CSE 3318 Notes 15: Shortest Paths

(Last updated 8/17/22 3:03 PM)

CLRS 22.3, 23.2

15.A. CONCEPTS

(Aside: `https://dl-acm-org.ezproxy.uta.edu/doi/10.1145/2530531`)

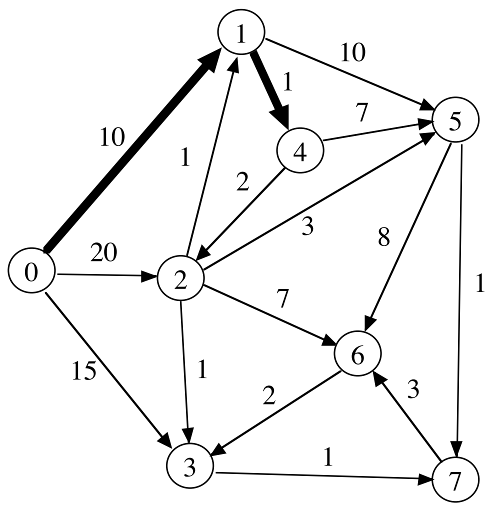
Input:

Directed graph with *non-negative* edge weights (stored as adj. matrix for Floyd-Warshall)
Dijkstra – source vertex

Output:

Dijkstra – tree that gives a shortest path from source to each vertex
Floyd-Warshall – shortest path between each pair of vertices ("all-pairs") as matrix

15.B. DIJKSTRA'S ALGORITHM – three versions



0	1	2	3	4	5	6	7
0(-)	∞	∞	∞	∞	∞	∞	∞
*	10(0)	20(0)	15(0)				
	*			11(1)	20(1)		
		13(4)		*	18(4)		
========	========	========	========	========	========	========	========
		*	14(2)		16(2)	20(2)	
			*				15(3)
						18(7)	*
					*		
						*	

Similar to Prim's MST:

S = vertices whose shortest path is known (initially just the source)

Length of path
Predecessor (vertex) on path (AKA shortest path tree)

T = vertices whose shortest path is not known

Each phase moves a T vertex to S by virtue of that vertex having the shortest path among all T vertices.

Third version may be viewed as being BFS with the FIFO queue replaced by a priority queue.

1. "Memoryless" – Only saves shortest path tree and current partition.
   (`https://ranger.uta.edu/~weems/NOTES3318/dijkstraMemoryless.c`)

Place desired source vertex $x \in V$ in S
T = V – {x}
x.distance = 0
x.pred = (-1)
while $T \neq \emptyset$
    Find the edge (s, t) over all $t \in T$ and all $s \in S$ with minimum value for s.distance + weight(s, t)
        (i.e. scan adj. list for each s)
    t.distance = s.distance + weight(s, t)
    t.pred = s
    T = T – {t}
    $S = S \cup \{t\}$

Since no substantial data structures are used, this takes $\Theta(EV)$ time.

2. Maintains T-table that provides the predecessor vertex in S for each vertex $t \in T$ to give the shortest possible path through S to t. (`https://ranger.uta.edu/~weems/NOTES3318/dijkstraTable.c`)

Eliminates scanning all S adjacency lists in every phase, but still scans the list of the last vertex moved from T to S.

Place desired source vertex $x \in V$ in S
T = V – {x}
x.distance = 0
x.pred = (-1)
for each $t \in T$
    Initialize t.distance with weight of (x, t) (or ∞ if non-existent) and t.pred = x

```
while T ≠ ∅
        Scan T entries to find vertex t with minimum value for t.distance
        T = T – {t}
        S = S ∪ {t}
        for each vertex x in adjacency list of t (i.e. (t, x))
                if x ∈ T and t.distance + weight(t, x) < x.distance
                        x.distance = t.distance + weight(t, x)
                        x.pred = t
```

Analysis:

Initializing the T-table takes Θ(V).
Scans of T-table entries contribute $\Theta(V^2)$.
Traversals of adjacency lists contribute Θ(E).
$\Theta\left(V^2 + E\right) = \Theta(V^2)$ overall worst-case.

3. Replace T-table by a min-heap.
(`https://ranger.uta.edu/~weems/NOTES3318/dijkstraHeap.cpp`)

The time for updating distances and predecessors increases, but the time for selection of the next vertex to move from T to S improves.

```
Place desired source vertex x ∈ V in S
T = V – {x}
x.distance = 0
x.pred = (-1)
for each t ∈ T
        Initialize T-heap with weight (as the priority) of (x, t) (or ∞ if non-existent) and t.pred = x
minHeapInit(T-heap)  // a fixDown at each parent node in heap
while T ≠ ∅
        Use heapExtractMin /* fixDown */ to obtain T-heap entry with minimum t.distance
        T = T – {t}
        S = S ∪ {t}
        for each vertex x in adjacency list of t (i.e. (t, x))
                if x ∈ T and t.distance + weight(t, x) < x.distance
                        x.distance = t.distance + weight(t, x)
                        x.pred = t
                        minHeapChange(T-heap)  // fixUp
```

Analysis:

Initializing the T-heap takes Θ(V).
Total cost for `heapExtractMin`s is Θ(V log V).
Traversals of adjacency lists and `minHeapChange`s contribute Θ(E log V).
$\Theta(E \log V)$ overall worst-case, since E > V.

Which version is the fastest?

	Theory	Sparse $(E = O(V))$	Dense $(E = \Omega(V^2))$
1.	$\Theta(EV)$	$\Theta(V^2)$	$\Theta(V^3)$
2.	$\Theta(V^2)$	$\Theta(V^2)$	$\Theta(V^2)$
3.	$\Theta(E \log V)$	$\Theta(V \log V)$	$\Theta(V^2 \log V)$

15.C. Floyd-Warshall Algorithm

Based on adjacency matrices. Will examine three versions:

Warshall's Algorithm – After $\Theta(V^3)$ preprocessing, processes each path *existence* query in $\Theta(1)$ time.

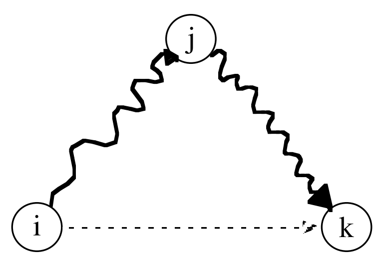
Warshall's Algorithm with Successors (or predecessors or transitive vertices) - After $\Theta(V^3)$ preprocessing, provides a path in response to a path existence query in O(V) time (similar to dynamic programming backtrace).

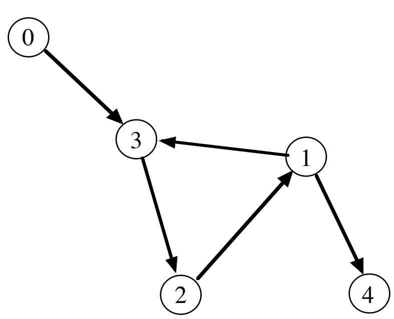
Floyd-Warshall Algorithm (with Successors) - After $\Theta(V^3)$ preprocessing, provides each *shortest* path in O(V) time.

Warshall's Algorithm:

```
for (j=0; j<V; j++)
  for (i=0; i<V; i++)
    if (A[i][j])
      for (k=0; k<V; k++)
        if (A[j][k])
          A[i][k]=1;
```

$$\begin{array}{c} \\ i \\ j \end{array} \left[\begin{array}{cc} j & k \\ 1 & \square \\ & 1 \end{array}\right]$$





	0	1	2	3	4
0				1	
1				1	1
2		1			
3			1		
4					

If zero-edge paths are useful for an application (i.e. reflexive, self-loops), the diagonal may be all ones.

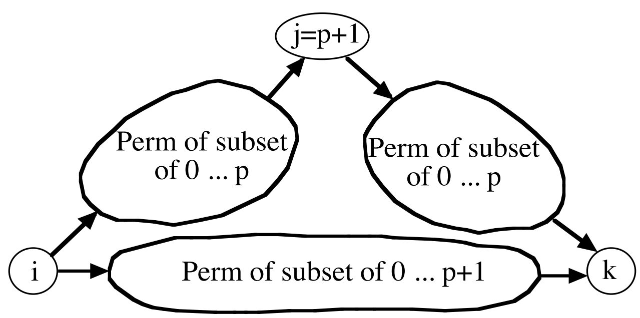
Why does it work?

a. *Correct* in use of transitivity.

b. Is it *complete*?

When	Paths That Can Be Detected
Before j=0	$x \to y$
After j=0	$x \to 0 \to y$
After j=1	$x \to 1 \to y$
	$x \to 0 \to 1 \to y$
	$x \to 1 \to 0 \to y$
After j=2	$x \to 2 \to y$
	$x \to 0 \to 2 \to y$
	$x \to 1 \to 2 \to y$
	$x \to 2 \to 0 \to y$
	$x \to 2 \to 1 \to y$
	$x \to 0 \to 1 \to 2 \to y$
	$x \to 0 \to 2 \to 1 \to y$
	$x \to 1 \to 0 \to 2 \to y$
	$x \to 1 \to 2 \to 0 \to y$
	$x \to 2 \to 0 \to 1 \to y$
	$x \to 2 \to 1 \to 0 \to y$
. . .	. . .
After j=p	$x \to$ Permutation of *subset* of 0 … p $\to y$
After j=V-1	ALL PATHS

Math. Induction:

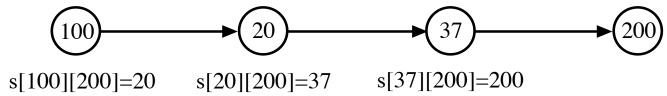


Suppose that there is only one path from vertex 5 to vertex 10 in a directed graph:
$5 \to 7 \to 4 \to 3 \to 2 \to 10$. During the scan of which column will Warshall's algorithm record the presence of this path?

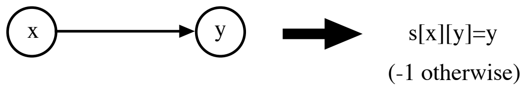
Warshall's Algorithm with Successors

Successor Matrix

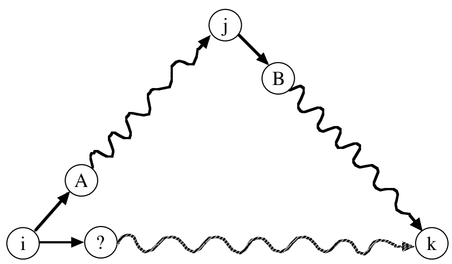
Buc-ee's directions:



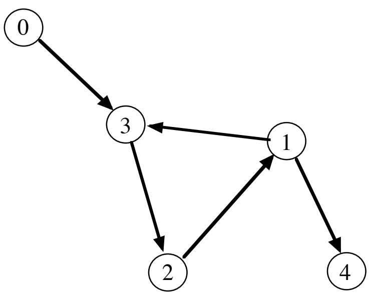
Initialize:



Warshall Matrix Update:



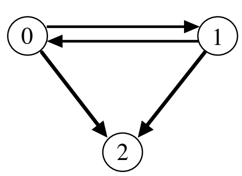
succ[i][j] = A   succ[j][k] = B   succ[i][k] = ?



	0	1	2	3	4
0				3	
1				3	4
2		1			
3			2		
4					

```
for (j=0; j<V; j++)
  for (i=0; i<V; i++)
    if (s[i][j] != (-1))
      for (k=0; k<V; k++)
        if (succ[i][k]==(-1) && succ[j][k]!=(-1))
          succ[i][k] = succ[i][j];
```

Suppose code in box is removed for this graph:



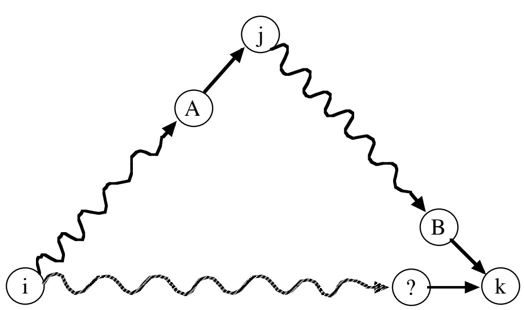
Complete Example (`https://ranger.uta.edu/~weems/NOTES3318/warshall.c`) saving paths using successors:

```
      0   1   2   3   4
 0   -1  -1  -1   3  -1
 1   -1  -1  -1   3   4
 2   -1   1  -1  -1  -1
 3   -1  -1   2  -1  -1
 4   -1  -1  -1  -1  -1
------------------------
      0   1   2   3   4
 0   -1  -1  -1   3  -1
 1   -1  -1  -1   3   4
 2   -1   1  -1  -1  -1
 3   -1  -1   2  -1  -1
 4   -1  -1  -1  -1  -1
------------------------
      0   1   2   3   4
 0   -1  -1  -1   3  -1
 1   -1  -1  -1   3   4
 2   -1   1  -1   1   1
 3   -1  -1   2  -1  -1
 4   -1  -1  -1  -1  -1
------------------------
```

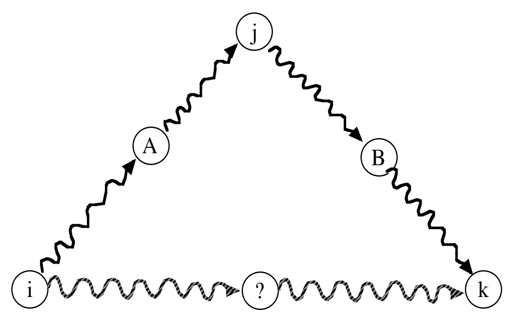
```
      0   1   2   3   4
 0   -1  -1  -1   3  -1
 1   -1  -1  -1   3   4
 2   -1   1  -1   1   1
 3   -1   2   2   2   2
 4   -1  -1  -1  -1  -1
------------------------
      0   1   2   3   4
 0   -1   3   3   3   3
 1   -1   3   3   3   4
 2   -1   1   1   1   1
 3   -1   2   2   2   2
 4   -1  -1  -1  -1  -1
------------------------
      0   1   2   3   4
 0   -1   3   3   3   3
 1   -1   3   3   3   4
 2   -1   1   1   1   1
 3   -1   2   2   2   2
 4   -1  -1  -1  -1  -1
------------------------
```

Other ways to save path information:

Predecessors (`warshallPred.c`)



Transitive/Intermediate/Column (`warshallCol.c`)

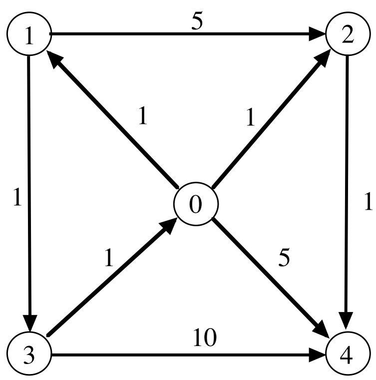



```
for (j=0;j<n;j++)
  for (i=0;i<n;i++)
    if (pred[i][j]!=(-1))
      for (k=0;k<n;k++)
        if (pred[i][k]==(-1) && pred[j][k]!=(-1))
          pred[i][k]=pred[j][k];
```

Floyd-Warshall Algorithm with Successors (`https://ranger.uta.edu/~weems/NOTES3318/floydWarshall.c`)

After j = p has been processed, the *shortest path* from each x to each y that uses *only* vertices in 0 . . . p as intermediate vertices is recorded in matrix.

```
for (j=0;j<n;j++)
{
  for (i=0;i<n;i++)
    if (dist[i][j]<oo)
      for (k=0;k<n;k++)
        if (dist[j][k]<oo)
        {
          newDist=dist[i][j]+dist[j][k];
          if (newDist<dist[i][k])
          {
            dist[i][k]=newDist;
            succ[i][k]=succ[i][j];
          }
        }
}
```



	0	1	2	3	4
0		1	1		5
1			5	1	
2					1
3	1				10
4					

	0	1	2	3	4
0	oo	1 1	1 2	oo	5 4
1	oo	oo	5 2	1 3	oo
2	oo	oo	oo	oo	1 4
3	1 0	oo	oo	oo	10 4
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	oo	1 1	1 2	oo	5 4
1	oo	oo	5 2	1 3	oo
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	oo	6 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	oo	1 1	1 2	2 1	5 4
1	oo	oo	5 2	1 3	oo
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	6 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	3 1	1 1	1 2	2 1	2 2
1	2 3	3 3	3 3	1 3	4 3
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	3 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	oo	1 1	1 2	2 1	2 2
1	oo	oo	5 2	1 3	6 2
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	3 0
4	oo	oo	oo	oo	oo

	0	1	2	3	4
0	3 1	1 1	1 2	2 1	2 2
1	2 3	3 3	3 3	1 3	4 3
2	oo	oo	oo	oo	1 4
3	1 0	2 0	2 0	3 0	3 0
4	oo	oo	oo	oo	oo

Note: In this example, zero-edge paths are not considered.