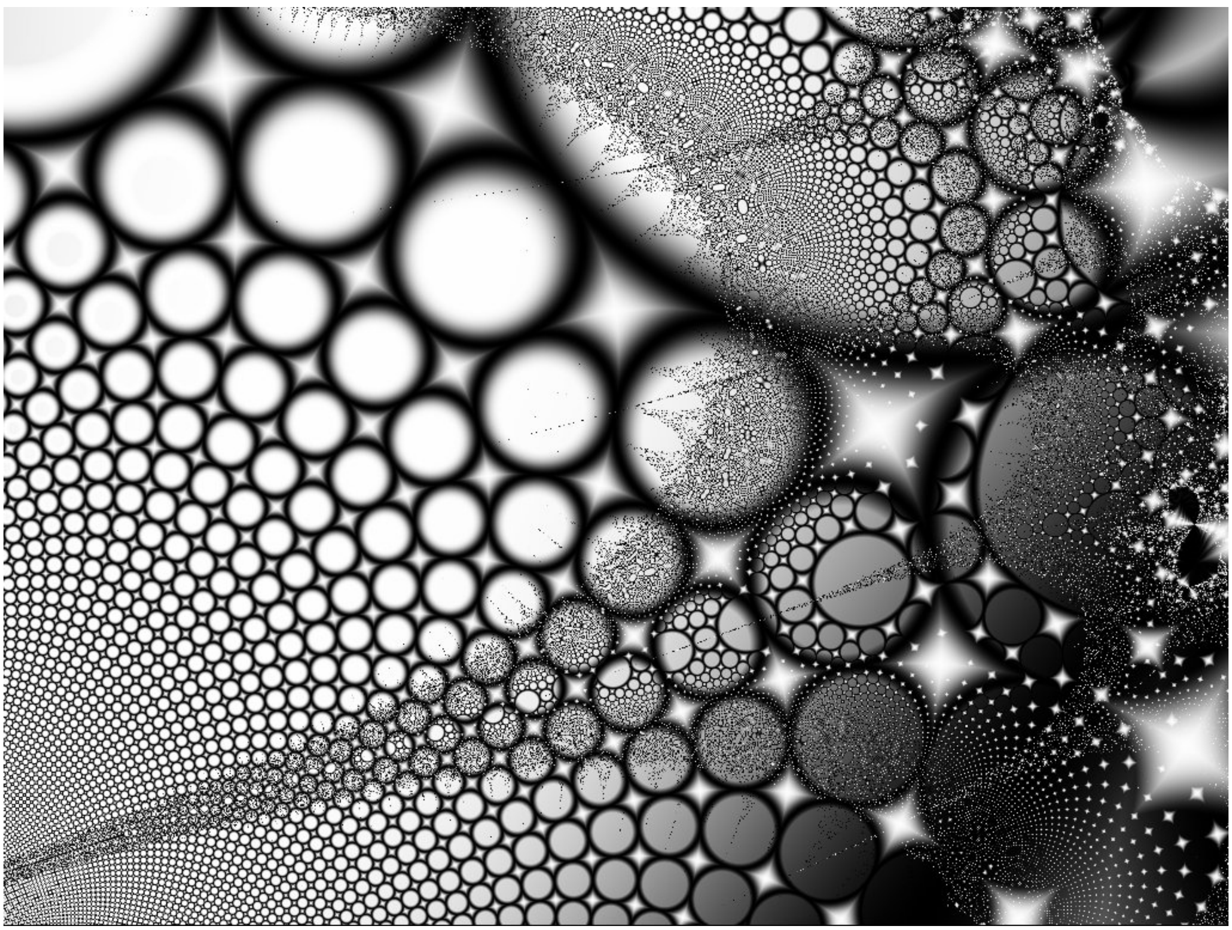
A (Moron's) Guide to Renormalization

Humberto Gilmer
Department of Physics
The Ohio State University



Preface

These notes constitute my attempts at constructing a comprehensive guide to renormalization, in particular its applications to quantum field theory in a particle physics context (thus making me the title's eponymous "moron"). I've not sought to make the notes self-contained; I assume a working knowledge of QFT, including topics such as deriving Feynman rules and diagrams from a Lagrangian, Yang-Mills theory, the Standard Model, and the superfield formalism.

Contents

Part I
Foundations

1 Introduction

1.1 Overview

In the context of particle physics, renormalization is first encountered as an esoteric set of tricks to evaluate loop integrals and to coax finite physical quantities out of the infinities these produce. Without further probing, the theory seems altogether magical and befuddling. In these notes, I attempt to show a logical development of the theory of renormalization, attempting to maintain as much contact with the physical principles involved as possible. In this first part, I will employ the ϕ^4 theory as toy model, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{1.1}$$

Upon establishing, in this section, that renormalization is an inextricable part of quantum field theory, I will then proceed to illustrate Ken Wilson's revolutionary approach to renormalization, elevating renormalization from its previous position of magical toolkit to indispensable part of physics. This will involve discussion of the Callan-Symanzik equations to describe renormalization group flows in theory space.

I will then return to QFT specifically, showing how the Wilsonian approach is employed in the context of particle physics (oftentimes called the Gell–Mann-Low or continuum approach), including expanding Lagrangians and couplings using counterterms. This will involve a brief discussion of symmetries and how they help simplify renormalization analysis. Next, I will consider various computational techniques employed in regularizing the loop integrals.

I will then illustrate the different renormalization schemes, ensuring that as much contact as possible is made between this oftentimes obscure mathematical formalism and the intuitive picture of flows in theory space. I will finally make a brief stop in effective field theories, showing how Wilson's picture of renormalization made these indispensable, whereas before they would have been considered useless.

In part II, I will go through explicit renormalization analyses for four different QFTs: QED, sQED, Yukawa theory and Yang-Mills theory. This essentially covers the gamut of theories used in particle physics, but is by no means exhaustive. Each theory requires a set of tricks to make the analysis as simple as possible, and I'll make sure to point them out to make building your toolkit as simple as possible. Real-world analysis will require using these tricks and skills in combination, but I figure using them in a simplified, well-understood context was easiest.

Lastly, in part III, I will tackle advanced topics. These include the quantum effective potential and its application to symmetries. I will also work through the Coleman-Weinberg potential, as an explicit example of quantum mechanically-induced spontaneous symmetry breaking. Another topic included is that of anomalies. Although not explicitly related to renormalization theory, many of the techniques used in analyzing anomalies are identical to those used in renormalization. Lastly, I will tackle supersymmetry, namely the theory of soft SUSY breaking and the non-renormalization theorems that make SUSY theories so attractive as theories for BSM physics.

1.2 Loop integrals

As mentioned above, the Lagrangian for this part of the notes will be the ϕ^4 Lagrangian. The Feynman rules for this Lagrangian are:

$$= \frac{i}{p^2 - m^2 + i\epsilon} \tag{1.2}$$

$$= -i\lambda \tag{1.3}$$

Remember that these are the Feynman rules in momentum space. Technically speaking, each vertex above has a momentum-conserving δ-function that for convenience is not explicitly written. At tree level, this makes things very simple and finite, since all momenta are thus "pinned down" by momentum conservation. At loop level, however, this implies that at least one momentum will be indeterminate by momentum conservation.

Since we want Lorentz invariance built into every aspect of the theory, this requires us to integrate over all such momenta (since failing to do so would 'privilege' a given segment of phase space, which would not, in general, be Lorentz invariant). Pay close attention: arguably speaking, this is where our troubles begin. Consider the one-loop corrections to propagator and interaction vertices:

The Feynman amplitude for these diagrams are

$$i\mathcal{M} = \frac{i}{p^2 - m^2}(i\lambda)i\Sigma(i\lambda)\frac{i}{p^2 - m^2} \tag{1.4}$$

and

$$i\mathcal{M} = (-i\lambda)^2 i\Pi(p_1, p_2) \tag{1.5}$$

where Σ is the integral

$$i\Sigma = \int \frac{i}{k^2 - m^2}\frac{\mathrm{d}^4k}{(2\pi)^4} \tag{1.6}$$

while $\Pi(p_1, p_2)$ is the integral

$$i\Pi(p_1, p_2) = \int \frac{i}{k^2 - m^2}\frac{i}{(k + p_1 + p_2)^2 - m^2}\frac{\mathrm{d}^4k}{(2\pi)^4} \tag{1.7}$$

Both of these integrals exhibit nasty divergences. These can be seen explicitly as follows. Suppose $p_1, p_2 \ll k$; consider Π in this case, and evaluate the integral using a cutoff Λ that will be taken to infinity (to satisfy Lorentz invariance):

$$\Pi(p_1, p_2) = -i\int \frac{1}{k^2 - m^2}\frac{1}{(k + p_1 + p_2)^2 - m^2}\frac{\mathrm{d}^4k}{(2\pi)^4} \sim -i\int_0^\Lambda \frac{1}{k^4}k^3\mathrm{d}k\int \mathrm{d}\Omega_4 \sim -i\pi^2\log\Lambda \tag{1.8}$$

Well, that's no good. Clearly there is a problem here. Our formalism for constructing Feynman amplitudes perturbatively has yielded a divergence. Notice that the problem arises when Λ becomes very large. This implies that so long as only low-momentum values of k are considered, everything is fine and no problem arises. But of course, we don't have the luxury of doing that, lest we sacrifice Lorentz invariance. Now what do we do?

1.3 The need for renormalization

In actuality, although the integral above is divergent, this is not in reality a problem, because we have made a (rather enormous assumption). Recall that the loop integral had to be done over all momenta to achieve Lorentz invariance. In doing this, however, we have assumed that the Lagrangian that yielded the Feynman rules used to construct this loop integral is an accurate description of physics at all momentum scales. This is quite a claim! And of course, the presence of the infinity indicates that we are amiss in doing so. Are we then to throw out the Lagrangian (1.1)? No, but we must be more careful in defining the domain of applicability of this theory; furthermore, we must find a way to account for the fact that our theory is an inadequate description of physics beyond some energy scale. Thus, in particle physics there is a fundamental tension between two core principles: (1) Lorentz invariance, requiring integration over all loop momenta and (2) describing the physics accurately, using a theory only within a momentum range where it does so.

2 The renormalization group

2.1 Wilsonian renormalization

As mentioned at the end of section 1.2, the problem with the integrals arises when loop momenta become too large and thus out of the descriptive range of our Lagrangian. What does this mean concretely and how can we account for these high-momentum modes? The following description is thanks to Ken Wilson. Prior to Wilson, all theories were thought or hoped to be 'fundamental', meaning they were adequate descriptions of Nature in all contexts. The presence of renormalization (including the pesky cutoff Λ) was thought to be a mathematical trick necessary to coax results out of the theory. Wilson's revolutionary insight was to realize that the cutoff Λ, far from being some sort of crutch or impediment to analysis, was rather a defining aspect of any theory. In other words, all theories come "equipped" with some sort of Λ, beyond which they are inadequate at describing the physics. The practical effect this had was to transform any theory from a fundamental theory to an effective field theory. Different effective field theories would describe different physics in different ranges of momentum space.

> This may seem like gobbledygook so here's a simple example. Newtonian mechanics (including the expression $E = \frac{p^2}{2m}$) is an adequate description of physical phenomena so long as the momenta considered are in the range $p \ll mc$. If you attempted to compute, say, the trajectory of a ball whizzing by with speed $v \sim c$ using Newtonian mechanics, you'd get nonsensical results. This is not because Newtonian mechanics is categorically wrong, but rather because you attempted to employ a physical theory *outside its domain of description*. Physics is more accurately governed by special relativity. When $p \sim mc$, it becomes necessary to use relativistic mechanics to describe physical phenomena. In other words, $\Lambda = mc$ constitutes the cutoff for Newtonian mechanics, beneath which it serves as an effective theory and beyond which we must switch to another effective theory that captures the physics. Notice that although special relativity is "more correct", in day-to-day computations it would be a nightmare to use to compute things such as kinematics and thus Newtonian mechanics suffices as an effective description.

Now let's return to our ϕ^4 example. What practical implications does the cutoff Λ have? Recall that any QFT can be defined in terms of its generating functional $Z[J]$:

$$Z[J] = \int e^{i\int \mathcal{L}+J\phi \mathrm{d}^4x}\mathcal{D}\phi \tag{2.1}$$

Of course, now we need to take the cutoff into account. Let's do a few things to make this simpler. First, let's work with $Z[J]$ in momentum space. We do this because that allows us to explicitly make ϕ vanish when it is evaluated at a momentum beyond Λ:

$$\phi(k) = \begin{cases} \phi & |k| \leq \Lambda \\ 0 & |k| > \Lambda \end{cases} \tag{2.2}$$

Furthermore, let's Euclideanize the theory (meaning $k_0 \to -ik_0$); remember that in Minkowski space a null vector $k = 0$ does not imply that all its components are equal to 0, only that for any space-like component, there is a corresponding time-like component to cancel it. This includes very large space-like momentum components. Since we're trying to avoid large momenta (whether space-like or time-like), it would behoove use to employ a metric where such cancellations are simply not possible. These assumptions have the effect of changing the path integral metric:

$$[\mathcal{D}\phi]_\Lambda = \Pi_{|k|<\Lambda}\mathrm{d}\phi(k) \tag{2.3}$$

Thus the actual generating functional, incorporating the cutoff is

$$Z[J] = \int e^{i\int \mathcal{L}+J\phi \mathrm{d}^4x}[\mathcal{D}\phi]_\Lambda \tag{2.4}$$

Now, let's recall what our goal is. We would like to parameterize the effect of high-momentum loop modes on our low energy theory. Once we've taken these effects into account, we can then proceed with the analysis using our low energy effective description at all energies, without fear that we're 'missing' some of the physics. How do we go about parameterizing these high energy modes? We will do so by integrating over a very thin 'shell' in momentum space. First split the domain of integration in momentum space into the 'bulk' and the aforementioned shell. To do so, let b be a real number in the range $0 < b < 1$. Consider the ranges $0 < |k| < b\Lambda$ (the bulk), $b\Lambda < |k| < \Lambda$ (the shell) and $|k| > \Lambda$ (the ultraviolet). We've already taken care of the ultraviolet by making $\phi = 0$ in this range. If we take $b \sim 1$, then the shell becomes extremely thin, as desired. Now, split ϕ into two fields, $\phi = \varphi + \hat{\phi}$, such that

$$\varphi = \begin{cases} \phi & 0 < |k| < b\Lambda \\ 0 & b\Lambda < |k| \leq \Lambda \end{cases} \qquad \hat{\phi} = \begin{cases} 0 & 0 < |k| < b\Lambda \\ \phi & b\Lambda < |k| \leq \Lambda \end{cases} \tag{2.5}$$

As such, $\hat{\phi}$ constitute the high-momentum modes we'll be integrating out (or getting rid of) to find their effect on the low energy theory written in terms of φ. Plugging this into the Lagrangian, expanding (using the fact that terms of the form $\varphi\hat{\phi}$ vanish identically, since they constitute orthogonal Fourier modes), and then collecting all terms independent of $\hat{\phi}$ (which constitutes our low-energy effective theory) into $\mathcal{L}[\varphi]$

gives

$$
\begin{aligned}
Z &= \int\int \exp\left(-\int\left[\frac{1}{2}\left(\partial_\mu\varphi+\partial_\mu\hat{\phi}\right)\left(\partial^\mu\varphi+\partial^\mu\hat{\phi}\right)+\frac{m^2}{2}\left(\varphi+\hat{\phi}\right)^2+\frac{\lambda}{4!}\left(\varphi+\hat{\phi}\right)^4\right]\mathrm{d}^Dx\right)\mathcal{D}\varphi\mathcal{D}\hat{\phi} \\
&= \int\int \exp\left(-\int\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi+\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi}+\frac{m^2}{2}\left(\varphi^2+\hat{\phi}^2\right)\right.\right. \\
&\qquad\qquad\left.\left.+\frac{\lambda}{4!}\left(\varphi^4+3\varphi^3\hat{\phi}+6\varphi^2\hat{\phi}^2+3\varphi\hat{\phi}^3+\hat{\phi}^4\right)\right]\mathrm{d}^Dx\right)\mathcal{D}\varphi\mathcal{D}\hat{\phi} \\
&= \int \exp\left(-\int\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi+\frac{m^2}{2}\varphi^2+\frac{\lambda}{4!}\varphi^4\right]\mathrm{d}^Dx\right) \\
&\qquad \times\int \exp\left(-\int\left[\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi}+\frac{m^2}{2}\hat{\phi}^2+\lambda\left(\frac{1}{6}\varphi^3\hat{\phi}+\frac{1}{4}\varphi^2\hat{\phi}^2+\frac{1}{6}\varphi\hat{\phi}^3+\frac{1}{4!}\hat{\phi}^4\right)\right]\mathrm{d}^Dx\right)\mathcal{D}\hat{\phi}\mathcal{D}\varphi \\
&= \int e^{\left(-\int\mathcal{L}[\varphi]\mathrm{d}^Dx\right)}\int \exp\left(-\int\left[\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi}+\frac{m^2}{2}\hat{\phi}^2+\lambda\left(\frac{1}{6}\varphi^3\hat{\phi}+\frac{1}{4}\varphi^2\hat{\phi}^2+\frac{1}{6}\varphi\hat{\phi}^3+\frac{1}{4!}\hat{\phi}^4\right)\right]\mathrm{d}^Dx\right)\mathcal{D}\hat{\phi}\mathcal{D}\varphi
\end{aligned}
$$

Let's take stock of what's happened. We have split the path integral into domains, where the high-momentum modes $\hat{\phi}$ have been split off from the low-momentum effective theory. Notice that the result is new interactions amongst the low-momentum modes that were not immediately obvious in the original Lagrangian. These new interactions constitute the effects of these high-momentum modes. Now, technically, all these wonderful new interactions are all wrapped up in the exponential. It is of course possible to Taylor expand, but let's do so intelligently. Recall that for a free massless scalar field theory, the 2-point Green function is given by

$$
\left\langle\hat{\phi}(k)\hat{\phi}(p)\right\rangle=\langle 0|\hat{\phi}(k)\hat{\phi}(p)|0\rangle=\frac{\int\exp\left(-\frac{1}{2}\int\partial_\mu\phi\partial^\mu\phi\mathrm{d}^4x\right)\phi(k)\phi(p)\mathcal{D}\phi}{\int\exp\left(-\frac{1}{2}\int\partial_\mu\phi\partial^\mu\phi\mathrm{d}^4x\right)\mathcal{D}\phi}=\frac{1}{k^2}\delta^{(D)}(k+p) \tag{2.6}
$$

Note that the $\hat{\phi}$ in the expectation values do not correspond to $\hat{\phi}$ as defined above; these simply indicate that the fields in the expectation value are quantum-mechanical rather than the classical objects present in the path integrals. Now let's examine our generating functional Z a little more closely and notice that it contains a "free" and "massless" scalar field $\hat{\phi}$, so long as we're concerned about $\hat{\phi}$ in the domain where $m \ll \Lambda$. In other words, let's reorganize Z so that it generates the dynamics and interactions of the $\hat{\phi}$ and φ fields, which can be done by treating all the m^2 and λ terms as perturbations. Thus the exponentials can be Taylor expanded, yielding a series of interaction terms (multiple integral signs have been dropped for compactness):

$$
\begin{aligned}
Z &= \int e^{\left(-\int \mathcal{L}[\varphi]\mathrm{d}^D x\right)} \int \exp\left(-\int\left[\frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + \frac{m^2}{2}\hat{\phi}^2 + \lambda\left(\frac{1}{6}\varphi^3\hat{\phi} + \frac{1}{4}\varphi^2\hat{\phi}^2 + \frac{1}{6}\varphi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\mathrm{d}^D x\right)\mathcal{D}\hat{\phi}\mathcal{D}\varphi \\
&= \int\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)} e^{\left(-\frac{\lambda}{4!}\int \varphi^4 \mathrm{d}^D x\right)} \\
&\qquad\qquad \times \exp\left(-\int\left[\frac{m^2}{2}\hat{\phi}^2 + \lambda\left(\frac{1}{6}\varphi^3\hat{\phi} + \frac{1}{4}\varphi^2\hat{\phi}^2 + \frac{1}{6}\varphi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\mathrm{d}^D x\right)\mathcal{D}\hat{\phi}\mathcal{D}\varphi \\
&= \int\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)} \left(1 - \frac{\lambda}{4!}\int \varphi^4 \mathrm{d}^D x + \dots\right) \\
&\quad\; \times \left(1 - \frac{m^2}{2}\int \hat{\phi}^2 \mathrm{d}^D x + \dots\right)\left(1 - \lambda\int\left(\frac{1}{6}\varphi^3\hat{\phi} + \frac{1}{4}\varphi^2\hat{\phi}^2 + \frac{1}{6}\varphi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\mathrm{d}^D x + \dots\right)\mathcal{D}\hat{\phi}\mathcal{D}\varphi \\
&= \int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)}\mathcal{D}\hat{\phi}\mathcal{D}\varphi - \frac{\lambda}{4!}\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)}\varphi^4\mathcal{D}\hat{\phi}\mathcal{D}\varphi\mathrm{d}^D x \\
&\quad - \frac{m^2}{2}\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)}\hat{\phi}^2\mathcal{D}\hat{\phi}\mathcal{D}\varphi\mathrm{d}^D x - \frac{\lambda}{4}\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)}\varphi^2\hat{\phi}^2\mathcal{D}\hat{\phi}\mathcal{D}\varphi\mathrm{d}^D x + \dots \\
&\quad - \left(\frac{\lambda}{4}\right)^2\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)}\left(\varphi^2\hat{\phi}^2\right)\left(\varphi^2\hat{\phi}^2\right)\mathcal{D}\hat{\phi}\mathcal{D}\varphi\mathrm{d}^D x\mathrm{d}^D y + \dots
\end{aligned}
\tag{2.7}
$$

As stated earlier, the first term of the fourth line of Eq. (2.7) corresponds to the generating functional for a "free", "massless" scalar field $\hat{\phi}$, which has a propagator:

$$
\overbracket{\hat{\phi}(k)\hat{\phi}(p)} = \frac{\int e^{-\int \mathcal{L}_0[\hat{\phi}]}\hat{\phi}(k)\hat{\phi}(p)\mathcal{D}\hat{\phi}}{\int e^{-\int \mathcal{L}_0[\hat{\phi}]}\mathcal{D}\hat{\phi}}
$$

This can be evaluated explicitly to give the propagator. To do so, consider the action integral and Fourier transform the fields (remembering that the fields are only defined for momenta inside our thin "shell"):

$$
\begin{aligned}
\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x &= \frac{1}{2}\int \partial_\mu\hat{\phi}\partial^\mu\hat{\phi}\mathrm{d}^D x \\
&= \frac{1}{2}\int\int p_\mu\hat{\phi}(p)e^{ip\cdot x}\frac{\mathrm{d}^D p}{(2\pi)^D}\int k^\mu e^{ik\cdot x}\hat{\phi}(k)\frac{\mathrm{d}^D k}{(2\pi)^D}\mathrm{d}^D x \\
&= \frac{1}{2}\int\int\left(\int e^{i(p+k)\cdot x}\mathrm{d}^D x\right)p\cdot k\hat{\phi}(p)\hat{\phi}(k)\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= \frac{1}{2}\int\int\left((2\pi)^D\delta^{(D)}(p+k)\right)p\cdot k\hat{\phi}(p)\hat{\phi}(k)\frac{\mathrm{d}^D p}{(2\pi)^D}\frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= \frac{1}{2}\int_{b\Lambda<|k|<\Lambda} k^2\hat{\phi}(k)\hat{\phi}(k)\frac{\mathrm{d}^D k}{(2\pi)^D}
\end{aligned}
$$

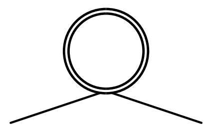
where, in the last step, I made the integration bound explicit. Thus the propagator (and its associated Feynman diagram, a double line) is

$$
\overbracket{\hat{\phi}(k)\hat{\phi}(p)} = \frac{1}{k^2}(2\pi)^D\delta^{(D)}(p+k)\Theta(k) =
$$

where the $\Theta(k)$ enforces the existence condition of the $\hat{\phi}$ fields in momentum space. Now let's return to Eq. (2.7). The propagator we just derived allows us to perturbatively evaluate the terms in the expansion. For example, the second term in the second line of Eq. (2.7) yields:

$$-\frac{\lambda}{4}\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)} \varphi^2 \hat{\phi}^2 \mathcal{D}\hat{\phi}\mathcal{D}\varphi \mathrm{d}^D x = -\frac{\lambda}{4}\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} \overbracket{\hat{\phi}\hat{\phi}}\varphi^2 \mathcal{D}\varphi \mathrm{d}^D x$$

This corresponds diagrammatically to a plain φ line with a closed loop of $\hat{\phi}$:



Remember that other, disconnected diagrams are also possible, but we end up getting rid of these by dividing out by $Z[0]$. The numeric contribution to $Z[J]$ is given by (noting the factor of two to compensate for the multiplicity of the diagram)

$$\begin{aligned}
-\frac{\lambda}{4}\int \overbracket{\hat{\phi}\hat{\phi}}\mathrm{d}^D x &= -(2)\frac{1}{2}\frac{\lambda}{4}\int\int_{b\Lambda<|k|<\Lambda} \frac{1}{k^2}\varphi^2 \frac{\mathrm{d}^D k}{(2\pi)^D}\mathrm{d}^D x \\
&= -\frac{\lambda}{4}\int\int_{b\Lambda<|k|<\Lambda} \frac{1}{k^2}\int \varphi(k_1) e^{ik_1\cdot x}\frac{\mathrm{d}^D k_1}{(2\pi)^D}\int \varphi(k_2) e^{ik_2\cdot x}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\frac{\mathrm{d}^D k}{(2\pi)^D}\mathrm{d}^D x \\
&= -\frac{\lambda}{4}\int_{b\Lambda<|k|<\Lambda}\int\int \frac{1}{k^2}\varphi(k_1)\varphi(k_2)\int e^{i(k_1+k_2)\cdot x}\mathrm{d}^D x \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= -\frac{\lambda}{4}\int_{b\Lambda<|k|<\Lambda}\int\int \frac{1}{k^2}\varphi(k_1)\varphi(k_2)(2\pi)^D\delta^{(D)}(k_1+k_2)\frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= -\frac{\lambda}{4}\int_{b\Lambda<|k|<\Lambda}\int \frac{1}{k^2}\varphi(k_1)\varphi(-k_1)\frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k}{(2\pi)^D} \\
&= -\frac{\lambda}{2}\int_{b\Lambda<|k|<\Lambda} \mu\varphi(k_1)\varphi(-k_1)\frac{\mathrm{d}^D k_1}{(2\pi)^D}
\end{aligned}$$

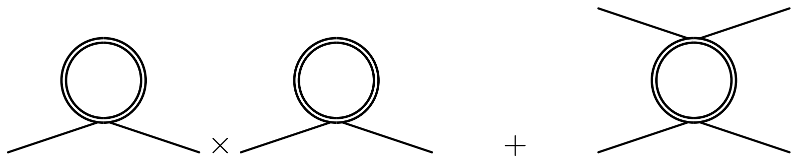
where μ is the contribution from the high-momentum modes running in the loop:

$$\mu = \frac{\lambda}{2}\int_{b\Lambda<|k|<\Lambda}\frac{1}{k^2}\frac{\mathrm{d}^D k_1}{(2\pi)^D} = \frac{\lambda}{(4\pi)^{\frac{D}{2}}\Gamma(\frac{D}{2})}\frac{1-b^{D-2}}{D-2}\Lambda^{D-2}$$

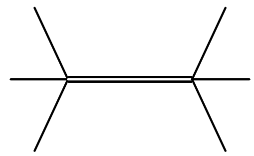
Thus, at order λ, we see that the divergent diagram has been split in such a way that the divergence is explicitly coming from the high momentum modes. Let's repeat this exercise at order λ^2 (showing only a couple of the possible contractions, for brevity):

$$\begin{aligned}
&-\left(\frac{\lambda}{4}\right)^2\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)} e^{\left(-\int \mathcal{L}_0[\hat{\phi}]\mathrm{d}^D x\right)}\left(\varphi^2\hat{\phi}^2\right)_x\left(\varphi^2\hat{\phi}^2\right)_y \mathcal{D}\hat{\phi}\mathcal{D}\varphi\mathrm{d}^D x\mathrm{d}^D y \\
&= -\left(\frac{\lambda}{4}\right)^2\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)}(\varphi^2\overbracket{\hat{\phi}\hat{\phi}})_x(\varphi^2\overbracket{\hat{\phi}\hat{\phi}})_y\mathcal{D}\varphi\mathrm{d}^D x\mathrm{d}^D y \\
&\qquad -\left(\frac{\lambda}{4}\right)^2\int e^{\left(-\int \mathcal{L}_0[\varphi]\mathrm{d}^D x\right)}(\varphi^2\hat{\phi}\hat{\phi})_x(\varphi^2\hat{\phi}\hat{\phi})_y\mathcal{D}\varphi\mathrm{d}^D x\mathrm{d}^D y
\end{aligned}$$

This corresponds diagrammatically to



Finally, there are terms of the form $\varphi^3\hat{\phi}$ at order λ. These correspond to completely new interactions between the φ, mediated by $\hat{\phi}$. This interaction could result in a diagram of the form



This diagram looks like a φ^6 interaction at low energy. This will correspond to what we call a non-renormalizable interaction; its appearance is mildly disturbing, given that we didn't write it down in our original Lagrangian. But we thus see how these high-momentum modes modify the low-energy Lagrangian in a systematic way, by providing corrections to the existing interactions and providing new ways for the low-momentum fields to interact. The Lagrangian thus goes from its original ϕ^4 form to

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + (\text{sum of connected diagrams})$$

where the "sum of connected diagrams" includes the loop corrections and the new interactions mediated by the high-momentum modes.

2.2 Theory space flows

We've thus seen the effects of integrating out a "shell" in momentum space on the low-momentum Lagrangian. Let's analyze the effects on the Lagrangian explicitly. The effective Lagrangian includes the original Lagrangian plus corrections to the original terms (listed as ΔZ, Δm^2, and $\Delta\lambda$) and all possible new interactions (that still respect the symmetries of the original):

$$\int \mathcal{L}_{\text{eff}}\mathrm{d}^D x = \int \left[\frac{1}{2}(1+\Delta Z)\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C(\partial\phi)^4 + \Delta D\phi^6 + \dots\right]\mathrm{d}^D x \tag{2.8}$$

Note that the kinetic, mass and interaction term all have their own corrections. A priori, we cannot assume that the kinetic term would scale the same as the mass term (they might still, but we want to leave things as general as possible). As we scale the momentum down, $k' = \frac{k}{b}$, which corresponds to a scaling *up* in real space, $x' = bx$. Inserting this scaling into the Lagrangian yields

$$\begin{aligned}
\int \mathcal{L}_{\text{eff}}\mathrm{d}^D x &= \int \left[\frac{1}{2}(1+\Delta Z)\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C(\partial\phi)^4 + \Delta D\phi^6 + \dots\right]\mathrm{d}^D x \\
&= \int \left[\frac{1}{2}(1+\Delta Z)b^2\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(m^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C b^4(\partial'\phi)^4 + \Delta D\phi^6 + \dots\right]b^{-D}\mathrm{d}^D x' \\
&= \int \left[\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi'^4 + C'(\partial'\phi')^4 + D'\phi'^6 + \dots\right]\mathrm{d}^D x'
\end{aligned}$$

Comparing the last and second-to-last lines, we see that

$$
\begin{aligned}
\phi' &= [b^{2-D}(1+\Delta Z)]^{\frac{1}{2}}\phi \\
m'^2 &= (m^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2} \\
\lambda' &= (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{D-4} \\
C' &= (C+\Delta C)(1+\Delta Z)^{-4}b^{D} \\
D' &= (D+\Delta D)(1+\Delta Z)^{-6}b^{2D-6}
\end{aligned}
\tag{2.9}
$$

There will be one such recursive relation for each coefficient of each term in the effective Lagrangian. Note that in the original Lagrangian, $C = D = 0$, but I included them here to keep things general. In effect, the various sets of values of m^2, λ, etc. constitute different "theories." Thus we see that integrating out a slice in momentum space has the effect of scaling the parameters of the Lagrangian, or, alternatively, moving to a new theory. This is a phenomenon known as the **running** of the couplings. We could concatenate two (or more!) such scalings, which would itself be a scaling, which looks an awful lot like the closure property of a group. Thus we see the genesis of the name **renormalization group**. Integrating out slices in momentum space corresponds very naturally to moving through theory space. Note also that the parameters no longer scale with their classical scaling dimension. This "anomalous scaling" is characteristic of renormalization procedures and will arise again in later sections.

Remember that the original ϕ^4 Lagrangian has a cutoff Λ, and computing Green functions perturbatively always has the risk of yielding divergent quantities when loops are involved. The effective Lagrangian will have a cutoff $\Lambda' = b\Lambda$, achieved by integrating out the high momentum modes. If we repeat the scalings down to the scale of the external momenta in the interactions so that $b\Lambda \sim p_i$, we'll have an effective Lagrangian that can be used to compute Green functions, without the risk of divergences appearing unexpectedly, since these have already been taken care of by rescaling the parameters. In the following section, we'll explore how to make this evolution from a high-cutoff theory to a low-cutoff theory concrete, but for now, let's explore qualitative behaviors.

Consider the free, massless theory whereby $m^2 = \lambda = C = D = \cdots = 0$. Thus the Lagrangian is simply $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. Since evolution of the parameters involves their initial values, we can see by Eq. (2.8) that the parameters remain at their initial values since $m^2 = 0 \Rightarrow \Delta m^2 = 0 \Rightarrow m'^2 = 0$. The field ϕ' itself may be rescaled, but no new interactions will arise (in other words, we won't move to a new spot in theory space). This special point in theory space is known as the **Gaussian fixed point**. Now, let's suppose we "turn on" the interactions by making m^2 and λ (very) small. If the interactions are very small, that will make the changes Δm^2 and $\Delta\lambda$ even smaller, which may thus be neglected, simplifying Eq. (2.8) to give

$$
m'^2 = m^2 b^{-2} \qquad \lambda' = \lambda b^{D-4} \qquad C' = Cb^{D} \qquad D' = Db^{2D-6} \qquad \ldots \tag{2.10}
$$

Let's specify to $D = 4$. Remember that $b < 1$, so we see that m'^2 grows by $\frac{1}{b^2}$; λ does not apparently scale. Lastly, C and D are multiplied by positive powers of b and thus these scale down. The three classes of scaling correspond to **relevant**, **marginal** and **irrelevant** interactions. Marginal interactions will sort themselves into **marginally relevant** and **marginally irrelevant** interactions depending on higher-order corrections. Figure 2 provides a schematic representation of theory space near the Gaussian fixed point the theory is rescaled toward the IR.

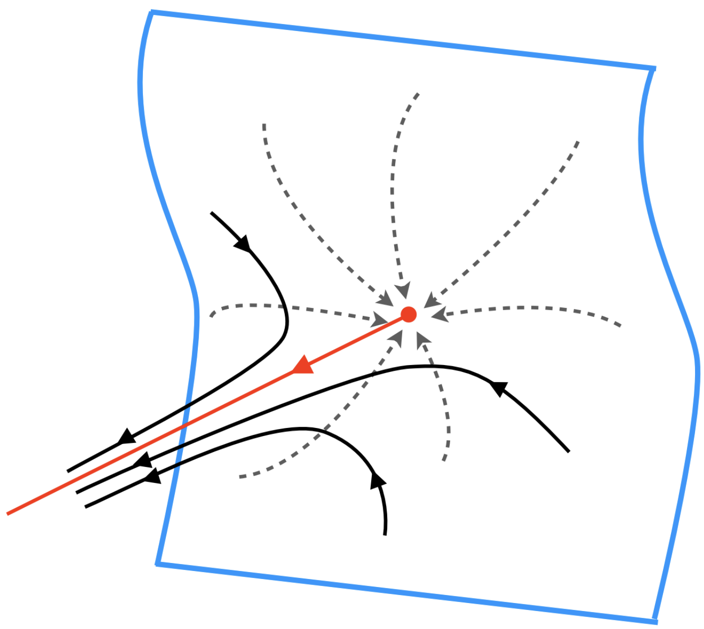


Figure 1: A subset of theory space illustrating the flows generated by the renormalization group.

The dashed lines represent the trajectories of the irrelevant parameters. The red line represents the trajectory of a relevant parameter (it is but one of many possible such **renormalized trajectories**, one for each relevant parameter, but we only show one for the sake of simplicity). The solid black lines are the trajectories of 'real' theories in theory space. Notice that as we proceed into the deep IR, the irrelevant parameters all die away, leaving behind a theory consisting of only relevant and marginal parameters. This has deep implications for developing theories. When constructing Lagrangians at low energy, it is only necessary to include those interactions consistent with the desired symmetries that are relevant or marginal; irrelevant operators can be neglected at low energies safely! Notice also that the trajectories 'focus' onto the renormalized trajectory; this focusing makes physical sense, since multiple high momentum theories are equally indistinguishable at low energies.

2.3 Polchinski renormalization group equation

In the previous section, we explored the effect on the Lagrangian of integrating out a slice of momentum space. Of course, this begs the question as to whether there is a differential or continuous version of this procedure. The answer is yes, and its provided by the Polchinski equation, which will be derived here. Using the same split between φ and $\hat{\phi}$, recall that there is a propagator for $\hat{\phi}$ (in the previous section we ignored the mass, I'm restoring it here). Instead of considering the range to be $k \in [b\Lambda, \Lambda]$, let's instead consider $\Lambda = \Lambda_1 + \delta\Lambda$

$$D_{\hat{\phi}}(x,y) = \int_{\Lambda_1 < |k| \leq \Lambda} \frac{e^{ik\cdot(x-y)}}{k^2+m^2} \frac{\mathrm{d}^D k}{(2\pi)^D} \tag{2.11}$$

This can be evaluated explicitly, assuming that $\delta\Lambda$ is so small that the integrand is essentially constant:

$$D_{\hat{\phi}}(x,y) = \frac{1}{(2\pi)^D} \frac{\Lambda^{D-1}\delta\Lambda}{\Lambda^2+m^2} e^{i\Lambda\hat{k}\cdot(x-y)} \int_{S^{D-1}} \mathrm{d}\Omega \tag{2.12}$$

where the integral over all angles arises due to the 'shell' in momentum space. Now, why does this matter? It matters solely due to that factor of $\delta\Lambda$. Since this factor is infinitesimally small, factors in our perturbative expansion can include at most one copy of $D_{\hat{\phi}}$. This breaks any diagram with n external legs into two classes: (1) those where this single allowed propagator links a given point to itself (with n external legs also attached for a total of $n+2$ couplings) and (2) those where this single propagator links two points, one with r external legs (with $r+1$ couplings) and the other with $n-r$ external legs (with $(n-r)+1$ couplings). In other words, as Λ_1 is varied, the effect on the action S is to introduce new diagrams with the structure described above; in other words:

$$-\Lambda\frac{\partial S[\varphi]}{\partial\Lambda} = \int \left[-D_{\hat{\phi}}(x,y)\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} + \frac{\delta S}{\delta\varphi(x)} D_{\hat{\phi}}(x,y)\frac{\delta S}{\delta\varphi(y)} \right] \mathrm{d}^D x \mathrm{d}^y \tag{2.13}$$

where the integrals are over all possible locations that can be connected by the high-momentum mode. The minus sign arises from the fact that we're expanding $e^{-S[\varphi]}$ when doing perturbation theory. This can be rewritten to yield some rather nice physics. Let's define a renormalization group 'time' by $t = \log\Lambda$. Therefore $\mathrm{d}t = \frac{1}{\Lambda}\mathrm{d}\Lambda$. The RHS of Eq. (2.12) can be rewritten as a second functional derivative of the $e^{-S[\varphi]}$ factor. Thus the Polchinski equation is

$$\frac{\partial}{\partial t} e^{-S[\varphi]} = -\int D_{\hat{\phi}}(x,y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} e^{-S[\varphi]} \mathrm{d}^D x \mathrm{d}^D y \tag{2.14}$$

If we define a 'Laplacian' on the space of fields as

$$\nabla^2 = \int D_{\hat{\phi}}(x,y)\frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} \mathrm{d}^D x \mathrm{d}^D y$$

then the Polchinski equation looks suspiciously like the heat equation:

$$\frac{\partial}{\partial t} e^{-S[\varphi]} = -\nabla^2 e^{-S[\varphi]} \tag{2.15}$$

Let's briefly recall the mathematics of the heat equation. A general solution $f(x,t)$ can be expanded as a series in eigenfunctions $u_k(x)$ such that

$$f(x,t) = \sum_k \tilde{f}_k(t) u_k(x) \qquad \text{where} \qquad \tilde{f}_k(t) = \tilde{f}_k(0) e^{-\lambda_k t}$$

As $t \to \infty$, those terms with $\lambda_k > 0$ will die away, leaving only those terms with $\lambda_k = 0$ (or, in the case of a pseudo-Riemannian manifold, those terms with $\lambda_k < 0$, which blow up).

What can this tell us about the renormalization group? First, it's nice to see a concrete mathematical expression of the behavior illustrated in Figure 2. Second, there exist eigen-operators of the functional Laplacian defined in Eq. (2.14); those with positive eigenvalue correspond to our relevant operators, whilst those with negative eigenvalue to our irrelevant operators.

2.4 Local potential approximation

This section lies outside the normal development and may be omitted on first reading
It turns out solving the Polchinski equation is an unmitigated nightmare (as you might guess), so various approximations have been developed to make it possible to extract useful information about the renormalization group flows from it. One such approximation is known as the local potential approximation

3 Renormalized perturbation theory

The Wilsonian renormalization group provided us with some wonderful intuition about flows in theory space as the scale at which the theory is defined is manipulated. Let's recall, however, that renormalization was a relevant question due to the fact that Lorentz invariance (among other symmetries) requires integration over all momenta; in other words, we still need $\Lambda \to \infty$. What can the tools we developed in the previous section tell us about actual calculations involving this limit? First, let's think about what taking this limit corresponds to in theory space. As Λ is taken to infinity, that leaves $b\Lambda$ behind in the IR; that is equivalent to shifting $b\Lambda$ several times into the IR from the UV (think Galilean relativity, except in momentum space). Recall that, regardless of how you view this shift, in the IR the only operators that we need to take into account are the marginal and relevant operators; irrelevant operators represent UV modes that decouple from our IR theory. Let's now recall some of the machinery developed to 'capture' the infinities that arise from taking such a limit naively, and then make contact with our Wilsonian understanding of what's actually happening.

3.1 1PI Green functions

Recall that at the heart of computing cross sections in perturbation theory lie $n-$point Green functions (or correlation functions) of the form

$$G(x_1, \ldots, x_n) = \langle \Omega | T \left\{ \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right\} | \Omega \rangle \tag{3.1}$$

One particularly important Green function is the two-point correlation function (also known as the Feynman propagator):

$$G_0(x, y) = \langle \Omega | T \left\{ \hat{\phi}_0(x) \hat{\phi}_0(y) \right\} | \Omega \rangle = -iD_F(x, y) = \int e^{ip \cdot (x-y)} \frac{i}{p^2 - m_0^2 + i\epsilon} \frac{\mathrm{d}^4 p}{(2\pi)^4} \tag{3.2}$$

Equivalently, the momentum-space representation of the Feynman propagator is

$$G_0(p) = \frac{i}{p^2 - m_0^2 + i\epsilon} \tag{3.3}$$

Notice that this expression is built out of the free fields (symbolized by the 0 subscript). Furthermore, notice that it has a pole at $p^2 = m_0^2$ (where m_0 is the value of the mass parameter appearing in the Lagrangian), with a residue of 1. These two features will be important later. The 2-point Green function built out of interacting fields

$$G(x, y) = \langle \Omega | T \left\{ \hat{\phi}(x) \hat{\phi}(y) \right\} | \Omega \rangle \tag{3.4}$$

is, of course much more complicated; it contains not only the possibility that a particle propagates from x to y, but also bound states that may form due to the interaction as well as two-particle states and other such complications. One useful representation is known as the **Källén-Lehmann spectral representation**:

$$G(p) = \int_0^\infty \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon} \frac{\mathrm{d}M}{2\pi} \tag{3.5}$$

where $\rho(M^2)$ is the spectral density function. In the free-field case, this function is simply a δ-function at the one-particle mass, $\rho(M^2) = \delta(m_0^2 - M^2)$. In an interacting theory, the spectral function becomes much more complicated to accommodate all the possible states listed earlier. A schematic plot of $\rho(M^2)$ is shown in Ch. 7 of P&S, reproduced below: Notice that, like before, there's a pole at the single-particle state,

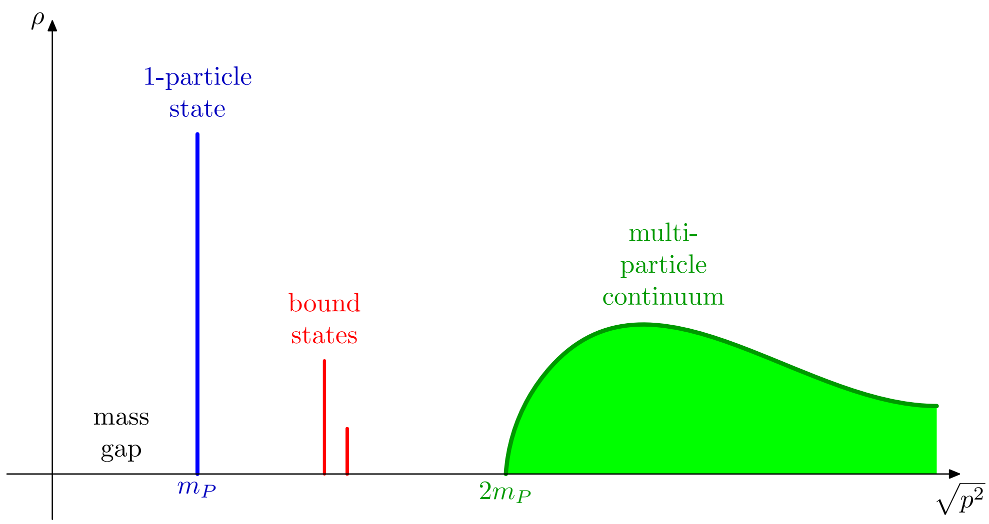


Figure 2: A plot illustrating the typical structure of $\rho(M^2)$ for an interacting field theory

followed by an undetermined spectrum starting at the two-particle state (with possible poles at bound states, which I'm going to ignore, without loss of generality). Thus the spectral representation becomes

$$G(p) = \frac{iZ}{p^2 - m_p^2 + i\epsilon} + \int_{(2m_p)^2}^\infty \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon} \frac{\mathrm{d}M}{2\pi} \tag{3.6}$$

Note that, like G_0, G has a pole at $p = m_p$ (hence the name **pole mass**), but the residue of that pole is Z, instead of 1. This factor of Z is known as the **wavefunction** or **field strength renormalization** and will later play a role in renormalization scheme calculations. This name will make more sense a moment. Notice also that m_p *does not necessarily equal* m_0! In fact, the parameter appearing in the Lagrangian will, in general, *not* be equal to the physical particle mass, as we will see.

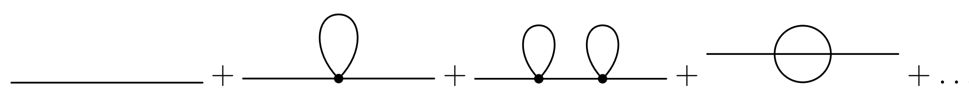
Why does this all matter? Let's recall what the aim of perturbation theory is. We are attempting to construct a full, interacting theory out of non-interacting pieces which are well-understood. In other words, we're trying to construct Eq. (3.6) out of pieces like Eq. (3.3). Remember that n-point Green functions can be built out of path integrals using generating functionals:

$$G^{(n)}(x_1 \dots x_n) = \frac{1}{Z[0]} \left(-i\frac{\delta}{\delta J(x_1)}\right) \dots \left(-i\frac{\delta}{\delta J(x_1)}\right) Z[J] \Bigg]_{J=0} = \frac{\int e^{i\int \mathcal{L}\mathrm{d}^4x} \phi(x_1) \dots \phi(x_n) \mathcal{D}\phi}{\int e^{i\int \mathcal{L}\mathrm{d}^4x} \mathcal{D}\phi} \tag{3.7}$$

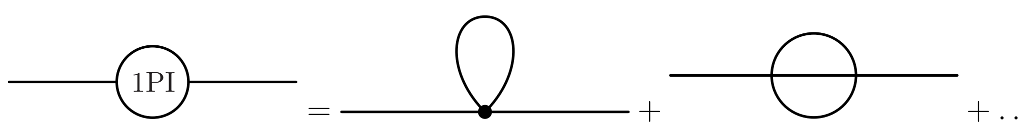
The Lagrangian can be split into the a 'free' part, $\mathcal{L}_0$, containing only the kinetic and mass terms (which are both quadratic in the fields) and the 'interacting part', $\mathcal{L}_{\text{int}}$. The exponential containing the interacting part can then be Taylor expanded, giving a series of terms consisting of interactions linked by 'free' 2-point Green functions. However, note that $G^{(2)}(x_1, x_2)$ is now built out of interacting fields, not free fields, so there is a slight modification that m_0 is replaced by m_r, known as the **renormalized mass**. For concreteness, let's return to our ϕ^4 model and consider the interacting 2-point Green function:

$$\begin{aligned}
G^{(2)}(x_1, x_2) &= \frac{1}{Z[0]} \int e^{i\int \mathcal{L} \mathrm{d}^4x} \phi(x_1)\phi(x_2)\mathcal{D}\phi \\
&= \frac{1}{Z[0]} \int e^{i\int \mathcal{L}_0 \mathrm{d}^4x} e^{\frac{i\lambda}{4!}\int \phi^4 \mathrm{d}^4x} \phi(x_1)\phi(x_2)\mathcal{D}\phi \\
&= \frac{1}{Z[0]} \int e^{i\int \mathcal{L}_0 \mathrm{d}^4x} \left(1 + \frac{i\lambda}{4!}\int \phi^4(x)\mathrm{d}^4x + \frac{1}{2}\left(\frac{i\lambda}{4!}\right)^2 \int\int \phi^4(x)\phi^4(y)\mathrm{d}^4x\mathrm{d}^4y + \dots\right) \phi(x_1)\phi(x_2)\mathcal{D}\phi
\end{aligned}$$

After ridding ourselves of the disconnected diagrams (which is done by dividing out by $Z[0]$), the above expression is diagrammatically given by

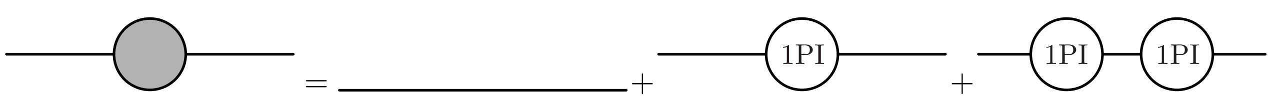
$G^{(2)}(x_1, x_2) =$  (3.8)

The first diagram is of order λ^0, the second of order λ and the last two of order λ^2. Notice, however, that of the last two diagrams, only the last one seems truly unique; the other seems to be composed of two order λ loops put back-to-back. We thus are introduced to the notion of 1PI or **1-particle irreducible diagrams**. These are best defined by what there are not. A 1-particle reducible diagram would be one where cutting one of the lines would result in the overall diagram falling apart into two, valid, subdiagrams. Thus if we cut the middle line between the two loops of the first $O(\lambda^2)$ diagram, we get two $O(\lambda)$ diagrams and thus this diagram is not 1PI. The expansion of the full Green function can be reorganized to be done in loop-by-loop order, rather than λ order. In other words, collecting all the 1PI diagrams at all orders of λ gives



(3.9)

Therefore the full Green function, represented by the gray blob, in terms of this loop expansion, is



(3.10)

This gives a nice visual representation of the notion that QFT consists of doing classical field theory, represented by the tree-level propagator, plus quantum corrections, represented by the loops. Let's put some math behind these diagrammatics. Let $-i\Pi(p)$ represent the loop contribution, shown diagrammatically as the 1PI blob. Thus, using the Feynman rules, the amplitude for any diagram in Eq. (3.9) is

$$iG^{(2)}(p) = \frac{i}{p^2 - m_r^2}(-i\Pi(p))\frac{i}{p^2 - m_r^2} \tag{3.11}$$

and therefore the full Green function is

$$\begin{aligned}
iG^{(2)}(p) &= \frac{i}{p^2-m_r^2} + \frac{i}{p^2-m_0^2}(-i\Pi(p))\frac{i}{p^2-m_r^2} + \frac{i}{p^2-m_r^2}(-i\Pi(p))\frac{i}{p^2-m_r^2}(-i\Pi(p))\frac{i}{p^2-m_r^2} + \dots \\
&= \frac{i}{p^2-m_r^2} + \frac{i}{p^2-m_r^2}\left(\frac{\Pi(p)}{p^2-m_r^2}\right) + \frac{i}{p^2-m_r^2}\left(\frac{\Pi(p)}{p^2-m_r^2}\right)^2 + \dots \\
&= \frac{i}{p^2-m_r^2}\left(1 + \left(\frac{\Pi(p)}{p^2-m_r^2}\right) + \left(\frac{\Pi(p)}{p^2-m_r^2}\right)^2 + \dots\right) \\
&= \frac{i}{p^2-m_r^2}\left(\frac{1}{1-\frac{\Pi(p)}{p^2-m_r^2}}\right) \\
&= \frac{i}{p^2-m_r^2-\Pi(p)}
\end{aligned}$$

where, in the second-to-last step, we employed the resummation of a geometric series. Note that the location of the pole $m_r^2 + \Pi(p)$ now exhibits momentum dependence, harkening back to Eq. (2.8) and the introduction of running parameters. I point out that we have not employed any explicit properties of the 1-loop diagram. The above resummation is equally valid for any $-i\Pi(p) < 1$, including any order of loop; at this point, our stamina in doing the calculations to compute the contribution to $-i\Pi(p)$ is the only limiting factor. Thus we see that the full Green function can be computed as an infinite series consisting of the tree-level propagator plus all the loop corrections, which can be computed to any order λ! In other words, when computing the 1PI blob, we get to choose how many loop orders (and thus how many powers of λ) are included in the calculation. Furthermore, note that the above has been done completely independently of infinite integrals. In other words, the above procedure would be necessary *even in a theory with no renormalization!*. Before moving on, let's note a few things about the analytic structure of this complete Green function. Notice first that there is a pole whenever the denominator vanishes. Solving this equation gives the location of the pole in the momentum plane and represents the physical mass of the particle, m_p:

$$f(p = m^2) = \left[p^2 - m_r^2 - \Pi(p)\right]_{p^2=m_p^2} = 0 \tag{3.12}$$

That leaves the question of the residue of this pole. Recall that the residue is nothing more than the coefficient of the $\frac{1}{z-z_0}$ term in the Laurent series of a function $f(z)$ in the vicinity of a pole z_0. Therefore, let's Taylor expand the denominator (equivalent, to first order, of expanding the Green function in a Laurent series), giving

$$\begin{aligned}
f(p^2) &= f(p^2 = m_p^2) + (p^2 - m_p^2)\left.\frac{\mathrm{d}f}{\mathrm{d}p^2}\right]_{p^2=m_p^2} \\
&= 0 + (p^2 - m_p^2)\left[\frac{\mathrm{d}}{\mathrm{d}p^2}\left(p^2 - m_r^2 - \Pi(p)\right)\right]_{p^2=m_p^2} \\
&= (p^2 - m_p^2)\left[1 - \frac{\mathrm{d}\Pi}{\mathrm{d}p^2}\right]_{p^2=m_p^2}
\end{aligned}$$

Thus we see that the residue is simply

$$\mathrm{Res}[G^{(2)}(p)] = \left(1 - \frac{\mathrm{d}\Pi}{\mathrm{d}p^2}\bigg]_{p^2=m_p^2}\right)^{-1} \tag{3.13}$$

Now, let's connect this to our earlier work involving the Källén-Lehmann spectral representation. Remember that the full, interacting Green function has the form

$$G^{(2)}(p) = \int e^{ip(x-y)} \langle\Omega| T\left\{\hat{\phi}(x)\hat{\phi}(y)\right\}|\Omega\rangle \,\mathrm{d}^4x = \frac{iZ}{p^2 - m_p^2 + i\epsilon} + (\text{terms regular at } p^2 = m_p^2) \tag{3.14}$$

Comparing the pole structures of the two forms of $G^{(2)}(p)$ we see that

$$\frac{iZ}{p^2 - m_p^2} \sim \frac{i}{p^2 - m_r^2 - \Pi(p)} \tag{3.15}$$

which gives

$$Z^{-1} = 1 - \frac{\mathrm{d}\Pi}{\mathrm{d}p^2}\bigg]_{p^2=m^2} \tag{3.16}$$

We can rid ourselves of the residue in the Green functions by rescaling the fields

$$\hat{\phi} = Z^{\frac{1}{2}}\hat{\phi}_r \tag{3.17}$$

thus explaining the name wavefunction renormalization. In the above notation, ϕ is known as a **bare field** while ϕ_r is known as a **renormalized field**. Let's summarize our findings thus far

1. The free Green function has a pole at $p^2 = m_0^2$, with residue 1
2. The mass parameter appearing in the Lagrangian, m_0, is not, in general, the same as the physical mass m
3. The full Green function can be expanded in a series of 1PI diagrams, which can then be resummed to yield a propagator that looks like the free Green function,

$$G^{(2)}(p) = \frac{i}{p^2 - (m_r^2 + \Pi(p))}$$

4. The full Green function has a pole at $p^2 = m_r^2 + \Pi(p)$, the renormalized mass, with residue Z
5. The physical mass m_p can be found by solving the equation $\left[p^2 - m_r^2 - \Pi(p)\right]_{p^2=m_p^2} = 0$
6. The pole residue is given by $Z^{-1} = 1 - \frac{\mathrm{d}\Pi}{\mathrm{d}p^2}\Big]_{p^2=m_p^2}$
7. The residue Z can be eliminated by rescaling the fields, $\phi = Z^{\frac{1}{2}}\phi_r$

3.2 Lagrangians & Counterterms

The previous sections have pointed us in the direction that parameters appearing in the Lagrangian are not the same as the physical values measured in the laboratory. In Eq. (2.8), we rescaled the positions/momenta and showed how the couplings in the effective Lagrangian reacted, leading to Eq. (2.9). In the previous section, we gave perturbative interpretations to two of these rescalings, namely the renormalized mass (which now depends on p) and the wavefunction renormalization. If we were smart enough we could just compute all correlation functions using the bare fields and couplings, but we're not, so instead, let's rewrite the Lagrangian in terms of renormalized fields and couplings. Our bare Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4 \tag{3.18}$$

where the 0 subscript now denotes bare fields/couplings. Now introduce the renormalized fields, $\phi_0 = \sqrt{Z}\phi_r$:

$$\mathcal{L} = \frac{1}{2}Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{m_0^2 Z}{2}\phi_r^2 + \frac{\lambda_0 Z^2}{4!}\phi_r^4 \tag{3.19}$$

which allows us to introduce the renormalized couplings; recall that the renormalized couplings are all written in the form $C' = C + \delta_C$ where 1 represents the theory at its initial point in theory space and δ_C represents the shift in the parameter as shells in momentum space are integrated out. Thus we see that

$$Z = 1 + \delta_Z \qquad\qquad m_0^2 Z = m_r^2 + \delta_m \qquad\qquad \lambda_0 Z^2 = \lambda_r + \delta_\lambda \tag{3.20}$$

Therefore, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_r\partial^\mu\phi_r - \frac{m_r^2}{2}\phi_r^2 + \frac{\lambda_r}{4!}\phi_r^4 + \frac{1}{2}\delta_Z\partial_\mu\phi_r\partial^\mu\phi_r - \frac{\delta_m}{2}\phi_r^2 + \frac{\delta_\lambda}{4!}\phi_r^4 \tag{3.21}$$

with Feynman rules

There is still an outstanding question: do the above counterterms suffice to 'capture' the effects of UV physics as we proceed into the IR? In other words, are there outstanding infinities we have not cancelled? We'll stick a pin in this question until we study divergences and when they arise. Another question is exactly are these counterterms and how do they relate to our earlier, Wilsonian picture? Recall that in the Wilsonian picture, a Lagrangian such as that in Eq. (3.19) is defined with some fundamental cutoff Λ_0, beyond which it is no longer an effective description of the physics, while Λ is the value of the cutoff in momentum space after an infinitesimal momentum shell has been integrated out. As shells of momenta are integrated out to bring $\Lambda \sim p$, where p are the external momenta, the theory traces out a path through theory space, bringing it close to a Gaussian fixed point (where a CFT lives, with no irrelevant interactions) before zooming off into the IR along a renormalized trajectory. Therefore, there exists a point along this trajectory that lies exceedingly close to the fixed point; let's call the value of Λ that corresponds to this point on the trajectory μ. By dimensional analysis, it is possible to write μ as

$$\mu = \Lambda_0 f(g_{01}, g_{02}, \dots g_{0n}) \tag{3.22}$$

where the g_{0i} are the bare values of the couplings and f is some function of these parameters, such that $f = 0$ when these parameters live on the plane of irrelevant couplings. Now let's recall that, due to

Lorentz invariance, we'll eventually send $\Lambda_0 \to \infty$, and thus it is necessary to tune the couplings such that μ remains finite. This is done by modifying the initial action to explicitly include the counterterms that depend on the fields and the cut-off Λ_0

$$S_{\Lambda_0}[\phi] \to S_{\Lambda_0}[\phi] + S_{\text{ct}}[\phi, \Lambda_0] \tag{3.23}$$

Remember that we found earlier that, technically, an effective Lagrangian should include all possible polynomials in the fields and their derivatives (even if these aren't present in the original Lagrangian, integrating out a shell in momentum space will introduce them) and thus, technically, we aren't introducing anything new. The counterterms were already there, we're just making them explicit. The counterterms (whose couplings comprise a subset of the g_{0i} used to compute f) are chosen by hand to render μ finite; thus changing the counterterms corresponds to changing the high-energy theory trajectory along we're evolving! This might seem like a catastrophe, but remember that in the deep IR, far along a renormalized trajectory, all theories focus onto that same trajectory and become virtually indistinguishable; by the time it matters, a new, UV-complete theory will be necessary, rendering the question moot. Furthermore, it's completely possible that some of the couplings diverge as well, but this is okay, since the bare fields/couplings are not physically measurable.

3.3 Counting divergences

Now that we have set up the infrastructure to study how divergences arise in perturbation theory and how to handle them, we now study to how to, at a glance, identify potential divergences in a diagram and the degree of that divergence, which informs the structure of the counterterms. Before doing this, let's establish a few facts about the mass dimension of fields and couplings. Recall that the action S should be unitless (in a system where $\hbar = c = 1$), since it is exponentiated when the generating functional is computed in the path integral formalism. Let's define the notation (for now) where $[\,\cdot\,]$ represents the spatial dimensionality of the object in question. The action S must be dimensionless, since it is exponentiated when computing the path integral and therefore $[S] = 0$ Since $S = \int \mathcal{L}\mathrm{d}^D x$, this implies that $[\mathcal{L}] + [\mathrm{d}^D x] = 0$ and therefore $[\mathcal{L}] = -[\mathrm{d}^D x] = -D$.

Although it is possible to continue this analysis in spatial dimensions, it is common to flip into momentum space; the switch is simple and is done simply by taking the negative of the spatial dimension, yielding what is known as the **mass dimension**. Thus in D dimensions, the mass dimension of the Lagrangian is simply $[\mathcal{L}] = D$ (where now $[\,\cdot\,]$ gives the mass dimensionality). By dimensional analysis, whatever product of fields, couplings and derivatives goes into $\mathcal{L}$, the sum of the mass dimensions of each term must be D. Before proceeding, let's define the mass dimension of the derivative:

$$[\partial_\mu] = \left[\frac{\partial}{\partial x^\mu}\right] = 1 \tag{3.24}$$

We can use this fact to derive the mass dimension of some common fields, by analyzing the kinetic term. Let's begin with a scalar field:

$$\begin{aligned} \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \qquad \to \qquad [\mathcal{L}] = D &= 2[\partial] + 2[\phi] \\ D &= 2(1) + 2[\phi] \\ [\phi] &= \frac{D-2}{2} \end{aligned} \tag{3.25}$$

Next a spinor field

$$\begin{aligned} \mathcal{L} = \bar{\psi}(i\partial\!\!\!/)\psi \qquad \rightarrow \qquad [\mathcal{L}] = D &= [\partial] + 2[\psi] \\ D &= 1 + 2[\psi] \\ [\psi] &= \frac{D-1}{2} \end{aligned} \tag{3.26}$$

And finally a vector field

$$\begin{aligned} \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \qquad \rightarrow \qquad [\mathcal{L}] = D &= 2[\partial] + 2[A] \\ D &= 2(1) + 2[A] \\ [A] &= \frac{D-2}{2} \end{aligned} \tag{3.27}$$

Let's return to our scalar field theory, but let's instead consider a general ϕ^n theory, to determine the mass dimension of a general coupling. A general term in the interaction Lagrangian for ϕ^n theory with n fields and m derivatives gives

$$\mathcal{L} = g_{n,m}\partial^m\phi^n \qquad \rightarrow \qquad D = [g_{n,m}] + m + n\left(\frac{D-2}{2}\right) \qquad \Rightarrow \qquad [g_{n,m}] = \frac{D(2-n) - 2m + 2n}{2} \tag{3.28}$$

Note that, at minimum, $n = 2$, $m = 0$ (since terms in the Lagrangian must be at least bilinear in the fields). In addition, derivatives must come in pairs (in other words $m = 2, 4, 6, \dots$), but this is only true for scalar fields. Using this, let's find the mass dimensionality of the mass. (Sanity check: we're expecting the mass dimension of m^2 to be $[m^2] = 2$, otherwise something has gone horribly wrong). Thus we see

$$[g_{2,0}] = [m^2] = \frac{D(2-2) - 2(0) + 2(2)}{2} = 2 \tag{3.29}$$

Note that this is true for all D, as expected! The mass dimensions for a few common couplings in $D = 4$ are listed below:

$$[g_{3,0}] = \frac{4(2-3) - 2(0) + 2(3)}{2} = 1 \tag{3.30}$$

$$[g_{4,0}] = \frac{4(2-4) - 2(0) + 2(4)}{2} = 0 \tag{3.31}$$

$$[g_{5,0}] = \frac{4(2-5) - 2(0) + 2(5)}{2} = -1 \tag{3.32}$$

$$[g_{6,0}] = \frac{4(2-6) - 2(0) + 2(6)}{2} = -2 \tag{3.33}$$

$$[g_{2,2}] = \frac{4(2-2) - 2(2) + 2(2)}{2} = 0 \tag{3.34}$$

$$[g_{3,2}] = \frac{4(2-3) - 2(2) + 2(3)}{2} = -1 \tag{3.35}$$

$$[g_{4,2}] = \frac{4(2-4) - 2(2) + 2(4)}{2} = -2 \tag{3.36}$$

$$[g_{2,4}] = \frac{4(2-2) - 2(4) + 2(2)}{2} = -2 \tag{3.37}$$

Thus we see that, in $D = 4$, only terms with at most two derivatives have positive or 0 mass dimension. This will play a role in discussing the renormalizability of this theory. We thus see how to find the mass dimension of a given operator. First find the mass dimension of the operator based on the number of fields and derivatives it contains, then multiply by a coupling whose dimension is positive, negative, or zero, depending what is needed to bring the total product's mass dimension equal to the dimension of the Lagrangian.

Again, why does this matter? It turns out there is a correspondence between the mass dimension of a given coupling, the behavior of loop integrals that correct that coupling, and its behavior under renormalization group flows. Let's now finally turn our attention to loop integrals (about bloody time!) in ϕ^n theory. Consider a general diagram with L loops, N external lines, V vertices, and P internal propagators. Finally, let d be the mass dimension of the integrand. Each loop requires an integral over loop momenta k_i, which brings in a differential $\mathrm{d}^D k_i$. The mass dimension of this element is $[\mathrm{d}^D k] = D$; thus each loop contributes to the mass dimension of the integrand a factor $d_L = DL$. In ϕ^4 theory, every internal propagator in a loop is proportional to k_i^{-2}, so each propagator contributes a factor $d_P = -2P$. The mass dimension of the integrand is thus

$$d = d_P + d_L = DL - 2P$$

Of course, we can do better than this. The number of loops and internal propagators will be related to the number of vertices and number of external lines. The number of loops is constrained by the number of vertices and propagators to be $L = P - V + 1$; since n lines meet a vertex, $nV = N + 2P$. Solving the first equation for P and inserting into the second equation gives L and P in terms of N and V:

$$\begin{aligned}
L = P - V + 1 \qquad \to \qquad P &= L + V - 1 \\
\Rightarrow \qquad nV &= N + 2P \\
nV &= N + 2(L + V - 1) \\
2L &= V(n-2) - N + 2 \\
L &= \frac{(n-2)V}{2} - \frac{N}{2} + 1 \\
\Rightarrow \qquad P &= \left(\frac{(n-2)V}{2} - \frac{N}{2} + 1\right) + V - 1
\end{aligned}$$

Thus we see that d is

$$\begin{aligned}
d &= DL - 2P \\
&= D\left(\frac{(n-2)V}{2} - \frac{N}{2} + 1\right) - 2\left(\frac{(n-2)V}{2} - \frac{N}{2} + V\right) \\
&= V\left(\frac{D(n-2)}{2} - \frac{2(n-2)}{2} - 2\right) + N\left(-\frac{D}{2} + 1\right) + D \qquad (3.38) \\
&= -V\left[D - n\left(\frac{D-2}{2}\right)\right] - N\left(\frac{D-2}{2}\right) + D \qquad (3.39)
\end{aligned}$$

So, what exactly is d? Earlier, we defined it as the mass dimension of the integrand. This means that as k_i, the loop momentum, becomes extremely large (say, $k_i \sim \Lambda$), the integrand will behave as k^d and thus the integral will behave as Λ^d. Thus provides the measure of the **superficial degree of divergence**. It gives us a handle on how badly the integral seemingly diverges (or converges). It may converge more rapidly than d (due to symmetry) or it may diverge more rapidly (due to divergent subdiagrams). If $d > 0$, this means that the integral will superficially diverge as Λ^d; if $d < 0$, the integral is superficially convergent. If $d = 0$, the integral is still divergent, but only logarithmically so.

Let's examine a few cases, for ϕ^4 theory. If $D = 2$, the number of external lines does not matter, while the coefficient of V is always < 0. If $V = 1$, then $d = 0$ and we have a logarithmically divergent diagram. If $V > 1$, then $d < 0$ and thus the diagrams are convergent, regardless of how many vertices or legs there are. Thus the total number of divergent diagrams is finite. We thus call ϕ^4 theory in $D = 2$ a **super-renormalizable theory**. Now, let's look at $D = 4$. The coefficient of V is 0, while the coefficient of N is -1. For $N = 2$, $d = 2$, while for $N = 4$, $d = 0$ and for $N = 6$, $d = -2$. Thus we see that a finite number of diagrams will diverge, but these diagrams can diverge at any order in perturbation theory (which increases the number of vertices). This is known as a **renormalizable theory**. Lastly, consider $N = 6$. In this case, as N and V both increase, d will become more and more positive and thus the total number of divergent diagrams is unbounded. This is known as a **non-renormalizable theory**. Notice the important factor here is the coefficient of the number of vertices. It turns out this factor is nothing more than the mass dimension of the coupling! To see this, recall Eq. (3.28), with $m = 0$:

$$[g_{n,0}] = \frac{D(2-n)+2n}{2} = D - n\left(\frac{D-2}{2}\right)$$

Thus if the mass dimension of the coupling is positive, we get diagrams that converge more rapidly the greater the number of vertices – this is nothing more than our super-renormalizable theory. If the mass dimension is 0, we get a finite number of divergent diagrams at every order in perturbation theory, the renormalizable theory. If the mass dimension is negative, we get an unbounded number of divergent diagrams at ever order in perturbation theory, the non-renormalizable theory. This fulfills the correspondence I promised earlier! But we can go a step further.

Recall from Ch. 2 that we found that as shells in momentum space are integrated out, new interactions are generated consisting of all possible combinations of fields and derivatives, consistent with the symmetries of the original Lagrangian. In other words, any effective Lagrangian has the schematic form

$$\mathcal{L} = \frac{Z}{2}(\partial\phi)^2 + \sum_{n,m} g_{n,m}\partial^m\phi^n \tag{3.40}$$

It is possible to redefine the couplings to make them dimensionless; this is done by dividing out by the only invariant momentum scale, the cutoff, raised to the necessary power, the mass dimension of $g_{n,m}$; let $[g_{n,m}] = g$. In other words, $g_{n,m} \to \Lambda^g g_{n,m}$, where now $[g_{n,m}] = 0$ since the mass dimensionality has been shunted onto the appropriate power of Λ. Now let $\Lambda \to \infty$. Notice that those operators with $g > 0$ blow up in this limit; these are nothing more than our relevant couplings from before. Those with $g < 0$ die away in the limit, corresponding to our irrelevant couplings. Lastly, those with $g = 0$ neither grow nor die away, at least superficially, corresponding to our marginal couplings. Thus we at last see the links between perturbation theory and our Wilsonian picture of renormalization. As we see, we are truly safe in writing Lagrangians only with operators with positive or zero mass dimension, since these correspond to the relevant and marginal couplings, which are truly only the ones that matter in the $\Lambda \to \infty$ limit. Of course, there is nothing restricting us from including negative mass dimension operators, but in doing so, we render the theory non-renormalizable, making it an effective field theory that does not have a continuum limit.

- The mass dimension of the coupling determines the behavior of the coupling under the renormalization group flows ...
  - ⋆ Positive mass dimension couplings correspond to relevant couplings and grow under the group flows
  - ⋆ Zero mass dimension couplings correspond to marginal couplings and could grow or decrease under group flows
  - ⋆ Negative mass dimension couplings correspond to irrelevant couplings and decrease under group flows
- A super-renormalizable theory:
  - ⋆ Has only a finite number of superficially divergent diagrams, which converge after a certain order in perturbation theory
  - ⋆ Has a Lagrangian built out of relevant couplings only
- A renormalizable theory:
  - ⋆ Has only a finite number of superficially divergent diagrams, but which may diverge at any order in perturbation theory
  - ⋆ Has a Lagrangian built out of relevant or marginal couplings
- A non-renormalizable theory:
  - ⋆ Has only an unbounded number of superficially divergent diagrams, the number of which grows at every order in perturbation theory
  - ⋆ Has a Lagrangian built out of any kind of coupling

4 The Callan-Symanzik equations

4.1 Subtraction points

Thus far we have established the viewpoint of group flows in theory space and have connected that notion to the Green functions so useful in perturbation theory. In this section, we will complete that connection by studying how Green functions defined at a given renormalization point evolve when that renormalization point is adjusted and deriving a differential equation describing this evolution. To do this, we must first define the theory at a scale. This means that we choose a **renormalization scheme** or **subtraction point**. This is a reference point in momentum space about which the theory is defined and from which it evolves. Remember: in sec 3.2 we stated formally that as $\Lambda \to \infty$, it was possible to choose the bare couplings so that μ, the value along which the parameters pass closest to the CFT, is finite. In terms of trajectories in theory space, we must choose a starting point in theory space from which the flows originate.

We will consider several possibilities for these schemes in chapter 6.

In the case of ϕ^4 theory, we will use a reference momentum $p^2 = -M^2$. This means that when the external momenta take on the value $-M^2$, the mass of the particle m_r^2 and its coupling λ will take on their tree-level values, so that quantum corrections vanish. This corresponds to to choosing values for the counterterms δ_Z, δ_m and δ_λ so that

$$\Pi(p^2 = -M^2) = 0 \qquad \left.\frac{\mathrm{d}\Pi}{\mathrm{d}p^2}\right]_{p^2=-M^2} = 0 \qquad \lambda_r(p^2 = -M^2) = \lambda \tag{4.1}$$

Of course, the M in $p^2 = -M^2$ is completely arbitrary. Defining the theory at M is physically equivalent to defining it at M'. In other words, choosing one point over another should make no difference to the evolution of the theory in theory space. There may be good reasons (such as experimental measurement) to define the theory at one point or another, but we would like to know how the theory evolves from a given point. The Callan-Symanzik equation will do just that.

4.2 Deriving the equations - Simplest case

In this simplest case, we will consider massless ϕ^4 theory defined at the above subtraction point. As stated above, defining the theory at one point M is physically equivalent to defining it another point M'. Recall that physical quantities in QFT are dependent on the n-point Green functions; we may construct these Green functions out of the bare fields ϕ_0, which couple with bare coupling λ_0 and are part of a Lagrangian with cutoff Λ. In other words

$$G_\Lambda^{(n)}(x_1, x_2, \ldots, x_n) = \langle\Omega|T\{\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\ldots\hat{\phi}_0(x_n)\}|\Omega\rangle \tag{4.2}$$

Note that this expression has no dependence on M. Subtraction point dependence only enters the picture when we rescale the bare fields to renormalized fields ($\phi_0 = Z^{\frac{1}{2}}\phi_r$) and replace the bare coupling with the renormalized coupling. As such

$$G^{(n)}(x_1, x_2, \ldots, x_n; M, \lambda) = \langle\Omega|T\{\hat{\phi}_r(x_1)\hat{\phi}_r(x_2)\ldots\hat{\phi}_r(x_n)\}|\Omega\rangle = Z^{-\frac{n}{2}}\langle\Omega|T\{\hat{\phi}_0(x_1)\hat{\phi}_0(x_2)\ldots\hat{\phi}_0(x_n)\}|\Omega\rangle \tag{4.3}$$

Doing this removes the Green function's cutoff dependence (making it safe to take $\Lambda \to \infty$) but trades it for subtraction point dependence; this dependence is both explicit and implicit (through the momentum-dependence the field rescaling Z and coupling λ exhibit). Now shift the subtraction point M by δM. In other words, let $M \to M + \delta M$. What effect does this have on the Green function in Eq. (4.3)? In other words, what is $\mathrm{d}G^{(n)}$? This seems like a very difficult question, given that ϕ_r is M-dependent, as are the couplings. But we can take advantage of the fact that the renormalized Green function is defined in terms of the bare Green function, which has no M-dependence at all:

$$\mathrm{d}G^{(n)} = \mathrm{d}\left(Z^{-\frac{n}{2}}\right)G_\Lambda^{(n)} + Z^{-\frac{n}{2}}\cancelto{0}{\mathrm{d}G_\Lambda^{(n)}} = \mathrm{d}\left(Z^{-\frac{n}{2}}\right)G_\Lambda^{(n)} \tag{4.4}$$

Thus we see that the Green function simply scales in terms of a multiplicative factor $\mathrm{d}\left(Z^{-\frac{n}{2}}\right)$ (as opposed to a more complicated derivative rule) and that this factor arises from the rescaling of the fields. All we need now is to find out how the fields rescale through $Z^{\frac{1}{2}}$. We'll put this explicit problem aside for now;

we'll just let the fields rescale as $\phi_r \to (1+\delta\eta)\phi_r$. Later, we'll relate $\delta\eta$ to $\mathrm{d}Z$ explicitly. To first order in $\delta\eta$, the n-point Green function behaves as

$$G^{(n)} \to (1+n\delta\eta)G^{(n)} = G^{(n)} + \mathrm{d}G^{(n)} \qquad \Rightarrow \qquad \mathrm{d}G^{(n)} = n\delta\eta G^{(n)} \tag{4.5}$$

Similarly, let the coupling rescale as $\lambda \to \lambda + \delta\lambda$. Thus $\mathrm{d}G^{(n)}$ can also be written

$$\mathrm{d}G^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda \tag{4.6}$$

Equating the two gives

$$\frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\delta\eta G^{(n)} \tag{4.7}$$

Rearranging this equation and multiplying by $\frac{M}{\delta M}$ gives

$$\begin{aligned}
\frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda &= n\delta\eta G^{(n)} \\
\frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda - n\delta\eta G^{(n)} &= 0 \\
\frac{M}{\delta M}\left(\frac{\partial}{\partial M}\delta M + \frac{\partial}{\partial \lambda}\delta\lambda - n\delta\eta\right) G^{(n)} &= 0 \\
\left(M\frac{\partial}{\partial M} + \frac{\partial}{\partial \lambda}M\frac{\delta\lambda}{\delta M} - nM\frac{\delta\eta}{\delta M}\right) G^{(n)} &= 0 \\
\left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial \lambda} + n\gamma\right) G^{(n)} &= 0
\end{aligned}$$

where, in the last step, I've introduced two definitions:

$$\beta = M\frac{\delta\lambda}{\delta M} \qquad\qquad \gamma = -M\frac{\delta\eta}{\delta M} \tag{4.8}$$

The first of these is the change in the coupling λ as the scale changes, whilst the second is the change in the field as the scale changes. Thus we see the explicit and implicit M-dependence terms I promised earlier. Since the Green function is renormalized, neither β nor γ can depend on Λ. Thus, by dimensional analysis, since we cannot form unitless quantities such as $\frac{M}{\Lambda}$, β and γ must also be independent of M; thus these two functions can only depend on the unitless coupling λ. This argument is valid here since λ is unitless; some subtleties arise when the coupling is not dimensionless and will be treated in the general case section. Thus, any Green function in massless ϕ^4 theory obeys the equation

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right) G^{(n)} = 0 \tag{4.9}$$

This is the vaulted Callan-Symanzik equation. It is a differential equation, whose describes the evolution of the Green function as M is varied. This looks suspiciously like a total derivative, which it is; the Callan-Symanzik equation can be equivalently written:

$$M\frac{\mathrm{d}G_0^{(n)}}{\mathrm{d}M} = M\frac{\mathrm{d}}{\mathrm{d}M}\left(Z^{\frac{n}{2}}G^{(n)}\right) = 0$$

In other words, as μ is varied, the bare Green function (and thus the physics) is independent of the subtraction point at which we choose to the define the theory. This is a perfectly reasonable conclusion; observations in physics should not depend on our definitions of the theory used to describe them! Another version of this equation can be derived by noting that if all momenta, fields and renormalization scale in a Green function are rescaled by the same factor then

$$\left(p\frac{\partial}{\partial p} - nd + M\frac{\partial}{\partial M}\right) G_R^{(n)} = 0 \tag{4.10}$$

which, when combined with the Callan-Symanzik equation gives

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} - n(d+\gamma(\lambda))\right) G_R^{(n)} = 0 \tag{4.11}$$

This version is interesting since it tells us that change in λ together with a rescaling of the fields according to the modified mass dimension $d + \gamma(\lambda)$ is equivalent to changing the scale of the external momenta! Next we'll move on to solving this equation and interpreting the results, but before we do that, we must compute β and γ.

4.3 Computing β and γ in perturbation theory

Recall that I promised that we would find an explicit relation for $\delta\eta$ to $\mathrm{d}Z$. I'll now fulfill that promise. Remember that $\phi_r = Z^{-\frac{1}{2}}(M)\phi_0$. Recall also that ϕ_0 does not depend at all on M and thus is unaffected by its variations. Let $M \to M + \delta M$ and thus we see

$$\begin{aligned}
\phi_r(p) &= Z^{-\frac{1}{2}}(M)\phi_0 \\
\phi_r(1+\delta\eta) &= Z^{-\frac{1}{2}}(M+\delta M)\phi_0 \\
\phi_r + \phi_r\delta\eta &= \left[Z^{-\frac{1}{2}}(M) + \delta M\frac{\partial}{\partial M}\left(Z^{-\frac{1}{2}}\right)\right]\phi_0 \\
\phi_r + \phi_r\delta\eta &= Z^{-\frac{1}{2}}(M)\phi_0 - \frac{1}{2}Z^{-\frac{3}{2}}\delta M\frac{\partial Z}{\partial M}\phi_0 \\
\phi_r\delta\eta &= -\frac{1}{2}Z^{-1}\delta M\frac{\partial Z}{\partial M}\left(Z^{-\frac{1}{2}}\phi_0\right) \\
\phi_r\delta\eta &= -\frac{1}{2}Z^{-1}\delta M\frac{\partial Z}{\partial M}\phi_r \\
\Rightarrow \qquad \delta\eta &= -\frac{1}{2}Z^{-1}\delta M\frac{\partial Z}{\partial M} \\
\frac{\delta\eta}{\delta M} &= -\frac{1}{2Z}\frac{\partial Z}{\partial M}
\end{aligned}$$

Combining this with our definition of γ gives

$$\gamma = \frac{1}{2}\frac{M}{Z}\frac{\partial Z}{\partial M} \tag{4.12}$$

This is known as the **anomalous dimension**. It in effect measures how badly a rescaling in momentum fails to be classical. All fields have a classical rescaling that depends on their mass dimension. As the scale is changed, due to quantum effects, this classical scaling is modified, and the modification is γ, thus explaining the name. This will be explicitly shown later. The other factor, β, can be similarly rewritten, taking into account the fact that $\beta = \beta(\lambda_r)$:

$$\beta(\lambda) = M\frac{\partial \lambda_r}{\partial M} \tag{4.13}$$

Of course, in QFT, we compute quantities such as Z and λ_r in perturbation theory, in terms of counterterms. How do we determine β and γ from the counterterms? Consider a general, n-point Green function with dimensionless coupling g. Schematically, this has the form

$$\begin{aligned}G^{(n)} &= (\text{tree-level diagram}) + (\text{1PI loop diagrams}) + (\text{vertex counterterm}) + (\text{external leg corrections}) \\ &= \left(\Pi_{i=1}^{n}\frac{i}{p_i^2}\right)\left[-ig - iB\log\frac{\Lambda^2}{-p^2} - i\delta_g + (-ig)\sum_i\left(A_i\log\frac{\Lambda^2}{-p_i^2} - \delta_{Z_i}\right)\right] + (\text{finite terms})\end{aligned} \tag{4.14}$$

where p_i are the external momenta and p^2 is some invariant built out of the momenta (like I said, this is very schematic; we'll make things more concrete in the next Part of these notes)[1]. Since this Green function must obey the Callan-Symanzik equations, and since the counterterms carry the explicit M-dependence, we see that

$$\begin{aligned}0 &= \left(M\frac{\partial}{\partial M} + \beta(g)\frac{\partial}{\partial g} + n\gamma_i(g)\right)G^{(n)} \\ 0 &= M\frac{\partial}{\partial M}\left(\delta_g - g\sum_i\delta_{Z_i}\right) + \beta(g) + \frac{1}{2}\frac{M}{Z_i}\frac{\partial Z_i}{\partial M}\end{aligned}$$

To leading order, since $Z_i = 1 + \delta_{Z_i}$, Eq. (4.10) becomes

$$\gamma_i = \frac{1}{2}\frac{M}{Z_i}\frac{\partial Z_i}{\partial M} \approx \frac{1}{2}M\frac{\partial}{\partial M}\delta_{Z_i} \tag{4.15}$$

and thus

$$\begin{aligned}0 &= M\frac{\partial}{\partial M}\left(\delta_g - g\sum_i\delta_{Z_i}\right) + \beta(g) + \frac{1}{2}\frac{M}{Z}\frac{\partial Z_i}{\partial M} \\ &= M\frac{\partial}{\partial M}\left(\delta_g - g\sum_i\delta_{Z_i}\right) + \beta(g) + g\sum_i\frac{1}{2}M\frac{\partial}{\partial M}\delta_{Z_i} \\ &= M\frac{\partial}{\partial M}\left(\delta_g - \frac{1}{2}g\sum_i\delta_{Z_i}\right) + \beta(g)\end{aligned}$$

So we that

$$\beta(g) = M\frac{\partial}{\partial M}\left(-\delta_g + \frac{1}{2}g\sum_i\delta_{Z_i}\right) \tag{4.16}$$

[1]The assumption that the contributions from the loop diagrams generate only logarithmic corrections is actually non-trivial, and stems from the fact that g is dimensionless. We will treat the general case (where leading divergences may not necessarily be logarithmic) later

Examining Eq. (4.12), we see that, at the subtraction point

$$\delta_g = B \log \frac{\Lambda^2}{M^2} + \text{finite} \qquad \text{and} \qquad \delta_{Z_i} = A_i \log \frac{\Lambda^2}{M^2} + \text{finite} \tag{4.17}$$

and thus

$$\beta(g) = -2B - g \sum_i A_i \qquad \text{and} \qquad \gamma_i(g) = A_i \tag{4.18}$$

We therefore see that β and γ can be computed from the counterterms simply by finding the coefficients of the divergent logarithms. Notice that the finite parts do not come into play, since they do not have M-dependence! For computation purposes, these are probably some of the most useful equations in this document, so I'll reproduce them below and box them:

The β-function and anomalous dimension are defined in general as

$$\beta(\lambda) = M \frac{\partial g}{\partial M} \qquad \qquad \gamma(\lambda) = \frac{1}{2} \frac{M}{Z} \frac{\partial Z}{\partial M}$$

and are defined in terms of counterterms as

$$\beta(g) = M \frac{\partial}{\partial M} \left(-\delta_g + \frac{1}{2} g \sum_i \delta_{Z_i} \right) \qquad \qquad \gamma_i = \frac{1}{2} M \frac{\partial}{\partial M} \delta_{Z_i} \tag{4.19}$$

The (relevant parts of the) counterterms are of the form

$$\delta_g = B \log \frac{\Lambda^2}{M^2} + \text{finite} \qquad \text{and} \qquad \delta_{Z_i} = A_i \log \frac{\Lambda^2}{M^2} + \text{finite} \tag{4.20}$$

Therefore β and γ are

$$\beta(g) = -2B - g \sum_i A_i \qquad \text{and} \qquad \gamma_i(g) = A_i \tag{4.21}$$

4.4 Solving the equations

Recall that the Callan-Symanzik equations specify how Green functions from the same theory evolve as they are specified at different points along the theory's renormalization group flow trajectory. In other words, as M changes, the Callan-Symanzik equations tell us how $G^{(n)}$ changes. Alternatively, we can solve these same equations to give Green functions that are properly scaled from the renormalization point M to the characteristic momentum scale p for a given process. As the Callan-Symanzik equation is a first-order linear PDE, it can be solved by means of the method of characteristics. Let $G_R^{(n)}$ be a renormalized n-point Green function, which satisfies the Callan-Symanzik equation

$$\left(M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + n\gamma_i(g) \right) G_R^{(n)} = 0 \tag{4.22}$$

Before we get to solving this, let's see if there's any information we can extract based on dimensional grounds alone. Let u be a dilatation parameter (given by $u = \frac{M}{p}$) so that

$$G_R^{(n)}\left(p;\lambda,\frac{M}{u}\right) = Z^{-\frac{n}{2}}(u)G_R^{(n)}(p;\bar{\lambda}(u),M) \tag{4.23}$$

where $\bar{\lambda}$ is a function of u and is known as the **running coupling**. It is defined as the solution to the equation

$$u\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}u} = \beta(\bar{\lambda}(u)) \tag{4.24}$$

with the initial condition $\bar{\lambda}(u=1) = \lambda$ where λ is the value of the renormalized coupling at the subtraction point M. For reference, this is often known as the **renormalization group equation**. Likewise, $Z(u)$ is defined as the solution to

$$\frac{1}{2}\frac{u}{Z}\frac{\mathrm{d}Z}{\mathrm{d}u} = -\gamma(\bar{\lambda}(u)) \tag{4.25}$$

with the initial condition $Z(u=1) = 1$. This ODE has the solution

$$Z(u) = \exp\left[-2\int_0^{\log u}\gamma(\bar{\lambda}(u'))\mathrm{d}\log u'\right] \tag{4.26}$$

On dimensional grounds, it is possible to write

$$G_R^{(n)}(p;\lambda,M) = M^{nd}\mathcal{G}\left(\lambda,\frac{p}{M}\right) \tag{4.27}$$

where d is the mass dimension of $G_R^{(n)}$ and $\mathcal{G}$ is a dimensionless function. Now let $M \to \frac{M}{u}$ which gives

$$G_R^{(n)}\left(p;\lambda,\frac{M}{u}\right) = \frac{M^{nd}}{u^{nd}}\mathcal{G}\left(\lambda,\frac{up}{M}\right) = u^{-nd}G_R^{(n)}(up;\lambda,M) \tag{4.28}$$

Combining this result with Eq. (4.23) yields

$$G_R^{(n)}\left(p;\lambda,\frac{M}{u}\right) = u^d Z^{-\frac{n}{2}}(u)G_R^{(n)}(p;\bar{\lambda}(u),M) \tag{4.29}$$

which, in turn, gives

$$G_R^{(n)}\left(p;\lambda,\frac{M}{u}\right) = u^d\exp\left[n\int_0^{\log u}\gamma(\bar{\lambda}(u'))\mathrm{d}\log u'\right]G_R^{(n)}(p;\bar{\lambda}(u),M) \tag{4.30}$$

We thus see one of the first major implications of the quantum corrections in renormalization. Recall that all fields (and Green functions, couplings, etc.) have a classical scaling dimension d (also known as the engineering dimension). We see that as the momenta are rescaled down from $p \to \frac{p}{u}$ (equivalently as $uM \to M$) the n-point Green function experiences a rescaling due to this classical dimension. But that is not the only rescaling it experiences; it also experiences a rescaling due to the anomalous dimension (hence the name), which in turn is a purely quantum mechanical effect and is contained in the exponential multiplicative factor. Remember that the anomalous dimension arises from the field renormalization, which itself is necessary due to Källèn-Lehmann spectral form of the full 2-point propagator.

Now let's find a complete solution. We introduced the running coupling earlier in Eq. (4.24); this is the solution to the differential equation

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda}\right)\bar{\lambda}(p,\lambda) = 0 \qquad \text{with} \qquad \bar{\lambda}(M,\lambda) = \lambda \tag{4.31}$$

On dimensional analysis grounds, we make an ansatz of the the Green function with dependence on $\bar{\lambda}$ that has the form

$$G_R^{(n)}(p;\lambda,M) = p^{-nd}\tilde{G}\left(\bar{\lambda}\left(\frac{p}{M},\lambda\right),\frac{p}{M}\right) \tag{4.32}$$

This allows us to rewrite the Callan-Symanzik equation as

$$\left(z\frac{\partial}{\partial z} - n\gamma(\lambda(z,\bar{\lambda}))\right)\tilde{G}(z,\bar{\lambda}) = 0 \tag{4.33}$$

where $z = \frac{p}{M}$ and $\lambda(z,\bar{\lambda})$ is the inverse of $\bar{\lambda}(z,\lambda)$. This equation has solution

$$\tilde{G}(z,\bar{\lambda}) = p^{-nd}\exp\left[\int_1^z n\gamma(\lambda(z',\bar{\lambda}))\frac{\mathrm{d}z'}{z'}\right]\mathcal{G}(\bar{\lambda}) \tag{4.34}$$

where $\mathcal{G}(\bar{\lambda})$ is a dimensionless function that depends on the dynamics of the QFT in question and the definition of the theory at the subtraction point. Thus the full solution for an n-point Green function is

$$G_R^{(n)}(p;\lambda,M) = p^{-nd}\mathcal{G}(\bar{\lambda}(p,\lambda))\exp\left[\int_1^z n\gamma(\lambda(z',\bar{\lambda}))\frac{\mathrm{d}z'}{z'}\right] \tag{4.35}$$

Equivalently, this may be written explicitly in terms of momenta as

$$G_R^{(n)}(p;\lambda,M) = p^{-nd}\mathcal{G}(\bar{\lambda}(p,\lambda))\exp\left[\int_{p'=M}^{p'=p} n\gamma(\bar{\lambda}(p',\lambda))\mathrm{d}\log\left(\frac{p'}{M}\right)\right] \tag{4.36}$$

Let's consider an explicit example for the 2-point Green function of this solution. The Callan-Symanzik equation for this function is (using the second form, in Eq. (4.11))

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} - 2(1+\gamma(\lambda))\right)G_R^{(2)} = 0 \tag{4.37}$$

whose solution, using Eq. (4.36), is

$$G_R^{(2)}(p;\lambda,M) = \frac{i}{p^2}\mathcal{G}(\bar{\lambda}(p,\lambda))\exp\left[2\int_{p'=M}^{p'=p}\gamma(\bar{\lambda}(p',\lambda))\mathrm{d}\log\left(\frac{p'}{M}\right)\right] \tag{4.38}$$

We'll now do some algebra to ensure that our ansatz in Eq. (4.38) actually solves the Callan-Symanzik equation Eq. (4.37). This will require the use of the following change-of-variables:

$$\int_M^p \mathrm{d}\log\left(\frac{p'}{M}\right) = \int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(p,\lambda)}\frac{\mathrm{d}\lambda'}{\beta(\lambda')}$$

And now for some algebra:

$$
\begin{aligned}
0 &= \left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2\left(1-\gamma(\lambda)\right)\right) G_R^{(2)}(p;\lambda,M) \\
&= \left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2\left(1-\gamma(\lambda)\right)\right) \mathcal{G}(\bar\lambda(p;\lambda)) \exp\left[-\int_{p'=M}^{p'=p} 2\left(1-\gamma(\bar\lambda(p';\lambda))\right) \mathrm{d}\log\left(\frac{p'}{M}\right)\right] \\
&= \left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2\left(1-\gamma(\lambda)\right)\right) \mathcal{G}(\bar\lambda(p;\lambda)) \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right) \frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]
\end{aligned}
$$

For the sake of clarity, I'll do each of the above terms separately. First, the derivative w.r.t. p:

$$
\begin{aligned}
\frac{\partial G_R^{(2)}}{\partial p} = \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar\lambda}\frac{\partial\bar\lambda}{\partial p} \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\
+ \mathcal{G}(\bar\lambda(p,\lambda)) \left(-\exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \frac{\partial}{\partial p}\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]\right)
\end{aligned}
$$

$$
= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar\lambda}\frac{\partial\bar\lambda}{\partial p} \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] - G_R^{(2)} \cdot \left(\frac{2\left(1-\gamma(\bar\lambda(p,\lambda))\right)}{\beta(\bar\lambda(p,\lambda))}\frac{\partial\bar\lambda}{\partial p}\right)
$$

$$
\frac{\partial G_R^{(2)}}{\partial p} + G_R^{(2)} \cdot \left(\frac{2\left(1-\gamma(\bar\lambda(p,\lambda))\right)}{\beta(\bar\lambda(p,\lambda))}\frac{\partial\bar\lambda}{\partial p}\right) = \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar\lambda}\frac{\partial\bar\lambda}{\partial p} \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]
$$

But we know from the renormalization group equation that $\frac{\beta(\bar\lambda(p,\lambda))}{p} = \frac{\partial\bar\lambda}{\partial p}$ and therefore

$$
\begin{aligned}
\frac{\partial G_R^{(2)}}{\partial p} + G_R^{(2)} \cdot \left(\frac{2\left(1-\gamma(\bar\lambda(p,\lambda))\right)}{\beta(\bar\lambda(p,\lambda))}\frac{\partial\bar\lambda}{\partial p}\right) &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar\lambda}\frac{\partial\bar\lambda}{\partial p} \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\
\frac{\partial G_R^{(2)}}{\partial p} + G_R^{(2)} \cdot \left(\frac{2\left(1-\gamma(\bar\lambda(p,\lambda))\right)}{\beta(\bar\lambda(p,\lambda))}\frac{\beta(\bar\lambda(p,\lambda))}{p}\right) &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar\lambda}\frac{\beta(\bar\lambda(p,\lambda))}{p} \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\
p\frac{\partial G_R^{(2)}}{\partial p} + G_R^{(2)} \cdot \left(2\left(1-\gamma(\bar\lambda(p,\lambda))\right)\right) &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar\lambda}\beta(\bar\lambda(p,\lambda)) \exp\left[-\int_{\bar\lambda(\lambda,M)}^{\bar\lambda(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]
\end{aligned}
\tag{4.39}
$$

Now perform the derivative w.r.t. λ:

$$\begin{aligned}\frac{\partial G_R^{(2)}}{\partial \lambda} &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar{\lambda}}\frac{\partial \bar{\lambda}}{\partial \lambda}\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\ &\qquad + \mathcal{G}(\bar{\lambda}(p,\lambda))\left(-\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]\frac{\partial}{\partial\lambda}\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]\right) \\ &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar{\lambda}}\frac{\partial \bar{\lambda}}{\partial \lambda}\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\ &\qquad - G_R^{(2)}\cdot\left(\frac{2\left(1-\gamma(\bar{\lambda}(p,\lambda))\right)}{\beta(\bar{\lambda}(p,\lambda))}\frac{\partial\bar{\lambda}(p,\lambda)}{\partial\lambda}\right) + G_R^{(2)}\cdot\left(\frac{2\left(1-\gamma(\bar{\lambda}(M,\lambda))\right)}{\beta(\bar{\lambda}(M,\lambda))}\frac{\partial\bar{\lambda}(M,\lambda)}{\partial\lambda}\right) \\ &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar{\lambda}}\frac{\partial \bar{\lambda}}{\partial \lambda}\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] - G_R^{(2)}\cdot\left(\frac{2\left(1-\gamma(\bar{\lambda}(p,\lambda))\right)}{\beta(\bar{\lambda}(p,\lambda))}\frac{\partial\bar{\lambda}}{\partial\lambda}\right) + G_R^{(2)}\cdot\left(\frac{2\left(1-\gamma(\lambda)\right)}{\beta(\lambda)}\right)\end{aligned}$$

where I've used the fact that $\bar{\lambda}(M,\lambda)=\lambda$. By the definition of the β-function, we can state

$$\frac{\beta(\bar{\lambda}(p,\lambda))}{\beta(\lambda)} = \frac{\partial\bar{\lambda}}{\partial\lambda} \qquad \Rightarrow \qquad \beta(\bar{\lambda}(p,\lambda)) = \beta(\lambda)\frac{\partial\bar{\lambda}}{\partial\lambda}$$

and thus the above expression can be rearranged:

$$\begin{aligned}\frac{\partial G_R^{(2)}}{\partial\lambda} + G_R^{(2)}\cdot\left(\frac{2\left(1-\gamma(\bar{\lambda}(p,\lambda))\right)}{\beta(\bar{\lambda}(p,\lambda))}\frac{\partial\bar{\lambda}}{\partial\lambda} - \frac{2\left(1-\gamma(\lambda)\right)}{\beta(\lambda)}\right) &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar{\lambda}}\frac{\partial \bar{\lambda}}{\partial \lambda}\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\ \beta(\lambda)\frac{\partial G_R^{(2)}}{\partial\lambda} + G_R^{(2)}\cdot\left(\frac{2\left(1-\gamma(\bar{\lambda}(p,\lambda))\right)}{\beta(\bar{\lambda}(p,\lambda))}\beta(\lambda)\frac{\partial\bar{\lambda}}{\partial\lambda} - 2\left(1-\gamma(\lambda)\right)\right) &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar{\lambda}}\frac{\partial \bar{\lambda}}{\partial \lambda}\beta(\lambda)\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right] \\ \beta(\lambda)\frac{\partial G_R^{(2)}}{\partial\lambda} + G_R^{(2)}\cdot\left(2\left(1-\gamma(\bar{\lambda})\right) - 2\left(1-\gamma(\lambda)\right)\right) &= \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}\bar{\lambda}}\beta(\bar{\lambda})\exp\left[-\int_{\bar{\lambda}(\lambda,M)}^{\bar{\lambda}(\lambda,p)} 2\left(1-\gamma(\lambda')\right)\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\right]\end{aligned} \tag{4.40}$$

Notice that the RHS of Eq. (4.39) and (4.40) are equal which allows us to write:

$$\begin{aligned}\beta(\lambda)\frac{\partial G_R^{(2)}}{\partial\lambda} + G_R^{(2)}\cdot\left(2\left(1-\gamma(\bar{\lambda})\right) - 2\left(1-\gamma(\lambda)\right)\right) &= p\frac{\partial G_R^{(2)}}{\partial p} + G_R^{(2)}\cdot\left(2\left(1-\gamma(\bar{\lambda}(p,\lambda))\right)\right) \\ \beta(\lambda)\frac{\partial G_R^{(2)}}{\partial\lambda} - 2\left(1-\gamma(\lambda)\right)G_R^{(2)} &= p\frac{\partial G_R^{(2)}}{\partial p} \\ -p\frac{\partial G_R^{(2)}}{\partial p} + \beta(\lambda)\frac{\partial G_R^{(2)}}{\partial\lambda} - 2\left(1-\gamma(\lambda)\right)G_R^{(2)} &= 0 \\ \left(p\frac{\partial G_R^{(2)}}{\partial p} - \beta(\lambda)\frac{\partial G_R^{(2)}}{\partial\lambda} + 2\left(1-\gamma(\lambda)\right)\right)G_R^{(2)} &= 0\end{aligned}$$

and thus we've recovered the Callan-Symanzik equation. Whew! That was quite a bit of work, but in doing it we've assured ourselves of the solution to the Callan-Symanzik equation and have learned quite a bit about the physics of running couplings that it encodes.

One last thing. There is still the matter of the undetermined function $\mathcal{G}$, corresponding to the initial condition. This is where renormalization deviates from a standard format into the realm of art. The initial condition is determined by the renormalization point, which in turn is decided based on the experimental data used to define the physical question being answered by the QFT. As an example, consider the 4-point Green function in ϕ^4 theory. As before, by dimensional analysis we can write the 4-point Green function at the subtraction point as

$$G_R^{(4)}(p;\lambda,M) = \left(\frac{i}{p^2}\right)^4 (-i\lambda)\Bigg]_{p^2=M^2} \tag{4.41}$$

which obeys the Callan-Symanzik equation:

$$0 = \left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + 4\gamma(\lambda)\right)G_R^{(4)} = \left(-p\frac{\partial}{\partial p} + \beta(\lambda)\frac{\partial}{\partial\lambda} - 8 + 4\gamma(\lambda)\right)G_R^{(4)}$$

As before, the solution will be

$$G_R^{(4)}(p;\lambda,M) = \left(\frac{i}{p^2}\right)^4 \mathcal{G}(\bar\lambda(p,\lambda))\exp\left[4\int_{p'=M}^{p'=p}\gamma(\bar\lambda(p',\lambda))\mathrm{d}\log\left(\frac{p'}{M}\right)\right] \tag{4.42}$$

Note that as we take the limit $p\to M$, Eq. (4.42) must approach Eq. (4.41) and thus

$$\mathcal{G}(\bar\lambda(M,\lambda)) = \mathcal{G}(\lambda) = -i\lambda \qquad\Rightarrow\qquad \mathcal{G}(\bar\lambda(p,\lambda)) = -i\bar\lambda + O(\bar\lambda^2) \tag{4.43}$$

Thus we see that the running coupling literally functions as an effective coupling at the relevant momentum scale. This section was rather long and contained several important points throughout. I've attempted to summarize these as best I could below

- The Callan-Symanzik equation can be viewed as a statement that the bare Green function is independent of the renormalization point (which itself is a statement that physics cares not about our arbitrary points at which we define the theory)
- As the momenta are rescaled from the subtraction point to the relevant scales for a given problem, the Green function evolves along a flow in theory space; in addition to its classical scaling, it receives a quantum mechanical contribution through the anomalous dimension
- The effective coupling (also known as the running coupling) is given by the expression

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\frac{p}{M}\right)}\bar g(g,p) = \beta(\bar g(g,p)) \qquad \text{with} \qquad \bar g(g,p=M) = g \tag{4.44}$$

- Be careful not to confuse functions of $\bar g$ for functions of g; *they are not in general the same!*

4.5 Alternatives for the running of the couplings

The β-function of a coupling contains information about the coupling under renormalization group flows. Let's first examine the weakly-coupled case, meaning we can safely assume $\beta(0) = 0$, and that it is possible to compute the first-order contribution to the β-funciton using perturbation theory. In this scenario, there are three distinct cases to consider.

- $\beta > 0$: Under the changing of scales (either decreasing μ or increasing p), the coupling becomes more and more positive. Eventually, this means that perturbative analysis breaks down and the theory becomes strongly coupled. The scale at which such a scenario takes place is known as the **Landau pole**, and it is unclear that a well-defined continuum limit (such that $\Lambda \to \infty$) truly exists. As such, these theories are known as **infrared safe** or **infrared free**. In this case, the origin represents an IR fixed point. Quantum electrodynamics is one example of such a theory; low-energy phenomena are well-described by the theory, but small-scale phenomena (on the order of the electron radius) are inaccessible due to the screening effects of virtual particles.

- $\beta < 0$: Under the changing of scales, the coupling gradually decreases in value and asymptotically approaches the fixed point at the origin. Thus the origin represents a UV fixed point. Theories that exhibit this type of running are known as **asymptotically free**, and their high-energy/small-scale physics are well-described using perturbation theory. On the other hand, their low-energy phenoma are non-perturbative and thus inaccessible using perturbation theory, a situation known as **infrared slavery**. Quantum chromodynamics is the clearest example of this phenomenon. At high energies, quarks behave as essentially free particles, but at low energies are confined within the proton.

- $\beta = 0$: In this case, the coupling does not run at all. Such theories are known as finite QFTs. The renormalized couplings are completely independent of momentum scale and thus equal to the bare couplings. Very few QFTs with this property are known to exist. In 2 dimensions, conformal field theories have this property, making them particularly tractable as toy models; in four dimensions, $\mathcal{N} = 4$ SUSY gauge theories also exhibit this behavior.

With the weakly coupled cases out of the way, let's now consider the strongly coupled cases. Of course, we cannot make quantitative predictions about the global nature of the couplings as we have in the previous sections, but we can make some predictions about the couplings in the vicinity of the fixed points these theories might generate. Suppose we (somehow) know $\beta(g)$ exactly, and suppose that at some g_*, $\beta(g_*) = 0$, constituting a fixed point. In the neighborhood of the fixed point, the β-function can be expanded in a polynomial, which, at its lowest order, can be written as a linear function:

$$\beta \approx -B(g - g_*) \tag{4.45}$$

which, in turn, implies

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\frac{p}{M}\right)}\bar{g} \approx -B(\bar{g} - g_*) \tag{4.46}$$

This has a solution

$$\bar{g}(p) = g_* + C\left(\frac{M}{p}\right)^B \tag{4.47}$$

which indicates that as $p \to \infty$, $\bar{g} \to g_*$, with the behavior governed by the slope of the β-function. The exact solution for the 2-point Green function in the vicinity of the fixed point can be computed simply, using the fact that as $p \to \infty$, $\bar{g} \sim g_*$ and thus

$$\begin{aligned}
G^{(2)}(p) &= \mathcal{G}(\bar{\lambda}(p;\lambda)) \exp\left[-\int_{p'=M}^{p'=p} 2\left(1-\gamma(\bar{\lambda}(p';\lambda))\right) \mathrm{d}\log\left(\frac{p'}{M}\right)\right] \\
&\approx \mathcal{G}(\lambda_*) \exp\left[-\int_{p'=M}^{p'=p} 2\left(1-\gamma(\lambda_*)\right) \mathrm{d}\log\left(\frac{p'}{M}\right)\right] \\
&\approx \mathcal{G}(\lambda_*) \exp\left[-2\left(1-\gamma(\lambda_*)\right) \log\left(\frac{p}{M}\right)\right] \\
&\approx C \cdot \left(\frac{1}{p^2}\right)^{1-\gamma(\lambda_*)}
\end{aligned} \tag{4.48}$$

Thus we see that the Green function in the vicinity of the fixed point looks almost identical to the propagator for a standard CFT, but with a slightly modified scaling law (the anomalous dimension shows up again!).

4.6 Deriving the equations - General case

In the simplest analysis we've considered above, the theory was constructed with one species of field with one dimensionless coupling. This is of course by no means a general treatment and several phenomena are not covered by this simple analysis. A simple generalization would be a theory with multiple species of fields, which may have several types of operators with dimensionless couplings (which would correspond to $d = 4$ products of fields). The generalization of the Callan-Symanzik equations in this case is very simple. Suppose the theory had i species of field and j dimensionless couplings. As you might suspect the generalization in this case consists of i, j copies of the corresponding terms in the Callan-Symanzik equations. In concrete terms, there are j β-functions and i anomalous dimensions. There is another generalization to consider. The n-point Green function can contain different species of field. Suppose a given n-point Green function has n_i copies of a given field i, such that

$$\sum_i n_i = n$$

where the sum is over all possible fields. Thus the Callan-Symanzik equation for such an n-point Green function is

$$\left(M\frac{\partial}{\partial M} + \sum_j \beta_j(\{\lambda_j\})\frac{\partial}{\partial \lambda_j} + \sum_i n_i\gamma_i(\{\lambda_j\})\right) G^{(n)}(p, \{\lambda_j\}, \{\varphi_i\}; M) = 0 \tag{4.49}$$

Note that the β-function and the anomalous dimension are general functions of the couplings, since all of them are dimensionless. There is one other generalization that should be considered. Suppose that a given theory contains multiple operators with the same quantum numbers (but not necessarily the same fields!). These operators may be written in terms of the bare fields and thus related to the renormalized fields using wavefunction renormalizations. However, whereas before these renormalizations were plain scalars (infinite though they may be...), now there is a new possibility, that they are matrices:

$$\varphi_0 = Z(M)\varphi \qquad \to \qquad \mathcal{O}_0^a = Z^{ab}(M)_{\mathcal{O}}\mathcal{O}^b \tag{4.50}$$

In other words, operators built out of multiple types of fields can mix under the renormalization flow. As such, the anomalous dimension must similarly be generalized to a matrix:

$$\gamma = \frac{M}{Z}\frac{\partial Z}{\partial M} \qquad \to \qquad \gamma^{ab}_{\mathcal{O}} = M\,(Z^{ac}_{\mathcal{O}})^{-1}\frac{\partial}{\partial M}Z^{cb}_{\mathcal{O}} \tag{4.51}$$

Thus Eq. (4.49) was actually the special case that the matrix above was diagonal; in this generalized case, the anomalous dimension matrix must diagonalized, meaning it is necessary to find the 'eigenoperators' of the renormalization group flow. For the purposes of simplicity, I will in subsequent discussion assume that such a basis has been found and the matrix appropriately diagonalized so that Eq. (4.49) applies.

Now that we have generalized the dimensionless coupling case, let's consider operators with non-zero mass dimension couplings. As before, let $\mathcal{O}_i$ be some operator consisting of a product of fields, with mass dimension d_i. As such, the Lagrangian of this theory is written

$$\mathcal{L} = \mathcal{L}_0 + C_k\mathcal{O}_k \tag{4.52}$$

where $\mathcal{L}_0$ is the massless Lagrangian (*not* the bare Lagrangian), and C_i is the coupling, with mass dimension $[C_k] = 4 - d_k$. For simplicity's sake, let's suppose this theory has a single dimensionless coupling λ. Since the C_k are couplings, terms *like* a β-function should appear in the Callan-Symanzik equation for each such coupling that appears in a given n-point Green function. Indeed, such terms will appear, but we will have to coax them out. The non-zero mass dimension of the C_k coupling means that now there is an additional anomalous dimension, that of the coupling. As such, the Callan-Symanzik equation takes the form

$$\left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \sum_i n_i\gamma_i(\lambda) + \sum_k \gamma_k(\lambda)C_k\frac{\partial}{\partial C_k}\right)G^{(n)}(p,\lambda,\{C_k\},\{\varphi_i\};M) = 0 \tag{4.53}$$

where the anomalous dimension for the coupling is computed using Eq. (4.51). To tease the β-function out of this we see that by dimensional analysis, it is possible to write C_k as $C_k = \rho_k M^{4-d_k}$ and thus the Callan-Symanzik equation becomes

$$0 = \left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \sum_i n_i\gamma_i(\lambda) + \sum_k \gamma_k(\lambda)C_k\frac{\partial}{\partial C_k}\right)G^{(n)}(p,\lambda,\{C_k\},\{\varphi_i\};M) \tag{4.54}$$

$$= \left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \sum_i n_i\gamma_i(\lambda) + \sum_k \left[\gamma_k(\lambda) + d_k - 4\right]\rho_k\frac{\partial}{\partial \rho_k}\right)G^{(n)}(p,\lambda,\{\rho_k\},\{\varphi_i\};M) \tag{4.55}$$

$$= \left(M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \sum_i n_i\gamma_i(\lambda) + \sum_k \beta_k\rho_k\frac{\partial}{\partial \rho_k}\right)G^{(n)}(p,\lambda,\{\rho_k\},\{\varphi_i\};M) \tag{4.56}$$

where, in the last step, I've identified the promised β-functions

$$\beta_k = \gamma_k(\lambda) + d_k - 4 \tag{4.57}$$

In this section, we have explored the behavior of Green functions under the renormalization group flow, and how these conclusions flow from the independence of physics from the choices made in defining the theory. In doing so, we have shown how running couplings arise and form a lynchpin in the study of renormalizable quantum field theories.

5 Regularization methods

The preceding sections have been heavy on theory, but (relatively) light on computation. This section will begin the exploration of the mechanics of doing renormalization calculations. When confronted with a divergent integral, various methods of taming these divergences have been developed. Invariably, these methods involve the parameterization of the integral, making the divergence 'tunable', in some sense. No two calculations are exactly alike, an experience will inform which of these methods is best used for a given case.

5.1 Momentum cutoff

5.2 Pauli-Villars regularization

5.3 Dimensional regularization

6 Renormalization schemes

6.1 On-shell scheme

6.2 MS & $\overline{\text{MS}}$ schemes

6.3 DR scheme

7 Computing physical quantities

8 Effective field theory

8.1 Ultraviolet completions

8.2 GSW vs. 4-Fermi theory

Part II

Examples

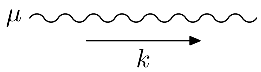
9 Quantum Electrodynamics

9.1 Preliminaries

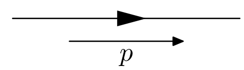
The QED Lagrangian is an Abelian gauge theory with a gauged $U(1)$ symmetry:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\bar{\psi}A\!\!\!/\psi \tag{9.1}$$

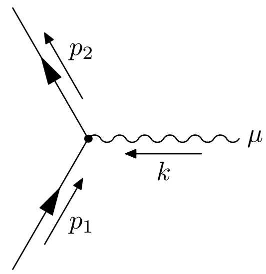
which corresponds to Feynman rules



$$= \frac{-i}{k^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$



$$= \frac{-i\left(k\!\!\!/ + m\right)}{k^2 + i\epsilon}$$



$$= -ie\gamma_\mu$$

In order to absorb the divergences of our theory, I'll now introduce the bare and renormalized fields. We are assuming multiplicative renormalization so that the fields are defined as

$$\psi_0 = \sqrt{Z_2}\psi \qquad\qquad A_0^\mu = \sqrt{Z_3}A^\mu \tag{9.2}$$

Plugging these into the Lagrangian gives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu,0}F_0^{\mu\nu} + \bar{\psi}_0\left(i\partial\!\!\!/ - m_0\right)\psi_0 - e_0\bar{\psi}_0A\!\!\!/_0\psi_0 = -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - m_0Z_2\bar{\psi}\psi - e_0Z_2Z_3^{\frac{1}{2}}\bar{\psi}A\!\!\!/\psi \tag{9.3}$$

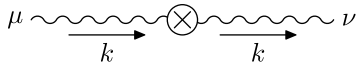
The couplings are renormalized as

$$m_0Z_2 = mZ_m \qquad\qquad e_0Z_2Z_3^{\frac{1}{2}} = eZ_1 \tag{9.4}$$

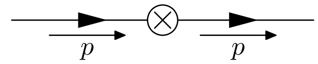
The multiplicative factors can be written in the form $Z_i = 1 + \delta_i$, thus allowing the Lagrangian to be written in terms of a renormalized Lagrangian and counterterms:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - m_0Z_2\bar{\psi}\psi - e_0Z_2Z_3^{\frac{1}{2}}\bar{\psi}A\!\!\!/\psi \\
&= -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - mZ_m\bar{\psi}\psi - eZ_1\bar{\psi}A\!\!\!/\psi \\
&= -\frac{1}{4}(1+\delta_3)F_{\mu\nu}F^{\mu\nu} + i(1+\delta_2)\bar{\psi}\partial\!\!\!/\psi - m(1+\delta_m)\bar{\psi}\psi - e(1+\delta_1)\bar{\psi}A\!\!\!/\psi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\bar{\psi}A\!\!\!/\psi - \frac{1}{4}\delta_3F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\delta_2\partial\!\!\!/ - \delta_m\right)\psi - e\delta_1\bar{\psi}A\!\!\!/\psi
\end{aligned} \tag{9.5}$$

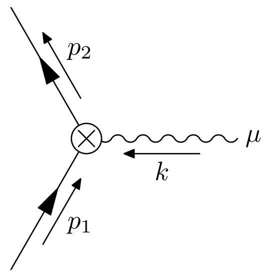
Therefore the counterterm vertex diagrams and Feynman rules are



$= -i\delta_3\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)$



$= i\left(\delta_2 \not p - \delta_m\right)$

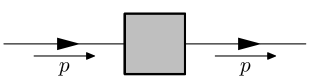


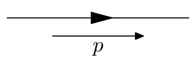
$= -ie\delta_1\gamma_\mu$

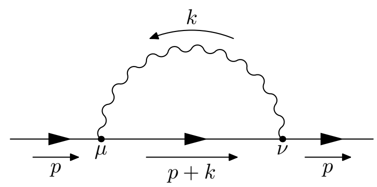
9.2 The Ward Identity

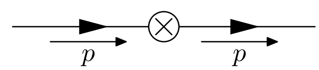
9.3 Electron Self-energy

We'll first compute the electron self-energy at one-loop order. Diagrammatically the full propagator, this takes the form



$=$


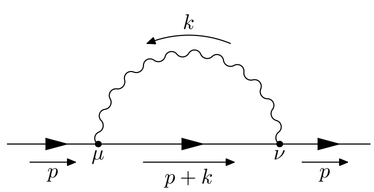
$+$


$+ \cdots +$


(9.6)

This may be reorganized into a 1-loop order

Let's focus on the 1-loop-corrected propagator; after amputating the external legs, the diagram gives

$i\Sigma(p) =$


$$= \int (-ie\gamma^\nu)\frac{i(\not p + \not k + m)}{(p+k)^2 - m^2}(-ie\gamma^\mu)\frac{-ig_{\mu\nu}}{k^2}\frac{\mathrm{d}^4k}{(2\pi)^4} \tag{9.7}$$

Since we are interested only in the divergent piece, and the vertices do not involve reference to the external momentum p, it is safe to set $p = 0$ and thus the integral simplifies to

$$\begin{aligned}
i\Sigma(p=0) &= -e^2 \int \gamma_\mu \frac{\not k + m}{\left(k^2 - m^2\right)k^2}\gamma^\mu \frac{\mathrm{d}^4k}{(2\pi)^4} \\
&= -e^2\left[\gamma_\mu\gamma^\alpha\gamma^\mu \int \frac{k_\alpha}{\left(k^2-m^2\right)k^2}\frac{\mathrm{d}^4k}{(2\pi)^4} + m\gamma_\mu\gamma^\mu \int \frac{1}{\left(k^2-m^2\right)k^2}\frac{\mathrm{d}^4k}{(2\pi)^4}\right]
\end{aligned}$$

We now employ dimensional regularization and replace the above integrals with their D-dimensional equivalents. Using the identities $(16.2-3)$, we can therefore simplify the γ-matrices:

$$\begin{aligned}
i\Sigma(p=0) &= -\frac{e^2}{(2\pi)^D}\left[\gamma_\mu\gamma^\alpha\gamma^\mu \int \frac{k_\alpha}{\left(k^2-m^2\right)k^2}\mathrm{d}^Dk + m\gamma_\mu\gamma^\mu \int \frac{1}{\left(k^2-m^2\right)k^2}\mathrm{d}^Dk\right] \\
&= -\frac{e^2}{(2\pi)^D}\left[-(D-2)\gamma^\alpha \int \frac{k_\alpha}{\left(k^2-m^2\right)k^2}\mathrm{d}^Dk + mD \int \frac{1}{\left(k^2-m^2\right)k^2}\mathrm{d}^Dk\right]
\end{aligned}$$

Now consolidate the denominators using the Feynman parameterization. Since the denominator of the integrand is the same for both integrals above, we need only do this once. Let $A = (k^2 - m^2)$ and $B = k^2$ and thus we see

$$\frac{1}{\left(k^2 - m^2\right) k^2} = \int_0^1 \frac{\mathrm{d}x}{\left[x\left(k^2 - m^2\right) + (1-x)k^2\right]^2} = \int_0^1 \frac{\mathrm{d}x}{\left[k^2 - xm^2\right]^2}$$

9.3.1 Custodial chiral symmetry

9.4 Vacuum polarization

9.5 Vertex correction

9.6 Renormalization group flows

10 Scalar Quantum Electrodynamics

10.1 Preliminaries

10.2 Electron self-energy

10.3 Vacuum polarization

10.4 3-point vertex correction

10.5 4-point vertex correction

10.6 The Ward Identity

10.7 Renormalization group flows

11 Yukawa theory

11.1 Preliminaries

11.2 Scalar self-energy

11.3 Electron Self-energy

11.4 Renormalization group flows

12 Yang-Mills theory

12.1 Preliminaries

12.2 Gauge boson self-energy

12.3 3-point vertex correction

12.4 4-point vertex correction

12.5 Renormalization group flows

Part III
Advanced topics

13 Effective potentials

13.1 Spontaneous symmetry breaking

13.2 Coleman-Weinberg potential

14 Anomalies

14.1 Chiral anomaly

14.2 Conformal anomaly

14.3 Anomaly cancellation in the Standard Model

15 SUSY

15.1 Soft SUSY breaking

15.2 Non-Renormalization theorems

Part IV
Appendix

16 Common formulae

16.1 D-dimensional formulae

Contractions of γ-matrices in D-dimensions are subject to subtle changes based on the fact that

$$g^{\mu\nu}g_{\mu\nu} = D \tag{16.1}$$

As such we see that

$$\begin{aligned}
\gamma^{\mu}\gamma_{\mu} &= D && (16.2)\\
\gamma^{\mu}\gamma^{\nu}\gamma_{\mu} &= -(D-2)\gamma^{\nu} && (16.3)\\
\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma_{\mu} &= 4g^{\nu\rho} - (4-D)\gamma^{\nu}\gamma^{\rho} && (16.4)\\
\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma_{\mu} &= -2\gamma^{\sigma}\gamma^{\rho}\gamma^{\nu} + (4-D)\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} && (16.5)
\end{aligned}$$

For reference, here is the solid angle in a D-dimensional sphere:

$$\Omega_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \tag{16.6}$$

and a table summarizing results:

dim	Ω_D
$D=1$	2
$D=2$	π
$D=3$	$\frac{4\pi}{3}$
$D=4$	$\frac{\pi^2}{2}$

(16.7)

16.2 Numerator replacements

When doing dimensional regularization, symmetry properties of the integrand and measure allow simplifications to be made. Namely:

$$\begin{aligned}
k^n &= 0 \quad \text{for any odd power of } n && (16.8)\\
k^{\mu}k^{\nu} &= \frac{1}{D}k^2 g^{\mu\nu} && (16.9)\\
k^{\mu}k^{\nu}k^{\rho}k^{\sigma} &= \frac{1}{D(D+2)}(k^2)^2\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}\right) && (16.10)
\end{aligned}$$

16.3 Taylor series expansions

In dimensional regularization, when taking the $D \to 4$ limit (equivalently, for $D = 4-2\epsilon$, taking the $\epsilon \to 0$ limit), the following expansions may be of use:

$$\left(\frac{1}{\Delta}\right)^{\epsilon} = 1 - \epsilon \log \Delta \tag{16.11}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon) \tag{16.12}$$

This combination frequently appears in calculations:

$$\begin{aligned} \frac{\Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}}\left(\frac{1}{\Delta}\right)^{2-\frac{D}{2}} &= \frac{\Gamma(\epsilon)}{(4\pi)^2}\left(\frac{4\pi}{\Delta}\right)^{\epsilon} \\ &= \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{\Delta}\right) - \gamma_E + +O(\epsilon)\right) \\ &= \frac{1}{(4\pi)^2}\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma_E}}{\Delta}\right) + O(\epsilon)\right) \end{aligned} \tag{16.13}$$

In the MS ($\overline{\text{MS}}$) scheme, the first (and second) term(s) represent the divergence to be cancelled by the relevant counterterms.

16.4 Miscellaneous formulae

Gamma function identities

$$\Gamma(n) \equiv \int_0^{\infty} x^{n-1} e^{-x} \mathrm{d}x \tag{16.14}$$

$$\Gamma(1) = 1 \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad x\Gamma(x) = \Gamma(x+1) \qquad \Gamma(x+1) = x! \tag{16.15}$$

Euler β-function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \equiv \int_0^1 (1-x)^{m-1} x^{n-1} \mathrm{d}x \tag{16.16}$$

17 Integration

17.1 Feynman parameterization

$$\frac{1}{AB} = \int_0^1 \frac{1}{[xA + (1-x)B]^2} \mathrm{d}x \tag{17.1}$$

$$\frac{1}{A_1 A_2 \ldots A_n} = \int_0^1 \delta\left(\sum_i x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \tag{17.2}$$

17.2 *D*-dimensional integrals

When doing dimensional regularization, the integration measure is separated into radial and angular pieces as follows:

$$\mathrm{d}^D k = k^{D-1} \mathrm{d}k \mathrm{d}\Omega_D \tag{17.3}$$

Integrating over all angles yields a factor of Ω_D, given in the table in the previous section. The radial portion, for an integrand with typical form, gives

$$\int \frac{k^m}{(k^2+\Delta)^n} \mathrm{d}k = \Delta^{\frac{m+1}{2}-n} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(n-\frac{m+1}{2}\right)}{2\Gamma(n)} \tag{17.4}$$

A handy set of common integrals is listed below:

$$\int \frac{1}{(k^2-\Delta)^n} \frac{\mathrm{d}^D k}{(2\pi)^D} = \frac{(-1)^n i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}} \tag{17.5}$$

$$\int \frac{k^2}{(k^2-\Delta)^n} \frac{\mathrm{d}^D k}{(2\pi)^D} = \frac{(-1)^{n-1} i}{(4\pi)^{\frac{D}{2}}} \frac{D}{2} \frac{\Gamma\left(n-\frac{D}{2}-1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-1} \tag{17.6}$$

$$\int \frac{k^\mu k^\nu}{(k^2-\Delta)^n} \frac{\mathrm{d}^D k}{(2\pi)^D} = \frac{(-1)^{n-1} i}{(4\pi)^{\frac{D}{2}}} \frac{g^{\mu\nu}}{2} \frac{\Gamma\left(n-\frac{D}{2}-1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-1} \tag{17.7}$$

$$\int \frac{(k^2)^2}{(k^2-\Delta)^n} \frac{\mathrm{d}^D k}{(2\pi)^D} = \frac{(-1)^n i}{(4\pi)^{\frac{D}{2}}} \frac{D(D+2)}{4} \frac{\Gamma\left(n-\frac{D}{2}-2\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-2} \tag{17.8}$$

$$\begin{aligned}\int \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2-\Delta)^n} \frac{\mathrm{d}^D k}{(2\pi)^D} &= \frac{(-1)^n i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(n-\frac{D}{2}-2\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{D}{2}-2} \\ &\quad \times \frac{1}{4}\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}\right)\end{aligned} \tag{17.9}$$

17.3 Passarino-Veltman integrals

This is a method in a larger class of methods known as **tensor reduction** methods. It essentially rewrites large, complicated integrals as a linear combination of master scalar integrals (that you can look up in a table) times tensorial factors, making the overall calculation easier. Some of the *D*-dimensional integrals listed above are Passarino-Veltman integrals. In this appendix, I'll step through the general procedure for massive fields. Massless fields bring some difficulties into the game (that's somewhat ironic) that I won't consider here.