Functional Analysis Assignment III

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Problem 1. Let $K = \{(x, y)| - 1 < x < 1, -1 < y < 1, \text{or } x = 1, -1 \leq y \leq 0\} \subset \mathbb{R}^2$, $(x_1, y_1) = (1, \frac{1}{2})$, and $(x_2, y_2) = (1, 1)$.

(a) Construct an explicit unique nonzero linear functional l satisfying $l(x_1, y_1) = 1$ and

$$l(x, y) \leq l(x_1, y_1) \quad \text{for all } (x, y) \in K$$

(b) Show that there are infinitely many linear functionals l satisfying $l(x_2, y_2) = 1$ and

$$l(x, y) \leq l(x_2, y_2) \quad \text{for all } (x, y) \in K$$

Proof. (a) For $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$, the linear functional of $\boldsymbol{x}$ takes form

$$l(\boldsymbol{x}) = \boldsymbol{a}\boldsymbol{x}^\top$$

Where $\boldsymbol{a} = (a_1, a_2)$.
(*Existence*): $\boldsymbol{a} = (1, 0)$ satisfies the condition. When $\boldsymbol{a} = (1, 0)$, we have $l(\boldsymbol{z}_1) = 1$. And for all $\boldsymbol{x} \in K$, $l(\boldsymbol{x}) = x \leq 1 = l(\boldsymbol{z}_1)$.
(*Uniqueness*): suppose there exists another $\boldsymbol{a}' = (1, a)$ such that $\forall \boldsymbol{x} \in K$,

$$l(\boldsymbol{x}) = x + ay \leq 1 + \frac{1}{2}a = l(\boldsymbol{z}_1)$$

We pick $\boldsymbol{x} = (1, 0) \in K \Rightarrow a \geq 0$. Then consider for all $a > 0$, there exists $n > 5$ (and sufficiently large), such that $-1 < 1 - \frac{a}{n} < 1$. Hence we have $\boldsymbol{x}' = (1 - \frac{a}{n}, \frac{4}{5}) \in K$, so

$$\begin{aligned} l(\boldsymbol{x}') = 1 - \frac{a}{n} + \frac{4}{5}a \leq 1 + \frac{a}{2} \\ \Rightarrow \frac{3}{5}a \leq -\frac{a}{n} + \frac{4}{5}a \leq \frac{a}{2} \end{aligned} \tag{1}$$

Which implies that $a \leq 0$. Contradicts the assumption that $a > 0$. So the only feasible linear functional is $l(\boldsymbol{x}) = (1, 0)\boldsymbol{x}^\top$
(b) $\boldsymbol{a} = (1, k)$ are all feasible linear functionals for $k \geq 0$. Because for all $\boldsymbol{x} = (x, y) \in K$, $x, y \leq 1$.

$$l(\boldsymbol{x}) = x + ky \leq 1 + k = l(\boldsymbol{z}_2) \tag{2}$$

□

Problem 2. Prove that any two norms on a finite dimensional linear space X are equivalent.

Proof. Let $\|\cdot\|$ be an arbitrary norm on X. For all $\boldsymbol{x} \in X$ we can write

$$\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i \tag{3}$$

Where $\boldsymbol{e}_i$ are basis of X with unit lenth under $\|\cdot\|$, i.e. $\|\boldsymbol{e}_i\| = 1$. Then it suffice to show that any norm on finite (n) dimensional linear space is equivalent to ∞-norm $\|\boldsymbol{x}\|_\infty = \max\{|x_1|, ..., |x_n|\}$. By *Triangle Ineq.* and *Homogeneity*:

$$\|\boldsymbol{x}\| \leq \sum_{i=1}^{n} \|x_i \boldsymbol{e}_i\| = \sum_{i=1}^{n} |x_i| \|\boldsymbol{e}_i\| = \sum_{i=1}^{n} |x_i| \leq n\|\boldsymbol{x}\|_\infty \tag{4}$$

Next we show the other direction. Let $\boldsymbol{I} : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|)$ Consider closed unit ball $B_1(\mathbf{0}) \subset (X, \|\cdot\|_\infty)$; clearly $B_1(\mathbf{0})$ is compact in $(X, \|\cdot\|_\infty)$.
Claim: $\boldsymbol{I}$ is continuous.
Proof of Claim: For any fixed $\boldsymbol{x} \in (X, \|\cdot\|_\infty)$, and for all $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{n}$, such that $\forall \boldsymbol{z} \in (X, \|\cdot\|_\infty)$ with $\|\boldsymbol{z} - \boldsymbol{x}\|_\infty \le \delta \Rightarrow$

$$\|\boldsymbol{z} - \boldsymbol{x}\| \le n\|\boldsymbol{z} - \boldsymbol{x}\|_\infty = n\delta = \epsilon \tag{5}$$

Therefore, $\|\boldsymbol{I}\|$ attains maximum and minimum on $B_1(\mathbf{0})$. Denote the minimum: $\min\limits_{\boldsymbol{x} \in B_1(\mathbf{0})} \|\boldsymbol{I}(\boldsymbol{x})\| = c$ for a constant $c > 0$. Let $\boldsymbol{z} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty} \in B_1(\mathbf{0})$, we have

$$\boldsymbol{I}(\boldsymbol{z}) = \left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_\infty} \right\| = \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\|_\infty} \ge c \tag{6}$$

So we have $c\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\| \le n\|\boldsymbol{x}\|_\infty$. If $c > \frac{1}{n}$, By $\frac{1}{n}\|\boldsymbol{x}\|_\infty \le c\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\| \le n\|\boldsymbol{x}\|_\infty$ we are done. Else, there exists $n' \ge n$, such that $0 < \frac{1}{n'} \le c \le \frac{1}{n}$, implies that

$$\frac{1}{n'}\|\boldsymbol{x}\|_\infty \le c\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\| \le n\|\boldsymbol{x}\|_\infty \le n'\|\boldsymbol{x}\|_\infty \tag{7}$$

For arbitrary norm $\|\cdot\|$, finished the proof. □

Problem 3. Prove the Holder inequality for l^p $(p \in (1, \infty))$. More precisely, for $x = (x_1, x_2, ...) \in l^p$ $(p \in (1, \infty))$ and $y = (y_1, y_2, ...) \in l^{\frac{p}{p-1}}$, show that

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \le \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}$$

Furthermore, the equality in (1) holds $\iff$ $\arg x_j y_j$ and $|x_j|^p / |y_j|^{\frac{p}{p-1}}$ are independent of j.

Proof. We first prove **Young's Ineq**: $p, q > 1$ such that $1/p + 1/q = 1$ then for any x, y:

$$xy \le \frac{1}{p}x^p + \frac{1}{q}y^q \tag{8}$$

Because e^x is convex,

$$LHS = e^{\log x + \log y} = e^{\frac{1}{p}\log x^p + \frac{1}{q}\log y^q} \le \frac{1}{p}e^{\log x^p} + \frac{1}{q}e^{\log y^q} = RHS \tag{9}$$

It is clear that the equility holds $\iff$ $y = x^{p/q}$.
Now denote $A := (\sum_{i \ge 1} |x_i|^p)^{1/p}$, $B := (\sum_{i \ge 1} |y_i|^q)^{1/q}$. Apply Young's Ineq:

$$\begin{aligned}
\frac{|\sum_{i \ge 1} x_i y_i|}{AB} &= \sum_{i \ge 1} \left| \frac{x_i}{A} \frac{y_i}{B} \right| \\
&\le \sum_{i \ge 1} \left| \frac{1}{p} \left(\frac{x_i}{A} \right)^p + \frac{1}{q} \left(\frac{y_i}{B} \right)^q \right| \\
&\le \frac{1}{p} \sum_{i \ge 1} \frac{|x_i|^p}{A^p} + \frac{1}{q} \sum_{j \ge 1} \frac{|y_i|^q}{B^q} \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned} \tag{10}$$

It can be seen directly from the second line (when Young's holds in equility) that equility holds $\iff$

$$\frac{|y_i|}{B} = \left(\frac{|x_i|}{A} \right)^{\frac{p}{q}} \iff \frac{|x_i|^p}{|y_i|^q} = \frac{A^p}{B^q} \text{ Independent of } i. \tag{11}$$

Finished the proof. □

Problem 4. (1) Verify that the composite of two linear maps is linear, and that the distributive law holds:

$$\boldsymbol{M}(\boldsymbol{N}+\boldsymbol{K}) = \boldsymbol{MN}+\boldsymbol{MK}$$

$$(\boldsymbol{M}+\boldsymbol{K})\boldsymbol{N} = \boldsymbol{MN}+\boldsymbol{KN}$$

(2) (*Thm.1*) Let $\boldsymbol{M}$ be a linear map $X \to U$,

1 The null space $N_{\boldsymbol{M}}$ is a linear subspace of X, the range $R_{\boldsymbol{M}}$ a linear subspace of U.
2 $\boldsymbol{M}$ is invertible iff $N_{\boldsymbol{M}} = \{0\}$ and $R_{\boldsymbol{M}} = U$.
3 $\boldsymbol{M}$ maps the quotient space $X/N_{\boldsymbol{M}}$ one-one onto $R_{\boldsymbol{M}}$.
4 If $\boldsymbol{M} : X \to U$ and $\boldsymbol{K} : U \to W$ are both invertible, so is their product and

$$(\boldsymbol{KM})^{-1} = \boldsymbol{M}^{-1}\boldsymbol{K}^{-1}$$

5 If $\boldsymbol{KM}$ is invertible, then

$$N_{\boldsymbol{M}} = \{0\}, R_{\boldsymbol{K}} = W$$

Proof. (1) Suppose $\boldsymbol{M} : U \to V$, $\boldsymbol{N} : X \to U$ are linear maps, $x, y \in X$. Linearity of composition and Distributive law are straightforward due to definitions

$$\begin{aligned}\boldsymbol{MN}(ax+by) &= \boldsymbol{M}(a\boldsymbol{N}(x)+b\boldsymbol{N}(y)) \\ &= a\boldsymbol{MN}(x)+b\boldsymbol{MN}(y)\end{aligned} \tag{12}$$

$$\begin{aligned}\boldsymbol{M}(\boldsymbol{N}+\boldsymbol{K})(x) &= \boldsymbol{M}(\boldsymbol{N}(x)+\boldsymbol{K}(x)) \\ &= \boldsymbol{MN}(x)+\boldsymbol{MK}(x)\end{aligned} \tag{13}$$

(2)

1 This is really trivial by definitions...
2 ($\Leftarrow$) It suffice to show $\boldsymbol{M}$ is bijective. $\boldsymbol{M}(X) = U$ implies $\forall x \in X$, $\exists u \in U$ such that $u = \boldsymbol{M}(x)$ (onto). Pick $x, x' \in X$ and suppose $\boldsymbol{M}(x) = \boldsymbol{M}(x')$; we can write $0 = \boldsymbol{M}(x) - \boldsymbol{M}(x) = \boldsymbol{M}(x'-x)$; since $N_M = \{0\} \Rightarrow x = x'$ (one-one). Finished the proof.
($\Rightarrow$) Very similar. $\boldsymbol{M}$ is onto $\Rightarrow \boldsymbol{M}(X) = U$. And assume $X \ni s, t \neq 0$, $s \in N_M$, we will have $\boldsymbol{M}(s+t) = 0 + \boldsymbol{M}(t)$, but $s+t \neq t$, contradicts the fact that $\boldsymbol{M}$ is one-one. Finished the other direction.
3 (*onto*): $\forall u \in R_M$, there exists $x \in X$ s.t. $\boldsymbol{M}(x) = u$. hence denote $r \in N_M$, $\boldsymbol{M}([x]) = \boldsymbol{M}(x) + \boldsymbol{M}(r) = \boldsymbol{M}(x)$.
(*one-one*): $\forall [x], [y] \in X/N_M$, by definition $[x]-[y] \notin N_M$. So

$$\boldsymbol{M}([x]-[y]) = \boldsymbol{M}([x]) - \boldsymbol{M}([y]) \neq 0 \tag{14}$$

Finished the proof.
4 By the alternative definition, $\boldsymbol{M}$ is invertible $\iff \exists$ *linear* map $\boldsymbol{M}^{-1}$, such that $\boldsymbol{MM}^{-1} = \boldsymbol{M}^{-1}\boldsymbol{M} = \boldsymbol{I}$. Where $\boldsymbol{I}$ is identity map. Clearly $\boldsymbol{IM} = \boldsymbol{M}$ for any linear map $\boldsymbol{M}$.
Claim: $\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}$ is a linear map, and $\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}\boldsymbol{KM} = \boldsymbol{KMM}^{-1}\boldsymbol{K}^{-1} = \boldsymbol{I}$.
Proof of Claim: Linearity is due to exercise 1. And

$$\begin{aligned}(\boldsymbol{M}^{-1}\boldsymbol{K}^{-1}\boldsymbol{KM})(x) &= \boldsymbol{M}^{-1}(\boldsymbol{K}^{-1}(\boldsymbol{K}(\boldsymbol{M}(x)))) \\ &= \boldsymbol{M}^{-1}(\boldsymbol{I}(\boldsymbol{M}(x))) \\ &= (\boldsymbol{M}^{-1}\boldsymbol{M})(x) = \boldsymbol{I}\end{aligned} \tag{15}$$

We proceed the same way for $\boldsymbol{KMM}^{-1}\boldsymbol{K}^{-1}$.
5 Assume $X \ni s, t \neq 0$, $s \in N_M$, We will have $\boldsymbol{KM}(t+s) = \boldsymbol{K}(\boldsymbol{M}(t)+0) = \boldsymbol{KM}(t)$, but $s+t \neq s$. Contradicts the fact $\boldsymbol{KM}$ is one-one. Hence $N_M = \{0\}$.
Since $\boldsymbol{KM} : X \to W$ is onto $\Rightarrow R_K = W$. If otherwise, there exists some $w \in W$, such that no element in U can be map onto w by $\boldsymbol{K}$; then clearly there is also no element in X that can be maped onto w.

□

Problem 5. (*Thm.3*) For any convex set $\boldsymbol{K}$,

$$p_K(x) \leq 1 \text{ If } x \in K.$$

$$p_K(x) < 1 \text{ Iff } x \text{ is an interior point of } K.$$

Proof. (1) is directly from definition, denote

$$p_K(x) := \inf\left\{a \middle| a > 0, \frac{x}{a} \in K\right\}$$

$p_K(x) \leq 1 \Rightarrow 1 \in \left\{a \middle| a > 0, \frac{x}{a} \in K\right\}$, i.e. $x \in K$.
(2) ($\Leftarrow$) x is an interior point of $K \iff \forall y \in X, \exists \epsilon(y) > 0$ such that $x + ty \in K$ as long as $|t| < \epsilon(y)$. We are free to pick $y = x$, then we have $(1+t)x \in K$ for all $0 \leq |t| < \epsilon(x)$. Therefore $p_K(x) \leq \frac{1}{1+|t|}$ for all $|t| \in [0, \epsilon(x))$, pick $|t| = \frac{\epsilon(x)}{2}$, yields $p_K(x) \leq \frac{1}{1+\epsilon(x)/2} < 1$. Finished the proof.
($\Rightarrow$) $p_K(x) < 1 \Rightarrow$ there exists $\epsilon > 0$ for which $(1+\epsilon)x \in K$ ($\triangle$).
Moreover, by definition of gauge function, $\forall y \in X$, $\frac{1}{\lambda} > p_K(y)$ we have $\lambda y \in K$ ($\dagger$).
Claim: For all $y \in X$, define $\epsilon(y) := \frac{\epsilon\lambda}{1+\epsilon}$, then as long as $|t| < \epsilon(y)$, we have $x + ty \in K$.
Proof of Claim: WLOG assume $t \geq 0$. Otherwise we just modify the claim as $|-t| < \epsilon(y)$. Since $t \leq \epsilon(y)$, exists $\delta > 0$, allowing us to write $t = \frac{\epsilon(\lambda-\delta)}{1+\epsilon}$. Hence

$$\begin{aligned} x + ty &= x + \frac{\epsilon(\lambda - \delta)}{1+\epsilon} y \\ &= \frac{1}{1+\epsilon} \cdot (1+\epsilon)x + \frac{\epsilon}{1+\epsilon} \cdot (\lambda - \delta) y \end{aligned} \tag{16}$$

By ($\triangle$) and ($\dagger$) $\Rightarrow$ $(1+\epsilon)x \in K$ and $(\lambda - \delta)y \in K$. Since K is convex, it follows that $x + ty \in K$. The *Claim* is equivalent to saying x in interior point of K, which finished the other direction of the proof.

□

Problem 6. For $(z, u) \in Z \oplus U := \{(z,u) | z \in Z, u \in U\}$, and Z, U normed linear space

a. Show that $\|(z,u)\| = \|z\|_Z + \|u\|_U$, $\max\{\|z\|_Z, \|u\|_U\}$, $\sqrt{\|z\|_Z^2 + \|u\|_U^2}$ are all norms.
b. Show that they are equivalent norms in terms of Def.(5).

Proof. (a) It suffices to check the 3 defining properties of norm. Base upon the fact that $\|\cdot\|_Z$ and $\|\cdot\|_U$ are norms

1 *Positivity* is trivial. $\|a(z,u)\| = \|az\|_Z + \|au\|_U = |a|(\|z\|_Z + \|u\|_U) = |a|\|(z,u)\|$ (*Homogeneity*)

$$\begin{aligned} \|(z_1,u_1) + (z_2,u_2)\| &= \|z_1 + z_2\|_Z + \|u_1 + u_2\|_U \\ &\leq \|z_1\|_Z + \|z_2\|_Z + \|u_1\|_U + \|u_2\|_U \\ &= \|(z_1,u_1)\| + \|(z_2,u_2)\| \quad (\textit{Triangle Ineq}) \end{aligned} \tag{17}$$

2 Since $\|\cdot\|_Z, \|\cdot\|_U$ are positive, $\max\{\|z\|_Z, \|u\|_U\} = 0 \iff \|z\|_Z = \|u\|_U = 0 \iff (z,u) = (0,0)$. (*Positivity*). *Homogeneity* is trivial.

$$\begin{aligned} \|(z_1,u_1) + (z_2,u_2)\| &= \max\{\|z_1 + z_2\|_Z, \|u_1 + u_2\|_U\} \\ &\leq \max\{\|z_1\|_Z + \|z_2\|_Z, \|u_1\|_U + \|u_2\|_U\} \\ &\leq \max\{\|z_1\|_Z + \|z_2\|_Z, \|u_1\|_U + \|u_2\|_U, \|z_1\|_Z + \|u_2\|_U, \|u_1\|_U + \|z_2\|_Z\} \\ &= \max\{\|z_1\|_Z, \|u_1\|_U\} + \max\{\|z_2\|_Z, \|u_2\|_U\} \\ &= \|(z_1,u_1)\| + \|(z_2,u_2)\| \quad (\textit{Triangle Ineq}) \end{aligned} \tag{18}$$

3 *Positivity* is trivial. $\|a(z,u)\| = \sqrt{\|az\|_Z^2 + \|au\|_U^2} = |a|\sqrt{\|z\|_Z^2 + \|u\|_U^2} = |a|\|(z,u)\|$ (*Homogeneity*). Denote $\langle x, y\rangle$ the scalar product of x, y,

$$\begin{aligned}
\|(z_1,u_1)+(z_2,u_2)\|^2 &= \|z_1+z_2\|_Z^2 + \|u_1+u_2\|_U^2 \\
&= (\|z_1\|_Z^2 + \|u_1\|_U^2) + (\|z_2\|_Z^2 + \|u_2\|_U^2) + 2(\|z_1\|\|z_2\|_Z + \|u_1\|\|u_2\|_U) \\
&= \|(z_1,u_1)\|^2 + \|(z_2,u_2)\|^2 + 2(\|z_1\|\|z_2\|_Z + \|u_1\|\|u_2\|_U) \\
&\le \|(z_1,u_1)\|^2 + \|(z_2,u_2)\|^2 + 2\sqrt{(\|z_1\|_Z^2 + \|u_1\|_U^2)(\|z_2\|_Z^2 + \|u_2\|_U^2)} \quad \text{(By Cauchy-Schwartz)} \\
&= (\|(z_1,u_1)\| + \|(z_2,u_2)\|)^2 \quad (\textit{Triangle Ineq})
\end{aligned} \tag{19}$$

(b) Firstly we show 1-norm and ∞-norm are equivalent. It is clear that

$$\frac{\|z\|_Z + \|u\|_U}{2} \le \max\{\|z\|_Z, \|u\|_U\} \le 2(\|z\|_Z + \|u\|_U) \tag{20}$$

Hence by definition of equivalence, $C = 2$, $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent.
Then we show ∞-norm and 2-norm are equivalent. Because

$$\begin{aligned}
\max\{\|z\|_Z, \|u\|_U\} &= \frac{1}{\sqrt{2}}\sqrt{\max\{\|z\|_Z, \|u\|_U\}^2 + \max\{\|z\|_Z, \|u\|_U\}^2} \\
\Rightarrow \|(z,u)\|_\infty &\ge \frac{1}{\sqrt{2}}\|(z,u)\|_2
\end{aligned} \tag{21}$$

And it's clear that $\|(z,u)\|_\infty \le \|(z,u)\|_2 \le \sqrt{2}\|(z,u)\|_2$. Hence by definition of equivalence, $C = \sqrt{2}$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Equivalence has transitivity, we finished the proof.

□

Problem 7. Let X be normed linear space, Y a linear subspace of X. The closure of Y is a linear subspace of X.

Proof. It suffice to show $\forall x, y \in \overline{Y}$, $a, b \in \mathbb{F}$, $ax + by \in \overline{Y}$.
If both $x, y \in Y$, then $ax + by \in Y \subseteq \overline{Y}$ is straightforward.
If $x \in Y$, $y \in \overline{Y} \setminus Y$, by definition, there is a $\{y_j\} \subset Y$, $y_j \to y$. Hence $\{ax + by_j\} \subset Y$ and $ax + by_j \to ax + by$ since the norm as a metric preserves linearity. It implies that $ax + by$ is a limit point of $Y \Rightarrow ax + by \in \overline{Y}$.
Thirdly, if $x, y \in \overline{Y} \setminus Y$, then $\{x_i\}, \{y_i\} \subset Y$; $x_i \to x, y_i \to y$. Again we construct new sequence $\{ax_i + by_i\} \subset Y$, and $ax_i + by_i \to ax + by$. Hence $ax + by$ is a limit point of $Y \Rightarrow ax + by \in \overline{Y}$. □

Problem 8. Show that if X is a Banach space, Y is a closed subspace of X, the quotient space X/Y is complete. (Hint: Use a Cauchy sequence $\{q_n\}$ in X/Y that satisfies $|q_n - q_{n+1}| < \frac{1}{n^2}$.)

Proof. (*Step.1*) Let $\{x_n + Y\} \subset X/Y$ be a Cauchy sequence. Then by definiton, there exists a subsequence $\{x_{n_k} + Y\}$, such that

$$\|(x_{n_k} + Y) - (x_{n_{k+1}} + Y)\| < \frac{1}{k} \tag{22}$$

Actually, $((x_{n_k} + Y) - (x_{n_{k+1}} + Y)) \sim [x_{n_k} - x_{n_{k+1}}]$, (denote $[\,\cdot\,]$ the equivalent class) so the inequility above is equivalent to

$$\|(x_{n_k} - x_{n_{k+1}}) + y)\| \le \frac{1}{k} \quad \text{For all } y \in Y.\ (\dagger) \tag{23}$$

(*Step.2*) Now, we start from $y_1 = 0$, then by (†), there exists $y^{[2]} = y_2 \in Y$ such that

$$\begin{aligned}
&\|(x_{n_1} - x_{n_2}) + y^{[2]}\| = \|(x_{n_1} - x_{n_2}) + (y_2 - 0)\| < \frac{1}{2} \\
&\Rightarrow \|(x_{n_1} - 0) - (x_{n_2} - y_2)\| < \frac{1}{2}
\end{aligned} \tag{24}$$

Further, there exists $y^{[3]} \in Y$, denote $y_3 = y^{[3]} + y_2 \in Y$

$$\begin{aligned} &\|(x_{n_2} - x_{n_3}) + y^{[3]}\| = \|(x_{n_2} - x_{n_3}) + (y_3 - y_2)\| < \frac{1}{3} \\ &\Rightarrow \|(x_{n_2} - y_2) - (x_{n_3} - y_3)\| < \frac{1}{3} \end{aligned} \tag{25}$$

Continue to proceed like this, we obtain a sequence $\{y_k\} \subset Y$, such that

$$\|(x_{n_k} - y_k) - (x_{n_{k+1}} - y_{k+1})\| < \frac{1}{k} \tag{26}$$

So by definition, $h_k := x_{n_k} - y_k$ is a Cauchy sequence in X. Since X is a Banach space, $h_k \to h \in X$. (*Step.3*) We show that $[x_{n_k}] \to [h]$.

$$\begin{aligned} \|(x_{n_k} + Y) - (h + Y)\| &= \|x_{n_k} - h + Y\| \\ &= \|x_{n_k} - y_k - h + Y\| \quad \text{(Since } y_k \in Y\text{)} \\ &= \|h_k - h + Y\| \xrightarrow{k \to \infty} 0 \end{aligned} \tag{27}$$

So for any Cauchy sequence $\{x_n + Y\} \in X/Y$, it has a convergent subsequence $\{x_{n_k} + Y\} \in X/Y$ that converges to $\{h + Y\} \in X/Y$. $\Rightarrow X/Y$ is complete. $\square$