Characteristic Function Evolution for Levy SDE

Let L_t^α denote an α-stable Levy processes, i.e., a process such that:

1. $L_0^\alpha = 0$ almost surely,
2. L_t^α has independent increments, and
3. For $t > s \geq 0$, $L_t^\alpha - L_s^\alpha \sim S_\alpha((t-s)^{1/\alpha}, 0, 0)$. That is, the increment over a time interval of length $t-s$ has an α-stable distribution with scale parameter $\sigma = (t-s)^{1/\alpha}$, skewness parameter $\beta = 0$, and location parameter $\mu = 0$. The characteristic function of this increment is:
$$E[\exp(is(L_t^\alpha - L_s^\alpha))] = \exp(-(t-s)|s|^\alpha). \tag{1}$$

Now consider the stochastic differential equation (SDE)

$$dX_t = f(X_t)dt + g(X_t)dL_t^\alpha. \tag{2}$$

Let $p(x,t)$ denote the probability density function (PDF) of X_t—note that p is the exact PDF of the exact solution of the SDE.

Suppose that $p(x,0)$ is given. Our goal is to compute $p(x,t)$ for $t > 0$.

To derive our method, we first discretize (2) in time via Euler-Maruyama with step $h > 0$:

$$x_{n+1} = x_n + f(x_n)h + g(x_n)\Delta L_{n+1}^\alpha, \tag{3}$$

where ΔL_{n+1}^α is independent of x_n and has characteristic function

$$\psi_{\Delta L_{n+1}^\alpha}(s) = \exp(-h|s|^\alpha). \tag{4}$$

The drift f and diffusion g functions can be assumed to be smooth. We can also assume that g is bounded away from zero, i.e., that there exists $\delta > 0$ such that $|g(x)| \geq \delta$ for all x. In fact, it is of interest to solve this problem (well) in the case where g is a positive constant.

We let $\tilde{p}(x, t_n)$ denote the exact PDF of x_n, itself an approximation to the exact solution at time t_n, $X(t_n)$.

Let us denote the conditional density of x_{n+1} given $x_n = y$ by $p_{n+1,n}(x|y)$. Applying this to (3), we obtain the following evolution equation for the marginal density of x_n:

$$\tilde{p}(x, t_{n+1}) = \int_{-\infty}^{\infty} p_{n+1,n}(x|y)\tilde{p}(y, t_n)\, dy. \tag{5}$$

From (3), we can show that the characteristic function of the conditional density $p_{n+1,n}(x|y)$ is

$$e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha).$$

Therefore, we can compute the characteristic function using

$$\psi_{n+1}(s) = \int_{x=-\infty}^{\infty} e^{isx}\tilde{p}(x, t_{n+1})dx.$$

The characteristic function is given by

$$\psi_{n+1}(s) = \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp\left(-h|sg(y)|^{\alpha}\right) \tilde{p}(y, t_n) dy. \tag{6}$$

Since

$$\tilde{p}(y, t_n) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} e^{-iuy} \psi_n(u) du \tag{7}$$

from (6) we get

$$\psi_{n+1}(s) = \int_{u=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp\left(-h|sg(y)|^{\alpha}\right) e^{-iuy} dy \right] \psi_n(u) du.$$

Let

$$K(s, u) = \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp\left(-h|sg(y)|^{\alpha}\right) e^{-iuy} dy.$$

and we get

$$\psi_{n+1}(s) = \int_{u=-\infty}^{\infty} K(s, u) \psi_n(u) du. \tag{8}$$

Up to quadrature, (8) is the algorithm. The idea now is to spatially discretize the characteristic function $\psi_0(s)$ in space and then apply quadrature repeatedly to advance forward in time via (8). If, at any point, we want to retrieve the PDF from the characteristic function, we use (7). We have several choices for how to carry out the spatial discretization. Among these, some leading candidates are:

1. A simple, finite-difference approach. Pick an equispaced grid in s space, truncate the integral, sample $K(s, u)$ on the equispaced grid, and convert (8) into simple matrix multiplication.

2. A collocation method in which we express $\psi_n(u)$ as a linear combination of basis functions. We then require (8) to hold at a number of points equal to the number of coefficients (or basis functions). Ultimately, if β_n are the coefficients of ψ_n, this should result in an update equation of the form $\beta_{n+1} = B^{-1} A B \beta_n$.