Lecture 5

Zed

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1 Newton's Method for Solving Nonlinear Equations

We want to find roots for nonlinear equation $g(x) = 0$.

Algo. **Newton's Method**

- Initialize a starting point x_0.
- Update x by $x_{n+1} \leftarrow x_n - \frac{g(x_n)}{g'(x_n)}$, suppose $g'(x_n) \neq 0$.
- Stop when $\|g(x_n)\| \leq thres$ (small), $\|x_{n+1} - x_n\| \leq thres$ (small) or $n \geq K$ (fail to converge).

The Newton's method only converges in a local sense. Suppose the real zero is α, s.t. $g(\alpha) = 0$, then using taylor expansion at x_n:

$$\begin{aligned} 0 = g(\alpha) &= g(x_n) + g'(x_n)(\alpha - x_n) + \frac{1}{2}g''(\xi_n)(\alpha - x_n)^2 \\ \Rightarrow 0 &= \left(\frac{g(x_n)}{g'(x_n)} - x_n\right) + \alpha + \frac{g''(\xi_n)}{2g'(x_n)}(\alpha - x_n)^2 \end{aligned} \tag{1}$$

Hence exist bound M such that

$$|\alpha - x_{n+1}| = \frac{g''(\xi_n)}{2g'(x_n)}|\alpha - x_n|^2 \leq M|\alpha - x_n|^2$$

That is, if the error at n-th iteration $|\alpha - x_n|$ is already small, the next error at $(n+1)$-th iteration will be the square of it. Which implies a (locally) quadratic convergence rate.

2 Implicit Methods

2.1 Implicit Runge-Kutta Method

We consider the Runge-Kutta method with table:

$$\begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & \cdots & a_{1,s-1} & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2,s-1} & a_{2s} \\ c_3 & a_{31} & a_{32} & \cdots & a_{3,s-1} & a_{3s} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array} = \begin{array}{c|c} 1/2 & 1/2 \\ \hline & 1 \end{array}$$

A table of this kind with a full matrix of entries $\boldsymbol{A}$ (instead of entries only in the lowertriangular area) suggests that this is an *implicit* RK method.

For the listed example, we have

$$\begin{cases} y_{n+1} = y_n + hb_1k_1 \\ k_1 = f(t_n + c_1h, y_n + ha_{11}k_1) \end{cases} \tag{2}$$

And insert the values of coefficients:

$$y_{n+1} = y_n + h\left(f(t_n + \tfrac{1}{2}h, \tfrac{y_{n+1}}{2} + \tfrac{y_n}{2})\right)$$

Which is usually referred to as the *Implicit Midpoint Method*. This method is the simplest implicit RK method, the simplest *Gauss-Legendre* method. And a *Symplectic method*, which is energy-preserving.

The Midpoint method is of order 2 (see the first question in HW1).

Ex. (*Energy Preserving*) Consider the Hamilton system: $p' = -q, q' = p$; with $p(0) = q(0) = 1$. We can find a Hamilton function H such that $p' = -\frac{\partial H}{\partial q}$, $q' = \frac{\partial H}{\partial p}$. And hence $\frac{\partial H}{\partial t} = 0$. However, when we use the explicit RK method to solve the system (for example, the `ode45` in `Matlab`), we will find that the method does not preserve energy, i.e. $\frac{\partial H}{\partial t} \neq 0$.