Electromagnetic Radiation

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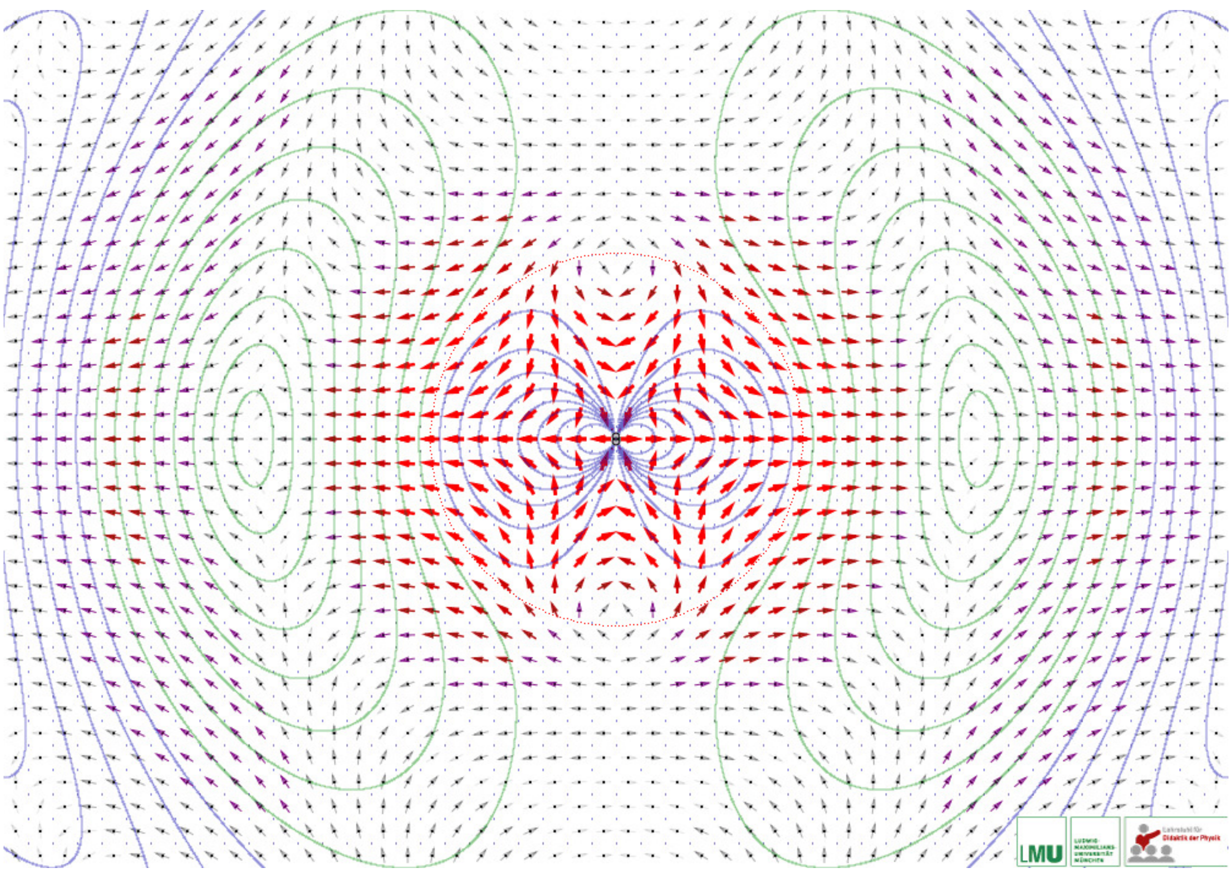

Introduction

Electromagnetic radiation plays a key role in many fields of physics today. Its treatment in many graduate courses, however, does not properly reflect this importance; an overarching vision of the field including many subtleties, connections to other fields, and useful applications is glossed over, and instead students are offered a rapid-fire treatment that effectively deals in a series of parlor tricks. As an example, most students upon finishing a Jackson-level E&M course will have a good handle on solving the Laplace and Poisson equations for the scalar potential. Ask them to do the same for the vector potential and few will be able to muster the same response. Ask them to solve for the electric and magnetic fields of radiation using the vector Helmholtz equation and you'll likely get blank stares. The reasons for this are myriad. There is only so much time in a semester and instructors are forced to drop certain topics; certainly if you can't fully cover radiation, then a quick exposure is better than none. The full machinery needed to solve the vector Helmholtz equation requires quite a bit more background than instructors are willing to invest (and for which, again, time is an issue).

In this set of notes, I hope to present a more cohesive vision for electromagnetic radiation, centered on liberal use of the vector spherical harmonics. Those sections denoted by *** may be omitted on first reading, but they're provided for completeness. In section 1 I'll step through the construction of these harmonics as generalizations of the scalar spherical harmonics (indeed, I'll actually show an overarching framework that enables the construction of several kinds of tensor spherical harmonics, of which the vector variety are just a subset). In section 2, I'll re-present the usual multipole expansions of the scalar and vector potentials and their associated electric and magnetic fields using this framework. In section 3, I'll muster the full might of the VSH to construct the solutions of the vector Helmholtz equations for the electric and magnetic fields, the full multipole solutions of which the static case is only a subset. Lastly in section 4, I'll consider a couple of useful applications: the dipole antenna, which provides fertile ground for comparing various approximation schemes and the resonant modes of a spherical cavity, to illustrate the use of the VSH for solving boundary value problems.

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1 Tensor spherical harmonics

1.1 Angular momentum addition

Suppose we have some system with orbital angular momentum $\vec{L}$ and spin angular moment $\vec{S}$. The total angular momentum is given by $\vec{J} = \vec{L} + \vec{S}$. In this system, there are product angular momentum states $|\ell, m_\ell; s, m_s\rangle = |\ell, m_\ell\rangle |s, m_s\rangle$ that satisfy the relations

$$\hat{L}^2 |\ell, m_\ell; s, m_s\rangle = \ell(\ell+1) |\ell, m_\ell; s, m_s\rangle \qquad \hat{L}_z |\ell, m_\ell; s, m_s\rangle = m_\ell |\ell, m_\ell; s, m_s\rangle \tag{1.1}$$

$$\hat{S}^2 |\ell, m_\ell; s, m_s\rangle = s(s+1) |\ell, m_\ell; s, m_s\rangle \qquad \hat{S}_z |\ell, m_\ell; s, m_s\rangle = m_s |\ell, m_\ell; s, m_s\rangle \tag{1.2}$$

thereby serving as eigenstates of the orbital and spin angular momentum operators. These states may be manipulated by means of raising and lowering operators, defined as

$$\hat{L}_\pm = \hat{L} \pm i\hat{L}_y \qquad \hat{S}_\pm = \hat{S} \pm i\hat{S}_y \tag{1.3}$$

and which have the effect

$$\hat{L}_\pm |\ell, m_\ell; s, m_s\rangle = \sqrt{(\ell \mp m_\ell)(\ell \pm m_\ell + 1)}\, |\ell, m_\ell \pm 1; s, m_s\rangle \tag{1.4}$$

$$\hat{S}_\pm |\ell, m_\ell; s, m_s\rangle = \sqrt{(s \mp m_s)(s \pm m_s + 1)}\, |\ell, m_\ell; s, m_s \pm 1\rangle \tag{1.5}$$

If we attempt to find the total angular momentum from these states (i.e, $J^2 |\ell, m_\ell; s, m_s\rangle = ?$), we are stymied by the fact that the product states do not have definite angular momentum; rather they are written as superpositions of total angular momentum states $\left|j, m_j; \ell, s\right\rangle$. This is done using Clebsch-Gordan coefficients:

$$|j, m; \ell, s\rangle = \sum_{m_\ell=-\ell}^{\ell} \sum_{m_s=-s}^{s} |\ell, m_\ell; s, m_s\rangle \langle \ell, m_\ell; s, m_s \,|\, j, m; \ell, s\rangle \tag{1.6}$$

where $\langle \ell, m_\ell; s, m_s \,|\, j, m; \ell, s\rangle$ is the CG coefficient. Angular momentum conservation tells us that $m_j = m_\ell + m_s$, and that $|\ell - s| \leq j \leq l + s$. These total angular momentum states will be eigenstates of the J^2, J_z, L^2, and S^2 operators, thus satisfying the relations

$$\hat{J}^2 |j, m; \ell, s\rangle = j(j+1) |j, m; \ell, s\rangle \qquad \hat{J}_z |j, m; \ell, s\rangle = m |j, m; \ell, s\rangle \tag{1.7}$$

$$\hat{L}^2 |j, m; \ell, s\rangle = \ell(\ell+1) |j, m; \ell, s\rangle \qquad \hat{S}^2 |j, m; \ell, s\rangle = s(s+1) |j, m; \ell, s\rangle \tag{1.8}$$

Note that $\hat{J}^2$ may be written as

$$\hat{J}^2 = \hat{\vec{J}} \cdot \hat{\vec{J}} = \left(\hat{\vec{L}} + \hat{\vec{S}}\right) \cdot \left(\hat{\vec{L}} + \hat{\vec{S}}\right) = \hat{L}^2 + \hat{S}^2 + 2\hat{L} \cdot \hat{S} \tag{1.9}$$

Let's apply this logic to the usual spherical harmonics, which from now on I'll refer to as the *scalar spherical harmonics* or SSH. The SSH actually constitute the position space representation of orbital angular momentum eigenstates, $Y_\ell^{m_\ell}(\theta, \phi) = \langle \theta, \phi \,|\, \ell, m \rangle$ and thus we can write

$$L^2 Y_\ell^{m_\ell} = -\ell(\ell+1) Y_\ell^{m_\ell}(\theta, \phi) \qquad\qquad L_z Y_\ell^{m_\ell} = m_\ell Y_\ell^{m_\ell}(\theta, \phi) \tag{1.10}$$

Similarly, let $\chi_s^{m_s}$ be the matrix representations of the spin eigenstates of S^2 and S_z. These will, therefore, take the form of arrays of dimension $2s+1$, and the operators matrices of dimension $(2s+1) \times (2s+1)$, which satisfy the relations

$$S^2 \chi_s^{m_s} = s(s+1) \chi_s^{m_s} \qquad\qquad S_z \chi_s^{m_s} = m_s \chi_s^{m_s} \tag{1.11}$$

We can therefore apply this machinery to a product of the form $Y_\ell^{m_\ell}(\theta, \phi)\chi_s^{m_s}$, to find total angular momentum states built out of these direct product states. Let $\mathcal{Y}_{jm}^{\ell s}(\theta, \phi)$ be the position space representation of the total angular momentum states. These will therefore satisfy the relations

$$J^2 \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) = j(j+1) \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) \qquad\qquad J_z \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) = m \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) \tag{1.12}$$

$$L^2 \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) = \ell(\ell+1) \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) \qquad\qquad S^2 \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) = s(s+1) \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) \tag{1.13}$$

Using the Clebsch-Gordan series, we can write

$$\begin{aligned} \mathcal{Y}_{jm}^{\ell s}(\theta, \phi) &= \sum_{m_\ell=-\ell}^{\ell} \sum_{m_s=-s}^{s} \langle \ell, m_\ell; s, m_s \,|\, j, m; \ell, s \rangle \, Y_\ell^{m_\ell}(\theta, \phi) \chi_s^{m_s} && (1.14) \\ &= \sum_{m_s=-s}^{s} \langle \ell, m-s; s, m_s \,|\, j, m; \ell, s \rangle \, Y_\ell^{m-m_s}(\theta, \phi) \chi_s^{m_s} && (1.15) \end{aligned}$$

where the last step has followed from the fact that $m = m_\ell + m_s$. General expressions for any ℓ and s are a mind-boggling in the number of indices, cases and summations; so I won't bother. Instead, let's find explicit expressions for the first two cases, where $s = \frac{1}{2}$, generating the spin spherical harmonics and $s = 1$, generating the vector spherical harmonics. These are much more tractable (and useful).

1.2 ***Spin spherical harmonics***

This section may be skipped on first reading. That said, it provides a good outline of the techniques we'll employ in the section on vector spherical harmonics, so it's not a bad idea to read this. For $s = \frac{1}{2}$, then j can take on the values $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$, so long as $\ell \neq 0$; that's it. If $\ell = 0$, then $j = \frac{1}{2}$ only. The CG coefficients for this are given by this table:

j	$m_s = \frac{1}{2}$	$m_s = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\left(\frac{\ell+m+\frac{1}{2}}{2\ell+1}\right)^{\frac{1}{2}}$	$\left(\frac{\ell-m+\frac{1}{2}}{2\ell+1}\right)^{\frac{1}{2}}$
$\ell - \frac{1}{2}$	$-\left(\frac{\ell-m+\frac{1}{2}}{2\ell+1}\right)^{\frac{1}{2}}$	$\left(\frac{\ell+m+\frac{1}{2}}{2\ell+1}\right)^{\frac{1}{2}}$

Thus the CG series becomes

$$\begin{aligned}\left|j, m; \ell, \tfrac{1}{2}\right\rangle &= \sum_{m_s=-\frac{1}{2}}^{\frac{1}{2}} \left|\ell, m - m_s; \tfrac{1}{2}, m_s\right\rangle \left\langle \ell, m - m_s; \tfrac{1}{2}, m_s \,\middle|\, j, m; \ell, \tfrac{1}{2}\right\rangle \\ &= \left|\ell, m - \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle \left\langle \ell, m - \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2} \,\middle|\, j, m; \ell, \tfrac{1}{2}\right\rangle + \left|\ell, m + \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \left\langle \ell, m + \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2} \,\middle|\, j, m; \ell, \tfrac{1}{2}\right\rangle\end{aligned}$$

Since $j = \ell \pm \frac{1}{2}$, we can write, using the table

$$\left|j = \ell \pm \tfrac{1}{2}, m\right\rangle = \frac{1}{\sqrt{2\ell+1}} \left[\pm\sqrt{\ell + \tfrac{1}{2} \pm m}\left|\ell, m - \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \sqrt{\ell + \tfrac{1}{2} \mp m}\left|\ell, m + \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle\right]$$

where I've suppressed the ℓ, s to clean up the notation. Since the matrix representation of the spin-$\frac{1}{2}$ states is pretty easy, in the form $\chi_{\frac{1}{2}}^{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Flipping the CG series into the position basis gives us

$$\mathcal{Y}^{\ell,\frac{1}{2}}_{j=\ell\pm\frac{1}{2},m}(\theta,\phi) = \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} \pm\sqrt{\ell \pm m + \frac{1}{2}}\,Y_\ell^{m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\ell \mp m + \frac{1}{2}}\,Y_\ell^{m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{1.16}$$

If $\ell = 0$, then

$$\mathcal{Y}^{0,\frac{1}{2}}_{j=\frac{1}{2},m}(\theta,\phi) = \begin{pmatrix} \pm\sqrt{\pm m + \frac{1}{2}}\,Y_0^{m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\mp m + \frac{1}{2}}\,Y_0^{m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{1.17}$$

Note that $m = \frac{1}{2}$ means that the upper element is a constant $Y_0^0 = \frac{1}{\sqrt{4\pi}}$ while the lower element vanishes, and vice-versa if $m = -\frac{1}{2}$. We'll now apply this logic to the vector spherical harmonics.

1.3 Vector spherical harmonics

For $s = 1$, then j can take on the values $j = \ell + 1, \ell, \ell - 1$, so long as $\ell \neq 0$. The CG coefficients are given by this table

j	$m_s = 1$	$m_s = 0$	$m_s = -1$
$\ell+1$	$\left[\frac{(\ell+m)(\ell+m+1)}{(2\ell+1)(2\ell+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(\ell-m+1)(\ell+m+1)}{(\ell+1)(2\ell+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\ell-m)(\ell-m+1)}{(2\ell+1)(2\ell+2)}\right]^{\frac{1}{2}}$
ℓ	$-\left[\frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\right]^{\frac{1}{2}}$	$\frac{m}{\sqrt{\ell(\ell+1)}}$	$\left[\frac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)}\right]^{\frac{1}{2}}$
$\ell-1$	$\left[\frac{(\ell-m)(\ell-m+1)}{2\ell(2\ell+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(\ell+m)(\ell-m)}{\ell(2\ell+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(\ell+m)(\ell+m+1)}{2\ell(2\ell+1)}\right]^{\frac{1}{2}}$

The simplest basis in which to express this result is the spherical basis, since this diagonalizes the spin matrices and makes the basis states trivial:

$$|1,1\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |1,0\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |1,-1\rangle \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.18}$$

in which case, the vector spherical harmonics (VSH) (and remapping $\ell = j \mp 1$) are

$$\mathcal{Y}^{j-1,1}_{j,m}(\theta,\phi) = \begin{pmatrix} \left[\frac{(j+m-1)(j+m)}{2j(2j-2)}\right]^{\frac{1}{2}} Y^{m-1}_{j-1}(\theta,\phi) \\ \left[\frac{(j-m)(j+m)}{j(2j-1)}\right]^{\frac{1}{2}} Y^{m}_{j-1}(\theta,\phi) \\ \left[\frac{(j-m)(j-m-1)}{2j(2j-1)}\right]^{\frac{1}{2}} Y^{m+1}_{j-1}(\theta,\phi) \end{pmatrix} \qquad \mathcal{Y}^{\ell+1,1}_{j,m}(\theta,\phi) = \begin{pmatrix} \left[\frac{(j-m+1)(j-m+2)}{2(j+1)(2j+3)}\right]^{\frac{1}{2}} Y^{m-1}_{j+1}(\theta,\phi) \\ -\left[\frac{(j-m+1)(j+m+1)}{(j+1)(2j+3)}\right]^{\frac{1}{2}} Y^{m}_{j+1}(\theta,\phi) \\ \left[\frac{(j+m+1)(j+m+2)}{2(j+1)(2j+3)}\right]^{\frac{1}{2}} Y^{m+1}_{j+1}(\theta,\phi) \end{pmatrix}$$

$$\mathcal{Y}^{\ell,1}_{j=\ell,m}(\theta,\phi) = \begin{pmatrix} -\left[\frac{(\ell-m+1)(\ell+m)}{2\ell(\ell+1)}\right]^{\frac{1}{2}} Y^{m-1}_{\ell}(\theta,\phi) \\ \frac{m}{\sqrt{\ell(\ell+1)}} Y^{m}_{\ell}(\theta,\phi) \\ \left[\frac{(\ell+m+1)(\ell-m)}{2\ell(\ell+1)}\right]^{\frac{1}{2}} Y^{m+1}_{\ell}(\theta,\phi) \end{pmatrix} \tag{1.19}$$

Now let's consider the last of these ($j = \ell$) and swap over to a Cartesian basis. In this basis, $(S_k)_{ij} = -i\epsilon_{ijk}$ (making this the adjoint representation, since ϵ_{ijk} are the structure constants of $SU(2)$) and thus the orthonormal basis vectors are

$$\chi_1^{\pm 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp 1 \\ -i \\ 0 \end{pmatrix} \qquad \chi_1^{0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.20}$$

In this basis the VSH $j = \ell$ becomes

$$\mathcal{Y}^{\ell,1}_{j=\ell,m}(\theta,\phi) = \frac{1}{2\sqrt{\ell(\ell+1)}} \begin{pmatrix} [(\ell-m+1)(\ell+m)]^{\frac{1}{2}} Y^{m-1}_{\ell}(\theta,\phi) + [(\ell+m+1)(\ell-m)]^{\frac{1}{2}} Y^{m+1}_{\ell}(\theta,\phi) \\ i\,[(\ell-m+1)(\ell+m)]^{\frac{1}{2}} Y^{m-1}_{\ell}(\theta,\phi) - i\,[(\ell+m+1)(\ell-m)]^{\frac{1}{2}} Y^{m+1}_{\ell}(\theta,\phi) \\ 2mY^{m}_{\ell}(\theta,\phi) \end{pmatrix} \tag{1.21}$$

Now, the prefactors in the first and second entries look suspiciously like the leftovers of raising and lowering operators, and in fact they are! Thus we can write:

$$\mathcal{Y}^{\ell,1}_{j=\ell,m}(\theta,\phi) = \frac{1}{2\sqrt{\ell(\ell+1)}}\begin{pmatrix} L_-Y_\ell^m(\theta,\phi) + L_+Y_\ell^m(\theta,\phi) \\ iL_-Y_\ell^m(\theta,\phi) - iL_+Y_\ell^m(\theta,\phi) \\ 2L_0Y_\ell^m(\theta,\phi)\end{pmatrix} = \frac{1}{2\sqrt{\ell(\ell+1)}}\begin{pmatrix} (L_- + L_+)\,Y_\ell^m(\theta,\phi) \\ i\,(L_- - L_+)\,Y_\ell^m(\theta,\phi) \\ 2L_0Y_\ell^m(\theta,\phi)\end{pmatrix}$$

$$\Rightarrow \qquad \mathcal{Y}^{\ell,1}_{j=\ell,m}(\theta,\phi) = \frac{1}{\sqrt{\ell(\ell+1)}}\vec{L}Y_\ell^m(\theta,\phi) \tag{1.22}$$

and we've thus written the total angular momentum state in terms of the scalar spherical harmonics and the angular momentum operator. This is such a nice result, we'll give it its own name. We'll call it $\mathbf{\Phi}_\ell^m(\theta,\phi) = \mathcal{Y}^{\ell,1}_{j=\ell,m}(\theta,\phi)$[1]. This function is what Jackson calls a vector spherical harmonic, $\mathbf{X}_\ell^m(\theta,\phi) = \frac{1}{i\sqrt{\ell(\ell+1)}}\vec{r}\times\nabla Y_\ell^m(\theta,\phi)$. Note that we can easily write

$$\vec{r}\cdot\mathcal{Y}^{\ell,1}_{j=\ell,m}(\theta,\phi) = \vec{r}\cdot\mathbf{\Phi}_\ell^m(\theta,\phi) = \frac{1}{\sqrt{\ell(\ell+1)}}\left(\vec{r}\cdot\vec{L}Y_\ell^m(\theta,\phi)\right) = 0 \tag{1.23}$$

Thus we see that $\mathbf{\Phi}_\ell^m(\theta,\phi)$ is a transverse vector field; keep this in mind, we'll use it again. Similarly, we also find that $\mathbf{\Phi}_\ell^m$ is a divergenceless vector field:

$$\nabla\cdot\mathbf{\Phi}_\ell^m = \frac{1}{i\sqrt{\ell(\ell+1)}}\nabla\cdot\left(\vec{r}\times\nabla Y_\ell^m\right) = \frac{1}{i\sqrt{\ell(\ell+1)}}\partial_a\left(\epsilon_{abc}x_b\partial_cY_\ell^m\right) = \frac{1}{i\sqrt{\ell(\ell+1)}}\epsilon_{abc}\left(\delta_{ab}\partial_cY_\ell^m + x_b\partial_a\partial_c\right) = 0 \tag{1.24}$$

Remember that a transverse dimension has two directions; since we've described one of them, let's find the other, using the cross product:

$$\hat{r}\times\mathbf{\Phi}_\ell^m = \frac{1}{i\sqrt{\ell(\ell+1)}}\hat{r}\times\left(\vec{r}\times\nabla Y_\ell^m\right) = \frac{1}{i\sqrt{\ell(\ell+1)}}\left[\cancelto{0}{\vec{r}(\hat{r}\cdot\nabla Y_\ell^m)} - \hat{r}\cdot\vec{r}\nabla Y_\ell^m\right] = -\frac{1}{i\sqrt{\ell(\ell+1)}}r\nabla Y_\ell^m$$

This turns out to be another useful relation that we'll name. First, let's clean it up:

$$i\hat{r}\times\mathbf{\Phi}_\ell^m = -\frac{1}{\sqrt{\ell(\ell+1)}}r\nabla Y_\ell^m = \mathbf{\Psi}_\ell^m \tag{1.25}$$

Note that $\mathbf{\Psi}_\ell^m$, like $\mathbf{\Phi}_\ell^m$ is transverse:

$$\vec{r}\cdot\mathbf{\Psi}_\ell^m = -\frac{1}{\sqrt{\ell(\ell+1)}}r\vec{r}\cdot\nabla Y_\ell^m = 0 \tag{1.26}$$

Given that we've thoroughly described the transverse direction, that leaves the longitudinal direction, which is covered by defining one more vector field $\mathbf{Y}_\ell^m = \hat{r}Y_\ell^m$. To summarize, we have

$$\mathbf{Y}_\ell^m(\theta,\phi) = \hat{r}Y_\ell^m(\theta,\phi) \qquad \mathbf{\Psi}_\ell^m(\theta,\phi) = -\frac{1}{\sqrt{\ell(\ell+1)}}r\nabla Y_\ell^m(\theta,\phi) \qquad \mathbf{\Phi}_\ell^m(\theta,\phi) = \frac{1}{i\sqrt{\ell(\ell+1)}}\vec{r}\times\nabla Y_\ell^m(\theta,\phi) \tag{1.27}$$

[1] Get it? Φ for $s\Phi rical!^{\text{hahahahahahelpme}}$

What the hell, you may be wondering? What do these vector fields have to do with the VSH we found earlier using the CG series? A good question, but one that, unfortunately, requires *a lot* of algebra to answer completely. It is possible to show, using the recursion relations in the Appendix or other methods[2] that

$$\mathbf{Y}_j^m(\theta,\phi) = -\sqrt{\frac{j+1}{2j+1}}\mathcal{Y}^{\ell,1}_{j=\ell-1,m}(\theta,\phi) + \sqrt{\frac{j}{2j+1}}\mathcal{Y}^{\ell,1}_{j=\ell+1,m}(\theta,\phi) \tag{1.28}$$

and

$$\mathbf{\Psi}_j^m(\theta,\phi) = \sqrt{\frac{j}{2j+1}}\mathcal{Y}^{\ell,1}_{j=\ell+1,m}(\theta,\phi) + \sqrt{\frac{j+1}{2j+1}}\mathcal{Y}^{\ell,1}_{j=\ell+1,m}(\theta,\phi) \tag{1.29}$$

For now, we'll content ourselves with these expressions, that they are related to the total angular momentum eigenstates found earlier by CG series and that, since they're written in terms of the SSH, they play more nicely with differential operators like the divergence and curl. We'll now turn to noting these properties, in anticipation of using them for physical problems. As an aside, from this point forward, when referring to *the* vector spherical harmonics, I'll be referring to the $(\mathbf{Y}, \mathbf{\Phi}, \mathbf{\Psi})$ triplet, instead of the total angular momentum states denoted by $\mathcal{Y}$.

[2]See e.g., the notes by Haber

1.4 Properties of the vector spherical harmonics

The VSH are all mutually orthogonal at any given point:

$$\begin{aligned} \mathbf{Y}_\ell^m \cdot \mathbf{\Psi}_\ell^m &= 0 \\ \mathbf{Y}_\ell^m \cdot \mathbf{\Phi}_\ell^m &= 0 \\ \mathbf{\Psi}_\ell^m \cdot \mathbf{\Phi}_\ell^m &= 0 \end{aligned} \tag{1.30}$$

As with the scalar spherical harmonics, they are also orthonormal to themselves:

$$\begin{aligned} \int \mathbf{Y}_\ell^m \cdot \left(\mathbf{Y}_{\ell'}^{m'}\right)^* \mathrm{d}\Omega &= \delta_{\ell\ell'}\delta_{mm'} \\ \int \mathbf{\Psi}_\ell^m \cdot \left(\mathbf{\Psi}_{\ell'}^{m'}\right)^* \mathrm{d}\Omega &= \delta_{\ell\ell'}\delta_{mm'} \\ \int \mathbf{\Phi}_\ell^m \cdot \left(\mathbf{\Phi}_{\ell'}^{m'}\right)^* \mathrm{d}\Omega &= \delta_{\ell\ell'}\delta_{mm'} \end{aligned} \tag{1.31}$$

and orthogonal to each other in Hilbert space:

$$\begin{aligned} \int \mathbf{Y}_\ell^m \cdot \left(\mathbf{\Psi}_{\ell'}^{m'}\right)^* \mathrm{d}\Omega &= 0 \\ \int \mathbf{\Psi}_\ell^m \cdot \left(\mathbf{\Phi}_{\ell'}^{m'}\right)^* \mathrm{d}\Omega &= 0 \\ \int \mathbf{\Psi}_\ell^m \cdot \left(\mathbf{\Phi}_{\ell'}^{m'}\right)^* \mathrm{d}\Omega &= 0 \end{aligned} \tag{1.32}$$

Now let's consider the divergence and curl properties of our VSH. First consider the divergences:

$$\begin{aligned} \nabla \cdot \mathbf{Y}_\ell^m(\theta,\phi) &= \nabla \cdot \left(\hat{r} Y_\ell^m(\theta,\phi)\right) \\ &= \left(\nabla \cdot \hat{r}\right) Y_\ell^m(\theta,\phi) + \hat{r} \cdot \nabla Y_\ell^m(\theta,\phi) \\ &= \frac{2}{r} Y_\ell^m(\theta,\phi) + \cancelto{0}{\nabla_r Y_\ell^m(\theta,\phi)} \\ &= \frac{2}{r} Y_\ell^m(\theta,\phi) \end{aligned} \tag{1.33}$$

$$\begin{aligned} \nabla \cdot \mathbf{\Psi}_\ell^m(\theta,\phi) &= -\frac{1}{\sqrt{\ell(\ell+1)}} \nabla \cdot \left(r \nabla Y_\ell^m(\theta,\phi)\right) \\ &= -\frac{1}{\sqrt{\ell(\ell+1)}} \left[\nabla r \cdot \nabla Y_\ell^m(\theta,\phi) + r \nabla^2 Y_\ell^m(\theta,\phi)\right] \\ &= -\frac{1}{\sqrt{\ell(\ell+1)}} \left[\cancelto{0}{\hat{r} \cdot \nabla Y_\ell^m(\theta,\phi)} - \frac{\ell(\ell+1)}{r} Y_\ell^m(\theta,\phi)\right] \\ &= \frac{\sqrt{\ell(\ell+1)}}{r} Y_\ell^m(\theta,\phi) \end{aligned} \tag{1.34}$$

$$
\begin{aligned}
\nabla \cdot \mathbf{\Phi}_\ell^m(\theta,\phi) &= \frac{1}{i\sqrt{\ell(\ell+1)}}\nabla \cdot \left(\vec{r} \times \nabla Y_\ell^m(\theta,\phi)\right) \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[\cancelto{0}{(\nabla \times \vec{r})} \cdot \nabla Y_\ell^m(\theta,\phi) - \vec{r} \cdot \cancelto{0}{\left(\nabla \times \nabla Y_\ell^m(\theta,\phi)\right)}\right] \\
&= 0
\end{aligned} \tag{1.35}
$$

Now consider the curl:

$$
\begin{aligned}
\nabla \times \mathbf{Y}_\ell^m(\theta,\phi) &= \nabla \times \left(\hat{r} Y_\ell^m(\theta,\phi)\right) \\
&= \left[\cancelto{0}{(\nabla \times \hat{r})} Y_\ell^m(\theta,\phi) + \hat{r} \times \left(\nabla Y_\ell^m(\theta,\phi)\right)\right] \\
&= i\frac{\sqrt{\ell(\ell+1)}}{r}\left[\frac{1}{i\sqrt{\ell(\ell+1)}}\vec{r} \times \left(\nabla Y_\ell^m(\theta,\phi)\right)\right] \\
&= \frac{i}{r}\sqrt{\ell(\ell+1)}\,\mathbf{\Phi}_\ell^m(\theta,\phi)
\end{aligned} \tag{1.36}
$$

$$
\begin{aligned}
\nabla \times \mathbf{\Psi}_\ell^m(\theta,\phi) &= -\frac{1}{\sqrt{\ell(\ell+1)}}\nabla \times \left(r \nabla Y_\ell^m(\theta,\phi)\right) \\
&= -\frac{1}{\sqrt{\ell(\ell+1)}}\left[(\nabla r) \times \nabla Y_\ell^m(\theta,\phi) + r\cancelto{0}{\left(\nabla \times \nabla Y_\ell^m(\theta,\phi)\right)}\right] \\
&= -\frac{1}{\sqrt{\ell(\ell+1)}}\hat{r} \times \nabla Y_\ell^m(\theta,\phi) \\
&= -\frac{i}{r}\mathbf{\Phi}_\ell^m(\theta,\phi)
\end{aligned} \tag{1.37}
$$

For this last one, it's easiest to flip into index notation:

$$
\begin{aligned}
\left[\nabla \times \mathbf{\Phi}_\ell^m(\theta,\phi)\right]_a &= \frac{1}{i\sqrt{\ell(\ell+1)}}\epsilon_{abc}\partial_b\left[\epsilon_{cde} x_d \partial_e \left(Y_\ell^m\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\epsilon_{abc}\epsilon_{dec}\left[\delta_{bd}\left(\partial_e Y_\ell^m\right) + x_d \partial_b \left(\partial_e Y_\ell^m\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[\epsilon_{abc}\epsilon_{dec}\delta_{bd}\left(\partial_e Y_\ell^m\right) + \epsilon_{abc}\epsilon_{dec} x_d \partial_b \left(\partial_e Y_\ell^m\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[\left(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}\right)\delta_{bd}\left(\partial_e Y_\ell^m\right) + \left(\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}\right) x_d \partial_b \left(\partial_e Y_\ell^m\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[\partial_a Y_\ell^m - 3\partial_a Y_\ell^m + x_a \partial^2 Y_\ell^m - x_b \partial_b \left(\partial_e Y_\ell^m\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad \nabla \times \mathbf{\Phi}_\ell^m(\theta,\phi) &= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[-2\nabla Y_\ell^m(\theta,\phi) + \vec{r}\,\nabla^2 Y_\ell^m(\theta,\phi) - \vec{r}\cdot\nabla\left(\nabla Y_\ell^m(\theta,\phi)\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[-2\nabla Y_\ell^m(\theta,\phi) - \frac{\vec{r}}{r^2}L^2 Y_\ell^m(\theta,\phi) - r\frac{\partial}{\partial r}\left(\nabla Y_\ell^m(\theta,\phi)\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[-2\nabla Y_\ell^m(\theta,\phi) + \ell(\ell+1)\frac{\hat{r}}{r}Y_\ell^m(\theta,\phi) - r\frac{\partial}{\partial r}\left(\frac{1}{r}\left[r\nabla Y_\ell^m(\theta,\phi)\right]\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[-2\nabla Y_\ell^m(\theta,\phi) + \ell(\ell+1)\frac{\hat{r}}{r}Y_\ell^m(\theta,\phi) + r\left(\frac{1}{r^2}\left[r\nabla Y_\ell^m(\theta,\phi)\right]\right)\right] \\
&= \frac{1}{i\sqrt{\ell(\ell+1)}}\left[-\nabla Y_\ell^m(\theta,\phi) + \ell(\ell+1)\frac{\hat{r}}{r}Y_\ell^m(\theta,\phi)\right] \\
&= \frac{i}{r}\left[-\sqrt{\ell(\ell+1)}\,\mathbf{Y}_\ell^m(\theta,\phi) + \mathbf{\Psi}_\ell^m(\theta,\phi)\right] \qquad (1.38)
\end{aligned}
$$

Lastly, let's consider the divergences and curls of the VSH times arbitrary radial functions. First consider the divergences:

$$
\begin{aligned}
\nabla\cdot\left(f(r)\mathbf{Y}_\ell^m(\theta,\phi)\right) &= \nabla f(r)\cdot\mathbf{Y}_\ell^m(\theta,\phi) + f(r)\nabla\cdot\mathbf{Y}_\ell^m(\theta,\phi) \\
&= \frac{\mathrm{d}f}{\mathrm{d}r}Y_\ell^m(\theta,\phi) + \frac{2f(r)}{r}Y_\ell^m(\theta,\phi) \\
&= \left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{2f(r)}{r}\right)Y_\ell^m(\theta,\phi) \qquad (1.39)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\left(f(r)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) &= \nabla f(r)\cdot\mathbf{\Psi}_\ell^m(\theta,\phi) + f(r)\nabla\cdot\mathbf{\Psi}_\ell^m(\theta,\phi) \\
&= \frac{\mathrm{d}f}{\mathrm{d}r}\cancelto{0}{\hat{r}\cdot\mathbf{\Psi}_\ell^m(\theta,\phi)} + \sqrt{\ell(\ell+1)}\frac{f(r)}{r}Y_\ell^m(\theta,\phi) \\
&= \sqrt{\ell(\ell+1)}\frac{f(r)}{r}Y_\ell^m(\theta,\phi) \qquad (1.40)
\end{aligned}
$$

$$
\nabla\cdot\left(f(r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right) = \nabla f(r)\cdot\mathbf{\Phi}_\ell^m(\theta,\phi) + f(r)\cancelto{0}{\nabla\cdot\mathbf{\Phi}_\ell^m(\theta,\phi)} = \frac{\mathrm{d}f}{\mathrm{d}r}\hat{r}\cdot\mathbf{\Phi}_\ell^m = 0 \qquad (1.41)
$$

Now consider the curl:

$$
\begin{aligned}
\nabla\times\left(f(r)\mathbf{Y}_\ell^m(\theta,\phi)\right) &= \nabla f(r)\times\mathbf{Y}_\ell^m(\theta,\phi) + f(r)\nabla\times\mathbf{Y}_\ell^m(\theta,\phi) \\
&= \frac{\mathrm{d}f}{\mathrm{d}r}\cancelto{0}{\hat{r}\times\hat{r}}Y_\ell^m(\theta,\phi) + i\sqrt{\ell(\ell+1)}\frac{f(r)}{r}\mathbf{\Phi}_\ell^m \\
&= i\sqrt{\ell(\ell+1)}\frac{f(r)}{r}\mathbf{\Phi}_\ell^m \qquad (1.42)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\left(f(r)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) &= \nabla f(r)\times\mathbf{\Psi}_\ell^m(\theta,\phi) + f(r)\nabla\times\mathbf{\Psi}_\ell^m(\theta,\phi) \\
&= \frac{\mathrm{d}f}{\mathrm{d}r}\hat{r}\times\mathbf{\Psi}_\ell^m(\theta,\phi) - i\frac{f(r)}{r}\mathbf{\Phi}_\ell^m(\theta,\phi) \\
&= -i\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{f(r)}{r}\right)\mathbf{\Phi}_\ell^m(\theta,\phi) \qquad (1.43)
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times (f(r)\mathbf{\Phi}_\ell^m(\theta,\phi)) &= \nabla f(r) \times \mathbf{\Phi}_\ell^m(\theta,\phi) + f(r)\nabla \times \mathbf{\Phi}_\ell^m(\theta,\phi) \\
&= \frac{\mathrm{d}f}{\mathrm{d}r}\hat{r} \times \mathbf{\Phi}_\ell^m(\theta,\phi) + f(r)\nabla \times \mathbf{\Phi}_\ell^m(\theta,\phi) \\
&= i\frac{\mathrm{d}f}{\mathrm{d}r}\mathbf{\Psi}_\ell^m + i\frac{f(r)}{r}\left[-\sqrt{\ell(\ell+1)}\mathbf{Y}_\ell^m(\theta,\phi) + \mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&= -i\sqrt{\ell(\ell+1)}\frac{f(r)}{r}\mathbf{Y}_\ell^m(\theta,\phi) - i\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{f(r)}{r}\right)\mathbf{\Psi}_\ell^m
\end{aligned} \tag{1.44}
$$

Lastly, we can use the VSH to decompose a vector field. Using the orthogonality conditions, any vector field may be written a sum of coefficients times the VSH as

$$
\vec{E} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[E_r^{\ell m}(r)\mathbf{Y}_\ell^m(\theta,\phi) + E_s^{\ell m}(r)\mathbf{\Psi}_\ell^m(\theta,\phi) + E_t^{\ell m}(r)\mathbf{\Phi}_{\ell m}(\theta,\phi)\right] \tag{1.45}
$$

As with the scalar spherical harmonics, the components E_r, E_s and E_t are found by 'projecting' the vector field onto each of the VSH:

$$
\begin{aligned}
E_r^{\ell m}(r) &= \int (\mathbf{Y}_\ell^m(\theta,\phi))^* \cdot \vec{E}\mathrm{d}\Omega \\
E_s^{\ell m}(r) &= \int (\mathbf{\Psi}_\ell^m(\theta,\phi))^* \cdot \vec{E}\mathrm{d}\Omega \\
E_t^{\ell m}(r) &= \int (\mathbf{\Phi}_\ell^m(\theta,\phi))^* \cdot \vec{E}\mathrm{d}\Omega
\end{aligned} \tag{1.46}
$$

Lastly, note that properties of the field, such as divergenceless-ness or curl-less-ness can have effects on the components. If the field is divergenceless, we can write

$$
\begin{aligned}
\nabla \cdot \vec{B} &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\nabla \cdot \left(B_r^{\ell m}(r)\mathbf{Y}_\ell^m(\theta,\phi)\right) + \nabla \cdot \left(B_s^{\ell m}(r)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + \nabla \cdot \left(B_t^{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right)\right] \\
0 &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 B_r^{\ell m}\right) - \frac{1}{r\sqrt{\ell(\ell+1)}}B_s^{\ell m}\right]Y_\ell^m(\theta,\phi)
\end{aligned}
$$

and this is only satisfied in general only if $B_r = B_s = 0$. Therefore a divergenceless field is proportional only to $\mathbf{\Phi}_\ell^m$:

$$
\vec{B} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} B_t^{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi) \tag{1.47}
$$

Thus we have a collection of the properties of the VSH. We can now proceed to actually use these to solve equations. We'll start with the electrostatic and magnetostatic multipoles.

2
Statics

2.1 The scalar potential and electrostatic field

The Maxwell equations for electrostatic fields are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad\qquad \nabla \times \vec{E} = 0 \tag{2.1}$$

The curl-less condition implies that $\vec{E} = -\nabla\Phi$, which when combined with the Gauß law yields the Poisson equation:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \tag{2.2}$$

For problems with spherical geometry, the Laplacian in spherical coordinates, when acting on a scalar function, is defined as

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}L^2\Phi \tag{2.3}$$

where L^2 is the angular momentum operator from quantum mechanics. Recall that the spherical harmonics form eigenfunctions of the angular momentum operator, with the action

$$L^2 Y_\ell^m(\theta,\phi) = -\ell(\ell+1)Y_\ell^m(\theta,\phi) \tag{2.4}$$

The spherical harmonics form a complete basis of functions and thus both sides of the equation may be expanded using spherical harmonics:

$$\Phi(\vec{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\phi_{\ell m}(r)Y_\ell^m(\theta,\phi) \qquad \text{and} \qquad \rho(\vec{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\rho_{\ell m}(r)Y_\ell^m(\theta,\phi) \tag{2.5}$$

As with the Fourier series, plugging both expansions into the Laplace equation gives

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi_{\ell m}(r)}{\partial r}\right)Y_\ell^m(\theta,\phi)+\frac{\phi_{\ell m}(r)}{r^2}L^2Y_\ell^m(\theta,\phi)\right] = -\frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\rho_{\ell m}(r)Y_\ell^m(\theta,\phi)$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{\mathrm{d}^2\phi_{\ell m}}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}\phi_{\ell m}}{\mathrm{d}r}-\frac{\ell(\ell+1)}{r^2}\phi_{\ell m}(r)\right]Y_\ell^m(\theta,\phi) = -\frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\rho_{\ell m}(r)Y_\ell^m(\theta,\phi)$$

Matching the coefficients of the spherical harmonics yields an ODE for the components of the potential expansion $\phi_{\ell m}$:

$$\frac{\mathrm{d}^2\phi_{\ell m}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\phi_{\ell m}}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}\phi_{\ell m}(r) = -\frac{\rho_{\ell m}}{\epsilon_0} \tag{2.6}$$

This is the Euler ODE, and the solution to this ODE using Green functions is

$$\phi_{\ell m}(r) = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{(2\ell+1)}\int_0^\infty \frac{r_<^\ell}{r_>^{\ell+1}}\rho_{\ell m}(r')r'^2\mathrm{d}r' \tag{2.7}$$

where $r_< = \min(r, r')$ and $r_> = \max(r, r')$. Now suppose that we observe the potential at a distance far away from the charge distribution generating it. In other words $r \gg r'$. If we expand the integrals appropriately

$$\phi_{\ell m}(r) = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{(2\ell+1)}\left[\int_0^r \frac{r'^\ell}{r^{\ell+1}}\rho_{\ell m}(r')r'^2\mathrm{d}r' + \int_r^\infty \frac{r^\ell}{r'^{\ell+1}}\rho_{\ell m}(r')r'^2\mathrm{d}r'\right] \tag{2.8}$$

and employ the fact that $\rho(r')$ is localized, the second integral vanishes, and the first may have the integration bound extended out to infinity, without cost:

$$\phi_{\ell m}(r) = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{(2\ell+1)r^{\ell+1}}\int_0^\infty r'^\ell\rho_{\ell m}(r')r'^2\mathrm{d}r' = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{(2\ell+1)r^{\ell+1}}\int\int_0^\infty r'^\ell\rho(\vec{r}')\left(Y_\ell^m(\theta',\phi')\right)^* r'^2\mathrm{d}r'\mathrm{d}\Omega' \tag{2.9}$$

$$\phi_{\ell m}(r) = \frac{q_{\ell m}}{4\pi\epsilon_0}\frac{4\pi}{(2\ell+1)r^{\ell+1}} \qquad \text{where} \qquad q_{\ell m} = \int\int_0^\infty r'^\ell\rho(\vec{r}')\left(Y_\ell^m(\theta',\phi')\right)^* r'^2\mathrm{d}r'\mathrm{d}\Omega' \tag{2.10}$$

This is the usual multipole expansion for the electrostatic potential. The full potential is just a sum over the ℓ, m moments. Now let's find the associated electric field. Taking the gradient of the expansion of Φ gives

$$\begin{aligned}
\vec{E} &= -\nabla\left(\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\phi_{\ell m}(r)Y_\ell^m(\theta,\phi)\right)\\
&= -\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\left[\frac{\mathrm{d}\phi_{\ell m}(r)}{\mathrm{d}r}\hat{r}Y_\ell^m(\theta,\phi) - \sqrt{\ell(\ell+1)}\frac{\phi_{\ell m}(r)}{r}\left(-\frac{1}{\sqrt{\ell(\ell+1)}}r\nabla Y_\ell^m(\theta,\phi)\right)\right]\\
&= \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\left[-\frac{\mathrm{d}\phi_{\ell m}(r)}{\mathrm{d}r}\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\ell(\ell+1)}\frac{\phi_{\ell m}(r)}{r}\mathbf{\Psi}_\ell^m(\theta,\phi)\right]\\
&= \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\frac{q_{\ell m}}{4\pi\epsilon_0}\frac{4\pi}{(2\ell+1)}\left[\frac{\ell+1}{r^{\ell+2}}\mathbf{Y}_\ell^m(\theta,\phi) - \frac{\sqrt{\ell(\ell+1)}}{r^{\ell+2}}\mathbf{\Psi}_\ell^m(\theta,\phi)\right]\\
&= \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}\frac{q_{\ell m}}{4\pi\epsilon_0 r^{\ell+2}}\frac{4\pi(\ell+1)}{(2\ell+1)}\left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}}\mathbf{\Psi}_\ell^m(\theta,\phi)\right]
\end{aligned} \tag{2.11}$$

Thus the electrostatic multipole field can be written as a sum over charge moments with the expected radial dependence and a well-determined angular dependence in terms of VSH.

2.2 The vector potential and magnetostatic field

The Maxwell equations in the magnetostatic case are

$$\nabla \cdot \vec{B} = 0 \qquad\qquad \nabla \times \vec{B} = \mu_0 \vec{J} \tag{2.12}$$

The Gauß law implies that the magnetic field may be written as the curl of a vector potential

$$\vec{B} = \nabla \times \vec{A} \tag{2.13}$$

Now, when speaking of the magnetic multipoles, we're interested in the vector Laplacian of the vector potential, which is defined as

$$\nabla^2 \vec{A} = \nabla \left(\nabla \cdot \vec{A}\right) - \nabla \times \nabla \times \vec{A} \tag{2.14}$$

meaning we need to find the action of the divergence and curl on a vector field decomposed in terms of the VSH. We can thus use these to compute the vector Laplacian of the vector potential. Recall that the gradient of a scalar function (which is what the divergence produces) creates a vector field consisting of the $\mathbf{Y}$ and $\mathbf{\Psi}$ VSH and their coefficients. Furthermore, the divergence of the vector potential depends only on A_r and A_s. It is possible, by fixing the gauge, to set A_r and A_s equal to 0, which is equivalent to choosing the Coulomb gauge. Therefore the vector potential is of the form

$$\vec{A} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_t^{\ell m}(r) \mathbf{\Phi}_\ell^m(\theta, \phi) \tag{2.15}$$

The associated $\vec{B}$-field is found by taking the curl; doing so gives

$$\vec{B} = \nabla \times \vec{A} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[-i\sqrt{\ell(\ell+1)} \frac{A_t^{\ell m}(r)}{r} \mathbf{Y}_\ell^m(\theta, \phi) - \frac{i}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r A_t^{\ell m}(r)\right) \mathbf{\Psi}_\ell^m(\theta, \phi) \right] \tag{2.16}$$

In this gauge, the vector Laplacian is simply

$$\nabla^2 \vec{A} = -\nabla \times \nabla \times \vec{A} = -\nabla \times \vec{B} \tag{2.17}$$

Computing the curl of the $\vec{B}$ field gives

$$\begin{aligned}
\nabla \times \nabla \times \vec{A} &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[-i\nabla \times \left(\sqrt{\ell(\ell+1)} \frac{A_t^{\ell m}(r)}{r} \mathbf{Y}_\ell^m(\theta, \phi) \right) - i\nabla \times \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r A_t^{\ell m}\right) \mathbf{\Psi}_\ell^m(\theta, \phi) \right) \right] \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\ell(\ell+1) \frac{A_t^{\ell m}(r)}{r^2} \mathbf{\Phi}_\ell^m(\theta, \phi) - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r A_t^{\ell m}\right)\right)\right] \mathbf{\Phi}_\ell^m(\theta, \phi) \right] \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\ell(\ell+1) \frac{A_t^{\ell m}(r)}{r^2} - \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(r A_t^{\ell m}(r)\right) \right] \mathbf{\Phi}_\ell^m(\theta, \phi) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\frac{\ell(\ell+1)}{r^2} A_t^{\ell m} - \frac{1}{r}\left(\frac{\mathrm{d}A_t^{\ell m}}{\mathrm{d}r} + \frac{\mathrm{d}A_t^{\ell m}}{\mathrm{d}r} + r\frac{\mathrm{d}^2 A_t^{\ell m}}{\mathrm{d}r^2} \right) \right] \mathbf{\Phi}_\ell^m(\theta, \phi) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\frac{\ell(\ell+1)}{r^2} A_t^{\ell m} - \frac{2}{r}\frac{\mathrm{d}A_t^{\ell m}}{\mathrm{d}r} - \frac{\mathrm{d}^2 A_t^{\ell m}}{\mathrm{d}r^2} \right] \mathbf{\Phi}_\ell^m(\theta, \phi)
\end{aligned} \tag{2.18}$$

Since the current density $\vec{J}$ is a vector field, it also has a VSH decomposition:

$$\vec{J}(\vec{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[J_r^{\ell m}(r)\mathbf{Y}_\ell^m(\theta,\phi) + J_s^{\ell m}(r)\mathbf{\Psi}_\ell^m(\theta,\phi) + J_t^{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{2.19}$$

We're considering only magnetostatics, then charge conservation gives $\nabla\cdot\vec{J} = 0$. A divergenceless vector field has no radial or solenoidal component, as we found above. In other words, $J_r = J_s = 0$. Therefore

$$\vec{J}(\vec{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} J_t^{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi) \qquad \text{where} \qquad J_t^{\ell m}(r) = \int \left(\mathbf{\Phi}_\ell^m(\theta,\phi)\right)^*\cdot\vec{J}(\vec{r})\mathrm{d}\Omega \tag{2.20}$$

Combining these two expansions into the vector Poisson equation gives

$$\begin{gathered}
\nabla^2\vec{A} = -\mu_0\vec{J} \\
-\nabla\times\nabla\times\vec{A} = -\mu_0\vec{J} \\
\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{\ell(\ell+1)}{r^2}A_t^{\ell m} - \frac{2}{r}\frac{\mathrm{d}A_t^{\ell m}}{\mathrm{d}r} - \frac{\mathrm{d}^2A_t^{\ell m}}{\mathrm{d}r^2}\right]\mathbf{\Phi}_\ell^m(\theta,\phi) = \mu_0\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} J_t^{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi) \\
\frac{\ell(\ell+1)}{r^2}A_t^{\ell m} - \frac{2}{r}\frac{\mathrm{d}A_t^{\ell m}}{\mathrm{d}r} - \frac{\mathrm{d}^2A_t^{\ell m}}{\mathrm{d}r^2} = \mu_0 J_t^{\ell m}
\end{gathered}$$

and matching the coefficients of the VSH yields an ODE for the $A_t^{\ell m}$:

$$\frac{\mathrm{d}^2A_t^{\ell m}}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}A_t^{\ell m}}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}A_t^{\ell m} = -\mu_0 J_t^{\ell m} \tag{2.21}$$

This ODE is identical to that for the electrostatic multipole moments $\phi_{\ell m}$, the solution of which is simply

$$A_t^{\ell m}(r) = \frac{\mu_0}{4\pi}\frac{1}{2\ell+1}\int_0^\infty\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right)J_t^{\ell m}(r')r'^2\mathrm{d}r' \tag{2.22}$$

Now suppose that we observe the potential at a distance far away from the charge distribution generating it. In other words $r \gg r'$. If we expand the integrals appropriately

$$A_t^{\ell m}(r) = \frac{\mu_0}{4\pi}\frac{1}{(2\ell+1)}\left[\int_0^r \frac{r'^\ell}{r^{\ell+1}}J_t^{\ell m}(r')r'^2\mathrm{d}r' + \int_r^\infty \frac{r^\ell}{r'^{\ell+1}}J_t^{\ell m}(r')r'^2\mathrm{d}r'\right]$$

and employ the fact that $\vec{J}(r')$ is localized, the second integral vanishes, and the first may have the integration bound extended out to infinity, without cost:

$$A_t^{\ell m}(r) = \frac{\mu_0}{4\pi}\frac{1}{(2\ell+1)}\frac{1}{r^{\ell+1}}\int_0^\infty r'^\ell J_t^{\ell m}(r')r'^2\mathrm{d}r' = \frac{\mu_0}{4\pi}\frac{1}{(2\ell+1)}\frac{1}{r^{\ell+1}}\int\int_0^\infty r'^\ell\left(\mathbf{\Phi}_\ell^m(\theta',\phi')\right)^*\cdot\vec{J}(\vec{r}')r'^2\mathrm{d}r'\mathrm{d}\Omega' \tag{2.23}$$

$$A_t^{\ell m}(r) = \frac{\mu_0}{4\pi(2\ell+1)}\frac{I_{\ell m}}{r^{\ell+1}} \qquad \text{where} \qquad I_{\ell m} = \int\int_0^\infty r'^\ell\left(\mathbf{\Phi}_\ell^m(\theta',\phi')\right)^*\cdot\vec{J}(\vec{r}')r'^2\mathrm{d}r'\mathrm{d}\Omega' \tag{2.24}$$

and I've introduced the notation $I_{\ell m}$ to denote the ℓ, m current moment. Note that the monopole moment automatically vanishes since $\mathbf{\Phi}_0^0 = 0$ identically. As with the electrostatic case, we can now find the associated magnetic field:

$$\begin{aligned}
\vec{B} &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[-i\sqrt{\ell(\ell+1)}\frac{A_t^{\ell m}(r)}{r}\mathbf{Y}_\ell^m(\theta,\phi) - \frac{i}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rA_t^{\ell m}(r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&= -i\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\mu_0 I_{\ell m}}{4\pi}\frac{1}{2\ell+1}\left[\frac{\sqrt{\ell(\ell+1)}}{r^{\ell+2}}\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^\ell}\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&= -i\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\mu_0 I_{\ell m}}{4\pi}\frac{1}{2\ell+1}\left[\frac{\sqrt{\ell(\ell+1)}}{r^{\ell+2}}\mathbf{Y}_\ell^m(\theta,\phi) - \frac{\ell}{r^{\ell+2}}\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&= -i\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\mu_0 I_{\ell m}}{4\pi r^{\ell+2}}\frac{\sqrt{\ell(\ell+1)}}{(2\ell+1)}\left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}}\mathbf{\Psi}_\ell^m(\theta,\phi)\right]
\end{aligned} \tag{2.25}$$

Except for the constants, this field has a form identical to that of the electric multipole field!

3
Radiation

3.1 Scalar spherical waves

Here we'll establish some of the mathematical relations we'll use in the rest of this section. Suppose we have a scalar field $\psi(\vec{r},t)$ satisfying the homogeneous wave equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = 0 \tag{3.1}$$

Now let ψ exhibit monochromatic harmonic time-dependence, so $\psi(\vec{r},t) = \psi(\vec{r},\omega)\exp[i\omega t]$. This gives

$$\begin{aligned}
& 0 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\psi(\vec{r},\omega)\exp[i\omega t]\right) - \nabla^2\left(\psi(\vec{r},\omega)\exp[i\omega t]\right) \\
& 0 = -\frac{\omega^2}{c^2}\psi(\vec{r},\omega)\exp[i\omega t] + \nabla^2\psi(\vec{r},\omega)\exp[i\omega t] \\
\Rightarrow \qquad & 0 = \left(\nabla^2 + k^2\right)\psi(\vec{r},\omega)
\end{aligned} \tag{3.2}$$

where $k^2 = \frac{\omega^2}{c^2}$, which is the homogeneous Helmholtz equation. Separation of variables yields a solution of the form

$$\psi(\vec{r},\omega) = \sum_{\ell,m} f_{\ell m}(r) Y_\ell^m(\theta,\phi) \tag{3.3}$$

with the radial equation satisfying the ODE:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + k^2 - \frac{\ell(\ell+1)}{r^2}\right] f_{\ell m}(r) = 0 \tag{3.4}$$

If we let $f(r) = \frac{u(r)}{r^{\frac{1}{2}}}$, then the ODE becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + k^2 - \frac{\left(\ell+\frac{1}{2}\right)^2}{r^2}\right] u_{\ell m}(r) = 0 \tag{3.5}$$

This is the Bessel ODE with $\nu = \ell + \frac{1}{2}$, with solutions

$$f_{\ell m}(r) = \frac{A_{\ell m}}{r^{\frac{1}{2}}} J_{\ell+\frac{1}{2}}(kr) + \frac{B_{\ell m}}{r^{\frac{1}{2}}} N_{\ell+\frac{1}{2}}(kr) \tag{3.6}$$

where J and N are the Bessel functions of the first and second kind, respectively. It is customary to define the *spherical Bessel functions*

$$j_\ell(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(x) \qquad\qquad n_\ell(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} N_{\ell+\frac{1}{2}}(x) \tag{3.7}$$

The spherical Bessel functions can be thought of as standing spherical waves. These can be combined to form propagating spherical waves, the *spherical Hankel functions*

$$h_\ell^{(1),(2)}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \left[J_{\ell+\frac{1}{2}}(x) \pm i N_{\ell+\frac{1}{2}}(x)\right]$$

The spherical Bessel functions have derivative relations

$$j_\ell(x) = (-x)^\ell \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \left(\frac{\sin x}{x}\right) \qquad\qquad n_\ell(x) = -(-x)^\ell \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^\ell \left(\frac{\cos x}{x}\right) \tag{3.8}$$

with small argument ($x \ll 1$) asymptotic form

$$j_\ell \to \frac{x^\ell}{(2\ell+1)!!}\left(1 - \frac{x^2}{2(2\ell+3)} + \dots\right) \qquad\qquad n_\ell(x) \to -\frac{(2\ell-1)!!}{x^{\ell+1}}\left(1 - \frac{x^2}{2(1-2\ell)} + \dots\right) \tag{3.9}$$

and with large argument ($x \gg 1$) asymptotic form

$$j_\ell(x) \to \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right) \qquad\qquad n_\ell(x) \to -\frac{1}{x}\cos\left(x - \frac{\ell\pi}{2}\right) \tag{3.10}$$

$$h_\ell^{(1)}(x) \to (-i)^{\ell+1}\frac{\exp[ix]}{x} \qquad\qquad h_\ell^{(2)}(x) = \left(h_\ell^{(1)}(x)\right)^* \tag{3.11}$$

and recursion relations

$$\frac{2\ell+1}{x} z_\ell(x) = z_{\ell-1}(x) + z_{\ell+1}(x) \tag{3.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} z_\ell(x) = \frac{1}{2\ell+1}\left[\ell z_{\ell-1}(x) - (\ell+1) z_{\ell+1}(x)\right] \tag{3.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x z_\ell(x)] = x z_{\ell-1}(x) - \ell z_\ell(x) \tag{3.14}$$

where z is any spherical radial solution. These allow us to write

$$z_{\ell-1}(x) = \frac{\mathrm{d}z_\ell(x)}{\mathrm{d}x} + \frac{\ell+1}{x} z_\ell(x) \qquad \text{and} \qquad -z_{\ell+1}(x) = \frac{\mathrm{d}z_\ell(x)}{\mathrm{d}x} - \frac{\ell}{x} z_\ell(x) \tag{3.15}$$

Using these radial solutions, the general solution for the scalar field is

$$\psi(\vec{r},\omega) = \sum_{\ell,m}\left[A_{\ell m}^{(1)} h_\ell^{(1)}(kr) + A_{\ell m}^{(2)} h_\ell^{(2)}(kr)\right] Y_\ell^m(\theta,\phi) \tag{3.16}$$

The Green function for the Helmholtz equation is equal to

$$\left(\nabla^2 + k^2\right) G\left(\vec{r}, \vec{r}'\right) = -\delta\left(\vec{r} - \vec{r}'\right) \qquad \Rightarrow \qquad G\left(\vec{r}, \vec{r}'\right) = \frac{\exp[ik\|\vec{r} - \vec{r}'\|]}{\|\vec{r} - \vec{r}'\|} \tag{3.17}$$

which can be expanded in terms of the spherical harmonics and radial solutions as

$$\frac{\exp[ik\|\vec{r} - \vec{r}'\|]}{\|\vec{r} - \vec{r}'\|} = ik \sum_{\ell=0}^{\infty} j_\ell(kr_<) h^{(1)}(kr_>) \sum_{m=-\ell}^{\ell} \left(Y_\ell^m(\theta', \phi')\right)^* Y_\ell^m(\theta, \phi) \tag{3.18}$$

These relations will be quoted in following sections.

3.2 Vacuum multipoles

For this section, we will consider only monochromatic harmonic time-dependence. In other words, we can write

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\exp[-i\omega t] \qquad\qquad \vec{H}(\vec{r},t) = \vec{H}(\vec{r})\exp[-i\omega t] \tag{3.19}$$

where I've introduced the H-field, $\vec{B} = \mu_0\vec{H}$, since this makes the expressions much cleaner throughout. In a source-free region of empty space, the electric and magnetic fields satisfy

$$\nabla\times\vec{E} = ikZ_0\vec{H} \qquad\qquad \nabla\times\vec{H} = -\frac{ik}{Z_0}\vec{E} \tag{3.20}$$

$$\nabla\cdot\vec{E} = 0 \qquad\qquad \nabla\cdot\vec{H} = 0 \tag{3.21}$$

where $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ is the *impedance of free space*. Recall that, assuming the Lorenz gauge condition

$$\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \nabla\cdot\vec{A} = 0 \tag{3.22}$$

the Maxwell equations may be put together to give the vector wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{E}(\vec{r},t) = 0 \qquad\qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{H}(\vec{r},t) = 0 \tag{3.23}$$

For monochromatic time-harmonic fields as we're considering, these reduce to the vector Helmholtz equation:

$$\left(\nabla^2 + k^2\right)\vec{E}(\vec{r}) = 0 \qquad\qquad \left(\nabla^2 + k^2\right)\vec{H}(\vec{r}) = 0 \tag{3.24}$$

Let's take a brief mathematical interlude. Suppose $\vec{A}$ is a vector field satisfying the vector wave equation. Suppose further that the field is divergenceless, $\nabla\cdot\vec{A} = 0$. Then it is perfectly admissible to write $\vec{A} = \nabla\times(u(\vec{r},t)\vec{r})$. Let's insert this expression into the wave equation and see what we can extract:

$$0 = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A}(\vec{r},t) = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left[\nabla\times(u(\vec{r},t)\vec{r})\right] = \nabla\times\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left[u(\vec{r},t)\vec{r}\right] \tag{3.25}$$

Now it's possible to write

$$\begin{aligned}
\left[\nabla\times\left(\nabla^2\left(u(\vec{r},t)\vec{r}\right)\right)\right]_i &= \epsilon_{ijk}\partial_j\left[\partial_\ell\partial_\ell\left(u(\vec{r},t)x_k\right)\right] \\
&= \epsilon_{ijk}\partial_j\left[\partial_\ell\left(x_k\partial_\ell u(\vec{r},t) + \delta_{k\ell}u(\vec{r},t)\right)\right] \\
&= \epsilon_{ijk}\partial_j\left[x_k\partial_\ell\partial_\ell u(\vec{r},t) + \delta_{k\ell}\partial_\ell u(\vec{r},t) + \delta_{k\ell}\partial_\ell u(\vec{r},t)\right] \\
&= \epsilon_{ijk}\partial_j\left[x_k\partial_\ell\partial_\ell u(\vec{r},t) + 2\partial_k u(\vec{r},t)\right] \\
&= \epsilon_{ijk}\partial_j\left(x_k\partial^2 u(\vec{r},t)\right) + 2\cancelto{0}{\epsilon_{ijk}\partial_j\partial_k} u(\vec{r},t) \\
&= \left[\nabla\times\left(\vec{r}\,\nabla^2 u(\vec{r},t)\right)\right]_i
\end{aligned}$$

Therefore, since the position vector can be moved freely past the time derivative, we can write

$$0 = \nabla\times\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left[u(\vec{r},t)\vec{r}\right] = \left[\nabla\times\vec{r}\right]\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)u(\vec{r},t) \tag{3.26}$$

which implies that $u(\vec{r}, t)$ satisfies a scalar wave equation (!). Since we're considering time-harmonic fields, the wave equation specializes to the Helmholtz equation and the same logic applies: a divergenceless vector field satisfying the Helmholtz equation may be written as the curl of a radial vector times a scalar field satisfying the Helmholtz equation. In fact there exists an important theorem regarding the decomposition of electromagnetic waves, known as the *Whittaker theorem*:

The Whittaker theorem

Let $\vec{A}$ be a divergenceless vector field, $\nabla \cdot \vec{A} = 0$ satisfying the wave equation. Since this implies that only two of three Cartesian components are independent, then only two scalar functions satisfying the wave equation are required to fully describe the vector field.

Now, both $\vec{E}$ and $\vec{B}$ are divergenceless in source-free space. Furthermore, the $\vec{E}$ and $\vec{B}$ fields are not independent of each other. Thus we only need two scalar functions to fully describe vector radiation fields! We're left with the question as to what scalar functions to choose and how to relate these to the electric and magnetic fields. A uniqueness theorem proved by Bouwkamp and Casimir gives the requisite answer. The proof that follows may be omitted, but is provided for completeness.

3.2.1 ***Uniqueness theorem***

Consider a region of space D between two concentric spheres. Let D be source-free, and let there be an electromagnetic field with vanishing radial components $\hat{r} \cdot \vec{E} = \hat{r} \cdot \vec{H} = 0$. In other words, the field is transverse electromagnetic. It then follows that $\vec{E} = \vec{H} = 0$, or in other words, that the radial components completely determine the field in this region[1].

Proof. If $E_r = H_r = 0$, then we can rewrite the Maxwell equations. For example, the Faraday law becomes

$$ikZ_0\vec{H} = \nabla \times \vec{E} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \frac{\partial E_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial E_r}{\partial\phi}} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \cancelto{0}{\frac{\partial E_r}{\partial\theta}}\right] \tag{3.27}$$

Therefore we can write

$$\frac{\partial}{\partial\theta}\left(\sin\theta E_\phi\right) - \frac{\partial E_\theta}{\partial\phi} = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) = -ikZ_0H_\theta \qquad \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = ikZ_0H_\phi \tag{3.28}$$

and similarly, the Ampère-Maxwell law gives

$$\frac{\partial}{\partial\theta}\left(\sin\theta H_\phi\right) - \frac{\partial H_\theta}{\partial\phi} = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = ikZ_0E_\theta \qquad \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) = -ikZ_0E_\phi \tag{3.29}$$

Consider the second and sixth equations. These can be rewritten to give

$$\frac{\partial}{\partial r}(rE_\phi) = -ikZ_0rH_\theta \qquad \text{and} \qquad \frac{\partial}{\partial r}(rH_\phi) = ikZ_0rE_\theta \tag{3.30}$$

[1] I believe this a manifestation of the hairy ball theorem from topology

Taking the second derivative allows the equations to be combined

$$\frac{\partial^2}{\partial r^2}(rE_\phi) = -ikZ_0\frac{\partial}{\partial r}(rH_\theta) \qquad \Rightarrow \qquad \frac{\partial^2}{\partial r^2}(rE_\phi) = k^2 Z_0^2 rE_\phi \tag{3.31}$$

and since this the simple harmonic oscillator equation, this has the immediate solution

$$rE_\phi = -A_2(\theta,\phi)\exp[ikZ_0r] - B_2(\theta,\phi)\exp[-ikZ_0r] \tag{3.32}$$

The other equations can be combined in similar manner to give, in summary,

$$rE_\theta = A_1(\theta,\phi)\exp[ikZ_0r] + B_1(\theta,\phi)\exp[-ikZ_0r] \tag{3.33}$$
$$rH_\phi = A_1(\theta,\phi)\exp[ikZ_0r] - B_1(\theta,\phi)\exp[-ikZ_0r] \tag{3.34}$$
$$rH_\theta = A_2(\theta,\phi)\exp[ikZ_0r] - B_2(\theta,\phi)\exp[-ikZ_0r] \tag{3.35}$$
$$rE_\phi = -A_2(\theta,\phi)\exp[ikZ_0r] - B_2(\theta,\phi)\exp[-ikZ_0r] \tag{3.36}$$

Now plug these expressions into the first equation of Eq. (3.28) to give

$$\begin{aligned}\frac{\partial}{\partial\theta}\big(\sin\theta(rE_\phi)\big) &= \frac{\partial(rE_\theta)}{\partial\phi}\\ -\frac{\partial}{\partial\theta}\left[\sin\theta\left(A_2\exp[ikZ_0r] + B_2\exp[-ikZ_0r]\right)\right] &= \frac{\partial A_1}{\partial\phi}\exp[ikZ_0r] + \frac{\partial B_1}{\partial\phi}\exp[-ikZ_0r]\end{aligned}$$

Matching coefficients of the exponentials yields

$$-\frac{\partial}{\partial\theta}\left(\sin\theta A_2\right) = \frac{\partial A_1}{\partial\phi} \qquad\qquad -\frac{\partial}{\partial\theta}\left(\sin\theta B_2\right) = \frac{\partial B_1}{\partial\phi} \tag{3.37}$$

Now plugging in the expressions into the first equation of Eq. (3.29) to give

$$\begin{aligned}\frac{\partial}{\partial\theta}\big(\sin\theta H_\phi\big) &= \frac{\partial H_\theta}{\partial\phi}\\ \frac{\partial}{\partial\theta}\left[\sin\theta\left(A_1\exp[ikZ_0r] - B_1\exp[-ikZ_0r]\right)\right] &= \frac{\partial A_2}{\partial\phi}\exp[ikZ_0r] - \frac{\partial B_2}{\partial\phi}\exp[-ikZ_0r]\end{aligned}$$

which yields

$$\frac{\partial}{\partial\theta}\left(\sin\theta A_1\right) = \frac{\partial A_2}{\partial\phi} \qquad\qquad \frac{\partial}{\partial\theta}\left(\sin\theta B_1\right) = \frac{\partial B_2}{\partial\phi} \tag{3.38}$$

These four equations in Eq. (3.37) and Eq. (3.38) can be combined to yield two complex ODEs. Let $C_1 = A_1 + iA_2$ and $C_2 = B_1 + iB_2$. Therefore, we can write

$$\begin{aligned}-\frac{\partial}{\partial\theta}\left(\sin\theta A_2\right) + i\frac{\partial}{\partial\theta}\left(\sin\theta A_1\right) &= \frac{\partial A_1}{\partial\phi} + i\frac{\partial A_2}{\partial\phi}\\ i\frac{\partial}{\partial\theta}\left[\sin\theta\left(A_1 + iA_2\right)\right] &= \frac{\partial}{\partial\phi}\left(A_1 + iA_2\right)\\ \frac{\partial}{\partial\theta}\left(\sin\theta C_1\right) &= -i\frac{\partial C_1}{\partial\phi}\end{aligned}$$

Similar arguments can be employed for C_2. This equation can be solved by change of variables. Let $\psi = -\log\left[\tan\left(\frac{\theta}{2}\right)\right]$. Therefore, the derivative maps to

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\psi}\frac{\partial\psi}{\partial\theta} = -\frac{1}{\tan\left(\frac{\theta}{2}\right)}\left(\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\right)\frac{\partial}{\partial\psi} = -\frac{1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\frac{\partial}{\partial\psi} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\psi}$$

and thus the differential equation becomes

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(\sin\theta C_1\right) &= -i\frac{\partial C_1}{\partial\phi}\\ \frac{1}{\sin\theta}\frac{\partial}{\partial\psi}(\sin\theta C_1) &= i\frac{\partial C_1}{\partial\phi}\\ \frac{\partial}{\partial\psi}(\sin\theta C_1) &= i\frac{\partial}{\partial\phi}(\sin\theta C_1)\\ \frac{\partial f}{\partial\psi} &= i\frac{\partial f}{\partial\phi}\end{aligned}$$

where $f = C_1\sin\theta$. This looks like a Cauchy-Riemann equation, with "real" coordinate ϕ and "imaginary" coordinate ψ , whose solution is a function of the form $f = g(\phi + i\psi)$. Thus we know that C_1 must be

$$C_1 = \frac{1}{\sin\theta}g\left(\phi - i\log\left[\tan\left(\frac{\theta}{2}\right)\right]\right) \tag{3.39}$$

Now we know that g must be single-valued and analytic in the "complex" plane $\phi + i\psi$ (since it satisfies the Cauchy-Riemann equations). Since $\sin\theta = 0$ when $\theta = 0, \pi$, then $g = 0$ at those points as well, which means $g(\phi \pm i\infty) = 0$. Furthermore, g must be periodic in ϕ with period 2π. Thus we see that g is uniformly bounded in the whole "complex" plane and by the Liouville theorem, it follows that g must be constant. And since g is identically zero at at least one point, then it must be zero everywhere, meaning $C_1 = 0$, meaning $A_1 = A_2 = 0$. Similar arguments apply for $B_1 = B_2 = 0$. In other words, specifying $\hat{r}\cdot\vec{E} = \hat{r}\cdot\vec{H} = 0$ has fixed $\vec{E} = \vec{H} = 0$, as desired. □

Therefore, our scalar functions are $u = \vec{r}\cdot\vec{E}$ and $w = \vec{r}\cdot\vec{H}$. In this case, the vector fields $\vec{\Pi}_E = u(\vec{r})\vec{r}$ and $\vec{\Pi}_M = w(\vec{r})\vec{r}$ are known as the *Debye potentials*. Now, as we saw in the proof, specifying both to be zero is tantamount to zeroing out the entire field. Therefore we cannot choose $\vec{r}\cdot\vec{E} = \vec{r}\cdot\vec{H} = 0$; incidentally, this means that spherically-symmetric transverse electromagnetic radiation does not exist, and hence why monopoles do not radiate!

However, we can specify one of the functions to be zero at a time. In the case of *transverse electric (TE)* (or magnetic multipole) fields, we choose

$$\vec{r}\cdot\vec{E}^{(\mathrm{TE})} = 0 \qquad\qquad \left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{H}^{(\mathrm{TE})}\right) = 0 \tag{3.40}$$

in the case of *transverse magnetic (TM)* (electric multipole) fields, we choose

$$\vec{r}\cdot\vec{H}^{(\mathrm{TM})} = 0 \qquad\qquad \left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{E}^{(\mathrm{TM})}\right) = 0 \tag{3.41}$$

We must therefore solve the scalar Helmholtz equation and use this to generate the rest of the electric/magnetic fields. Using the relations from the previous section, we can write

$$\vec{r}\cdot\vec{E}^{(\mathrm{TM})}_{\ell m} = Z_0\frac{\sqrt{\ell(\ell+1)}}{k}f_\ell(kr)Y_\ell^m(\theta,\phi) \qquad \vec{r}\cdot\vec{H}^{(\mathrm{TE})}_{\ell m} = -\frac{\sqrt{\ell(\ell+1)}}{k}g_\ell(kr)Y_\ell^m(\theta,\phi) \tag{3.42}$$

where $f_\ell(kr), g_\ell(kr) = A_\ell^{(1)}h_\ell^{(1)}(kr)+A_\ell^{(2)}h_\ell^{(2)}(kr)$. The factors of ℓ, Z_0 and k were chosen to make subsequent expressions cleaner; there's no loss of generality in doing this, it just amounts to a redefinition of the constants $A^{(1)}$ and $A^{(2)}$. Now recall that previously we found that a divergenceless vector field is of the form

$$\nabla\cdot\vec{A} = 0 \qquad \Rightarrow \qquad \vec{A} = \sum_{\ell,m} A_t^{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.43}$$

and furthermore, the curl of such a field is

$$\nabla\times\vec{A} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[-i\sqrt{\ell(\ell+1)}\frac{A_t^{\ell m}(r)}{r}\mathbf{Y}_\ell^m(\theta,\phi) - \frac{i}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rA_t^{\ell m}(r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \tag{3.44}$$

Thus we'd expect $\vec{E}^{(\mathrm{TM})}$, $\vec{B}^{(\mathrm{TE})}$ to be of the form in Eq. (3.43):

$$\vec{E}^{(\mathrm{TE})} = \sum_{\ell,m} E^{(\mathrm{TE})}_{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi) \qquad \vec{B}^{(\mathrm{TM})} = \sum_{\ell,m} B^{(\mathrm{TM})}_{\ell m}(r)\mathbf{\Phi}_\ell^m(\theta,\phi)$$

and their associated fields to be of the from in Eq. (3.44):

$$\vec{H}^{(\mathrm{TE})} = \frac{1}{ikZ_0}\nabla\times\vec{E}^{(\mathrm{TE})} = -\frac{1}{kZ_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\sqrt{\ell(\ell+1)}\frac{E^{(\mathrm{TE})}_{\ell m}(r)}{r}\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rE^{(\mathrm{TE})}_{\ell m}(r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \tag{3.45}$$

$$\vec{E}^{(\mathrm{TM})} = -\frac{Z_0}{ik}\nabla\times\vec{H}^{(\mathrm{TM})} = \frac{Z_0}{k}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\sqrt{\ell(\ell+1)}\frac{H^{(\mathrm{TM})}_{\ell m}(r)}{r}\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rH^{(\mathrm{TM})}_{\ell m}(r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \tag{3.46}$$

Recall that $\mathbf{\Phi}_\ell^m = \frac{1}{i\sqrt{\ell(\ell+1)}}\vec{r}\times\nabla Y_\ell^m(\theta,\phi)$, meaning $\hat{r}\cdot\mathbf{\Phi}_\ell^m = 0$. Furthermore, recall that $\hat{r}\cdot\mathbf{\Psi}_\ell^m(\theta,\phi) = \hat{r}\cdot\mathbf{\Phi}_\ell^m(\theta,\phi) = 0$ and $\hat{r}\cdot\mathbf{Y}_\ell^m(\theta,\phi) = Y_\ell^m(\theta,\phi)$. Thus we can write

$$\begin{aligned}\vec{r}\cdot\left(\nabla\times\vec{A}\right) &= \sum_{\ell,m}\left[-i\sqrt{\ell(\ell+1)}A_t^{\ell m}(r)\hat{r}\cdot\mathbf{Y}_\ell^m(\theta,\phi) - \frac{i}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rA_t^{\ell m}(r)\right)\cancelto{0}{\hat{r}\cdot\mathbf{\Psi}_\ell^m(\theta,\phi)}\right] \\ &= -\sum_{\ell,m} i\sqrt{\ell(\ell+1)}A_t^{\ell m}(r)Y_\ell^m(\theta,\phi)\end{aligned}$$

We can combine this fact with the Ampère-Maxwell law to give

$$\begin{aligned}
\nabla \times \vec{H}_{\ell m}^{(\mathrm{TM})} &= -\frac{ik}{Z_0}\vec{E}_{\ell m}^{(\mathrm{TM})} \\
\vec{r}\cdot\left(\nabla \times \vec{H}_{\ell m}^{(\mathrm{TM})}\right) &= -\frac{ik}{c}\vec{r}\cdot\vec{E}_{\ell m}^{(\mathrm{TM})} \\
-i\sqrt{\ell(\ell+1)}H_t^{\ell m}Y_\ell^m(\theta,\phi) &= -\frac{ik}{Z_0}\left(Z_0\frac{\sqrt{\ell(\ell+1)}}{k}f_\ell(kr)Y_\ell^m(\theta,\phi)\right) \\
\Rightarrow \qquad H_t^{\ell m} &= f_\ell(kr)
\end{aligned} \tag{3.47}$$

Similarly, combining with the Faraday law gives

$$\begin{aligned}
\nabla \times \vec{E}_{\ell m}^{(\mathrm{TE})} &= ikZ_0\vec{B}_{\ell m}^{(\mathrm{TE})} \\
\vec{r}\cdot\left(\nabla \times \vec{E}_{\ell m}^{(\mathrm{TE})}\right) &= ikZ_0\vec{r}\cdot\vec{B}_{\ell m}^{(\mathrm{TE})} \\
-i\sqrt{\ell(\ell+1)}E_t^{\ell m}Y_\ell^m(\theta,\phi) &= -ikZ_0\left(\epsilon_0\frac{\sqrt{\ell(\ell+1)}}{k}g_\ell(kr)Y_\ell^m(\theta,\phi)\right) \\
\Rightarrow \qquad E_t^{\ell m} &= Z_0 g_\ell(kr)
\end{aligned} \tag{3.48}$$

The total electric/magnetic field is the sum of the TE and TM fields for each. Putting these all together, the total electric and magnetic fields are

$$\vec{E} = Z_0\sum_{\ell,m}\left[\Lambda_E(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{kr}f_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rf_\ell(kr)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + \Lambda_M(\ell,m)g_\ell(kr)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{3.49}$$

$$\vec{H} = \sum_{\ell,m}\left[-\Lambda_M(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{kr}g_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rg_\ell(kr)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + \Lambda_E(\ell,m)f_\ell(kr)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{3.50}$$

Note that all three VSH are present in the decomposition, as would be expected. The coefficients Λ_E and Λ_M, also known as the electric and magnetic multipole moments, and the ingoing/outgoing factors $A^{(1)}$, $A^{(2)}$ are determined by boundary condition. Recall that the uniqueness theorem indicated that specifying the fields (in fact, just the radial components of the fields) at two radii was enough to determine the field everywhere in that region. To that end, using the orthogonality of the VSH, it can be shown that

$$\Lambda_M(\ell,m)g_\ell(kr) = -\frac{k}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^m(\theta,\phi)\right)^*\vec{r}\cdot\vec{H}\mathrm{d}\Omega \tag{3.51}$$

$$Z_0\Lambda_E(\ell,m)f_\ell(kr) = \frac{k}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^m(\theta,\phi)\right)^*\vec{r}\cdot\vec{E}\mathrm{d}\Omega \tag{3.52}$$

These equations make the at-first contradictory names for the coefficients make sense. Note, lastly, that the expressions for $\vec{E}$ are stronger than $\vec{H}$ by a factor of Z_0.

To summarize thus far, we have TE waves with components

$$\vec{r}\cdot\vec{E}^{(\mathrm{TE})} = 0 \qquad\qquad \vec{r}\cdot\vec{H}^{(\mathrm{TE})} = -\sum_{\ell,m}\Lambda_M(\ell,m)\frac{\sqrt{\ell(\ell+1)}}{k}g_\ell(kr)Y_\ell^m(\theta,\phi) \tag{3.53}$$

and thus fields

$$\vec{E}^{(\mathrm{TE})} = Z_0\sum_{\ell,m}\Lambda_M(\ell,m)g_\ell(kr)\mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.54}$$

$$\vec{H}^{(\mathrm{TE})} = -\sum_{\ell,m}\Lambda_M(\ell,m)\left[\frac{\sqrt{\ell(\ell+1)}}{kr}g_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rg_\ell(kr)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \tag{3.55}$$

and TM waves with components

$$\vec{r}\cdot\vec{H}^{(\mathrm{TM})} = 0 \qquad\qquad \vec{r}\cdot\vec{E}^{(\mathrm{TM})} = Z_0\sum_{\ell,m}\Lambda_E(\ell,m)\frac{\sqrt{\ell(\ell+1)}}{k}f_\ell(kr)Y_\ell^m(\theta,\phi) \tag{3.56}$$

and thus fields

$$\vec{H}^{(\mathrm{TM})} = \sum_{\ell,m}\Lambda_E(\ell,m)f_\ell(kr)\mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.57}$$

$$\vec{E}^{(\mathrm{TM})} = Z_0\sum_{\ell,m}\Lambda_E(\ell,m)\left[\frac{\sqrt{\ell(\ell+1)}}{kr}f_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rf_\ell(kr)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \tag{3.58}$$

where Λ_E and Λ_M are the coefficients specifying the amount of electric and magnetic multipole fields, respectively, given by the expressions

$$\Lambda_M(\ell,m)g_\ell(kr) = -\frac{k}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^m(\theta,\phi)\right)^*\vec{r}\cdot\vec{B}\mathrm{d}\Omega \tag{3.59}$$

$$Z_0\Lambda_E(\ell,m)f_\ell(kr) = \frac{k}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^m(\theta,\phi)\right)^*\vec{r}\cdot\vec{E}\mathrm{d}\Omega \tag{3.60}$$

3.3 ***Hansen multipoles***

Now in principle, we could stop here. But I'd like to go a bit further in order to clean the above expressions up. The fact that the electric and magnetic fields in vacuum satisfy such nice, symmetric equations implies that the solutions should somehow also be symmetric. In addition, we'd like to relate our solutions back to the vector Helmholtz equation which the fields also satisfy. Indeed, the full expressions for the multipole fields suggest such expressions. These combinations of the VSH and spherical Bessel functions are known as the *Hansen multipoles*. Note the expressions for the TM electric field and TE magnetic field:

$$\vec{E}^{(\mathrm{TM})} = Z_0 \sum_{\ell,m} \Lambda_E(\ell,m) \left[\frac{\sqrt{\ell(\ell+1)}}{kr} f_\ell(kr) \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r f_\ell(kr)\right) \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.61}$$

$$\vec{H}^{(\mathrm{TE})} = \sum_{\ell,m} \Lambda_M(\ell,m) \left[\frac{\sqrt{\ell(\ell+1)}}{kr} g_\ell(kr) \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r g_\ell(kr)\right) \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.62}$$

We can rename the term in brackets wholesale as a new function $\mathbf{N}_{\ell m}$:

$$\mathbf{N}_{\ell m} = \frac{\sqrt{\ell(\ell+1)}}{kr} g_\ell(kr) \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r g_\ell(kr)\right) \mathbf{\Psi}_\ell^m(\theta,\phi) \tag{3.63}$$

Using the recursion relations in Appendix A, these can be related to the total angular momentum states by the relation

$$\mathbf{N}_{\ell m} = \sqrt{\frac{\ell+1}{2\ell+1}} g_{\ell-1}(kr) \mathcal{Y}_{\ell,m}^{\ell-1,1} - \sqrt{\frac{\ell}{2\ell+1}} g_{\ell+1}(kr) \mathcal{Y}_{\ell,m}^{\ell+1,1} \tag{3.64}$$

Now the expressions for the TM electric field and TE magnetic field were obtained by taking the curls of the TM magnetic field and the TE electric field, respectively, and thus we can also define

$$\mathbf{M}_{\ell m} = g_\ell(kr) \mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.65}$$

which can also be related to the total angular momentum states by the relation

$$\mathbf{M}_{\ell m} = g_\ell(kr) \mathcal{Y}_{\ell,m}^{\ell,1}(\theta,\phi) \tag{3.66}$$

As stated, this must satisfy the relation

$$\nabla \times \mathbf{M}_{\ell m} = -ik\mathbf{N}_{\ell m} \tag{3.67}$$

The Faraday and Ampère-Maxwell laws, in turn imply

$$\nabla \times \mathbf{N}_{\ell m} = ik\mathbf{M}_{\ell m} \tag{3.68}$$

Thus the multipole fields take the form

$$\vec{E} = Z_0 \sum_{\ell,m} \left[\Lambda_E(\ell,m)\mathbf{N}_{\ell m} + \Lambda_M(\ell,m)\mathbf{M}_{\ell m} \right] \tag{3.69}$$

$$\vec{H} = \sum_{\ell,m} \left[-\Lambda_M(\ell,m)\mathbf{N}_{\ell m} + \Lambda_E(\ell,m)\mathbf{M}_{\ell m} \right] \tag{3.70}$$

Since the fields are all divergence-free as the multipole fields are, then we can write

$$\nabla \cdot \mathbf{M}_{\ell m} = 0 \qquad\qquad \nabla \cdot \mathbf{N}_{\ell m} = 0 \tag{3.71}$$

Clearly $\mathbf{M}_{\ell m} \cdot \mathbf{N}_{\ell m} = 0$; we thus require one more, mutually orthogonal vector field and we'll have fully described the space again. Since $\mathbf{M}_{\ell m}$ is proportional to $\mathbf{\Phi}_\ell^m$ and $\mathbf{N}_{\ell m}$ is proportional to a sum of $\mathbf{Y}_\ell^m$ and $\mathbf{\Psi}_\ell^m$; thus it stands to reason that the final Hansen multipole should be a similar combination of $\mathbf{Y}_\ell^m$ and $\mathbf{\Psi}_\ell^m$, and indeed, it can be shown that

$$\mathbf{L}_{\ell m} = -\frac{i}{k}\nabla(g_\ell(kr)Y_\ell^m(\theta,\phi)) = \frac{\sqrt{\ell(\ell+1)}}{kr}g_\ell(kr)\mathbf{\Psi}_\ell^m(\theta,\phi) - \frac{\mathrm{d}g_\ell(kr)}{\mathrm{d}(kr)}\mathbf{Y}_\ell^m(\theta,\phi) \tag{3.72}$$

which can be related to the total angular momentum states by the relation

$$\mathbf{L}_{\ell,m} = \sqrt{\frac{\ell}{2\ell+1}}g_{\ell-1}(kr)\mathcal{Y}_{\ell,m}^{\ell-1,1} + \sqrt{\frac{\ell+1}{2\ell+1}}g_{\ell+1}(kr)\mathcal{Y}_{\ell,m}^{\ell+1,1} \tag{3.73}$$

Now, we might've guessed this expression; it looks *a lot* like the electric field generated by in the electrostatic case! Now since it's a gradient, it's manifestly curl-less:

$$\nabla \times \mathbf{L}_{\ell m} = 0 \tag{3.74}$$

So let's find the divergence

$$\begin{aligned}
\nabla \cdot \mathbf{L}_{\ell m} &= -\frac{i}{k}\nabla^2(f_\ell(kr)Y_\ell^m(\theta,\phi)) \\
&= -\frac{i}{k}\left[\frac{1}{(kr)^2}\frac{\mathrm{d}}{\mathrm{d}(kr)}\left[(kr)^2\frac{\mathrm{d}f_\ell}{\mathrm{d}(kr)}\right]Y_\ell^m(\theta,\phi) + \frac{f_\ell(kr)}{r^2}L^2Y_\ell^m(\theta,\phi)\right] \\
&= -\frac{i}{k}\left[\frac{1}{(kr)^2}\left(2(kr)\frac{\mathrm{d}f_\ell}{\mathrm{d}(kr)} + (kr)^2\frac{\mathrm{d}^2 f_\ell}{\mathrm{d}(kr)^2}\right) - \frac{\ell(\ell+1)}{r^2}f_\ell(kr)\right]Y_\ell^m(\theta,\phi) \\
&= -\frac{i}{k}\left[\frac{\mathrm{d}^2 f_\ell}{\mathrm{d}(kr)^2} + \frac{2}{kr}\frac{\mathrm{d}f_\ell}{\mathrm{d}(kr)} - \frac{\ell(\ell+1)}{r^2}f_\ell(kr)\right]Y_\ell^m(\theta,\phi) \\
&= -\frac{i}{k}(-k^2 f_\ell(kr))Y_\ell^m(\theta,\phi) \\
&= ikf_\ell(kr)Y_\ell^m(\theta,\phi)
\end{aligned} \tag{3.75}$$

With these expressions, we can write the effects of the vector Laplacian on each Hansen multipole:

$$\nabla^2\mathbf{L}_{\ell m} = \nabla\left(\nabla \cdot \mathbf{L}_{\ell,m}\right) - \cancelto{0}{\nabla \times \nabla \times \mathbf{L}_{\ell,m}} = ik\nabla(f_\ell(kr)Y_\ell^m(\theta,\phi)) = k^2\mathbf{L}_{\ell m} \tag{3.76}$$

$$\nabla^2\mathbf{M}_{\ell m} = \cancelto{0}{\nabla\left(\nabla \cdot \mathbf{M}_{\ell,m}\right)} - \nabla \times \nabla \times \mathbf{M}_{\ell,m} = -\nabla \times \left(-ik\mathbf{N}_{\ell,m}\right) = k^2\mathbf{M}_{\ell m} \tag{3.77}$$

$$\nabla^2\mathbf{N}_{\ell m} = \cancelto{0}{\nabla\left(\nabla \cdot \mathbf{N}_{\ell,m}\right)} - \nabla \times \nabla \times \mathbf{N}_{\ell,m} = -\nabla \times \left(ik\mathbf{M}_{\ell,m}\right) = k^2\mathbf{N}_{\ell m} \tag{3.78}$$

Since the Hansen multipoles form a complete basis, they can be used in a resolution of the identity, and can thus be used to find the Green function for the vector Helmholtz equation:

$$\begin{aligned}
\tilde{G}_\pm(\vec{r},\vec{r}') &= \mp ik\sum_{j,\ell,m} h_\ell^\pm(kr_>)j_\ell(kr_<)\mathcal{Y}_{j,m}^{\ell,1}(\theta,\phi)\left(\mathcal{Y}_{j,m}^{\ell,1}(\theta',\phi)'\right)^* \\
&= \mp ik\sum_{\ell,m}\left[\mathbf{L}_{\ell m}^+(\vec{r}_>)\left(\mathbf{L}_{\ell m}^0(\vec{r}_<)\right)^* + \mathbf{M}_{\ell m}^+(\vec{r}_>)\left(\mathbf{M}_{\ell m}^0(\vec{r}_<)\right)^* + \mathbf{N}_{\ell m}^+(\vec{r}_>)\left(\mathbf{N}_{\ell m}^0(\vec{r}_<)\right)^*\right]
\end{aligned} \tag{3.79}$$

Here the (0)/(+) superscript refers to the spherical Bessel function or spherical Hankel function that should be used in the Hansen multipole. Using the Hansen multipoles, a full vector field satisfying the homogeneous vector Helmholtz equation

$$\left(\nabla^2 + k^2\right) \vec{A}(\vec{x}) = 0$$

will be of the form

$$\vec{A} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_L^{\ell m} \mathbf{L}_{\ell m}(\vec{r}) + A_M^{\ell m} \mathbf{M}_{\ell m}(\vec{r}) + A_N^{\ell m} \mathbf{N}_{\ell m}(\vec{r}) \right] \tag{3.80}$$

Note that the expansion coefficients are constants, independent of position, since the Hansen multipoles on their own are good enough to satisfy the Helmholtz equation. In summary, we write

The Hansen multipoles are defined as combinations of the VSH and spherical Bessel functions (i.e., solutions to the scalar Helmholtz equation)

$$\mathbf{L}_{\ell m} = -\frac{i}{k} \nabla (f_\ell(kr) Y_\ell^m(\theta,\phi)) = \frac{\sqrt{\ell(\ell+1)}}{kr} g_\ell(kr) \mathbf{\Psi}_\ell^m(\theta,\phi) - \frac{\mathrm{d}g_\ell(kr)}{\mathrm{d}(kr)} \mathbf{Y}_\ell^m(\theta,\phi) \tag{3.81}$$

$$\mathbf{M}_{\ell m} = g_\ell(kr) \mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.82}$$

$$\mathbf{N}_{\ell m} = \frac{i}{k} \nabla \times \mathbf{M}_\ell^m = \frac{\sqrt{\ell(\ell+1)}}{kr} g_\ell(kr) \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr} \frac{\mathrm{d}}{\mathrm{d}r} \left(r g_\ell(kr) \right) \mathbf{\Psi}_\ell^m(\theta,\phi) \tag{3.83}$$

Their divergences are

$$\nabla \cdot \mathbf{L}_{\ell m} = ik f_\ell(kr) Y_\ell^m(\theta,\phi) \tag{3.84}$$

$$\nabla \cdot \mathbf{M}_{\ell m} = 0 \tag{3.85}$$

$$\nabla \cdot \mathbf{N}_{\ell m} = 0 \tag{3.86}$$

and their curls are

$$\nabla \times \mathbf{L}_{\ell m} = 0 \tag{3.87}$$

$$\nabla \times \mathbf{M}_{\ell m} = -ik \mathbf{N}_{\ell m} \tag{3.88}$$

$$\nabla \times \mathbf{N}_{\ell m} = ik \mathbf{M}_{\ell m} \tag{3.89}$$

In terms of these expressions, the multipole fields take the form

$$\vec{E} = Z_0 \sum_{\ell,m} \left[\Lambda_E(\ell,m) \mathbf{N}_{\ell m} + \Lambda_M(\ell,m) \mathbf{M}_{\ell m} \right] \tag{3.90}$$

$$\vec{H} = \sum_{\ell,m} \left[-\Lambda_M(\ell,m) \mathbf{N}_{\ell m} + \Lambda_E(\ell,m) \mathbf{M}_{\ell m} \right] \tag{3.91}$$

3.4 ***Implications of gauge invariance***

The Hansen multipoles provide a rather elegant playground to explore some aspects of gauge invariance as they related to multipole fields. The electric and magnetic fields are generated by the scalar and vector potentials $(\phi, \vec{A})$ through the relations

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A} \tag{3.92}$$

If we assume monochromatic time harmonic potentials constrained by the Lorenz gauge condition

$$\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = 0 \tag{3.93}$$

then the potentials obey the scalar and vector Helmholtz equations, respectively:

$$\left(\nabla^2 + k^2\right)\Phi(\vec{x}) = -\frac{\rho}{\epsilon_0} \tag{3.94}$$

$$\left(\nabla^2 + k^2\right)\vec{A}(\vec{x}) = -\mu_0 \vec{J} \tag{3.95}$$

Both may be solved using their respective Green functions. In particular, the scalar potential has the form

$$\begin{aligned}
\Phi(\vec{r}) &= \frac{ik}{\epsilon_0}\int \sum_{\ell,m} j_\ell(kr_<)h^{(1)}(kr_>)\left(Y_\ell^m(\theta',\phi')\right)^* Y_\ell^m(\theta,\phi)\rho(\vec{r}')\mathrm{d}^3x' \\
&= \frac{ik}{\epsilon_0}\sum_{\ell,m}\int\left[h_\ell^{(1)}(kr)\int_0^r j_\ell(kr')\rho(\vec{r}')r'^2\mathrm{d}r' + j_\ell(kr)\int_r^\infty h_\ell^{(1)}(kr')\rho(\vec{r}')r'^2\mathrm{d}r'\right]\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}\Omega' Y_\ell^m(\theta,\phi) \\
&= \frac{ik}{\epsilon_0}\sum_{\ell,m}\left[h_\ell^{(1)}(kr)\Phi_{S,\mathrm{int}}^{\ell m}(\theta,\phi) + j_\ell(kr)\Phi_{S,\mathrm{ext}}^{\ell m}(\theta,\phi)\right]
\end{aligned}$$

As usual, if we assume the source is localized and small, we can extend the interior moment integral to infinity without much consequence and thus the potential gives

$$\Phi(\vec{r}) = \frac{ik}{\epsilon_0}\sum_{\ell,m} h_\ell^{(1)}(kr)\Phi_S^{\ell m}(\theta,\phi) \tag{3.96}$$

where I've dropped the [ext] subscript to clean up the notation. Repeating this exercise with the vector potential gives

$$\vec{A}(\vec{r}) = ik\mu_0\sum_{\ell,m}\left[A_L^{\ell m}\mathbf{L}_{\ell m}^+(\vec{r}) + A_M^{\ell m}\mathbf{M}_{\ell m}^+(\vec{r}) + A_N^{\ell m}\mathbf{N}_{\ell m}^+(\vec{r})\right] \tag{3.97}$$

where

$$A_L^{\ell m} = \int \vec{J}(\vec{r}')\cdot\left(\mathbf{L}_{\ell m}^0(\vec{r}')\right)^*\mathrm{d}^3x' \tag{3.98}$$

$$A_M^{\ell m} = \int \vec{J}(\vec{r}')\cdot\left(\mathbf{M}_{\ell m}^0(\vec{r}')\right)^*\mathrm{d}^3x' \tag{3.99}$$

$$A_N^{\ell m} = \int \vec{J}(\vec{r}')\cdot\left(\mathbf{N}_{\ell m}^0(\vec{r}')\right)^*\mathrm{d}^3x' \tag{3.100}$$

These four sets of numbers, $\Phi_S^{\ell m}, A_L^{\ell m}, A_M^{\ell m}, A_N^{\ell m}$, as you might guess, are not independent. The vector field of a massless particle is over-specified by two degrees of freedom and as such only two of these are truly independent. The other two are related by the gauge condition. Since we've assumed the Lorenz gauge condition, let's see what effect this has on the constants. Plugging in our expansions gives

$$\begin{aligned}
0 &= \frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} \\
&= \sum_{\ell,m} \left[\frac{ik}{c^2 \epsilon_0} \left(-i\omega h_\ell^{(1)}(kr) \Phi_S^{\ell m} \right) - k^2 \mu_0 A_L^{\ell m} \right] \\
&= \sum_{\ell,m} \left[\frac{k^2}{\epsilon_0 c} \Phi_S^{\ell m} - k^2 \mu_0 A_L^{\ell m} \right] h_\ell^{(1)}(kr) \\
&= -\mu_0 k^2 \sum_{\ell,m} \left[c\Phi_S^{\ell m} - A_L^{\ell m} \right] h_\ell^{(1)}(kr)
\end{aligned}$$

For the above to be true in general, the term in brackets must be zero identically and thus we can write

$$c\Phi_S^{\ell m} = A_L^{\ell m} \tag{3.101}$$

Thus we see that the scalar moments Φ_S are tied to the longitudinal moments A_L by the Lorenz condition! Now that's all well and good, but what about the fields themselves? The magnetic field is given by $\vec{B} = \nabla \times \vec{A}$:

$$\begin{aligned}
\vec{H} &= \nabla \times \left(ik\mu_0 \sum_{\ell,m} \left[A_L^{\ell m} \mathbf{L}^+_{\ell m}(\vec{r}) + A_M^{\ell m} \mathbf{M}^+_{\ell m}(\vec{r}) + A_N^{\ell m} \mathbf{N}^+_{\ell m}(\vec{r}) \right] \right) \\
&= ik\mu_0 \sum_{\ell,m} \left[A_L^{\ell m} \cancelto{0}{\nabla \times \mathbf{L}^+_{\ell m}(\vec{r})} + A_M^{\ell m} \nabla \times \mathbf{M}^+_{\ell m}(\vec{r}) + A_N^{\ell m} \nabla \times \mathbf{N}^+_{\ell m}(\vec{r}) \right] \\
&= ik\mu_0 \sum_{\ell,m} \left[-ikA_M^{\ell m} \mathbf{N}^+_{\ell m}(\vec{r}) + ikA_N^{\ell m} \mathbf{M}^+_{\ell m}(\vec{r}) \right] \\
&= k^2\mu_0 \sum_{\ell,m} \left[A_M^{\ell m} \mathbf{N}^+_{\ell m}(\vec{r}) - A_N^{\ell m} \mathbf{M}^+_{\ell m}(\vec{r}) \right]
\end{aligned} \tag{3.102}$$

and the associated electric field by the Ampère-Maxwell law $\nabla \times \vec{B} = -i\omega\mu_0\epsilon_0 \vec{E}$:

$$\begin{aligned}
-i\omega\mu_0\epsilon_0 \vec{E} &= \nabla \times \left(k^2\mu_0 \sum_{\ell,m} \left[A_M^{\ell m} \mathbf{N}^+_{\ell m}(\vec{r}) - A_N^{\ell m} \mathbf{M}^+_{\ell m}(\vec{r}) \right] \right) \\
\vec{E} &= i\frac{k^2}{\omega\epsilon_0} \sum_{\ell,m} \left[A_M^{\ell m} \nabla \times \mathbf{N}^+_{\ell m}(\vec{r}) - A_N^{\ell m} \nabla \times \mathbf{M}^+_{\ell m}(\vec{r}) \right] \\
&= i\frac{k}{c\epsilon_0} \sum_{\ell,m} \left[ikA_M^{\ell m} \mathbf{M}^+_{\ell m}(\vec{r}) + ikA_N^{\ell m} \mathbf{N}^+_{\ell m}(\vec{r}) \right] \\
&= -\frac{k^2\mu_0}{c} \sum_{\ell,m} \left[A_M^{\ell m} \mathbf{M}^+_{\ell m}(\vec{r}) + A_N^{\ell m} \mathbf{N}^+_{\ell m}(\vec{r}) \right]
\end{aligned} \tag{3.103}$$

Thus we see that the electric and magnetic fields only depend on the transverse multipoles! But what of the other expression for the electric field, using the potentials? Let's find out; plugging in the expansions into $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$ gives

$$\begin{aligned}
\vec{E} &= -\nabla \left[\frac{ik}{\epsilon_0} \sum_{\ell,m} h_\ell^{(1)}(kr) \Phi_S^{\ell m}(\theta, \phi) \right] + i\omega \left(ik\mu_0 \sum_{\ell,m} \left[A_L^{\ell m} \mathbf{L}_{\ell m}^+(\vec{r}) + A_M^{\ell m} \mathbf{M}_{\ell m}^+(\vec{r}) + A_N^{\ell m} \mathbf{N}_{\ell m}^+(\vec{r}) \right] \right) \\
&= \frac{ik}{\epsilon_0} \sum_{\ell,m} \nabla \left[h_\ell^{(1)}(kr) \Phi_S^{\ell m}(\theta, \phi) \right] - \frac{k^2 \mu_0}{c} \sum_{\ell,m} \left[A_L^{\ell m} \mathbf{L}_{\ell m}^+(\vec{r}) + A_M^{\ell m} \mathbf{M}_{\ell m}^+(\vec{r}) + A_N^{\ell m} \mathbf{N}_{\ell m}^+(\vec{r}) \right] \\
&= \sum_{\ell,m} \left[\frac{k^2}{c\epsilon_0} \Phi_S^{\ell m}(\theta, \phi) - \frac{k^2 \mu_0}{c} A_L^{\ell m} \right] \mathbf{L}_{\ell m}^+(\vec{r}) - \frac{k^2 \mu_0}{c} \sum_{\ell,m} \left[A_M^{\ell m} \mathbf{M}_{\ell m}^+(\vec{r}) + A_N^{\ell m} \mathbf{N}_{\ell m}^+(\vec{r}) \right] \\
&= \frac{k^2}{\epsilon_0} \sum_{\ell,m} \left[\Phi_S^{\ell m}(\theta, \phi) - \frac{1}{c} A_L^{\ell m} \right] \mathbf{L}_{\ell m}^+(\vec{r}) - \frac{k^2 \mu_0}{c} \sum_{\ell,m} \left[A_M^{\ell m} \mathbf{M}_{\ell m}^+(\vec{r}) + A_N^{\ell m} \mathbf{N}_{\ell m}^+(\vec{r}) \right] \qquad (3.104)
\end{aligned}$$

And thus we see that the electric field is proportional only to the transverse multipoles if $c\Phi_S^{\ell m}(\theta, \phi) = A_L^{\ell m}$, as we found before! Thus the Lorenz gauge condition is deeply connected to the cancellation of the longitudinal and scalar parts of the electric field, which we've shown relatively cleanly using the Hansen multipoles.

3.5 Properties of multipole fields

3.5.1 Spatial regions of radiation

Radiation fields of course carry energy as they propagate through space. The fields themselves structure space into regions where the energy flow exhibits different behavior. In order to see this, consider the complex Poynting vector $\vec{S} = \vec{E} \times \vec{H}$. If we consider a sphere of radius r surrounding the source of radiation. The power, corresponding to the energy flux per unit time, is given by

$$\frac{\mathrm{d}U}{\mathrm{d}t} = P = \int \vec{S} \cdot \hat{r} \mathrm{d}A \tag{3.105}$$

Now the electric and magnetic fields are complex at this point; recall that they're both multiplied by an exponential time factor $\exp[-i\omega t]$. Thus to extract the physics we need to massage the above expression a little bit:

$$\begin{aligned}
\vec{S} &= \vec{E} \times \vec{H} \\
&= \mathrm{Re}\left[\vec{E} \exp[-i\omega t]\right] \times \mathrm{Re}\left[\vec{H} \exp[-i\omega t]\right] \\
&= \frac{1}{2}\left(\vec{E} \exp[-i\omega t] + \vec{E}^* \exp[i\omega t]\right) \times \frac{1}{2}\left(\vec{H} \exp[-i\omega t] + \vec{H}^* \exp[i\omega t]\right) \\
&= \frac{1}{4}\left[\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H} + \vec{E} \times \vec{H} \exp[-2i\omega t] + \vec{E}^* \times \vec{H}^* \exp[2i\omega t]\right] \\
&= \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right] + \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H} \exp[-2i\omega t]\right]
\end{aligned} \tag{3.106}$$

Time-averaging over a complete cycle would make the second term vanish due to the oscillations from the exponential, but we can still consider the contribution to the instantaneous energy flow due to this term. We thus need to consider the products $\left(\vec{E} \times \vec{H}^*\right) \cdot \hat{r}$ and $\left(\vec{E} \times \vec{H}\right) \cdot \hat{r}$ and these can be rewritten using a vector identity to give

$$\left(\vec{E} \times \vec{H}^*\right) \cdot \hat{r} = \left(\hat{r} \times \vec{E}\right) \cdot \vec{H}^* \tag{3.107}$$

Recalling that the VSH satisfy the relations

$$\hat{r} \times \mathbf{Y}_\ell^m = 0 \qquad \hat{r} \times \mathbf{\Psi}_\ell^m = i\mathbf{\Phi}_\ell^m \qquad \hat{r} \times \mathbf{\Phi}_\ell^m = -i\mathbf{\Psi}_\ell^m \tag{3.108}$$

then, if we replace f_ℓ, g_ℓ with spherical Hankel functions to correspond to propagating radiation, we can write

$$\hat{r} \times \vec{E} = Z_0 \sum_{\ell,m} \left(i\Lambda_E(\ell,m) \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r h_\ell^{(1)}(kr)\right) \mathbf{\Phi}_\ell^m(\theta,\phi) - i\Lambda_M(\ell,m) h_\ell^{(1)}(kr) \mathbf{\Psi}_\ell^m(\theta,\phi) \right) \tag{3.109}$$

Recall the form of the magnetic field:

$$\vec{H} = \sum_{\ell,m} \left[-\Lambda_M(\ell,m) \left(\frac{\sqrt{\ell(\ell+1)}}{kr} h_\ell^{(1)}(kr) \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r h_\ell^{(1)}(kr)\right) \mathbf{\Psi}_\ell^m(\theta,\phi) \right) + \Lambda_E(\ell,m) h_\ell^{(1)}(kr) \mathbf{\Phi}_\ell^m(\theta,\phi) \right] \tag{3.110}$$

and, for our purposes, the complex conjugate:

$$\vec{H}^* = \sum_{\ell,m}\left[-\Lambda_M^*(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{kr}h_\ell^{(2)}(kr)\left(\mathbf{Y}_\ell^m(\theta,\phi)\right)^* + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(2)}(kr)\right)\left(\mathbf{\Psi}_\ell^m(\theta,\phi)\right)^*\right) + \Lambda_E^*(\ell,m)h_\ell^{(2)}(kr)\left(\mathbf{\Phi}_\ell^m(\theta,\phi)\right)^*\right] \tag{3.111}$$

Exploiting the fact that $\hat{r}\cdot\mathbf{\Phi}_\ell^m = \hat{r}\cdot\mathbf{\Psi}_\ell^m = 0$, upon integration the cross-terms corresponding to $\mathbf{\Phi}_\ell^m\cdot\mathbf{\Psi}_{\ell'}^{m'}$ will vanish thanks the orthogonality of the VSH. And thus we can write

$$\int\left(\hat{r}\times\vec{E}\right)\cdot\vec{H}r^2\mathrm{d}\Omega = \frac{iZ_0}{k^2}\sum_{\ell,m}\left[\Lambda_M(\ell,m)^2 + \Lambda_E(\ell,m)^2\right]krh_\ell^{(1)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(1)}(kr)\right) \tag{3.112}$$

and

$$\int\left(\hat{r}\times\vec{E}\right)\cdot\vec{H}^*r^2\mathrm{d}\Omega = \frac{iZ_0}{k^2}\sum_{\ell,m}\left[\left|\Lambda_M(\ell,m)\right|^2 krh_\ell^{(1)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(2)}(kr)\right) + \left|\Lambda_E(\ell,m)\right|^2 krh_\ell^{(2)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(1)}(kr)\right)\right] \tag{3.113}$$

Since we're interested in the real part of the Poynting vector, we need to consider the sums

$$\begin{aligned}&\frac{1}{4}\int\left(\hat{r}\times\vec{E}\right)\cdot\vec{H}r^2\mathrm{d}\Omega + \frac{1}{4}\int\left(\hat{r}\times\vec{E}^*\right)\cdot\vec{H}^*r^2\mathrm{d}\Omega\\ &= \frac{iZ_0}{4k^2}\sum_{\ell,m}\mathrm{Re}\left[\left(\Lambda_M(\ell,m)^2 + \Lambda_E(\ell,m)^2\right)krh_\ell^{(1)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(1)}(kr)\right)\right]\end{aligned} \tag{3.114}$$

and

$$\begin{aligned}&\frac{1}{4}\int\left(\hat{r}\times\vec{E}\right)\cdot\vec{H}^*r^2\mathrm{d}\Omega + \frac{1}{4}\int\left(\hat{r}\times\vec{E}^*\right)\cdot\vec{H}r^2\mathrm{d}\Omega\\ &= \frac{iZ_0}{4k^2}\sum_{\ell,m}\left(\left|\Lambda_M(\ell,m)\right|^2 + \left|\Lambda_E(\ell,m)\right|^2\right)\left[krh_\ell^{(1)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(2)}(kr)\right) - krh_\ell^{(2)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(1)}(kr)\right)\right]\\ &= \frac{iZ_0}{4k^2}\sum_{\ell,m}\left(\left|\Lambda_M(\ell,m)\right|^2 + \left|\Lambda_E(\ell,m)\right|^2\right)[-2i]\\ &= \frac{Z_0}{2k^2}\sum_{\ell,m}\left(\left|\Lambda_M(\ell,m)\right|^2 + \left|\Lambda_E(\ell,m)\right|^2\right)\end{aligned} \tag{3.115}$$

Thus the power through our sphere of radius r is

$$P = \frac{Z_0}{2k^2}\sum_{\ell,m}\left(\left|\Lambda_M(\ell,m)\right|^2 + \left|\Lambda_E(\ell,m)\right|^2\right) + \frac{iZ_0}{4k^2}\sum_{\ell,m}\mathrm{Re}\left[\left(\Lambda_M(\ell,m)^2 + \Lambda_E(\ell,m)^2\right)krh_\ell^{(1)}(kr)\frac{\mathrm{d}}{\mathrm{d}r}\left(rh_\ell^{(1)}(kr)\right)\exp[-2i\omega t]\right] \tag{3.116}$$

Note that the first term is independent of both time and r. Upon averaging over a cycle this is the only term that remains and it exhibits no radial dependence. This is very reassuring and in keeping with our intuition, since it means radiating fields carry away a finite amount of energy, regardless of distance from the source. The second term can be interpreted as as the non-radiating part of the field, which 'sloshes' back and forth across our sphere of measurement and thus on average carries away no energy. It does exhibit a non-trivial radial dependence, but this isn't a huge problem since it does not carry energy away, merely moving it around.

3.5.2 Radiation zone fields

A significant application of the multipole vector fields is in the treatment of radiation zone fields. The radiation zone corresponds to the limit $kr \gg 1$. Suppose we consider only outgoing waves, which correspond to setting f_ℓ, g_ℓ to spherical Hankel functions of the first kind. These functions have asymptotic form

$$f_\ell(kr), g_\ell(kr) \to (-i)^{\ell+1}\frac{\exp[ikr]}{kr} \tag{3.117}$$

The TM magnetic field becomes

$$\vec{H}^{(\mathrm{TM})} = \frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.118}$$

Note that the field has the form of a spherical wave multiplied by an angular function or (sum of angular functions) describing the vector dependence. The associated TM electric field is thus

$$\begin{aligned}
\vec{E}^{(\mathrm{TM})} &= Z_0\sum_{\ell,m}\Lambda_E(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{kr}(-i)^{\ell+1}\frac{\exp[ikr]}{kr}\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(r(-i)^{\ell+1}\frac{\exp[ikr]}{kr}\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) \\
&= Z_0\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\left[\sqrt{\ell(\ell+1)}\frac{\exp[ikr]}{(kr)^2}\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k^2 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\exp[ikr]\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&= Z_0\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\left[\sqrt{\ell(\ell+1)}\frac{\exp[ikr]}{(kr)^2}\mathbf{Y}_\ell^m(\theta,\phi) + \frac{i\exp[ikr]}{kr}\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&= Z_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\left[\frac{\sqrt{\ell(\ell+1)}}{kr}\mathbf{Y}_\ell^m(\theta,\phi) + i\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \\
&\approx iZ_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\mathbf{\Psi}_\ell^m(\theta,\phi)
\end{aligned} \tag{3.119}$$

where, in the last step, only the term proportional to $\frac{1}{r}$ has been kept as all other terms would be suppressed. As with the magnetic field, the electric field has assumed the form of a spherical wave multiplied by a sum of angular functions describing the vector dependence. Now note that, at a given point in space,

$$\begin{aligned}
\vec{H}^{(\mathrm{TM})}\cdot\vec{E}^{(\mathrm{TM})} &= \left(\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi)\right)\cdot\left(iZ_0\frac{\exp[ikr]}{kr}\sum_{\ell',m'}(-i)^{\ell'+1}\Lambda_E(\ell',m')\mathbf{\Psi}_{\ell'}^{m'}(\theta,\phi)\right) \\
&= i\frac{\exp[2ikr]}{(kr)^2}\sum_{\ell,m}\sum_{\ell',m'}(-1)^{\ell+\ell'+2}\Lambda_E(\ell,m)\Lambda_E(\ell',m')\cancelto{0}{\mathbf{\Phi}_\ell^m(\theta,\phi)\cdot\mathbf{\Psi}_{\ell'}^{m'}}(\theta,\phi) = 0
\end{aligned} \tag{3.120}$$

where we exploited the vector orthogonality of the VSH in the last step. Since the coefficients in the sum are the same in both the electric and magnetic fields, we can exploit this orthogonality to write $\vec{E}^{(\mathrm{TM})}$ as

$$\begin{aligned}\vec{E}^{(\mathrm{TM})} &= Z_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\left(i\mathbf{\Psi}_\ell^m(\theta,\phi)\right)\\ &= Z_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\left(-\hat{r}\times\mathbf{\Phi}_\ell^m(\theta,\phi)\right)\\ &= -Z_0\hat{r}\times\left(\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi)\right)\\ &= -Z_0\hat{r}\times\vec{H}^{(\mathrm{TM})} \qquad (3.121)\end{aligned}$$

In other words, radiation zone fields are transverse electromagnetic. This makes sense as spherical waves, far from the source, appear approximately as plane waves which exhibit such properties. Lastly, note that TE and TM fields are orthogonal to each other, again thanks to the orthogonality properties of the VSH. Putting these all together, we can write the radiation zone fields as

$$\vec{E} = Z_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\left[\Lambda_M(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi)+i\Lambda_E(\ell,m)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \qquad (3.122)$$

$$\vec{H} = \frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\left[\Lambda_E(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi)-i\Lambda_M(\ell,m)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \qquad (3.123)$$

Thus, a summary of radiation zone results is

TM (or electric multipole) fields have the form

$$\vec{H}^{(\mathrm{TM})} = \frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_E(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi) \qquad \vec{E}^{(\mathrm{TM})} = -Z_0\hat{r}\times\vec{H}^{(\mathrm{TM})} \qquad (3.124)$$

TE (or magnetic multipole) fields have the form

$$\vec{E}^{(\mathrm{TE})} = Z_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\Lambda_M(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi) \qquad \vec{H}^{(\mathrm{TE})} = \frac{1}{Z_0}\hat{r}\times\vec{E}^{(\mathrm{TE})} \qquad (3.125)$$

In the radiation zone, electric and magnetic multipole fields become transverse electromagnetic

$$\vec{E}^{(\mathrm{TM})}\cdot\vec{H}^{(\mathrm{TM})} = 0 \qquad \vec{E}^{(\mathrm{TE})}\cdot\vec{H}^{(\mathrm{TE})} = 0 \qquad (3.126)$$

And lastly, electric multipole electric/magnetic fields are orthogonal to magnetic multipole electric/magnetic fields, at a fixed point in space

$$\left.\vec{E}^{(\mathrm{TM})}\cdot\vec{E}^{(\mathrm{TE})}\right]_{\vec{r}} = 0 \qquad \left.\vec{H}^{(\mathrm{TM})}\cdot\vec{H}^{(\mathrm{TE})}\right]_{\vec{r}} = 0 \qquad (3.127)$$

3.5.3 Energy radiation

Recall that the energy density u in a region of space filled by monochromatic time-harmonic fields is given by

$$u = \frac{\epsilon_0}{4}\left(\vec{E}\cdot\vec{E}^* + Z_0^2\vec{H}\cdot\vec{H}^*\right) \tag{3.128}$$

A general radiation field is comprised of both TM and TE waves, with the forms given in the previous section

$$\vec{E} = \vec{E}^{(\mathrm{TE})}+\vec{E}^{(\mathrm{TM})} = Z_0\frac{\exp[ikr]}{kr}\sum_{\ell,m}(-i)^{\ell+1}\left[\Lambda_M(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi) + i\Lambda_E(\ell,m)\mathbf{\Psi}_\ell^m(\theta,\phi)\right] \qquad \vec{H} = \frac{1}{Z_0}\hat{r}\times\vec{E} \tag{3.129}$$

Plugging these decompositions into the energy density gives

$$\begin{aligned} u &= \frac{\epsilon_0}{4}\left(\vec{E}\cdot\vec{E}^* + Z_0^2\vec{H}\cdot\vec{H}^*\right) \\ &= \frac{\epsilon_0}{2}\vec{E}\cdot\vec{E}^* \\ &= \frac{\epsilon_0}{2}\left[\left(\vec{E}^{(\mathrm{TM})}+\vec{E}^{(\mathrm{TE})}\right)\cdot\left(\vec{E}^{(\mathrm{TM})}+\vec{E}^{(\mathrm{TE})}\right)^*\right] \\ &= \frac{\epsilon_0}{2}\left[\vec{E}^{(\mathrm{TM})}\cdot\left(\vec{E}^{(\mathrm{TM})}\right)^* + \vec{E}^{(\mathrm{TE})}\cdot\left(\vec{E}^{(\mathrm{TE})}\right)^* + \vec{E}^{(\mathrm{TE})}\cdot\left(\vec{E}^{(\mathrm{TM})}\right)^* + \vec{E}^{(\mathrm{TM})}\cdot\left(\vec{E}^{(\mathrm{TE})}\right)^*\right] \end{aligned} \tag{3.130}$$

where we've exploited the fact that $\|\vec{H}\| = \frac{1}{Z_0}\|\vec{E}\|$ in the radiation zone. The energy $\mathrm{d}U$ in a spherical shell of radius $\mathrm{d}r$ is given by integrating this expression over all angles:

$$\mathrm{d}U = \left(\int u\mathrm{d}\Omega\right)r^2\mathrm{d}r \tag{3.131}$$

Integrating over all angles allows us to exploit the orthogonality of the VSH and thus cancel the cross-terms. This leaves the first and second terms in the brackets. Let's consider the first term in the above expression:

$$\begin{aligned} &\int \vec{E}^{(\mathrm{TM})}\cdot\left(\vec{E}^{(\mathrm{TM})}\right)^*\mathrm{d}\Omega \\ &= Z_0^2\frac{\exp[ikr]}{kr}\int\sum_{\ell,m}(-i)^{\ell+1}c\Lambda_E(\ell,m)\left(i\mathbf{\Psi}_\ell^m(\theta,\phi)\right)\cdot\left(\frac{\exp[ikr]}{kr}\sum_{\ell',m'}(-i)^{\ell'+1}\Lambda_E(\ell',m')\left(i\mathbf{\Psi}_{\ell'}^{m'}(\theta,\phi)\right)\right)^*\mathrm{d}\Omega \\ &= \frac{Z_0^2}{k^2r^2}\sum_{\ell,\ell',m,m'}i^{\ell'-\ell}\Lambda_E(\ell,m)\Lambda_E^*(\ell',m')\int\mathbf{\Psi}_\ell^m(\theta,\phi)\cdot\left(\mathbf{\Psi}_{\ell'}^{m'}(\theta,\phi)\right)^*\mathrm{d}\Omega \\ &= \frac{Z_0^2}{k^2r^2}\sum_{\ell,\ell',m,m'}i^{\ell'-\ell}\Lambda_E(\ell,m)\Lambda_E^*(\ell',m')\delta_{\ell\ell'}\delta_{mm'} \\ &= \frac{Z_0^2}{k^2r^2}\sum_{\ell,m}\Lambda_E(\ell,m)\Lambda_E^*(\ell,m) \\ &= \frac{Z_0^2}{k^2r^2}\sum_{\ell,m}|\Lambda_E(\ell,m)|^2 \end{aligned} \tag{3.132}$$

By symmetry, we can write

$$\int \vec{E}^{(\mathrm{TE})} \cdot \left(\vec{E}^{(\mathrm{TE})}\right)^* \mathrm{d}\Omega = \frac{Z_0^2}{k^2 r^2} \sum_{\ell,m} |\Lambda_M(\ell,m)|^2 \tag{3.133}$$

Therefore the energy in a spherical shell of radius r is

$$\mathrm{d}U = \frac{Z_0}{2k^2} \sum_{\ell,m} \left(|\Lambda_E(\ell,m)|^2 + |\Lambda_M(\ell,m)|^2\right) \mathrm{d}r \tag{3.134}$$

Note that this energy is independent of the radius, as we would expect for radiating fields.

3.5.4 Angular distribution

Recall that the time-averaged power is related to the Poynting vector by the relation $\langle \mathrm{d}P \rangle = \langle \vec{S} \rangle \cdot \mathrm{d}\vec{A}$. The Poynting vector is given by $\vec{S} = \vec{E} \times \vec{H}$; if the fields are monochromatic time-harmonic, then we can write $\langle \vec{S} \rangle = \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right]$. As in the previous section, we can write a general radiation field as a combination of TE and TM waves. Plugging this into the time-averaged Poynting vector gives

$$\begin{aligned}
\langle \vec{S} \rangle &= \frac{1}{2}\mathrm{Re}\left[\vec{E} \times \vec{H}^*\right] \\
&= \frac{1}{2Z_0}\mathrm{Re}\left[\vec{E} \times \hat{r} \times \vec{E}^*\right] \\
&= \frac{1}{2Z_0}\mathrm{Re}\left[\hat{r}\left(\vec{E} \cdot \vec{E}^*\right) + \vec{E}^* \cancelto{0}{\left(\hat{r} \cdot \vec{E}\right)}\right] \\
&= \frac{1}{2Z_0}\mathrm{Re}\left[\left(\vec{E}^{(\mathrm{TE})} + \vec{E}^{(\mathrm{TM})}\right) \cdot \left(\vec{E}^{(\mathrm{TE})} + \vec{E}^{(\mathrm{TM})}\right)^*\right] \\
&= \frac{1}{Z_0 \epsilon_0} u \hat{r} \\
&= \hat{r} c u
\end{aligned} \tag{3.135}$$

As expected, the Poynting vector is proportional to the energy density, flowing at the speed of light. Consider a pure multipole field (i.e., a single value for ℓ, m); employing the expressions for the multipole fields derived in the previous chapter, then we can write

$$\begin{aligned}
&\vec{E}^{(\mathrm{TE})}_{\ell,m} \cdot \left(\vec{E}^{(\mathrm{TE})}_{\ell,m}\right)^* \\
&= Z_0^2 \frac{\exp[ikr]}{kr}(-i)^{\ell+1}\Lambda_M(\ell,m)\mathbf{\Phi}_\ell^m(\theta,\phi) \cdot \left[\frac{\exp[-ikr]}{kr}(i)^{\ell+1}\Lambda_M^*(\ell,m)\left(\mathbf{\Phi}_\ell^m(\theta,\phi)\right)^*\right] \\
&= \frac{Z_0^2}{k^2 r^2}|\Lambda_M(\ell,m)|^2 \|\mathbf{\Phi}_\ell^m(\theta,\phi)\|^2
\end{aligned} \tag{3.136}$$

and similarly

$$\vec{E}^{(\mathrm{TM})}_{\ell,m} \cdot \left(\vec{E}^{(\mathrm{TM})}_{\ell,m}\right)^* = \frac{Z_0^2}{k^2 r^2} |\Lambda_E(\ell,m)|^2 \|\mathbf{\Psi}^m_\ell(\theta,\phi)\|^2$$
$$= \frac{Z_0^2}{k^2 r^2} |\Lambda_E(\ell,m)|^2 \|\hat{r} \times \mathbf{\Phi}^m_\ell(\theta,\phi)\|^2$$
$$= \frac{Z_0^2}{k^2 r^2} |\Lambda_E(\ell,m)|^2 \|\mathbf{\Phi}^m_\ell(\theta,\phi)\|^2 \tag{3.137}$$

With $\mathrm{d}\vec{A} = \hat{r} r^2 \mathrm{d}\Omega$, we can write

$$\langle \mathrm{d}P \rangle = \langle \vec{S} \rangle \cdot \mathrm{d}\vec{A}$$
$$= (\hat{r} c u) \cdot \hat{r} r^2 \mathrm{d}\Omega$$
$$\Rightarrow \qquad \left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{1}{2Z_0} \left[\vec{E}^{(\mathrm{TM})} \cdot \left(\vec{E}^{(\mathrm{TM})}\right)^* + \vec{E}^{(\mathrm{TE})} \cdot \left(\vec{E}^{(\mathrm{TE})}\right)^* \right] r^2 \mathrm{d}\Omega$$
$$= \frac{Z_0}{2k^2} \left[|\Lambda_M(\ell,m)|^2 + |\Lambda_E(\ell,m)|^2 \right] \|\mathbf{\Phi}^m_\ell(\theta,\phi)\|^2 \tag{3.138}$$

To proceed further, we therefore need an explicit expression for $\|\mathbf{\Phi}^m_\ell(\theta,\phi)\|^2$:

$$\|\mathbf{\Phi}^m_\ell\|^2 = \left\| \frac{1}{\sqrt{\ell(\ell+1)}} \vec{L} Y^m_\ell \right\|^2$$
$$= \frac{1}{\ell(\ell+1)} \left\| \left(\frac{1}{\sqrt{2}} L_+ \hat{e}_+ + \frac{1}{\sqrt{2}} L_- \hat{e}_- + L_0 \hat{e}_0 \right) Y^m_\ell \right\|^2$$
$$= \frac{1}{\ell(\ell+1)} \left\| \frac{1}{\sqrt{2}} \sqrt{\ell(\ell+1) - m(m+1)} Y^{m+1}_\ell \hat{e}_+ + \frac{1}{\sqrt{2}} \sqrt{\ell(\ell+1) - m(m-1)} Y^{m-1}_\ell \hat{e}_- + m Y^m_\ell \hat{e}_0 \right\|^2$$
$$= \frac{1}{\ell(\ell+1)} \left(\frac{\ell(\ell+1) - m(m+1)}{2} |Y^{m+1}_\ell|^2 + \frac{\ell(\ell+1) - m(m-1)}{2} |Y^{m-1}_\ell|^2 + m^2 |Y^m_\ell|^2 \right) \tag{3.139}$$

This gives the angular distribution as

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{Z_0}{2k^2} \frac{1}{\ell(\ell+1)} \left[|\Lambda_M(\ell,m)|^2 + |\Lambda_E(\ell,m)|^2 \right]$$
$$\times \left(\frac{\ell(\ell+1) - m(m+1)}{2} |Y^{m+1}_\ell|^2 + \frac{\ell(\ell+1) - m(m-1)}{2} |Y^{m-1}_\ell|^2 + m^2 |Y^m_\ell|^2 \right) \tag{3.140}$$

If we instead consider a general multipole field (i.e., a superposition of several values for ℓ, m), then we can still write the angular distribution, it just won't be as clean, since it involves cross-terms:

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{Z_0}{2k^2} \sum_{\ell,m} \left\| (-i)^{\ell+1} \left(\Lambda_M(\ell,m) \mathbf{\Phi}^m_\ell(\theta,\phi) + i \Lambda_E(\ell,m) \mathbf{\Psi}^m_\ell(\theta,\phi) \right) \right\|^2 \tag{3.141}$$

and that's as far as we can go. To find the total power, in either case integrate over all angles. Since the spherical harmonics are normalized to unity, this makes

$$
\begin{aligned}
\langle P \rangle &= \int \left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle \mathrm{d}\Omega \\
&= \frac{Z_0}{2k^2} \sum_\ell \frac{1}{\ell(\ell+1)} \left[|\Lambda_M(\ell,m)|^2 + |\Lambda_E(\ell,m)|^2 \right] \left(\frac{\ell(\ell+1) - m(m+1)}{2} + \frac{\ell(\ell+1) - m(m-1)}{2} + m^2 \right) \\
&= \frac{Z_0}{2k^2} \sum_\ell \frac{1}{\ell(\ell+1)} \left[|\Lambda_M(\ell,m)|^2 + |\Lambda_E(\ell,m)|^2 \right] \left(\frac{2\ell(\ell+1) - 2m^2 + 2m^2}{2} \right) \\
&= \frac{Z_0}{2k^2} \sum_\ell \left[|\Lambda_M(\ell,m)|^2 + |\Lambda_E(\ell,m)|^2 \right] \qquad (3.142)
\end{aligned}
$$

The total power will not change between the pure multipole and general cases, since it involves an integral over all variables and thus the orthogonality of the VSH ensures that the cross-terms vanish, restoring the expression shown above. As we've seen, the electric and magnetic multipole moments play a key role in the above expressions. There is a way to connect these moments to the charges and currents generating the radiation.

3.6 Radiation from a localized source

Let's suppose that there exists a collection of charges and currents exhibiting some arbitrary (but relatively well-behaved) time-dependence. We can consider each Fourier component of this collection independently as monochromatic time-harmonic distributions:

$$\rho(\vec{r},t) = \rho(\vec{r})\exp[-i\omega t] \qquad \vec{J}(\vec{r},t) = \vec{J}(\vec{r})\exp[-i\omega t] \tag{3.143}$$

In this regime, the Maxwell equations are

$$\nabla\times\vec{E} = ikZ_0\vec{H} \qquad \nabla\times\vec{H} = \vec{J} - \frac{ik}{Z_0}\vec{E} \tag{3.144}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\cdot\vec{B} = 0 \tag{3.145}$$

Thanks to this time-dependence, the continuity equation takes the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \qquad \Rightarrow \qquad i\omega\rho = \nabla\cdot\vec{J} \tag{3.146}$$

Using this we can write the Gauß law in terms of $\vec{E}' = \vec{E} + \frac{i}{\epsilon_0\omega}\vec{J}$ making it a divergence-free field. And thus the Maxwell equations become

$$\nabla\times\vec{E}' - ikZ_0\vec{H} = \frac{i}{\epsilon_0\omega}\nabla\times\vec{J} \qquad \nabla\times\vec{H} + \frac{ik}{Z_0}\vec{E}' = 0 \tag{3.147}$$

$$\nabla\cdot\vec{E}' = 0 \qquad \nabla\cdot\vec{B} = 0 \tag{3.148}$$

Combining the Ampère-Maxwell and Faraday laws and introducing Z_0 gives inhomogeneous Helmholtz equations:

$$\left(\nabla^2 + k^2\right)\vec{H} = -\nabla\times\vec{J} \qquad \left(\nabla^2 + k^2\right)\vec{E}' = -\frac{iZ_0}{k}\nabla\times\nabla\times\vec{J} \tag{3.149}$$

These fields are divergenceless and therefore the fields are completely determined by the scalars $u = \vec{r}\cdot\vec{E}'$ and $w = \vec{r}\cdot\vec{H}$. Dotting $\vec{r}$ on both sides and using the identity $\vec{r}\cdot\left(\nabla\times\vec{A}\right) = (\vec{r}\times\nabla)\cdot\vec{A}$ gives

$$\left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{H}\right) = -\vec{r}\cdot\left(\nabla\times\vec{J}\right) = -\left(\vec{r}\times\nabla\right)\cdot\vec{J} \tag{3.150}$$

$$\left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{E}'\right) = -\frac{iZ_0}{k}\vec{r}\cdot\nabla\times\nabla\times\vec{J} = -\frac{iZ_0}{k}\left(\vec{r}\times\nabla\right)\cdot\left(\nabla\times\vec{J}\right) \tag{3.151}$$

Then, using the vector identities

$$\left(\vec{r}\times\nabla\right)\cdot\vec{J} = \nabla\cdot\left(\vec{r}\times\vec{J}\right) \tag{3.152}$$

$$\left(\vec{r}\times\nabla\right)\cdot\left(\nabla\times\vec{J}\right) = \nabla^2\left(\vec{r}\cdot\vec{J}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\nabla\cdot\vec{J}\right) \tag{3.153}$$

and the continuity equation and introduce the notation $J_\parallel = \vec{r}\cdot\vec{J}$ and $\vec{J}_\perp = \vec{r}\times\vec{J}$, then we can at last write

$$\left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{H}\right) = -\nabla\cdot\vec{J}_\perp \tag{3.154}$$

$$\left(\nabla^2 + k^2\right)\left(\vec{r}\cdot\vec{E}'\right) = -\frac{iZ_0}{k}\nabla^2 J_\parallel + \frac{cZ_0}{r}\frac{\partial}{\partial r}\left(r^2\rho\right) \tag{3.155}$$

This is nothing more than an inhomogeneous scalar Helmholtz equation, the solution for which we can write using a Green function:

$$\begin{aligned}\vec{r}\cdot\vec{H} &= -ik\int\sum_{\ell,m} j_\ell(kr_<)h^{(1)}(kr_>)\left(Y_\ell^m(\theta',\phi')\right)^* Y_\ell^m(\theta,\phi)\nabla\cdot\vec{J}_\perp(\vec{r}')\mathrm{d}^3x' \\ &= -ik\sum_{\ell,m}\Bigg[h_\ell^{(1)}(kr)\int\int_0^r \nabla\cdot\vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^* r'^2\mathrm{d}r'\mathrm{d}\Omega' \\ &\qquad + j_\ell(kr)\int\int_r^\infty \nabla\cdot\vec{J}_\perp h_\ell^{(1)}(kr')\left(Y_\ell^m(\theta',\phi')\right)^* r'^2\mathrm{d}r'\mathrm{d}\Omega'\Bigg] Y_\ell^m(\theta,\phi)\end{aligned} \tag{3.156}$$

Now we assume that the current and charge distributions are very localized and therefore we can, analogously to the static multipole expansion, expand the interior integral to all space and discard the exterior integral to give

$$\vec{r}\cdot\vec{H} = -ik\sum_{\ell,m} h_\ell^{(1)}(kr)\left[\int\int_0^\infty \nabla\cdot\vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^* r'^2\mathrm{d}r'\mathrm{d}\Omega'\right] Y_\ell^m(\theta,\phi) \tag{3.157}$$

Analogously, we can also write

$$\vec{r}\cdot\vec{E}' = ik\sum_{\ell,m} h_\ell^{(1)}(kr)\left[\int\int_0^\infty \left(-\frac{iZ_0}{k}\nabla^2 J_\parallel + \frac{cZ_0}{r}\frac{\partial}{\partial r}\left(r^2\rho\right)\right) j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^* r'^2\mathrm{d}r'\mathrm{d}\Omega'\right] Y_\ell^m(\theta,\phi) \tag{3.158}$$

Earlier, we derived expressions for Λ_E and Λ_M using the projection properties of the VSH:

$$\Lambda_M(\ell,m)g_\ell(kr) = -\frac{k}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^m(\theta,\phi)\right)^*\vec{r}\cdot\vec{B}\mathrm{d}\Omega$$

$$Z_0\Lambda_E(\ell,m)f_\ell(kr) = \frac{k}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^m(\theta,\phi)\right)^*\vec{r}\cdot\vec{E}\mathrm{d}\Omega$$

Plugging in the expression for $\vec{r}\cdot\vec{B}$ into the first gives

$$\begin{aligned}
\Lambda_M(\ell,m)h_\ell^{(1)}(kr) &= -\frac{k}{\sqrt{\ell(\ell+1)}}\int \left(Y_\ell^m(\theta,\phi)\right)^* \vec{r}\cdot\vec{B}\mathrm{d}\Omega \\
&= -\frac{ik^2}{\sqrt{\ell(\ell+1)}}\int \left(Y_\ell^m(\theta,\phi)\right)^* \sum_{\ell,m} h_\ell^{(1)}(kr)\left[\int \nabla\cdot\vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}^3x'\right] Y_\ell^m(\theta,\phi)\mathrm{d}\Omega \\
&= -\frac{ik^2}{\sqrt{\ell(\ell+1)}}\sum_{\ell,m} h_\ell^{(1)}(kr)\left[\int \nabla\cdot\vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}^3x'\right]\int \left(Y_\ell^m(\theta,\phi)\right)^* Y_\ell^m(\theta,\phi)\mathrm{d}\Omega \\
&= -\frac{ick^2}{\sqrt{\ell(\ell+1)}} h_\ell^{(1)}(kr)\left[\int \nabla\cdot\vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}^3x'\right]
\end{aligned}$$

$$\Rightarrow \qquad \Lambda_M(\ell,m) = -\frac{ik^2}{\sqrt{\ell(\ell+1)}}\int \nabla\cdot\vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}^3x' \tag{3.159}$$

Repeating the process with Λ_E gives

$$\Lambda_E(\ell,m) = \frac{ik^2}{\sqrt{\ell(\ell+1)}}\int\left(-\frac{i}{k}\nabla^2 J_\parallel - \frac{c}{r'}\frac{\partial}{\partial r'}\left(r'^2\rho\right)\right) j_\ell(kr')\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}^3x' \tag{3.160}$$

Now these expressions can be cleaned up; using Green's second identity, and using the fact that spherical waves vanish at infinity, the Laplacian in the first term can be replaced by a factor of $-k^2$. For the second term, using integration by parts and the assumption that the charge is sufficiently localized that the surface term vanishes, we can write

$$\begin{aligned}
\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\rho\right) j_\ell(kr')r^2\mathrm{d}r &= \int_0^\infty r j_\ell(kr)\frac{\partial}{\partial r}\left(r^2\rho\right)\mathrm{d}r \\
&= \cancelto{0}{r^3\rho j_\ell(kr)\Big]_0^\infty} - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left[r j_\ell(kr)\right]\rho r^2\mathrm{d}r \\
&= -\int_0^\infty \rho\frac{\mathrm{d}}{\mathrm{d}r}\left[r j_\ell(kr)\right] r^2\mathrm{d}r
\end{aligned}$$

Thus the electric multipole becomes

$$\Lambda_E(\ell,m) = \frac{ik^2}{\sqrt{\ell(\ell+1)}}\int\left(ikJ_\parallel j_\ell(kr') + c\rho\frac{\mathrm{d}}{\mathrm{d}r'}\left[r' j_\ell(kr')\right]\right)\left(Y_\ell^m(\theta',\phi')\right)^*\mathrm{d}^3x' \tag{3.161}$$

Notice that the magnetic multipole is determined solely by $\vec{J}_\perp$, the transverse current density oscillations, while the electric multipole is determined by $J_\parallel$, the radial current density oscillations and the oscillations of the charge density. In summary, we write

The inhomogeneous Maxwell equations with monochromatic time harmonic fields are of the form

$$\nabla \times \vec{E} = ikZ_0\vec{H} \qquad \nabla \times \vec{H} = \vec{J} - \frac{ik}{Z_0}\vec{E} \tag{3.162}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{H} = 0 \tag{3.163}$$

Together with the continuity equation

$$\nabla \cdot \vec{J} = i\omega\rho \tag{3.164}$$

these imply the inhomogeneous Helmholtz equations obeyed by the fields

$$\left(\nabla^2 + k^2\right)\vec{H} = -\nabla \times \vec{J} \qquad \left(\nabla^2 + k^2\right)\vec{E}' = -\frac{iZ_0}{k}\nabla \times \nabla \times \vec{J} \tag{3.165}$$

These can be solved by means of (several) vector identities, to ultimately yield the full electric and magnetic multipoles

$$\Lambda_E(\ell, m) = \frac{ik^2}{\sqrt{\ell(\ell+1)}}\int \left(ikJ_{\parallel}j_\ell(kr') + c\rho\frac{\mathrm{d}}{\mathrm{d}r'}\left[r'j_\ell(kr')\right]\right)\left(Y_\ell^m(\theta', \phi')\right)^* \mathrm{d}^3x' \tag{3.166}$$

$$\Lambda_M(\ell, m) = -\frac{ik^2}{\sqrt{\ell(\ell+1)}}\int \nabla \cdot \vec{J}_\perp j_\ell(kr')\left(Y_\ell^m(\theta', \phi')\right)^* \mathrm{d}^3x' \tag{3.167}$$

As a final note, I point out that the current contribution to Λ_E is smaller than the charge contribution by a factor of $\frac{1}{c}$, and we established earlier that the Λ_E terms are stronger than the Λ_M terms by a factor of Z_0, meaning that the charge distribution dominates multipole radiation, followed by the perpendicular current distribution, followed by the parallel current distribution. This establishes the usual $E1 - M1 - E2 - \ldots$ hierarchy of multipole radiation usually seen in textbooks.

3.7 Static approximations of multipole fields

Fields in the near zone ($kr \to 0$) arise in two separate cases. The first is the literal case where approximations are applied to the radiating fields in the near vicinity of a radiating source of charges/currents; this is equivalent to $\lambda \to \infty$, which gives rise to calling this the *long wavelength approximation*. The second is as a recovery of the static case considered in the previously. Since $k \to 0$ reduces the Helmholtz operator to the Laplacian, then we would expect our multipole fields to reduce to the static fields we encountered earlier as solutions to the Poisson equation. As a reminder, recall that electrostatic fields have no curl $\nabla \times \vec{E} = 0$, and are of the form

$$\vec{E} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{q_{\ell m}}{4\pi\epsilon_0 r^{\ell+2}} \frac{4\pi}{(2\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.168}$$

where

$$q_{\ell m} = \int r'^{\ell} \rho(\vec{r}') \left(Y_\ell^m(\theta',\phi') \right)^* \mathrm{d}^3 x' \tag{3.169}$$

Magnetostatic fields are the curl of a divergence free field $\vec{B} = \nabla \times \vec{A}$, $\nabla \cdot \vec{A} = 0$, and therefore have the form

$$\vec{H} = -i \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\mu_0 I_{\ell m}}{4\pi r^{\ell+2}} \frac{\sqrt{\ell(\ell+1)}}{2\ell+1} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} r^{\ell+2} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.170}$$

where

$$I_{\ell m} = \int r'^{\ell} \left(\mathbf{\Phi}_\ell^m(\theta',\phi') \right)^* \cdot \vec{J}(\vec{r}') \mathrm{d}^3 x' \tag{3.171}$$

For $kr \to 0$, we may consider the asymptotic forms of f_ℓ and g_ℓ; in this limit we can write

$$f_\ell, g_\ell \sim n_\ell \to -\frac{(2\ell-1)!!}{x^{\ell+1}} \qquad \text{and} \qquad j_\ell \to \frac{x^\ell}{(2\ell+1)!!} \tag{3.172}$$

Note that we're explicitly considering a region that does not include the origin (since we're close to a radiating source, not inside it), so we need not worry about regularity of the solution. The TE electric field/TM magnetic field become

$$\vec{E}^{(\mathrm{TE})} = -Z_0 \sum_{\ell,m} \Lambda_M(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+1}} \mathbf{\Phi}_\ell^m(\theta,\phi) \qquad \text{and} \qquad \vec{H}^{(\mathrm{TM})} = -\sum_{\ell,m} \Lambda_E(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+1}} \mathbf{\Phi}_\ell^m(\theta,\phi) \tag{3.173}$$

and the affiliated TE magnetic field is

$$
\begin{aligned}
\vec{H}^{(\mathrm{TE})} &= \sum_{\ell,m} \Lambda_M(\ell,m) \left[\frac{\sqrt{\ell(\ell+1)}}{kr} \frac{(2\ell-1)!!}{(kr)^{\ell+1}} \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr} \frac{\mathrm{d}}{\mathrm{d}(kr)} \left(kr \frac{(2\ell-1)!!}{(kr)^{\ell+1}} \right) \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&= \sum_{\ell,m} \Lambda_M(\ell,m)(2\ell-1)!! \left[\frac{\sqrt{\ell(\ell+1)}}{kr} \frac{1}{(kr)^{\ell+1}} \mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr} \left(-\frac{\ell}{(kr)^{\ell+1}} \right) \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&= \sum_{\ell,m} \Lambda_M(\ell,m)(2\ell-1)!! \left[\frac{\sqrt{\ell(\ell+1)}}{(kr)^{\ell+2}} \mathbf{Y}_\ell^m(\theta,\phi) - \frac{\ell}{(kr)^{\ell+2}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&= \sum_{\ell,m} \Lambda_M(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right]
\end{aligned}
\tag{3.174}
$$

and by symmetry, the affiliated TM electric field is

$$
\vec{E}^{(\mathrm{TM})} = Z_0 \sum_{\ell,m} \Lambda_E(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.175}
$$

These fields are of the form expected above for electrostatic and magnetostatic fields! As we can see, the TE magnetic field is stronger than the TE electric field by a factor of $\frac{1}{kr}$ and therefore the TE magnetic field dominates in the near zone; it is safe to say that in the near zone only TE magnetic multipole fields are relevant. Conversely, the TM electric field is stronger than the TM magnetic field by a factor of $\frac{1}{kr}$ and therefore the TM electric field dominates in the near zone:

$$
\vec{E} \approx \vec{E}^{(\mathrm{TM})} = Z_0 \sum_{\ell,m} \Lambda_E(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.176}
$$

$$
\vec{H} \approx \vec{H}^{(\mathrm{TE})} = \sum_{\ell,m} \Lambda_M(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \tag{3.177}
$$

Now this is not the only approximation we need to make; we must also approximate the full electric and magnetic multipoles Λ_E and Λ_M and see how they approximate the static multipoles $q_{\ell m}$ and $I_{\ell m}$. Using the approximation for the spherical Bessel function we find that the electric multipole becomes

$$
\begin{aligned}
\Lambda_E(\ell,m) &= \frac{ik^2}{\sqrt{\ell(\ell+1)}} \int \left(ikJ_\parallel j_\ell(kr') + c\rho \frac{\mathrm{d}}{\mathrm{d}r'} \left[r' j_\ell(kr') \right] \right) \left(Y_\ell^m(\theta',\phi') \right)^* \mathrm{d}^3x' \\
&\approx \frac{ik^2}{\sqrt{\ell(\ell+1)}} \int \left(ikJ_\parallel \frac{(kr')^\ell}{(2\ell+1)!!} + c\rho \frac{\mathrm{d}}{\mathrm{d}(kr')} \left[kr' \frac{(kr')^\ell}{(2\ell+1)!!} \right] \right) \left(Y_\ell^m(\theta',\phi') \right)^* \mathrm{d}^3x' \\
&= \frac{ik^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \int r'^\ell \left(ikJ_\parallel + c(\ell+1)\rho(\vec{r}') \right) \left(Y_\ell^m(\theta',\phi') \right)^* \mathrm{d}^3x' \\
&= \frac{ik^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \left[c(\ell+1) q_{\ell m} + ikr \int r'^\ell \hat{r} \cdot \vec{J} \left(Y_\ell^m(\theta',\phi') \right)^* \mathrm{d}^3x' \right]
\end{aligned}
\tag{3.178}
$$

This is very nice; we've recovered the electrostatic charge moments $q_{\ell m}$ so that the full electric multipole is approximately equal to the electrostatic multipole. Now note that if $kr \to 0$, then the second term arising from the current density is comfortably negligible; plugging this multipole into the approximate electric field gives

$$\begin{aligned}
\vec{E} &\approx Z_0 \sum_{\ell,m} \Lambda_E(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&\approx Z_0 \sum_{\ell,m} \frac{ik^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} c(\ell+1) q_{\ell m} \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&= \sum_{\ell,m} \frac{(\ell+1)}{2\ell+1} icZ_0 \frac{q_{\ell m}}{r^{\ell+2}} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&= i \sum_{\ell,m} \frac{4\pi(\ell+1)}{2\ell+1} \frac{q_{\ell m}}{4\pi\epsilon_0 r^{\ell+2}} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \qquad (3.179)
\end{aligned}$$

Comparing this to the original electric field, we see that the long-wavelength approximate multipole field is incredibly close to the electrostatic multipole field, up to a multiplicative factor term-by-term. Doing this same approximation, using the first Green identity and assuming the charge density is sufficiently localized, the magnetic multipole gives

$$\begin{aligned}
\Lambda_M(\ell,m) &= -\frac{ik^2}{\sqrt{\ell(\ell+1)}} \int \nabla \cdot \vec{J}_\perp j_\ell(kr') \left(Y_\ell^m(\theta',\phi')\right)^* \mathrm{d}^3x' \\
&\approx -\frac{ik^2}{\sqrt{\ell(\ell+1)}} \int \nabla \cdot \left(\vec{r} \times \vec{J}\right) \frac{(kr')^\ell}{(2\ell+1)!!} \left(Y_\ell^m(\theta',\phi')\right)^* \mathrm{d}^3x' \\
&= -\frac{ik^2}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \left[\oint \left(\hat{r} \cdot \vec{J}_\perp\right) (kr')^\ell \left(Y_\ell^m(\theta',\phi')\right)^* \mathrm{d}A' - \int \vec{J}_\perp \cdot \nabla \left[(kr')^\ell \left(Y_\ell^m(\theta',\phi')\right)^* \right] \mathrm{d}^3x' \right] \\
&= \frac{ik^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \int \left(\vec{r} \times \vec{J}\right) \cdot \left[\ell r'^{\ell-1} \hat{r} \left(Y_\ell^m(\theta',\phi')\right)^* + r'^\ell \left(\nabla Y_\ell^m(\theta',\phi')\right)^* \right] \mathrm{d}^3x' \\
&= \frac{ik^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \int r'^\ell \left(\vec{r} \times \vec{J}\right) \cdot \left(\nabla Y_\ell^m(\theta',\phi')\right)^* \mathrm{d}^3x' \\
&= \frac{k^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \int r'^\ell \vec{J} \cdot \left(\frac{1}{i} \vec{r} \times \nabla Y_\ell^m(\theta',\phi') \right)^* \mathrm{d}^3x' \\
&= \frac{k^{\ell+2}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \int r'^\ell \vec{J} \cdot \left(\mathbf{\Phi}_\ell^m(\theta',\phi')\right)^* \mathrm{d}^3x' \\
&= \frac{k^{\ell+2} I_{\ell m}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \qquad (3.180)
\end{aligned}$$

As with the electric field, let's plug this into the approximate magnetic field to give

$$
\begin{aligned}
\vec{H} &\approx \sum_{\ell,m} \Lambda_M(\ell,m) \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&\approx \sum_{\ell,m} \frac{k^{\ell+2} I_{\ell m}}{(2\ell+1)!!\sqrt{\ell(\ell+1)}} \frac{(2\ell-1)!!}{(kr)^{\ell+2}} \sqrt{\ell(\ell+1)} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right] \\
&= \sum_{\ell,m} \frac{1}{(2\ell+1)} \frac{I_{\ell m}}{r^{\ell+2}} \left[\mathbf{Y}_\ell^m(\theta,\phi) - \sqrt{\frac{\ell}{\ell+1}} \mathbf{\Psi}_\ell^m(\theta,\phi) \right]
\end{aligned} \tag{3.181}
$$

and as with the electric field, we've (almost) recovered the magnetostatic multipole field, up to a multiplicative factor, but the functional dependence is all in place.

4 Applications

4.1 Dipole antenna

Now that we've fully laid out the machinery to describe radiation fields, their approximations and their properties, it's time to consider an illustrative example. The example I'll work through here is the center-fed linear antenna, better-known as the dipole antenna. This example is nice for several reasons. The first is that it has a very nice current density that can be constructed from simple assumptions; furthermore this current density is nice enough that it can be used to compute an exact radiation zone solution to the Maxwell equations, no approximation required; this is equivalent to saying that the sum over multipoles adds up to something expressible in closed form. The current density also plays nicely with the projection onto VSH and thus easy comparison between the exact and approximate solutions is possible.

Let's begin by constructing the current density. Without loss of generality, we can orient the antenna along the z-axis. Since the current density only occupies a finite length along a long, its current density will involve a pair of δ-functions in the x and y directions, along with a Heaviside Θ-function to ensure the finiteness of the segment. Lastly, to make the field oscillate spatially, we require a sinusoidal function in z. Putting these together, we write

$$\vec{J} = I_0 \exp[-i\omega t]\delta(x)\delta(y)\Theta\left(\tfrac{d}{2} - |z|\right)\sin\left(\tfrac{kd}{2} - k|z|\right)\hat{z} \tag{4.1}$$

In spherical coordinates, this can equivalently be written

$$\vec{J} = \frac{I(r)}{2\pi r^2}\exp[-i\omega t]\left[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)\right]\hat{r} \tag{4.2}$$

Using the fact that the current and charge densities are time-harmonic, the continuity equation thus gives

$$i\omega\rho = \nabla\cdot\vec{J} \qquad \Rightarrow \qquad \rho = \frac{\exp[-i\omega t]}{i\omega}\frac{\mathrm{d}I}{\mathrm{d}r}\left[\frac{\delta(\cos\theta - 1) - \delta(\cos\theta + 1)}{2\pi r^2}\right] \tag{4.3}$$

Now recall that the electric and magnetic fields are generated by the scalar and vector potentials, which in the Lorenz gauge, satisfy the wave equation. If we assume monochromatic time-harmonic sources, these yield the usual Helmholtz equation:

$$\left(\nabla^2 + k^2\right)\Phi = -\frac{\rho}{\epsilon_0} \qquad\qquad \left(\nabla^2 + k^2\right)\vec{A} = \mu_0\vec{J} \tag{4.4}$$

The solution to these can be written using Green functions:

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{r}',t')\frac{\exp[ik\|\vec{r}-\vec{r}'\|]}{\|\vec{r}-\vec{r}'\|}\mathrm{d}^3x' \quad \text{and} \quad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}',t')\frac{\exp[ik\|\vec{r}-\vec{r}'\|]}{\|\vec{r}-\vec{r}'\|}\mathrm{d}^3x' \tag{4.5}$$

In the radiation zone, only the vector potential contributes meaningfully to the electric and magnetic fields and thus by Taylor expanding the separation $\|\vec{r}-\vec{r}'\|$ we can write

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{\exp[ikr]}{r}\int \vec{J}(\vec{r}',t)\exp[-ik\hat{n}\cdot\vec{r}']\mathrm{d}^3x' \tag{4.6}$$

$$\vec{H} = -\frac{1}{Z_0}\hat{r}\times\frac{\partial\vec{A}}{\partial t} \qquad \text{and} \qquad \vec{E} = -Z_0\hat{r}\times\vec{H} \tag{4.7}$$

Thus the time-averaged angular distribution of power can be written as

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{1}{2Z_0}\left\|\vec{r}\times\tfrac{\partial\vec{A}}{\partial t}\right\|^2 \tag{4.8}$$

Thus to find the exact distribution, we need to compute $\vec{A}$:

$$\begin{aligned}
\vec{A}\left(\vec{r}\right) &= \frac{\mu_0}{4\pi}\frac{\exp[ikr]}{r}\int \vec{J}\left(\vec{r}'\right)\exp[-ik\hat{n}\cdot\vec{r}']\mathrm{d}^3x' \\
&= \frac{\mu_0 I_0}{4\pi}\frac{\exp[ikr]}{r}\hat{z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x')\delta(y')\Theta\left(\tfrac{d}{2}-|z'|\right)\sin\left(\tfrac{kd}{2}-k|z'|\right)\exp\left[-ik\left(\tfrac{xx'}{r}+\tfrac{yy'}{r}+\tfrac{zz'}{r}\right)\right]\mathrm{d}z'\mathrm{d}y'\mathrm{d}x' \\
&= \frac{\mu_0 I_0}{4\pi}\frac{\exp[ikr]}{r}\hat{z}\int_{-\frac{d}{2}}^{\frac{d}{2}}\sin\left(\tfrac{kd}{2}-k|z'|\right)\exp\left[-ik\tfrac{zz'}{r}\right]\mathrm{d}z' \\
&= \frac{\mu_0 I_0}{4\pi}\frac{\exp[ikr]}{r}\hat{z}\frac{1}{2i}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(\exp\left[i\left(\tfrac{kd}{2}-k|z'|\right)\right]-\exp\left[-i\left(\tfrac{kd}{2}+k|z|\right)\right]\right)\exp[-ikz'\cos\theta]\mathrm{d}z' \\
&= \frac{\mu_0 I_0}{8\pi i}\frac{\exp[ikr]}{r}\hat{z}\left(\int_{-\frac{d}{2}}^{\frac{d}{2}}\exp\left[i\left(\tfrac{kd}{2}-k|z'|-kz'\cos\theta\right)\right]\mathrm{d}z' - \int_{-\frac{d}{2}}^{\frac{d}{2}}\exp\left[i\left(-\tfrac{kd}{2}+k|z'|-kz'\cos\theta\right)\right]\mathrm{d}z'\right) \\
&= \frac{\mu_0 I_0}{8\pi i}\frac{\exp[ikr]}{r}\hat{z}\left(\int_{-\frac{d}{2}}^{0}\exp\left[i\left(\tfrac{kd}{2}+kz'-kz'\cos\theta\right)\right]\mathrm{d}z' + \int_{0}^{\frac{d}{2}}\exp\left[i\left(\tfrac{kd}{2}-kz'-kz'\cos\theta\right)\right]\mathrm{d}z'\right. \\
&\qquad\left. - \int_{-\frac{d}{2}}^{0}\exp\left[i\left(-\tfrac{kd}{2}-kz'-kz'\cos\theta\right)\right]\mathrm{d}z' - \int_{0}^{\frac{d}{2}}\exp\left[i\left(-\tfrac{kd}{2}+kz'-kz'\cos\theta\right)\right]\mathrm{d}z'\right)
\end{aligned}$$

Realizing a change of variables in the first and third terms allows us to recombine the integrals into easy-to-do sines:

$$\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_0 I_0}{8\pi i}\frac{\exp[ikr]}{r}\hat{z}\left(\int_{\frac{d}{2}}^{0}\exp\left[i\left(\tfrac{kd}{2}-kz'+kz'\cos\theta\right)\right](-\mathrm{d}z') + \int_0^{\frac{d}{2}}\exp\left[i\left(\tfrac{kd}{2}-kz'-kz'\cos\theta\right)\right]\mathrm{d}z'\right.\\ &\qquad\left.-\int_{\frac{d}{2}}^{0}\exp\left[i\left(-\tfrac{kd}{2}+kz'+kz'\cos\theta\right)\right](-\mathrm{d}z') - \int_0^{\frac{d}{2}}\exp\left[i\left(-\tfrac{kd}{2}+kz'-kz'\cos\theta\right)\right]\mathrm{d}z'\right)\\ &= \frac{\mu_0 I_0}{8\pi i}\frac{\exp[ikr]}{r}\hat{z}\left(\int_0^{\frac{d}{2}}\exp\left[i\left(\tfrac{kd}{2}-kz'+kz'\cos\theta\right)\right]\mathrm{d}z' + \int_0^{\frac{d}{2}}\exp\left[i\left(\tfrac{kd}{2}-kz'-kz'\cos\theta\right)\right]\mathrm{d}z'\right.\\ &\qquad\left.-\int_0^{\frac{d}{2}}\exp\left[-i\left(\tfrac{kd}{2}-kz'-kz'\cos\theta\right)\right]\mathrm{d}z' - \int_0^{\frac{d}{2}}\exp\left[-i\left(\tfrac{kd}{2}-kz'+kz'\cos\theta\right)\right]\mathrm{d}z'\right)\\ &= \frac{\mu_0 I_0}{4\pi}\frac{\exp[ikr]}{r}\hat{z}\left(\int_0^{\frac{d}{2}}\sin\left(\tfrac{kd}{2}-kz'(1-\cos\theta)\right)\mathrm{d}z' + \int_0^{\frac{d}{2}}\sin\left(\tfrac{kd}{2}-kz'(1+\cos\theta)\right)\mathrm{d}z'\right)\end{aligned}$$

Evaluating the integrals and combining over like denominators gives

$$\begin{aligned}\vec{A}(\vec{r}) &= \frac{\mu_0 I_0}{4\pi}\frac{\exp[ikr]}{r}\hat{z}\left(\frac{\cos\left(\tfrac{kd}{2}-kz'(1-\cos\theta)\right)}{k(1-\cos\theta)}\Bigg]_0^{\frac{d}{2}} + \frac{\cos\left(\tfrac{kd}{2}-kz'(1+\cos\theta)\right)}{k(1+\cos\theta)}\Bigg]_0^{\frac{d}{2}}\right)\\ &= \frac{\mu_0 I_0}{4\pi}\frac{\exp[ikr]}{r}\hat{z}\left(\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)}{k(1-\cos\theta)} + \frac{\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)}{k(1+\cos\theta)}\right)\\ &= \frac{\mu_0 I_0}{2\pi}\frac{\exp[ikr]}{kr}\hat{z}\left(\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)}{\sin^2\theta}\right)\end{aligned}\tag{4.9}$$

Therefore the angular distribution is

$$\begin{aligned}\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle &= \frac{1}{2Z_0}\left\|\vec{r}\times\tfrac{\partial\vec{A}}{\partial t}\right\|^2\\ &= \frac{\omega^2\mu_0^2 I_0^2}{8\pi^2 Z_0}\frac{1}{k^2r^2}\left\|\vec{r}\times\hat{z}\right\|^2\left|\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)}{\sin^2\theta}\right|^2\\ &= \frac{c^2\mu_0^2 I_0^2}{8\pi^2 Z_0}\left\|\hat{r}\times\left(\hat{r}\cos\theta-\hat{\theta}\sin\theta\right)\right\|^2\left|\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)}{\sin^2\theta}\right|^2\\ &= \frac{Z_0 I_0^2}{8\pi^2}\left|\frac{\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)}{\sin\theta}\right|^2\end{aligned}\tag{4.10}$$

The total power is found by integrating the above expression over all angles:

$$P = \frac{Z_0 I_0^2}{4\pi}\int_0^{\pi}\frac{\left(\cos\left(\tfrac{kd}{2}\cos\theta\right)-\cos\left(\tfrac{kd}{2}\right)\right)^2}{\sin\theta}\mathrm{d}\theta \tag{4.11}$$

This must be done numerically, so I won't bother here; we'll plot this distribution against those obtained by our multipole methods in a bit.

Now we employ the multipole expansion. Recall that we derived an expression relating the charge and current densities to the multipole moments Λ_E and Λ_M. Since $\vec{J} \propto \hat{r}$, then $\vec{J}_\perp = \vec{r} \times \vec{J} = 0$ and therefore $\Lambda_M = 0$ for all ℓ, m. Thus we need to only consider Λ_E:

$$
\begin{aligned}
\Lambda_E(\ell, m) &= \frac{ik^2}{\sqrt{\ell(\ell+1)}} \int \left(ikJ_\parallel j_\ell(kr') + c\rho \frac{\mathrm{d}}{\mathrm{d}r'} \left[r' j_\ell(kr') \right] \right) \left(Y_\ell^m(\theta', \phi') \right)^* \mathrm{d}^3x' \\
&= \frac{ik^2}{\sqrt{\ell(\ell+1)}} \int \left(ik \frac{I(r')}{2\pi r'} \exp[-i\omega t] \left[\delta(\cos\theta' - 1) - \delta(\cos\theta' + 1) \right] j_\ell(kr') \right. \\
&\qquad \left. + c \frac{\exp[-i\omega t]}{i\omega} \frac{\mathrm{d}I}{\mathrm{d}r'} \left[\frac{\delta(\cos'\theta - 1) - \delta(\cos'\theta + 1)}{2\pi r'^2} \right] \frac{\mathrm{d}}{\mathrm{d}r'} \left[r' j_\ell(kr') \right] \right) \left(Y_\ell^m(\theta', \phi') \right)^* \mathrm{d}^3x' \\
&= \frac{k^2 \exp[-i\omega t]}{2\pi \sqrt{\ell(\ell+1)}} \int_0^{\frac{d}{2}} \left(-kr' I(r') j_\ell(kr') + \frac{1}{k} \frac{\mathrm{d}I}{\mathrm{d}r'} \frac{\mathrm{d}}{\mathrm{d}r'} \left[r' j_\ell(kr') \right] \right) \mathrm{d}r' \\
&\qquad \times \int \left[\delta(\cos\theta' - 1) - \delta(\cos\theta' + 1) \right] \left(Y_\ell^m(\theta', \phi') \right)^* \mathrm{d}\Omega'
\end{aligned}
$$

The absence of ϕ-dependence in the charge/current distribution (due to the axial symmetry of the problem) means that $\Lambda_E(\ell, m) = 0$ for $m \neq 0$. Therefore, the angular integral yields

$$
\int \left[\delta(\cos\theta' - 1) - \delta(\cos\theta' + 1) \right] \left(Y_\ell^m(\theta', \phi') \right)^* \mathrm{d}\Omega' = 2\pi \left[Y_\ell^0(0) - Y_\ell^0(\pi) \right]
$$

Then using the explicit expressions for the SSH in Appendix A, and the fact that the Legendre polynomials are odd for odd ℓ, we can write

$$
\begin{aligned}
Y_\ell^0(0) - Y_\ell^0(\pi) &= \sqrt{\frac{2\ell+1}{4\pi}} P_\ell^0(1) - \sqrt{\frac{2\ell+1}{4\pi} \frac{\ell!}{(\ell+1)!}} P_\ell^0(-1) \\
&= \sqrt{\frac{2\ell+1}{4\pi}} \left(P_\ell(1) - (-1)^\ell P_\ell(1) \right) \\
&= 2\sqrt{\frac{2\ell+1}{4\pi}}
\end{aligned}
$$

Thus for the angular integral we get

$$
\int \left[\delta(\cos\theta' - 1) - \delta(\cos\theta' + 1) \right] \left(Y_\ell^m(\theta', \phi') \right)^* \mathrm{d}\Omega' = \sqrt{4\pi(2\ell+1)} \tag{4.12}
$$

We can clean up the radial piece by integrating by parts

$$
\begin{aligned}
&\int_0^{\frac{d}{2}} \left(-kr' I(r') j_\ell(kr') + \frac{1}{k} \frac{\mathrm{d}I}{\mathrm{d}r'} \frac{\mathrm{d}}{\mathrm{d}r'} \left[r' j_\ell(kr') \right] \right) \mathrm{d}r' \\
&= \int_0^{\frac{d}{2}} \left(-kr' I(r') j_\ell(kr') + \frac{1}{k} \frac{\mathrm{d}}{\mathrm{d}r'} \left[r' \frac{\mathrm{d}I}{\mathrm{d}r'} j_\ell(kr') \right] - \frac{1}{k} \frac{\mathrm{d}^2 I}{\mathrm{d}r^2} r' j_\ell(kr') \right) \mathrm{d}r' \\
&= \frac{1}{k} \int_0^{\frac{d}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}r'} \left[r' \frac{\mathrm{d}I}{\mathrm{d}r'} j_\ell(kr') \right] - \left(\frac{\mathrm{d}^2 I}{\mathrm{d}r^2} + k^2 I(r')^2 \right) r' j_\ell(kr') \right) \mathrm{d}r'
\end{aligned}
$$

Putting these all together yields

$$\Lambda_E(\ell,0) = \exp[-i\omega t]\frac{k}{2\pi}\sqrt{\frac{4\pi(2\ell+1)}{\ell(\ell+1)}}\int_0^{\frac{d}{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r'}\left[r'\frac{\mathrm{d}I}{\mathrm{d}r'}j_\ell(kr')\right] - \left(\frac{\mathrm{d}^2 I}{\mathrm{d}r^2} + k^2 I(r')^2\right)r' j_\ell(kr')\right)\mathrm{d}r' \quad (4.13)$$

Now we know that $I(r') = I_0 \sin\left(\frac{kd}{2} - kr|\cos\theta|\right)$; but we've restricted ourselves to $\theta = 0, \pi$ so $I(r') = I_0 \sin\left(\frac{kd}{2} - kr\right)$ and thus the second term vanishes identically and we're left with

$$\begin{aligned}
\Lambda_E(\ell,0) &= \exp[-i\omega t]\frac{k}{2\pi}\sqrt{\frac{4\pi(2\ell+1)}{\ell(\ell+1)}}\int_0^{\frac{d}{2}}\frac{\mathrm{d}}{\mathrm{d}r'}\left[r'\frac{\mathrm{d}I}{\mathrm{d}r'}j_\ell(kr')\right]\mathrm{d}r' \\
&= -\exp[-i\omega t]\frac{kI_0}{2\pi}\sqrt{\frac{4\pi(2\ell+1)}{\ell(\ell+1)}}\left[kr'\cos\left(\tfrac{kd}{2} - kr'\right)j_\ell(kr')\right]_0^{\frac{d}{2}} \\
&= -\exp[-i\omega t]\frac{kI_0}{2\pi}\sqrt{\frac{4\pi(2\ell+1)}{\ell(\ell+1)}}\left[\tfrac{kd}{2}j_\ell\left(\tfrac{kd}{2}\right)\right] \\
&= -\exp[-i\omega t]\frac{I_0}{\pi d}\sqrt{\frac{4\pi(2\ell+1)}{\ell(\ell+1)}}\left[\left(\tfrac{kd}{2}\right)^2 j_\ell\left(\tfrac{kd}{2}\right)\right] \qquad (4.14)
\end{aligned}$$

We've thus found the expression we could plug in to find the electric and magnetic fields everywhere in space as a multipole expansion. We'd like to make comparisons amongst the various orders; the best way to do this is via the angular distributions and the power radiated. This requires only appeals to the radiation zone fields. The angular distribution is given by the expression

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{Z_0}{2k^2}\sum_{\ell,\mathrm{odd}}\left\|(-i)^\ell\Lambda_E(\ell,0)\mathbf{\Psi}_\ell^0(\theta,\phi)\right\|^2 \qquad (4.15)$$

which, when expanded becomes an ungodly sum due to the cross terms; we won't expand it (b/c I'm not a total masochist...) so we can write:

$$\begin{aligned}
\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle &= \frac{Z_0}{2k^2}\sum_{\ell,\mathrm{odd}}\left\|\Lambda_E(1,0)\mathbf{\Psi}_1^0(\theta,\phi) - \Lambda_E(3,0)\mathbf{\Psi}_3^0(\theta,\phi) + \ldots\right\|^2 \\
&= \frac{Z_0}{2k^2}\Lambda_E(1,0)|^2\left\|\mathbf{\Psi}_1^0(\theta,\phi) - \frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}\mathbf{\Psi}_3^0(\theta,\phi) + \ldots\right\|^2 \\
&= \frac{Z_0}{2k^2}\frac{I_0^2}{\pi^2 d^2}\frac{12\pi}{2}\left[\left(\tfrac{kd}{2}\right)^4 j_1\left(\tfrac{kd}{2}\right)^2\right]\left\|\mathbf{\Psi}_1^0(\theta,\phi) - \frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}\mathbf{\Phi}_3^0(\theta,\phi) + \ldots\right\|^2 \\
&= \frac{3Z_0 I_0^2}{4\pi}\left[\left(\tfrac{kd}{2}\right)^2 j_1\left(\tfrac{kd}{2}\right)^2\right]\left\|\mathbf{\Psi}_1^0(\theta,\phi) - \frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}\mathbf{\Psi}_3^0(\theta,\phi) + \ldots\right\|^2 \qquad (4.16)
\end{aligned}$$

For the purposes of explicit computation, we'll truncate at $\ell = 3$. To proceed further, we'll need explicit expressions for the angular distributions of VSH; the first few are tabulated in Appendix B[1] and we can

[1]Yeah, yeah, I know, the angular distributions in Appendix B are for $\mathbf{\Phi}$, not for $\mathbf{\Psi}$. But since we're taking magnitudes and $\mathbf{\Phi} \propto \hat{r} \times \mathbf{\Phi}$ then the distributions are identical

thus write

$$\left\|\mathbf{\Psi}_1^0(\theta,\phi)\right\|^2 = \frac{3}{8\pi}\sin^2\theta \qquad \left\|\mathbf{\Psi}_3^0(\theta,\phi)\right\|^2 = \frac{21}{64\pi}\sin^2\theta\left(5\cos^2\theta - 1\right)^2 \tag{4.17}$$

$$\left(\mathbf{\Psi}_1^0(\theta,\phi)\right)^* \cdot \left(\mathbf{\Psi}_3^0(\theta,\phi)\right) = \frac{3}{16\pi}\sqrt{\frac{7}{2}}\sin^2\theta\left(5\cos^2\theta - 1\right) \tag{4.18}$$

Computing the ratio $\frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}$, we can write

$$\left|\frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}\right| = \frac{I_0}{\pi d}\sqrt{\frac{7\pi}{3}}\left[\left(\tfrac{kd}{2}\right)^2 j_3\left(\tfrac{kd}{2}\right)\right]\frac{\pi d}{I_0}\sqrt{\frac{1}{6\pi}}\left[\left(\tfrac{2}{kd}\right)^2\frac{1}{j_1\left(\frac{kd}{2}\right)}\right] = \frac{1}{3}\sqrt{\frac{7}{2}}\frac{j_3\left(\frac{kd}{2}\right)}{j_1\left(\frac{kd}{2}\right)} \tag{4.19}$$

Thus we arrive at

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{3Z_0I_0^2}{4\pi}\left(\frac{3}{8\pi}\sin^2\theta\right)\left[\left(\tfrac{kd}{2}\right)^2 j_1\left(\tfrac{kd}{2}\right)^2\right]\left|1 - \sqrt{\frac{7}{8}}\frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}(5\cos^2\theta - 1)\right|^2 \tag{4.20}$$

Fortunately, the interference terms drop out of the total power, once integration is performed over all angles, leaving

$$P = \frac{Z_0}{2k^2}\sum_{\ell,\mathrm{odd}}|\Lambda_E(\ell,0)|^2 = \frac{Z_0}{2k^2}|\Lambda_E(1,0)|^2\left(1 + \left(\frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}\right)^2 + \dots\right) \approx \frac{3Z_0I_0^2}{4\pi}\left[\left(\frac{kd}{2}\right)^2 j_1\left(\frac{kd}{2}\right)^2\right] \tag{4.21}$$

where in the last step, I've made the dipole approximation. Now let's do some comparisons. Let's take stock of what we have. The exact distribution and power in the radiation zone were found above to be

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{Z_0I_0^2}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2 \qquad \text{and} \qquad P = \frac{Z_0I_0^2}{4\pi}\int_0^\pi\frac{\left(\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right)^2}{\sin\theta}\mathrm{d}\theta \tag{4.22}$$

The full dipole approximate distribution and power were found to be

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{3Z_0I_0^2}{4\pi}\left(\frac{3}{8\pi}\sin^2\theta\right)\left[\left(\tfrac{kd}{2}\right)^2 j_1\left(\tfrac{kd}{2}\right)^2\right] \qquad \text{and} \qquad P = \frac{3Z_0I_0^2}{4\pi}\left[\left(\tfrac{kd}{2}\right)^2 j_1\left(\tfrac{kd}{2}\right)^2\right] \tag{4.23}$$

and the full dipole+octopole approximate distribution and power were found to be

$$\left\langle\frac{\mathrm{d}P}{\mathrm{d}\Omega}\right\rangle = \frac{3Z_0I_0^2}{4\pi}\left(\frac{3}{8\pi}\sin^2\theta\right)\left[\left(\tfrac{kd}{2}\right)^2 j_1\left(\tfrac{kd}{2}\right)^2\right]\left|1 + \sqrt{\frac{7}{8}}\frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}(5\cos^2\theta - 1)\right|^2 \tag{4.24}$$

$$P = \frac{3Z_0I_0^2}{4\pi}\left[\left(\tfrac{kd}{2}\right)^2 j_1\left(\tfrac{kd}{2}\right)^2\right]\left[1 + \left(\frac{\Lambda_E(3,0)}{\Lambda_E(1,0)}\right)^2\right] \tag{4.25}$$

We can illustrate the above formulae with a few choice plots.

For $kd = \pi$ (i.e., the half-wavelength antenna), the exact, full dipole+octopole and full dipole angular distributions look like this:

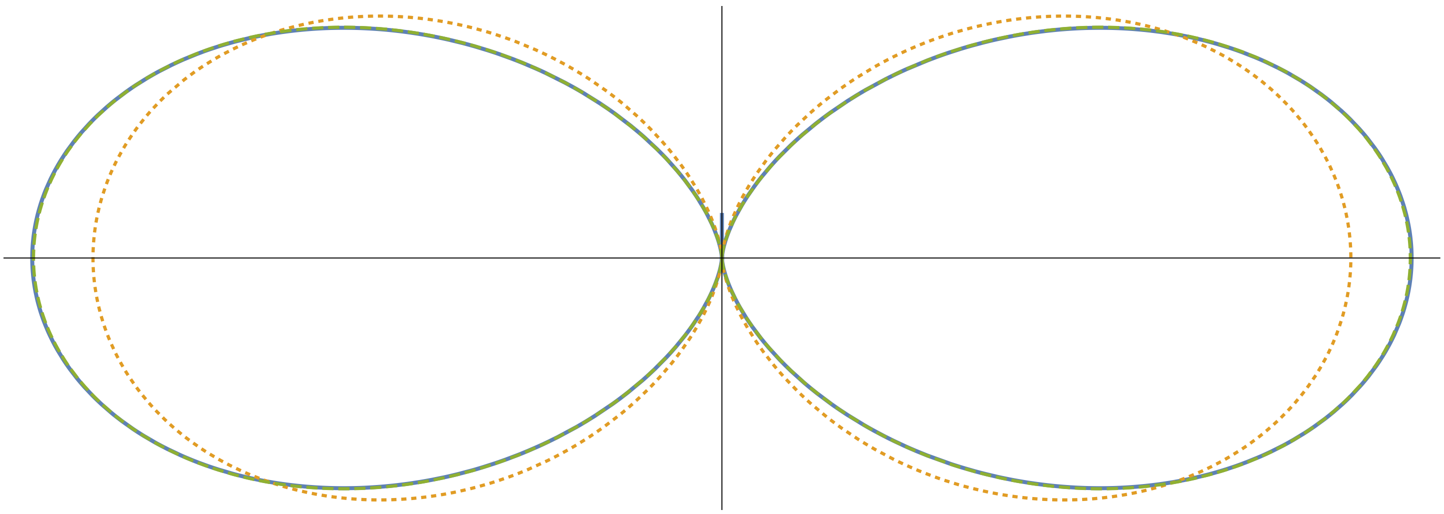

Figure 4.1: Cross-section of the radiation pattern for the the half-wavelength antenna; the z-axis is oriented upward. The solid line is the exact pattern, the dashed is the dipole+octopole and the dotted line is the dipole.

The agreement between the exact and dipole+octopole is so good that you have to zoom in to the very tip of the distribution to see the disagreement!

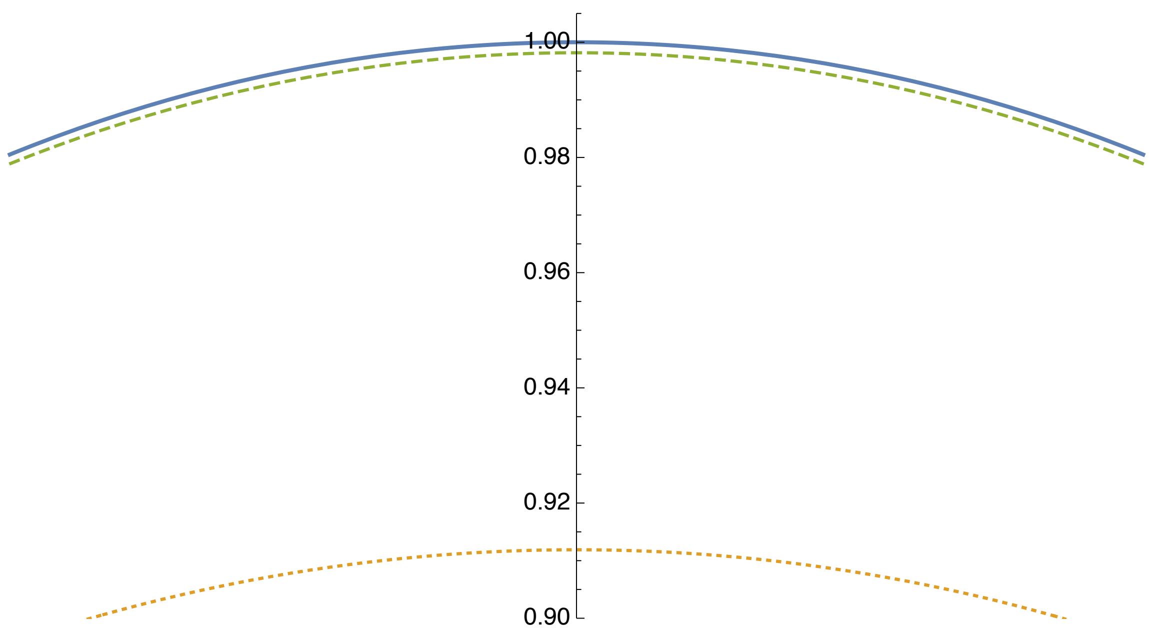

Figure 4.2: The tip of the lobe of the radiation pattern, zoomed in and rotated (z-axis is to the right), showing the extremely close agreement between the exact and dipole+octopole patterns.

Things don't look quite as good for the full-wavelength antenna $kd = 2\pi$:

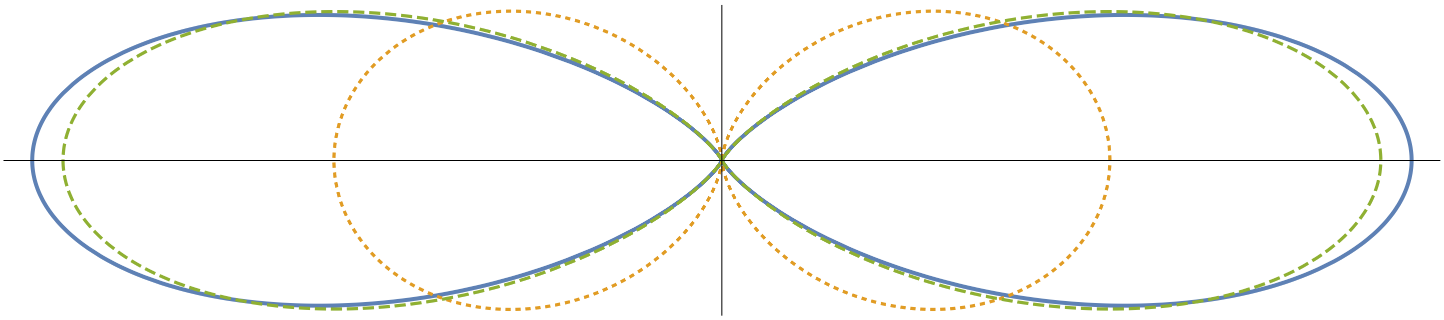

Figure 4.3: Cross-section of the radiation pattern for the the half-wavelength antenna; the z-axis is oriented upward. The solid line is the exact pattern, the dashed is the dipole+octopole and the dotted line is the dipole.

Yikes! Not good at all; but that's sort of to be expected. The approximation is breaking down, but we could shore it up by including more multipole terms to compensate for the factors of kd becoming relevant. This point is poignantly illustrated by looking at a plot of the power as a function of kd:

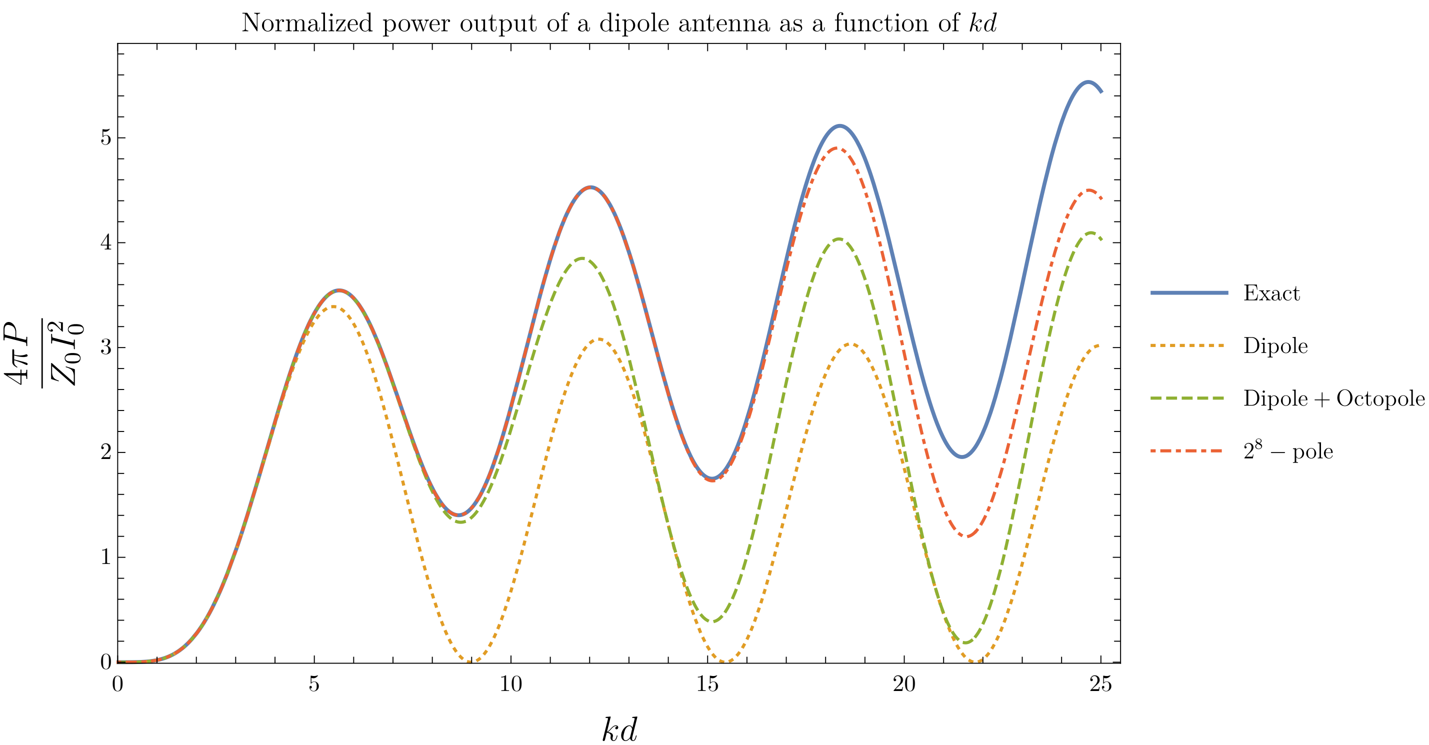

Figure 4.4

For small kd, as expected there's great agreement, but as kd grows both approximations depart from the exact answer by varying amounts. It's pretty significant that the octopole approximation agrees with the exact solution up to around $kd \sim \frac{5\pi}{2}$, and this agreement extends all the up $kd \sim 5\pi$ if we go up to the 2^8-pole solution; this is a testament to the power of the multipole expansion!

4.2 Spherical resonant cavities

Whereas in the previous section we used propagating waves to study the solutions of a radiating system, here we'll study the standing wave solutions of a resonant conducting cavity. We'll find that employing the multipole expansion makes this problem almost trivial! The Maxwell equations that govern such modes are the usual set:

$$\nabla \times \vec{E} = i\omega\mu\vec{H} \qquad \nabla \times \vec{H} = -i\omega\mu\epsilon\vec{E} \tag{4.26}$$

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{H} = 0 \tag{4.27}$$

where note that now Z is the impedance of the cavity material. Together, these imply the Helmholtz equation:

$$\left(\nabla^2 + \epsilon\mu\omega^2\right)\vec{E} = 0 \qquad \left(\nabla^2 + \epsilon\mu\omega^2\right)\vec{H} = 0$$

These equations, as studied previously, have solution:

$$\vec{E}^{(\mathrm{TM})} = Z\sum_{\ell,m}\Lambda_E(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{k^{(\mathrm{TM})}r}f_\ell(k^{(\mathrm{TM})}r)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k^{(\mathrm{TM})}r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rf_\ell(k^{(\mathrm{TM})}r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) \tag{4.28}$$

$$\vec{E}^{(\mathrm{TE})} = Z\sum_{\ell,m}\Lambda_M(\ell,m)g_\ell(k^{(\mathrm{TE})}r)\mathbf{\Phi}_\ell^m(\theta,\phi) \tag{4.29}$$

Note that the transverse electric and transverse magnetic fields have different possible modes wavenumbers, which arise due to the boundary conditions we'll study shortly. Now in this case, since we're studying standing modes including the origin, this would entail replacing the radial terms with the spherical Bessel function, $f_\ell, g_\ell \to j_\ell$; the spherical Neumann function is irregular at the origin and thus must be excluded categorically. In addition, we need to ensure that the interface conditions are satisfied:

$$\hat{r} \times \left(\vec{E}_{\mathrm{out}} - \vec{E}_{\mathrm{in}}\right) = 0 \qquad \hat{r} \cdot \left(\vec{D}_{\mathrm{out}} - \vec{D}_{\mathrm{in}}\right) = \sigma_{\mathrm{f}} \tag{4.30}$$

Since we're interested in just studying the modes of the cavity, we can assume $\vec{E}_{\mathrm{out}} = 0$ and thus $\hat{r}\times\vec{E}_{\mathrm{in}} = 0$. This is tantamount to saying that the tangential components of the fields must vanish identically. This condition, together with the divergence-free condition establishes a spectrum of discrete solutions, spanned by a set of eigenfunctions $\vec{E}_\lambda$ with eigenfrequencies ω_λ. To establish orthogonality, we begin with this vector identity:

$$\int_{\partial\Omega}\left[\vec{a}\times\left(\nabla\times\vec{b}\right) + \left(\nabla\cdot\vec{b}\right)\vec{a}\right]\cdot\mathrm{d}\vec{A} = \int_\Omega\left[(\nabla\times\vec{a})\cdot\left(\nabla\times\vec{b}\right) + (\nabla\cdot\vec{a})\left(\nabla\cdot\vec{b}\right) + \vec{a}\cdot\nabla^2\vec{b}\right]\mathrm{d}^3x \tag{4.31}$$

Let $\vec{a} = \vec{E}_\lambda$ and $\vec{b} = \vec{E}_\nu$. Then we can write, using the Maxwell equations:

$$\int_{\partial\Omega}\left[\vec{E}_\lambda\times\left(\nabla\times\vec{E}_\nu\right) + \cancelto{0}{\left(\nabla\cdot\vec{E}_\nu\right)}\vec{E}_\lambda\right]\cdot\mathrm{d}\vec{A} = \int_\Omega\left[\left(\nabla\times\vec{E}_\lambda\right)\cdot\left(\nabla\times\vec{E}_\nu\right) + \cancelto{0}{\left(\nabla\cdot\vec{E}_\lambda\right)\left(\nabla\cdot\vec{E}_\nu\right)} + \vec{E}_\lambda\cdot\nabla^2\vec{E}_\nu\right]\mathrm{d}^3x$$

$$\int_{\partial\Omega}\left[\vec{E}_\lambda\times\left(i\omega_\nu\mu\vec{H}_\nu\right)\right]\cdot\hat{r}\mathrm{d}A = \int_\Omega\left[\left(\nabla\times\vec{E}_\lambda\right)\cdot\left(\nabla\times\vec{E}_\nu\right) - \vec{E}_\lambda\cdot\left(\omega_\nu\mu\epsilon\vec{E}_\nu\right)\right]\mathrm{d}^3x$$

$$i\omega_\nu\mu\int_{\partial\Omega}\cancelto{0}{\left[\hat{r}\times\vec{E}_\lambda\right]}\cdot\vec{H}_\nu\mathrm{d}\vec{A} = \int_\Omega\left(\nabla\times\vec{E}_\lambda\right)\cdot\left(\nabla\times\vec{E}_\nu\right)\mathrm{d}^3x - \omega_\nu^2\mu\epsilon\int\vec{E}_\lambda\cdot\vec{E}_\nu\mathrm{d}^3x$$

$$\int_\Omega\left(\nabla\times\vec{E}_\lambda\right)\cdot\left(\nabla\times\vec{E}_\nu\right)\mathrm{d}^3x = \omega_\nu^2\mu\epsilon\int\vec{E}_\lambda\cdot\vec{E}_\nu\mathrm{d}^3x$$

If we now swap $\lambda \leftrightarrow \nu$ and exploit the the fact that the dot product is symmetric, we can write:

$$\int_\Omega \left(\nabla \times \vec{E}_\lambda\right) \cdot \left(\nabla \times \vec{E}_\nu\right) \mathrm{d}^3x = \omega_\lambda^2 \mu\epsilon \int \vec{E}_\lambda \cdot \vec{E}_\nu \mathrm{d}^3x$$

Subtracting these two equations gives

$$0 = \mu\epsilon \left(\omega_\nu^2 - \omega_\lambda^2\right) \int \vec{E}_\lambda \cdot \vec{E}_\nu \mathrm{d}^3x$$

which implies, so long as $\omega_\mu \neq \omega_\lambda$ (i.e., the eigenfrequencies aren't degenerate) that

$$\int \vec{E}_\lambda \cdot \vec{E}_\nu \mathrm{d}^3x = 0 \tag{4.32}$$

which is the desired orthogonality. This in turn implies

$$\int_\Omega \left(\nabla \times \vec{E}_\lambda\right) \cdot \left(\nabla \times \vec{E}_\nu\right) \mathrm{d}^3x = -\mu^2 \omega_\lambda \omega_\nu \int_\Omega \vec{H}_\lambda \cdot \vec{H}_\nu \mathrm{d}^3x = 0 \tag{4.33}$$

and thus the associated magnetic fields are also orthogonal. Since both $\mathbf{\Phi}_\ell^m$ and $\mathbf{\Psi}_\ell^m$ are tangential, we can establish the requisite condition on the modes by requiring that the radial coefficients of $\mathbf{\Phi}_\ell^m$ and $\mathbf{\Psi}_\ell^m$ vanish at $r = R$. But since we've already decomposed our fields into vector components, this becomes extremely easy!

$$j_\ell(k_{\lambda\ell}^{(\mathrm{TE})} R) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}r}\left(r j_\ell(k_{\lambda\ell}^{(\mathrm{TM})} r)\right)\bigg]_{r=R} = 0 \tag{4.34}$$

where we write $k_\lambda^{(\mathrm{TE})/(\mathrm{TM})}$ to denote the λ^{th} wavenumber corresponding to the λ^{th} zero of the ℓ^{th} spherical Bessel function. This in turn gives the eigenfrequency by the relation $\omega_\ell = ck_\ell$. We've thus established the spectrum of solutions in a spherical conducting cavity. Their form is of the spherical solutions to the Helmholtz equation considered previously, with a discrete spectrum of frequencies allowed by the geometry and boundary conditions. Thus for a given multipole field, we can expand the time-dependence in the eigenbasis, as a sum over eigenfrequencies:

$$\vec{E}_{\ell m}^{(\mathrm{TM})}(\vec{r}, t) = \sum_\lambda \exp[i\omega_\lambda^{(\mathrm{TM})} t] \vec{E}_{\ell m}^{(\mathrm{TM})}\left(\vec{r}, \omega_\lambda^{(\mathrm{TM})}\right) \tag{4.35}$$

$$\vec{E}^{(\mathrm{TE})} = \sum_\lambda \exp[i\omega_\lambda^{(\mathrm{TE})} t] \vec{E}_{\ell m}^{(\mathrm{TE})}\left(\vec{r}, \omega_\lambda^{(\mathrm{TE})}\right) \tag{4.36}$$

4.3 ***Scattering***

As a final application, we consider the use of the VSH in studying systems involving scattering. We'll consider the problem of a plane electromagnetic wave incident on a dielectric sphere of radius R. Such a problem is interesting because it requires matching of solutions across a boundary; we got a taste of this when considering resonant cavities previously and now we'll extend our efforts. The solution of the Maxwell equations across a dielectric sphere is also known as the *Mie solution*, which has imparted this theory of scattering with the name *Lorenz-Mie scattering*.

Far away from the sphere, the total electric and magnetic fields will be given by a sum of the incident and scattered electric and magnetic fields:

$$\vec{E} = \vec{E}_{\text{inc}} + \vec{E}_{\text{sc}} \qquad \text{and} \qquad \vec{B} = \vec{B}_{\text{inc}} + \vec{B}_{\text{sc}} \tag{4.37}$$

The incoming wave we've taken to be a plane wave; the expansion of a vector plane wave in terms of spherical waves is given in Appendix C by

$$\vec{E}_{\text{inc}} = \mathcal{E}\sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)}\left[i\left(\frac{\sqrt{\ell(\ell+1)}}{k_1 r} j_\ell(k_1 r)\mathbf{Y}_\ell^{\pm1}(\theta,\phi) + \frac{1}{k_1 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r j_\ell(k_1 r)\right)\mathbf{\Psi}_\ell^{\pm1}(\theta,\phi)\right) \mp j_\ell(k_1 r)\mathbf{\Phi}_\ell^{\pm1}(\theta,\phi)\right] \tag{4.38}$$

$$\vec{H}_{\text{inc}} = \frac{\mathcal{E}}{Z_1}\sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)}\left[\pm\left(\frac{\sqrt{\ell(\ell+1)}}{k_1 r} j_\ell(k_1 r)\mathbf{Y}_\ell^{\pm1}(\theta,\phi) + \frac{1}{k_1 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r j_\ell(k_1 r)\right)\mathbf{\Psi}_\ell^{\pm1}(\theta,\phi)\right) + i j_\ell(k_1 r)\mathbf{\Phi}_\ell^{\pm1}(\theta,\phi)\right] \tag{4.39}$$

This leaves the scattered fields and the fields within the sphere itself. Employing the expansion in VSH found in Chapter 3, we can thus write:

$$\vec{E}_{\text{sc}} = \sum_{\ell,m}\left[A_{\ell m}\left(\frac{\sqrt{\ell(\ell+1)}}{k_1 r} h_\ell^{(1)}(k_1 r)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k_1 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r h_\ell^{(1)}(k_1 r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + B_{\ell m} h_\ell^{(1)}(k_1 r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{4.40}$$

$$\vec{H}_{\text{sc}} = \frac{1}{Z_1}\sum_{\ell,m}\left[-B_{\ell m}\left(\frac{\sqrt{\ell(\ell+1)}}{k_1 r} h_\ell^{(1)}(k_1 r)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k_1 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r h_\ell^{(1)}(k_1 r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + A_{\ell m} h_\ell^{(1)}(k_1 r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{4.41}$$

and

$$\vec{E}_{\text{sph}} = \sum_{\ell,m}\left[C_{\ell m}\left(\frac{\sqrt{\ell(\ell+1)}}{k_2 r} j_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k_2 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r j_\ell(k_2 r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + D_{\ell m} j_\ell(k_2 r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{4.42}$$

$$\vec{H}_{\text{sph}} = \frac{1}{Z_2}\sum_{\ell,m}\left[-D_{\ell m}\left(\frac{\sqrt{\ell(\ell+1)}}{k_2 r} j_\ell(k_2 r)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k_2 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r j_\ell(k_2 r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + C_{\ell m} j_\ell(k_2 r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{4.43}$$

Here we've introduced the notation

$$Z_i = \sqrt{\frac{\mu_i}{\epsilon_i}} \qquad k_i = \frac{n_i\omega}{c} \qquad n_i = c\sqrt{\mu_i\epsilon_i} \tag{4.44}$$

Z_i is the impedance of the respective dielectric material, k_i is the wavenumber corresponding to the frequency ω_i in that same material and n_i is the index of refraction of the material. Note that A, B, C, D

have replaced Λ_E, Λ_M for the various fields. Note also that the solutions in the sphere are spherical Bessel functions, not Hankel functions; this reflects the fact that the solutions in the sphere are resonant standing waves, not the propagating waves exterior to the sphere. Our task therefore is to 'stitch' these solutions together across the boundary of the sphere, thus determining A, B, C, D, using the matching conditions:

$$\hat{r} \times \left(\vec{E}_{\text{out}} - \vec{E}_{\text{in}}\right) = 0 \qquad \hat{r} \times \left(\vec{H}_{\text{out}} - \vec{H}_{\text{in}}\right) = \vec{J}_{\text{f}} \tag{4.45}$$

We're assuming no free currents or charges so the above expressions simplify; the total exterior field is the sum of the incident and scattered fields. Thus we write:

$$\hat{r} \times \left(\vec{E}_{\text{sc}} + \vec{E}_{\text{inc}}\right)\Big]_{r=R} = \hat{r} \times \vec{E}_{\text{sph}}\Big]_{r=R} \qquad \hat{r} \times \left(\vec{H}_{\text{sc}} + \vec{H}_{\text{inc}}\right)\Big]_{r=R} = \hat{r} \times \vec{H}_{\text{sph}}\Big]_{r=R} \tag{4.46}$$

Recalling that the VSH satisfy these relations

$$\hat{r} \times \mathbf{Y}_\ell^m = 0 \qquad \hat{r} \times \mathbf{\Psi}_\ell^m = i\mathbf{\Phi}_\ell^m \qquad \hat{r} \times \mathbf{\Phi}_\ell^m = -i\mathbf{\Psi}_\ell^m \tag{4.47}$$

then we can write on the surface of the sphere for the incident wave:

$$\hat{r} \times \vec{E}_{\text{inc}} = \mathcal{E} \sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \left(-\frac{1}{k_1 R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r j_\ell(k_1 r)\right) \Big]_{r=R} \mathbf{\Phi}_\ell^{\pm 1}(\theta, \phi) \pm i j_\ell(k_1 R) \mathbf{\Psi}_\ell^{\pm 1}(\theta, \phi) \right) \tag{4.48}$$

$$\hat{r} \times \vec{H}_{\text{inc}} = \frac{\mathcal{E}}{Z_1} \sum_{\ell=1}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)} \left(\pm \frac{i}{k_1 R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r j_\ell(k_1 r)\right) \Big]_{r=R} \mathbf{\Phi}_\ell^{\pm 1}(\theta, \phi) + j_\ell(k_1 R) \mathbf{\Psi}_\ell^{\pm 1}(\theta, \phi) \right) \tag{4.49}$$

for the scattered wave:

$$\hat{r} \times \vec{E}_{\text{sc}} = \sum_{\ell,m} \left(\frac{iA_{\ell m}}{k_1 R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r h_\ell^{(1)}(k_1 r)\right) \Big]_{r=R} \mathbf{\Phi}_\ell^m(\theta, \phi) - iB_{\ell m} h_\ell^{(1)}(k_1 R) \mathbf{\Psi}_\ell^m(\theta, \phi) \right) \tag{4.50}$$

$$\hat{r} \times \vec{H}_{\text{sc}} = \frac{1}{Z_1} \sum_{\ell,m} \left(-\frac{iB_{\ell m}}{k_1 R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r h_\ell^{(1)}(k_1 r)\right) \mathbf{\Phi}_\ell^m(\theta, \phi) - iA_{\ell m} h_\ell^{(1)}(k_1 R) \mathbf{\Psi}_\ell^m(\theta, \phi) \right) \tag{4.51}$$

and lastly the sphere waves:

$$\hat{r} \times \vec{E}_{\text{sph}} = \sum_{\ell,m} \left(\frac{iC_{\ell m}}{k_2 R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r j_\ell(k_2 r)\right) \Big]_{r=R} \mathbf{\Phi}_\ell^m(\theta, \phi) - iD_{\ell m} j_\ell(k_2 R) \mathbf{\Psi}_\ell^m(\theta, \phi) \right) \tag{4.52}$$

$$\hat{r} \times \vec{H}_{\text{sph}} = \frac{1}{Z_2} \sum_{\ell,m} \left(-\frac{iD_{\ell m}}{k_2 R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r j_\ell(k_2 r)\right) \mathbf{\Phi}_\ell^m(\theta, \phi) - iC_{\ell m} j_\ell(k_2 R) \mathbf{\Psi}_\ell^m(\theta, \phi) \right) \tag{4.53}$$

For the sake of brevity, let

$$X_\ell^f(x_i) = \frac{1}{x_i} \frac{\mathrm{d}}{\mathrm{d}x_i} \left(x_i f_\ell(x_i)\right) \Big]_{x_i = a_i} \qquad \text{where} \qquad x_i = k_i r \qquad \text{and} \qquad a_i = k_i R \tag{4.54}$$

and f can be either h or j, depending on whether a spherical Bessel or Hankel function is being represented. Examining the electric field matching condition Eq. (4.46) and matching the coefficients of $\mathbf{\Phi}_\ell^m$, we see two separate cases, for $m = 1$:

$$-\mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} X_\ell^j(x_1) + iA_{\ell 1} X_\ell^h(x_1) = iC_{\ell 1} X_\ell^j(x_2) \tag{4.55}$$

and for $m \neq 1$:

$$A_{\ell m} X_\ell^h(x_1) = C_{\ell m} X_\ell^j(x_2) \tag{4.56}$$

Now match the coefficients of $\mathbf{\Psi}_\ell^m$, and we again see two cases, for $m = 1$:

$$\pm\, \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} j_\ell(a_1) - B_{\ell 1} h_\ell^{(1)}(a_1) = -D_{\ell 1} j_\ell(a_2) \tag{4.57}$$

and for $m \neq 1$:

$$B_{\ell m} h_\ell^{(1)}(a_1) = D_{\ell m} j_\ell(a_2) \tag{4.58}$$

Repeating the process for the magnetic field matching condition Eq. (4.46) and matching coefficients of $\mathbf{\Phi}_\ell^m$ for $m = 1$:

$$\pm\, \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} X_\ell^j(x_1) - B_{\ell 1} X_\ell^h(x_1) = -\frac{Z_1}{Z_2} D_{\ell 1} X_\ell^j(x_2) \tag{4.59}$$

and for $m \neq 1$:

$$B_{\ell m} X_\ell^h(x_1) = \frac{Z_1}{Z_2} D_{\ell m} X_\ell^j(x_2) \tag{4.60}$$

And now match the coefficients of $\mathbf{\Psi}_\ell^m$ for $m = 1$:

$$\mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} j_\ell(a_1) - i A_{\ell 1} h_\ell^{(1)}(a_1) = -i \frac{Z_1}{Z_2} C_{\ell 1} j_\ell(a_2) \tag{4.61}$$

and lastly for $m \neq 1$:

$$A_{\ell m} h_\ell^{(1)}(k_1 R) = \frac{Z_1}{Z_2} C_{\ell m} j_\ell(k_2 R) \tag{4.62}$$

Let's collect all the above equations so we can see what we're working with. The $m = 1$ case equations are:

$$iA_{\ell 1} X_\ell^h(x_1) - iC_{\ell 1} X_\ell^j(x_2) = \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} X_\ell^j(x_1) \tag{4.63}$$

$$iA_{\ell 1} h_\ell^{(1)}(a_1) - i\frac{Z_1}{Z_2} C_{\ell 1} j_\ell(a_2) = \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} j_\ell(a_1) \tag{4.64}$$

$$B_{\ell 1} h_\ell^{(1)}(a_1) - D_{\ell 1} j_\ell(a_2) = \pm \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} j_\ell(a_1) \tag{4.65}$$

$$B_{\ell 1} X_\ell^h(x_1) - \frac{Z_1}{Z_2} D_{\ell 1} X_\ell^j(x_2) = \pm \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} X_\ell^j(x_1) \tag{4.66}$$

and the $m \neq 1$ case equations are:

$$A_{\ell m} X_\ell^h(x_1) = C_{\ell m} X_\ell^j(x_2) \tag{4.67}$$

$$A_{\ell m} h_\ell^{(1)}(k_1 R) = \frac{Z_1}{Z_2} C_{\ell m} j_\ell(k_2 R) \tag{4.68}$$

$$B_{\ell m} h_\ell^{(1)}(a_1) = D_{\ell m} j_\ell(a_2) \tag{4.69}$$

$$B_{\ell m} X_\ell^h(x_1) = \frac{Z_1}{Z_2} D_{\ell m} X_\ell^j(x_2) \tag{4.70}$$

To proceed further, examine Eq. (4.67) and (4.68). Invert (4.68) to solve for $C_{\ell m}$; then plug it into (4.67) to give

$$\begin{aligned}
A_{\ell m} X_\ell^h(x_1) &= C_{\ell m} X_\ell^j(x_2) \\
A_{\ell m} X_\ell^h(x_1) &= \left(A_{\ell m} \frac{Z_2}{Z_1} \frac{h_\ell^{(1)}(k_1 R)}{j_\ell(k_2 R)} \right) X_\ell^j(x_2) \\
Z_1 A_{\ell m} j_\ell(a_2) X_\ell^h(x_1) &= Z_2 A_{\ell m} h_\ell^{(1)}(a_1) X_\ell^j(x_2) \\
\Rightarrow \qquad A_{\ell m} &\left(Z_1 j_\ell(a_2) X_\ell^h(x_1) - Z_2 h_\ell^{(1)}(a_1) X_\ell^j(x_2) \right) = 0
\end{aligned}$$

This expression has two solutions:

$$\text{either} \quad A_{\ell m} = 0 \qquad \text{or} \qquad Z_1 j_\ell(a_2) X_\ell^h(x_1) - Z_2 h_\ell^{(1)}(a_1) X_\ell^j(x_2) = 0$$

The second condition is *extremely* restrictive, and it must be satisfied for every single ℓ. For a given value of ℓ, it may be possible to satisfy, by choosing R carefully. But since R can't be tuned and it's therefore impossible to satisfy the second condition in general, then $A_{\ell m} = 0$ which therefore implies $C_{\ell m} = 0$. It's possible to perform similar analysis to show that $B_{\ell m} = D_{\ell m} = 0$. Now this makes sense from a physical perspective. The problem of scattering off a sphere by a plane wave is axisymmetric, and thus it stands to reason that the solution should also be axisymmetric, which restricts us to the axisymmetric VSH, $m = 1$.

Now keep in mind that the above analysis is only for $m \neq 1$. The coefficients in the $m = 1$ case are still undetermined; as such we'll relabel $A_{\ell 1}, B_{\ell 1}, C_{\ell 1}, D_{\ell 1}$ to $A_\ell, B_\ell, C_\ell, D_\ell$. Using this notation, we can write the $m = 1$ equations into a matrix equation:

$$\begin{pmatrix} i X_\ell^h(x_1) & 0 & -i X_\ell^j(x_2) & 0 \\ i h_\ell^{(1)}(a_1) & 0 & -i \frac{Z_1}{Z_2} j_\ell(a_2) & 0 \\ 0 & X_\ell^h(x_1) & 0 & -\frac{Z_1}{Z_2} X_\ell^j(x_2) \\ 0 & h_\ell^{(1)}(a_1) & 0 & -j_\ell(a_2) \end{pmatrix} \begin{pmatrix} A_\ell \\ B_\ell \\ C_\ell \\ D_\ell \end{pmatrix} = \mathcal{E} i^\ell \sqrt{4\pi(2\ell+1)} \begin{pmatrix} X_\ell^j(x_1) \\ j_\ell(a_1) \\ \pm X_\ell^j(x_1) \\ \pm j_\ell(a_1) \end{pmatrix} \tag{4.71}$$

Now we're left to solve a 4×4 system of equations. This can be done relatively easily, but it's a rather laborious task[2] so we'll just use a CAS to turn the crank for us and give:

$$A_\ell = -i^{\ell+1} \sqrt{4\pi(2\ell+1)} \frac{\frac{Z_1}{Z_2} j_\ell(a_2) X_\ell^j(x_1) - j_\ell(a_1) X_\ell^j(a_2)}{\frac{Z_1}{Z_2} j_\ell(x_2) X_\ell^h(x_1) - h_\ell^{(1)}(a_1) X_\ell^j(x_2)} \mathcal{E} \tag{4.72}$$

$$B_\ell = \pm i^\ell \sqrt{4\pi(2\ell+1)} \frac{j_\ell(a_2) X_\ell^j(x_1) - \frac{Z_1}{Z_2} j_\ell(a_1) X_\ell^j(x_2)}{j_\ell(a_2) X_\ell^h(x_1) - \frac{Z_1}{Z_2} h_\ell^{(1)}(a_1) X_\ell^j(x_2)} \mathcal{E} \tag{4.73}$$

$$C_\ell = i^{\ell+1} \sqrt{4\pi(2\ell+1)} \frac{-j_\ell(a_1) X_\ell^h(x_1) + h_\ell^{(1)}(a_1) X_\ell^j(x_1)}{-\frac{Z_1}{Z_2} j_\ell(a_2) X_\ell^h(x_1) + h_\ell^{(1)}(a_1) X_\ell^j(x_2)} \mathcal{E} \tag{4.74}$$

$$D_\ell = \pm i^\ell \sqrt{4\pi(2\ell+1)} \frac{h_\ell^{(1)}(a_1) X_\ell^j(x_1) - j_\ell(a_1) X_\ell^h(x_1)}{j_\ell(a_2) X_\ell^h(x_1) - \frac{Z_1}{Z_2} h_\ell^{(1)}(a_1) X_\ell^j(x_2)} \mathcal{E} \tag{4.75}$$

[2] and I'm lazy and error-prone

Let's clean these up a bit. First multiply throughout by $\frac{Z_2}{Z_2}$; then multiply by $\frac{a_1 a_2}{a_1 a_2}$ and introduce the notation

$$Q_\ell(x_i) = x_i h_\ell^{(1)}(x_i) \qquad \text{and} \qquad S_\ell(x) = x j_\ell(x) \tag{4.76}$$

where we use primes to mark derivatives. These functions are known as the *Riccati-Bessel functions*. Thus we can write:

$$A_\ell = i^{\ell+1}\sqrt{4\pi(2\ell+1)}\frac{Z_2 S_\ell(a_1)S'_\ell(a_2) - Z_1 S_\ell(a_2)S'_\ell(a_1)}{Z_1 S_\ell(a_2)Q'_\ell(a_1) - Z_2 Q_\ell(a_1)S'_\ell(a_2)}\mathcal{E} \tag{4.77}$$

$$B_\ell^{(\pm)} = \pm i^{\ell}\sqrt{4\pi(2\ell+1)}\frac{Z_2 S_\ell(a_2)S'_\ell(a_1) - Z_1 S_\ell(a_1)S'_\ell(a_2)}{Z_2 S_\ell(a_2)Q'_\ell(a_1) - Z_1 Q_\ell(a_1)S'_\ell(a_2)}\mathcal{E} \tag{4.78}$$

$$C_\ell = i^{\ell+1}\sqrt{4\pi(2\ell+1)}\frac{Z_2 S_\ell(a_1)Q'_\ell(a_1) - Z_2 Q_\ell(a_1)S'_\ell(a_1)}{Z_1 S_\ell(a_2)Q'_\ell(a_1) - Z_2 Q_\ell(a_1)S'_\ell(a_2)}\mathcal{E} \tag{4.79}$$

$$D_\ell^{(\pm)} = \pm i^{\ell}\sqrt{4\pi(2\ell+1)}\frac{Z_2 Q_\ell(a_1)S'_\ell(a_1) - Z_2 S_\ell(a_1)Q'_\ell(a_1)}{Z_2 S_\ell(a_2)Q'_\ell(a_1) - Z_1 Q_\ell(a_1)S'_\ell(a_2)}\mathcal{E} \tag{4.80}$$

Lastly, there's a lot of common factors floating around, so let's define dimensionless quantities α_ℓ, β_ℓ, γ_ℓ and ζ_ℓ as

$$A_\ell = i^{\ell+1}\sqrt{4\pi(2\ell+1)}\mathcal{E}\alpha_\ell \qquad B_\ell^{(\pm)} = \pm i^{\ell}\sqrt{4\pi(2\ell+1)}\mathcal{E}\beta_\ell \qquad C_\ell = i^{\ell+1}\sqrt{4\pi(2\ell+1)}\mathcal{E}\gamma_\ell \qquad D_\ell^{(\pm)} = \pm i^{\ell}\sqrt{4\pi(2\ell+1)}\mathcal{E}\gamma_\ell \tag{4.81}$$

so that

$$\alpha_\ell = \frac{Z_2 S_\ell(a_1)S'_\ell(a_2) - Z_1 S_\ell(a_2)S'_\ell(a_1)}{Z_1 S_\ell(a_2)Q'_\ell(a_1) - Z_2 Q_\ell(a_1)S'_\ell(a_2)} \tag{4.82}$$

$$\beta_\ell = \frac{Z_2 S_\ell(a_2)S'_\ell(a_1) - Z_1 S_\ell(a_1)S'_\ell(a_2)}{Z_2 S_\ell(a_2)Q'_\ell(a_1) - Z_1 Q_\ell(a_1)S'_\ell(a_2)} \tag{4.83}$$

$$\gamma_\ell = \frac{Z_2 S_\ell(a_1)Q'_\ell(a_1) - Z_2 Q_\ell(a_1)S'_\ell(a_1)}{Z_1 S_\ell(a_2)Q'_\ell(a_1) - Z_2 Q_\ell(a_1)S'_\ell(a_2)} \tag{4.84}$$

$$\zeta_\ell = \frac{Z_2 Q_\ell(a_1)S'_\ell(a_1) - Z_2 S_\ell(a_1)Q'_\ell(a_1)}{Z_2 S_\ell(a_2)Q'_\ell(a_1) - Z_1 Q_\ell(a_1)S'_\ell(a_2)} \tag{4.85}$$

And thus, we've done it! We've described the incident and scattered waves, coupled through the resonant modes of a dielectric sphere. Our final answer for the scattered fields is therefore

$$\vec{E}_{\text{sc}} = \mathcal{E}\sum_{\ell,m} i^{\ell}\sqrt{4\pi(2\ell+1)}\left[i\alpha_\ell\left(\frac{\sqrt{\ell(\ell+1)}}{k_1 r}h_\ell^{(1)}(k_1 r)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k_1 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r h_\ell^{(1)}(k_1 r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) \pm \beta_\ell h_\ell^{(1)}(k_1 r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{4.86}$$

$$\vec{H}_{\text{sc}} = \frac{\mathcal{E}}{Z_1}\sum_{\ell,m} i^{\ell}\sqrt{4\pi(2\ell+1)}\left[\mp\beta_\ell\left(\frac{\sqrt{\ell(\ell+1)}}{k_1 r}h_\ell^{(1)}(k_1 r)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{k_1 r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r h_\ell^{(1)}(k_1 r)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + i\alpha_\ell h_\ell^{(1)}(k_1 r)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{4.87}$$

Let's stop and appreciate what we have here before going on. We've established an expression for the scattered electric and magnetic fields off a dielectric sphere, for arbitrary length scale $a = kR$, meaning we can freely manipulate both the size of the scatterer and the wavelength of the scattering and still have a rigorous expression. Although the algebra was a little messy, at its heart this was done by simply matching coefficients of the VSH, just like with the resonant modes in the previous section. What remains now is to find the angular distribution of the scattered radiation, the total scattered power and of course the cross section. Fortunately, almost all of the heavy lifting is over, so now we just have to turn the crank. Using the radiation zone formulae from Chapter 3, we can write

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle = \frac{Z_i}{2k_i^2} \sum_{\ell,m} \left\| (-i)^{\ell+1} \left(\Lambda_M(\ell,m) \mathbf{\Phi}_\ell^m(\theta,\phi) + i\Lambda_E(\ell,m) \mathbf{\Psi}_\ell^m(\theta,\phi) \right) \right\|^2$$

and that the total power is

$$\langle P \rangle = \frac{Z_1}{2k_1^2} \sum_{\ell,m} \left[|\Lambda_M(\ell,m)|^2 + |\Lambda_E(\ell,m)|^2 \right]$$

which is a general expression, since multipoles don't interfere with each other when averaged over all angles. One last expression we'll need for the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\text{scattered power radiated into a solid angle 'patch'}}{\text{incident power per unit area}} = \frac{1}{P_{\text{inc}}} \frac{\mathrm{d}P_{\text{sc}}}{\mathrm{d}\Omega} \tag{4.88}$$

which when integrated gives the total cross section:

$$\sigma_{\text{tot}} = \frac{P_{\text{sc}}}{P_{\text{inc}}} \tag{4.89}$$

Now, we found, for the scattered wave

$$\Lambda_E(\ell,\pm 1) = A_\ell = i^{\ell+1}\sqrt{4\pi(2\ell+1)}\mathcal{E}\alpha_\ell \qquad \text{and} \qquad \Lambda_M(\ell,\pm 1) = B_\ell^{(\pm)} = \pm i^\ell \sqrt{4\pi(2\ell+1)}\mathcal{E}\beta_\ell \tag{4.90}$$

Going back to the plane wave representation we can show that the incident power is

$$\langle P \rangle_{\text{inc}} = \frac{1}{2}\pi r^2 \mathcal{E}^2 \tag{4.91}$$

The angular distribution of scattered power is

$$\begin{aligned}
\left\langle \frac{\mathrm{d}P}{\mathrm{d}\Omega} \right\rangle &= \frac{Z_0}{2k^2} \sum_{\ell,m} \left\| (-i)^{\ell+1} \left(\Lambda_M(\ell,m) \mathbf{\Phi}_\ell^m(\theta,\phi) + i\Lambda_E(\ell,m) \mathbf{\Psi}_\ell^m(\theta,\phi) \right) \right\|^2 \\
&= \frac{Z_1}{2k_1^2} \sum_{\ell} \left\| (-i)^{\ell+1} i^{\ell+1} \sqrt{4\pi(2\ell+1)}\mathcal{E} \left(\alpha_\ell \mathbf{\Phi}_\ell^{\pm 1}(\theta,\phi) \pm i\beta_\ell \mathbf{\Psi}_\ell^{\pm 1}(\theta,\phi) \right) \right\|^2 \\
&= \mathcal{E}^2 \frac{2\pi Z_1}{k^2} \left\| \sum_\ell \sqrt{2\ell+1} \left[\alpha_\ell \mathbf{\Phi}_\ell^{\pm 1}(\theta,\phi) \pm i\beta_\ell \mathbf{\Psi}_\ell^{\pm 1}(\theta,\phi) \right] \right\|^2
\end{aligned} \tag{4.92}$$

Note that the polarization of the radiation, represented by the relative strengths of the $\mathbf{\Phi}$ and $\mathbf{\Psi}$ VSH is in general elliptical; if $\alpha_\ell = \beta_\ell$ then we would recover circular polarization, and so on for other special cases. Integrating over all angles and exploiting the orthogonality of the VSH gives the total scattered power:

$$\sigma_{\text{tot}} = \frac{4\pi Z_1}{k^2} \sum_\ell (2\ell+1) \left[\|\alpha_\ell\|^2 + \|\beta_\ell\|^2 \right] \tag{4.93}$$

Thus we see we need to only worry about α_ℓ and β_ℓ.

A few sanity checks are in order. We can check some simple limits and see if they line up with physical intuition. The two simplest checks are manipulating the dielectric properties of the sphere. In the limit that $Z_2 = Z_1$, it's as if the sphere weren't there at all. In that case, the scattered wave should vanish and the interior modes should be identical to the propagating plane wave. In this limit, it is necessarily true that $\mu_1 = \mu_2$ and $\epsilon_1 = \epsilon_2$ and therefore, $a_1 = a_2$ since $a_i = k_i R$ and k_i depends on the index of refraction. It thus becomes trivial to see that

$$\alpha_\ell = \beta_\ell = 0 \qquad \text{and} \qquad \gamma_\ell = -\zeta_\ell = 1 \tag{4.94}$$

Plugging these into $\vec{E}_{\text{sc}}$ and $\vec{E}_{\text{sph}}$ we find

$$\vec{E}_{\text{sc}} = 0 \qquad \text{and} \qquad \vec{E}_{\text{sph}} = \vec{E}_{\text{inc}} \tag{4.95}$$

as expected. Another limit is treating the sphere as a perfect conductor for which $Z_2 = 0$. In this limit, the properties of the sphere isolate the interior solutions from the exterior, meaning that the incident wave should decouple entirely from the scattered wave and no sphere modes should be present. We get

$$\alpha_\ell = -\frac{S'(a_1)}{Q'_\ell(a_1)} \qquad \beta_\ell = \frac{S_\ell(a_1)}{Q_\ell(a_1)} \qquad \text{and} \qquad \gamma_\ell = \zeta_\ell = 0 \tag{4.96}$$

Since $S_\ell(a_1) = a_1 j_\ell(a_1)$ and $Q_\ell(a_1) = a_1 h_\ell^{(1)}(a_1)$, and since $h_\ell^{(1)}(a_1) = j_\ell(a_1) + i n_\ell(a_1)$ then this allows us to write

$$\alpha_\ell = \frac{[x j_\ell(x)]'_{x=a_1}}{[x h_\ell(x)]'_{x=a_1}} = -\frac{[x j_\ell(x)]'_{x=a_1}}{[x j_\ell(x)]'_{x=a_1} + i[x n_\ell(x)]'_{x=a_1}} = -1 + i\frac{[x j_\ell(x)]'_{x=a_1}}{[x n_\ell(x)]'_{x=a_1}} \tag{4.97}$$

and

$$\beta_\ell = \frac{a_1 j_\ell(a_1)}{a_1 h_\ell(a_1)} = \frac{a_1 j_\ell(a_1)}{a_1 j_\ell(a_1) + i a_1 n_\ell(a_1)} = 1 - i\frac{j_\ell(a_1)}{n_\ell(a_1)} \tag{4.98}$$

this strongly suggests that the coefficients can be written like a complex number

$$z = x + iy = re^{i\theta} \qquad \text{where} \qquad \theta = \arctan\frac{y}{x}.$$

which gives

$$\alpha_\ell = -1 + \exp[i\delta'_\ell] \qquad \text{and} \qquad \beta_\ell = 1 - \exp[i\delta_\ell] \tag{4.99}$$

where

$$\delta_\ell = \arctan\left[\frac{[x j_\ell(x)]'_{x=a_1}}{[x n_\ell(x)]'_{x=a_1}}\right] \qquad \text{and} \qquad \delta'_\ell = \arctan\left[\frac{j_\ell(a_1)}{n_\ell(a_1)}\right] \tag{4.100}$$

These angles are none other than the scattering phase shifts often encountered in quantum mechanical scattering. If we considered the case $Z_2 \to \infty$, the above would still be true, but with $\delta_\ell \leftrightarrow \delta'_\ell$.

Lastly, let's consider a particular limiting case. Let's suppose that the sphere is non-magnetic and sits in a vacuum. Let's also assume that $\lambda \gg R$. What implications does this have for our parameters? The non-magnetic nature of the sphere makes $\mu_2 = \mu_0$; the material properties of the sphere are entirely determined by $\epsilon_2 = \epsilon$. Since we're in vacuum, this makes $Z_1 = Z_0$ and we can designate $Z_2 = Z_S = \frac{Z_0}{n}$, where $n = \sqrt{\frac{\epsilon}{\epsilon_0}}$ is the index of refraction of the sphere. Now let $k = k_1$. Since $\lambda_2 = \frac{n}{\lambda}$ then we can write $k_2 = nk$. Therefore, we can write $a = a_1 = k_1 R = kR$ and $a_2 = k_2 R = nkR = na$. Now if $\lambda \gg R$ (meaning either the wavelength is very large or the radius of the sphere is very small) this gives $a \ll 1$, and therefore we can approximate the spherical Bessel functions in the scattering coefficients by their limiting forms as we do in the long-wavelength approximation:

$$j_\ell(x) \to \frac{x^\ell}{(2\ell+1)!!}\left(1 - \frac{x^2}{2(2\ell+3)} + \dots\right) \qquad \text{and} \qquad h_\ell^{(1)}(x) \to -i\frac{(2\ell-1)!!}{x^{\ell+1}}\left(1 - \frac{x^2}{2(1-2\ell)} + \dots\right) \tag{4.101}$$

which allows us to simplify the coefficients. First consider α_ℓ

$$\begin{aligned}
\alpha_\ell &= \frac{Z_S S_\ell(a) S'_\ell(na) - Z_0 S_\ell(na) S'_\ell(a)}{Z_0 S_\ell(na) Q'_\ell(a) - Z_S Q_\ell(a) S'_\ell(na)} \\
&= \frac{Z_S\left(\frac{a^{\ell+1}}{(2\ell+1)!!}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\ell+1}}{(2\ell+1)!!}\right)\Big]_{x=na} - Z_0\left(\frac{(na)^{\ell+1}}{(2\ell+1)!!}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\ell+1}}{(2\ell+1)!!}\right)\Big]_{x=a}}{Z_0\left(\frac{(na)^{\ell+1}}{(2\ell+1)!!}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-i\frac{(2\ell-1)!!}{x^\ell}\right)\Big]_{x=a} - Z_S\left(-i\frac{(2\ell-1)!!}{a^\ell}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^{\ell+1}}{(2\ell+1)!!}\right)\Big]_{x=na}} \\
&= \frac{Z_S\left(\frac{a^{\ell+1}}{(2\ell+1)!!}\right)\left((\ell+1)\frac{(na)^\ell}{(2\ell+1)!!}\right) - Z_0\left(\frac{(na)^{\ell+1}}{(2\ell+1)!!}\right)\left((\ell+1)\frac{a^\ell}{(2\ell+1)!!}\right)}{Z_0\frac{(na)^{\ell+1}}{2\ell+1}\left(\frac{i\ell}{a^{\ell+1}}\right) - Z_S\left(-\frac{i}{a^\ell}\right)\left((\ell+1)\frac{(na)^\ell}{2\ell+1}\right)} \\
&= \frac{(\ell+1)\frac{n^\ell}{((2\ell+1)!!)^2}a^{2\ell+1}[Z_S - nZ_0]}{\frac{in^\ell}{2\ell+1}[nZ_0\ell + Z_S(\ell+1)]} \\
&= -i\frac{(2\ell+1)(\ell+1)}{((2\ell+1)!!)^2}a^{2\ell+1}\frac{Z_S - nZ_0}{nZ_0\ell + Z_S(\ell+1)} \\
&= -i\frac{(2\ell+1)(\ell+1)}{((2\ell+1)!!)^2}a^{2\ell+1}\frac{\frac{Z_0}{n} - nZ_0}{nZ_0\ell + \frac{Z_0}{n}(\ell+1)} \\
&= i\frac{(2\ell+1)(\ell+1)}{((2\ell+1)!!)^2}a^{2\ell+1}\frac{n^2-1}{n^2\ell+\ell+1}
\end{aligned} \tag{4.102}$$

and we can obtain β_ℓ by a change of sign and by replacing $n \to \frac{1}{n}$ to give:

$$\beta_\ell = -i\frac{(2\ell+1)(\ell+1)}{2((2\ell+1)!!)^2}a^{2\ell+1}\frac{n^2-1}{n^2\ell+\ell+1} \tag{4.103}$$

In the long-wavelength limit, only the dipole term $\ell = 1$ is really relevant, so we get

$$\alpha_{\ell=1} = \frac{2i}{3}a^3\frac{n^2-1}{n^2+2} \tag{4.104}$$

and

$$\beta_{\ell=1} = -\frac{i}{3}a^3\frac{n^2-1}{n^2+2} \tag{4.105}$$

and thus the differential scattering cross section is:

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{P_{\text{inc}}}\frac{\mathrm{d}P}{\mathrm{d}\Omega} &= \frac{2\pi Z_0}{k^2}\left\|\frac{2i\sqrt{3}}{3}a^3\left(\frac{n^2-1}{n^2+2}\right)\left[\boldsymbol{\Phi}_\ell^{\pm1}(\theta,\phi) \mp 2i\boldsymbol{\Psi}_\ell^{\pm1}(\theta,\phi)\right]\right\|^2 \\
&= \frac{2\pi Z_0}{k^2}\frac{4}{3}\left(\frac{n^2-1}{n^2+2}\right)^2 a^6\left\|\boldsymbol{\Phi}_1^{\pm1}(\theta,\phi) \mp 2i\boldsymbol{\Psi}_1^{\pm1}(\theta,\phi)\right\|^2 \\
&= \frac{8\pi Z_0}{3}\left(\frac{n^2-1}{n^2+2}\right)^2 k^4R^6\left\|\boldsymbol{\Phi}_1^{\pm1}(\theta,\phi) \mp 2i\boldsymbol{\Psi}_1^{\pm1}(\theta,\phi)\right\|^2
\end{aligned} \tag{4.106}$$

Working out the VSH in the above expression, we can write:

$$\left\|\boldsymbol{\Phi}_1^{\pm1}\right\|^2 = \left\|\boldsymbol{\Psi}_1^{\pm1}\right\|^2 = \frac{3}{16\pi}(1+\cos^2\theta) \qquad \text{and} \qquad \left[\pm i\left(\boldsymbol{\Psi}_1^{\pm1}\right)^* \cdot \boldsymbol{\Phi}_1^{\pm1}\right] = -\frac{3}{8\pi}\cos\theta \tag{4.107}$$

and thus the final expression is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{8\pi Z_0}{3}\left(\frac{n^2-1}{n^2+2}\right)^2 k^4R^6\left[\frac{15}{16\pi}\left(1+\cos^2\theta\right) - \frac{3}{2\pi}\cos\theta\right] = 4Z_0\left(\frac{n^2-1}{n^2+2}\right)^2 k^4R^6\left[\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right] \tag{4.108}$$

When we integrate over all angles, we get the total cross section

$$\sigma = \frac{4\pi}{3}Z_0\left(\frac{n^2-1}{n^2+2}\right)^2 k^4R^6 \tag{4.109}$$

which, given the λ^{-4} dependence, is nothing more than the the famous formula for Rayleigh scattering! It's no surprise that this should arise, given that Rayleigh's derivation made the assumption that the scatterer is *much* smaller than the incident wavelength, meaning the radiation is nothing more than an oscillating dipole. We can plot σ for comparison. Below is the log-log plot for the cross section as a function of $a = kR$

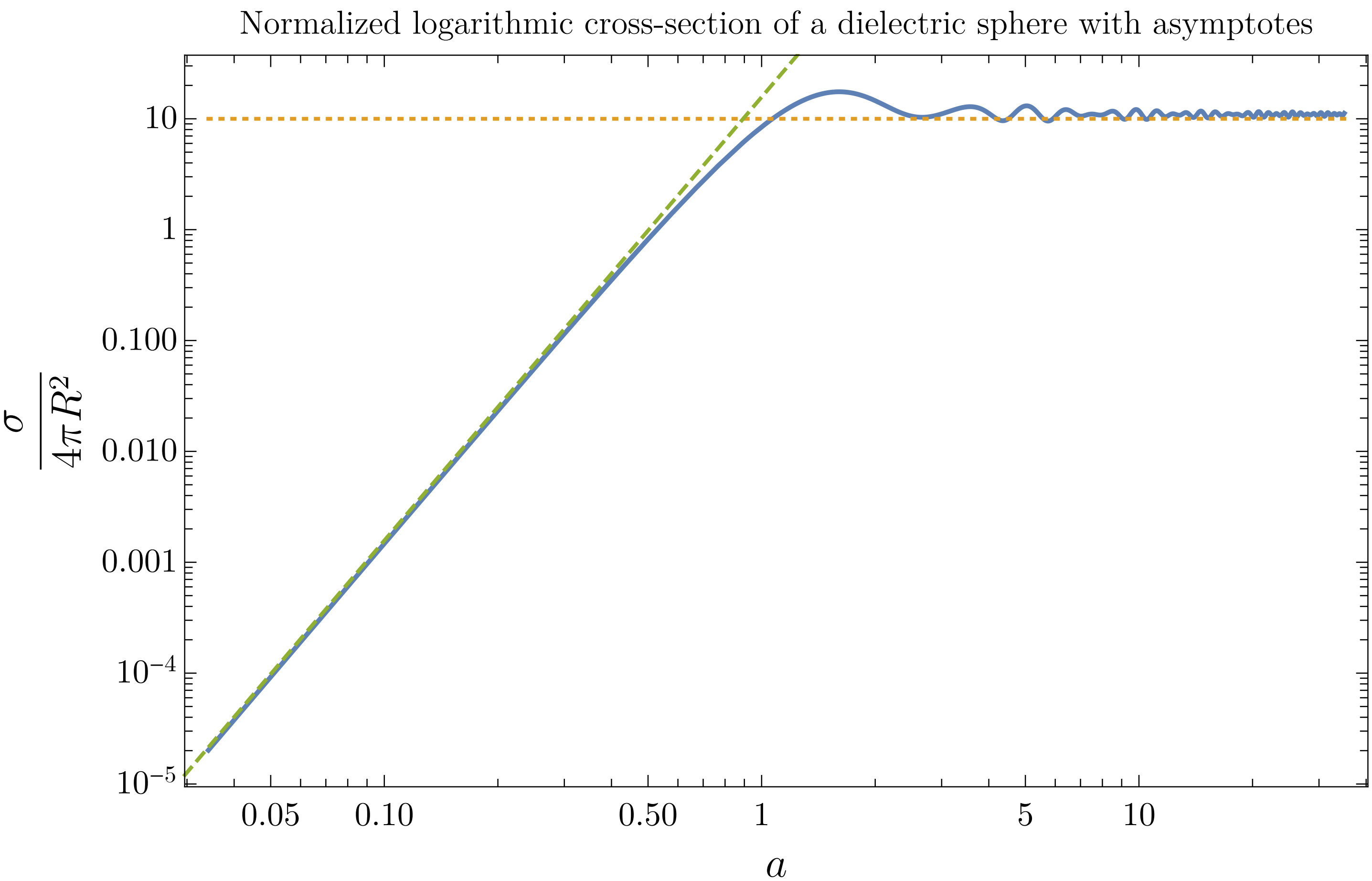

Figure 4.5: Normalized cross-section of a non-magnetic dielectric sphere of radius $R = 1$ mm along with the long-wavelength Rayleigh approximation (green) and the geometric asymptote (orange)

Note that for large λ, the cross section is quartic and thus exhibits the Rayleigh behavior; as λ shrinks the total cross section departs from the Rayleigh law and then oscillates about a constant value. That constant value is nothing more than the geometric cross section.

The cross section itself exhibits similar behavior:

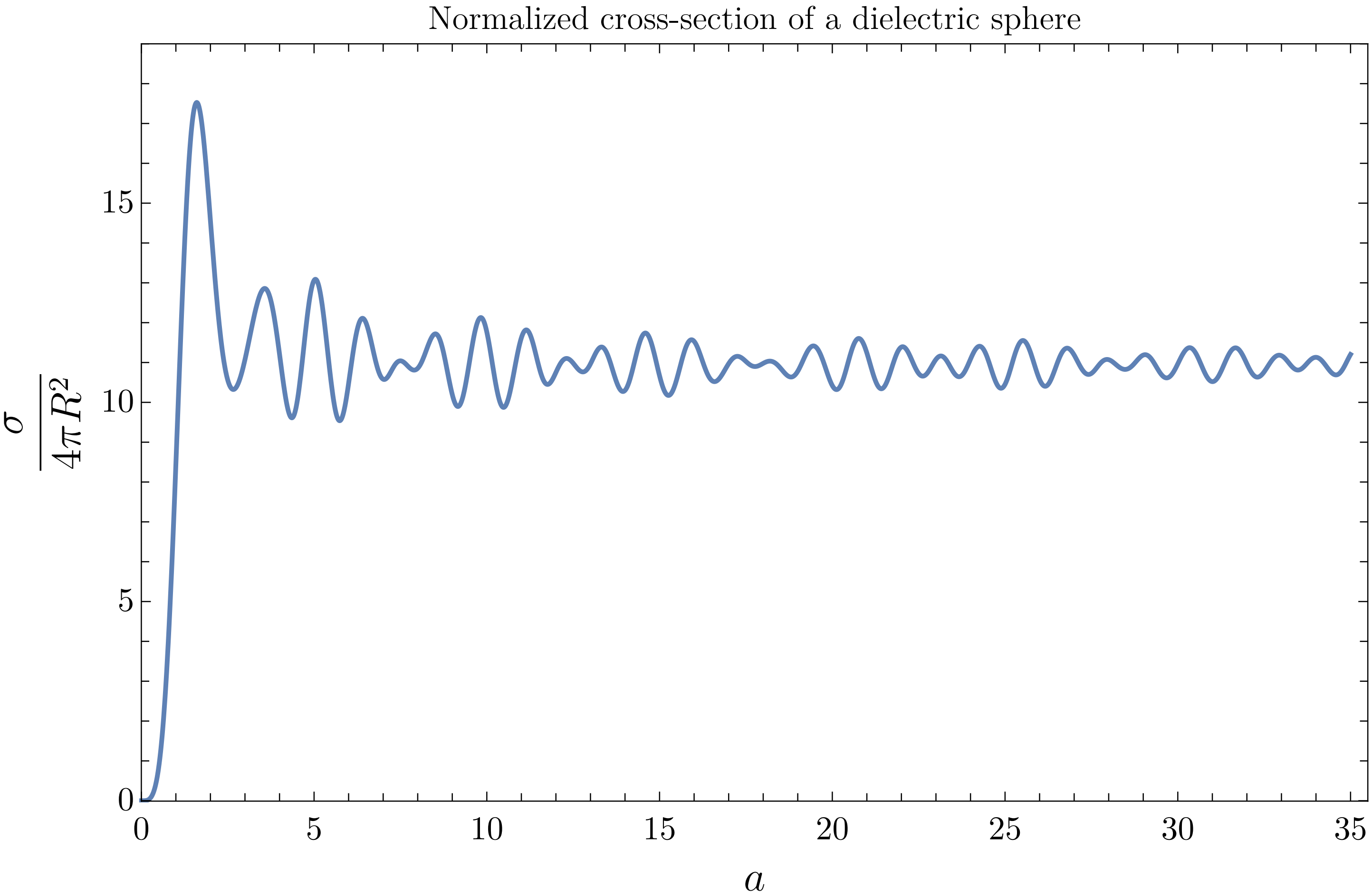

Figure 4.6

Note that quartic behavior near the origin, followed by oscillation about a constant, the geometric cross-section. Note also the presence of many little wiggles, which correspond to the resonances excited in the sphere by the incident radiation. Such resonances can be further explored using the *Debye series* decomposition, which rewrites the fields as combination of reflection and transmission coefficients, which allows for interpretation in terms of singly, doubly, etc. internally-reflected beams. As these beams emerge from the sphere they correspond to the spikes seen in the above pattern.

Conclusion

In this document I've presented a unified approach to electromagnetic radiation using the vector spherical harmonics, along with some useful approximation schemes and applications. Of course, there is much more in the rich world of electrodynamics that I could not cover (I had to stop writing at some point!), such as dielectric resonant cavities, coupled modes, guided waves and more! That said, I believe the formalism contained in these pages lends itself to easy application to these areas, and the student who understands these notes will be well-prepared to tackle these challenges.

A
Scalar spherical harmonics

Defintions

The scalar spherical harmonics (SSH) are defined as

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} \sqrt{\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta) \exp[im\phi] \tag{A.1}$$

where $P_\ell^m(x)$ are the associated Legendre polynomials defined as

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} \left(1 - x^2\right)^{\frac{m}{2}} \left[\frac{\mathrm{d}}{\mathrm{d}x}\right]^{\ell+m} \left(x^2 - 1\right)^\ell \tag{A.2}$$

The SSH are orthornormal and form a complete basis:

$$\int \left(Y_{\ell'}^{m'}(\theta, \phi)\right)^* Y_\ell^m(\theta, \phi) \mathrm{d}\Omega = \delta_{\ell\ell'} \delta_{mm'} \tag{A.3}$$

The SSH are eigenstates of the angular momentum operator, which has position-space representation The position space representation of the orbital angular momentum operator is, of course

$$\vec{L} = \frac{1}{i} \vec{r} \times \nabla = \frac{r}{i} \hat{r} \times \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) = \frac{1}{i}\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right) \tag{A.4}$$

The raising and lowering operators may be written in terms of the components of the angular momentum operator as

$$L_\pm = L_x \pm iL_y \qquad \text{which implies} \qquad \vec{L} = \frac{1}{2}\begin{pmatrix} L_+ + L_- \\ -i(L_+ - L_-) \\ 2L_0 \end{pmatrix} \tag{A.5}$$

and have effect on the SSH

$$L_\pm Y_\ell^{m_\ell}(\theta, \phi) = \sqrt{(\ell \mp m_\ell)(\ell \pm m_\ell + 1)} Y_\ell^{m_\ell}(\theta, \phi) \tag{A.6}$$

Recursion relations

$$Y_\ell^{m-1}(\theta,\phi) = \frac{\exp[-i\phi]}{\sqrt{(\ell-m)(\ell+m+1)}}\left[\frac{\partial}{\partial\theta} - m\cot\theta\right] Y_\ell^m(\theta,\phi) \tag{A.7}$$

$$Y_\ell^{m+1}(\theta,\phi) = \frac{\exp[i\phi]}{\sqrt{(\ell+m)(\ell-m+1)}}\left[-\frac{\partial}{\partial\theta} - m\cot\theta\right] Y_\ell^m(\theta,\phi) \tag{A.8}$$

$$Y_{\ell+1}^{m}(\theta,\phi) = \sqrt{\frac{2\ell+3}{(2\ell+1)(\ell+m+1)(\ell-m+1)}}\sin\theta\left[\frac{\partial}{\partial\theta} + (\ell+1)\cot\theta\right] Y_\ell^m(\theta,\phi) \tag{A.9}$$

$$Y_{\ell-1}^{m}(\theta,\phi) = \sqrt{\frac{2\ell-1}{(2\ell+1)(\ell+m)(\ell-m)}}\sin\theta\left[-\frac{\partial}{\partial\theta} + \ell\cot\theta\right] Y_\ell^m(\theta,\phi) \tag{A.10}$$

$$Y_{\ell+1}^{m+1}(\theta,\phi) = \exp[i\phi]\sqrt{\frac{2\ell+3}{(2\ell+1)(\ell+m+1)(\ell+m+2)}}\left[\cos\theta\frac{\partial}{\partial\theta} - (\ell+1)\sin\theta - \frac{m}{\sin\theta}\right] Y_\ell^m(\theta,\phi) \tag{A.11}$$

$$Y_{\ell+1}^{m-1}(\theta,\phi) = \exp[-i\phi]\sqrt{\frac{2\ell+3}{(2\ell+1)(\ell-m+1)(\ell-m+2)}}\left[-\cos\theta\frac{\partial}{\partial\theta} + (\ell+1)\sin\theta - \frac{m}{\sin\theta}\right] Y_\ell^m(\theta,\phi) \tag{A.12}$$

$$Y_{\ell-1}^{m+1}(\theta,\phi) = \exp[i\phi]\sqrt{\frac{2\ell-1}{(2\ell+1)(\ell-m-1)(\ell-m)}}\left[\cos\theta\frac{\partial}{\partial\theta} + \ell\sin\theta - \frac{m}{\sin\theta}\right] Y_\ell^m(\theta,\phi) \tag{A.13}$$

$$Y_{\ell-1}^{m-1}(\theta,\phi) = \exp[-i\phi]\sqrt{\frac{2\ell-1}{(2\ell+1)(\ell+m)(\ell+m-1)}}\left[-\cos\theta\frac{\partial}{\partial\theta} - \ell\sin\theta - \frac{m}{\sin\theta}\right] Y_\ell^m(\theta,\phi) \tag{A.14}$$

B
Vector spherical harmonics

Explicit forms

Here we'll give a few explicit expressions for $\mathbf{\Psi}_\ell^m(\theta,\phi)$; note that $\mathbf{\Phi}_\ell^m(\theta,\phi)$ is easily obtained by taking the relevant cross product. For $\ell = 1$

$$\mathbf{\Psi}_1^0(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\hat{\theta} \qquad \mathbf{\Psi}_1^{\pm1}(\theta,\phi) = -\sqrt{\frac{3}{16\pi}}\exp[\pm i\phi]\left[\cos\theta\hat{\theta} + i\hat{\phi}\right] \tag{B.1}$$

For $\ell = 2$

$$\mathbf{\Psi}_2^0(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\hat{\theta} \qquad \mathbf{\Psi}_2^{\pm1}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\exp[\pm i\phi]\left[\left(1-2\cos^2\theta\right)\hat{\theta} \mp i\cos\theta\hat{\phi}\right] \tag{B.2}$$

$$\mathbf{\Psi}_2^{\pm2}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\exp[\pm 2i\phi]\sin\theta\left[\cos\theta\hat{\theta} \pm i\hat{\phi}\right] \tag{B.3}$$

For $\ell = 3$

$$\mathbf{\Psi}_3^0(\theta,\phi) = -\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)\hat{\theta} \tag{B.4}$$

$$\mathbf{\Psi}_3^{\pm1}(\theta,\phi) = \sqrt{\frac{7}{256\pi}}\exp[\pm i\phi]\left[\cos\theta\left(5\cos^2\theta - 9\right)\hat{\theta} \mp i\left(5\cos^2\theta - 1\right)\hat{\phi}\right] \tag{B.5}$$

$$\mathbf{\Psi}_3^{\pm2}(\theta,\phi) = \sqrt{\frac{35}{32\pi}}\exp[\pm 2i\phi]\sin\theta\left[\frac{1}{2}\left(3\cos^2\theta + 1\right)\hat{\theta} \mp i\cos\theta\hat{\phi}\right] \tag{B.6}$$

$$\mathbf{\Psi}_3^{\pm3}(\theta,\phi) = -\sqrt{\frac{105}{128\pi}}\exp[\pm 3i\phi]\sin^2\theta\left[\cos\theta\hat{\theta} \pm i\hat{\phi}\right] \tag{B.7}$$

Angular distributions

Here we'll give a few angular distributions used in the text for $\left\|\mathbf{\Phi}_\ell^m(\theta,\phi)\right\|^2$. For $\ell = 1$:

$$\left\|\mathbf{\Phi}_1^0(\theta,\phi)\right\|^2 = \frac{3}{8\pi}\sin^2\theta \qquad \left\|\mathbf{\Phi}_1^{\pm 1}(\theta,\phi)\right\|^2 = \frac{3}{16\pi}(1+\cos^2\theta) \tag{B.8}$$

For $\ell = 2$:

$$\left\|\mathbf{\Phi}_2^0(\theta,\phi)\right\|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta \qquad \left\|\mathbf{\Phi}_2^{\pm 1}(\theta,\phi)\right\|^2 = \frac{5}{16\pi}(1-3\cos^2+4\cos^4\theta) \tag{B.9}$$

$$\left\|\mathbf{\Phi}_2^{\pm 2}(\theta,\phi)\right\|^2 = \frac{5}{16\pi}(1-\cos^4\theta) \tag{B.10}$$

C Plane wave expansion

One of the most common uses for the multipole expansion is in its application to spherically symmetric problems, such as scattering off a dielectric sphere. A usual approximation is to treat the incident radiation as planar; if the material is linear, we can expand the plane wave in a series of spherical waves, which do not couple to each other upon scattering, since spherically-symmetric problems do not couple angular momentum modes. As such, I'll present a brief derivation of this expansion here, both for scalar and vector cases.

Scalar waves

Suppose we have a scalar field ψ. A plane wave propagating in the z-direction in this field will be of the form $\psi(\vec{r}, t) = A\exp[-i\omega t + ikz]$. This can be expanded in a series of spherical waves as

$$\exp[ikz] = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}j_{\ell}(kr)P_{\ell}(\cos\theta) \tag{C.1}$$

To obtain this, we begin with the integral representation of the spherical Bessel function

$$j_{\ell}(x) = \frac{1}{2\ell!}\left(\frac{x}{2}\right)^{\ell}\int_{-1}^{1}\exp[ixt]\left(1-t^2\right)^{\ell}\mathrm{d}t \tag{C.2}$$

If we rewrite this slightly in terms of a derivative and integrate by parts we obtain:

$$\begin{aligned} j_{\ell}(x) &= \frac{1}{2\ell!}\left(\frac{x}{2}\right)^{\ell}\int_{-1}^{1}\frac{1}{ix}\frac{\mathrm{d}}{\mathrm{d}t}\exp[ixt]\left(1-t^2\right)^{\ell}\mathrm{d}t \\ &= \frac{1}{2\ell!}\left(\frac{x}{2}\right)^{\ell}\left(\cancelto{0}{\left[\exp[ixt]\left(1-t^2\right)^{\ell}\right]_{-1}^{1}} - \int_{-1}^{1}\frac{1}{ix}\exp[ixt]\frac{\mathrm{d}}{\mathrm{d}t}\left(1-t^2\right)^{\ell}\mathrm{d}t\right) \\ &= -\frac{1}{2\ell!}\left(\frac{x}{2}\right)^{\ell}\int_{-1}^{1}\frac{1}{ix}\exp[ixt]\frac{\mathrm{d}}{\mathrm{d}t}\left(1-t^2\right)^{\ell}\mathrm{d}t \end{aligned}$$

If we repeat this process ℓ times, we obtain

$$j_{\ell}(x) = \frac{1}{2\ell!}\left(\frac{x}{2}\right)^{\ell}\int_{-1}^{1}\frac{1}{(ix)^{\ell}}\exp[ixt]\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell}\left(1-t^2\right)^{\ell}\mathrm{d}t = \frac{1}{2}\int_{-1}^{1}(-i)^{\ell}\exp[ixt]\left[\frac{1}{2^{\ell}\ell!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\ell}\left(t^2-1\right)^{\ell}\right]\mathrm{d}t \tag{C.3}$$

The term in brackets is the definition of the Legendre polynomials:

$$P_\ell(t) = \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell \left(t^2-1\right)^\ell \tag{C.4}$$

and thus the spherical Bessel function is written

$$j_\ell(x) = \frac{(-i)^\ell}{2}\int_{-1}^{1} \exp[ixt] P_\ell(t)\mathrm{d}t \tag{C.5}$$

The Legendre polynomials form a complete orthonormal basis on the interval $[-1,1]$ which allows us to write

$$\int_{-1}^{1} P_\ell(t)P_m(t)\mathrm{d}t = \frac{2}{2\ell+1}\delta_{\ell m} \qquad \text{and} \qquad \sum_{\ell=0}^{\infty}\frac{2\ell+1}{2}P_\ell(t)P_\ell(t') = \delta(t-t') \tag{C.6}$$

Therefore if we multiply $j_\ell(x)$ by $\frac{2\ell+1}{2}P_\ell(t')$ and sum over ℓ we can write

$$\begin{aligned}
2(-i)^{-\ell}j_\ell(x) &= \int_{-1}^{1}\exp[ixt]P_\ell(t)\mathrm{d}t \\
\sum_{\ell=0}^{\infty} 2i^\ell \frac{2\ell+1}{2}P_\ell(t')j_\ell(x) &= \int_{-1}^{1}\exp[ixt]\sum_{\ell=0}^{\infty}\frac{2\ell+1}{2}P_\ell(t')P_\ell(t)\mathrm{d}t \\
\sum_{\ell=0}^{\infty} 2i^\ell (2\ell+1)P_\ell(t')j_\ell(x) &= \int_{-1}^{1}\exp[ixt]\sum_{\ell=0}^{\infty}\delta(t-t')\mathrm{d}t \\
\sum_{\ell=0}^{\infty} i^\ell (2\ell+1)P_\ell(t')j_\ell(x) &= \exp[ixt']
\end{aligned} \tag{C.7}$$

If we let $t' = \cos\theta$ and $x = kr$ then we get

$$\exp[ikr\cos\theta] = \exp[ikz] = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)P_\ell(\cos\theta)j_\ell(kr) \tag{C.8}$$

as desired. This formula can be generalized for the case of a plane wave propagating in an arbitrary direction by means of the spherical harmonic addition theorem to give

$$\exp[i\vec{k}\cdot\vec{r}] = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} i^\ell j_\ell(kr)\left(Y_\ell^m(\theta_{\vec{k}},\phi_{\vec{k}})\right)^* Y_\ell^m(\theta_{\vec{r}},\phi_{\vec{r}}) \tag{C.9}$$

Vector waves

We can derive an analogous formula to the above for vector waves. Suppose we have a circularly polarized plane wave propagating along the z axis which is given by

$$\vec{E}(\vec{r}) = \frac{\mathcal{E}}{\sqrt{2}}\left(\hat{x} \pm i\hat{y}\right)\exp[ikz] \qquad \text{and} \qquad c\vec{B}(\vec{r}) = \hat{z}\times\vec{E} = \mp i\vec{E} \tag{C.10}$$

where the $(\pm)$ refers to the helicity of the wave. Quoting from the text, the general expansion of an EM wave is

$$\vec{E} = \sum_{\ell,m}\left[\Lambda_E^{(\pm)}(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{kr}j_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rj_\ell(kr)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + \Lambda_M^{(\pm)}(\ell,m)j_\ell(kr)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{C.11}$$

$$c\vec{B} = \sum_{\ell,m}\left[-\Lambda_M^{(\pm)}(\ell,m)\left(\frac{\sqrt{\ell(\ell+1)}}{kr}j_\ell(kr)\mathbf{Y}_\ell^m(\theta,\phi) + \frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rj_\ell(kr)\right)\mathbf{\Psi}_\ell^m(\theta,\phi)\right) + \Lambda_E^{(\pm)}(\ell,m)j_\ell(kr)\mathbf{\Phi}_\ell^m(\theta,\phi)\right] \tag{C.12}$$

Using the orthogonality of the VSH we can write:

$$\Lambda_M^{(\pm)}(\ell,m)j_\ell(kr) = \int \left(\mathbf{\Phi}_\ell^m(\theta,\phi)\right)^* \cdot \vec{E}\mathrm{d}\Omega \tag{C.13}$$

$$\Lambda_E^{(\pm)}(\ell,m)j_\ell(kr) = c\int \left(\mathbf{\Phi}_\ell^m(\theta,\phi)\right)^* \cdot \vec{B}\mathrm{d}\Omega \tag{C.14}$$

Let's analyze the Λ_M term explicitly. Before doing so, recall that we can write the combination $\hat{x} \pm i\hat{y}$ as

$$\hat{x} \pm i\hat{y} = \mp\left(\mp\hat{x} - i\hat{y}\right) = \mp\sqrt{2}\hat{e}_\pm$$

Therefore the electric field becomes

$$\vec{E}(\vec{r}) = \frac{\mathcal{E}}{\sqrt{2}}\left(\hat{x} \pm i\hat{y}\right)\exp[ikz] = \mp\mathcal{E}\hat{e}_\pm\exp[ikz] \tag{C.15}$$

Using the explicit form of the $\mathbf{\Phi}_\ell^m$ in terms of the angular momentum operator $\vec{L}$, we can write

$$\begin{aligned}
\Lambda_M^{(\pm)}(\ell,m)j_\ell(kr) &= \frac{1}{\sqrt{\ell(\ell+1)}}\int \left(\vec{L}Y_\ell^m(\theta,\phi)\right)^* \cdot \vec{E}\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{1}{\sqrt{\ell(\ell+1)}}\int \left(\vec{L}Y_\ell^m(\theta,\phi)\right)^* \cdot \hat{e}_\pm\exp[ikz]\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{1}{\sqrt{\ell(\ell+1)}}\int \left(L_\mp Y_\ell^m(\theta,\phi)\right)^*\exp[ikz]\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{\sqrt{(\ell\pm m)(\ell\mp m+1)}}{\sqrt{\ell(\ell+1)}}\int \left(Y_\ell^{m\mp1}(\theta,\phi)\right)^*\exp[ikz]\mathrm{d}\Omega
\end{aligned} \tag{C.16}$$

Now recall that

$$Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta) \qquad \Rightarrow \qquad P_\ell(\cos\theta) = \sqrt{\frac{4\pi}{2\ell+1}}Y_\ell^0(\theta,\phi) \tag{C.17}$$

which when combined with scalar plane wave expansion derived above gives

$$\begin{aligned}
\Lambda_M^{(\pm)}(\ell,m)j_\ell(kr) &= \mp\mathcal{E}\frac{\sqrt{(\ell\pm m)(\ell\mp m+1)}}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^{m\mp1}(\theta,\phi)\right)^*\exp[ikz]\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{\sqrt{(\ell\pm m)(\ell\mp m+1)}}{\sqrt{\ell(\ell+1)}}\int\left(Y_\ell^{m\mp1}(\theta,\phi)\right)^*\sum_{\ell'=0}^{\infty}i^{\ell'}(2\ell'+1)P_{\ell'}(\cos\theta)j_{\ell'}(kr)\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{\sqrt{(\ell\pm m)(\ell\mp m+1)}}{\sqrt{\ell(\ell+1)}}\sum_{\ell'=0}^{\infty}i^{\ell'}(2\ell'+1)j_{\ell'}(kr)\int\left(Y_\ell^{m\mp1}(\theta,\phi)\right)^*P_{\ell'}(\cos\theta)\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{\sqrt{(\ell\pm m)(\ell\mp m+1)}}{\sqrt{\ell(\ell+1)}}\sum_{\ell'=0}^{\infty}i^{\ell'}(2\ell'+1)\sqrt{\frac{4\pi}{2\ell'+1}}j_{\ell'}(kr)\int\left(Y_\ell^{m\mp1}(\theta,\phi)\right)^*Y_{\ell'}^0(\theta,\phi)\mathrm{d}\Omega \\
&= \mp\mathcal{E}\frac{\sqrt{(\ell\pm m)(\ell\mp m+1)}}{\sqrt{\ell(\ell+1)}}\sum_{\ell'=0}^{\infty}i^{\ell'}\sqrt{4\pi(2\ell'+1)}j_{\ell'}(kr)\delta_{\ell\ell'}\delta_{m\mp1,0} \\
&= \mp\mathcal{E}\frac{\sqrt{(\ell+1)(\ell-1+1)}}{\sqrt{\ell(\ell+1)}}i^\ell\sqrt{4\pi(2\ell+1)}j_\ell(kr)\delta_{m,\pm1}
\end{aligned}$$

$$\Rightarrow \qquad \Lambda_M^{(\pm)}(\ell,m) = \mp\mathcal{E}i^\ell\sqrt{4\pi(2\ell+1)}\delta_{m,\pm1} \tag{C.18}$$

Using the relation $\vec{B} = \mp i\vec{E}$, then we can easily write

$$\Lambda_E^{\pm}(\ell,m) = \mp i\Lambda_M(\ell,m) = \mathcal{E}i^{\ell+1}\sqrt{4\pi(2\ell+1)}\delta_{m,\pm1} \tag{C.19}$$

Thus the multipole expansion of the plane wave propagating in the z-direction is

$$\vec{E} = \mathcal{E}\sum_{\ell=1}^{\infty}i^\ell\sqrt{4\pi(2\ell+1)}\left[i\left(\frac{\sqrt{\ell(\ell+1)}}{kr}j_\ell(kr)\mathbf{Y}_\ell^{\pm1}(\theta,\phi)+\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rj_\ell(kr)\right)\mathbf{\Psi}_\ell^{\pm1}(\theta,\phi)\right)\mp j_\ell(kr)\mathbf{\Phi}_\ell^{\pm1}(\theta,\phi)\right] \tag{C.20}$$

$$c\vec{B} = \mathcal{E}\sum_{\ell=1}^{\infty}i^\ell\sqrt{4\pi(2\ell+1)}\left[\pm\left(\frac{\sqrt{\ell(\ell+1)}}{kr}j_\ell(kr)\mathbf{Y}_\ell^{\pm1}(\theta,\phi)+\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}\left(rj_\ell(kr)\right)\mathbf{\Psi}_\ell^{\pm1}(\theta,\phi)\right)+ij_\ell(kr)\mathbf{\Phi}_\ell^{\pm1}(\theta,\phi)\right] \tag{C.21}$$

as desired.