

CSE 3318 Notes 3: Summations

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CLRS, appendix A

3.A. GEOMETRIC SERIES (review)

$$\sum_{k=0}^t x^k = \frac{x^{t+1} - 1}{x - 1} \quad \text{when } x \neq 1 \quad [\text{Not hard to verify by math induction}]$$

$$\sum_{k=0}^t x^k \leq \sum_{k=0}^{\infty} x^k = \lim_{k \rightarrow \infty} \frac{x^k - 1}{x - 1} = \frac{1}{1 - x} \quad \text{when } 0 < x < 1$$

3.B. HARMONIC SERIES

$$\ln n \leq H_n = \sum_{k=1}^n \frac{1}{k} \leq \ln n + .577 \dots \leq \ln n + 1$$

As n approaches ∞ , $\frac{H_n}{H_{2n}}$ approaches?

- A. $1/2$ B. 1 C. 2 D. $\ln n$ E. n F. $n!$ G. ∞

3.C. APPROXIMATION BY INTEGRALS (p. 1150-1151)

For a monotonically increasing ($x \leq y \Rightarrow f(x) \leq f(y)$) function:

$$\int_{m-1}^n f(x) dx \leq \sum_{k=m}^n f(k) \leq \int_m^{n+1} f(x) dx$$

Since:

$$\int_{k-1}^k f(x) dx \leq f(k) \leq \int_k^{k+1} f(x) dx$$

in this situation.

3.D. BOUNDING SUMMATIONS USING MATH INDUCTION AND INEQUALITIES

[Techniques are especially important for recurrences in Notes 04]

Show $\sum_{i=1}^n i^2 = \Theta(n^3)$ [Trivial to show using integration or $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$.]

a. Show $O(n^3)$

$$(i) \quad \sum_{i=1}^{n=1} i^2 = 1 \leq cn^3 \text{ using any constant } c \geq 1$$

(ii) Suppose this holds for n :

$$\sum_{i=1}^n i^2 \leq cn^3$$

Now go on to $n + 1$ and show that the bound *still holds*

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^n i^2 + (n+1)^2 \\ &= \sum_{i=1}^n i^2 + n^2 + 2n + 1 \\ &\leq cn^3 + n^2 + 2n + 1 \\ &= ??? \\ &\leq c(n+1)^3 \end{aligned}$$

The bridging step (???) separates the bounding term ($c(n+1)^3$) from everything else (x):

$$\begin{aligned} c(n+1)^3 + x &= cn^3 + n^2 + 2n + 1 \\ x &= cn^3 + n^2 + 2n + 1 - cn^3 - 3cn^2 - 3cn - c = (1-3c)n^2 + (2-3c)n + 1 - c \\ \text{So } ??? \text{ is now } &c(n+1)^3 + \left[(1-3c)n^2 + (2-3c)n + 1 - c \right] \end{aligned}$$

Can drop [. . .] (through \leq) if it cannot become positive. Happens if $c \geq 1$

b. Show $\Omega(n^3)$

$$(i) \quad \sum_{i=1}^{n=1} i^2 = 1 \geq cn^3 \text{ using any constant } 0 < c \leq 1$$

(ii) Suppose this holds for n :

$$\sum_{i=1}^n i^2 \geq cn^3$$

Now go on to $n + 1$ and show that the bound *still holds*

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^n i^2 + (n+1)^2 \\ &= \sum_{i=1}^n i^2 + n^2 + 2n + 1 \\ &\geq cn^3 + n^2 + 2n + 1 \\ &= ??? \\ &\geq c(n+1)^3 \end{aligned}$$

The bridging step (???) involves the same algebra as before.

Can drop [. . .] (through \geq) if it cannot become negative. Happens if $0 < c \leq 1/3$

Suppose we attempt to show $\sum_{i=1}^n i^2 = \Theta(n^2)$

a. Show $O(n^2)$

$$(i) \quad \sum_{i=1}^{n=1} i^2 = 1 \leq cn^2 \text{ using any constant } c \geq 1$$

(ii) Suppose this holds for n :

$$\sum_{i=1}^n i^2 \leq cn^2$$

Now attempt to go on to $n + 1$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^n i^2 + (n+1)^2 \\
&= \sum_{i=1}^n i^2 + n^2 + 2n + 1 \\
&\leq cn^2 + n^2 + 2n + 1 \\
&= ??? \\
&\leq c(n+1)^2
\end{aligned}$$

The bridging step (???) separates the bounding term ($c(n+1)^2$) from everything else (x):

$$\begin{aligned}
c(n+1)^2 + x &= cn^2 + n^2 + 2n + 1 \\
x &= cn^2 + n^2 + 2n + 1 - cn^2 - 2cn - c = n^2 + (2-2c)n + 1 - c
\end{aligned}$$

$$\text{So ??? is now } c(n+1)^2 + \left[n^2 + (2-2c)n + 1 - c \right]$$

Can drop $[\dots]$ (through \leq) if it cannot become positive. ***Fails as n grows.***

b. Can still show $\Omega(n^2)$

$$(i) \quad \sum_{i=1}^{n=1} i^2 = 1 \geq cn^2 \text{ using any constant } 0 < c \leq 1$$

(ii) Suppose this holds for n :

$$\sum_{i=1}^n i^2 \geq cn^2$$

Now go on to $n+1$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^n i^2 + (n+1)^2 \\
&= \sum_{i=1}^n i^2 + n^2 + 2n + 1 \\
&\geq cn^2 + n^2 + 2n + 1 \\
&= ??? \\
&\geq c(n+1)^2
\end{aligned}$$

The bridging step separates the bounding term ($c(n+1)^2$) from everything else (x):

$$\begin{aligned}
c(n+1)^2 + x &= cn^2 + n^2 + 2n + 1 \\
x &= cn^2 + n^2 + 2n + 1 - cn^2 - 2cn - c = n^2 + (2-2c)n + 1 - c
\end{aligned}$$

So ??? is now $c(n+1)^2 + \left[n^2 + (2-2c)n + 1 - c \right]$

Can drop $[\dots]$ (through \geq) if it cannot become negative.

Happens if $0 < c \leq 1$ (or for “sufficiently large” n).