CSE 3318 Notes 14: Minimum Spanning Trees

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CLRS 19.3, 21.1-21.2

14.A. CONCEPTS

Given a weighted, connected, undirected graph, find a minimum (total) weight free tree connecting the vertices. (AKA bottleneck shortest path tree)

Cut Property: Suppose S and T partition V such that

1. $S \cap T = \emptyset$
2. $S \cup T = V$
3. $|S| > 0$ and $|T| > 0$

then there is some MST that includes a minimum weight edge {s, t} with $s \in S$ and $t \in T$.

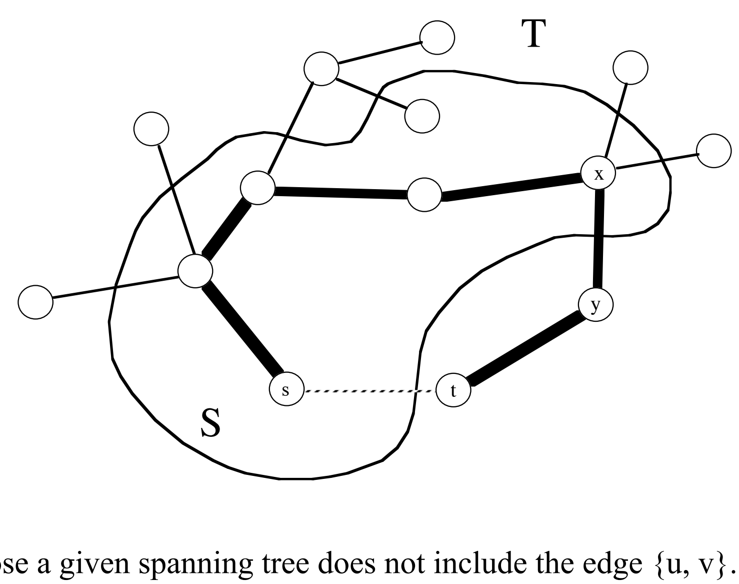
Proof:

Suppose there is a partition with a minimum weight edge {s, t}.

A spanning tree without {s, t} must still have a path between s and t.

Since $s \in S$ and $t \in T$, there must be at least one edge {x, y} on this path with $x \in S$ and $y \in T$.

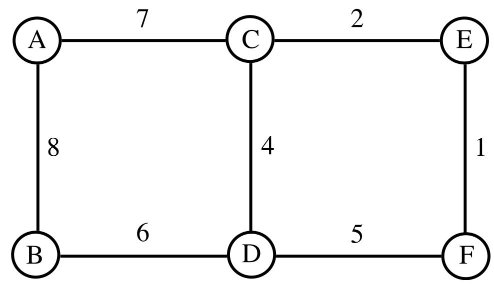
By removing {x, y} and including {s, t}, a spanning tree whose total weight is no larger is obtained. •••



Cycle Property: Suppose a given spanning tree does not include the edge {u, v}. If the weight of {u, v} is no larger than the weight of an edge {x, y} on the unique spanning tree path between u and v, then replacing {x, y} with {u, v} yields a spanning tree whose weight does not exceed that of the original spanning tree.

Proof: Including {u, v} in the set of chosen edges introduces a cycle, but removing {x, y} will remove the cycle to yield a modified tree whose weight is no larger.

The proof suggests a slow approach - iteratively find and remove a maximum weight edge from some remaining cycle:



14.B. PRIM'S ALGORITHM – Three versions

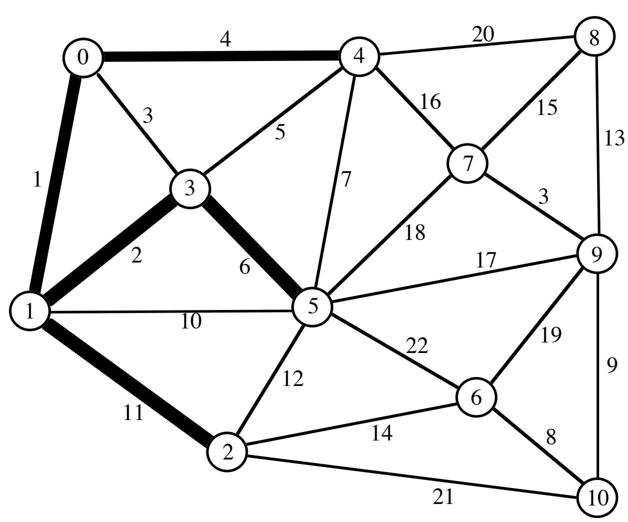
Prim's algorithm applies the cut property by having S include those vertices connected by a subtree of the eventual MST and T contains vertices that have not yet been included. A minimum weight edge from S to T will be used to move one vertex from T to S

1. "Memoryless" – Only saves partial MST and current partition.
   (`https://ranger.uta.edu/~weems/NOTES3318/primMemoryless.c`)

Place any vertex $x \in V$ in S.
$T = V - \{x\}$
while $T \neq \emptyset$
    Find the minimum weight edge {s, t} over all $t \in T$ and all $s \in S$. (Scan adj. list for each t)
    Include {s, t} in MST.
    $T = T - \{t\}$
    $S = S \cup \{t\}$

Since no substantial data structures are used, this takes $\Theta(EV)$ time.

Which edge does Prim's algorithm select next?

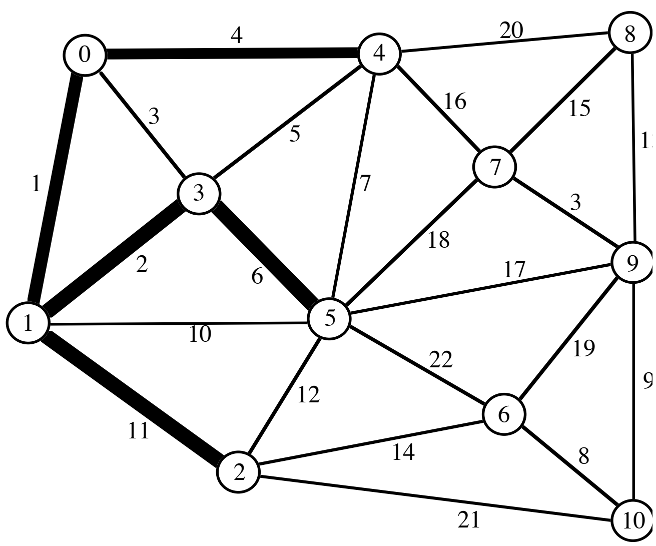


2. Maintains T-table that provides the closest vertex in S for each vertex in T.
   (`https://ranger.uta.edu/~weems/NOTES3318/primTable.c` traverses adjacency lists)

Eliminates scanning all T adjacency lists in every phase, but still scans the adjacency list of the last vertex moved from T to S.

```
Place any vertex x ∈ V in S.
T = V – {x}
for each t ∈ T
        Initialize T-table entry with weight of {t, x} (or ∞ if non-existent) and x as best-S-neighbor
while T ≠ ∅
        Scan T-table entries for the minimum weight edge {t, best-S-neighbor[t]}
                over all t ∈ T and all s ∈ S.
        Include edge {t, best-S-neighbor[t]} in MST.
        T = T – {t}
        S = S ∪ {t}
        for each vertex x in adjacency list of t
                if x ∈ T and weight of {x, t} < T-weight[x]
                        T-weight[x] = weight of {x, t}
                        best-S-neighbor[x] = t
```

What are the T-table contents before and after the next MST vertex is selected?



6	14 (2)
7	16 (4)
8	20 (4)
9	17 (5)
10	21 (2)

Analysis:

Initializing the T-table takes Θ(V).
Scans of T-table entries contribute $\Theta(V^2)$.
Traversals of adjacency lists contribute Θ(E).
$\Theta\left(V^2 + E\right) = \Theta(V^2)$ overall worst-case.

3. Replace T-table by a min-heap.
   (`https://ranger.uta.edu/~weems/NOTES3318/primHeap.cpp`)

The time for updating for best-S-neighbor increases, but the time for selection of the next vertex to move from T to S improves.

```
Place any vertex x ∈ V in S.
T = V – {x}
for each t ∈ T
        Load T-heap entry with weight (as the priority) of {t, x} (or ∞ if non-existent) and x as
                best-S-neighbor
minHeapInit(T-heap)  // a fixDown at each parent node in heap
while T ≠ ∅
        Use heapExtractMin /* fixDown */ to obtain T-heap entry with the minimum weight edge
                over all t ∈ T and all s ∈ S.
        Include edge {t, best-S-neighbor[t]} in MST.
        T = T – {t}
        S = S ∪ {t}
        for each vertex x in adjacency list of t
                if x ∈ T and weight of {x, t} < T-weight[x]
                        T-weight[x] = weight of {x, t}
                        best-S-neighbor[x] = t
                        minHeapChange(T-heap)  // fixUp
```

Analysis:

Initializing the T-heap takes $\Theta(V)$.
Total cost for `heapExtractMins` is $\Theta(V \log V)$.
Traversals of adjacency lists and `minHeapChanges` contribute $\Theta(E \log V)$.
$\Theta(E \log V)$ overall worst-case, since $E > V$.

Which version is the fastest?

	Theory	Sparse $\left(E = O(V)\right)$	Dense $\left(E = \Omega\left(V^2\right)\right)$
1.	$\Theta(EV)$	$\Theta\left(V^2\right)$	$\Theta\left(V^3\right)$
2.	$\Theta(V^2)$	$\Theta\left(V^2\right)$	$\Theta\left(V^2\right)$
3.	$\Theta(E \log V)$	$\Theta(V \log V)$	$\Theta\left(V^2 \log V\right)$

14.C. Union-Find Trees to Represent Disjoint Subsets

Abstraction:

Set of *n* elements: 0 . . *n* - 1

Initially all elements are in *n* different subsets

`find(i)` - Returns integer ("leader") indicating which subset includes `i`

`i` and `j` are in the same subset ⇔ `find(i)==find(j)`

`union(i,j)` - Takes the set union of the subsets with leaders `i` and `j`.

Results of previous `find`s are invalid after a `union`.

Implementation 1: (https://ranger.uta.edu/~weems/NOTES3318/uf1.c)

Initialization:

```
for (i=0; i<n; i++)
  id[i]=i;
```

```
find(i):

      return id[i];

unionFunc(i,j):

      for (k=0; k<n; k++)
        if (id[k]==i)
          id[k]=j;
```

0	1	2	3	4
0	1	2	3	4

Implementation 2: (https://ranger.uta.edu/~weems/NOTES3318/uf2.c)

```
find(i):

      while (id[i]!=i)
        i=id[i];
      return i;

unionFunc(i,j):

      id[i]=j;
```

0	1	2	3	4
0	1	2	3	4

Implementation 3: (https://ranger.uta.edu/~weems/NOTES3318/uf3.c)

Initialization:

```
for (i=0; i<n; i++)
{
  id[i]=i;
  sz[i]=1;
}
```

```
find(x):

      for (i=x;
           id[i]!=i;
           i=id[i])
        ;
      root=i;
      // path compression - make all nodes on path
      // point directly at the root
      for (i=x;
           id[i]!=i;
           j=id[i],id[i]=root,i=j)
        ;
      return root;

unionFunc(i,j):

      if (sz[i]<sz[j])
      {
        id[i]=j;
        sz[j]+=sz[i];
      }
      else
      {
        id[j]=i;
        sz[i]+=sz[j];
      }
```

Best-case (shallow tree) and worst-case (deep tree) for a sequence of unions?

14.D. KRUSKAL'S ALGORITHM – A Simple Method for MSTs Based on Union-Find Trees

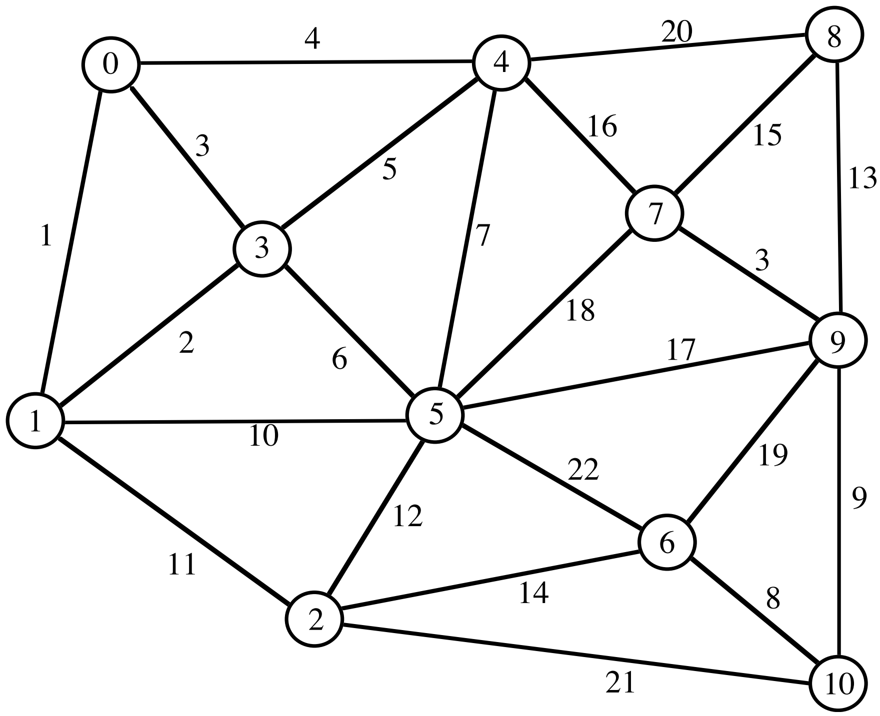
(`https://ranger.uta.edu/~weems/NOTES3318/kruskal.c`)

Sort edges in ascending weight order.

Place each vertex in its own set.

Process each edge {x, y} in sorted order:

> a=FIND(x)
> b=FIND(y)
> if a $\neq$ b
> > UNION(a,b)
> > Include {x, y} in MST



1	{0, 1}	12	{2, 5}
2	{1, 3}	13	{8, 9}
3	{0, 3}	14	{2, 6}
3	{7, 9}	----------------	
4	{0, 4}	15	{7, 8}
5	{3, 4}	16	{4, 7}
6	{3, 5}	17	{5, 9}
7	{4, 5}	18	{5, 7}
8	{6, 10}	19	{6, 9}
9	{9, 10}	20	{4, 8}
10	{1, 5}	21	{2, 10}
11	{1, 2}	22	{5, 6}

Time to sort, $\Theta(E \log V)$, dominates computation