Quick and Dirty SUSY

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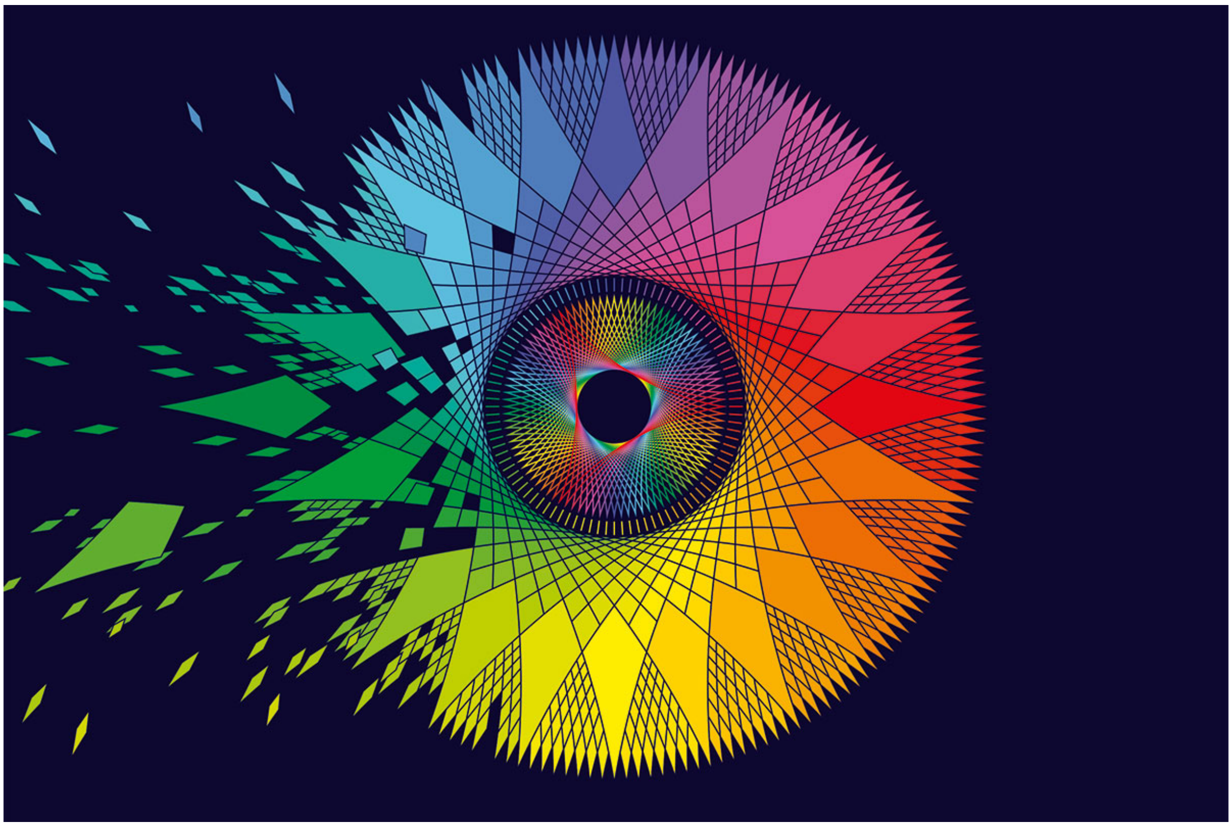

Preface

These notes constitute my collection of handy SUSY relations in the superfield formalism for $\mathcal{N} = 1$ supersymmetry. I may perhaps expand on them for higher supersymmetries in the future.

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Part I
General Superfields

1 Definitions

A $\mathcal{N}=1$ superfield is a function of spacetime coordinates x_μ and Grassmann spinor coordinates θ_α and their conjugates $\theta^\dagger_{\dot{\alpha}}$. As expansions in Grassmann coordinates naturally truncate, such a field has an expansion

$$\begin{aligned}\mathcal{S}(x,\theta,\theta^\dagger) = a(x) + \theta\cdot\xi(x) + \theta^\dagger\cdot\chi^\dagger(x) + \theta\cdot\theta b(x) + \theta^\dagger\cdot\theta^\dagger c(x) \\ + \theta^\dagger\bar{\sigma}^\mu\theta v_\mu(x) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x) + \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x) + \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger d(x)\end{aligned} \tag{1.1}$$

where a,b,c,d are complex scalars, v_μ is a complex vector and ξ,χ,ζ,η are complex Weyl spinors. Thus there are

$$2\times 4 + 2\times 4 = 16 \text{ bosonic d.o.f.} \qquad \text{and} \qquad 2\times 2\times 4 = 16 \text{ fermionic d.o.f.} \tag{1.2}$$

meaning the number of bosonic and fermionic degrees of freedom match as desired/expected. Note that this is a scalar-valued superfield. There can be such things as spinor-valued and vector-valued superfields where the terms in the expansion must have identical Lorentz index structure. The mass dimensions of the fields are as follows. Since all factors in the expansion must have the same dimension and scalar fields (the $a(x)$ field in this case) have $[a]=1$, this must mean that a product like $[\theta\cdot\xi]=1$. But since $[\xi]=\frac{3}{2}$, then $[\theta]=[\theta^\dagger]=-\frac{1}{2}$. Thus we see that the bosonic fields have mass dimensions

$$[a]=1 \qquad [b]=[c]=[v_\mu]=2 \qquad [d]=3 \tag{1.3}$$

whilst the fermionic fields have mass dimension

$$[\xi]=[\chi]=\frac{3}{2} \qquad [\eta]=[\zeta]=\frac{5}{2} \tag{1.4}$$

1.1 Translations in Superspace

As given in the appendix, the generators of the SUSY algebra are

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\theta^\dagger)_\alpha\partial_\mu \qquad Q^{\dagger\dot{\alpha}} = i\frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}} - (\bar{\sigma}^\mu\theta)^{\dot{\alpha}}\partial_\mu \tag{1.5}$$

Recall that an ordinary quantum field behaves as follows under the Poincaré group, for example:

$$e^{ia_\mu\hat{P}^\mu}\hat{\phi}(x)e^{-ia_\mu\hat{P}^\mu} = \hat{\phi}(x+a) \qquad \phi(x+a) = e^{ia_\mu P^\mu}\phi(x) \tag{1.6}$$

or

$$[a_\mu\hat{P}^\mu, \hat{\phi}(x)] = a_\mu P^\mu\hat{\phi}(x) = -ia_\mu\partial^\mu\phi(x) \tag{1.7}$$

Thus we see that

$$\delta\phi(x) = ia_\mu\partial^\mu\phi(x) \tag{1.8}$$

Similarly, under the SUSY charges, a superfield transforms as

$$[\mathcal{S}, \epsilon\cdot\hat{Q} + \epsilon^\dagger\cdot\hat{Q}^\dagger] = (\epsilon\cdot Q + \epsilon^\dagger\cdot Q^\dagger)\mathcal{S} \tag{1.9}$$

$$
\begin{aligned}
\sqrt{2}\delta_\epsilon \mathcal{S} &= -i\left(\epsilon\cdot Q + \epsilon^\dagger\cdot Q^\dagger\right)\mathcal{S} && (1.10)\\
&= \left(\epsilon\cdot\frac{\partial}{\partial\theta} + \epsilon^\dagger\cdot + i\left[\epsilon\sigma^\mu\theta^\dagger + \epsilon^\dagger\bar{\sigma}^\mu\theta\right]\partial_\mu\right)\mathcal{S} && (1.11)\\
&= \mathcal{S}(x^\mu + i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\bar{\sigma}^\mu\theta, \theta+\epsilon, \theta^\dagger+\epsilon^\dagger) - \mathcal{S}(x,\theta,\theta^\dagger) && (1.12)
\end{aligned}
$$

Thus the action of the supercharges is to execute translation in superspace

$$
\theta^\alpha \to \theta^\alpha + \epsilon^\alpha \qquad \theta^\dagger_{\dot\alpha} \to \theta^\dagger_{\dot\alpha} + \epsilon^\dagger_{\dot\alpha} \qquad x^\mu \to x^\mu + \underbrace{i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\bar{\sigma}^\mu\theta}_{\Delta^\mu} \tag{1.13}
$$

These transformations can be used to easily implement the SUSY transformations among the components. The $a(x)$ rotates into the $\xi(x)$ and $\chi(x)$ fields:

$$
\begin{aligned}
a(x) &\to a(x+\Delta)\\
&= a(x) + \Delta^\mu\partial_\mu a(x)\\
&= a(x) + \left(i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\bar{\sigma}^\mu\theta\right)\partial_\mu a(x)\\
&= a(x) + \theta^\dagger\cdot\left(-i\bar{\sigma}^\mu\epsilon\partial_\mu a(x)\right) + \theta\cdot\left(-i\sigma^\mu\epsilon^\dagger\partial_\mu a(x)\right)
\end{aligned}
$$

The ξ field rotates into the $a(x)$ field, the $b(x)$ field and the $v_\mu(x)$ field:

$$
\begin{aligned}
\theta\cdot\xi(x) &\to (\theta+\epsilon)\cdot\xi(x+\Delta)\\
&= \theta\cdot\xi(x) + \theta\cdot\Delta^\mu\partial_\mu\xi(x) + \epsilon\cdot\xi(x) + \epsilon\cdot\Delta^\mu\partial_\mu\xi(x)\\
&= \theta\cdot\xi(x) + \theta\cdot\left(i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\bar{\sigma}^\mu\theta\right)\partial_\mu\xi(x) + \epsilon\cdot\xi(x) + O(\epsilon^2)\\
&= \theta\cdot\xi(x) + \epsilon\cdot\xi(x) + i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\theta^{\dagger\dot\alpha}\theta^\beta\partial_\mu\xi_\beta(x) + i\epsilon^\dagger_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}\theta_\alpha\theta^\beta\partial_\mu\xi_\beta(x)\\
&= \theta\cdot\xi(x) + \epsilon\cdot\xi(x) + i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\left(-\frac{1}{2}\theta\sigma^\nu\theta^\dagger\bar{\sigma}^{\dot\alpha\beta}_\nu\right)\partial_\mu\xi_\beta(x) + i\epsilon^\dagger_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\theta^\gamma\theta^\beta\partial_\mu\xi_\beta(x)\\
&= \theta\cdot\xi(x) + \epsilon\cdot\xi(x) + \theta^\dagger\bar{\sigma}^\nu\theta\left(\frac{i}{2}\epsilon\sigma^\mu\bar{\sigma}_\nu\partial_\mu\xi(x)\right) + i\epsilon^\dagger_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\left(-\frac{1}{2}\epsilon^{\gamma\beta}\theta\cdot\theta\right)\partial_\mu\xi_\beta(x)\\
&= \theta\cdot\xi(x) + \epsilon\cdot\xi(x) + \theta^\dagger\bar{\sigma}^\nu\theta\left(\frac{i}{2}\epsilon\sigma^\mu\bar{\sigma}_\nu\partial_\mu\xi(x)\right) - \theta\cdot\theta\left(\frac{i}{2}\epsilon^\dagger_{\dot\alpha}\bar{\sigma}^{\mu\dot\alpha\alpha}\delta^\beta_\alpha\partial_\mu\xi_\beta(x)\right)\\
&= \theta\cdot\xi(x) + \epsilon\cdot\xi(x) + \theta^\dagger\bar{\sigma}^\nu\theta\left(\frac{i}{2}\epsilon\sigma^\mu\bar{\sigma}_\nu\partial_\mu\xi(x)\right) + \theta\cdot\theta\left(-\frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi(x)\right)
\end{aligned}
$$

The $\chi^\dagger$ field rotates into the $a(x)$ field, the $c(x)$ field and the $v_\mu(x)$ field.

$$\begin{aligned}
\theta^\dagger \cdot \chi^\dagger(x) &\to (\theta^\dagger + \epsilon^\dagger) \cdot \xi^\dagger(x+\Delta) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \theta^\dagger \cdot \Delta^\mu \partial_\mu \xi(x) + \epsilon^\dagger \cdot \chi^\dagger(x) + \epsilon^\dagger \cdot \Delta^\mu \partial_\mu \chi^\dagger(x) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \theta^\dagger \cdot \left(i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\bar{\sigma}^\mu\theta \right) \partial_\mu \chi^\dagger(x) + \epsilon^\dagger \cdot \chi^\dagger(x) + O(\epsilon^2) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \epsilon^\dagger \cdot \chi^\dagger(x) + i\epsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \theta^{\dagger\dot{\alpha}} \theta^\dagger_{\dot{\beta}} \partial_\mu \chi^{\dagger\dot{\beta}}(x) + i\epsilon^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \theta_\alpha \theta^\dagger_{\dot{\beta}} \partial_\mu \chi^{\dagger\dot{\beta}}(x) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \epsilon^\dagger \cdot \chi^\dagger(x) + i\epsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\gamma}} \theta^\dagger_{\dot{\gamma}} \theta^\dagger_{\dot{\beta}} \partial_\mu \chi^{\dagger\dot{\beta}}(x) + i\epsilon^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \left(-\frac{1}{2} \theta^\dagger \bar{\sigma}_\nu \theta \sigma^\nu_{\alpha\dot{\beta}} \right) \partial_\mu \chi^{\dagger\dot{\beta}}(x) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \epsilon^\dagger \cdot \chi^\dagger(x) + i\epsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\gamma}} \left(-\frac{1}{2} \epsilon_{\dot{\gamma}\dot{\beta}} \theta^\dagger \cdot \theta^\dagger \right) \partial_\mu \chi^{\dagger\dot{\beta}}(x) - \theta^\dagger \bar{\sigma}_\nu \theta \left(\frac{i}{2} \epsilon^\dagger \bar{\sigma}^\mu \sigma^\nu \partial_\mu \chi^\dagger(x) \right) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \epsilon^\dagger \cdot \chi^\dagger(x) - \theta^\dagger \cdot \theta^\dagger \left(\frac{i}{2} \epsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \delta^{\dot{\alpha}}_{\dot{\beta}} \partial_\mu \chi^{\dagger\dot{\beta}}(x) \right) - \theta^\dagger \bar{\sigma}_\nu \theta \left(\frac{i}{2} \epsilon^\dagger \bar{\sigma}^\mu \sigma^\nu \partial_\mu \chi^\dagger(x) \right) \\
&= \theta^\dagger \cdot \chi^\dagger(x) + \epsilon^\dagger \cdot \chi^\dagger(x) + \theta^\dagger \cdot \theta^\dagger \left(-\frac{i}{2} \epsilon \sigma^\mu \partial_\mu \chi^\dagger(x) \right) + \theta^\dagger \bar{\sigma}_\nu \theta \left(-\frac{i}{2} \epsilon^\dagger \bar{\sigma}^\mu \sigma^\nu \partial_\mu \chi^\dagger(x) \right)
\end{aligned}$$

The $b(x)$ field rotates into the ξ field and the ζ field

$$\begin{aligned}
\theta \cdot \theta b(x) &\to (\theta + \epsilon) \cdot (\theta + \epsilon)\, b(x+\Delta) \\
&= \theta \cdot \theta b(x+\Delta) + 2\epsilon \cdot \theta b(x+\Delta) + O(\epsilon^2) \\
&= \theta \cdot \theta \left(b(x) + \Delta^\mu \partial_\mu b(x) \right) + 2\epsilon \cdot \theta \left(b(x) + \Delta^\mu \partial_\mu b(x) \right) \\
&= \theta \cdot \theta b(x) + \theta \cdot \theta \Delta^\mu \partial_\mu b(x) + 2\epsilon \cdot \theta b(x) + O(\epsilon^2) \\
&= \theta \cdot \theta b(x) + i\theta \cdot \theta \epsilon \sigma^\mu \theta^\dagger \partial_\mu b(x) + 2\epsilon \cdot \theta b(x) + f(\theta^3) \\
&= \theta \cdot \theta b(x) + \theta \cdot (2\epsilon b(x)) + \theta \cdot \theta \theta^\dagger \cdot \left(-i\bar{\sigma}^\mu \epsilon \partial_\mu b(x) \right)
\end{aligned}$$

The $c(x)$ field rotates into the χ field and the η field:

$$\begin{aligned}
\theta^\dagger \cdot \theta^\dagger c(x) &\to \left(\theta^\dagger + \epsilon^\dagger \right) \cdot \left(\theta^\dagger + \epsilon^\dagger \right) c(x+\Delta) \\
&= \theta \cdot \theta c(x+\Delta) + 2\epsilon^\dagger \cdot \theta^\dagger c(x+\Delta) + O(\epsilon^2) \\
&= \theta^\dagger \cdot \theta^\dagger \left(c(x) + \Delta^\mu \partial_\mu c(x) \right) + 2\epsilon^\dagger \cdot \theta^\dagger \left(c(x) + \Delta^\mu \partial_\mu c(x) \right) \\
&= \theta^\dagger \cdot \theta^\dagger c(x) + i\theta^\dagger \cdot \theta^\dagger \epsilon^\dagger \bar{\sigma}^\mu \theta \partial_\mu c(x) + 2\epsilon^\dagger \cdot \theta^\dagger c(x) + f(\theta^3) \\
&= \theta^\dagger \cdot \theta^\dagger c(x) - i\theta^\dagger \cdot \theta^\dagger \theta \sigma^\mu \epsilon^\dagger \partial_\mu c(x) + 2\epsilon^\dagger \cdot \theta^\dagger c(x) \\
&= \theta^\dagger \cdot \theta^\dagger c(x) + \theta^\dagger \cdot (2\epsilon^\dagger c(x)) + \theta^\dagger \cdot \theta^\dagger \theta \cdot \left(-i\sigma^\mu \epsilon^\dagger \partial_\mu c(x) \right)
\end{aligned}$$

The $v_\mu(x)$ field rotates into the ξ, χ, η and ζ fields:

$$\begin{aligned}
\theta^\dagger\bar\sigma^\mu\theta v_\mu(x) &\to (\theta^\dagger+\epsilon^\dagger)\bar\sigma^\mu(\theta+\epsilon)v_\mu(x+\Delta)\\
&= \theta^\dagger\bar\sigma^\mu\theta v_\mu(x+\Delta)+\theta^\dagger\bar\sigma^\mu\epsilon v_\mu(x+\Delta)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x+\Delta)+O(\epsilon^2)\\
&= \theta^\dagger\bar\sigma^\mu\theta\left[v_\mu(x)+\Delta^\nu\partial_\nu v_\mu(x)\right]+\theta^\dagger\bar\sigma^\mu\epsilon v_\mu(x)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x)+O(\epsilon^2)\\
&= \theta^\dagger\bar\sigma^\mu\theta\left[v_\mu(x)+i\epsilon\sigma^\nu\theta^\dagger\partial_\nu v_\mu(x)+i\epsilon^\dagger\bar\sigma^\nu\theta\partial_\nu v_\mu(x)\right]-\epsilon\sigma^\mu\theta^\dagger v_\mu(x)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x)+O(\epsilon^2)\\
&= \theta^\dagger\bar\sigma^\mu\theta v_\mu(x)+\left[i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\theta^{\dagger\dot\alpha}\theta^\dagger_{\dot\beta}\bar\sigma^{\nu\dot\beta\beta}\epsilon_\beta-i\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\theta^\beta\sigma^\nu_{\beta\dot\beta}\epsilon^{\dagger\dot\beta}\right]\partial_\nu v_\mu(x)-\epsilon\sigma^\mu\theta^\dagger v_\mu(x)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x)\\
&= \theta^\dagger\bar\sigma^\mu\theta v_\mu(x)+\left[i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\theta^\dagger_{\dot\gamma}\theta^\dagger_{\dot\beta}\bar\sigma^{\nu\dot\beta\beta}\epsilon_\beta-i\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\theta^\gamma\theta^\beta\sigma^\nu_{\beta\dot\beta}\epsilon^{\dagger\dot\beta}\right]\partial_\nu v_\mu(x)-\epsilon\sigma^\mu\theta^\dagger v_\mu(x)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x)\\
&= \theta^\dagger\bar\sigma^\mu\theta v_\mu(x)-\epsilon\sigma^\mu\theta^\dagger v_\mu(x)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x)\\
&\qquad+\left[i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\left(-\frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\theta^\dagger\cdot\theta^\dagger\right)\bar\sigma^{\nu\dot\beta\beta}\epsilon_\beta-i\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\left(-\frac{1}{2}\epsilon^{\gamma\beta}\theta\cdot\theta\right)\sigma^\nu_{\beta\dot\beta}\epsilon^{\dagger\dot\beta}\right]\partial_\nu v_\mu(x)\\
&= \theta^\dagger\bar\sigma^\mu\theta v_\mu(x)-\epsilon\sigma^\mu\theta^\dagger v_\mu(x)+\epsilon^\dagger\bar\sigma^\mu\theta v_\mu(x)\\
&\qquad+\left[-\frac{i}{2}\theta^\dagger\cdot\theta^\dagger\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\delta^{\dot\alpha}_{\dot\beta}\bar\sigma^{\nu\dot\beta\beta}\epsilon_\beta+\frac{i}{2}\theta\cdot\theta\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\delta^\beta_\alpha\sigma^\nu_{\beta\dot\beta}\epsilon^{\dagger\dot\beta}\right]\partial_\nu v_\mu(x)\\
&= \theta^\dagger\bar\sigma^\mu\theta v_\mu(x)+\theta^\dagger\cdot\left(\bar\sigma^\mu\epsilon v_\mu(x)\right)+\theta\cdot\left(-\sigma^\mu\epsilon^\dagger v_\mu(x)\right)\\
&\qquad+\left[\theta^\dagger\cdot\theta^\dagger\theta\cdot\left(-\frac{i}{2}\sigma^\mu\bar\sigma^\nu\epsilon\right)+\theta\cdot\theta\theta^\dagger\cdot\left(\frac{i}{2}\bar\sigma^\mu\sigma^\nu\epsilon^\dagger\right)\right]\partial_\nu v_\mu(x)
\end{aligned}$$

The η field rotates into the $c(x)$ field, the $v_\mu(x)$ and the $d(x)$ field:

$$\begin{aligned}
\theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x) &\to (\theta^\dagger+\epsilon^\dagger)\cdot(\theta^\dagger+\epsilon^\dagger)(\theta+\epsilon)\cdot\eta(x+\Delta)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\left(\eta(x)+\Delta^\mu\partial_\mu\eta(x)\right)+2\theta^\dagger\cdot\epsilon^\dagger\theta\cdot\eta(x)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)+O(\epsilon^2)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)+i\theta^\dagger\cdot\theta^\dagger\left(\epsilon^\dagger\bar\sigma^\mu\theta\right)\theta\cdot\partial_\mu\eta(x)+2\epsilon^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)+f((\theta^\dagger)^3)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)+i\theta^\dagger\cdot\theta^\dagger\epsilon^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\theta^\beta\partial_\mu\eta_\beta(x)+2\epsilon^\dagger_{\dot\alpha}\theta^{\dagger\dot\alpha}\theta^\alpha\eta_\alpha(x)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)+i\theta^\dagger\cdot\theta^\dagger\epsilon^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\theta^\gamma\theta^\beta\partial_\mu\eta_\beta(x)+2\epsilon^\dagger_{\dot\alpha}\left(-\frac{1}{2}\theta\sigma^\nu\theta^\dagger\bar\sigma_\nu^{\dot\alpha\alpha}\right)\eta_\alpha(x)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)+i\theta^\dagger\cdot\theta^\dagger\epsilon^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\left(-\frac{1}{2}\epsilon^{\gamma\beta}\theta\cdot\theta\right)\partial_\mu\eta_\beta(x)+\theta^\dagger\bar\sigma^\nu\theta\left(\epsilon^\dagger\bar\sigma_\nu\eta_\alpha(x)\right)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)-\frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\epsilon^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\delta^\beta_\alpha\partial_\mu\eta_\beta(x)+\theta^\dagger\bar\sigma^\nu\theta\left(\epsilon^\dagger\bar\sigma_\nu\eta_\alpha(x)\right)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)\\
&= \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x)+\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(-\frac{i}{2}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\eta(x)\right)+\theta^\dagger\bar\sigma^\nu\theta\left(\epsilon^\dagger\bar\sigma_\nu\eta_\alpha(x)\right)+\theta^\dagger\cdot\theta^\dagger\epsilon\cdot\eta(x)
\end{aligned}$$

The $\zeta^\dagger$ field rotates into the $b(x)$ field, the $v_\mu(x)$ field and the $d(x)$ field:

$$\begin{aligned}
\theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x) &\to (\theta+\epsilon)\cdot(\theta+\epsilon)(\theta^\dagger+\epsilon^\dagger)\cdot\zeta^\dagger(x+\Delta)\\
&= \theta\cdot\theta\theta^\dagger\cdot\left(\zeta^\dagger(x)+\Delta^\mu\partial_\mu\zeta^\dagger(x)\right)+2\theta\cdot\epsilon\theta^\dagger\cdot\zeta^\dagger(x)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)+O(\epsilon^2)\\
&= \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)+i\theta\cdot\theta\left(\epsilon\sigma^\mu\theta^\dagger\right)\theta^\dagger\cdot\partial_\mu\zeta^\dagger(x)+2\epsilon\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)\\
&= \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)+i\theta\cdot\theta\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\theta^{\dagger\dot\alpha}\theta^\dagger_{\dot\beta}\partial_\mu\zeta^{\dagger\dot\beta}(x)+2\epsilon^\alpha\theta_\alpha\theta^\dagger_{\dot\alpha}\zeta^{\dagger\dot\alpha}(x)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)\\
&= \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)+i\theta\cdot\theta\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\theta^\dagger_{\dot\gamma}\theta^\dagger_{\dot\beta}\partial_\mu\zeta^{\dagger\dot\beta}(x)+2\epsilon^\alpha\left(-\frac{1}{2}\theta^\dagger\bar\sigma_\mu\theta\sigma^\mu_{\alpha\dot\alpha}\right)\zeta^{\dagger\dot\alpha}(x)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)\\
&= \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)+i\theta\cdot\theta\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\left(-\frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\theta^\dagger\cdot\theta^\dagger\right)\partial_\mu\zeta^{\dagger\dot\beta}(x)-\theta^\dagger\bar\sigma_\mu\theta\left(\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\zeta^{\dagger\dot\alpha}(x)\right)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)\\
&= \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)-\frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\delta^{\dot\alpha}_{\dot\beta}\partial_\mu\zeta^{\dagger\dot\beta}(x)-\theta^\dagger\bar\sigma_\mu\theta\left(\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\zeta^{\dagger\dot\alpha}(x)\right)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)\\
&= \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x)+\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(-\frac{i}{2}\epsilon\sigma^\mu\partial_\mu\zeta^\dagger(x)\right)+\theta^\dagger\bar\sigma_\mu\theta\left(-\epsilon\sigma^\mu\zeta^\dagger(x)\right)+\theta\cdot\theta\epsilon^\dagger\cdot\zeta^\dagger(x)
\end{aligned}$$

Lastly, the $d(x)$ field rotates into the η and ζ fields

$$\begin{aligned}
\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger d(x) &\to (\theta+\epsilon)\cdot(\theta+\epsilon)(\theta^\dagger+\epsilon^\dagger)\cdot(\theta^\dagger+\epsilon^\dagger)d(x+\Delta)\\
&= \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(d(x)+\Delta^\mu\partial_\mu d(x)\right)+2\theta\cdot\theta\theta^\dagger\cdot\epsilon^\dagger d(x)+2\theta\cdot\epsilon\theta^\dagger\cdot\theta^\dagger d(x)+O(\epsilon^2)\\
&= \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger d(x)+2\theta\cdot\theta\theta^\dagger\cdot\epsilon^\dagger d(x)+2\theta\cdot\epsilon\theta^\dagger\cdot\theta^\dagger d(x)+f(\theta^3)+f((\theta^\dagger)^3)\\
&= \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger d(x)+\theta\cdot\theta\theta^\dagger\cdot\left(2\epsilon^\dagger d(x)\right)+\theta^\dagger\cdot\theta^\dagger\theta\cdot\left(2\epsilon d(x)\right)
\end{aligned}$$

Collecting the above results, we see that the fields transform as

$$\sqrt{2}\delta_\epsilon a(x) = \epsilon\cdot\xi(x)+\epsilon^\dagger\cdot\chi^\dagger(x) \tag{1.14}$$

$$\sqrt{2}\delta_\epsilon\xi_\alpha(x) = 2\epsilon_\alpha b(x)-(\sigma^\mu\epsilon^\dagger)_\alpha\left(i\partial_\mu a(x)+v_\mu(x)\right) \tag{1.15}$$

$$\sqrt{2}\delta_\epsilon\chi^{\dagger\dot\alpha} = 2\epsilon^{\dagger\dot\alpha}c(x)-(\bar\sigma^\mu\epsilon)^{\dot\alpha}\left(i\partial_\mu a(x)-v_\mu(x)\right) \tag{1.16}$$

$$\sqrt{2}\delta_\epsilon b(x) = \epsilon^\dagger\cdot\zeta^\dagger(x)-\frac{i}{2}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\xi(x) \tag{1.17}$$

$$\sqrt{2}\delta_\epsilon c(x) = \epsilon\cdot\eta(x)-\frac{i}{2}\epsilon\sigma^\mu\partial_\mu\chi^\dagger(x) \tag{1.18}$$

$$\sqrt{2}\delta_\epsilon v_\mu(x) = \frac{i}{2}\epsilon\sigma_\nu\bar\sigma_\mu\partial^\nu\xi(x)-\frac{i}{2}\epsilon^\dagger\bar\sigma_\nu\sigma_\mu\partial^\nu\chi^\dagger(x)+\epsilon^\dagger\bar\sigma_\mu\eta_\alpha(x)-\epsilon\sigma_\mu\zeta^\dagger(x) \tag{1.19}$$

$$\sqrt{2}\delta_\epsilon\eta_\alpha(x) = 2\epsilon_\alpha d(x)-i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu c(x)-\frac{i}{2}\left(\sigma^\mu\bar\sigma^\nu\epsilon\right)_\alpha\partial_\nu v_\mu(x) \tag{1.20}$$

$$\sqrt{2}\delta_\epsilon\zeta^{\dagger\dot\alpha}(x) = 2\epsilon^{\dagger\dot\alpha}d(x)-i(\bar\sigma^\mu\epsilon)^{\dot\alpha}\partial_\mu b(x)+\frac{i}{2}(\bar\sigma^\mu\sigma^\nu\epsilon^\dagger)^{\dot\alpha}\partial_\nu v_\mu(x) \tag{1.21}$$

$$\sqrt{2}\delta_\epsilon d(x) = -\frac{i}{2}\epsilon^\dagger\bar\sigma^\mu\partial_\mu\eta(x)-\frac{i}{2}\epsilon\sigma^\mu\partial_\mu\zeta^\dagger(x) \tag{1.22}$$

Note that since each term has only one ϵ or $\epsilon^\dagger$, each bosonic fields are rotated into fermionic field and vice-versa.

2 Chiral covariant derivatives

Recall that the ordinary derivative ∂_μ does not commute with gauge transformations δ and must be promoted to a gauge covariant derivative so that

$$\delta(\partial_\mu \psi(x)) \neq \partial_\mu(\delta\psi) \qquad \Rightarrow \qquad \delta(\nabla_\mu \psi(x)) = \nabla_\mu(\delta\psi) \tag{2.1}$$

A similar 'promotion' for the superspace derivative $\frac{\partial}{\partial\theta^\alpha}$ must be executed in order to make superfields invariant under SUSY transformations

$$\delta_\epsilon\left(\frac{\partial \mathcal{S}}{\partial\theta^\alpha}\right) \neq \frac{\partial}{\partial\theta^\alpha}(\delta_\epsilon \mathcal{S}) \qquad \Rightarrow \qquad \delta_\epsilon\left(D_\alpha \mathcal{S}\right) = D_\alpha(\delta_\epsilon \mathcal{S}) \tag{2.2}$$

where the chiral covariant derivative D_α is defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \qquad\qquad D^\alpha = -\frac{\partial}{\partial\theta_\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \tag{2.3}$$

and similarly, under complex conjugation

$$\bar D^{\dot\alpha} = \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} - i(\bar\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \qquad\qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\theta^{\dagger\dot\alpha}} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \tag{2.4}$$

Thus the chiral covariant derivative anticommutes with the SUSY generators (in the superfield representation):

$$\left\{Q_\alpha, D_\beta\right\} = \left\{Q^\dagger_{\dot\alpha}, D_\beta\right\} = \left\{Q_\alpha, \bar D_{\dot\beta}\right\} = \left\{Q^\dagger_{\dot\alpha}, \bar D_{\dot\beta}\right\} = 0 \tag{2.5}$$

and furthermore satisfy the following identities

$$\left\{D_\alpha, D_\beta\right\} = \left\{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\right\} = 0 \qquad\qquad \left\{D_\alpha, \bar D_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{2.6}$$

Lastly, note the important and useful fact that

$$\int \mathrm{d}^2\theta D_\alpha(\text{anything}) \propto \partial_\mu(\text{stuff}) \qquad \text{and} \qquad \int \mathrm{d}^2\theta^\dagger \bar D_{\dot\alpha}(\text{anything}) \propto \partial_\mu(\text{stuff}) \tag{2.7}$$

which will be useful for constructing Lagrangians out of superfields. Along with this, note that triple application of D or $\bar D$ gives identically 0:

$$D_\alpha D_\beta D_\gamma(\text{anything}) = 0 \qquad \text{and} \qquad \bar D_{\dot\alpha}\bar D_{\dot\beta}\bar D_{\dot\gamma}(\text{anything}) = 0 \tag{2.8}$$

Part II
Matter fields

3 Chiral Superfields

A general superfield is usually a highly redundant description for a physical theory. It is necessary and more wieldy to specialize to 'smaller' superfields that contain particular fields before constructing a Lagrangian. Chiral superfields are the workhorses of $\mathcal{N} = 1$ SUSY. They contain the matter fields and as such appear in all SUSY theories in some capacity. The definition of a chiral superfield Φ is that it satisfy the relation

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \tag{3.1}$$

Such a field is also known as a left-chiral superfield. This implies, using Eq. (1.30), that chiral and general superfields are related by the expression

$$\bar{D} \cdot \bar{D}\mathcal{S} = \Phi \tag{3.2}$$

Although it is possible to implement the constraint in Eq. (2.1) by brute force, this is very painful and time-consuming. An alternative lies in changing variables such that Φ only depends on chiral coordinates $\{y, \theta\}$ so that $\bar{D}y = 0$, thus making any function $\Phi(y, \theta)$ automatically satisfy Eq. (2.1). Such a coordinate is given by

$$y^\mu = x^\mu + i\theta^\dagger\bar{\sigma}^\mu\theta \tag{3.3}$$

in terms of which the chiral covariant derivative becomes

$$D_\alpha = \frac{\partial}{\partial\theta^\theta} - 2i(\sigma^\mu\theta^\dagger)_\alpha\frac{\partial}{\partial y^\mu} \qquad\qquad D^\alpha = -\frac{\partial}{\partial\theta_\theta} + 2i(\theta^\dagger\bar{\sigma}^\mu)^\alpha\frac{\partial}{\partial y^\mu} \tag{3.4}$$

$$\bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}} \qquad\qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\theta^{\dagger\dot{\alpha}}} \tag{3.5}$$

Based on these coordinates, the expansion of a chiral superfield is

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta \cdot \psi(y) + \theta \cdot \theta F(y) \tag{3.6}$$

Note that such a field contains

$$2 \times 2 = 4 \text{ bosonic d.o.f.} \qquad \text{and} \qquad 2 \times 2 = 4 \text{ fermionic d.o.f.} \tag{3.7}$$

again providing the requisite parity between bosons and fermions. Similarly, for the complex conjugate of a left-chiral superfield (giving a right-chiral superfield), the chiral coordinate is $y^*_\mu = x_\mu - i\theta^\dagger\bar{\sigma}_\mu\theta$ and the derivatives are

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} \qquad\qquad D^\alpha = -\frac{\partial}{\partial\theta_\alpha} \tag{3.8}$$

$$\bar{D}^{\dot{\alpha}} = \frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}} - 2i(\bar{\sigma}_\mu\theta)^{\dot{\alpha}}\frac{\partial}{\partial y^*_\mu} \qquad\qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\theta^{\dagger\dot{\alpha}}} + 2i(\theta\sigma_\mu)_{\dot{\alpha}}\frac{\partial}{\partial y^*_\mu} \tag{3.9}$$

while the expansion of the right-chiral superfield is

$$\Phi^*(y^*, \theta^\dagger) = \phi^*(y^*) + \sqrt{2}\theta^\dagger \cdot \psi^\dagger(y^*) + \theta^\dagger \cdot \theta^\dagger F^*(y^*) \tag{3.10}$$

Expanding the functions of y in a power series returns us to the original coordinates, allowing for direct comparison to the general superfield:

$$\begin{aligned}
\Phi(x,\theta,\theta^\dagger) &= \phi(x^\mu + i\theta^\dagger\bar\sigma^\mu\theta) + \sqrt{2}\theta\cdot\psi(x^\mu + i\theta^\dagger\bar\sigma^\mu\theta) + \theta\cdot\theta F(x^\mu + i\theta^\dagger\bar\sigma^\mu\theta) \\
&= \phi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \frac{1}{2}(i\theta^\dagger\bar\sigma^\mu\theta)(i\theta^\dagger\bar\sigma^\nu\theta)\partial_\mu\partial_\nu\phi(x) \\
&\qquad + \sqrt{2}\theta\cdot\psi(x) + \sqrt{2}\theta\cdot\left(i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\psi(x)\right) + \theta\cdot\theta F(x) + f(\theta^3) \\
&= \phi(x) + \sqrt{2}\theta\cdot\psi(x) - \frac{1}{2}\left(-\theta\sigma^\mu\theta^\dagger\theta^\dagger\bar\sigma^\nu\theta\right)\partial_\mu\partial_\nu\phi(x) + i\sqrt{2}\theta^\dagger\bar\sigma^\mu\theta\theta\cdot\partial_\mu\psi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \theta\cdot\theta F(x) \\
&= \phi(x) + \sqrt{2}\theta\cdot\psi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \theta\cdot\theta F(x) + i\sqrt{2}\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\theta^\beta\partial_\mu\psi_\beta(x) \\
&\qquad + \frac{1}{2}\left(\sigma^\mu_{\alpha\dot\alpha}\theta^{\dagger\dot\alpha}\theta^\dagger_{\dot\beta}\bar\sigma^{\nu\dot\beta\beta}\theta^\alpha\theta_\beta\right)\partial_\mu\partial_\nu\phi(x) \\
&= \phi(x) + \sqrt{2}\theta\cdot\psi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \theta\cdot\theta F(x) + i\sqrt{2}\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\theta^\gamma\theta^\beta\partial_\mu\psi_\beta(x) \\
&\qquad + \frac{1}{2}\left(\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\theta^\dagger_{\dot\gamma}\theta^\dagger_{\dot\beta}\bar\sigma^{\nu\dot\beta\beta}\epsilon^{\alpha\lambda}\theta_\lambda\theta_\beta\right)\partial_\mu\partial_\nu\phi(x) \\
&= \phi(x) + \sqrt{2}\theta\cdot\psi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \theta\cdot\theta F(x) + i\sqrt{2}\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\epsilon_{\alpha\gamma}\left(-\frac{1}{2}\epsilon^{\gamma\beta}\theta\cdot\theta\right)\partial_\mu\psi_\beta(x) \\
&\qquad + \frac{1}{2}\left(\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dot\alpha\dot\gamma}\left(-\frac{1}{2}\epsilon_{\dot\gamma\dot\beta}\theta^\dagger\cdot\theta^\dagger\right)\bar\sigma^{\nu\dot\beta\beta}\epsilon^{\alpha\lambda}\left(\frac{1}{2}\epsilon_{\lambda\beta}\theta\cdot\theta\right)\right)\partial_\mu\partial_\nu\phi(x) \\
&= \phi(x) + \sqrt{2}\theta\cdot\psi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \theta\cdot\theta F(x) - i\frac{\sqrt{2}}{2}\theta\cdot\theta\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\delta^\beta_\alpha\partial_\mu\psi_\beta(x) \\
&\qquad - \frac{1}{8}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\sigma^\mu_{\alpha\dot\alpha}\delta^{\dot\alpha}_{\dot\beta}\bar\sigma^{\nu\dot\beta\beta}\delta^\alpha_\beta\right)\partial_\mu\partial_\nu\phi(x) \\
&= \phi(x) + \sqrt{2}\theta\cdot\psi(x) + i\theta^\dagger\bar\sigma^\mu\theta\partial_\mu\phi(x) + \theta\cdot\theta F(x) - i\frac{\sqrt{2}}{2}\theta\cdot\theta\theta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\delta^\beta_\alpha\partial_\mu\psi_\beta(x) \\
&\qquad - \frac{1}{8}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\mathrm{Tr}[\sigma^\mu\bar\sigma^\nu]\partial_\mu\partial_\nu\phi(x) \\
&= \phi(x) + \theta\cdot\left(\sqrt{2}\psi(x)\right) + \theta^\dagger\bar\sigma^\mu\theta(i\partial_\mu\phi(x)) + \theta\cdot\theta F(x) + \theta\cdot\theta\theta^\dagger\cdot\left(-i\frac{\sqrt{2}}{2}\bar\sigma^\mu\partial_\mu\psi(x)\right) \\
&\qquad + \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\frac{1}{4}\partial_\mu\partial^\mu\phi(x)\right)
\end{aligned}$$

Thus we can identify the components of the general superfield that comprise the chiral superfield:

$$\begin{array}{lcr} a=\phi & \xi_\alpha=\sqrt{2}\psi_\alpha & \chi^\dagger=0 \\ b=F & c=0 & v_\mu=i\partial_\mu\phi \\ \eta=0 & \zeta^{\dagger\dot{\alpha}}=-\frac{i}{\sqrt{2}}(\bar{\sigma}^\mu\partial_\mu\psi)^{\dot{\alpha}} & d=\frac{1}{4}\partial_\mu\partial^\mu\phi \end{array} \tag{3.11}$$

A similar expansion for a right-chiral field gives

$$\Phi^*(x,\theta,\theta^\dagger)=\phi^*(x)+\theta^\dagger\cdot\left(\sqrt{2}\psi^\dagger(x)\right)+\theta^\dagger\bar{\sigma}^\mu\theta(-i\partial_\mu\phi^*(x))$$
$$+\theta^\dagger\cdot\theta^\dagger F^*(x)+\theta^\dagger\cdot\theta^\dagger\theta\cdot\left(-i\frac{\sqrt{2}}{2}\sigma^\mu\partial_\mu\psi^\dagger(x)\right)+\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\frac{1}{4}\partial_\mu\partial^\mu\phi^*(x)\right)$$

Under SUSY rotations, the chiral superfield transforms as

$$\delta_\epsilon\phi(x)=\epsilon\cdot\psi(x) \tag{3.12}$$
$$\delta_\epsilon\psi_\alpha(x)=\epsilon_\alpha F(x)-(\sigma^\mu\epsilon^\dagger)_\alpha\left(i\partial_\mu\phi(x)\right) \tag{3.13}$$
$$\delta_\epsilon F(x)=-i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi(x) \tag{3.14}$$

All other fields rotate to 0, as expected. Lastly, any product of chiral superfields will itself be a chiral superfield. This follows from the linearity of the (anti-)chiral covariant derivative. For example, with a product of two chiral superfields, we see

$$\bar{D}^{\dot{\alpha}}(\Phi_i\Phi_i)=(\bar{D}^{\dot{\alpha}}\Phi_i)\Phi_j+\Phi_i(\bar{D}^{\dot{\alpha}}\Phi_j)=0 \tag{3.15}$$

since $\bar{D}^{\dot{\alpha}}\Phi_i=\bar{D}^{\dot{\alpha}}\Phi_j=0$. Thus a function $W(\Phi_i)$ (but not Φ_i^*) that is holomorphic in the fields (treated as complex variables) will also be a chiral superfield. Suppose there is a chiral superfield charged under some global group:

$$\Phi_i\to U_i^{\ j}\Phi_j=\left[\exp\left(ig\phi_a t^a\right)\right]_i^{\ j}\Phi_j \tag{3.16}$$

Thus a product of charged fields, such as $\Phi_i\Phi_j$ or $\Phi_i\Phi_j\Phi_k$ will also be a chiral superfield. This fact about chiral superfields can also be used to implement gauge transformations. Suppose a chiral superfield is charged under some gauge group:

$$\Phi_i\to U_i^{\ j}\Phi_j=\left[\exp\left(ig\Lambda_a(x)t^a\right)\right]_i^{\ j}\Phi_j \tag{3.17}$$

In order for the above product to remain a chiral superfield, the gauge phase $\Lambda_a(x)$ must itself be promoted to a chiral superfield. Such a superfield may be expanded as usual

$$\Lambda_a(y,\theta)=\phi_a(y)+\theta\cdot\left(\sqrt{2}\chi_a(y)\right)+\theta\cdot\theta F_a(y) \tag{3.18}$$

but now the mass dimensions are different:

$$[\phi_a]=0 \qquad [\chi_a]=\frac{1}{2} \qquad [F_a]=1 \tag{3.19}$$

We now have enough formalism to construct the simplest supersymmetric Lagrangian.

3.1 SUSY-invariant Lagrangians from chiral superfields

We can use the transformation properties of chiral superfields to simply construct manifestly SUSY-invariant Lagrangians. Take note of the transformation of the effect of a SUSY rotation on the coefficient of the $\theta \cdot \theta$ term in the chiral superfield, as given in Eq. (3.14):

$$\delta_\epsilon F = i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi \tag{3.20}$$

We see that the effect of a SUSY-rotation on this F-term is to simply generate a total derivative. Recall that Lagrangians must be integrated over all spacetime to give the action. Any Lagrangian whose behavior under some symmetry transformation is to return the Lagrangian up to a total derivative can be considered classically invariant under that symmetry, since the Euler-Lagrange equations do not take total derivatives into account.

Thus a Lagrangian consisting of F-terms pulled out of chiral superfields will be invariant up to such total derivatives under SUSY! Thus, all we need to do is take a product of chiral superfields, project out the F-term using Grassmannian integration and we'll automatically obtain a SUSY-invariant Lagrangian. This is much simpler and efficient that the brute-force method. It also severely constrains the terms that may be considered (assuming we're interested in a renormalizable Lagrangian). Recall that projecting out the F-term means integrating against $\mathrm{d}^2\theta$, which has mass dimension $[\mathrm{d}^2\theta] = 1$. Since a chiral superfield has mass dimension $[\Phi] = 1$, we can only consider terms with one, two or at most three chiral superfields before non-renormalizable terms appear. Let's consider an explicit example by constructing the product of two chiral superfields and finding the behavior of the F-term. Recall that terms of the form θ^3 vanish identically, and thus the product becomes

$$\begin{aligned}
\Phi(y,\theta)\Phi(y,\theta) &= \left(\phi(y) + \sqrt{2}\theta \cdot \psi(y) + \theta \cdot \theta F(y)\right)\left(\phi(y) + \sqrt{2}\theta \cdot \psi(y) + \theta \cdot \theta F(y)\right) \\
&= \phi^2(y) + 2\sqrt{2}\theta \cdot \psi(y)\phi(y) + 2\theta \cdot \theta\phi(y)F(y) + 2\theta \cdot \psi(y)\theta \cdot \psi(y) \\
&= \phi^2(y) + 2\sqrt{2}\theta \cdot \psi(y)\phi(y) + 2\theta \cdot \theta\phi(y)F(y) - (\theta \cdot \theta)(\psi(y) \cdot \psi(y)) \\
&= \phi^2(y) + 2\sqrt{2}\theta \cdot \psi(y)\phi(y) + \theta \cdot \theta(2\phi(y)F(y) - \psi(y) \cdot \psi(y))
\end{aligned}$$

The F-term is thus simply

$$\Phi\Phi]_F = 2\phi(y)F(y) - \psi(y) \cdot \psi(y) \tag{3.21}$$

Let's now check the transformation; if it returns a total derivative, we will have succeeded in constructing a term acceptable in a SUSY-invariant Lagrangian:

$$\begin{aligned}
\delta_\epsilon(\,\Phi\Phi]_F) &= 2(\delta_\epsilon\phi)F + 2\phi(\delta_\epsilon F) - 2\psi \cdot (\delta_\epsilon \psi) \\
&= 2(\epsilon \cdot \psi)F + 2\phi(-i\epsilon^\dagger \bar{\sigma}_\mu \partial^\mu \psi) - 2(\psi \cdot \epsilon F - i\psi\sigma^\mu \epsilon^\dagger \partial_\mu \phi) \\
&= -2i\phi\epsilon^\dagger \bar{\sigma}_\mu \partial^\mu \psi + 2i\psi\sigma^\mu\epsilon^\dagger\partial_\mu\phi
\end{aligned}$$

Integrating the first term by parts, we see that

$$-2i\phi\epsilon^\dagger\bar{\sigma}_\mu\partial^\mu\psi = -2i\partial^\mu\left(\phi\epsilon^\dagger\bar{\sigma}_\mu\psi\right) + 2i(\partial^\mu\phi)\epsilon^\dagger\bar{\sigma}_\mu\psi$$

and thus

$$\begin{aligned}
\delta_\epsilon(\,\Phi\Phi]_F) &= -2i\phi\epsilon^\dagger\bar{\sigma}_\mu\partial^\mu\psi + 2i\psi\sigma^\mu\epsilon^\dagger\partial_\mu\phi \\
&= -2i\partial^\mu\left(\phi\epsilon^\dagger\bar{\sigma}_\mu\psi\right) + 2i(\partial^\mu\phi)\epsilon^\dagger\bar{\sigma}_\mu\psi + 2i\psi\sigma^\mu\epsilon^\dagger\partial_\mu\phi \\
&= -2i\partial^\mu\left(\phi\epsilon^\dagger\bar{\sigma}_\mu\psi\right) - 2i(\partial^\mu\phi)\psi^\dagger\sigma_\mu\epsilon + 2i\psi\sigma^\mu\epsilon^\dagger\partial_\mu\phi = \partial^\mu\left(-2i\phi\epsilon^\dagger\bar{\sigma}_\mu\psi\right)
\end{aligned}$$

Therefore the variation of the F-term from a product of two chiral superfields simply yields a total derivative, making it an admissible, SUSY-invariant term to the Lagrangian. This greatly simplifies the construction of SUSY-invariant theories, as we need only consider the terms that result with a coefficient $\theta\cdot\theta$, rather than considering the entire superfield.

3.2 SUSY-invariant Lagrangians from non-chiral superfields

Of course, the above discussion doesn't quite suffice, since this will not be able to generate terms such as $\partial_\mu\phi^*\partial^\mu\phi$. In general, constructing SUSY-invariant Lagrangians will require both chiral superfields and non-chiral superfields. Recall the SUSY transformation for the $\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger$ component of a a general superfield, given by Eq. (1.22) is:

$$\sqrt{2}\delta_\epsilon d(x) = -\frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\eta(x) - \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\zeta^\dagger(x) \tag{3.22}$$

Note that the effect of a SUSY-rotation is to generate two terms, both of which are total derivatives. As with the F-terms, this means they are admissible as terms in the Lagrangian, since they will not contribute to the classical equations of motion. Let's check the effect of something simple, like $\Phi^*\Phi$, where Φ is a chiral superfield. Since the product involves the complex conjugate Φ^*, this product is clearly not a chiral superfield. We will concern ourselves only with the $\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger$ term (also known as a D-term), since it transforms as a total derivative under SUSY rotations:

$$\begin{aligned}
\Phi(x,\theta,\theta^\dagger)\Phi(x,\theta,\theta^\dagger) = &\left(\frac{1}{4}\phi^*(x)\partial_\mu\partial^\mu\phi(x) + \frac{1}{4}\phi(x)\partial_\mu\partial^\mu\phi^*(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger \\
&+ \theta\cdot\theta\theta^\dagger\left(-i\bar{\sigma}^\mu\partial_\mu\psi(x)\right)\theta^\dagger\cdot\psi^\dagger(x) + \theta^\dagger\cdot\theta^\dagger\theta\left(-i\sigma^\mu\partial_\mu\psi^\dagger(x)\right)\theta\cdot\psi(x) \\
&+ \theta^\dagger\bar{\sigma}^\mu\theta\theta^\dagger\bar{\sigma}^\nu\theta(i\partial_\mu\phi(x))(-i\partial_\nu\phi^*(x)) \\
\\
= &\left(\frac{1}{4}\phi^*(x)\partial_\mu\partial^\mu\phi(x) + \frac{1}{4}\phi(x)\partial_\mu\partial^\mu\phi^*(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger \\
&- i\theta\cdot\theta\theta^\dagger_{\dot{\alpha}}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha(x)\right)\theta^\dagger_{\dot{\beta}}\psi^{\dagger\dot{\beta}}(x) - i\theta^\dagger\cdot\theta^\dagger\theta^\alpha\left(\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^{\dagger\dot{\alpha}}(x)\right)\theta^\beta\psi_\beta(x) \\
&- \theta^\dagger\bar{\sigma}^\mu\theta\theta\sigma^\nu\theta^\dagger\partial_\mu\phi(x)\partial_\nu\phi^*(x) \\
\\
= &\left(-\frac{1}{4}\partial_\mu\phi^*(x)\partial^\mu\phi(x) - \frac{1}{4}\partial_\mu\phi^*(x)\partial^\mu\phi(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger \\
&+ i\theta\cdot\theta\theta^\dagger_{\dot{\alpha}}\theta^\dagger_{\dot{\beta}}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha(x)\right)\psi^{\dagger\dot{\beta}}(x) + i\theta^\dagger\cdot\theta^\dagger\theta^\alpha\theta^\beta\left(\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^{\dagger\dot{\alpha}}(x)\right)\psi_\beta(x) \\
&- \theta^\dagger_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\theta_\alpha\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\theta^{\dagger\dot{\beta}}\partial_\mu\phi(x)\partial_\nu\phi^*(x)
\end{aligned}$$

We can consolidate terms by integrating by parts; this leaves behind a total derivative and a derivative moved to another term at the cost of a minus sign. Doing so, we can rewrite the factors in parenthesis for a start; other terms will be consolidated using this trick (total derivatives will be ruthlessly ignored):

$$\begin{aligned}
\Phi(x,\theta,\theta^\dagger)\Phi(x,\theta,\theta^\dagger) = &\left(-\frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger - \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha(x)\right)\epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dagger\dot{\beta}}(x) \\
&- \frac{i}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\left(\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^{\dagger\dot{\alpha}}(x)\right)\epsilon^{\alpha\beta}\psi_\beta(x) + \bar{\sigma}^{\mu\dot{\alpha}\alpha}\epsilon_{\alpha\gamma}\theta^\gamma\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\theta^\dagger_{\dot{\gamma}}\theta^\dagger_{\dot{\alpha}}\partial_\mu\phi(x)\partial_\nu\phi^*(x) \\
= &\left(-\frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger + \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\psi^\dagger_{\dot{\alpha}}(x)\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha(x)\right) \\
&+ \frac{i}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\left(\psi^\alpha(x)\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^{\dagger\dot{\alpha}}(x)\right) - \frac{1}{4}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\bar{\sigma}^{\mu\dot{\alpha}\alpha}\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta}\sigma^\nu_{\beta\dot{\beta}}\epsilon^{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\gamma}\dot{\alpha}}\partial_\mu\phi(x)\partial_\nu\phi^*(x) \\
= &\left(-\frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger + \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\psi^\dagger(x)\bar{\sigma}^\mu\partial_\mu\psi(x)\right) \\
&- \frac{i}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\left((\partial_\mu\psi(x))\sigma^\mu\psi^\dagger(x)\right) - \frac{1}{4}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\mathrm{Tr}[\bar{\sigma}^\mu\sigma^\nu]\partial_\mu\phi(x)\partial_\nu\phi^*(x) \\
= &\left(-\frac{1}{2}\partial_\mu\phi^*(x)\partial^\mu\phi(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger + \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\psi^\dagger(x)\bar{\sigma}^\mu\partial_\mu\psi(x)\right) \\
&+ \frac{i}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\left(\psi^\dagger(x)\bar{\sigma}^\mu\partial_\mu\psi(x)\right) - \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi^*(x) \\
= &\left(-\partial_\mu\phi^*(x)\partial^\mu\phi(x) + i\psi^\dagger(x)\bar{\sigma}^\mu\psi(x) + F^*(x)F(x)\right)\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger
\end{aligned}$$

Thus we see that the D-term, the coefficient of the $\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger$ is nothing more than the kinetic term for the component fields in the chiral superfield. And as noted, the D-term transforms as a sum of total derivatives under SUSY rotations and thus we see that, up to total derivatives, the kinetic terms are admissible terms (as they should!) When considering renormalizable interactions, only products such as $\Phi^*\Phi$ will be considered; this is due to the requirement of hermiticity, since a product of three fields could not be hermitian. At minimum, a product of four fields would be hermitian, but this would yield terms of mass dimension of at least 6.

4 Wess-Zumino model

We now have all the tools to construct the simplest supersymmetric theory for a single, uncharged chiral superfield with interactions. The components of the superfield are The Lagrangian in this model is

$$\mathcal{L} = \Phi^*\Phi]_D + W(\Phi)]_F + h.c. \tag{4.1}$$

4.1 Kinetic terms

The kinetic terms are simply the D-terms computed in section 3.2:

$$\Phi^*\Phi]_D = -\partial_\mu\phi^*(x)\partial^\mu\phi(x) + i\psi^\dagger(x)\bar{\sigma}^\mu\psi(x) + F^*(x)F(x) \tag{4.2}$$

Note that the auxiliary fields do not propagate, as they do not have the appropriate derivative terms. We'll return to this expression once we have the potential, thus enabling us to rewrite the theory without the auxiliary fields.

4.2 Masses and interactions

4.2.1 Superpotential

Before constructing the explicit superpotential for this theory, let's consider a general few facts about superpotentials. The superpotential W is a product of chiral superfields. It therefore can be expanded in a Taylor series. However, this is a very particular Taylor series, in that it is an expansion about the origin in superspace. This corresponds to setting $\theta = \theta^\dagger = 0$ and expanding about this point. Equivalently, this corresponds to setting $\Phi = \phi$, since this is the only component that does not vanish when the supercoordinates are suppressed. Thus we see

$$W(\Phi) = W(\phi) + \underbrace{(\Phi - \phi)}_{=\theta\cdot\theta F}\left.\frac{\partial W}{\partial\phi}\right]_{\Phi=\phi} + \frac{1}{2}\underbrace{(\Phi-\phi)^2}_{=2(\theta\cdot\psi)(\theta\cdot\psi)}\left.\frac{\partial^2 W}{\partial\phi^2}\right]_{\Phi=\phi}$$

Recall that eventually we will project out the $\theta\cdot\theta$ component of the superfield, and thus all we need from the factors $(\Phi-\phi)$ and $(\Phi-\phi)^2$ can be found in the brackets beneath each expression above. As such, the F-terms from the superpotential are

$$W(\Phi)]_F = F\frac{\partial W(\phi)}{\partial\phi} - \frac{1}{2}\psi\cdot\psi\frac{\partial^2 W(\phi)}{\partial\phi^2} \tag{4.3}$$

Thus the full Lagrangian may be written using the above information and the kinetic terms computed before, giving:

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\psi + F^*F + F\frac{\partial W(\phi)}{\partial\phi} - \frac{1}{2}\psi\cdot\psi\frac{\partial^2 W(\phi)}{\partial\phi^2} + F^*\frac{\partial W^*(\phi^*)}{\partial\phi^*} - \frac{1}{2}\psi^\dagger\cdot\psi^\dagger\frac{\partial^2 W^*(\phi^*)}{\partial\phi^{*2}} \tag{4.4}$$

We can use the equation of motion of the F field to write it completely out of the theory:

$$\frac{\partial L}{\partial F^*} = F + \frac{\partial W^*}{\partial\phi^*} = 0 \qquad \Rightarrow \qquad F = -\frac{\partial W^*}{\partial\phi^*} \tag{4.5}$$

and thus the Lagrangian takes the form

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \psi - \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2}\psi \cdot \psi \frac{\partial^2 W}{\partial \phi^2} - \frac{1}{2}\psi^\dagger \cdot \psi^\dagger \frac{\partial^2 W^*}{\partial \phi^{*2}} \tag{4.6}$$

Meanwhile, the superpotential takes the form

$$\begin{aligned} W(\Phi) &= F\frac{\partial W(\phi)}{\partial \phi} - \frac{1}{2}\psi \cdot \psi \frac{\partial^2 W(\phi)}{\partial \phi^2} \\ &= \left(\frac{\partial W^*(\phi^*)}{\partial \phi^*} \right) \left(\frac{\partial W(\phi)}{\partial \phi} \right) - \frac{1}{2}\psi \cdot \psi \frac{\partial^2 W(\phi)}{\partial \phi^2} \\ &= \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2}\psi \cdot \psi \frac{\partial^2 W(\phi)}{\partial \phi^2} \end{aligned} \tag{4.7}$$

We can thus see that the derivatives of the superpotential act as a potential $V(\phi)$ for the scalar field (the first derivative) and as a potential for the spinor fields (the second derivative). In other words,

$$V(\phi) = \left| \frac{\partial W}{\partial \phi} \right|^2 \tag{4.8}$$

This term will play a role in spontaneous SUSY breaking.

4.2.2 Computing the Lagrangian

As stated in section 3.1, products of chiral superfields can only consist of one, two or three such fields before non-renormalizable terms appear. Therefore, the most general superpotential for this theory is of the form

$$W(\Phi) = c\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{6}y\Phi^3 \tag{4.9}$$

Note that $[y] = 0$. We can now use Eq. (4.7) and the superpotential above to find the Wess-Zumino Lagrangian. Beginning with Eq. (4.7), we first need $\frac{\partial W}{\partial \phi}$

$$\frac{\partial W}{\partial \phi} = \frac{\partial}{\partial \phi} \left(c\phi + \frac{1}{2}m\phi^2 + \frac{1}{6}y\phi^3 \right) = c + m\phi + \frac{1}{2}y\phi^2$$

and $\frac{\partial^2 W}{\partial \phi^2}$

$$\frac{\partial^2 W}{\partial \phi^2} = \frac{\partial^2}{\partial \phi^2} \left(\phi + \frac{1}{2}m\phi^2 + \frac{1}{6}y\phi^3 \right) = m + y\phi$$

We thus claim that the superpotential for this theory is

$$\begin{aligned} W(\Phi)]_F &= \left| \frac{\partial W}{\partial \phi} \right|^2 - \frac{1}{2}\psi \cdot \psi \frac{\partial^2 W(\phi)}{\partial \phi^2} \\ &= \left(c + m\phi^* + \frac{1}{2}y\phi^{*2} \right) \left(c + m\phi + \frac{1}{2}y\phi^2 \right) - \frac{1}{2}\psi \cdot \psi \left(m + y\phi \right) \\ &= c^2 + m^2|\phi|^2 + cm\left(\phi + \phi^*\right) + \frac{1}{2}cy\left(\phi^{*2} + \phi^2\right) + \frac{1}{2}my|\phi|^2\left(\phi + \phi^*\right) + \frac{1}{4}y^2|\phi|^4 - \frac{1}{2}m\psi \cdot \psi - \frac{1}{2}y\phi\psi \cdot \psi \end{aligned}$$

Thus the Lagrangian is

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \psi + m^2|\phi|^2 + \frac{1}{4}y^2|\phi|^4 + c^2 - \frac{1}{2}m\left(\psi\cdot\psi + \psi^\dagger\cdot\psi^\dagger\right) + cm\left(\phi+\phi^*\right)$$
$$+ \frac{1}{2}cy\left(\phi^{*2}+\phi^2\right) + \frac{1}{2}my|\phi|^2\left(\phi+\phi^*\right) - \frac{1}{2}y\left(\phi\psi\cdot\psi + \phi^*\psi^\dagger\cdot\psi^\dagger\right) \tag{4.10}$$

Whew! That was quite the exercise! But we now have found the simplest, supersymmetric Lagrangian that incorporates interactions. Note that almost every class of renormalizable spin-0 and spin-$\frac{1}{2}$ interactions are present, from ϕ^4 to Yukawa couplings. Note furthermore that the scalars and spinors have the same mass, as expected.

4.2.3 Loop calculations

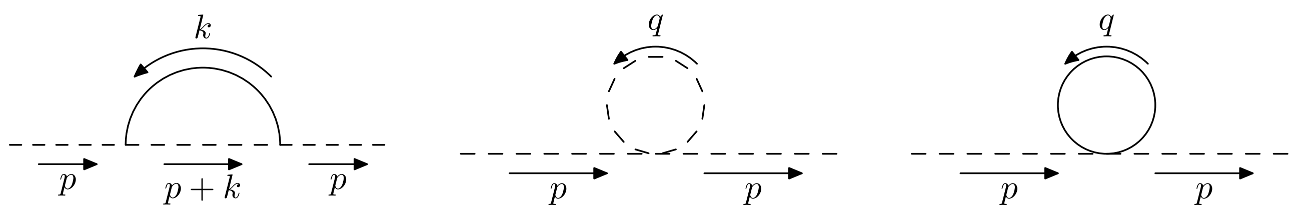
We will now proceed with explicitly showing the cancellation of divergences arising from loop diagrams. Before proceeding, let's clean up the above Lagrangian; first, we'll set $c = 0$ and also introduce $\phi = \frac{1}{\sqrt{2}}(A+iB)$ to write the linear terms slightly more conveniently:

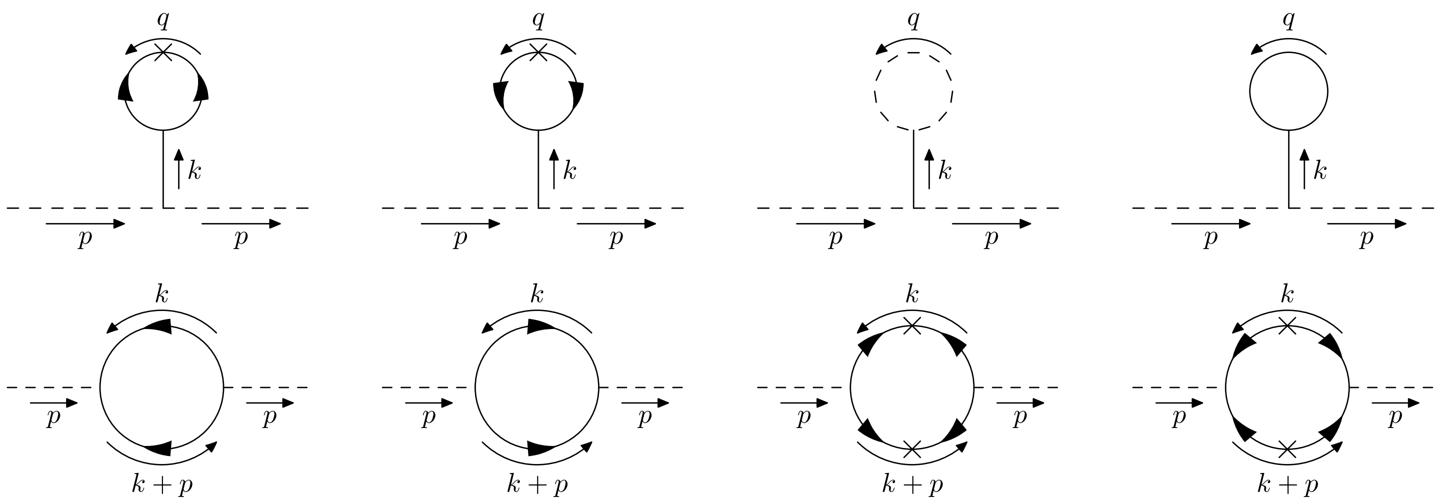
$$|\phi|^2 = \frac{1}{2}\left(A^2+B^2\right)$$
$$|\phi|^4 = \frac{1}{4}\left(A^2+B^2\right)^2 = \frac{1}{4}\left(A^4+2A^2B^2+B^4\right)$$
$$|\phi|^2\left(\phi+\phi^*\right) = \frac{1}{2}\left(A^2+B^2\right)\left[\frac{1}{\sqrt{2}}(A+iB) + \frac{1}{\sqrt{2}}(A-iB)\right] = \frac{1}{2\sqrt{2}}\left(A^2+B^2\right)(2A) = \frac{\sqrt{2}}{2}\left(A^3+AB^2\right)$$

With these substitutions, and setting $g = \frac{y}{\sqrt{8}}$, the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A\partial^\mu A + \frac{1}{2}m^2A^2 - \frac{1}{2}\partial_\mu B\partial^\mu B + \frac{1}{2}m^2B^2 + i\psi^\dagger\bar{\sigma}^\mu\psi - \frac{1}{2}m\left(\psi\cdot\psi+\psi^\dagger\cdot\psi^\dagger\right)$$
$$+\frac{1}{2}g^2A^4 + \frac{1}{2}g^2B^4 + g^2A^2B^2 + mgA^3 + mgAB^2 - gA\left(\psi\cdot\psi+\psi^\dagger\cdot\psi^\dagger\right)$$
$$- igB\left(\psi\cdot\psi - \psi^\dagger\cdot\psi^\dagger\right) \tag{4.11}$$

Now consider the two-point B function at first-loop order. Ordinarily, we'd expect a scalar two-point function to exhibit a quadratic divergence at one-loop order. However, as we'll see, SUSY enforces the cancellation of these divergences, rendering the diagrams finite. Using the Lagrangian, the following diagrams contribute at this order:





where the solid line represents an A propagator, a dashed line a B propagator and an arrowed line a fermionic propagator. The amplitude of each diagram above has the form

$$i\mathcal{M} = \frac{i}{p^2 - m^2} i\Pi(p) \frac{i}{p^2 - m^2}$$

and thus what remains is to compute $\Pi(p)$ for each of the above cases. The first diagram gives

$$\begin{aligned}
\Pi_1(p) &= (2ig)^2 \int \frac{i}{(p+k)^2 - m^2} \frac{i}{k^2 - m^2} \frac{\mathrm{d}^4 k}{(2\pi)^4} \\
&= 4g^2 \int \int_0^1 \frac{1}{\left[x\left(p^2 + 2p\cdot k + k^2 - m^2\right) + (1-x)\left(k^2 - m^2\right)\right]^2} \mathrm{d}x \frac{\mathrm{d}^4 k}{(2\pi)^4} \\
&= 4g^2 \int_0^1 \int \frac{1}{\left[k^2 + 2p\cdot kx + p^2 x^2 - \left(p^2 x^2 + m^2(1-x)\right)\right]^2} \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{d}x \\
&= 4g^2 \int_0^1 \int \frac{1}{\left[(k+px)^2 - \Delta\right]^2} \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{d}x \\
&= 4g^2 \int_0^1 \int_0^\Lambda \frac{1}{\left[k^2 - \Delta\right]^2} \frac{k^3 \mathrm{d}k}{(2\pi)^4} \int \mathrm{d}\Omega_3 \mathrm{d}x \\
&= \frac{4g^2}{16\pi^4} \int_0^1 () \left(\frac{\pi^2}{2}\right) \mathrm{d}x
\end{aligned}$$

The second diagram gives

$$
\begin{aligned}
\Pi_2(p) &= 12ig^2 \int \frac{i}{q^2 - m^2} \frac{\mathrm{d}^D q}{(2\pi)^D} \\
&= -12g^2 \frac{(-1)^2 i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(1 - \frac{D}{2}\right)}{\Gamma(1)} \left(\frac{1}{m^2}\right)^{1-\frac{D}{2}} \\
&= -12g^2 \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + \text{finite}\right) \\
&= -\frac{3ig^2}{4\pi^2\epsilon} + \dots
\end{aligned}
$$

Similarly, the third diagram gives

$$
\begin{aligned}
\Pi_3(p) &= 12ig^2 \int \frac{i}{q^2 - m^2} \frac{\mathrm{d}^D q}{(2\pi)^D} \\
&= -12g^2 \frac{(-1)^2 i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(1 - \frac{D}{2}\right)}{\Gamma(1)} \left(\frac{1}{m^2}\right)^{1-\frac{D}{2}} \\
&= -12g^2 \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + \text{finite}\right) \\
&= -\frac{3ig^2}{4\pi^2\epsilon} + \dots
\end{aligned}
$$

Leaving the tadpole diagrams aside for now, the next diagram to consider is the eighth. This can be evaluated using Passarino-Veltman decomposition:

$$
\begin{aligned}
\Pi_8(p) &= (-1)g^2 \int \mathrm{Tr}\left[\frac{ik\cdot\bar{\sigma}}{k^2 - m^2} \frac{i(k+p)\cdot\sigma}{(k+p)^2 - m^2}\right] \frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= g^2 \mathrm{Tr}[\bar{\sigma}^\mu \sigma^\nu] \int \frac{k^\mu k^\nu + k^\mu p^\nu}{(k^2 - m^2)((k+p)^2 - m^2)} \frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= 2g^2 g^{\mu\nu} \int_{\mathrm{F}} \left[\int \frac{k^\mu k^\nu}{[(k+xp)^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D} + \int \frac{k^\mu p^\nu}{[(k+xp)^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D}\right] \\
&= 2g^2 g^{\mu\nu} \int_{\mathrm{F}} \left[\int \frac{(k-xp)^\mu (k-xp)^\nu}{[k^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D} + \int \frac{(k-xp)^\mu p^\nu}{[k^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D}\right] \\
&= g^2 \int_{\mathrm{F}} \left[\int \frac{k^2 + x^2 p^2}{[k^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D} - \int \frac{xp^2}{[k^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D}\right] \\
&= g^2 \int_{\mathrm{F}} \left[\int \frac{k^2}{[k^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D} - x(1-x)p^2 \int \frac{1}{[k^2 + \Delta]^2} \frac{\mathrm{d}^D k}{(2\pi)^D}\right] \\
&= g^2 \int_{\mathrm{F}} \left[\frac{i\Delta}{(4\pi)^2 \varepsilon} - x(1-x)p^2 \frac{i}{(4\pi)^2 \varepsilon}\right]
\end{aligned}
$$

Part III
Gauge fields

5 Real (Vector) Superfields

Whereas chiral superfields contain the matter fields of physical theories, vector superfields contain the gauge fields. Vector superfields satisfy the relation

$$\mathcal{V} = \mathcal{V}^* \tag{5.1}$$

Comparing the components of the field $\mathcal{V}$ and its complex conjugate

$$\begin{aligned}\mathcal{V} = a(x) + \theta\cdot\xi(x) + \theta^\dagger\cdot\chi^\dagger(x) + \theta\cdot\theta b(x) + \theta^\dagger\cdot\theta^\dagger c(x)\\ + \theta^\dagger\bar{\sigma}^\mu\theta v_\mu(x) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\eta(x) + \theta\cdot\theta\theta^\dagger\cdot\zeta^\dagger(x) + \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger d(x)\end{aligned}$$

$$\begin{aligned}\mathcal{V}^\dagger = a^*(x) + \theta^\dagger\cdot\xi^\dagger(x) + \theta\cdot\chi(x) + \theta^\dagger\cdot\theta^\dagger b^*(x) + \theta\cdot\theta c^*(x)\\ + \theta^\dagger\bar{\sigma}^\mu\theta v_\mu^*(x) + \theta\cdot\theta\theta^\dagger\cdot\eta^\dagger(x) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\zeta(x) + \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger d^*(x)\end{aligned}$$

we see that

$$a = a^* \qquad \xi^\dagger = \chi^\dagger \qquad b = c^* \qquad v_\mu = v_\mu^* \qquad \zeta^\dagger = \eta^\dagger \qquad d = d^* \tag{5.2}$$

Employing additionally the convenient expansions

$$\eta_\alpha = \lambda_\alpha - \frac{i}{2}(\sigma^\mu\partial_\mu\xi^\dagger)_\alpha \qquad v_\mu = A_\mu \qquad d = \frac{1}{2}D - \frac{1}{4}\partial_\mu\partial^\mu a \tag{5.3}$$

the vector superfield is thus

$$\begin{aligned}\mathcal{V}(x,\theta,\theta^\dagger) = a(x) + \theta\cdot\xi(x) + \theta^\dagger\cdot\xi^\dagger(x) + \theta\cdot\theta b(x) + \theta^\dagger\cdot\theta^\dagger b^*(x) + \theta^\dagger\bar{\sigma}^\mu\theta A_\mu(x)\\ + \theta^\dagger\cdot\theta^\dagger\theta\cdot\left(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^\dagger\right) + \theta\cdot\theta\theta^\dagger\cdot\left(\lambda^\dagger - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi\right) + \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\frac{1}{2}D(x) - \frac{1}{4}\partial_\mu\partial^\mu a(x)\right)\end{aligned}$$

Deriving the transformation of these components under SUSY rotations is a bit more work than for the chiral superfield, but not nearly the same as the general superfield. Quoting the results from Eq. (1.14)-(1.22), begin with the a field:

$$\sqrt{2}\delta_\epsilon a = \epsilon\cdot\xi + \epsilon^\dagger\cdot\chi^\dagger = \epsilon\cdot\xi + \epsilon^\dagger\cdot\xi^\dagger$$

The ξ_α field, which also gives the $\chi^\dagger$ field:

$$\sqrt{2}\delta_\epsilon\xi_\alpha = 2\epsilon_\alpha b(x) - (\sigma^\mu\epsilon^\dagger)_\alpha\left(i\partial_\mu a(x) + v_\mu(x)\right) = 2\epsilon_\alpha b(x) - (\sigma^\mu\epsilon^\dagger)_\alpha\left(i\partial_\mu a(x) + A_\mu(x)\right)$$

The b field and the c^* field:

$$\begin{aligned}\sqrt{2}\delta_\epsilon b(x) &= \epsilon^\dagger\cdot\zeta^\dagger(x) - \frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi(x)\\ &= \epsilon^\dagger\cdot\eta^\dagger(x) - \frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi(x)\\ &= \epsilon^\dagger\cdot\left(\lambda^\dagger(x) - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi(x)\right) - \frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi(x)\\ &= \epsilon^\dagger\cdot\lambda(x) - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\xi(x)\end{aligned}$$

The A^μ field:

$$\begin{aligned}
\sqrt{2}\delta_\epsilon A_\mu &= \frac{i}{2}\epsilon\sigma_\nu\bar{\sigma}_\mu\partial^\nu\xi(x) - \frac{i}{2}\epsilon^\dagger\bar{\sigma}_\nu\sigma_\mu\partial^\nu\chi^\dagger(x) + \epsilon^\dagger\bar{\sigma}_\mu\eta_\alpha(x) - \epsilon\sigma_\mu\zeta^\dagger(x) \\
&= \frac{i}{2}\epsilon\sigma_\nu\bar{\sigma}_\mu\partial^\nu\xi(x) - \frac{i}{2}\epsilon^\dagger\bar{\sigma}_\nu\sigma_\mu\partial^\nu\xi^\dagger(x) + \epsilon^\dagger\bar{\sigma}_\nu\left(\lambda_\alpha(x) - \frac{i}{2}(\sigma^\mu\partial_\mu\xi^\dagger(x))_\alpha\right) - \epsilon\sigma^\mu\left(\lambda^{\dagger\dot{\alpha}}(x) - \frac{i}{2}(\bar{\sigma}^\mu\partial_\mu\xi(x))^{\dot{\alpha}}\right) \\
&= i\epsilon\sigma_\nu\bar{\sigma}_\mu\partial^\nu\xi(x) - i\epsilon^\dagger\bar{\sigma}_\nu\sigma_\mu\partial^\nu\xi^\dagger(x) + \epsilon^\dagger\bar{\sigma}_\nu\lambda_\alpha(x) - \epsilon\sigma^\mu\lambda^{\dagger\dot{\alpha}}(x)
\end{aligned}$$

The λ field, in lieu of the η field:

$$\begin{aligned}
\sqrt{2}\delta_\epsilon\lambda_\alpha &= \sqrt{2}\delta_\epsilon\eta_\alpha(x) + \frac{i}{2}\sigma^\mu\partial_\mu(\sqrt{2}\delta_\epsilon\xi^\dagger_\alpha(x)) \\
&= 2\epsilon_\alpha d(x) - i(\sigma^\mu\epsilon^\dagger)_\alpha\partial_\mu c(x) - \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\epsilon\right)_\alpha\partial_\nu v_\mu(x) + \frac{i}{2}\sigma^\mu\partial_\mu\left(2\epsilon^{\dagger\dot{\alpha}}c(x) - \left(\bar{\sigma}^\nu\epsilon\right)^{\dot{\alpha}}\left(i\partial_\nu a(x) - v_\nu(x)\right)\right) \\
&= 2\epsilon_\alpha\left(\frac{1}{2}D(x) - \frac{1}{4}\partial_\mu\partial^\mu a(x)\right) + \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) + \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha\partial_\mu\partial_\nu a(x) \\
&= \epsilon_\alpha D(x) - \frac{1}{2}\epsilon_\alpha\partial_\mu\partial^\mu a(x) + \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) + \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\epsilon_\beta\partial_\mu\partial_\nu a(x) \\
&= \epsilon_\alpha D(x) - \frac{1}{2}\epsilon_\alpha\partial_\mu\partial^\mu a(x) + \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) + \frac{1}{2}(g^{\mu\nu}\delta^\beta_\alpha - 2i(\sigma^{\mu\nu})_\alpha{}^\beta)\epsilon_\beta\partial_\mu\partial_\nu a(x) \\
&= \epsilon_\alpha D(x) - \frac{1}{2}\epsilon_\alpha\partial_\mu\partial^\mu a(x) + \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) + \frac{1}{2}\epsilon_\alpha\partial_\mu\partial^\mu a(x) \\
&= \epsilon_\alpha D(x) + \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\epsilon)_\alpha\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)
\end{aligned}$$

And lastly the D field, in lieu of the d field:

$$\begin{aligned}
\sqrt{2}\delta_\epsilon D &= 2\sqrt{2}\delta_\epsilon d(x) + \frac{\sqrt{2}}{2}\partial_\mu\partial^\mu(\delta_\epsilon a(x)) \\
&= 2\left(-\frac{i}{2}\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\eta(x) - \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\zeta^\dagger(x)\right) + \frac{1}{2}\partial_\mu\partial^\mu\left(\epsilon\cdot\xi(x) + \epsilon^\dagger\cdot\xi^\dagger(x)\right) \\
&= -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\left(\lambda(x) - \frac{i}{2}\sigma^\nu\partial_\nu\xi^\dagger(x)\right) - i\epsilon\sigma^\mu\partial_\mu\left(\lambda^\dagger(x) - \frac{i}{2}\bar{\sigma}^\nu\partial_\nu\xi(x)\right) + \frac{1}{2}\epsilon\cdot(\partial_\mu\partial^\mu\xi) + \frac{1}{2}\epsilon^\dagger\cdot\left(\partial_\mu\partial^\mu\xi^\dagger(x)\right) \\
&= -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\lambda(x) - i\epsilon\sigma^\mu\partial_\mu\lambda^\dagger(x) - \frac{1}{2}\epsilon^\dagger\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu\xi^\dagger(x) - \frac{1}{2}\epsilon\sigma^\mu\bar{\sigma}^\nu\partial_\mu\partial_\nu\xi(x) + \frac{1}{2}\epsilon\cdot(\partial_\mu\partial^\mu\xi) + \frac{1}{2}\epsilon^\dagger\cdot\left(\partial_\mu\partial^\mu\xi^\dagger(x)\right) \\
&= -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\lambda(x) - i\epsilon\sigma^\mu\partial_\mu\lambda^\dagger(x) - \frac{1}{2}\epsilon^\dagger\cdot\partial_\mu\partial^\mu\xi^\dagger(x) - \frac{1}{2}\epsilon\cdot\partial_\mu\partial^\nu\xi(x) + \frac{1}{2}\epsilon\cdot(\partial_\mu\partial^\mu\xi) + \frac{1}{2}\epsilon^\dagger\cdot\left(\partial_\mu\partial^\mu\xi^\dagger(x)\right) \\
&= -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\lambda(x) - i\epsilon\sigma^\mu\partial_\mu\lambda^\dagger(x)
\end{aligned}$$

Thus, in summary, the components of a vector superfield transform as

$$\begin{aligned}
\sqrt{2}\delta_\epsilon a &= \epsilon \cdot \xi(x) + \epsilon^\dagger \cdot \xi^\dagger(x) \\
\sqrt{2}\delta_\epsilon \xi_\alpha &= 2\epsilon_\alpha b(x) - (\sigma^\mu \epsilon^\dagger)_\alpha \left(i\partial_\mu a(x) + A_\mu(x)\right) \\
\sqrt{2}\delta_\epsilon b &= \epsilon^\dagger \cdot \lambda(x) - i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \xi(x) \\
\sqrt{2}\delta_\epsilon A_\mu &= i\epsilon\sigma_\nu \bar{\sigma}_\mu \partial^\nu \xi(x) - i\epsilon^\dagger \bar{\sigma}_\nu \sigma_\mu \partial^\nu \xi^\dagger(x) + \epsilon^\dagger \bar{\sigma}_\nu \lambda_\alpha(x) - \epsilon\sigma^\mu \lambda^{\dagger\dot{\alpha}}(x) \\
\sqrt{2}\delta_\epsilon \lambda_\alpha &= \epsilon_\alpha D(x) + \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \epsilon)_\alpha \left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) \\
\sqrt{2}\delta_\epsilon D &= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \lambda(x) - i\epsilon\sigma^\mu \partial_\mu \lambda^\dagger(x)
\end{aligned}$$

This notation implies that a vector superfield contains the gauge boson field A_μ, a gaugino field λ and an auxiliary field D. There are extraneous degrees of freedom in a, ξ and b that can be 'supergauged' away. It is possible to construct vector superfields out of chiral superfields. If Φ is a chiral superfield, then the combinations $\Phi^*\Phi$, $\Phi + \Phi^*$ and $i(\Phi - \Phi^*)$ are vector superfields. This can be used to implement interactions between vector and chiral superfields.

6 Abelian gauge theory

6.1 Supergauge transformations

Recall that a chiral superfield with a gauged symmetry transforms as

$$\Phi_i \to \exp(2igq_i\Lambda(x))\Phi_i \qquad\qquad \Phi^{*i} \to \exp(-2igq_i\Lambda(x))\Phi^{*i} \tag{6.1}$$

This presents no issue for the superpotential since it must be constructed out of products of chiral superfields that are gauge singlets. This leaves the kinetic term $\Phi^*\Phi$ which transforms as

$$\Phi^{*i}\Phi_i \to \Phi^{*i}\exp(-2igq_i\Lambda^*(x))\exp(2igq_i\Lambda(x))\Phi_i = \Phi^{*i}\exp(2igq_i(\Lambda(x)-\Lambda^*(x)))\Phi_i \tag{6.2}$$

and this is not gauge invariant. We need something to 'soak up' the effects of the gauge transformation. As we do in standard QFT, we thus replace the kinetic term with a gauge covariant term, which in superfield language amounts to inserting a factor with a vector superfield

$$\Phi^{*i}\exp(2gq_i\mathcal{V})\Phi_i \to \Phi^{*i}\exp(2gq_i\left[\mathcal{V}+i(\Lambda(x)-\Lambda^*(x))\right])\Phi_i \tag{6.3}$$

In order for this to be invariant, the vector superfield must transform as

$$\mathcal{V} \to \mathcal{V} + i(\Lambda^*(x)-\Lambda(x)) \tag{6.4}$$

The gauge transformation superfield $i(\Lambda^*(x)-\Lambda(x))$ contains components:

$$\begin{aligned}
i(\Lambda^*(x)-\Lambda(x)) = {}& i\phi^*(x) - \theta^\dagger\cdot\left(i\sqrt{2}\psi^\dagger(x)\right) + \theta^\dagger\bar\sigma^\mu\theta(\partial_\mu\phi^*(x)) + \theta^\dagger\cdot\theta^\dagger iF^*(x) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\left(\frac{\sqrt{2}}{2}\theta\sigma^\mu\partial_\mu\psi^\dagger(x)\right)\\
&+\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\frac{i}{4}\partial_\mu\partial^\mu\phi^*(x)\right) - i\phi(x) - \theta\cdot\left(i\sqrt{2}\psi(x)\right) + \theta^\dagger\bar\sigma^\mu\theta(\partial_\mu\phi(x)) - \theta\cdot\theta iF(x)\\
&-\theta\cdot\theta\theta^\dagger\cdot\left(\frac{\sqrt{2}}{2}\bar\sigma^\mu\partial_\mu\psi(x)\right) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\frac{i}{4}\partial_\mu\partial^\mu\phi(x)\right)\\[1em]
={}& i(\phi^*(x)-\phi(x)) - \theta\cdot\left(i\sqrt{2}\psi(x)\right) - \theta^\dagger\cdot\left(i\sqrt{2}\psi^\dagger(x)\right) - \theta\cdot\theta iF(x) + \theta^\dagger\cdot\theta^\dagger iF^*(x)\\
&+\theta^\dagger\bar\sigma^\mu\theta(\partial_\mu\phi(x)+\partial_\mu\phi^*(x)) + \theta\cdot\theta\theta^\dagger\left(\frac{\sqrt{2}}{2}\bar\sigma^\mu\partial_\mu\psi(x)\right) + \theta^\dagger\cdot\theta^\dagger\theta\left(\frac{\sqrt{2}}{2}\sigma^\mu\partial_\mu\psi^\dagger(x)\right)\\
&-\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(\frac{i}{4}\partial_\mu\partial^\mu\left(\phi^*(x)-\phi(x)\right)\right)
\end{aligned}$$

and thus the components of the vector superfield transform as

$$a(x) \to a(x) + i(\phi^*(x)-\phi(x)) \tag{6.5}$$
$$\xi_\alpha(x) \to \xi_\alpha(x) - i\sqrt{2}\psi_\alpha(x) \tag{6.6}$$
$$b(x) \to b(x) - iF(x) \tag{6.7}$$
$$A_\mu(x) \to A_\mu(x) + \partial_\mu\left(\phi^*(x)+\phi(x)\right) \tag{6.8}$$
$$\lambda_\alpha(x) \to \lambda_\alpha(x) \tag{6.9}$$
$$D(x) \to D(x) \tag{6.10}$$

Thus we see that it is possible by appropriate choice of $i(\phi - \phi^*) = \mathrm{Im}\phi$, ψ_α and F to completely cancel the a, ξ_α and b fields. Such a choice of supergauge is known as the Wess-Zumino gauge and leaves the vector superfield with terms

$$\mathcal{V}_{\mathrm{WZ}}(x, \theta, \theta^\dagger) = \theta^\dagger \bar{\sigma}_\mu \theta A^\mu(x) + \theta^\dagger \cdot \theta^\dagger \theta \cdot \lambda(x) + \theta \cdot \theta \theta^\dagger \cdot \lambda^\dagger(x) + \frac{1}{2} \theta \cdot \theta \theta^\dagger \cdot \theta^\dagger D(x) \tag{6.11}$$

Note that this choice of supergauge has a residual symmetry corresponding to the ordinary gauge symmetry:

$$A_\mu(x) \to A_\mu(x) + 2\mathrm{Re}\left[\partial_\mu \phi(x)\right] \tag{6.12}$$

As in ordinary QFT, physical results are independent of gauge choice; thus it is perfectly fine to use the Wess-Zumino gauge when computing physical results. The gauge choice, however, is not invariant under SUSY transformations. In other words, $\delta_\epsilon(\mathcal{V}_{\mathrm{WZ}})$ will not necessarily be in Wess-Zumino gauge. This is not a problem, however, as a another supergauge transformation can restore the SUSY-rotated field to the Wess-Zumino gauge.

6.2 Field strength superfields

The vector superfield is not gauge invariant. This is plainly evident since it contains the gauge field A_μ and which by itself is not gauge invariant. This is a problem if we wish to construct gauge invariant SUSY Lagrangians. We must therefore construct a new object that exhibits a transformation rule

$$\mathcal{F} \to e^{2iq_i\Lambda} \mathcal{F} e^{-2iq_i\Lambda} \tag{6.13}$$

so that products such as $\mathrm{Tr}[\mathcal{V}^2]$ are gauge invariant. Clearly, $\mathcal{F}$ must be a chiral superfield, since it is multiplied by a function of Λ, a chiral superfield. As such it must satisfy the chiral superfield condition:

$$\bar{D}_{\dot{\alpha}} \mathcal{F} = 0 \tag{6.14}$$

which therefore implies that whatever $\mathcal{F}$ is, it must include a factor of $-\frac{1}{4}\bar{D} \cdot \bar{D}$. Lastly, since we know from ordinary QFT that $\partial_\mu A_\nu$ can be used to construct the field strength tensor, which is gauge invariant, our result must include a chiral covariant derivative of the vector superfield $D_\alpha \mathcal{V}$. Thus we define the Abelian field strength superfield

$$\mathcal{F}_\alpha = -\frac{1}{4} \bar{D} \cdot \bar{D} D_\alpha \mathcal{V} \qquad\qquad \mathcal{F}^\dagger_{\dot{\alpha}} = -\frac{1}{4} D \cdot D \bar{D}_{\dot{\alpha}} \mathcal{V} \tag{6.15}$$

Unlike other superfields, this one has a spinor index and is anticommuting. This superfield is supergauge invariant:

$$\begin{aligned}
\mathcal{F}_\alpha &\to -\frac{1}{4} \bar{D} \cdot \bar{D} D_\alpha \left(\mathcal{V} + i(\Lambda^* - \Lambda)\right) \\
&= \mathcal{F}_\alpha + \frac{i}{4} \bar{D} \cdot \bar{D} D_\alpha \Lambda \\
&= \mathcal{F}_\alpha - \frac{i}{4} \bar{D}^{\dot{\beta}} \left(\{\bar{D}_{\dot{\beta}}, D_\alpha\} - D_\alpha \bar{D}_{\dot{\beta}}\right) \Lambda \\
&= \mathcal{F}_\alpha + \frac{1}{2} \sigma^\mu_{\alpha\dot{\beta}} \bar{D}^{\dot{\beta}} \partial_\mu \Lambda \\
&= \mathcal{F}_\alpha
\end{aligned}$$

Now let's compute the actual components of the field strength superfield. Before proceeding, let's convert the vector superfield to a function of chiral coordinates (for ease of computation). We thus perform an expansion about $x_\mu = y_\mu - i\theta^\dagger\bar{\sigma}^\mu\theta$:

$$\begin{aligned}
\mathcal{V}_{\rm WZ}(y,\theta,\theta^\dagger) &= \theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y - i\theta^\dagger\bar{\sigma}^\nu\theta) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y - i\theta^\dagger\bar{\sigma}^\mu\theta) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y - i\theta\sigma^\mu\theta^\dagger) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D(y - i\theta^\dagger\bar{\sigma}^\mu\theta)\\
&= \theta^\dagger\bar{\sigma}_\mu\theta\left(A^\mu - i\theta^\dagger\bar{\sigma}^\nu\theta\partial_\nu A^\mu\right) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D(y)\\
&= \theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y) + i\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\theta_\alpha\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\theta^\dagger_{\dot{\alpha}}\theta^{\dagger\dot{\beta}}\partial_\nu A^\mu(y) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D(y)\\
&= \theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y) + i\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\epsilon_{\alpha\gamma}\theta^\gamma\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\gamma}}\theta^{\dagger\dot{\gamma}}\theta^{\dagger\dot{\beta}}\partial_\nu A^\mu(y) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D(y)\\
&= \theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y) - \frac{i}{4}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\delta^\beta_\alpha\sigma^\nu_{\beta\dot{\beta}}\delta^{\dot{\beta}}_{\dot{\alpha}}\partial_\nu A^\mu(y) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D(y)\\
&= \theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y) - \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger g^{\mu\nu}\partial_\nu A^\mu(y) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D(y)\\
&= \theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(D(y) - i\partial_\mu A^\mu(y)\right)
\end{aligned}$$

This was done because the anti-chiral covariant derivative takes a particularly simple form with these coordinates, namely $\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\theta^{\dagger\dot{\alpha}}}$. Thus when we compute $D_\alpha\mathcal{V}$, we need only retain those terms that have two factors of $\theta^\dagger$. Let's thus compute $D_\alpha\mathcal{V}$, retaining only those terms with such a factor:

$$\begin{aligned}
D_\alpha\mathcal{V}_{\rm WZ} &= \left(\frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\nu\theta^\dagger)_\alpha\frac{\partial}{\partial y^\nu}\right)\left(\theta^\dagger\bar{\sigma}_\mu\theta A^\mu(y) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda(y) + \theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger(y) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(D(y) - i\partial_\mu A^\mu(y)\right)\right)\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - 2i\sigma^\nu_{\alpha\dot{\alpha}}\theta^{\dagger\dot{\alpha}}\theta^\dagger_{\dot{\beta}}\bar{\sigma}_\mu^{\dot{\beta}\beta}\theta_\beta\partial_\nu A^\mu(y) - 2i\theta\cdot\theta\sigma^\nu_{\alpha\dot{\alpha}}\theta^{\dagger\dot{\alpha}}\theta^{\dagger\dot{\beta}}\partial_\nu\lambda^\dagger_{\dot{\beta}}(y) + \ldots\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) + i\theta^\dagger\cdot\theta^\dagger\sigma^\nu_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\gamma}\dot{\beta}}\bar{\sigma}_\mu^{\dot{\beta}\beta}\theta_\beta\partial_\nu A^\mu(y) - i\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\sigma^\nu_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\partial_\nu\lambda^\dagger_{\dot{\beta}}(y)\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - i\sigma^\nu_{\alpha\dot{\alpha}}\delta^{\dot{\alpha}}_{\dot{\beta}}\bar{\sigma}_\mu^{\dot{\beta}\beta}\theta_\beta\partial_\nu A^\mu(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - i\theta^\dagger\cdot\theta^\dagger(\sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta\theta_\beta\partial_\nu A_\mu(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha\\
&\qquad - i\theta^\dagger\cdot\theta^\dagger\left[\frac{1}{2}\left(\sigma^\nu\bar{\sigma}^\mu + \sigma^\mu\bar{\sigma}^\nu\right)\theta + \frac{1}{2}\left(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu\right)\theta\right]_\alpha\partial_\nu A_\mu(y)\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha - i\theta^\dagger\cdot\theta^\dagger\left[g^{\nu\mu}\theta + \frac{1}{2}\left(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu\right)\theta\right]_\alpha\partial_\nu A_\mu(y)\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha - i\theta^\dagger\cdot\theta^\dagger\left[\frac{1}{2}\left(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu\right)\theta\right]_\alpha\partial_\nu A_\mu(y)\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha - i\theta^\dagger\cdot\theta^\dagger\left[\frac{1}{2}\left(\sigma^\nu\bar{\sigma}^\mu\partial_\nu A_\mu(y) - \sigma^\mu\bar{\sigma}^\nu\partial_\nu A_\mu(y)\right)\theta\right]_\alpha\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha - i\theta^\dagger\cdot\theta^\dagger\left[\frac{1}{2}\left(\sigma^\mu\bar{\sigma}^\nu\partial_\mu A_\nu(y) - \sigma^\mu\bar{\sigma}^\nu\partial_\nu A_\mu(y)\right)\theta\right]_\alpha\\
&= \theta^\dagger\cdot\theta^\dagger\lambda_\alpha(y) + \theta_\alpha\theta^\dagger\cdot\theta^\dagger D(y) - \theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha - \frac{i}{2}\theta^\dagger\cdot\theta^\dagger(\sigma^\mu\bar{\sigma}^\nu\theta)\left(\partial_\mu A_\nu(y) - \partial_\nu A_\mu(y)\right)
\end{aligned}$$

where we've ignored the factor of $\partial_\mu A^\mu$ since it constitutes a total derivative. Thus, after applying the derivative $-\frac{1}{4}\bar{D}\cdot\bar{D}$, the Abelian field strength superfield in component form is

$$\mathcal{F}_\alpha = \lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu}(y) - \theta\cdot\theta\left(i\sigma^\nu\partial_\nu\lambda^\dagger(y)\right)_\alpha \tag{6.16}$$

$$\mathcal{F}^{\dagger\dot{\alpha}} = \lambda^{\dagger\dot{\alpha}}(y) + \theta^{\dagger\dot{\alpha}}D(y) + \frac{i}{2}(\bar{\sigma}^\nu\sigma^\mu\theta^\dagger)^{\dot{\alpha}}F_{\mu\nu} - \theta^\dagger\cdot\theta^\dagger\left(i\bar{\sigma}^\nu\partial_\nu\lambda(y)\right)^{\dot{\alpha}} \tag{6.17}$$

Although we employed the Wess-Zumino gauge in deriving the above results, since the field strength superfield is supergauge invariant, the above results hold irrespective of gauge choice. Now, it may seem necessary to return to the x coordiantes. But, recall that this superfield is used only to find the F-term; expanding again in $y = x + i\theta^\dagger\bar{\sigma}\theta$ would bring about terms with coefficients of the form $\theta^\dagger\cdot\theta^\dagger\theta$ which would not survive when computing the F-term.

7 Gauged Wess-Zumino model

Here we'll consider the simplest gauged Wess-Zumino model consisting of n chiral superfields Φ_i, $i = \{1, \ldots, n\}$. Each field is charged under a $U(1)$ symmetry and thus has a charge q_i. As such, the Lagrangian will receive contributions from the F-factors of uncharged chiral fields Φ_i, and of the term $\Phi_i\Phi_j$ (and its complex conjugate), which has mass dimension $[\Phi_i\Phi_j] = 2$, meaning it must be multiplied by a mass matrix M^{ij}. Since the product $\Phi_i\Phi_j$ is symmetric, then M_{ij} must also be symmetric. The Lagrangian will also receive contributions from the F-factors of the term $\Phi_i\Phi_j\Phi_k$, which has mass dimension $[\Phi_i\Phi_j\Phi_k] = 3$, meaning it must be multiplied by a Yukawa coupling matrix y_{ijk}. Since the product $\Phi_i\Phi_j\Phi_k$ is symmetric under exchange of any index, y_{ijk} must also be fully symmetric. Gauge invariance requires that, for given i, j, k,

$$q_i \neq 0 \qquad \Rightarrow \qquad c_i = 0 \tag{7.1}$$

$$q_i + q_j \neq 0 \qquad \Rightarrow \qquad M_{ij} = 0 \tag{7.2}$$

$$q_i + q_j + q_k \neq 0 \qquad \Rightarrow \qquad y_{ijk} = 0 \tag{7.3}$$

In other words, only those terms whose charges add up such that the product of superfields is a gauge singlet will contribute. This exhausts all possible chiral superfield contributions, and thus the superpotential is

$$W(\Phi_i) = \frac{1}{2}M_{ij}\Phi_i\Phi_j + \frac{1}{6}y_{ijk}\Phi_i\Phi_j\Phi_k \tag{7.4}$$

where the linear term has been dropped; any field satisfying this condition would reduce to the ungauged example in the previous chapter. Now examine vector superfields, which in this model come from the kinetic term $\Phi^{*i}e^{2q_i\mathcal{V}}\Phi_i$ and, since the gauge group is Abelian, so that the gauge superfield is not charged under the gauge group, from the gauge superfield itself. Thus the Lagrangian is

$$\mathcal{L} = \Phi^{*i}e^{2q_i\mathcal{V}}\Phi_i\Big]_D - 2\,\kappa\mathcal{V}]_D + \frac{1}{4}\left(\mathcal{F}^\alpha\mathcal{F}_\alpha]_F + \text{c.c.}\right) + \left(c_i\Phi_i]_F + \frac{1}{2}M^{ij}\;\Phi_i\Phi_j\big]_F + \frac{1}{6}y^{ijk}\;\Phi_i\Phi_j\Phi_k\big]_F + \text{c.c.}\right) \tag{7.5}$$

In the following sections we'll consider the explicit components the Lagrangian corresponding to the above model.

7.1 Matter kinetic terms

First let's consider the coupling of the chiral superfields to the vector superfields, $\Phi^{*i}e^{2q_i\mathcal{V}}\Phi_i$. Normally the exponential factor such as $e^{2q_i\mathcal{V}}$ would be worrisome (i.e. non-renormalizable). However, in the Wess-Zumino gauge, this is not a problem because $\mathcal{V}^n$ for $n \geq 3$ contains factors of θ^3 and $(\theta^\dagger)^3$, which vanish identically. Let's work out the components of this expansion. Expanding the exponential, we see

$$\begin{aligned}
\Phi^{*i}e^{2q_i\mathcal{V}}\Phi_i\Big]_D &= \Phi^{*i}\left(1 + 2q_i\mathcal{V} + \frac{1}{2}(4q_i^2\mathcal{V}^2)\right)\Phi_i\Big]_D \\
&= \Phi^{*i}\Phi\big]_D + 2q_i\Phi^{*i}\mathcal{V}\Phi_i\big]_D + 2q_i^2\Phi^{*i}\mathcal{V}^2\Phi_i\big]_D
\end{aligned}$$

The first term is simply the usual Wess-Zumino kinetic term, worked out in Chapter 5:

$$\Phi^*\Phi_i]_D = \partial^\mu\phi^{*i}\partial_\mu\phi_i + i\psi^{i\dagger}\bar{\sigma}^\mu\partial_\mu\psi_i + F^{*i}F_i \tag{7.6}$$

For the other two terms, note the fact that, in WZ gauge, the vector superfield has terms that have at minimum one θ and one $\theta^\dagger$. Thus the two other terms, keeping only those terms that have two θ and two $\theta^\dagger$ (which will contribute to the D-term) give

$$\begin{aligned}
\Phi^{*i}\mathcal{V}\Phi_i &= 2\theta^\dagger\cdot\psi^{\dagger i}\theta^\dagger\bar\sigma_\mu\theta A^\mu\theta\cdot\psi_i + \sqrt{2}\phi^{*i}\theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda\theta\cdot\psi_i + \sqrt{2}\theta^\dagger\cdot\psi^{\dagger i}\theta\cdot\theta\theta^\dagger\cdot\lambda^\dagger\phi_i \\
&\quad + i\theta^\dagger\bar\sigma_\mu\theta\partial^\mu\phi^{*i}\theta^\dagger\bar\sigma_\nu\theta A^\nu\phi_i - i\phi^{*i}\theta^\dagger\bar\sigma_\nu\theta A^\nu\theta^\dagger\bar\sigma_\mu\theta\partial^\mu\phi_i \\
&\quad + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\phi^{*i}\phi_i D + \dots
\end{aligned}$$

$$\begin{aligned}
&= 2\theta^\dagger_{\dot\alpha}\psi^{\dagger\dot\alpha i}\theta^\dagger_{\dot\beta}\bar\sigma_\mu^{\dot\beta\beta}\theta_\beta\theta^\alpha\psi_{\alpha i}A^\mu - \frac{\sqrt{2}}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\phi^{*i}\psi_i\cdot\lambda - \frac{\sqrt{2}}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\lambda^\dagger\cdot\psi^{i\dagger}\phi_i \\
&\quad + \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger g_{\mu\nu}\partial^\mu\phi^{*i}A^\nu\phi_i - \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger g_{\nu\mu}\phi^{*i}A^\nu\partial^\mu\phi_i \\
&\quad + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\phi^{*i}\phi_i D + \dots
\end{aligned}$$

$$\begin{aligned}
&= -2\psi^{\dagger\dot\alpha i}\left(-\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\theta^\dagger\cdot\theta^\dagger\right)\bar\sigma_\mu^{\dot\beta\beta}\epsilon_{\beta\gamma}\left(-\frac{1}{2}\epsilon^{\gamma\alpha}\theta\cdot\theta\right)\psi_{\alpha i}A^\mu - \frac{\sqrt{2}}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\phi^{*i}\psi_i\cdot\lambda \\
&\quad - \frac{\sqrt{2}}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\lambda^\dagger\cdot\psi^{i\dagger}\phi_i - \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger A_\mu\left(\phi^{*i}\partial^\mu\phi_i - \phi_i\partial^\mu\phi^{*i}\right) \\
&\quad + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\phi^{*i}\phi_i D + \dots
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\psi^{\dagger i}\bar\sigma_\mu\psi_i A^\mu - \frac{\sqrt{2}}{2}\theta^\dagger\cdot\theta^\dagger\theta\cdot\theta\left(\phi^{*i}\psi_i\cdot\lambda + \lambda^\dagger\cdot\psi^{i\dagger}\phi_i\right) \\
&\quad - \frac{i}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger A_\mu\left(\phi^{*i}\partial^\mu\phi_i - \phi_i\partial^\mu\phi^{*i}\right) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger\phi^{*i}\phi_i D + \dots
\end{aligned}$$

and

$$\begin{aligned}
\Phi^{*i}\mathcal{V}^2\Phi_i &= \phi^{*i}\theta^\dagger\bar\sigma_\mu\theta A^\mu\theta^\dagger\bar\sigma_\nu\theta A^\nu\phi_i \\
&= \phi^{*i}\phi_i A^\mu A^\nu\theta^\dagger\bar\sigma_\mu\theta\theta^\dagger\bar\sigma_\nu\theta \\
&= -\frac{1}{2}\phi^{*i}\phi_i A^\mu A_\mu\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger
\end{aligned}$$

Thus the D-terms are

$$\begin{aligned}
\Phi^{*i}e^{2q_i\mathcal{V}}\Phi_i\Big]_D &= -\partial^\mu\phi^{*i}\partial_\mu\phi_i + i\psi^{i\dagger}\bar\sigma^\mu\partial_\mu\psi_i + F^{*i}F_i - \frac{q_i}{2}\psi^{\dagger i}\bar\sigma_\mu\psi_i A^\mu - q_i\sqrt{2}\left(\phi^{*i}\psi_i\cdot\lambda + \lambda^\dagger\cdot\psi^{i\dagger}\phi_i\right) \\
&\quad - iq_i A_\mu\left(\phi^{*i}\partial^\mu\phi_i - \phi_i\partial^\mu\phi^{*i}\right) + q_i\phi^{*i}\phi_i D - q_i^2\phi^{*i}\phi_i A^\mu A_\mu
\end{aligned}$$

It is notable that the kinetic terms simplify by introducing the covariant derivative:

$$\nabla_\mu\phi_i = \partial_\mu\phi_i - iA_\mu\phi_i \qquad\qquad \nabla_\mu\psi_i = \partial_\mu\psi_i - iA_\mu\psi_i \tag{7.7}$$

and therefore the SUSY kinetic terms are

$$\Phi^{*i}e^{2q_i\mathcal{V}}\Phi_i\Big]_D = -\left(\nabla^\mu\phi^i\right)^*\nabla_\mu\phi_i + i\psi^{i\dagger}\bar{\sigma}^\mu\nabla_\mu\psi_i + F^{*i}F_i - q_i\sqrt{2}\left(\phi^{*i}\psi_i\cdot\lambda + \lambda^\dagger\cdot\psi^{i\dagger}\phi_i\right) + q_i\phi^{*i}\phi_i D \quad (7.8)$$

and thus, as expected, a SUSY gauge theory involves promoting the ordinary derivatives to covariant derivatives; in addition, however, it requires new interactions amongst the scalars, gauginos, and fermions, as well as new interactions between the D field and the scalars

7.2 Fayet-Illipoulos term

The presence of the term $2\kappa\mathcal{V}$, known as a *Fayet-Illipoulos term* may seem unusual. It can be included due to the fact that the gauge fields of an Abelian theory are uncharged; in a non-Abelian theory such a term would be forbidden. Its contribution is simply

$$\mathcal{L} \subset -2\,\kappa\mathcal{V}]_D =$$

7.3 Gauge kinetic terms

The gauge kinetic terms are given by the F-terms from $\mathcal{F}^\alpha\mathcal{F}_\alpha$ and its complex conjugate. Since we want the F-terms, we need keep only terms with two θ coordinates. Doing this, and aggressively integrating by parts while ruthlessly ignoring total derivatives gives

$$\begin{aligned}
\mathcal{F}^\alpha\mathcal{F}_\alpha &= \left[\lambda^\alpha(y) + \theta^\alpha D(y) - \frac{i}{2}(\theta\sigma^\rho\bar{\sigma}^\beta)^\alpha F_{\mu\nu}(y) + \theta\cdot\theta\left(i\partial_\mu\lambda^\dagger(y)\bar{\sigma}^\mu\right)^\alpha\right] \\
&\quad\times\left[\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu}(y) - \theta\cdot\theta\left(i\sigma^\mu\partial_\mu\lambda^\dagger(y)\right)_\alpha\right] \\
&= \theta\cdot\theta D^2 - \frac{1}{4}\theta\sigma^\alpha\bar{\sigma}^\beta\sigma^\mu\bar{\sigma}^\nu\theta F_{\mu\nu}F_{\alpha\beta} + \theta\cdot\theta\left(i\partial_\mu\lambda^\dagger\bar{\sigma}^\mu\lambda - i\lambda\sigma^\mu\partial_\mu\lambda^\dagger\right) + \ldots \\
&= \theta\cdot\theta D^2 + \frac{1}{8}\theta\cdot\theta\mathrm{Tr}\left[\sigma^\alpha\bar{\sigma}^\beta\sigma^\mu\bar{\sigma}^\nu\right]F_{\mu\nu}F_{\alpha\beta} + \theta\cdot\theta\left(i\partial_\mu\lambda^\dagger\bar{\sigma}^\mu\lambda + i\partial_\mu\lambda^\dagger\bar{\sigma}^\mu\lambda\right) + \ldots \\
&= \theta\cdot\theta D^2 + \frac{1}{4}\theta\cdot\theta\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - i\epsilon^{\alpha\beta\mu\nu}\right)F_{\mu\nu}F_{\alpha\beta} + \theta\cdot\theta\left(2i\partial_\mu\lambda^\dagger\bar{\sigma}^\mu\lambda\right) + \ldots \\
&= \theta\cdot\theta D^2 + \frac{1}{4}\theta\cdot\theta\left(F^\mu{}_\mu F^\alpha{}_\alpha - F^{\alpha\beta}F_{\alpha\beta} + F^{\beta\alpha}F_{\alpha\beta} - i\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}F_{\alpha\beta}\right) + \theta\cdot\theta\left(-2i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda\right) + \ldots \\
&= \theta\cdot\theta D^2 + \theta\cdot\theta\left(-\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta} - \frac{i}{4}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}F_{\alpha\beta}\right) + \theta\cdot\theta\left(-2i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda\right) + \ldots
\end{aligned}$$

Thus the F-terms are

$$\mathcal{F}^\alpha\mathcal{F}_\alpha]_F = D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{i}{4}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}F_{\alpha\beta} - 2i\lambda^\dagger\bar{\sigma}^\mu\partial_\mu\lambda \quad (7.9)$$

By symmetry, the F-terms of the complex conjugate are simply

$$\mathcal{F}^\dagger_{\dot{\alpha}}\mathcal{F}^{\dagger\dot{\alpha}}\Big]_F = D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}F_{\alpha\beta} + 2i\lambda\sigma^\mu\partial_\mu\lambda^\dagger \quad (7.10)$$

When combined, we see

$$\mathcal{F}^\alpha \mathcal{F}_\alpha]_F + \text{c.c.} = 2D^2 - F^{\mu\nu}F_{\mu\nu} - 4i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda$$

Note that to get the appropriate normalization for the gauge field and gaugino kinetic terms, we need to multiply a factor of $\frac{1}{4}$, giving

$$\frac{1}{4}\ \mathcal{F}^\alpha \mathcal{F}_\alpha]_F + \text{c.c.} = \frac{1}{2}D^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda^\dagger \bar{\sigma}^\mu \partial_\mu \lambda \tag{7.11}$$

Notice the presence of the CP-violating term. For a gauge theory involving an Abelian symmetry, this can be written as a total derivative and thus ignored. For a non-Abelian gauge symmetry, such a term cannot be ignored due to instanton effects. We will not consider such theories in these notes. Note also that the gauginos do not couple directly to the gauge field. This is because they are the superpartners of the abelian gauge boson, which is uncharged. In a non-Abelian case, the gauginos would live in adjoint representation of the gauge group and thus carry a gauge charge, enabling a coupling to the gauge bosons.

7.4 Superpotential

Recall that the superpotential is simply

$$W(\Phi_i) = \frac{1}{2}M_{ij}\Phi_i\Phi_j + \frac{1}{6}y_{ijk}\Phi_i\Phi_j\Phi_k \tag{7.12}$$

where the linear term has been dropped since it is not a gauge singlet

8 sQED

Here we'll consider the supersymmetric version of quantum electrodynamics, consisting of two chiral superfields $\mathcal{E}$ and $\bar{\mathcal{E}}$; note that the bar in $\bar{\mathcal{E}}$ does not indicate conjugation, but merely a differentiation in the name. The field content of $\mathcal{E}$ is that of a chiral superfield, with fields $\tilde{e}$, e and $\bar{F}$. Note that the tilde on e marks the scalar partner of the electron, the selectron. Likewise, the field content of $\bar{\mathcal{E}}$ is the fields $\tilde{\bar{e}}$, $\bar{e}$, $\bar{F}$. Lastly, the field content of the vector superfield in WZ-gauge are the auxiliary field D, the gaugino λ and the field strength $F_{\mu\nu}$.

Each chiral superfield is charged under a $U(1)$ symmetry, with $\mathcal{E}$ having a charge $-q$ and $\bar{\mathcal{E}}$ having a charge q. As such, the Lagrangian will receive contributions from the F-factors of the term $\bar{\mathcal{E}}\mathcal{E}$ (and its complex conjugate), which has mass dimension $[\bar{\mathcal{E}}\mathcal{E}] = 2$, meaning it must be multiplied by a mass m (since projecting out the F-term increases the mass dimension by 1). Only those terms whose charges add up such that the product of superfields is a gauge singlet will contribute. As such, this mass term exhausts all possible chiral superfield contributions, due to the charges, and thus the superpotential is

$$W(\mathcal{E}, \bar{\mathcal{E}}) = \frac{1}{2} m \mathcal{E}\mathcal{E} + \frac{1}{2} m \bar{\mathcal{E}}\bar{\mathcal{E}} \tag{8.1}$$

Now examine vector superfields, which in this model come from the kinetic term $\mathcal{E}^\dagger e^{2q_i \mathcal{V}} \mathcal{E}$ and, since the gauge group is Abelian, so that the gauge superfield is not charged under the gauge group, from the gauge superfield itself, in the form of a Fayet-Illiopoulos term. Thus the Lagrangian is

$$\mathcal{L} = \mathcal{E}^\dagger e^{-2q\mathcal{V}} \mathcal{E}\Big]_D + \bar{\mathcal{E}}^\dagger e^{2q\mathcal{V}} \bar{\mathcal{E}}\Big]_D - 2\,\kappa \mathcal{V}]_D + \frac{1}{4}\left(\mathcal{F}^\alpha \mathcal{F}_\alpha]_F + \text{c.c.}\right) + \frac{1}{2} m \left(\mathcal{E}\mathcal{E}]_F + \bar{\mathcal{E}}\bar{\mathcal{E}}\big]_F\right) + \text{c.c.} \tag{8.2}$$

In the following sections we'll consider the explicit components the Lagrangian corresponding to the above model, liberally quoting results from the previous model.

8.1 Kinetic terms

8.2 Superpotential

9 Non-Abelian gauge theory

9.1 Supergauge transformations

Consider a chiral superfield with a gauged non-Abelian symmetry; this transforms as

$$\Phi_i \to \left[\exp\left(2ig_a\Lambda^a T^a\right)\right]_i{}^j \Phi_j \qquad \Phi^{*i} \to \Phi^{*j}\left[\exp\left(2ig_a\Lambda^{\dagger a}T^a\right)\right]_j{}^i \tag{9.1}$$

Analogously to the Abelian case, the ungauged kinetic term is replaced by the gauged kinetic term

$$\Phi^{*i}\Phi_i \to \Phi^{*i}\left[\exp(2g_aT^a\mathcal{V}^a)\right]_i{}^j\Phi_j \tag{9.2}$$

Note that the vector superfield and the gauge parameter superfield now take on values in the non-Abelian algebra (indicated by the a index). To clean up the notation, it is possible to define matrix-valued superfields in a given representation R:

$$\mathcal{V}_i^j = 2g_a[T^a]_i{}^j\mathcal{V}^a \qquad \Lambda_i^j = 2g_a[T^a]_i{}^j\Lambda^a \tag{9.3}$$

which gives the chiral field transformations as

$$\Phi_i \to \left[\exp(i\Lambda)\right]_i{}^j\Phi_j \qquad \Phi^{*i} = \Phi^{*j}\left[\exp(i\Lambda^\dagger)\right]_j{}^i \tag{9.4}$$

and the kinetic term transformation as

$$\Phi^{*i}\left[\exp(\mathcal{V})\right]_i{}^j\Phi_j \to \Phi^{*i}\left[\exp(i\Lambda^\dagger)\right]_i{}^k\left[\exp(\mathcal{V}^a)\right]_k{}^\ell\left[\exp(i\Lambda)\right]_\ell{}^j\Phi_j \tag{9.5}$$

In order for this to be invariant, the vector superfield must transform as

$$\exp(\mathcal{V}) \to \exp(i\Lambda^*)\exp(\mathcal{V})\exp(i\Lambda) \tag{9.6}$$

Unlike the Abelian case, this is a non-linear transformation, brought about by the fact that the superfields do not commute past each other due to the fact that they take on values in a non-Abelian algebra. This expression can expanded, using the Baker-Campbell-Hausdorff lemma, keeping only those terms linear in Λ and Λ^*:

$$\mathcal{V} \to \mathcal{V} + i\left(\Lambda^\dagger - \Lambda\right) - \frac{i}{2}\left[\mathcal{V}, \Lambda + \Lambda^\dagger\right] + i\sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left[\mathcal{V},\left[\mathcal{V},\ldots\left[\mathcal{V},\Lambda^\dagger-\Lambda\right]\ldots\right]\right] \tag{9.7}$$

where the k^{th} term involvs k commutators of $\mathcal{V}$ and B_{2k} are the Bernoulli numbers given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \tag{9.8}$$

Eq. (8.7) can be recomputed using Eq. (8.4) and the Lie algebra commutator $[T^a, T^b] = if^{abc}T^c$:

$$\mathcal{V}^a \to \mathcal{V}^a + i\left(\Lambda^{a^*} - \Lambda^a\right) + g_af^{abc}\mathcal{V}^b\left(\Lambda^{*c} + \Lambda^c\right) - \frac{i}{3}g_a^2f^{abc}f^{cde}\mathcal{V}^b\mathcal{V}^d\left(\Lambda^{*e} - \Lambda^e\right) + \ldots \tag{9.9}$$

As in the Abelian case, an appropriate choice of $\Lambda^{a^*} - \Lambda^a$ (as before, the Wess-Zumino gauge) gives a very simple expression for the vector superfield

$$\mathcal{V}^a_{\text{WZ}}(x,\theta,\theta^\dagger) = \theta^\dagger\bar{\sigma}^\mu\theta A^a_\mu(x) + \theta^\dagger\cdot\theta^\dagger\theta\cdot\lambda^a(x) + \theta\cdot\theta\theta^\dagger\cdot\lambda^{a\dagger}(x) + \frac{1}{2}\theta\cdot\theta\theta^\dagger\cdot\theta^\dagger D^a(x) \tag{9.10}$$

9.2 Field strength superfields

As in the Abelian case, the non-Abelian field strength superfield must be a chiral superfield with transformation rule

$$\mathcal{F}_\alpha \to e^{i\Lambda}\mathcal{F}_\alpha e^{-i\Lambda} \tag{9.11}$$

which is identical to Eq. (7.13). The generalization of Eq. (7.15) is a little less straightforward:

$$\mathcal{F}_\alpha = -\frac{1}{4}\bar{D}\cdot\bar{D}\left(e^{-\mathcal{V}}D_\alpha e^{\mathcal{V}}\right) \tag{9.12}$$

As with the vector and gauge parameter superfields, the field strength superfield takes on values in the R representation of the corresponding Lie algebra and therefore we can write

$$\mathcal{F}_\alpha = 2g_a T_a \mathcal{F}^a_\alpha \tag{9.13}$$

Expanding the term in parenthesis gives

$$e^{-\mathcal{V}}D_\alpha e^{\mathcal{V}} = D_\alpha\mathcal{V} - \frac{1}{2}[\mathcal{V}, D_\alpha\mathcal{V}] + \frac{1}{6}[\mathcal{V},[\mathcal{V},D_\alpha]] + \ldots \tag{9.14}$$

In WZ gauge, only the first two terms contribute, and thus the field strength superfield components are given by the expression

$$\mathcal{F}^a_\alpha = -\frac{1}{4}\bar{D}\cdot\bar{D}\left(D_\alpha\mathcal{V}^a - ig_a f^{abc}\mathcal{V}^b D_\alpha\mathcal{V}^c\right)$$

which yields

$$(\mathcal{F}_{\mathrm{WZ}})^a_\alpha = \lambda^a_\alpha(x) + \theta_\alpha D^a(x) + i(\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu} - i\theta\cdot\theta(\sigma^\mu\nabla_\mu\lambda^{\dagger a})_\alpha \tag{9.15}$$

Note that in this case, unlike in the Abelian case, the field strength superfield is gauge-covariant (instead of gauge invariant) and thus the gauge choice must be noted

Part IV
Spotaneous SUSY Breaking

10 F-term breaking

10.1 O'Raifeartaigh model

The simplest model exhibiting F-term breaking consists of a triplet of chiral superfields Φ_1, Φ_2, Φ_3 for which the superpotential is

$$W = g\Phi_1(\Phi_3^2 - m^2) + M\Phi_2\Phi_3 \tag{10.1}$$

where $M \gg m$. The scalar potential of this theory is

$$V = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 = g^2 \left|\phi_3^2 - m^2\right|^2 + M^2 \left|\phi_3\right|^2 + \left|2g\phi_1\phi_3 + M\phi_2\right|^2 \tag{10.2}$$

The first two terms only involve ϕ_3; therefore write ϕ_3 in terms of two real fields A and B as $\phi_3 = \frac{1}{\sqrt{2}}(A + iB)$ and thus

$$\begin{aligned}
\left|\phi_3^2 - m^2\right|^2 &= \left|\frac{1}{2}A^2 - \frac{1}{2}B^2 - m^2 + \frac{2i}{\sqrt{2}}AB\right|^2 \\
&= \left(\frac{1}{2}A^2 - \frac{1}{2}B^2 - m^2 - \frac{2i}{\sqrt{2}}AB\right)\left(\frac{1}{2}A^2 - \frac{1}{2}B^2 - m^2 + \frac{2i}{\sqrt{2}}\right) \\
&= \left(\frac{1}{2}A^2 - \frac{1}{2}B^2 - m^2\right)^2 + 2A^2B^2 \\
&= \frac{1}{4}A^4 - \frac{1}{2}A^2B^2 + \frac{1}{4}B^4 + m^4 - m^2(A^2 - B^2) + 2A^2B^2 \\
&= \frac{1}{4}\left(A^4 + B^2\right) + \frac{1}{2}A^2B^2 + m^4 - m^2(A^2 - B^2) \\
&= \frac{1}{4}\left(A^2 + B^2\right)^2 + m^4 - m^2(A^2 - B^2)
\end{aligned}$$

$$\left|\phi_3\right|^2 = \frac{1}{2}\left(A^2 + B^2\right)$$

Combining these two terms, the relevant potential is:

$$\begin{aligned}
V &= \frac{g^2}{4}\left(A^4 + B^2\right)^2 + g^2m^4 - g^2m^2(A^2 - B^2) + \frac{M^2}{2}\left(A^2 + B^2\right) \\
&= \frac{g^2}{4}\left(A^4 + B^2\right)^2 + g^2m^4 + \frac{1}{2}A^2\left(M^2 - 2g^2m^2\right) + \frac{1}{2}\left(M^2 + 2g^2m^2\right)
\end{aligned}$$

Minimizing this potential

11 D-term breaking

12 Soft SUSY breaking

Part V
The Minimally-Supersymmetric Standard Model

13 Field contents

14 Matter kinetic terms

15 Gauge kinetic terms

16 Higgs kinetic terms

17 Soft SUSY breaking

Part VI
Appendices

18 Two-component Spinors

An element of the LH spinor representation of the Lorentz group $\left(\frac{1}{2},0\right)$ will be represented by χ_α with a lower, undotted index; conversely, an element of the RH spinor representation $\left(0,\frac{1}{2}\right)$ will be represented by $\eta^{\dagger\dot\alpha}$ with upper, dotted indices. Recall that the LH and RH representations are related by Hermitian conjugation; thus η^α takes on values in the LH representation, but the dagger moves it into the RH representation. The height of the indices may be manipulated by the 'spinor metric' $\epsilon_{\alpha\beta} = -i\sigma^2_{\alpha\beta}$ following the rules

$$\chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta \qquad \chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta \qquad \chi^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\chi^{\dagger\dot\beta} \qquad \chi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\chi^\dagger_{\dot\beta} \tag{18.1}$$

Furthermore, these satisfy the relations

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma_\alpha \qquad \text{and} \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}_{\dot\alpha} \tag{18.2}$$

Repeated spinor indices contracted like

$$^\alpha{}_\alpha \qquad \text{or} \qquad {}_{\dot\alpha}{}^{\dot\alpha} \tag{18.3}$$

are suppressed by convention. This has some quirky results, namely:

$$\chi\cdot\eta = \chi^\alpha\eta_\alpha = \chi^\alpha\epsilon_{\alpha\beta}\eta^\beta = -\eta^\beta\epsilon_{\alpha\beta}\chi^\alpha = \eta^\beta\epsilon_{\beta\alpha}\chi^\alpha = \eta^\beta\chi_\beta = \eta\cdot\chi \tag{18.4}$$

and

$$\eta^\dagger\bar\sigma^\mu\chi = \eta^\dagger_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\chi_\alpha = -\chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\eta^{\dagger\dot\alpha} = -\chi\sigma^\mu\eta^\dagger \tag{18.5}$$

Under complex conjugation spinor products behave as

$$(\chi\cdot\eta)^\dagger = \eta^\dagger\cdot\chi^\dagger = \chi^\dagger\cdot\eta^\dagger \tag{18.6}$$

Meanwhile, products involving σ matrices behave differently:

$$(\chi^\dagger\bar\sigma^\mu\eta)^\dagger = \eta^\dagger\bar\sigma^\mu\chi \qquad \text{and} \qquad (\chi\sigma^\mu\bar\sigma^\nu\eta)^\dagger = \eta^\dagger\bar\sigma^\nu\sigma^\mu\chi^\dagger \tag{18.7}$$

Products of the form $\theta_\alpha\theta_\beta$ can be simplified as follows

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\cdot\theta \qquad \theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\cdot\theta \qquad \theta^\dagger_{\dot\alpha}\theta^\dagger_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\theta^\dagger\cdot\theta^\dagger \qquad \theta^{\dagger\dot\alpha}\theta^{\dagger\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\theta^\dagger\cdot\theta^\dagger \tag{18.8}$$

Lastly, there is the Fierz rearrangement identity

$$\chi_\alpha(\xi\cdot\eta) + \xi_\alpha(\eta\cdot\chi) + \eta_\alpha(\chi\cdot\xi) = 0 \tag{18.9}$$

which imply

$$(\theta\cdot\eta)(\theta\cdot\chi) = -\frac{1}{2}(\eta\cdot\chi)(\theta\cdot\theta) \tag{18.10}$$

$$(\chi^\dagger\bar\sigma^\mu\eta)(\lambda\sigma_\mu\psi^\dagger) = -2(\chi^\dagger\cdot\psi^\dagger)(\eta\cdot\lambda) \tag{18.11}$$

$$\psi^{\dot\alpha}\chi^\alpha = -\frac{1}{2}\chi^\beta\sigma_{\mu\beta\dot\beta}\psi^{\dagger\dot\beta}\bar\sigma^{\mu\dot\alpha\alpha} \tag{18.12}$$

19 Grassmann algebra & calculus

A generic Grassmann variable η satisfies the relation $\eta^2 = 0$ and thus a Taylor expansion of a function of such a variable truncates at the linear term

$$f(\eta) = a + b\eta \tag{19.1}$$

Derivatives act the usual way on such functions:

$$\frac{\partial}{\partial\eta} f(\eta) = \frac{\partial}{\partial\eta}(a + b\eta) = b \tag{19.2}$$

Integrals are a bit weirder:

$$\int \mathrm{d}\eta = 0 \qquad\qquad \int \mathrm{d}\eta\eta = 1 \tag{19.3}$$

and thus we see

$$\int \mathrm{d}\eta f(\eta) = \int \mathrm{d}\eta(a + b\eta) = a\int \mathrm{d}\eta + b\int \mathrm{d}\eta\eta = b \tag{19.4}$$

Now generalize these expressions to deal with spinor coordinates. Derivatives become

$$\frac{\partial}{\partial\theta^\alpha}(\theta^\beta) = \delta_\alpha^\beta \qquad \frac{\partial}{\partial\theta^\alpha}(\theta^\dagger_{\dot\beta}) = 0 \qquad \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}}(\theta^\beta) = 0 \qquad \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}}(\theta^\dagger_{\dot\beta}) = \delta^{\dot\alpha}_{\dot\beta} \tag{19.5}$$

which give (for example)

$$\frac{\partial}{\partial\theta^\alpha}(\psi\theta) = \frac{\partial}{\partial\theta^\alpha}(\psi^\beta\theta_\beta) = \frac{\partial}{\partial\theta^\alpha}(\theta^\beta\psi_\beta) = \delta_\alpha^\beta\psi_\beta = \psi_\alpha \tag{19.6}$$

or

$$\frac{\partial}{\partial\theta_\alpha}(\psi\theta) = \frac{\partial}{\partial\theta_\alpha}(\psi^\beta\theta_\beta) = -\psi^\beta\frac{\partial}{\partial\theta_\alpha}\theta_\beta = -\delta_\beta^\alpha\psi^\beta = -\psi^\alpha \tag{19.7}$$

Note that the cost of lowering the spinor index in the derivative is a minus sign in the final product. One final example is

$$\partial_\alpha(\theta\cdot\theta) = \partial_\alpha(\theta^\beta\theta_\beta) = \partial_\alpha(\theta^\beta\epsilon_{\beta\rho}\theta^\rho) = \delta_\alpha^\beta\epsilon_{\beta\rho}\theta^\rho - \theta^\beta\epsilon_{\beta\rho}\delta_\alpha^\rho = \epsilon_{\alpha\rho}\theta^\rho - \epsilon_{\beta\alpha}\theta^\beta = \theta_\alpha + \epsilon_{\alpha\beta}\theta^\beta = 2\theta_\alpha \tag{19.8}$$

and, as suspected

$$\partial^\alpha(\theta\cdot\theta) = -2\theta^\alpha \tag{19.9}$$

To define the corresponding integrals, we define the integration measures as

$$\mathrm{d}^2\theta = -\frac{1}{4}\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta\epsilon_{\alpha\beta} \qquad\qquad \mathrm{d}^2\theta^\dagger = -\frac{1}{4}\mathrm{d}\theta^\dagger_{\dot\alpha}\mathrm{d}\theta^\dagger_{\dot\beta}\epsilon^{\dot\alpha\dot\beta} \tag{19.10}$$

Thus the integrals are

$$\int \mathrm{d}^2\theta\theta\cdot\theta = 1 \qquad\qquad \int \mathrm{d}^2\theta^\dagger\theta^\dagger\cdot\theta^\dagger = 1 \tag{19.11}$$

It is important to note that, since superspace coordinates have mass dimension $[\theta] = -\frac{1}{2}$, that the measures also carry mass dimension $[\mathrm{d}^2\theta] = 1$. Lastly, integrals of total derivatives w.r.t. the fermionic coordinates vanish identically

$$\int \mathrm{d}^2\theta\frac{\partial}{\partial\theta^\alpha}(\text{anything}) = 0 \qquad\qquad \int \mathrm{d}^2\theta^\dagger\theta\frac{\partial}{\partial\theta^\dagger_{\dot\alpha}}(\text{anything}) = 0 \tag{19.12}$$

20 Algebra

The Poincaré algebra is generated by the momentum operator $\hat{P}_\mu$ and the 4D angular momentum operators $\hat{M}_{\mu\nu}$, with Lie brackets

$$\left[\hat{P}_\mu, \hat{P}_\nu\right] = 0 \tag{20.1}$$

$$\left[\hat{M}_{\mu\nu}, \hat{P}_\sigma\right] = i\left(g_{\mu\sigma}\hat{P}_\nu - g_{\nu\sigma}P_\mu\right) \tag{20.2}$$

$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = i\left(g_{\mu\rho}\hat{M}_{\nu\sigma} + g_{\nu\sigma}\hat{M}_{\mu\rho} - g_{\mu\sigma}\hat{M}_{\nu\rho} - g_{\nu\rho}\hat{M}_{\mu\sigma}\right) \tag{20.3}$$

where $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. The σ matrices are defined as

$$\sigma^\mu = (\mathbf{1}_{2\times 2}, \vec{\sigma}) \qquad\qquad \bar{\sigma}^\mu = (\mathbf{1}_{2\times 2}, -\vec{\sigma}) \tag{20.4}$$

and satisfy the Clifford algebra

$$\left\{\sigma_\mu, \sigma_\nu\right\} = \left\{\bar{\sigma}_\mu, \bar{\sigma}_\nu\right\} = 0 \qquad\qquad \left\{\sigma_\mu, \bar{\sigma}_\nu\right\} = 2g_{\mu\nu}\mathbf{1} \tag{20.5}$$

Using these matrices, it is possible to define the LH and RH spinor representations of the generators of the Lorentz algebra:

$$\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta = \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right) \qquad\qquad \left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right) \tag{20.6}$$

These relations often prove useful in simplifying expressions involving symmetric/anti-symmetric tensors:

$$\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha{}^\beta = g^{\mu\nu}\delta^\beta_\alpha - 2i(\sigma^{\mu\nu})_\alpha{}^\beta \qquad\qquad \left(\bar{\sigma}^\mu\sigma^\nu\right)^{\dot{\alpha}}{}_{\dot{\beta}} = g^{\mu\nu}\delta^\beta_\alpha - 2i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \tag{20.7}$$

The $\mathcal{N} = 1$ SUSY algebra extends the Poincaré algebra by adding spinor-valued generators $\hat{Q}_\alpha$, $\hat{Q}^\dagger_{\dot{\alpha}}$, along with the additional brackets:

$$\begin{aligned}
\left[\hat{Q}_\alpha, \hat{P}_\mu\right] = \left[\hat{Q}^\dagger_{\dot{\alpha}}, \hat{P}_\mu\right] &= 0 \\
\left[\hat{M}_{\mu,\nu}, \hat{Q}_\alpha\right] &= i\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta\hat{Q}_\beta \\
\left[\hat{M}_{\mu,\nu}, \hat{Q}^{\dagger\dot{\alpha}}\right] &= i\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\hat{Q}^{\dagger\dot{\beta}} \\
\left\{\hat{Q}_\alpha, \hat{Q}_\beta\right\} = \left\{\hat{Q}^\dagger_{\dot{\alpha}}, \hat{Q}^\dagger_{\dot{\beta}}\right\} &= 0 \\
\left\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot{\alpha}}\right\} &= 2\sigma^\mu_{\alpha\dot{\alpha}}\hat{P}_\mu
\end{aligned} \tag{20.8}$$

The explicit representation of the above generators of course depends on the nature of the field on which it is operating. For example, $\hat{M}_{\mu\nu}$ when operating on a scalar field gives

$$\hat{M}_{\mu\nu} \qquad \to \qquad M_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \tag{20.9}$$

The representation of the supercharges when acting on superfields is

$$Q_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu\theta^\dagger)_\alpha\partial_\mu \qquad\qquad Q^{\dagger\dot{\alpha}} = i\frac{\partial}{\partial\theta^\dagger_{\dot{\alpha}}} - \left(\bar{\sigma}^\mu\theta\right)^{\dot{\alpha}}\partial_\mu \tag{20.10}$$

or, alternatively

$$Q^\alpha = -i\frac{\partial}{\partial\theta_\alpha} + (\theta^\dagger\bar{\sigma}^\mu)^\alpha\partial_\mu \qquad\qquad Q^\dagger_{\dot{\alpha}} = i\frac{\partial}{\partial\theta^{\dagger\dot{\alpha}}} + \left(\theta\sigma^\mu\theta\right)_{\dot{\alpha}}\partial_\mu \tag{20.11}$$