Lecture 1

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1 Euler Method

In this section we consider the numerical methods for solving the 1st order IVP (†):

$$\begin{cases} y' = f(t, y(t)), t \in [0, T] \\ y(0) = y_0 \end{cases}$$

We specify a lattice $0 < t_1 < t_2 < ... < t_n = T$ with $t_{k+1} - t_k = h$, $t_n = nh$. Thus the integral form of the IVP is:

$$y(t_{k+1}) - y(t_k) = \int_{t_k}^{t_{k+1}} f(t, y(t))dt$$

Thm. (**Existence and Uniqueness of Solution**) If $f(t, y(t))$ satisfies Lipschitz condition, i.e.

$$|f(t, y) - f(t, z)| \le L|y - z|$$

for $\forall y, z, t$, constant L; and $y(t)$ is continuous in $t \Rightarrow \exists$ 1 solution for (†).

Def. (**Euler Method**): We define $y_0, y_1, ..., y_n$ as the approximations to the values of y on the lattice, i.e. $y(0), y(t_1), ..., y(t_n)$. The *Explicit* Euler Method is the iterative procedure using rectangle approximation of the integral to the RHS of the integral form:

$$y_{k+1} - y_k = f(t_k, y_k)h$$

And the *Implicit* Euler Method use the right side of the rectangle:

$$y_{k+1} - y_k = f(t_{k+1}, y_{k+1})h$$

Def. **Convergence**: A method to solve (†) is said to be convergent if for any ODE with Lipschitz f, $T > 0$, it is true that

$$\lim_{h \to 0} \max_{k=0,1,...,\lfloor \frac{T}{h} \rfloor} |y_k^{[h]} - y(t_k)| = 0$$

in which the superscript $[h]$ is to distinguish that the step size is h when getting $\{y_k\}$. In the following text we just say y_k.

Thm. The Euler Method is convergent.

Proof. Define error $|e_k| := |y_k - y(t_k)|$, we are going to show $e_k \to 0$ as $h \to 0$.
Consider the Taylor expansion of $y(t_{k+1})$ at $y(t_k)$:

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + y'(t_k)(t_{k+1} - t_k) + \frac{1}{2}y''(\xi_k)(t_{k+1} - t_k)^2 \\ &= y(t_k) + f(t_k, y(t_k))h + \frac{1}{2}y''(\xi_k)h^2 \end{aligned} \tag{1}$$

And the Euler scheme:

$$y_{k+1} = y_k + f(t_k, y_k)h \tag{2}$$

(2) - (1) $\Rightarrow$

$$\begin{aligned}
e_{k+1} &= e_k + h\left[f(t_k, y_k) - f(t_k, y(t_k))\right] - \frac{1}{2}y''(\xi_k)h^2 \\
|e_{k+1}| &\le |e_k| + hL|e_k| + ch^2 \\
&\le ch^2(1 + ... + (1+hL)^{k-1}) \\
&= ch^2\frac{(1+hL)^k - 1}{hL} \\
&\le \frac{ch}{L}(1+hL)^{\frac{T}{h}} \\
&\le h \cdot \left(\frac{c}{L}e^{LT}\right) = O(h) \quad \blacksquare
\end{aligned} \tag{3}$$

Ex.

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}$$

The analytical solution is $y = (\frac{2}{3}t)^{3/2}$, but using explicit Euler one can only find $y \equiv 0$ (which is another solution). The problem is that Lipschitz condition is not satisfied at $t = 0$, so it has multiple solutions.

Def. **Truncation Error**: For Explicit Euler,

$$R_k = \frac{y(t_{k+1}) - y(t_k)}{h} - f(t_k, y(t_k))$$

is the difference between the finite difference approx and the true value of y' at t_k, it measures how much does the finite difference deviate from the ODE.

Def. A method is said to be of order p if $|R_k| = O(h^p)$.

2 Trapezoid Method

Def. **Trapezoid Method**: We can use trapezoid formula to approximate the value of integral

$$y(t_{k+1}) - y(t_k) = \int_{t_k}^{t_{k+1}} f(t, y(t))dt$$

which is:

$$y_{k+1} = y_k + \frac{1}{2}h\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$

Thm. Trapezoid method is of order 2.

Proof.

$$\begin{aligned}
R_k &= \frac{y(t_{k+1}) - y(t_k)}{h} - \frac{1}{2}[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))] \\
&= \frac{1}{h}\left[y'(t_k)h + \frac{1}{2}y''(t_k)h^2 + \frac{1}{6}y'''(t_k)h^3 + O(h^4)\right] - \frac{1}{2}[y'(t_k) + y'(t_{k+1})] \\
&= \left[y'(t_k) + \frac{1}{2}y''(t_k)h + \frac{1}{6}y'''(t_k)h^2 + O(h^3)\right] - \frac{1}{2}\left[y'(t_k) + \left(y'(t_k) + y''(t_k)h + \frac{1}{2}y'''(t_k)h^2 + O(h^3)\right)\right] \\
&= -\frac{1}{12}y'''(t_k)h^2 + O(h^3) \\
&= O(h^2) \quad \blacksquare
\end{aligned} \tag{4}$$

Thm. Trapezoid method is convergent.

Proof.

$$y_{k+1} = y_k + \frac{1}{2}h\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$

By definition of trunc error:

$$y(t_{k+1}) = y(t_k) + \frac{1}{2}h\left[f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))\right] + hR_k$$

The first equiation minus the second one:

$$\begin{aligned}
e_{k+1} &= e_k + \frac{h}{2}[f(t_k, y_k) - f(t_k, y(t_k))] + \frac{h}{2}[f(t_{k+1}, y_{k+1}) - f(t_{k+1}, y(t_{k+1}))] + O(h^3) \\
|e_{k+1}| &\leq |e_k| + \frac{hL}{2}|e_k| + \frac{hL}{2}|e_{k+1}| + O(h^3) \\
&\leq \frac{1+\frac{hL}{2}}{1-\frac{hL}{2}}|e_k| + \frac{C}{1-\frac{hL}{2}}h^3 \\
&\leq C(T)\cdot h^2 = O(h^2)
\end{aligned} \tag{5}$$

Where $C(T)$ is a constant on T, L is the constant in Lipschitz condition. ■

3 Theta Method

We can generalize the trapezoid method to any linear combination of k and $k+1$ sides:

$$y_{k+1} = y_k + h(\theta f(t_k, y_k) + (1-\theta)f(t_{k+1}, y_{k+1}))$$

Note that

- $\theta = 1$, Euler method, and it is the only emplicit method.
- $\theta = 0$, Implicit Euler.
- $\theta = 0.5$, Trapezoid.