Project 1
Convertible Bonds

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1 Documentation

1.1 Introduction and basic definitions

Convertible bonds are a hybrid form of financing, which combines traditional bonds with stock options [2]. The holder of a convertible bond has the right but not the obligation to exchange the bond for a predetermined number of shares of the company, which issued the bond. This number is called the **conversion ratio**, it tells the holder the amount of shares she receives after converting one such bond. The price of a convertible bond can be decomposed into two parts. Suppose the holder never convert the bond, in this case the contract behaves as if it was a plain corporate bond. The present value of such bond is called the **Bond Floor**. The **Investment Premium** is the second component of the price CB, it shows the amount the investor would pay in order to get the option part of the contract. It is important to note that Vanilla CBs are considered to be American derivatives, in a sense that at every time step, the buyer has chance to do the conversion. Since the CB is a richer security in terms of choice, than a vanilla corporate bond, its price with the same specifications (maturity, coupons, yield) should be higher than the corporate one. CBs are priced similarly as corporate bonds if the equity price is low, in this case the conversion does not happen. If the equity's price is high, a convertible bond is likely to be converted and the price behaviour is similar to the underlying stock. In the followings, we will discuss different types of convertible bonds and some of its properties, based on a paper on Convertible Bond Pricing [3].

1.1.1 Callable convertible bonds

Callable convertible bonds the issuer has the right to buy the CB back, in the so-called call periods, at predetermined price. A holder will still have the possibility to convert the bond, she will do so, if she would earn more than the call price. This is called a forced conversion. Similar to callables, the putable version gives the right to the investor to give the CB back to the investor and terminate the contract.

1.1.2 Mandatory Convertibles

Mandatory Convertibles must be converted onto shares on maturity date. Before converting, the bond guarantees a return in form of coupons, but at maturity the investor should convert it to shares of stocks. In this scenario the holder will no longer enjoy the coupons but she may be benefiting from higher returns from the stocks. Mandatory CBs are closer to European derivatives, in a sense that the buyer can only decide to convert at the terminal date.

1.1.3 Contingent Conversion

Contingent Conversion is a variant where the investor can only convert the bond if the share price exceeds a predetermined conversion threshold. Similarly Soft Callability means that the issuer can call back the bond, when the stock price is above a predetermined threshold.

1.1.4 Exercise priorities

In the case of the standard convertible bond, only the buyer has an option that he may choose to exercise (i.e. priority is given to the bondholder, rather than the issuer).

In the case of the callable convertible bond, there are two options: the buyer is given a conversion option (idem as with the vanilla CB) and the issuer is given a call option. Just like in the previous case, the bondholder may decide between receiving a fixed cash flow (either the face value at maturity, or the call price at any "call" date). However, the date at which this occurs will most like depend on the issuer. If the issuer determines that terminating the bond at an earlier date might be beneficial for him, he will do so. The bondholder will then be forced to make a choice between receiving the call price or the pre-specified number of shares. Note that the bondholder, although forced to make the decision, still holds some sort of "freedom" (regarding the exercise of his option). It is worth noting that the bondholder still has the choice to exercise the conversion option at any "conversion" date (regardless of the exercise of the call option). We assume that both the buyer and the issuer of the CB are rational agents, meaning they both seek to maximize their profits and minimize losses when converting or calling the bond.

1.2 Pros and cons of convertible bonds

Apart from the equity-like return potential that global CB markets have demonstrated, convertibles have the advantage of typically performing positively if interest rates rise and therefore often work well for hedging duration risk. But also compared to equities, convertibles can be attractive: their downside risk is lower due to the bond floor's protective nature. From an investor point of view, she can gain profits even in case of defaults, but on the other hand she can enjoy the stock returns if the underlying company performs well. From a company point of view, if the company wants to raise money, CBs can substitute corporate bonds, since CBs can be sold with lower interest rates at the same price because of the convertibility. However a company cannot issue too many CBs. if most buyers convert to shares, share dilution can happen, which means the investors will loose some proportion of their ownership in the company.
Having CBs as a core portfolio component can provide diversification for different levels of risk tolerance. This kind of mixed-asset portfolio enables us to have increased return for the same amount of risk, because the correlation between global government and conventional bonds is low.

1.3 Overview of the Convertible Bond Market

The global CB market today includes over 380 billion of securities from 750 companies [4]. U.S. companies represent approximately two-thirds of the CB market, with the rest represented by issuers in Europe , Japan and mostly China)[5]. Convertible issuers include technology and pharmaceutical corporations, as well as established industrial companies.

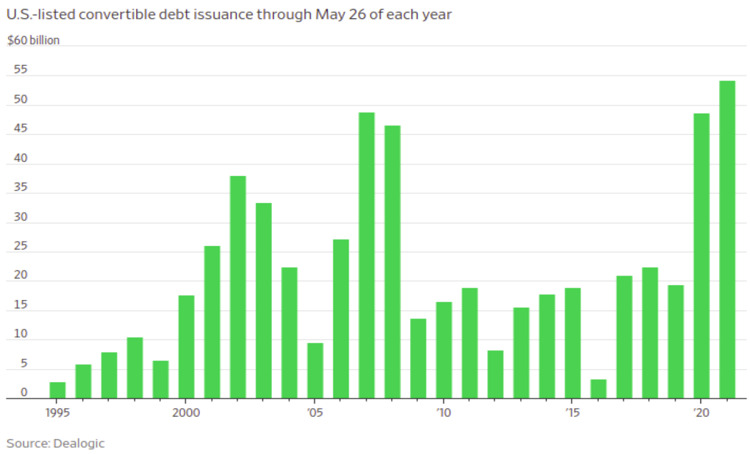


Figure 1: Capitalization of CB issuances through years, source: The Wall Street Journal

Due to the COVID-19 pandemic, many issuers sought cheaper ways for borrowing money. The solution was CBs, which attracted many investors. As it can be seen on the figure the US capitalization of the market almost tripled from 2019 to 2020. In 2021 the trend continued and the market reached an all time high with almost 55 million issued CBs just in the US as of 26 May.

1.4 Callables versus Vanilla/Mandatory: buyer's point of view

If market interest rates are low, an issuer may choose to call back its CBs in order to change its form of debts. This means it gives one extra dimension of freedom to the issuer, so the buyer can easily loose her incomes from the coupons. On the other hand this comes with the advantage that the investors will demand higher coupons compared to vanilla and mandatory CBs (we will see that this is indeed the case in Analysis part 5.5). [1]

1.5 Benefits of the cCBs: issuer's point of view

Companies seeking financing might end up preferring cCBs due to the many benefits that the aforementioned offer (when compared with their counterparts). Some of those benefits are listed below.

1.5.1 Callable CBs vs. Standard bonds

Interest rates fluctuate over the life of the bond. This can lead to situations in which the yield of the bond is too high (i.e. issuer borrowing at a much higher rate than others). Given the immutable "nature" of plain coupon bonds, the issuing company won't be able to terminate the bond at an earlier date (i.e. borrow at a lower rate). To avoid such situations, the cCBs offer the issuer the option to call the bond at given pre-specified dates. If interest rates decrease dramatically, the issuer will just have to wait until the next "call" date in order to terminate the contract.

1.5.2 Callable CBs vs. Standard CBs

Just like interest rates (and even more so), the stock price of the underlying fluctuates over the life of the bond. In the case in which the former is following a steep (unexpected) upward trend, the bondholder will have the option to convert. The issuer will then incur in high losses. In order limit losses, the issuer may decide to call the bond (before the stock price reaches too high values). This "freedom" of terminating the contract at given dates, can limit/reduce unexpected losses. Standard CBs do not offer such a benefit.

1.5.3 Callable CBs vs. Mandatory CBs

Much like in the case of standard CBs, an unexpected increase in the stock value might cause the issuer to incur in significant losses. The aforementioned situation can, once again, be avoided by giving the issuing company the option to call the bond at any pre-specified dates.

2 Preliminary analysis

2.1 Mandatory Convertible Bonds (mCB)

The mandatory CBs must be converted to to stocks at time T, where its conversion rate is given in terms of two constants $\alpha \geq \beta$ as

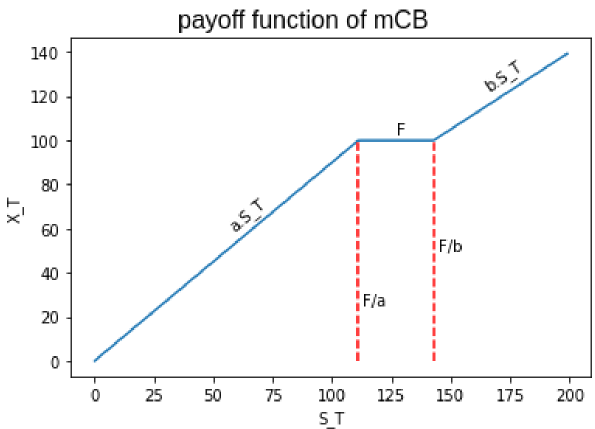
$$M_T = \alpha \mathbb{1}_{\{\alpha S_T < F\}} + \frac{F}{S_T} \mathbb{1}_{\{\frac{F}{\alpha} \leq S_T \leq \frac{F}{\beta}\}} + \beta \mathbb{1}_{\{\beta S_T > F\}}.$$

We decompose the security into 3 key components:

- we go long in a coupon bond with face value F, that pays cF coupons
- we go long in β amount of european call options written on the underlying stock
- we go short in α amount of european put options written on the underlying stock.

The terminal value of this portfolio is exactly the terminal payoff of the conversion. If we denote the value of the portfolio with X_T, we have $X_T = P_T(H) = M_T \cdot S_T$, and

$$X_T(S_T) = \beta \left(S_T - \frac{F}{\beta} \right)^+ - \alpha \left(-S_T + \frac{F}{\alpha} \right)^+ + F$$



possible terminal payoff function with $\alpha = 0.9$, $\beta = 0.7$ and $F = 100$

Using this replicating porfolio and the fact that the market is complete and arbitrage-free, we can conclude that the price of the mCB at any time must be equal to the value of the replicating portfolio. Since the discounted value of the replicating portfolio is a martingale under the risk neutral measure and using the fact that the conditional expectation is linear, we have

$$P_0(H) = e^{-rT\Delta}\mathbb{E}_0^{\mathbb{Q}}\left[P_T(H)\right] = e^{-rT\Delta}\left(\mathbb{E}_0^{\mathbb{Q}}\left[P_T(H_{CB}) + \beta P_T(H_{call}) + \alpha P_T(H_{put})\right]\right) =$$
$$F\mathbb{E}_0^{\mathbb{Q}}\left[\sum_{i=1}^{T} c_i e^{-ri\Delta}\right] + e^{-rT\Delta}\beta\mathbb{E}_0^{\mathbb{Q}}\left[\left(S_T - \frac{F}{S}\right)^+\right] - e^{-rT\Delta}\alpha\mathbb{E}_0^{\mathbb{Q}}\left[\left(-S_T + \frac{F}{\alpha}\right)^+\right]$$

where $c_i = c + \mathbb{1}_{i=T}$ is the payoff function of the coupon bond. In practise, the CB valuation is straightforward, the value of the put and call options should be calculated using the backward recursion in the binomial tree.

2.2 Plain vanilla convertible bond (CB)

Let's denote the value of the CB with X_T at time T. At maturity date, this equals the payoff.

$$V_T = X_T(S_T) = c \cdot F + \max\{F; S_T \cdot \gamma\}$$

At any other coupon time on condition of exercise we will get $S_t \cdot \gamma + c \cdot F$, otherwise we will get only the coupon, so $c \cdot F$. On any other dates the payoff is zero.
We get the same payoff with the portfolio of:

- Coupon bond with maturity T, with coupons $c \cdot F$ with the same coupon dates and a face value F
- American derivative with payoff function

$$G_t = \begin{cases} S_T\gamma - F & t = T \\ S_t\gamma - V_t^B & t \neq T \text{ and } t \text{ is a coupon date} \\ 0 & \text{otherwise} \end{cases}$$

Where V_t^B is the value of the coupon bond at time t. If we have this coupon bond and American derivative, we end up having the same payoff at every t.
We can attain the prices using the pricing formula for the coupon bond and the American derivative since it is a replicating portfolio for the CB.
The Snell Envelope of the discounted payoff process of the American derivative is

$$\hat{Z}_t = \max_{\theta \in \mathcal{S}_t} \mathbb{E}_t^{\mathbb{Q}}[\mathbb{1}_{\{\theta \leq T\}}\hat{G}_\theta]$$

We can use the result of the lecture that the SE satisfies

$$\hat{Z}_t = \mathbb{1}_{\{t=T\}}\hat{G}_T^+ + \mathbb{1}_{\{t<T\}}\max\{\hat{G}_t; \mathbb{E}_t^{\mathbb{Q}}[\hat{Z}_{t+1}]\}$$

and the stopping time

$$\tau_t^* = \inf\{s \geq t : Z_s = G_s\} \in \mathcal{S}_t$$

attains the maximum in SE, so the price of the derivative is $\forall t : Z_t$ almost surely.

$$P_t(H_A) = \mathbb{1}_{\{t=T\}}(S_T\gamma - F)^+ + \mathbb{1}_{\{t<T\}}\max\{\mathbb{1}_{\{t\in C\}}((S_T\gamma - V_t^B); e^{-r\Delta}\mathbb{E}_t^{\mathbb{Q}}[P_{t+1}(H_A)]\}$$

The pricing of the coupon bond is simpler, we can get it by

$$P_t(H_{CB}) = F\sum_{i=1}^{T} c_i e^{-ri\Delta}$$

where $c_i = c + \mathbb{1}_{i=T}$
As a result, the price of the CB is the following

$$P_0(H) = \mathbb{E}_0^{\mathbb{Q}}\left[P_T(H_A) + P_T(H_{CB})\right]$$

As for the optimality, it is optimal to convert, when the continuation value is smaller than the value received at exercise, where

$$C_t = e^{-r\Delta}\mathbb{E}_t^{\mathbb{Q}}[P_{t+1}]$$

So it is optimal to convert when P_t equals the payoff at t.
These dates are the same for the American derivative, so it is optimal to stop when it is optimal to stop in the American derivative part.

2.3 Callable convertible bond (cCB)

Question: 2.4.
In the case of a call, the bondholder has the option to either receive the call price or the shares of stock. Under the assumption that the issuer wants to maximize his earnings, he will call the bond only when the amount to be paid to the bondholder (by the of fact of calling the bond) is lower than the continuation value. Therefore, the issuer will call the bond at date t if and only if $C_t > \gamma \cdot S_t$ (if the buyer chooses the shares. Note that the shares received are ex-dividend) and $C_t > F_c$ (if the buyer chooses the call price), i.e.

$$C_t > max\{\gamma \cdot S_t; \quad F_c\}$$

Question: 2.5.
If $t \in \mathcal{C}$, then the issuer is allowed to call the bond. In such a case, we will have $B_t = max\{\gamma \cdot S_t; \quad F_c\}$ (under the assumption that the bondholder wants to maximize his earnings). On the other hand, if the issuer doesn't call the bond, we will have $B_t = max\{\gamma \cdot S_t; \quad C_t\}$ (the buyer will only exercise the conversion option in cases in which the value of the shares received exceeds the continuation value). Therefore, if $t \in \mathcal{C}$, we have that:

$$B_t = \begin{cases} max\{\gamma \cdot S_t, F_c\}, & \text{if } C_t > max\{\gamma \cdot S_t, F_c\} \\ max\{\gamma \cdot S_t, C_t\}, & \text{otherwise.} \end{cases}$$

What's more, when $t \notin \mathcal{C}$ the issuer cannot call the bond and the buyer cannot exercise the conversion option. Hence, in such cases the bond behaves as "plain" coupon bond. We conclude that:

$$B_t = \begin{cases} max\{\gamma \cdot S_t, F_c\}, & \text{if } t \in \mathcal{C} \text{ and } C_t > max\{\gamma \cdot S_t, F_c\} \\ max\{\gamma \cdot S_t, C_t\}, & \text{if } t \in \mathcal{C} \text{ and } C_t \leq max\{\gamma \cdot S_t, F_c\} \\ C_t, & \text{otherwise.} \end{cases}$$

N.B. $C_t = e^{-r_{t+1}\Delta}E_t^{\mathbb{Q}}[B_{t+1} + \mathbf{1}_{\{t+1\in\mathcal{C}\}}cF] \cdot \mathbf{1}_{\{t<T\}} + F \cdot \mathbf{1}_{\{t=T\}}$.

3 Implementation and analysis

Apart from the attached zip file, our code is available on github via the link https: `https://github.com/hbenedek/convertible-bonds`
'model.py' contains our implementation of the BinomialModel along with intermediate calculations of the evolution of the risky asset, the calculation of the EMM probabilities and arbitrage checking. In order to initialize a BM, one should instantiate a BM object with its parameters. This instance is then passed to the corresponding derivative. All derivatives are implemented in the same file as well. A derivative should get an underlying model and a function calculate_price() which returns the price as of today and sets the class attribute price_tree to a np.ndarray matrix, representing the price at different nodes of the BM. We implemented the pricing of simple derivatives (Zero Coupon Bonds, Vanilla Bonds, European Options and American Options), which we can use as building blocks for pricing more complicated securities such as Mandatory Convertible Bonds, (vanilla) Convertible Bonds and Callable Convertible Bonds.

4 Model calibration

Question: 4.1

- Let Q_s be the number of units (of the underlying asset) at date $s \geq 0$.
- Let $X_s = Q_s \cdot S_s$ (i.e. value of the position at date s).
- Let $C_s = \sum_{i=0}^{s} \frac{\delta}{1-\delta} \cdot S_i \cdot \mathbf{1}_{\{i \in \mathbb{D}\}}$ (i.e. cumulative cash flow process).

Considering the strategy by which we re-invest all cash flows in the asset, we have

$$\begin{aligned}
\Delta Q_s &= \frac{Q_s \cdot \Delta C_s}{S_s} \\
&= Q_s \frac{\delta \cdot \mathbf{1}_{\{s \in \mathbb{D}\}}}{1-\delta} \\
\Rightarrow Q_s &= \left(1 - \frac{\delta}{1-\delta} \cdot \mathbf{1}_{\{s \in \mathbb{D}\}}\right)^{-1} Q_{s-1} \\
\Rightarrow Q_s &= Q_0 \cdot \prod_{i=1}^{s} \left(1 - \frac{\delta}{1-\delta} \cdot \mathbf{1}_{\{i \in \mathbb{D}\}}\right)^{-1}
\end{aligned}$$

Assuming $\Delta = 1/12$ (i.e. monthly frequency) and $\mathbb{D} = \bigcup_{n \in \mathbb{N}} \{12 \cdot n\}$ (i.e. yearly dividend payments), implies that

$$\begin{aligned}
Q_{12 \cdot n} &= \left(1 - \frac{\delta}{1-\delta}\right)^{-n} \cdot Q_0 \\
\Rightarrow X_{12 \cdot n} &= \left(1 - \frac{\delta}{1-\delta}\right)^{-n} \cdot Q_0 \cdot S_{12 \cdot n}
\end{aligned}$$

Furthermore, by choosing Q_0 such that $X_{12 \cdot n} = S_{12 \cdot n}$, we obtain

$$\begin{aligned}
Q_0 &= \left(1 - \frac{\delta}{1-\delta}\right)^{n} \\
\Rightarrow \mathcal{P}_0(12 \cdot n) &= X_0 = Q_0 \cdot S_0 \\
&= \left(1 - \frac{\delta}{1-\delta}\right)^{n} \cdot S_0 \\
\Rightarrow \mathcal{D}_0(12 \cdot n) &= S_0 - \mathcal{P}_0(12 \cdot n) \\
&= \left(1 - \left(1 - \frac{\delta}{1-\delta}\right)^{n}\right) \cdot S_0
\end{aligned}$$

With $\mathcal{P}_0(12 \cdot n)$ the present value of receiving the payoff $S_{12 \cdot n}$ at date $12 \cdot n$, and $\mathcal{D}_0(12 \cdot n)$ the present value of the dividends paid by the stock over the next $n \in \mathbb{N}$ years.
N.B. $\eta = \left(1 - \frac{\delta}{1-\delta}\right)$

Question: 4.2

To calculate δ and r, we used the formula for put-call parity.

$$c_0(k) - p_0(k) = S_0 - \mathcal{D}_0(12) - e^{-r} k$$

Since we have 16 different equations, we used linear regression to best approximate $\mathcal{D}_0(12)$ and e^{-r}. After the regression we got $\mathcal{D}_0(12) = 5.17$ and $e^{-r} = 0.9512$. From this we could get r directly, and we can get δ by substituting in the formula from the previous part.

$$\mathcal{D}_0(12) = S_0 \left(1 - \left(1 - \frac{\delta}{1-\delta}\right)\right)$$

Our results are $r = 0.5$ and $\delta = 0.0287$.

Question: 4.3

For calibrating the binomial model, we even need to find U and D. Since $q = \frac{1}{2}$, we can use the formula of $q = \frac{e^{r\Delta} - D}{U - D}$ and $D = 1/U$. Our results are $U = 1.0957$ and $D = 0.9127$.

5 Analysis

Question: 5.1

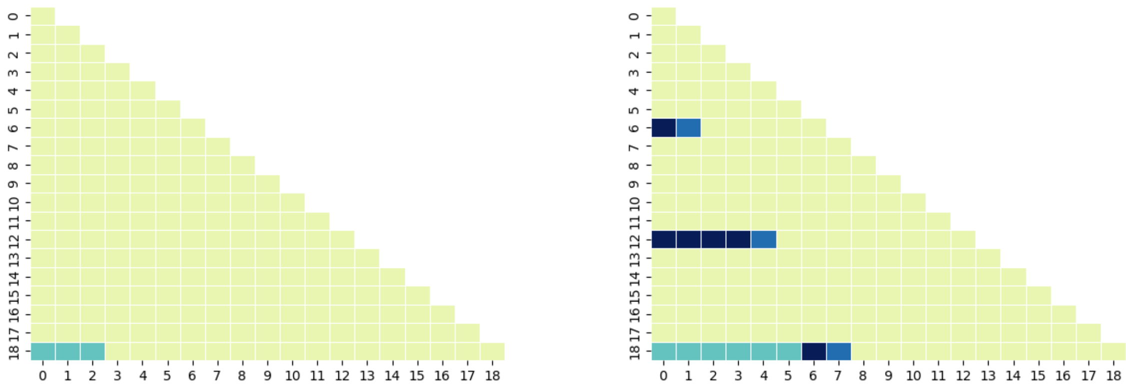
The inital prices for the different types of convertible bonds.

mCB	1070.5045
CB	1075.0511
cCB	1009.2178

In case of the mandatory CB, since $\alpha = 1000$ it is almost sure that the stock price won't fall below 1, making the final payoff almost surely bigger than 1000. On the other hand the holder can benefit at maturity from the stock price if the stock goes above 250, which is not impossible. To have a better understanding: in our model it happens 28 states out of 61 terminal states (we cannot give probabilities, since in pricing the actual probabilities do not matter, only the risk neutral ones). For comparison the plain vanilla bond with the same parameters costs 953.557.

Question: 5.2 and 5.3

For visual purposes, in the Figure below we can see the dates when it is optimal for the bondholder to exercise the conversion option (light blue squares). Sea blue squares denote when the issuer should call the bond, and dark blue denotes when both should exercise their respective options. The vertical axis denotes the date t (in monthly periods), and the horizontal shows the possible (different) values of the stock price.



(a) Vanilla convertible bond

(b) Callable convertible bond

Figure 2: Optimal exercises (description above)

The exact dates are the following for the exercises for the two different bonds.

Vanilla CB

At the third coupon date:

- if $S_t = 880.7271$
- if $S_t = 733.6311$
- if $S_t = 611.1026$

Callable CB
At the first coupon date:

- if $S_t = 252.25$, the issuer should exercise the call option.
- if $S_t = 302.82$, the issuer should call the bond and the bondholder should exercise the conversion option.

At the second coupon date:

- if $S_t = 245.04$, the issuer should exercise the call option.
- if $S_t = 294.17$, the issuer should call the bond and the bondholder should exercise the conversion option.
- if $S_t = 353.15$, the issuer should call the bond and the bondholder should exercise the conversion option.
- if $S_t = 423.96$, the issuer should call the bond and the bondholder should exercise the conversion option.
- if $S_t = 508.97$, the issuer should call the bond and the bondholder should exercise the conversion option.

At the third coupon date:

- if $S_t = 245.07$, the issuer should exercise the call option.
- if $S_t = 294.21$, the issuer should call the bond and the bondholder should exercise the conversion option.
- if $S_t = 353.20$, the bondholder should exercise the conversion option.
- if $S_t = 424.02$, the bondholder should exercise the conversion option.
- if $S_t = 509.04$, the bondholder should exercise the conversion option.
- if $S_t = 611.10$, the bondholder should exercise the conversion option.
- if $S_t = 733.63$, the bondholder should exercise the conversion option.
- if $S_t = 880.73$, the bondholder should exercise the conversion option.

As opposed to the vanilla bond, there are states (prior to the third coupon date) in which it is optimal to exercise the conversion option. However, one must keep in mind that this conversion only happens in cases in which the issuer calls the bond. Otherwise, the bondholder would (in principle) wait until the third coupon date.
As opposed to the bondholder, there are states during the first coupon date in which it is optimal to exercise their option (i.e. call the bond).
Additionally, it is worth noting that during the third coupon date, there are "more" states (in the case of the cCB) in which it is optimal to exercise the conversion option (when compared with its vanilla counterpart). This could derive from a "devaluation" of the continuation value, due to a potential exercise of the call option in future dates.

Question: 5.4

In the followings we analyse how the initial price of the CBs behave as a function of coupon rate, conversion rate (upper conversion in the case of mandatory CB), and call price in case of the callable.

Coupon rate c:

- In the case of the callable CB, even high coupon payments do not drive the price up. A coupon rate that is too high would probably result in an issuer call.
- Although barely noticeable, there's a difference in the price of the vanilla and mandatory CB (when the coupon rate is too low). In particular, the mandatory CB has a lower price than the vanilla CB, which coincides with our expectations. The conversion option gives the buyer more "freedom", therefore the bond's price should be higher.

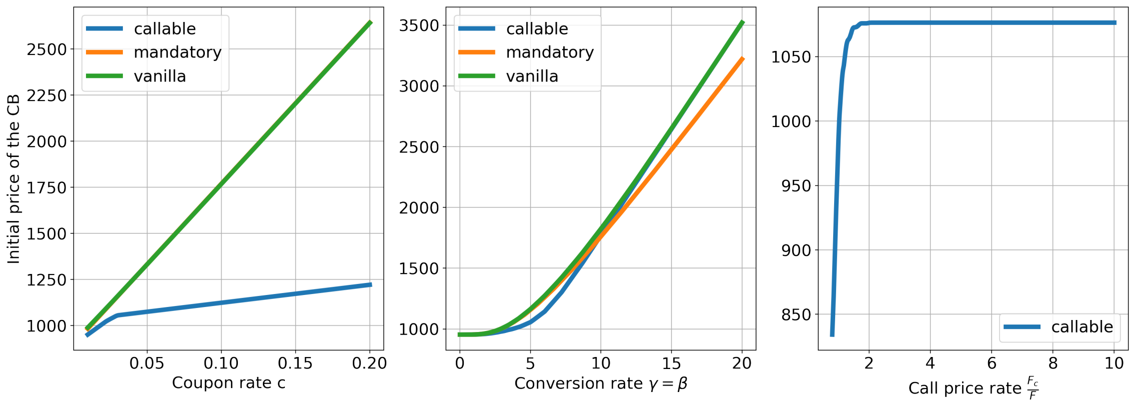


Figure 3: Prices of CBs depending on coupon, conversion and call price

- The aforementioned difference in price (between the vanilla and mandatory CB) decreases as the coupon rate increases, since with high coupon payments it is not optimal to convert early.
- The size of the coupon has a linear effect on the price both mandatory and vanilla, since the bigger the coupon is, the derivative is more likely to behave like the underlying bond and not the option, and the price of the bond is linear in the coupon.

Conversion rate $\gamma = \beta$:

- When the conversion rate is too low, it is never optimal for the buyer to exercise their option (the conversion option becomes "worthless"). Therefore, all three bonds have similar prices.
- Regarding the mCB, since conversion is not an option (but an obligation), an increase in the conversion rate will always lead to an increase in its value. In the case of the vanilla CB, increasing the conversion rate can also increase the potential earnings of the bondholder (which translates into a higher initial price), given the "added" freedom to choose between conversion or continuing. As for the callable CB, an increase in the conversion rate also presents a possible increase in the buyer's profit. However, if γ remains relatively low (e.g. $\gamma \in \{5, ..., 10\}$), the call option (of the issuer) eclipses the advantages brought by the conversion option. More concretely, a possible call of the bond might force the bondholder to exercise conversion, even in cases in which waiting was the optimal choice. This could explain the relatively low price of the cCB when $\gamma \in \{5, ..., 10\}$ (as opposed to the other bonds).
- When the conversion rate is too high, conversion at earlier dates might always be optimal. Therefore, the call option stops being a problem for the bondholder. In other words, the callable bond starts "behaving" like its vanilla counterpart (hence the similar prices). As for the mandatory CB, the lack of "freedom" (to convert at earlier dates) limits the potential profits of the bondholder (which in turn lowers the initial price).

Call price rate $\frac{F_c}{F}$:

- When the call price rate reaches two (i.e. call price $F_c = 2 \cdot F$), it stops having an effect on the price, since the issuer can no longer benefit from the call. The initial converges to ≈ 1076, which is slightly above the price of the CB with the same parameter (≈ 1074).

Question: 5.5

Another metric to compare the bonds is their par coupon rate. We fix the bond's parameters (except c) and we determine the value of the coupon rate for which the initial price is equal to the face value (i.e. determine c for which the bond is priced at par). We used grid search in order to determine the par coupons. We priced the bonds for

200 coupon rate values (uniformly distributed) between 0 and 0.2. Then, we chose the parameter which minimizes $|P_0 - 1000|$. For all bonds with $\gamma = 6$, and for the vanilla CB with $\gamma = 5$, all calculated prices were bigger than 1000. For the other cases the results are summarized in the figure below. The implementation can be found in the function solve_for_c() in 'run.py'.

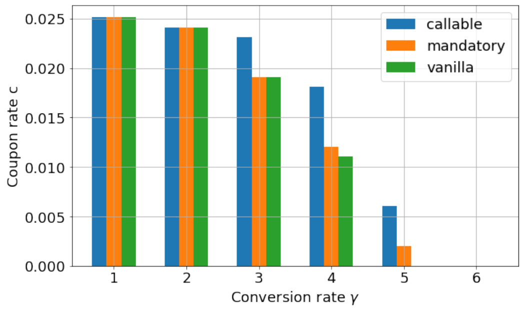


Figure 4: Par coupon of the bonds

As observed before, when the conversion ratio increases, so does the price of the bond. Therefore, in order to make its value par, the coupons should decrease as a function of gamma. The callable CB with the same price should offer a higher coupon in order to compensate for an early termination of the contract (in case of an exercise of the call option, by the issuer). As for the mandatory CB, since investors don't have the possibility to convert at earlier dates, they require higher yields. We can also see that, with $\gamma = 1$, all bonds roughly offer the same coupons, which is really close to the par coupon of the corresponding plain corporate bonds ($c = 0.0253$). This means that, with such low conversion rate, the CBs behave like a plain bond offering the same coupons, the option to convert is not likely to be used (the same phenom can be observed while looking at Figure 3 (price against conversion rate), the CB prices are almost identical, really close to the vanilla bond's price at lower conversion rates).

Question: 5.6

Vanilla Convertible Bond

We first find the coupon rate for a plain coupon bond with the same setup to initially worth the face value by grid search. For this we got $c = 0.0253$. This value is important because for coupon rates bigger than this value the is not interpretable since we would need a negative gamma to make the derivative par.

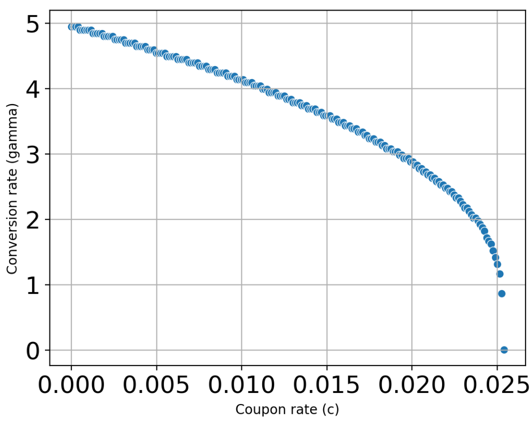


Figure 5: Par conversion rates for the vanilla CB for different coupon rates

In the Figure 5 we calculated the calibrated gammas for the the different values of coupon rates by grid search. We visualized the values only for the coupon rates smaller than the value mentioned above. The pricing is implemented in a function called solve_for_gamma() in 'run.py'.

Callable Convertible Bond

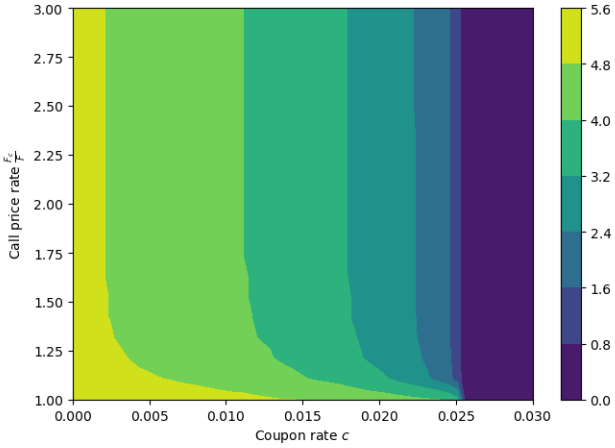


Figure 6: Par conversion rate γ as a function of the coupon rate and the call price rate

Comments on Figure 6:

- When the coupon rate is too high (around 0.025), it is impossible to choose gamma such that the bond is traded at par (regardless of the value of the call price rate, because this is the par coupon of the plain bond and we expect the callable to be more expensive).
- When the call price reaches a certain threshold (i.e. around 1.75), the value of gamma (required for the bond to be traded at par) stops depending on the former. This is in accordance with our comments on Figure 3 (initial price as a function of the call price rate). As explained in question 5.4, the cCB starts behaving like its vanilla counterpart. In fact, by the fixing call price rate (with a value that is higher than 1.75) and plotting the conversion rate γ as a function of the coupon rate c, we arrive to the same curve as in Figure 5.
- When the call price rate is relatively low (i.e. call price is relatively close to the face value), a low decrease of the call price rate will require a high increase of the conversion factor (in order to keep the initial price constant and equal to the face value). Another interpretation is that, a low decrease in the call price rate will require a high increase in the coupon rate (in order to keep both the conversion rate and the initial price constant).

Term sheet
JNJ CONVERTIBLE BOND

PRODUCT DESCRIPTION: Johnson & Johnson introduces its new financial product in a form of convertible bond. Three types of the product are available, vanilla, mandatory and callable convertible bond. The bondholder can get coupons at predetermined dates, if he exercise his conversion option, he receives a specific number of stocks of the company nad otherwise the face value.

Issuer	Johnson & Johnson
Issue date	January 1, 2022
Issue price	Traded at Par
Face value	1000 USD
Maturity	5 years

Type of Security	**Vanilla Convertible Bond**
Coupons	Coupons paid in every 6 month with 2% coupon rate until conversion.
Conversion right	Conversion rate is 4 and conversion is possible at every coupon date.
Risk for investors	Value could decrease with rising interest rates. Offer lower coupons compared to mandatory CBs.

Type of Security	**Mandatory Convertible Bond**
Coupons	Coupons paid in every 6 month with 2% coupon rate.
Conversion right	Lower conversion rate is 1000 (equal to the face value), the upper conversion rate is 4 and conversion is possible at maturity date.
Risk for investors	If the stock does not perform well, the derivative will behave like a plain bond, but on a higher price.

Type of Security	**Callable Convertible Bond**
Coupons	Coupons paid in every 6 month with 2% coupon rate.
Conversion right	Conversion rate is 4 and conversion is possible at every coupon date.
Risk for investors	If interest rates are declining, the issuer will likely call, meaning the investor's returns are uncertain.

References

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