CSE 3318 Notes 13: Graph Representations and Search

(Last updated 8/17/22 3:01 PM)

CLRS 20.1-20.5

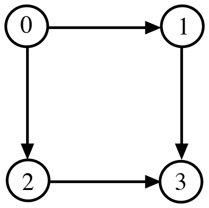
13.A. Graph Representations

Adjacency Matrices – for dense $\left(E = \Omega\left(V^2\right)\right)$ classes of graphs

("A sparse graph is one whose number of edges is reasonably viewed as being proportional to its number of vertices" `https://dl-acm-org.ezproxy.uta.edu/doi/10.1145/2492007.2492029`)

Directed Graph

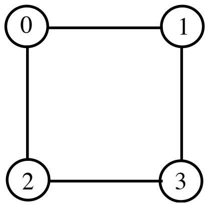
$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \end{array}$$



Diagonal: Zero edges allowed for paths? (reflexive, assumed self-loops)

Undirected Graph

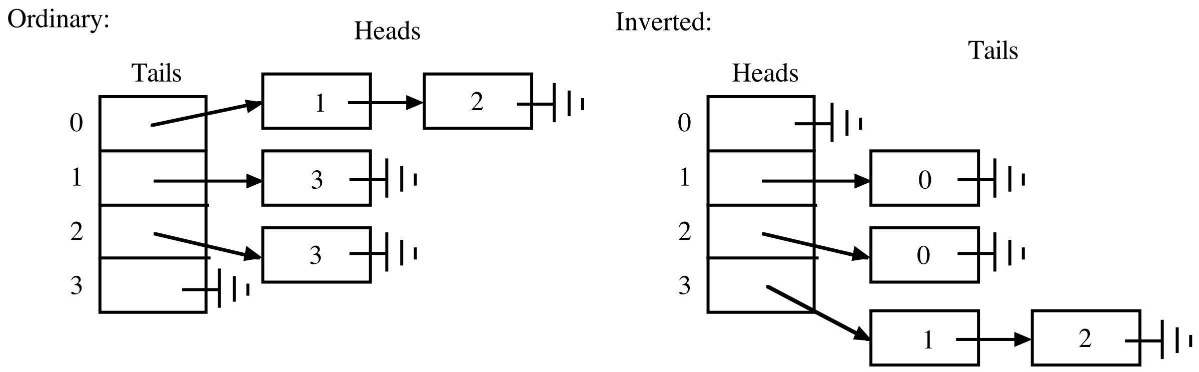
$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 & 0 \end{array}$$



Which is more general? Time to query for presence of an edge?

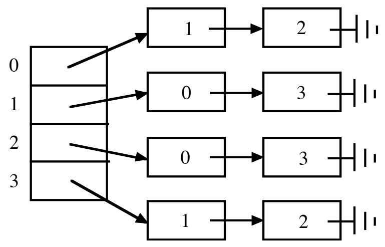
Adjacency Lists – for sparse $\left(E = O(V)\right)$ classes of graphs

Directed



1. Time to query for presence of an edge?
2. Can convert between ordinary and inverted in Θ(V + E) time, assuming unordered lists.
3. These two structures can be integrated using both tables and a common set of nodes with two linked lists through each node.

Undirected:

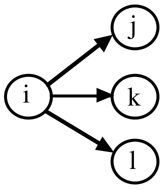


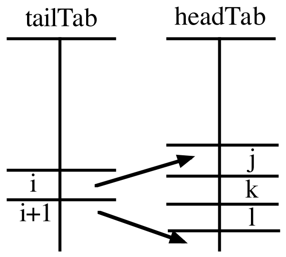
Weights – Used to represent distances, capacities, or costs.

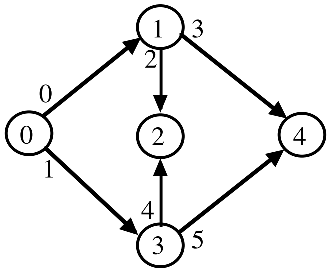
Entries in adjacency matrix.

Field in nodes of adjacency list.

Compressed Adjacency Lists – useful "pointerless" representation for sparse, static graphs (not in book, `https://dl-acm-org.ezproxy.uta.edu/doi/10.1145/3230485` discusses similar Compressed Sparse Row format). (code and examples at `https://ranger.uta.edu/~weems/NOTES3318/COMPADJLIST/`)

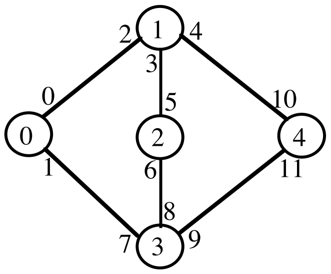






tailTab	
0	0
1	2
2	4
3	4
4	6
5	6

headTab	
0	1
1	3
2	2
3	4
4	2
5	4



tailTab	
0	0
1	2
2	5
3	7
4	10
5	12

headTab	
0	1
1	3
2	0
3	2
4	4
5	1
6	3
7	0
8	2
9	4
10	1
11	3

To process the edges with vertex `i` as the tail:

```
for (j=tailTab[i]; j<tailTab[i+1]; j++)
     < Process edge i → headTab[j] >
```

Time to query for presence of an edge?

13.B. BREADTH-FIRST SEARCH (Traversal) – Queue-Based

1. Assume input is directed graph with path from source to every vertex

   Source vertex is designated (assume 0)

   Vertex colors and interpretations

   a. White – undiscovered/unvisited

   b. Gray – presently in queue

   c. Black – completely processed (all adjacent vertices have been discovered)

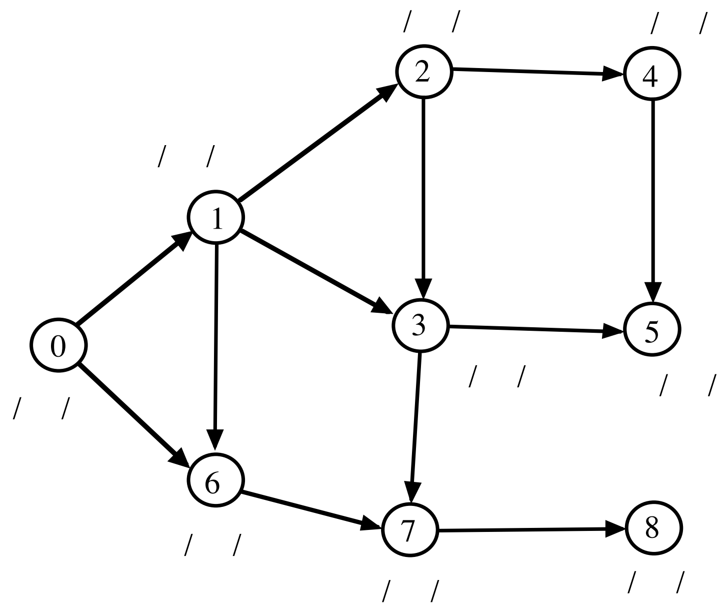
   Possible outputs:

   a. BFS number (assigned sequentially)

   b. Distance (hops) from source

   c. Predecessor on BFS tree

   Label node with a/b/c



   Queue:

Time:

   a. Initialization (Θ(V))   b. Process each edge once (Θ(E))

(An implementation of BFS is included in the Ford-Fulkerson network flow code on webpage.)

2. Remove assumption regarding path from source to every vertex:

```
Initialize all vertices as white
for (i=0; i<V; i++)
    if vertex i is white
        Call BFS with i as source
```

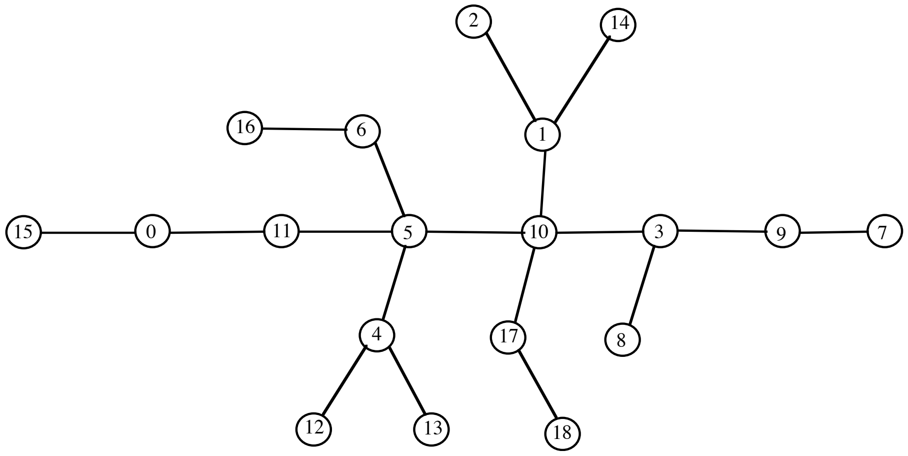
Can also use on undirected graph.

Number of BFS calls ("restarts") is the number of connected components.

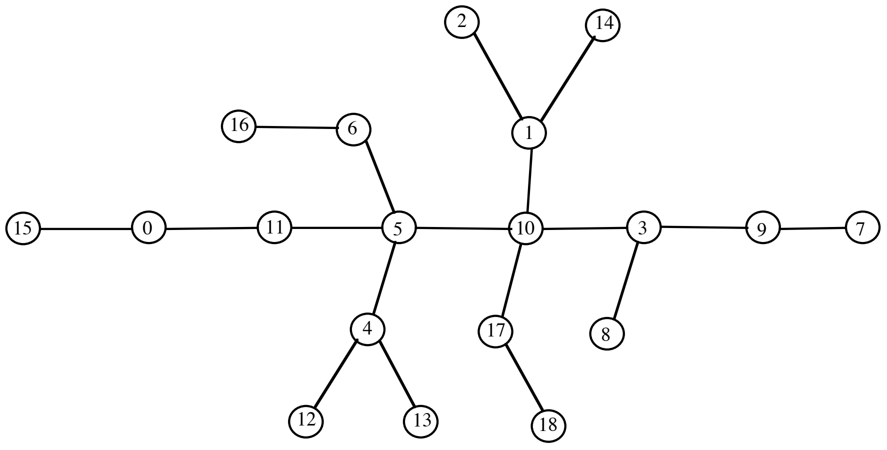
Each edge is processed *twice*, but each vertex is discovered *once*.

Diameter of Tree – Application of BFS

1. Choose arbitrary source for BFS. Run BFS and select any vertex X at maximum distance ("hops") from source (e.g. last vertex removed from queue).



2. Run second BFS using X as source. X will be at one end of a diameter and any vertex at maximum distance from X can be the other end of the diameter.



Takes $\Theta(V + E)$ time.

13.C. DEPTH-FIRST SEARCH (Traversal) – Stack/Recursion-Based

Usually applied to a directed graph.

Vertex colors and interpretations

a. White – undiscovered (neither time assigned, i.e. value of both is still `-1`)

b. Gray – presently in stack (only discovery time assigned)

c. Black – completely processed (all adjacent vertices have been discovered, both times assigned)

Possible outputs:

a. Discovery (preorder) time

b. Finish (postorder) time

c. Predecessor on DFS tree

d. Edge types (these reflect ancestry possibilities among vertices)

Processing:

a. Change vertex from white → gray the first time it enters stack and assign discovery time (using counter).

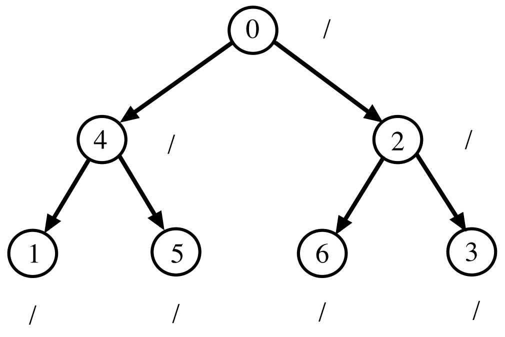
b. When a vertex (and pointer to its adjacency list) is popped, check for next adjacent vertex and push this vertex again.

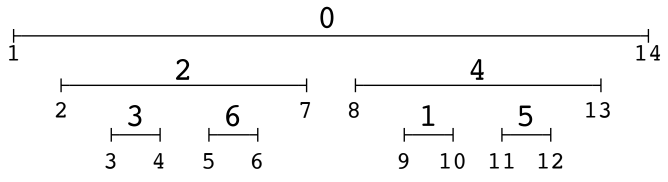
c. If no remaining adjacent vertices, then change vertex from gray → black and assign finish time.

Like BFS, DFS takes Θ(V + E) time.

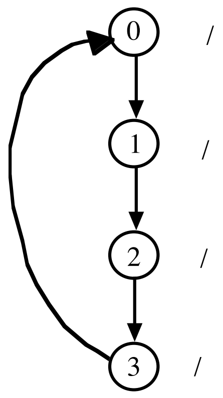
Relationship between vertex and adjacent vertex determines the *edge type*.

a. Unvisited (white) ⇒ *tree edge*





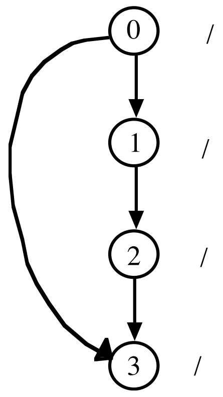
b. On the stack (gray indicating ancestor) ⇒ *back edge*



c. Previously visited, not on stack (black), but known to be descendant ⇒ *forward edge* (AKA *down edge*)

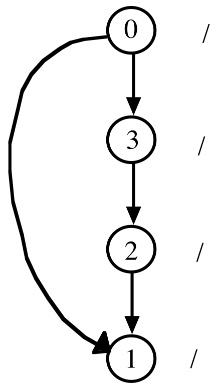
1. Find path of tree edges? TEDIOUS

2. discovery(tail) < discovery(head)

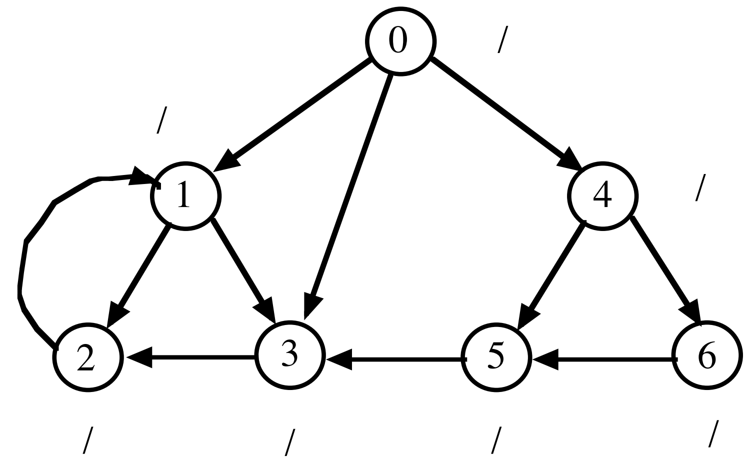


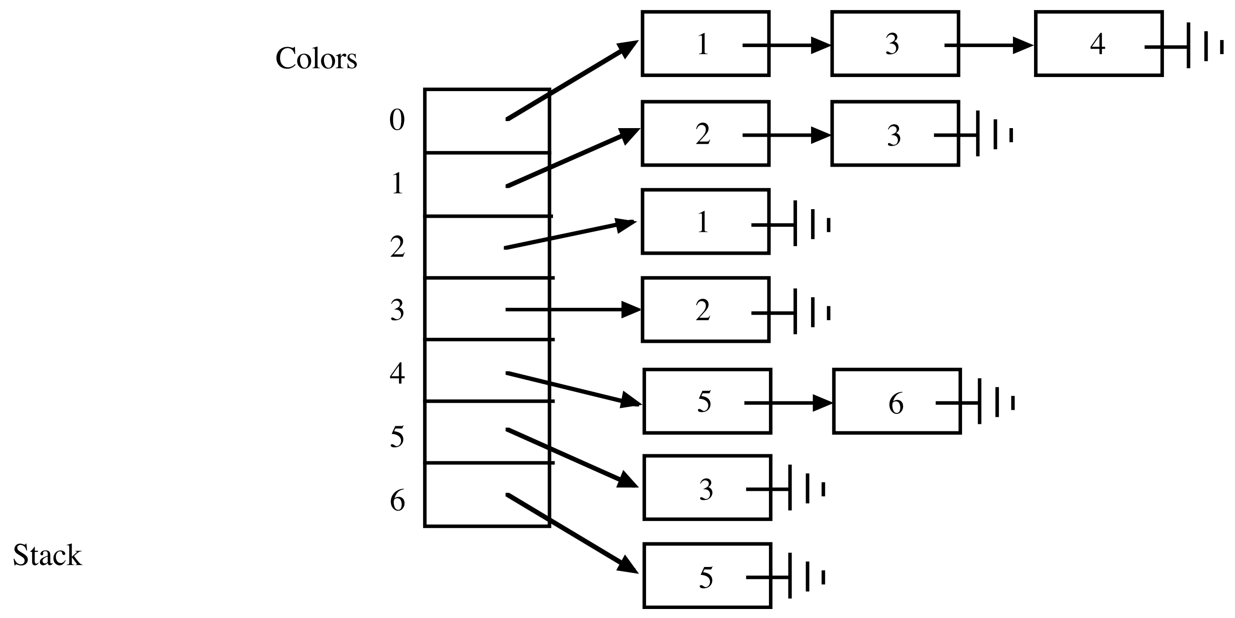
d. None of the above . . . Not on stack (black) and not a descendant ⇒ *cross edge*

Test using discovery(tail) > discovery(head)

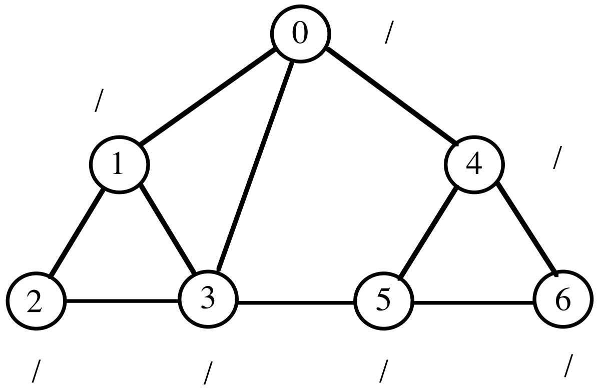


Example:

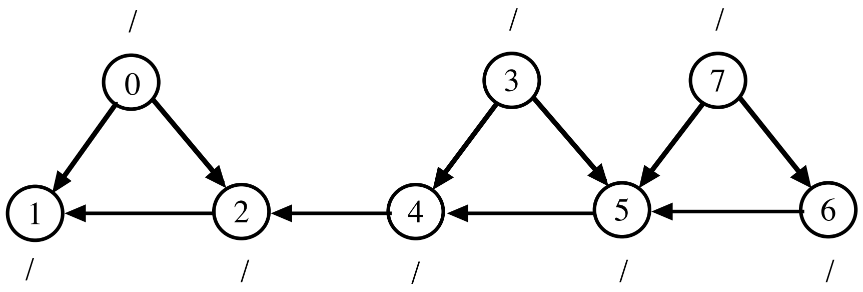




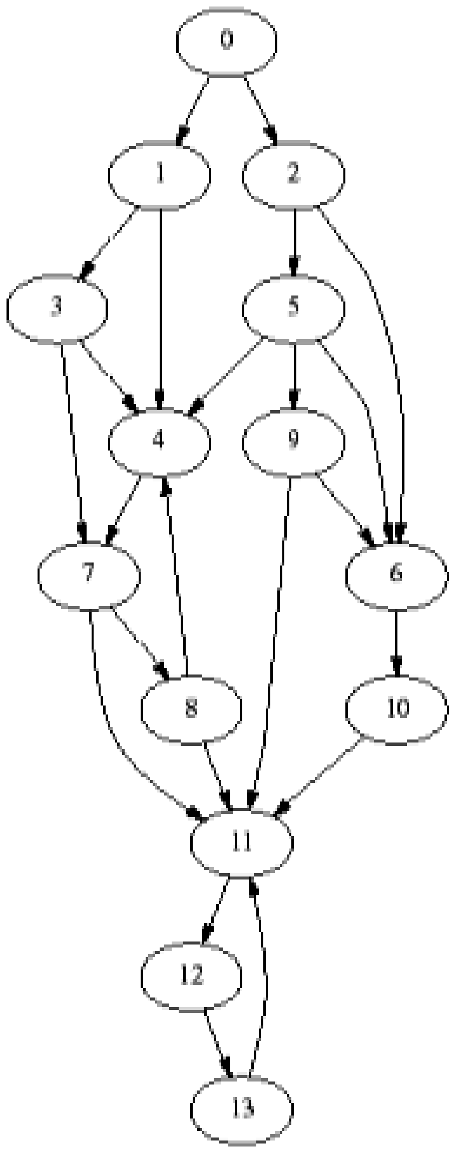
Undirected – Can't have cross or forward edges:



Restarts – handled like BFS



Example – available from course web page (`https://ranger.uta.edu/~weems/NOTES3318/dfsDir.c` `https://ranger.uta.edu/~weems/NOTES3318/dfs1.dat`)



```
Vertex  discovery    finish     predecessor
   0         1         28           -1
   1         2         17            0
   2        18         27            0
   3         3         16            1
   4         4         15            3
   5        19         26            2
   6        20         23            5
   7         5         14            4
   8         6         13            7
   9        24         25            5
  10        21         22            6
  11         7         12            8
  12         8         11           11
  13         9         10           12
Edge Tail Head Type
   0    0    1  tree
   1    0    2  tree
   2    1    3  tree
   3    1    4  forward
   4    2    5  tree
   5    2    6  forward
   6    3    4  tree
   7    3    7  forward
   8    4    7  tree
   9    5    4  cross
  10    5    6  tree
  11    5    9  tree
  12    6   10  tree
  13    7    8  tree
  14    7   11  forward
  15    8    4  back
  16    8   11  tree
  17    9    6  cross
  18    9   11  cross
  19   10   11  cross
  20   11   12  tree
  21   12   13  tree
  22   13   11  back
```

13.D. Topological Sort of a Directed Graph

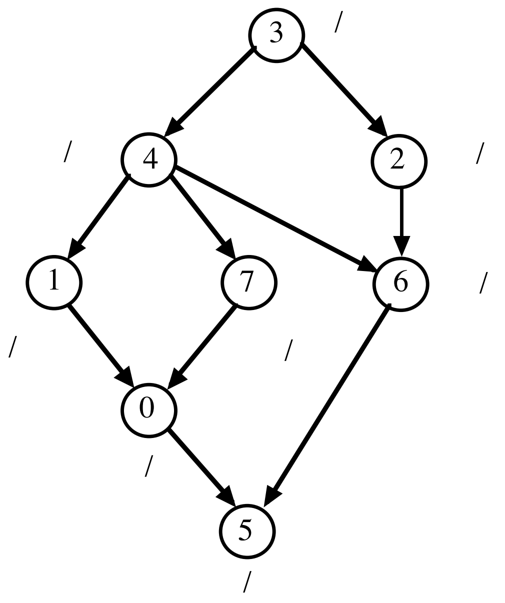
Linear ordering of all vertices in a graph.

Vertex x precedes y in ordering if there is a path from x to y in graph.

Apply DFS:

1. Back edge ⇔ graph has a cycle (no topological ordering).

2. When vertex turns black, insert at beginning of ordering (ordering is reverse of finish times).

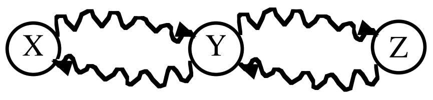


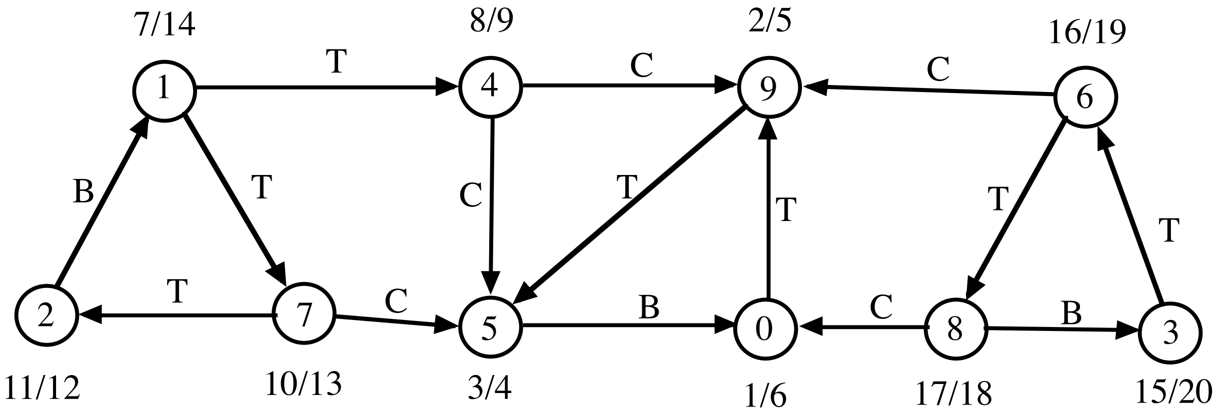
3 4 7 2 6 1 0 5

13.E. Strongly Connected Components

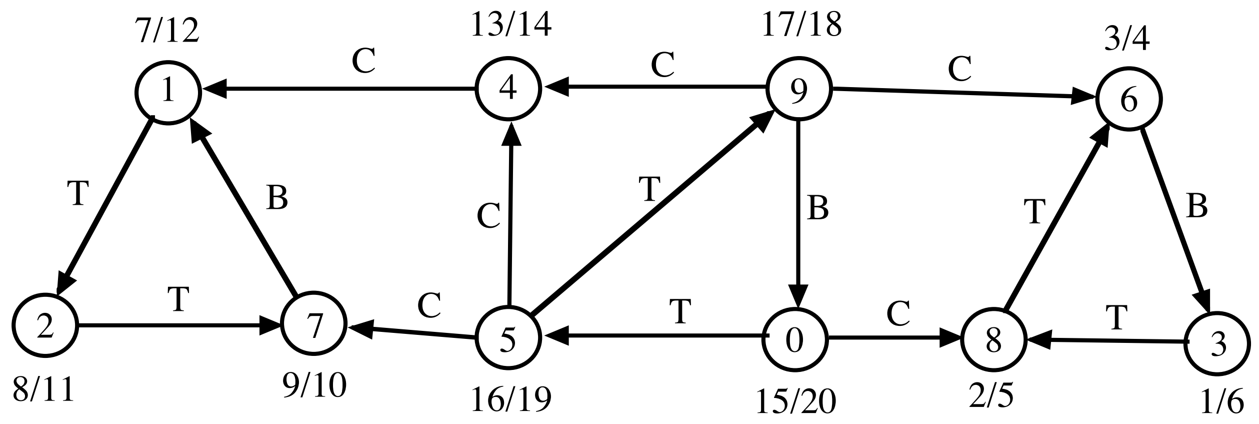
(Kosaraju's method, `https://ranger.uta.edu/~weems/NOTES3318/dfsSCC.c`)

Equivalence Relation – definition (reflexive, symmetric, transitive)





1. Perform DFS. When vertex turns black ⇒ insert at beginning of list. (3 6 8 1 7 2 4 0 9 5)
2. Reverse edges. (Does not change the strongly connected equivalence relation)



3. Perform DFS, but each restart chooses the first white vertex in list from 1. Vertices discovered within the same restart are in the same strong component.

Observation: If there is a path from x to y and no path from y to x, then finish(x) > finish(y) (first DFS).

This implies that the reverse edge (y, x) corresponding to an original edge (x, y) without a "return path" will be a cross edge during 2nd DFS. The head vertex y will be in a SCC that has already been output.

Takes Θ(V + E) time.