Symmetric Positive Definite Matrices

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1 Preliminaries

1.1 Inner Products

- Inner product on $\mathbb{R}^n$: $\langle \boldsymbol{v}, \boldsymbol{u} \rangle = \boldsymbol{v}^\top \boldsymbol{u}$. Has 3 properties:
  1. Positivity: $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$; $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \iff \boldsymbol{v} = \boldsymbol{0}$.
  2. Bilinearity: $\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle = a\langle \boldsymbol{x}, \boldsymbol{z} \rangle + b\langle \boldsymbol{y}, \boldsymbol{z} \rangle$; $\langle \boldsymbol{z}, a\boldsymbol{x} + b\boldsymbol{y} \rangle = a\langle \boldsymbol{z}, \boldsymbol{x} \rangle + b\langle \boldsymbol{z}, \boldsymbol{y} \rangle$.
  3. Symmetry: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$.
- Norm on $\mathbb{R}^n$: $\|\boldsymbol{v}\|^2 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle$
- Inner product on $\mathbb{C}^n$: $\langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\mathbb{C}} = \boldsymbol{v}^H \boldsymbol{u}$. Where $\boldsymbol{v}^H = (\bar{v}_1, ..., \bar{v}_n)$ is conjugate transpose of col vector $\boldsymbol{v}$, $\bar{v}$ is complex conjugate of entry v, i.e.

  $$\langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\mathbb{C}} = \sum_{j=1}^{n} u_j \bar{v}_j$$

  Also 3 properties:
  1. Positivity: $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} \geq 0$; $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} = 0 \iff \boldsymbol{v} = \boldsymbol{0}$.
  2. Sesquilinearity:

     $$\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle_{\mathbb{C}} = a\langle \boldsymbol{x}, \boldsymbol{z} \rangle_{\mathbb{C}} + b\langle \boldsymbol{y}, \boldsymbol{z} \rangle_{\mathbb{C}}$$
     $$\langle \boldsymbol{z}, a\boldsymbol{x} + b\boldsymbol{y} \rangle_{\mathbb{C}} = \bar{a}\langle \boldsymbol{z}, \boldsymbol{x} \rangle_{\mathbb{C}} + \bar{b}\langle \boldsymbol{z}, \boldsymbol{y} \rangle_{\mathbb{C}}$$

     Proof. Use conjugate symmetry. $\langle \boldsymbol{z}, a\boldsymbol{x} + b\boldsymbol{y} \rangle_{\mathbb{C}} = \overline{\langle a\boldsymbol{x} + b\boldsymbol{y}, \boldsymbol{z} \rangle_{\mathbb{C}}} = \bar{a}\overline{\langle \boldsymbol{x}, \boldsymbol{z} \rangle_{\mathbb{C}}} + \bar{b}\overline{\langle \boldsymbol{y}, \boldsymbol{z} \rangle_{\mathbb{C}}} = \bar{a}\langle \boldsymbol{z}, \boldsymbol{x} \rangle_{\mathbb{C}} + \bar{b}\langle \boldsymbol{z}, \boldsymbol{y} \rangle$
  3. Conjugate Symmetry: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$.
- Norm on $\mathbb{C}^n$: $\|v\|_{\mathbb{C}}^2 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} = \boldsymbol{v}^H \boldsymbol{v}$.
- Let $\boldsymbol{A}$ be a matrix with complex entries, its conjugate transpose (hermitian): $\boldsymbol{A}^H = \overline{\boldsymbol{A}}^\top$. We have $(\boldsymbol{AB})^H = \boldsymbol{B}^H \boldsymbol{A}^H$.
  And by definition of inner product on complex field, $\langle \boldsymbol{Au}, \boldsymbol{v} \rangle_{\mathbb{C}} = \boldsymbol{v}^H \boldsymbol{Au} = \langle \boldsymbol{u}, (\boldsymbol{v}^H \boldsymbol{A})^H \rangle_{\mathbb{C}} = \langle \boldsymbol{u}, \boldsymbol{A}^H \boldsymbol{v} \rangle_{\mathbb{C}}$. Similarly $\langle \boldsymbol{u}, \boldsymbol{Bv} \rangle = \langle \boldsymbol{B}^H \boldsymbol{u}, \boldsymbol{v} \rangle$.

2 Properties

2.1 Basics

- Symmetric matrix: $\boldsymbol{A} = \boldsymbol{A}^\top$. Let $\boldsymbol{X}$ be an arbitrary matrix, $\boldsymbol{X}^\top \boldsymbol{X}$ and $(\boldsymbol{X} + \boldsymbol{X}^\top)$ are symmetric.
- $\langle \boldsymbol{Au}, \boldsymbol{v} \rangle = \boldsymbol{v}^\top \boldsymbol{Au} = \langle \boldsymbol{u}, \boldsymbol{A}^\top \boldsymbol{v} \rangle$. And $\langle \boldsymbol{u}, \boldsymbol{Bv} \rangle = \boldsymbol{u}^\top \boldsymbol{Bv} = \langle \boldsymbol{B}^\top \boldsymbol{u}, \boldsymbol{v} \rangle$.
- $\boldsymbol{u} \perp \boldsymbol{v} \iff \langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$.

· A matrix $\boldsymbol{Q}$ is orthogonal iff any two different columns of it are orthonormal. (orthogonal and unit norm); or any two different rows of it are orthonormal.

Thm. A square matrix $\boldsymbol{Q}$ is orthogonal iff $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^\top$.

Proof. We have $\boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}$.

$$\boldsymbol{Q}^\top \boldsymbol{Q} = \begin{pmatrix} \boldsymbol{q}_1^\top \\ \boldsymbol{q}_2^\top \\ \vdots \\ \boldsymbol{q}_n^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_n \end{pmatrix} = \begin{pmatrix} \|\boldsymbol{q}_1\|^2 & \langle \boldsymbol{q}_1, \boldsymbol{q}_2 \rangle & \cdots & \langle \boldsymbol{q}_1, \boldsymbol{q}_n \rangle \\ \langle \boldsymbol{q}_2, \boldsymbol{q}_1 \rangle & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \langle \boldsymbol{q}_n, \boldsymbol{q}_1 \rangle & \cdots & \cdots & \|\boldsymbol{q}_n\|^2 \end{pmatrix}$$

Hence $\boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I} \iff \|\boldsymbol{q}_i\| = 1$ for all i and $\langle \boldsymbol{q}_j, \boldsymbol{q}_k \rangle = 0$ for $k \neq j$. □

2.2 Eigvals and Eigvecs

Thm. Any eigval of symmetric matrix is real number.

Proof. We have $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$. $\boldsymbol{A}$ symmetric and real, hence $\boldsymbol{A} = \boldsymbol{A}^H$. Consider

$$\langle \boldsymbol{A}\boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} = \boldsymbol{v}^H \boldsymbol{A}\boldsymbol{v} = \langle \boldsymbol{v}, \boldsymbol{A}^H \boldsymbol{v} \rangle_{\mathbb{C}} = \langle \boldsymbol{v}, \boldsymbol{A}\boldsymbol{v} \rangle_{\mathbb{C}}$$

And $\langle \boldsymbol{A}\boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} = \langle \lambda \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathbb{C}} = \lambda \|\boldsymbol{v}\|_{\mathbb{C}}^2$; $\langle \boldsymbol{v}, \boldsymbol{A}\boldsymbol{v} \rangle_{\mathbb{C}} = \langle \boldsymbol{v}, \lambda \boldsymbol{v} \rangle_{\mathbb{C}} = \bar{\lambda} \|\boldsymbol{v}\|_{\mathbb{C}}^2$
$\Rightarrow \lambda = \bar{\lambda}$, which implies that λ is real. □

· Eigvecs corresponding to different eigvals of symmetric matrix are orthogonal.
Proof. $\langle \boldsymbol{A}\boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \langle \lambda_1 \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \lambda_1 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$.
$\langle \boldsymbol{A}\boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \langle \boldsymbol{v}_1, \boldsymbol{A}^\top \boldsymbol{v}_2 \rangle = \langle \boldsymbol{v}_1, \boldsymbol{A}\boldsymbol{v}_2 \rangle = \langle \boldsymbol{v}_1, \lambda_2 \boldsymbol{v}_2 \rangle = \lambda_2 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$.
Hence $\lambda_2 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \lambda_1 \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle$; $\lambda_1 \neq \lambda_2 \Rightarrow \boldsymbol{v}_1 \perp \boldsymbol{v}_2$. □