$$

Hence $\mathbb{C}\text{ov}\left[N(t), S_N\right] = \mathbb{E}\left[N(t) S_N\right] - \mathbb{E}\left[S_N\right] \mathbb{E}\left[N(t)\right] = \lambda t \mu$

Problem 11.

Solution. (a) Since $\{X_i\}$ are i.i.d Exponential, we have $\mathbb{E}\left[\prod_1^n X_i\right] = \prod_1^n \mathbb{E}\left[X_i\right] = \frac{1}{\mu^n}$.

$$\begin{aligned}
\mathbb{E}\left[S(t)\right] &= s\sum_{n\geq 0}\mathbb{E}\left[\prod_1^{N(t)} X_i \middle| N(t)=n\right]\cdot\mathbb{P}\left(N(t)=n\right) \\
&= s\sum_{n\geq 0}\frac{1}{\mu^n}\cdot\frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= se^{-\lambda t}\sum_{n\geq 0}\cdot\frac{(\lambda t/\mu)^n}{n!} = se^{-\lambda t+\frac{\lambda t}{\mu}}
\end{aligned} \tag{36}$$

(b) Similarly we have $\mathbb{E}\left[\prod_1^n X_i^2\right] = \prod_1^n \mathbb{E}\left[X_i^2\right] = \left(\frac{2}{\mu^2}\right)^n$

$$\begin{aligned}
\mathbb{E}\left[S^2(t)\right] &= s^2\sum_{n\geq 0}\mathbb{E}\left[\prod_1^{N(t)} X_i^2 \middle| N(t)=n\right]\cdot\mathbb{P}\left(N(t)=n\right) \\
&= s^2\sum_{n\geq 0}\frac{2}{\mu^{2n}}\cdot\frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= s^2e^{-\lambda t}\sum_{n\geq 0}\cdot\frac{(2\lambda t/\mu^2)^n}{n!} = se^{-\lambda t+\frac{2\lambda t}{\mu^2}}
\end{aligned} \tag{37}$$

Problem 12. For a Poisson process show that for $s < t$,

$$\mathbb{P}\left(N(s)=k|N(t)=n\right) = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}$$

Proof. Define $\Delta_{s,t} := N(t)-N(s)$, it is clear that following two events are equivalent:

$$\{N(s)=k, N(t)=n\} \iff \{N(s)=k, \Delta_{s,t}=n-k\} \quad (\dagger)$$

Therefore we have

$$\begin{aligned}
\mathbb{P}\left(N(s)=k|N(t)=n\right) &= \frac{\mathbb{P}\left(N(s)=k, N(t)=n\right)}{\mathbb{P}\left(N(t)=n\right)} \\
&= \frac{\mathbb{P}\left(N(s)=k, \Delta_{s,t}=n-k\right)}{\mathbb{P}\left(N(t)=n\right)} \\
&= \frac{\mathbb{P}\left(N(s)=k\right)\mathbb{P}\left(\Delta_{s,t}=n-k\right)}{\mathbb{P}\left(N(t)=n\right)} \quad \text{(By independent increments.)} \\
&= \frac{e^{\lambda s}(\lambda s)^k}{k!}\cdot\frac{e^{\lambda(t-s)}(\lambda(t-s))^{n-k}}{(n-k)!}\bigg/\frac{e^{\lambda t}(\lambda t)^n}{n!} \\
&= \frac{n!}{k!(n-k)!}\frac{s^k(t-s)^{n-k}}{t^n} = \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}
\end{aligned} \tag{38}$$

$\square$

Problem 13.

Solution. (c) By definition of non-homogeneous Poisson process, we known that $N(t)$ is a Poisson RV with rate $m(t) = \int_0^t \lambda(x)dx$; and $\Delta_{s,s+t}$ is a Poisson RV with rate $m(t+s)-m(s) = \int_s^{t+s}\lambda(x)dx$ hence

$$\mathbb{P}\left(T_1 > t\right) = \mathbb{P}\left(N(t)=0\right) = e^{-m(t)} \tag{39}$$

Which implies $F_{T_1}(t) = 1 - e^{-m(t)}$, and $f_{T_1}(t) = \lambda(t)e^{-m(t)}$ for $t \geq 0$.
(a,b) Then we derive the distribution of T_2

$$\begin{aligned}\mathbb{P}(T_2 > t) &= \int_0^\infty \mathbb{P}(T_2 > t | T_1 = s) f_{T_1}(s) ds \\ &= \int_0^\infty \mathbb{P}(\Delta_{s,s+t} = 0) f_{T_1}(s) ds \\ &= \int_0^\infty e^{m(s+t)-m(s)} \lambda(s) e^{-m(s)} ds\end{aligned} \tag{40}$$

(TODO)

Problem 14.

Solution. (a) By the meaning of X, we can define it explicitly as

$$X := \begin{cases} 0 & N(t) = 0, \\ \sum_{i=1}^{N(t)} (t - S_i) & \text{Otherwise.} \end{cases} \tag{41}$$

Where $N(t)$ is counting at t, S_i is waiting time of event {The arrival of i^{th} person}. By theorem, we know $S_i | N(t) \sim i.i.d.\mathcal{U}(0, t)$; hence $\mathbb{E}[S_i | N(t)] = t/2$ for all $i \geq 1$.

$$\mathbb{E}[X | N(t)] = \sum_{i=1}^{N(t)} (t - \mathbb{E}[S_i | N(t)]) = \frac{tN(t)}{2} \tag{42}$$

(b) $\mathbb{V}\text{ar}[S_i | N(t)] = (t-0)^2/12 = t^2/12$

$$\mathbb{V}\text{ar}[X | N(t)] = \sum_{i=1}^{N(t)} \mathbb{V}\text{ar}[-S_i | N(t)] = \frac{t^2 N(t)}{12} \tag{43}$$

(3) $N(t) \sim \text{Pois}(\lambda t)$

$$\begin{aligned}\mathbb{V}\text{ar}[X] &= \mathbb{E}[\mathbb{V}\text{ar}[X | N(t)]] + \mathbb{V}\text{ar}[\mathbb{E}[X | N(t)]] \\ &= \mathbb{E}\left[\frac{t^2 N(t)}{12}\right] + \mathbb{V}\text{ar}\left[\frac{tN(t)}{2}\right] \\ &= \frac{t^2 \lambda t}{12} + \frac{t^2 \lambda t}{4} = \frac{t^3 \lambda}{3}\end{aligned} \tag{44}$$

Problem 15. Calculate $\mathbb{C}\text{ov}[X(t), X(s)]$ for compound Poisson: for $\{Y_i\}$ i.i.d and independent of $\{N(t) : t \geq 0\}$

$$X(t) := \sum_{i=1}^{N(t)} Y_i$$

Solution. WLOG assume $s \leq t \Rightarrow N(s) \leq N(t)$. And suppose Poisson process accociated with $X(t)$ has rate λ, then by **Wald's Identity**: $\mathbb{E}[X(t)] = \mathbb{E}[N(t)]\mathbb{E}[Y_1]$. It suffices to compute $\mathbb{E}[X(t)X(s)]$

$$\begin{aligned}\mathbb{E}[X(t)X(s)] &= \mathbb{E}\left[\sum_{i=1}^{N(s)} Y_i^2 + \sum_{(i,j), i \neq j}^{(N(s),N(t))} Y_i Y_j\right] \\ &= \mathbb{E}[N(s)]\mathbb{E}\left[Y_1^2\right] + \mathbb{E}\left[\mathbb{E}\left[\sum_{(i,j), i \neq j}^{(N(s),N(t))} Y_i Y_j \middle| (N(s), N(t))\right]\right] \\ &= \mathbb{E}[N(s)]\mathbb{E}\left[Y_1^2\right] + \mathbb{E}\left[\sum_{(i,j), i \neq j}^{(N(s),N(t))} \mathbb{E}[Y_i Y_j | (N(s), N(t))]\right] \\ &= \mathbb{E}[N(s)]\mathbb{E}\left[Y_1^2\right] + \mathbb{E}\left[(N(s)N(t) - N(s))\,\mathbb{E}^2[Y_1]\right] \\ &= \mathbb{E}[N(s)]\mathbb{V}\text{ar}[Y_1] + \mathbb{E}^2[Y_1]\mathbb{E}[N(s)N(t)] \quad (\dagger)\end{aligned} \tag{45}$$

Since $N(t) = N(s) + \Delta_{s,t}$, $\Delta_{s,t} \sim \text{Pois}(\lambda(t-s))$ and independent wrt $N(s)$. We have $\mathbb{E}\left[N(s)N(t)\right] = \mathbb{E}\left[N^2(s)\right] + \mathbb{E}\left[N(s)\right]\mathbb{E}\left[N(t-s)\right]$. Therefore

$$\begin{aligned}
(\dagger) - \mathbb{E}\left[X(t)\right]\mathbb{E}\left[X(s)\right] &= \lambda s \mathbb{V}\text{ar}\left[Y_1\right] + \mathbb{E}^2\left[Y_1\right]\left(\lambda^2 s^2 + \lambda s + \lambda s(\lambda t - \lambda s)\right) - \lambda s \lambda t \mathbb{E}^2\left[Y_1\right] \\
&= \lambda s(\mathbb{V}\text{ar}\left[Y_1\right] + \mathbb{E}^2\left[Y_1\right]) \\
&= \lambda s \mathbb{E}\left[Y_1^2\right]
\end{aligned} \tag{46}$$

Generalize this result to arbitrary s, t, we conclude that

$$\mathbb{C}\text{ov}\left[X(t), X(s)\right] = \min\{\lambda s, \lambda t\} \cdot \mathbb{E}\left[Y_1^2\right] \tag{47}$$
