Stochastic Process Assignment I

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Problem.1 Denote $A_i := \{\text{The player fails at } i^{th} \text{ round}\}$. Then clearly $\{A_i : i \geq 1\}$ are (mutually) independent. $E_i := \{\text{The first player wins at } i^{th} \text{ round}\}$, i odd, are disjoint. We have

$$E_n = \left(\bigcap_{i=1}^{n-1} A_i\right) \cap A_n^{\complement}; \qquad E = \bigcup_{n\geq 1,\text{odd}} E_n$$

Hence,

$$\mathbb{P}(E) = \sum_{n\geq 1,\text{odd}} p(1-p)^{n-1} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}$$

$$\mathbb{P}\left(E^{\complement}\right) = \frac{1-p}{2-p}$$

Problem.2 Let $S_{n,m}$ be a string consisting of n copies of A and m copies of B which represents the stack of votes. Say, a possible version of $S_{3,2}$ can be $S_* = AABAB$.
Denote $S_{n,m}^{[k]}$ be the prefix of $S_{n,m}$ that has k characters. For example, the above S_* has prefixes: $S_*^{[1]} = A$, $S_*^{[2]} = AA$, $S_*^{[3]} = AAB$, $S_*^{[4]} = AABA$, $S_*^{[5]} = AABAB$.
Then we have a equivalent problem:

$$\begin{aligned} &E := \{A \text{ always in the lead}\} \\ &\iff \{k^{th} \text{ prefix of } S_{n,m} \text{ contains more } A \text{ than } B,\ \forall 1 \leq k \leq (n+m)\} \end{aligned} \tag{1}$$

Denote $K(n,m)$ be the number of solutions (i.e. the string $S_{n,m}$) that solve the problem. Observing that we can recursively remove the last character of $S_{n,m}$, we have:

$$K(n,m) = K(n-1,m) + K(n,m-1)$$

$$P_{n,m} = \frac{K(n,m)}{\binom{m+n}{n}} \quad (\dagger)$$

Which implies that this problem has a *Dynamic Programming* solution. Trivially, $K(n,0) = 1$ for all $n \geq 1$. The value of $K(n,m)$ for all $0 \leq n, m \leq 6$ are given in following matrix, with $\boldsymbol{K}_{ij} = K(i,j)$.

$$\begin{array}{cc} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \; : m \\ \boldsymbol{K} = & \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ 1 & 1 & 0 & & & & \\ 1 & 2 & 2 & 0 & & & \\ 1 & 3 & 5 & 5 & 0 & & \\ 1 & 4 & 9 & 14 & 14 & 0 & \\ 1 & 5 & 14 & 28 & 42 & 42 & 0 \end{pmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \; : n \end{array}$$

By $(\dagger)$:

$$P_{2,1} = 1; \quad P_{3,1} = \frac{1}{2}; \quad P_{3,2} = \frac{1}{5}; \quad P_{4,2} = \frac{1}{3}; \quad P_{4,3} = \frac{1}{7}; \quad P_{5,3} = \frac{1}{4}$$

Look at the first column,

$$P_{n,1} = \frac{n-1}{\binom{1+n}{n}} = \frac{n-1}{n+1}$$

Look at the second column, by the algorithm, we know $K(n,2) = \sum_{j=3}^{n} K(j,1)$ for $n \geq 3$,

$$P_{n,2} = \frac{\sum_{j=3}^{n}(j-1)}{\binom{2+n}{n}} = \frac{\frac{1}{2}(n+1)(n-2)}{\frac{1}{2}(n+2)(n+1)} = \frac{n-2}{n+2}$$

Claim. $P_{n,m} = \frac{n-m}{n+m}$.

Proof of Claim. (**by induction**) Suppose $P_{n,m-1} = \frac{n-m+1}{n+m-1}$, $P_{n-1,m} = \frac{n-1-m}{n-1+m}$. We choose the *boundary cases* as the second column and the diagonal in $\boldsymbol{K}$, which is already ensured by $P_{n,1} = \frac{n-1}{n+1}$ and $P_{n,n} = 0$, $\forall n \geq 1$. Therefore

$$\begin{aligned} P_{n,m} &= \mathbb{P}\left(E|\{\text{last char is A}\}\right)\mathbb{P}\left(\text{last char is A}\right) \\ &\quad + \mathbb{P}\left(E|\{\text{last char is B}\}\right)\mathbb{P}\left(\text{last char is B}\right) \\ &= P_{n-1,m}\frac{n}{m+n} + P_{n,m-1}\frac{m}{m+n} \\ &= \frac{n-1-m}{n-1+m}\frac{n}{m+n} + \frac{n-m+1}{n+m-1}\frac{m}{m+n} \\ &= \frac{(m+n)(m-n)+m-n}{(m+n-1)(m+n)} = \frac{m-n}{m+n} \quad \blacksquare \end{aligned} \tag{2}$$

Problem.3 Denote $A_i := \{\text{Person } i \text{ get his hat correctly.}\}$, note that $\{A_i : 0 \leq i \leq n\}$ are *not* disjoint.

$$A := \{\text{At least one person get his hat.}\} = \bigcup_{i=1}^{n} A_i$$

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \\ &= \sum_{i=1}^{n}\mathbb{P}(A_i) + (-1)^{2-1}\sum_{i_1<i_2}\mathbb{P}(A_{i_1}\cap A_{i_2}) + \ldots \\ &\quad + (-1)^{k-1}\sum_{i_1<\ldots<i_k}\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) + \ldots \\ &\quad + (-1)^{n-1}\mathbb{P}\left(\bigcap_{j=1}^{n} A_{i_j}\right) \end{aligned} \tag{3}$$

We look into the representative term $\sum_{i_1<\ldots<i_k}\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right)$. There are $\binom{n}{k}$ choices of men (who pick hat correctly) inside the summation. For each choice of $\{i_j : 1 \leq j \leq k\}$, there are n-permutations of possible outcomes, in which k hats are already matched. So among all cases, there $(n-k)$-permutations of feasible outcomes.

$$\sum_{i_1<\ldots<i_k}\mathbb{P}\left(\bigcap_{j=1}^{k} A_{i_j}\right) = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$$

Therefore

$$\mathbb{P}\left(A^{\complement}\right) = 1 - \mathbb{P}(A) = 1 - \frac{1}{1!} + \frac{1}{2!} + \ldots + (-1)^n\frac{1}{n!} \xrightarrow{n\to\infty} \frac{1}{e} \quad \blacksquare$$

Problem.4 Denote $W := \{\text{This is a woman.}\}$

$$\begin{aligned}\mathbb{P}(C|W) &= \frac{\mathbb{P}(W|C)\,\mathbb{P}(C)}{\mathbb{P}(W)} = \frac{\mathbb{P}(W|C)\,\mathbb{P}(C)}{\mathbb{P}(W|A)\,\mathbb{P}(A)+\mathbb{P}(W|B)\,\mathbb{P}(B)+\mathbb{P}(W|C)\,\mathbb{P}(C)} \\ &= \frac{0.7\cdot\frac{4}{9}}{0.5\cdot\frac{2}{9}+0.6\cdot\frac{1}{3}+0.7\cdot\frac{4}{9}} = \frac{1}{2}\end{aligned} \tag{4}$$

Problem.5 (a)

$$1 = \int_{\mathbb{R}} f(x)dx = \int_0^2 c(4x-2x^2)dx = c\cdot\frac{8}{3} \quad \Rightarrow\; c = \frac{3}{8} \tag{5}$$

(b)

$$\mathbb{P}\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{3}{8}(4x-2x^2)dx = \frac{11}{16} \tag{6}$$

Problem.6 (a)

$$\begin{aligned}\mathbb{P}(Y=j) &= \sum_{i=0}^{j}\binom{j}{i}\frac{e^{-2\lambda}\lambda^j}{j!} = \frac{e^{-2\lambda}}{j!}\sum_{i=0}^{j}\binom{j}{i}\lambda^{j-i}\lambda^j \\ &= \frac{e^{-2\lambda}(2\lambda)^j}{j!}\end{aligned} \tag{7}$$

(b)

$$\begin{aligned}\mathbb{P}(X=i) &= \sum_{j=i}^{\infty}\binom{j}{i}\frac{e^{-2\lambda}\lambda^j}{j!} = \sum_{j=i}^{\infty}\frac{j!}{i!(j-i)!}\frac{e^{-2\lambda}\lambda^j}{j!} \\ &= \sum_{j=i}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!}\frac{e^{-\lambda}\lambda^{(j-i)}}{(j-i)!} \quad (\text{Let } t := j-i) \\ &= \frac{e^{-\lambda}\lambda^i}{i!}\sum_{t=0}^{\infty}\frac{e^{-\lambda}\lambda^t}{t!} = \frac{e^{-\lambda}\lambda^i}{i!}\end{aligned} \tag{8}$$

(c)

$$\begin{aligned}\mathbb{P}(Y-X=k) &= \mathbb{P}(X=i, Y=i+k) \\ &= \sum_{i=0}^{\infty}\binom{i+k}{i}\frac{e^{-2\lambda}\lambda^{i+k}}{(i+k)!} \\ &= \sum_{i=0}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!}\frac{e^{-\lambda}\lambda^k}{k!} \\ &= \frac{e^{-\lambda}\lambda^k}{k!}\sum_{i=0}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!} = \frac{e^{-\lambda}\lambda^k}{k!}\end{aligned} \tag{9}$$

Problem.7 (a) Since $\{X_i : 1 \le i \le 10\}$ are independent Possion with mean 1. But for Possion RV, $\mathbb{E}[X] = \lambda = 1 =: \mu$. Hence the distribution is fully parametrized. $\{X_i\} \sim$ i.i.d Possion(1). So $S_{10} := \sum_1^{10} X_i \sim$ Possion(10). By Markov

$$\mathbb{P}(S_{10} \ge 15) \le \frac{\mathbb{E}[S_{10}]}{15} = \frac{2}{3} \tag{10}$$

(2) $\mathbb{V}\mathrm{ar}[X] = \lambda = 1 =: \sigma^2$. By CLT: $\frac{S_n}{n} \xrightarrow{D} W \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$. I.e. $\frac{(S_n/n)-1}{1/\sqrt{n}} \xrightarrow{a} Z \sim \mathcal{N}(0,1)$

$$\mathbb{P}(S_{10} \ge 15) = \mathbb{P}\left(\frac{S_{10}}{10} \ge 1.5\right) = \mathbb{P}\left(\frac{\frac{S_{10}}{10}-1}{1/\sqrt{10}} \ge \frac{\sqrt{10}}{2}\right) \approx 1-\Phi\left(\frac{\sqrt{10}}{2}\right) = 0.057 \tag{11}$$

Problem.8 Use same notations as of problem 3. $A_i := \{\text{Person } i \text{ picks right hat.}\}$. It is clear that $\mathbb{P}(A_i) = \frac{1}{n} = \mathbb{E}[\mathbb{1}_{A_i}]$ for all i. Moreover, we have

$$X = \sum_{i=1}^{n} \mathbb{1}_{A_i}; \quad \mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} \mathbb{1}_{A_i}\right] = 1$$

Clearly $\mathbb{1}_{A_i} \sim \text{Bernoulli}(\frac{1}{n})$. Hence $\mathbb{V}\text{ar}[\mathbb{1}_{A_i}] = \frac{1}{n}(1-\frac{1}{n})$.
For any $i \neq j$, $\mathbb{E}\left[\mathbb{1}_{A_i \cap A_j}\right] = \mathbb{P}\left(\mathbb{1}_{A_i \cap A_j} = 1\right) = \mathbb{P}(A_i A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j|A_i) = \frac{1}{n} \cdot \frac{1}{n-1}$. So we have

$$\begin{aligned}
\mathbb{C}\text{ov}\left[\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right] &= \mathbb{E}\left[(\mathbb{1}_{A_i} - \mathbb{E}[A_i])(\mathbb{1}_{A_j} - \mathbb{E}[A_j])\right] \\
&= \mathbb{E}\left[\mathbb{1}_{A_i \cap A_j} - \frac{1}{n}(\mathbb{1}_{A_j} + \mathbb{1}_{A_i}) + \frac{1}{n^2}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{A_i \cap A_j}\right] - \frac{1}{n^2} = \frac{1}{n(n-1)} - \frac{1}{n} = \frac{1}{n^2(n-1)}
\end{aligned} \tag{12}$$

Therefore

$$\begin{aligned}
\mathbb{V}\text{ar}[X] &= \sum_{i=1}^{n} \mathbb{V}\text{ar}[\mathbb{1}_{A_i}] + \sum_{i \neq j} \mathbb{C}\text{ov}\left[\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right] \\
&= n \cdot \frac{n-1}{n^2} + (n^2 - n) \cdot \frac{1}{n^2(n-1)} \\
&= \frac{n-1}{n} + \frac{1}{n} = 1 \quad \blacksquare
\end{aligned} \tag{13}$$

Problem.9 It suffices to check moment generating function. By thm. $\phi_{2X}(t) = \phi_{X+Y}(t)\phi_{X-Y}(t) \iff X+Y$ and $X-Y$ are independent.
Note that as linear combinations of gaussian, $2X \sim \mathcal{N}(2\mu, 4\sigma^2)$; $X+Y \sim \mathcal{N}(2\mu, 2\sigma^2)$; $X-Y \sim \mathcal{N}(0, 2\sigma^2)$

$$LHS = \exp\left(2\mu t + \frac{1}{2} \cdot 4\sigma^2 t^2\right)$$

$$RHS = \exp\left(2\mu t + \frac{1}{2} \cdot 2\sigma^2 t^2\right) \exp\left(0 + \frac{1}{2} \cdot 2\sigma^2 t^2\right) = LHS \quad \blacksquare$$

Problem.10 (a) The first equality:

$$\begin{aligned}
\mathbb{E}[N] &:= \sum_{j=0}^{\infty} j \cdot \mathbb{P}(N=j) \quad (\text{Note that } j = \sum_{k=1}^{j} 1) \\
&= \sum_{j=0}^{\infty} \left(\sum_{k=1}^{j} 1\right) \mathbb{P}(N=j) = \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} \mathbb{1}_{\{k \leq j\}}(k)\right) \mathbb{P}(N=j) \\
&= \sum_{k=1}^{\infty} \left(\sum_{j=0}^{\infty} \mathbb{1}_{\{k \leq j\}}(k) \cdot \mathbb{P}(N=j)\right) = \sum_{k=1}^{\infty} \left(\sum_{j=k}^{\infty} \mathbb{P}(N=j)\right) = \sum_{k=1}^{\infty} \mathbb{P}(N \geq k)
\end{aligned} \tag{14}$$

The second is pretty much the same,

$$\begin{aligned}\mathbb{E}\left[N\right] &:= \sum_{j=0}^{\infty} j\cdot\mathbb{P}\left(N=j\right) \quad \text{(Note that } j=\sum_{k=0}^{j-1}1\text{)}\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j-1}1\right)\mathbb{P}\left(N=j\right) = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\mathbb{1}_{\{k<j\}}(k)\right)\mathbb{P}\left(N=j\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\mathbb{1}_{\{k<j\}}(k)\cdot\mathbb{P}\left(N=j\right)\right) = \sum_{k=1}^{\infty}\left(\sum_{j=k+1}^{\infty}\mathbb{P}\left(N=j\right)\right)\\ &= \sum_{k=1}^{\infty}\mathbb{P}\left(N\geq k+1\right) = \sum_{k=1}^{\infty}\mathbb{P}\left(N>k\right) \quad \blacksquare\end{aligned} \tag{15}$$

(c) But I want to show general case directly...

$$\begin{aligned}\mathbb{E}\left[g(X)\right] &= \int_{\Omega} g(X)d\mathbb{P} = \int_{\Omega}\left(\int_0^{g(X)}1dt\right)d\mathbb{P}\\ &= \int_{\Omega}\left(\int_0^{\infty}\mathbb{1}_{\{t<g(X)\}}dt\right)d\mathbb{P} = \int_0^{\infty}\left(\int_{\Omega}\mathbb{1}_{\{t<g(X)\}}d\mathbb{P}\right)dt \quad \text{(By \textbf{Tonelli.})}\\ &= \int_0^{\infty}\mathbb{P}\left(X>g^{-1}(t)\right)dt \quad \text{(Define } z:=g^{-1}(t)\text{, then } t=g(z)\text{)}\\ &= \int_0^{\infty}\mathbb{P}\left(X>z\right)g'(z)dz \quad (\dagger) \quad \blacksquare\end{aligned} \tag{16}$$

The equations in (b) are implied by (†). Take $g(X) := X$, we have (b-1), take $g(X) := X^n$, we obtain (b-2). Both also satisfy $g(0) = 0$.

Problem.11 (a) CDF is defined as the mapping $F_X(x) := \mathbb{P}\left(X \leq x\right)^1$. Hence the CDF of RV $F_X(X)$ is

$$F_{F_X(X)}(z) := \mathbb{P}\left(F_X(X)\leq z\right) = \mathbb{P}\left(X\leq F_X^{-1}(z)\right) =: F_X(F_X^{-1}(z)) = z \tag{17}$$

for all $z \in (0,1)$; implies that $F_X(X) \sim \mathcal{U}(0,1)$.
(b) Now given $U \sim \mathcal{U}(0,1)$, we have $F_U(u) = \frac{u-0}{1-0}$.

$$\begin{aligned}F_{F_X^{-1}(U)}(z) &= \mathbb{P}\left(F_X^{-1}(U)\leq z\right) = \mathbb{P}\left(U\leq F_X(z)\right)\\ &= F_U(F_X(z)) = \frac{F_X(z)-0}{1-0} = F_X(z) \quad \blacksquare\end{aligned} \tag{18}$$

Problem.12 (a) It suffices to determine joint-pmf:

$$\begin{aligned}p_{N_1,\ldots,N_r}(n_1,\ldots,n_r) &:= \mathbb{P}\left(N_1=n_1, N_2=n_2, \ldots, N_r=n_r\right)\\ &= \mathbb{P}\left(N_1=n_1\right)\mathbb{P}\left(N_2=n_2|N_1=n_1\right)\cdot\ldots\cdot\mathbb{P}\left(N_r=n_r|N_1=n_1,\ldots,N_{r-1}=n_{r-1}\right)\\ &= \binom{n}{n_1}p_1^{n_1}\binom{n-n_1}{n_2}p_2^{n_2}\cdot\ldots\cdot\binom{n-n_1-\ldots-n_{r-1}}{n_r}p_r^{n_r}\\ &= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\ldots-n_{r-1})!}{n_r!0!}\prod_{k=1}^{r}p_k^{n_k}\\ &= \frac{n!}{\prod_{k=1}^{r}n_k!}\prod_{k=1}^{r}p_k^{n_k}\end{aligned} \tag{19}$$

[1]To avoid duplicate notations we specify $F_X(\cdot) \equiv F(\cdot)$ in this problem, but not using $F(\cdot)$ on its own.

When $\sum_{k=1}^{r} n_k = 1$. Otherwise $p_{N_1,...,N_r}(n_1,...,n_r) = 0$ $\blacksquare$.
(b) Denote $A_i^{[m]} := \{i^{th}$ outcome appears at m^{th} trial$\}$. $1 \leq i \leq r$, $1 \leq m \leq n$. Then $A_i^{[p]}, A_j^{[q]}$ are independent as long as $p \neq q$. Moreover $\mathbb{1}_{A_i^{[m]} \cap A_j^{[m]}} \equiv 0$ for $i \neq j$.

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{A_i^{[m]}}\right] &= \mathbb{P}\left(\mathbb{1}_{A_i^{[m]}} = 1\right) = p_i; \qquad \mathbb{V}\mathrm{ar}\left[\mathbb{1}_{A_i^{[m]}}\right] = p_i(1-p_i) \\
\mathbb{E}\left[N_i\right] &= \mathbb{E}\left[\sum_{m=1}^{n} \mathbb{1}_{A_i^{[m]}}\right] = np_i; \qquad \mathbb{V}\mathrm{ar}\left[N_i\right] = np_i(1-p_i)
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\mathbb{E}\left[N_i N_j\right] = \mathbb{E}\left[\sum_{m=1}^{n} \mathbb{1}_{A_i^{[m]}} \sum_{m=1}^{n} \mathbb{1}_{A_j^{[m]}}\right] = \mathbb{E}\left[\sum_{m=1}^{n} \mathbb{1}_{A_i^{[m]} \cap A_j^{[m]}} + \sum_{1 \leq p \neq q \leq n} \mathbb{1}_{A_i^{[p]} \cap A_j^{[q]}}\right] = (n^2 - n)p_i p_j \\
&\mathbb{C}\mathrm{ov}\left[N_i, N_j\right] = \mathbb{E}\left[(N_i - np_i)(N_j - np_j)\right] = \mathbb{E}\left[N_i N_j\right] - n^2 p_i p_j = -np_i p_j \quad \blacksquare
\end{aligned} \tag{21}$$

(c) Denote $B_i := \{$Outcome i does not occur throughout all trials$\}$. $\mathbb{E}\left[\mathbb{1}_{B_i}\right] = \mathbb{P}\left(B_i\right) = (1-p_i)^n$. $K := \sum_{i=1}^{r} \mathbb{1}_{B_i}$ is #outcomes that do not occur.

$$\begin{aligned}
\mathbb{E}\left[K\right] &= \sum_{i=1}^{r} (1-p_i)^n \\
\mathbb{E}\left[K^2\right] &= \sum_{i=1}^{r} (1-p_i)^{2n} + \sum_{i \neq j} \mathbb{E}\left[\mathbb{1}_{B_i \cap B_j}\right]
\end{aligned} \tag{22}$$

And $\mathbb{P}\left(B_i \cap B_j\right) = \mathbb{P}\left(B_i\right)\mathbb{P}\left(B_j | B_i\right) = (1-p_i)^n \left(1 - \frac{p_j}{1-p_i}\right)^n = (1 - p_i - p_j)^n$. Hence **TODO**

Problem.13 (a) X_1, X_2 indep. $\iff \phi_{X_1+X_2}(t) = \phi_{X_1}(t)\phi_{X_2}(t)$. Hence

$$\phi_{X_1+X_2}(t) = \exp[\lambda_1(e^t - 1)]\exp[\lambda_2(e^t - 1)] = \exp[(\lambda_1 + \lambda_2)(e^t - 1)] \tag{23}$$

Therefore $X_1 + X_2 \sim$ Possion$(\lambda_1 + \lambda_2)$
(b)

$$\begin{aligned}
p_{X_1|X_1+X_2}(x|z) = \mathbb{P}\left(X_1 = x | X_1 + X_2 = z\right) &= \frac{\mathbb{P}\left(X_1 = x, X_1 + X_2 = z\right)}{\mathbb{P}\left(X_1 + X_2 = z\right)} \\
&= \frac{\frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!}}{e^{-\lambda_1-\lambda_2}(\lambda_1+\lambda_2)^z / z!} = \frac{z!}{x!(z-x)!}\frac{\lambda_1^x \lambda_2^{z-x}}{(\lambda_1+\lambda_2)^z} \\
&= \binom{z}{x}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-x}
\end{aligned} \tag{24}$$

Hence, $X_1 | \{X_1 + X_2 = n\} \sim$ Binomial$(\frac{\lambda_1}{\lambda_1+\lambda_2}, n)$. $\blacksquare$

Problem.14 Due to (**Jensen**), for *concave* function $\phi(\cdot)$ and RV X, we have $\phi(\mathbb{E}\left[X\right]) \geq \mathbb{E}\left[\phi(X)\right]$. Take X be the discrete RV that takes value $\{x_1, x_2, ..., x_n\}$ with uniform probability. Take $\phi(\cdot) = \log(\cdot)$. Then

$$\log\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \geq \frac{1}{n}\sum_{i=1}^{n}\log(x_i) = \log\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right) \tag{25}$$

And log is monotone transform. We are done with required equation. $\blacksquare$

Problem.15 (a)

$$LHS = \mathbb{E}\left[X^2\right] - \mathbb{E}^2\left[X\right] \tag{26}$$

$$\begin{aligned} RHS &= \mathbb{E}\left[\mathbb{E}\left[X^2 \middle| Y\right] - \mathbb{E}^2\left[X|Y\right]\right] + \mathbb{E}\left[\mathbb{E}^2\left[X|Y\right] - \mathbb{E}^2\left[\mathbb{E}\left[X|Y\right]\right]\right] \\ &= \mathbb{E}\left[X^2\right] - \mathbb{E}\left[\mathbb{E}^2\left[X|Y\right]\right] + \mathbb{E}\left[\mathbb{E}^2\left[X|Y\right]\right] - \mathbb{E}^2\left[X\right] \\ &= LHS \end{aligned} \tag{27}$$

(b) By **Wald's Identity**, since $\{X_i\}$ are i.i.d, indep. of N, denote $S_N := \sum_{i=1}^{N} X_i$

$$\mathbb{E}\left[S_N\right] = \mathbb{E}\left[\mathbb{E}\left[S_N|N\right]\right] = \mathbb{E}\left[N\mathbb{E}\left[X_1\right]\right] = \mu\mathbb{E}\left[N\right] \tag{28}$$

$$\begin{aligned} \mathbb{V}\text{ar}\left[S_N\right] &= \mathbb{E}\left[\mathbb{V}\text{ar}\left[S_N|N\right]\right] + \mathbb{V}\text{ar}\left[\mathbb{E}\left[S_N|N\right]\right] \\ &= \mathbb{E}\left[N\sigma^2\right] + \mathbb{V}\text{ar}\left[\mu N\right] \\ &= \sigma^2\mathbb{E}\left[N\right] + \mu^2\mathbb{V}\text{ar}\left[N\right] \end{aligned} \tag{29}$$