This collection of exam questions and solutions is intended to help you study for this semester's Math 340 courses.

Chapter 1

2025 Spring Exams

1.1 Math340Sp25-Midterm1

Notation: $\mathbf{0}$ denotes the zero vector, O denotes the zero matrix, and I_n denotes the $n \times n$ identity matrix.

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 + 2x_3 \\ 2x_1 \\ 2x_1 + 4x_3 \end{bmatrix}$.

(a) Evaluate $T\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right)$.

(b) Find the (induced) matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$.

(c) Is there a **nonzero** vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $T(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$? Fully justify your answer.

2. Suppose A is a 3×3 matrix and $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue -2, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue 3. Compute each of the following (note: your final answers will be vectors in $\mathbb{R}^3$). Show all work.

(a) $A \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} =$

(b) $A^3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} =$

(c) $A \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix} =$

Hint. Can you write this vector as a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

3. Clearly mark the correct answer for each of the following by **completely** filling in the appropriate bubble. **No justification is needed.**

(a) **(True/False)** If A and B are both 3×3 matrices, then $(A + B)^2 = A^2 + 2AB + B^2$.

A. True

B. False

(b) **(True/False)** $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ 0 \end{bmatrix}$ is a linear transformation.

A. True

B. False

(c) **(True/False)** The matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is a reduced row echelon form matrix.

A. True

B. False

(d) **(True/False)** If A is a 2×2 matrix such that for every 2×2 matrix B, we have $AB = BA$, then A must be I_2.

A. True

B. False

(e) **(True/False)** If $T : \mathbb{R}^5 \to \mathbb{R}^4$ is a matrix transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then A is a 5×4 matrix.

A. True

B. False

(f) **(Multiple Choice-choose one)** For $M = \begin{bmatrix} 2 & 3 & 1 \\ -3 & c & 2 \\ 1 & 0 & 2 \end{bmatrix}$, for which values(s) of c would M be **not invertible**?

A. $c = \frac{12}{5}$

B. $c = -8$

C. All values of c except $c = -8$

D. $c = 0$

E. None of the above.

(g) **(Multiple Choice-choose one)** What are the eigenvalues of the matrix $A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}$?

A. $\lambda = 0, 2$

B. $\lambda = 1, 3$

C. $\lambda = -1, 3$

D. $\lambda = -3, 1$

E. None of the above.

(h) **(Multiple Choice-choose one)**

For which s and t is the matrix $M = \begin{bmatrix} 0 & t & s \\ 3t & 0 & s+t \\ 4 & 4 & t \end{bmatrix}$ **symmetric**?

A. $s = 0, t = 0$

B. $s = -4, t = 0$

C. $s = 4, t = 0$

D. $s = 0$, t can be any value.

E. None of the above.

4.

(a) Suppose A and B are 4×4 matrices with $2A^T B^{-1} + I_4 = O$. If $\det(A) = 10$, find the determinant of B.

(b) Suppose C, D are 4×4 matrices, and D is obtained from C by the following elementary row operations:

- Add $3r_2$ to r_4 (and replace r_4)
- Multiply r_3 by k (for some unknown scalar k)
- Swap r_3 and r_4
- Add $-r_1$ to r_2 (and replace r_2)
- Multiply r_2 by 5

If $\det(C) = 10$ and $\det(D) = -4$, find the scalar k. Show all work.

5. Suppose a system of three equations in four unknowns (x_1, x_2, x_3, x_4) is represented by the follow augmented matrix:

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & -1 & 3 \\ -1 & -2 & 1 & -3 & 5 \\ 2 & 4 & 2 & -2 & 6 \end{array}\right]$$

(a) Write out the system of linear equations corresponding to this augmented matrix.

(b) Through elementary row operations, we have the following reduction to reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 3 \\ -1 & -2 & 1 & -3 & 5 \\ 2 & 4 & 2 & -2 & 6 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Using the reduced row echelon form, write out the parametrized solution for x_1, x_2, x_3, x_4 for this linear system.

6. Consider 2×2 matrices A, B, C, D, where $B = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$.

(a) Find D^{-1}. Show all work.

(b) If $(2A + B)^{-1}C = D$, find A. Show all work.

Hint. You can use your answer to part (a)

7. Suppose A is a 4×4 matrix that is row-equivalent to the matrix $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Answer each of the following questions about the matrix A, if possible. Justification is not required. (Another reminder: You are answering questions about A, not B).

(a) **(Multiple Choice-choose one)** Is A invertible?

A. Yes

B. No

C. Not enough information to answer.

(b) **(Multiple Choice-choose one)** What is the rank of A?

A. 0

B. 1

C. 2

D. 3

E. 4

F. Not enough information to answer.

(c) **(Multiple Choice-choose one)** For $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$, does the linear system $A\mathbf{x} = \mathbf{b}$ have...

A. No solutions

B. One solution

C. Infinitely many solutions

D. Not enough information to answer.

(d) (Multiple Choice-choose one) For $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, does the linear system $A\mathbf{x} = \mathbf{b}$ have...

A. No solutions

B. One solution

C. Infinitely many solutions

D. Not enough information to answer.

1.2 Math340Sp25-Midterm2

Notation reminders: M_{mn} is the vector space of $m \times n$ real matrices with standard matrix addition and scalar multiplication, P_d is the vector space of polynomials (with single variable x) of degree at most d with standard polynomial addition and scalar multiplication, $\mathbf{0}$ denotes the zero vector in a vector space V. $\mathbb{R}^n$ uses the standard vector addition/scalar multiplication unless otherwise noted.

Additional notation reminders: O denotes the zero matrix, and I_n denotes the $n \times n$ identity matrix. $\mathrm{tr}(A)$ denotes the trace of a matrix A.

1. Suppose $T : \mathbb{R}^3 \to M_{22}$ is a linear transformation with $T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\-1 & 1\end{bmatrix}$, $T\left(\begin{bmatrix}2\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}3 & 1\\4 & 0\end{bmatrix}$, and $T\left(\begin{bmatrix}0\\0\\4\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$.

(a) Compute $T\left(\begin{bmatrix}4\\-5\\2\end{bmatrix}\right)$. Show all work. You do not need to simplify your final numerical answer.

(b) Is T onto? Justify your answer.

2. Let $A = \begin{bmatrix} 8 & 0 & -5\\ -20 & -2 & 10\\ 10 & 0 & -7\end{bmatrix}$. You are given the following information about A: A has exactly two eigenvalues $\lambda = -2$ and $\lambda = 3$, and a basis for the eigenspace for $\lambda = -2$ is $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$.

(a) Find a basis for the eigenspace of A for $\lambda = 3$. Show all work.

(b) Is A diagonalizable? Justify your answer.

3. Clearly mark the correct answer for each of the following by **completely** filling in the appropriate bubble. **No justification is needed.**

(a) **(True/False)** If U is a subspace of V and $\mathbf{u} + \mathbf{v}$ is in U, then $\mathbf{u}$ and $\mathbf{v}$ are both in U.

A. True

B. False

(b) **(True/False)** Suppose A and B are invertible 2×2 matrices with $A + B = O$. Then $\{A, B\}$ is a linearly independent set.

A. True

B. False

(c) **(True/False)** Suppose A and B are invertible 2×2 matrices with $A + B = O$. Then the columns of A is a linearly independent set, and the columns of B is a linearly independent set.

A. True

B. False

(d) **(True/False)** The set of noninvertible 3×3 matrices is closed under scalar multiplication.

A. True

B. False

(e) **(True/False)** Any set of 7 matrices in M_{23} must span M_{23}.

A. True

B. False

(f) **(Multiple Choice-choose one)** Suppose A is similar to $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{bmatrix}$. Which of the following must be true about A?

A. A has trace 0.

B. A is diagonalizable.

C. The eigenvectors of A are the same as the eigenvectors of B.

D. A is similar to I_3.

E. All of the above.

(g) **(Multiple Choice-choose one)** Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for a vector space V. Which of the following is also a basis for V?

A. $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}\}$

B. $\{\mathbf{u}, \mathbf{u}+\mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}\}$

C. $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}, \mathbf{v}\}$

D. $\{\mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}, \mathbf{w}\}$

E. All of the above.

F. None of the above.

(h) **(Multiple Choice-choose one)** Which vector space has the same dimension as M_{24}?

A. $\mathbb{R}^6$

B. P_8

C. M_{42}

D. All of the above.

E. None of the above.

4. Consider P_2 and the following subset W of polynomials:

$$W = \{p(x) \text{ in } P_2 \mid p'(0) = 0\}.$$

(Reminder: $p'(x)$ is the derivative of $p(x)$.)

(a) Show that W is nonempty.

(b) Show that for $p_1(x), p_2(x)$ in W, $p_1(x) + p_2(x)$ is in W. Fully justify your answer.

(c) Show that for $p(x)$ in W and c in $\mathbb{R}$, $cp(x)$ is in W. Fully justify your answer.

5. Let V be the set of vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where x and y are both positive real numbers.

We define vector addition $\oplus$ and scalar multiplication $\odot$ as:

$$\begin{bmatrix} x \\ y \end{bmatrix} \oplus \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} xw \\ yz \end{bmatrix}, \quad c \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^c \\ y^c \end{bmatrix}$$

(a) Find the zero vector $\mathbf{0}$ for this vector addition $\oplus$. Justify your answer.

(b) For $\mathbf{v}$ in V, find the additive inverse of $\mathbf{v}$ (Recall: the additive inverse of $\mathbf{v}$ is a vector $\mathbf{u}$ in V such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{0} = \mathbf{v} \oplus \mathbf{u}$). Fully justify your answer.

(c) Show that the following vector space property is true for this vector addition and scalar multiplication: For any c, d in $\mathbb{R}$ and any vector $\mathbf{u}$ in V, $(c+d) \odot \mathbf{u} = c \odot \mathbf{u} \oplus d \odot \mathbf{u}$.

(d) For an arbitrary vector $\mathbf{u}$ in V, compute $0 \odot \mathbf{u}$. Does the vector space property $0 \odot \mathbf{u} = \mathbf{0}$ hold?

6. Consider the vector space P_1 and the set $S = \{2x + 10, c^2 - 41 - 5x\}$. Find all values of c so that the set S is linearly dependent.

7. Each of the following statements is **false**. Show each statement is false by providing explicit counterexamples. You must fully justify your answers.

(a) Let V be $\mathbb{R}^2$ with the operation $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ y_1 + y_2 \end{bmatrix}$. Then $\oplus$ satisfies the vector space property $\mathbf{v} \oplus \mathbf{u} = \mathbf{u} \oplus \mathbf{v}$.

(b) $F : M_{22} \to \mathbb{R}$ by $F(A) = \det(A) + \operatorname{tr}(A)$ is a linear transformation.

(c) $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ in } \mathbb{R}^2 : \ |x| \geq |y| \right\}$ is a subspace of $\mathbb{R}^2$.

8. Suppose A is a 3×5 matrix whose reduced row echelon form $RREF(A)$ is

$$RREF(A) = \begin{bmatrix} 1 & 3 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Answer each of the following questions about the matrix A, if possible. Justification is not required.

(a) **(Multiple Choice-choose one)** What is the dim(null A)?

A. 0
B. 1
C. 2
D. 3
E. 4
F. Not enough information to answer.

(b) **(Multiple Choice-choose one)** Is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the column space of A?

A. Yes
B. No
C. Not enough information to answer.

(c) **(Multiple Choice-choose one)** For $T : \mathbb{R}^5 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, what is the dimension of the image of T?

A. 0
B. 1
C. 2
D. 3
E. 4

F. 5

G. Not enough information to answer.

(d) **(Multiple Choice-choose one)** For $T : \mathbb{R}^5 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, is T one-to-one (also called injective)?

A. Yes

B. No

C. Not enough information to answer.

(e) **(Multiple Choice-choose one)** Are the (five) column vectors making up A a linearly independent set in $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information to answer.

1.3 Math340Sp25-FinalExam

Vector space notation reminders: M_{mn} is the vector space of $m \times n$ real matrices with standard matrix addition and scalar multiplication, P_d is the vector space of polynomials (in x) of degree at most d with standard polynomial addition and scalar multiplication, $\mathbf{0}$ denotes the zero vector in a vector space V. $\mathbb{R}^n$ uses the standard vector addition/scalar multiplication and dot product unless otherwise noted.

Matrix notation reminders: O denotes the zero matrix, and I_n denotes the $n \times n$ identity matrix. $\text{tr}(A)$ denotes the trace of a matrix A.

Additional notation reminders: $M_{DB}(T)$ denotes the matrix representation of the linear transformation $T : V \to W$ using ordered bases B for V and D for W. $M_B(T)$ denotes the matrix representation of the linear operator T with respect to the ordered basis B. $C_B(\mathbf{v})$ denotes the coordinate representation of the vector $\mathbf{v}$ with respect to the ordered basis B.

1. Given $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 2 & -1 \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 2 & 4 & 1 \\ 2 & 6 & 4 \\ 1 & 2 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 4 \\ 3 & 2 & 9 \end{bmatrix}$, find B if $A + BC = D$. Show all work.

2. Let W be the vector space of 2×2 symmetric matrices (using standard matrix addition/scalar multiplication), and define $T : P_2 \to W$ by $T(p(x)) = \begin{bmatrix} p'(0) & p(0) \\ p(0) & 0 \end{bmatrix}$. You can assume (without proof) that T is a linear transformation.

(a) Using ordered basis $B = (4x+1, x^2, x^2-2)$ for P_2 and ordered basis $D = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$ for W, find the matrix representation $M_{DB}(T)$. Show all work.

(b) Is T an isomorphism? Justify your answer.

3. Clearly mark the correct answer for each of the following by **completely** filling in the appropriate bubble. **No justification is needed.**

(a) **(True/False)** If $\mathbf{v}$ is a 3×1 vector, and A is a 3×3 matrix, then $\mathbf{v}A$ is a 3×1 vector.

A. True
B. False

(b) **(True/False)** If A is a 4×4 matrix with $\text{tr}(A) = 0$, then A is invertible.

A. True
B. False

(c) **(True/False)** The subspace $W = \left\{ p \text{ in } P_{10} \middle| \int_0^1 p(x)dx = 0 \right\}$ of P_{10} is finite-dimensional.

A. True
B. False

(d) **(True/False)** The function $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = 2x + 1$ is an isomorphism.

A. True

B. False

(e) **(True/False)** Suppose A is a 3×3 matrix with $A = A^{-1}$. Then $\det(A) = 1$ or $\det(A) = -1$.

A. True

B. False

(f) **(True/False)** If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in an inner product space, then $\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$ must also be orthogonal.

A. True

B. False

(g) **(Multiple Choice-choose one)** Which vector space is isomorphic to P_3?

A. $\mathbb{R}^4$

B. The nullspace of $A = \begin{bmatrix} 1 & 2 & 4 & 3 & -1 \\ 2 & 4 & 8 & 6 & -2 \end{bmatrix}$

C. M_{22}

D. All of the above.

(h) **(Multiple Choice-choose one)** Suppose A is a 3×3 matrix which is row-equivalent to $B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Which of the following must be true about A?

A. $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ has no solutions.

B. $\lambda = 1$ is an eigenvalue of A.

C. $\lambda = 0$ is an eigenvalue of A.

D. The columns of A must be linearly independent.

E. All of the above.

F. None of the above.

(i) **(Multiple Choice-choose one)** Suppose $\{\mathbf{u}, \mathbf{v}\}$ is a basis for a vector space V. Which of the following is also a basis for V?

A. $\{\mathbf{u} + \mathbf{v}\}$

B. $\{\mathbf{u} + \mathbf{v}, 2\mathbf{u} - \mathbf{v}\}$

C. $\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{u}\}$

D. $\{\mathbf{u} + \mathbf{v}, \mathbf{u} + 2\mathbf{v}, \mathbf{v}\}$

E. All of the above.

F. None of the above.

(j) **(Multiple Choice-choose one)** Let V be $\mathbb{R}^2$ with the following vector addition $\oplus$ and scalar multiplication $\odot$:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 3 \\ y_1 + y_2 \end{bmatrix}, \quad c \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 + 3c - 3 \\ cy_1 \end{bmatrix}$$

Which of the following is the zero vector $\mathbf{0}$ for this vector addition $\oplus$?

A. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

B. $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

C. $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

D. $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

E. None of the above.

4. Consider the vector space P_1, and define the function

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$$

for $p(x), q(x)$ in P_1.
You can assume that $\langle p(x), q(x) \rangle$ is an inner product for P_1.

(a) With respect to this inner product, $\langle p(x), q(x) \rangle$, find the distance between $3x + 1$ and $x - 3$. Show all work, but you do not need to simplify your final numerical answer.

(b) With respect to this inner product, $\langle p(x), q(x) \rangle$, show that the polynomials
$p = 3 - 3x$ and $q = 6x - 2$ are orthogonal. Show all work.

(c) With respect to this inner product,$\langle p(x), q(x) \rangle$, using $p = 3 - 3x$ and $q = 6x - 2$, create an orthonormal basis for P_1. Show all work.

5. Let $T : P_1 \to P_1$ be a linear transformation with$M_B(T) = \begin{bmatrix} 6 & 2 \\ -2 & 10 \end{bmatrix}$ with respect to the ordered basis $B = (x + 1, x - 3)$.

(a) Compute $T(2x - 1)$. Show all work.

(b) Is T diagonalizable? Justify your answer.

Again, let $T : P_1 \to P_1$ be a linear transformation with$M_B(T) = \begin{bmatrix} 6 & 2 \\ -2 & 10 \end{bmatrix}$ with respect to the ordered basis $B = (x + 1, x - 3)$.

(c) Suppose $P_{B \leftarrow D} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is the change matrix from unknown basis $D = (p_1, p_2)$ to B. Find p_1 and p_2.

(d) For the same basis D as in part (c), use the Similarity Theorem find $M_D(T)$.

Hint. You do NOT need to know D to solve this problem

6. Consider the vector space $\mathbb{R}^3$, and consider the set U of vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 + 3x_2 = x_3$ and $x_1x_2 \geq 0$, i.e.
$U = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, x_1 + 3x_2 = x_3, \, x_1x_2 \geq 0 \right\}$. Is U a subspace of $\mathbb{R}^3$? Fully justify your answer.

7. Consider $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^2$, and define the function

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2.$$

(a) Verify the following inner product property for $\langle \mathbf{u}, \mathbf{v} \rangle$:
$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$.
For part (b), assume the remaining inner product properties are true for
$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$.

(b) With respect to this inner product $\langle \mathbf{u}, \mathbf{v} \rangle$, find a basis for W, where W is the subspace of $\mathbb{R}^2$ of vectors orthogonal to $\begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Show all work.

8. Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -1 & 1 \\ 3 & -3 & 1 \\ c & -10 & 3 \end{bmatrix}$. If A and B are similar matrices, find c. Show all work.

9. Suppose A is a 3×7 matrix with $\text{rank}(A) = 3$. Answer the following questions about A, if possible. Justification is not required.

(a) **(Multiple Choice-choose one)** What is the $\dim(\text{null } A)$?

A. 0
B. 1
C. 2
D. 3
E. 4
F. Not enough information to answer.

(b) **(Multiple Choice-choose one)** Is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the column space of A?

A. Yes
B. No
C. Not enough information to answer.

(c) **(Multiple Choice-choose one)** Does the linear system $A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ have...

A. No solution
B. One solution
C. Infinitely many solutions
D. Not enough information to answer.

(d) **(Multiple Choice-choose one)** For $T : \mathbb{R}^7 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, is T onto?

A. Yes
B. No
C. Not enough information to answer.

(e) **(Multiple Choice-choose one)** For $T : \mathbb{R}^7 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, is T one-to-one (also called injective)?

A. Yes
B. No
C. Not enough information to answer.

Chapter 2

2024 Fall Exams

2.1 Math340Fa24-Midterm1

Notation: $\mathbf{0}$ denotes the zero vector, O denotes the zero matrix, and I_n denotes the $n \times n$ identity matrix.

1. Clearly mark the correct answer(s) for each of the following by completely filling in the appropriate bubble. **No justification is needed.**

(a) **(True/False)** $\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon matrix.

A. True

B. False

(b) **(True/False)** If A is an 3×3 matrix such that $A^2 = O$, then $A = O$.

A. True

B. False

(c) **(True/False)** If two square matrices A and B are row equivalent, then $\det(A) = \det(B)$.

A. True

B. False

(d) **(True/False)** If A is a 3×3 matrix with $\det(A) = 6$, then $\det(3A) = 18$.

A. True

B. False

(e) **(True/False)** $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ are row equivalent.

A. True

B. False

(f) **(Multiple Choice-choose one)** Suppose A is a 5×5 matrix. If the rank of A is 3, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ will have...

A. No solutions.
B. Exactly one solution.
C. Exactly two solutions.
D. Exactly three solutions.
E. Infinitely many solutions.

(g) **(Multiple Choice-choose one)** If A is 2×3, B is 4×4, and C is 4×3, then what is the size of the matrix AC^TB?

A. 4×4
B. 2×3
C. 3×4
D. 4×3
E. None of the above.

(h) **(Multiple Choice-choose one)** Suppose A is a 3×3 skew-symmetric matrix. Consider the following statements:

- $A + A^T = O$
- $A^T - A = O$
- $A + I_3 = I_3 + A$
- $A\mathbf{x} = -A^T\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^3$.
- $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^3$.

Which of these statements is true about A?

A. (a), (b), and (d)
B. (a), (c), and (d)
C. (b) and (c)
D. (a) and (e)
E. None of the above.

2.

(a) Suppose A, B are 4×4 matrices, and B is obtained from A by the following elementary row operations:

- Multiply r_3 by $\frac{1}{6}$
- Add $7r_1$ to r_2 (and replace r_2)
- Swap r_2 and r_4
- Add $-r_3$ to r_4 (and replace r_4)

- Multiply r_3 by 10

If $\det(B) = -4$, find $\det(A)$. Show all work.

(b) Suppose C, D, E are 3×3 matrices, where $\det(C) = 5$ and $\det(D) = -2$. If $C^{-1}D^TE = I_3$, find $\det(E)$. Show all work.

3. Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$.

(a) Find all the eigenvalues of A. Show all work.

Hint. One of the eigenvalues is 5

(b) Find an eigenvector of A corresponding to $\lambda = 5$. Show all work.

4. Suppose A, B, C, D are 2×2 matrices, where $D^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$, and $D(B+C) = A$.

(a) Which of the following is a correct expression for C? (Choose one. No justification is required).

A. $C = AD^{-1} - B$
B. $C = D^{-1}A - B$

(b) Using your answer from part (a), calculate matrix C. Show all work.

5. Suppose a linear system of 3 equations and 3 unknowns (x, y, z) reduces to the following form via elementary row operations:

$$\left[\begin{array}{ccc:c} 1 & 3 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & (k-3) & (k+1) \end{array}\right]$$

(a) For what value(s) of k (if any) will the system have **no solutions**? Briefly explain your answer.

(b) If $k = 5$, finish reducing the augmented matrix to **reduced row echelon form**, then solve for x, y, z. Show all work.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 2x_2 \\ 2x_1 - x_2 \end{bmatrix}$.

(a) Evaluate $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$.

(b) Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(c) Is there an $\mathbf{x}$ in $\mathbb{R}^2$ such that $T(\mathbf{x}) = \begin{bmatrix} -2 \\ -6 \\ 11 \end{bmatrix}$? Fully justify your answer.

7. Find A, if $(I_2 + 4A)^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Show all work.

8. For which value(s) of c does $A = \begin{bmatrix} 3 & 0 & 1 \\ c & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix}$ have a **nontrivial** solution for $A\mathbf{x} = \mathbf{0}$? Show all work, and justify your answer.

2.2 Math340Fa24-Midterm2

Notation reminders: M_{mn} is the vector space of $m \times n$ real matrices with standard matrix addition and scalar multiplication, P_d is the vector space of polynomials (with single variable x) of degree at most d with standard polynomial addition and scalar multiplication, $\mathbf{0}$ denotes the zero vector in a vector space V. $\mathbb{R}^n$ uses the standard vector addition/scalar multiplication unless otherwise noted.

Additional notation reminders: O denotes the zero matrix, and I_n denotes the $n \times n$ identity matrix. $\operatorname{tr}(A)$ denotes the trace of a matrix A.

1. Clearly mark the correct answer(s) for each of the following by completely filling in the appropriate bubble. **No justification is needed.**

(a) Let A be an $n \times m$ matrix. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of linearly independent vectors in $\mathbb{R}^m$, then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is a set of linearly independent vectors in $\mathbb{R}^n$.

A. True

B. False

(b) For the matrix $A = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$, the image of A, $\operatorname{im} A$, is a subspace of $\mathbb{R}^2$ with $\dim(\operatorname{im} A) = 2$.

A. True

B. False

(c) If A is 5×5 diagonalizable matrix, then A must have 5 distinct eigenvalues.

A. True

B. False

(d) Let V be a vector space, and $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subspace of V. If $\mathbf{v}$ is a vector in V which is not in W, then $\mathbf{v}$ is not a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

A. True

B. False

(e) In P_2, the vectors $4x^2 - 2x + 1$ and $2x^2 - x + 3$ are linearly dependent.

A. True

B. False

(f) **(Multiple Choice-choose one)** Suppose A is a 2×2 matrix where $AP = PD$ for $P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Note that $P^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$. What is A^{10}?

A. $\begin{bmatrix} -2\,a^{10} + 3\,b^{10} & 3\,a^{10} - 3\,b^{10} \\ -2\,a^{10} + 2\,b^{10} & 3\,a^{10} - 2\,b^{10} \end{bmatrix}$

B. $\begin{bmatrix} -2\,a^{10} + 3\,b^{10} & -6\,a^{10} + 6\,b^{10} \\ a^{10} - b^{10} & 3\,a^{10} - 2\,b^{10} \end{bmatrix}$

C. $\begin{bmatrix} a^{10} & 0 \\ 0 & b^{10} \end{bmatrix}$

D. None of the above.

(g) **(Multiple Choice-choose one)** Consider a set $S = \{p_1, p_2, p_3, p_4, p_5\}$ of 5 polynomials in P_3. Which of the following is true?

A. S spans P_3.

B. S is a linearly independent set.

C. S is a basis for P_3.

D. At least one of the polynomials in S is a linear combination of the others.

E. None of the above.

(h) **(Multiple Choice-choose one)** For what value(s) of c do the vectors $\begin{bmatrix} 3 \\ 0 \\ c \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

A. $c = 9$

B. All values of c except $c = 9$

C. No values of c make this a basis.

D. All values of c make this a basis.

E. None of the above.

2. Suppose A is a 6×4 matrix of rank 4, and consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^6$ by $T(\mathbf{x}) = A\mathbf{x}$. Answer each of the following questions. Justification is not required.

(a) What is the dimension of the kernel of T? Fill in the blank below.

$\dim \ker(T) = \boxed{}$

(b) **(Multiple Choice-choose one)** Is T one-to-one?

A. Yes

B. No

C. Not enough information to answer.

(c) What is the dimension of the image of T? Fill in the blank below.

$\dim \operatorname{im}(T) = \boxed{}$

(d) **(Multiple Choice-choose one)** Is T onto?

A. Yes

B. No

C. Not enough information to answer.

3. Suppose $T : P_1 \to M_{22}$ is a linear transformation with $T(4x+3) = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}, T(x+1) = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$. Find $T(1)$. Show all work.

4. Consider $A = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}$, and let $W = \{X \text{ in } M_{22} |\ AX = XA\}$.

(a) Show that W is nonempty.

(b) Is W closed under addition? Fully justify your answer using complete sentences.

(c) Is W closed under scalar multiplication? Fully justify your answer using complete sentences.

5. Let V be $\mathbb{R}^2$ with the following vector addition $\oplus$ and scalar multiplication $\odot$:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 4 \\ y_1 + y_2 - 5 \end{bmatrix}, \quad c \odot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 + 4c - 4 \\ cy_1 - 5c + 5 \end{bmatrix}$$

(a) Compute $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

(b) Find the zero vector $\mathbf{0}$ for this vector addition $\oplus$. You must justify that$\mathbf{v} \oplus \mathbf{0} = \mathbf{v} = \mathbf{0} \oplus \mathbf{v}$ for all $\mathbf{v}$ in $\mathbb{R}^2$.

(c) For $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, find the additive inverse of $\mathbf{v}$ (i.e. find $\mathbf{u}$ such that $\mathbf{u} \oplus \mathbf{v} = \mathbf{0} = \mathbf{v} \oplus \mathbf{u}$). Fully justify your answer.

(d) Show the following vector space property holds:

$1 \odot \mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in $\mathbb{R}^2$.

6. For $A = \begin{bmatrix} 10 & 6 & 12 \\ -3 & 1 & -6 \\ -3 & -3 & -2 \end{bmatrix}$:

(a) Find a basis for the eigenspace associated to the eigenvalue $\lambda = 4$ (in other words, find a set of basic eigenvectors associated to $\lambda = 4$). Show all work.

(b) Given that $\lambda = 1$ is the only other eigenvalue of A, and a basis for the eigenspace of $\lambda = 1$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$, answer the following: Is A diagonalizable? Fully justify your answer using complete sentences.

7. For this problem, let A be a 3×5 matrix whose reduced row echelon form $RREF(A)$ is given as $RREF(A) = \begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Note: This is the **reduced row echelon form of** A, not A. Answer each of the following questions about the matrix A, if possible. Justification is not required.

(a) **(Multiple Choice-choose one)** What is the rank of A?

A. 1
B. 2
C. 3
D. 4
E. 5
F. Not enough information to answer.

(b) **(Multiple Choice-choose one)** For which value of c is $\begin{bmatrix} c \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ in the nullspace of A?

A. $c = -3$
B. $c = 0$
C. $c = 1$
D. $c = 3$
E. Not enough information to answer.

(c) **(Multiple Choice-choose one)** Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the column space of A?

A. Yes
B. No
C. Not enough information to answer.

(d) **(Multiple Choice-choose one)** Are the (five) column vectors making up A a linearly independent set in $\mathbb{R}^3$?

A. Yes

B. No

C. Not enough information to answer.

8. Each of the following statements is **false**. Show each statement is false by providing explicit counterexamples. You must fully justify your answers.

(a) The function $F : M_{22} \to \mathbb{R}$ by $F(A) = (\mathrm{tr}(A))^2$ is a linear transformation.

(b) The set of vectors $W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \; a + b \geq -1 \right\}$ is a subspace of $\mathbb{R}^2$.

(c) Let V be $\mathbb{R}^2$ with the operation $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 \end{bmatrix}$. $\oplus$ satisfies the vector space property $\mathbf{v} \oplus \mathbf{u} = \mathbf{u} \oplus \mathbf{v}$.

2.3 Math340Fa24-FinalExam

Notation reminders: M_{mn} is the vector space of $m \times n$ real matrices with standard matrix addition and scalar multiplication, P_d is the vector space of polynomials (in x) of degree at most d with standard polynomial addition and scalar multiplication, $\mathbf{0}$ denotes the zero vector in a vector space V. $\mathbb{R}^n$ uses the standard vector addition/scalar multiplication and dot product unless otherwise noted.

Additional notation reminders: O denotes the zero matrix, and I_n denotes the $n \times n$ identity matrix. $\operatorname{tr}(A)$ denotes the trace of a matrix A.

1. Consider $T : P_2 \to P_1$ by $T(p(x)) = p(0) + 2p'(x)$. You can assume that T is a linear transformation.

(a) Find $M_{DB}(T)$, where $B = (5, x^2 + 3, x)$ and $D = (1, x)$. Show all work.

Hint. your final matrix will be 2×3

2. Consider the subspace W of $\mathbb{R}^4$ spanned by the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 6 \end{bmatrix} \right\}$$

(a) What is the dimension of W? Show your work, and briefly justify your answer.

(b) For which d is W isomorphic to P_d? Justify your answer in one sentence.

(c) Is W isomorphic to M_{22}? Justify your answer in one sentence.

3. Fill in the bubble next to the correct answer. Justification is not necessary.

(a) Let A be an $n \times n$ matrix with $\operatorname{rank}(A) = n$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$.

A. True

B. False

(b) Let A be an $n \times n$ matrix. If A^2 is invertible, then A is invertible.

A. True

B. False

(c) Let V be a finite-dimensional vector space where $\dim(V) = n$. Let B and D be ordered bases for V, and let $T : V \to V$ be a linear operator on V. Then $\det(M_B(T)) = \det(M_D(T))$.

A. True

B. False

(d) Consider P_1 with the following inner product: $\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\mathrm{d}x$. Using this inner product, x is a unit vector.

A. True

B. False

(e) Suppose A and B are 3×3 matrices. Then $\mathrm{rank}(AB) = \mathrm{rank}(A)$.

A. True

B. False

(f) If $\mathbf{u}, \mathbf{v}$ are linearly dependent vectors in $\mathbb{R}^2$, then $|\langle \mathbf{u}, \mathbf{v}\rangle| = \|\mathbf{u}\|\,\|\mathbf{v}\|$.

A. True

B. False

(g) **(Multiple Choice-choose one)** If $T : P_2 \to \mathbb{R}^3$ is an isomorphism, which of the following is true?

A. $\ker(T) = P_2$

B. $\mathrm{im}(T) = P_2$

C. T is onto

D. The inverse of T does not exist.

E. None of the above.

(h) **(Multiple Choice-choose one)** Consider $\mathbb{R}^3$ with the standard inner product, and consider the subspace W of vectors in $\mathbb{R}^3$ which are orthogonal to both $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}$. What is the dimension of W?

A. $\dim(W) = 0$
B. $\dim(W) = 1$
C. $\dim(W) = 2$
D. $\dim(W) = 3$

(i) **(Multiple Choice-choose one)** Suppose A is a 3×3 matrix with eigenvalue 2 and eigenvalue 3. Which of the following is true?

A. If $\mathbf{v}$ is in both the eigenspace of $\lambda = 2$ and the eigenspace of $\lambda = 3$, then $\mathbf{v} = \mathbf{0}$.
B. If $\mathbf{w}$ is an eigenvector of A associated to $\lambda = 2$, then $-\mathbf{w}$ is an eigenvector of A associated to $\lambda = -2$.
C. A is not diagonalizable.
D. A is similar to I_3.
E. All of the above.

(j) **(Multiple Choice-choose one)** Suppose $A = \begin{bmatrix} -1 & x \\ -2 & y \end{bmatrix}$, and you know that A is similar to $\begin{bmatrix} -3 & -2 \\ 12 & 8 \end{bmatrix}$. Find x and y.

A. $x = 6$, $y = 3$
B. $x = -4$, $y = 22$
C. $x = 12$, $y = 24$
D. $x = 3$, $y = 6$
E. None of the above.

4. Given $A = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, and $D = \begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix}$, find B if $(A + B)C = D$. Show all work.

5. Let $T : P_1 \to P_1$ be a linear transformation with $M_B(T) = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ with respect to the ordered basis $B = (x + 3, x - 1)$. (NOTE: B is NOT the standard basis for P_1).

(a) Compute $T(9 - x)$. Show all work. (Note: your final answer should be in P_1).

(b) Find a basis for the kernel of T. (Note: your final answer should be in P_1).

(c) Is T diagonalizable? Justify your answer.

Again, let $T : P_1 \to P_1$ be a linear transformation with$M_B(T) = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$ with respect to the ordered basis $B = (x + 3, x - 1)$.

(d) Is T an isomorphism? Justify your answer.

(e) Suppose $P_{B \leftarrow D} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$ is the change matrix from unknown basis $D = (p_1, p_2)$ to B. Find p_1 and p_2.

(f) For the same basis D as in part (e), find $M_D(T)$.

Hint. You do NOT need to know D to solve this problem.

6. Consider the vector space P_1, and define the function

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$$

for $p(x), q(x)$ in P_1.
You can assume that $\langle p(x), q(x) \rangle$ is an inner product for P_1.

(a) Using this inner product $\langle p(x), q(x) \rangle$, find the distance between $2x + 3$ and $x - 4$. Show all work. You do not need to simplify your final numerical answer.

(b) Still using this inner product $\langle p(x), q(x) \rangle$, for which c is $cx + 2$, $2x - 2$ orthogonal? Show all work.

7. Is $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_2 = 2x_1 + 3x_3, \; x_1x_3 \geq 0 \right\}$ a subspace of $\mathbb{R}^3$? If it is a subspace, prove it using the subspace test. If it is not a subspace, provide an explicit counterexample showing which of the subspace criteria is violated. Fully justify your answer.

8. Consider M_{22}, and let A, B be matrices in M_{22}. Let $\langle A, B \rangle = \mathrm{tr}(AB^T)$. You can assume this is an inner product for M_{22}.

Consider $M = \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix}$ and $N = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$. Consider $W = \mathrm{span}\{M, N\}$.

(a) Verify that M and N are orthogonal using this inner product. Show all work.

(b) Assuming that $\{M, N\}$ is a basis for W, use M and N to build an orthonormal basis for W using this inner product. Show all work.

9. Suppose a linear system of 3 equations and 3 unknowns (x, y, z) reduces to the following form via elementary row operations:

$$\left[\begin{array}{ccr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & (a^2 - 9) & (a + 3) \end{array}\right]$$

(a) For which value(s) of a will the system have **no solutions**? Briefly explain your answer.

(b) For which value(s) of a will the system have **infinitely many solutions**? Briefly explain your answer.

(c) If $a = 2$, finish reducing the augmented matrix to **reduced row echelon form**, then solve for x, y, z. Show all work.

10. Suppose A is a 5×5 matrix with

$$RREF(A) = \begin{bmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the reduced row echelon form of A, not A. Answer each question. Justification is not required.

(a) Is A invertible?

A. Yes

B. No

C. Not enough information to determine.

(b) $\left\{ \begin{bmatrix} j \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ k \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for the nullspace of A. Find j and k. Write your answers in the provided boxes.

$j = \boxed{}$ $k = \boxed{}$

(c) Is 0 an eigenvalue of A?

A. Yes
B. No
C. Not enough information to determine.

(d) Is 1 an eigenvalue of A?

A. Yes
B. No
C. Not enough information to determine.

(e) If $T : \mathbb{R}^5 \to \mathbb{R}^5$ is the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$, what is the dimension of the image of T?

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5
G. Not enough information to determine.