

Characteristic Function Evolution for Lévy SDE

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Problem Statement. Let L_t^α denote an α -stable Lévy processes, i.e., a process such that:

1. $L_0^\alpha = 0$ almost surely,
2. L_t^α has independent increments, and
3. For $t > s \geq 0$, $L_t^\alpha - L_s^\alpha \sim S_\alpha((t-s)^{1/\alpha}, 0, 0)$. That is, the increment over a time interval of length $t-s$ has an α -stable distribution with scale parameter $\sigma = (t-s)^{1/\alpha}$, skewness parameter $\beta = 0$, and location parameter $\mu = 0$. The characteristic function of this increment is:

$$E[\exp(is(L_t^\alpha - L_s^\alpha))] = \exp(-(t-s)|s|^\alpha). \quad (1)$$

Now consider the stochastic differential equation (SDE)

$$dX_t = f(X_t)dt + g(X_t)dL_t^\alpha. \quad (2)$$

Let $p(x, t)$ denote the probability density function (PDF) of X_t —note that p is the exact PDF of the exact solution of the SDE.

Suppose that $p(x, 0)$ is given. Our goal is to compute $p(x, t)$ for $t > 0$.

Derivation of Method (Temporal Discretization). To derive our method, we first discretize (2) in time via Euler-Maruyama with step $h > 0$:

$$x_{n+1} = x_n + f(x_n)h + g(x_n)\Delta L_{n+1}^\alpha, \quad (3)$$

where ΔL_{n+1}^α is independent of x_n and has characteristic function

$$\psi_{\Delta L_{n+1}^\alpha}(s) = \exp(-h|s|^\alpha). \quad (4)$$

The drift f and diffusion g functions can be assumed to be smooth. We can also assume that g is bounded away from zero, i.e., that there exists $\delta > 0$ such that $|g(x)| \geq \delta$ for all x . In fact, it is of interest to solve this problem (well) in the case where g is a positive constant.

We let $\tilde{p}(x, t_n)$ denote the exact PDF of x_n , itself an approximation to the exact solution at time t_n , $X(t_n)$.

Let us denote the conditional density of x_{n+1} given $x_n = y$ by $p_{n+1,n}(x|y)$. Applying this to (3), we obtain the following evolution equation for the marginal density of x_n :

$$\tilde{p}(x, t_{n+1}) = \int_{-\infty}^{\infty} p_{n+1,n}(x|y)\tilde{p}(y, t_n) dy. \quad (5)$$

From (3), we can show that the characteristic function of the conditional density $p_{n+1,n}(x|y)$ is

$$e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha).$$

Therefore, we can compute the characteristic function using

$$\psi_{n+1}(s) = \int_{x=-\infty}^{\infty} e^{isx} \tilde{p}(x, t_{n+1}) dx. \quad (6)$$

The characteristic function is given by

$$\psi_{n+1}(s) = \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) \tilde{p}(y, t_n) dy. \quad (7)$$

Since

$$\tilde{p}(y, t_n) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} e^{-iuy} \psi_n(u) du \quad (8)$$

from (7) we get

$$\psi_{n+1}(s) = \int_{u=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy \right] \psi_n(u) du.$$

Let

$$K(s, u) = \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy.$$

and we get

$$\psi_{n+1}(s) = \int_{u=-\infty}^{\infty} K(s, u) \psi_n(u) du. \quad (9)$$

Up to quadrature, (9) is the algorithm. By repeatedly applying (9) we evolve the characteristic function forward in time. If, at any point, we want to retrieve the PDF from the characteristic function, we use (8).

Numerical Analysis (Spatial Discretization). The idea now is to spatially discretize both the characteristic function $\psi_0(s)$, the kernel $K(s, u)$, and the integral in (9). Numerical evaluation of the kernel is the most expensive and critical part, and so we focus on that first.

Before doing any numerical computation, we first note that the kernel is singular at $s = 0$:

$$K(0, u) = \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{-iuy} dy = \delta(u). \quad (10)$$

If we now use this result in (9), we obtain

$$\psi_{n+1}(0) = \int_{u=-\infty}^{\infty} \delta(u) \psi_n(u) dy = \psi_n(0). \quad (11)$$

By (6), we know that

$$\psi_n(0) = \int_{x=-\infty}^{\infty} \tilde{p}(x, t_n) dx.$$

Hence (11) is conservation of probability. As long as our initial characteristic function satisfies $\psi_0(0) = 1$, all subsequent characteristic functions will preserve this property exactly.

In what follows, we assume $s \neq 0$. We split the domain of integration:

$$\begin{aligned} K(s, u) = & \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy \\ & + \frac{1}{2\pi} \int_{y=-L/2}^{L/2} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy \\ & + \frac{1}{2\pi} \int_{y=L/2}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy \end{aligned}$$

The inner integral over the finite domain $[-L/2, L/2]$ is the one we will compute using a quadrature rule. We set up an equispaced grid with $N > 0$ grid points. Then $\Delta y = L/N$ and $y_j = -L/2 + (\Delta y)j$ for $j = 0, 1, 2, \dots, N-1$. The inner integral then becomes a sum of terms of the form

$$\int_{y_j}^{y_{j+1}} e^{iky} \phi(y) dy.$$

We now begin the derivation of the quadrature rule. Assume ϕ is sufficiently smooth so that we can approximate it well using the Lagrange interpolant:

$$\phi(y) \approx \frac{y - y_j}{\Delta y} \phi(y_j + \Delta y) - \frac{y - y_{j+1}}{\Delta y} \phi(y_j).$$

Using this approximation, we derive the quadrature rule:

$$\begin{aligned} \int_{y_j}^{y_{j+1}} e^{iky} \phi(y) dy & \approx \int_{y_j}^{y_{j+1}} e^{iky} \frac{y - y_j}{\Delta y} \phi(y_j + \Delta y) dy - \int_{y_j}^{y_{j+1}} e^{iky} \frac{y - y_{j+1}}{\Delta y} \phi(y_j) dy \\ & = \frac{m_1(y_j) - y_j m_0(y_j)}{\Delta y} \phi(y_j + \Delta y) - \frac{m_1(y_j) - y_{j+1} m_0(y_j)}{\Delta y} \phi(y_j) \end{aligned} \quad (12)$$

In this derivation, we have

$$m_0(y_j) = \int_{y_j}^{y_{j+1}} e^{iky} dy = \begin{cases} \Delta y & k = 0 \\ (ik)^{-1} (e^{iky_{j+1}} - e^{iky_j}) & k \neq 0 \end{cases}$$

and

$$m_1(y_j) = \int_{y_j}^{y_{j+1}} y e^{iky} dy = \begin{cases} \frac{1}{2} (y_{j+1}^2 - y_j^2) & k = 0 \\ (ik)^{-1} (y_{j+1} e^{iky_{j+1}} - y_j e^{iky_j}) + k^{-2} (e^{iky_{j+1}} - e^{iky_j}) & k \neq 0. \end{cases}$$

Equipped with the quadrature rule, we set

$$\phi(x) = \exp[h(isf(x) - |sg(x)|^\alpha)]$$

and complete the calculation:

$$\begin{aligned} & \frac{1}{2\pi} \int_{y=-L/2}^{L/2} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy \\ & = \frac{1}{2\pi} \sum_{j=0}^{N-1} \int_{y_j}^{y_{j+1}} e^{i(s-u)y} \phi(y) dy \\ & \approx \frac{1}{2\pi} \sum_{j=0}^{N-1} \frac{m_1(y_j) - y_j m_0(y_j)}{\Delta y} \phi(y_j + \Delta y) - \frac{m_1(y_j) - y_{j+1} m_0(y_j)}{\Delta y} \phi(y_j). \end{aligned} \quad (13)$$

Asymptotics. We now turn to the integrals over the unbounded domains $(-\infty, -L/2)$ and $(L/2, +\infty)$. Our strategy will be to use asymptotic approximations of f and g to compute the integrals. We will explain this strategy by example.

1. Suppose that $\alpha = 1$ and

$$\begin{aligned} f(x) &= \tan^{-1} x \\ g(x) &= \sqrt{1+x^2} \end{aligned}$$

We have

$$f(x) \approx \pm\pi/2, \text{ as } x \rightarrow \pm\infty, \text{ respectively,}$$

and

$$g(x) \approx |x|, \text{ as } |x| \rightarrow \infty.$$

We assume L is sufficiently large and apply these approximations:

$$\begin{aligned} \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy &\approx \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{i(s-u)y} e^{-ish\pi/2} e^{h|s|y} dy \\ &= -\frac{i}{2\pi} \frac{\exp(-(i/2)(h\pi s + L(s-u)) - (1/2)(hL|s|))}{s-u-ih|s|} \end{aligned}$$

and

$$\begin{aligned} \int_{y=L/2}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy &\approx \int_{y=L/2}^{\infty} e^{i(s-u)y} e^{ish\pi/2} e^{-h|s|y} dy \\ &= \frac{i}{2\pi} \frac{\exp((i/2)(h\pi s + L(s-u)) - (1/2)(hL|s|))}{s-u+ih|s|}. \end{aligned}$$

The two asymptotic contributions are complex conjugates of one another—due to the symmetries of f and g —and hence we obtain

$$\Re \left[\frac{i}{\pi} \frac{\exp((i/2)(h\pi s + L(s-u)) - (1/2)(hL|s|))}{s-u+ih|s|} \right] \quad (14)$$

as the total asymptotic contribution.

2. Suppose that $\alpha = 1$ and

$$\begin{aligned} f(x) &= \tan^{-1} x \\ g(x) &= g > 0 \end{aligned}$$

We apply the same approximation for f as above. This yields

$$\begin{aligned} \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy &\approx \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{i(s-u)y} e^{-ish\pi/2} e^{-h|s|g} dy \\ &= e^{-ish\pi/2} e^{-h|s|g} \left(\frac{e^{-iL(s-u)/2}}{2\pi i(s-u)} + \frac{1}{2} \delta(s-u) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_{y=L/2}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy &\approx \frac{1}{2\pi} \int_{y=L/2}^{\infty} e^{i(s-u)y} e^{ish\pi/2} e^{-h|s|g} dy \\ &= e^{ish\pi/2} e^{-h|s|g} \left(-\frac{e^{iL(s-u)/2}}{2\pi i(s-u)} + \frac{1}{2} \delta(s-u) \right) \end{aligned}$$

Putting these two contributions together, we obtain

$$e^{-h|s|g} \left(\cos(sh\pi/2) \delta(s-u) - \frac{\sin(L(s-u)/2 + sh\pi/2)}{\pi(s-u)} \right).$$

Note: I am perfectly happy to integrate the Dirac delta against $\psi_n(u)$, but I am not entirely sure how to compute the integral of the second part against $\psi_n(u)$. Even if I interpret $1/(s-u)$ as a distribution—using Cauchy principal value—it is not clear how to evaluate $\int_u (1/(s-u)) \psi_n(u) du$.

3. The above two examples are hard ones. Let us try to find a general class of examples that are both useful and more immediately compatible with our method. Suppose we have a potential $V(x)$ —here we think of harmonic or anharmonic, single- or multiple-well potentials. Then we can define

$$f(x) = -\frac{dV}{dx}.$$

Suppose that the potential saturates for large arguments, i.e., for $|x| \geq L/2$, we have

$$V(x) = C.$$

This will imply that for $|x| \geq L/2$,

$$f(x) = 0.$$

Further assume that the diffusion coefficient is constant, i.e.,

$$g(x) = g > 0.$$

Then the asymptotic contributions will be:

$$\begin{aligned} \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy &\approx \frac{1}{2\pi} \int_{y=-\infty}^{-L/2} e^{i(s-u)y} e^{-h|s|^\alpha g^\alpha} dy \\ &= e^{-h|s|^\alpha g^\alpha} \left(\frac{e^{-iL(s-u)/2}}{2\pi i(s-u)} + \frac{1}{2} \delta(s-u) \right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_{y=L/2}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) e^{-iuy} dy &\approx \frac{1}{2\pi} \int_{y=L/2}^{\infty} e^{i(s-u)y} e^{-h|s|^\alpha g^\alpha} dy \\ &= e^{-h|s|^\alpha g^\alpha} \left(-\frac{e^{iL(s-u)/2}}{2\pi i(s-u)} + \frac{1}{2} \delta(s-u) \right) \end{aligned}$$

Putting the two together, we obtain

$$e^{-h|s|^\alpha g^\alpha} \left(\delta(s-u) - \frac{\sin(L(s-u)/2)}{\pi(s-u)} \right).$$

It is easy to see how we can integrate this against $\psi_n(u)$. We take care of the Dirac delta using pencil and paper. As for the remaining function, it is equivalent to

$$\frac{L}{2\pi} \operatorname{sinc} \left[\frac{L}{2}(s-u) \right],$$

a smooth function with no singularities. The integral of this function against $\psi_n(u)$ is a convolution integral. Convolution of the characteristic function with the sinc function is equivalent to multiplying the PDF by a rectangular, low-pass filter.

Brief Bit of Analysis. Given any random variable X with density $p(x)$, we can define the characteristic function as the Fourier transform:

$$\psi(s) = \int_{x=-\infty}^{\infty} e^{isx} p(x) dx.$$

Hence

$$\|\psi(s)\| = \left| \int_{x=-\infty}^{\infty} e^{isx} p(x) dx \right| \leq \int_{x=-\infty}^{\infty} p(x) dx = 1.$$

because $|e^{isx}| = 1$ and $p(x) \geq 0$. Note that $\psi(0) = 1$, and so $\|\psi\|_\infty = 1$. Now suppose that $\|\psi_n\|_\infty = 1$. Then we have

$$\begin{aligned} \|\psi_{n+1}(s)\| &= \left| \int_u K(s, u) \psi_n(u) du \right| \\ &\leq \int_u |K(s, u)| |\psi_n(u)| du \\ &\leq \int_u |K(s, u)| du. \end{aligned}$$

Hence in order to guarantee $\|\psi_{n+1}(s)\| \leq 1$, it is sufficient that

$$\leq \int_u |K(s, u)| du \leq 1.$$

After computing $K(s, u)$ numerically, it is easy to check this condition. Typically, we form a matrix κ by setting $\kappa_{ij} = K(s_i, u_j)$. One numerical approximation of the above sufficient condition is then

$$\sum_j |\kappa_{ij}| (\Delta u) \leq 1$$

or

$$\|\kappa\|_\infty \leq (\Delta u)^{-1}.$$

Another Way of Looking At Things. Let's reexamine K :

$$K(s, u) = \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{i(s-u)y} \exp(h[isf(y) - |sg(y)|^\alpha]) dy.$$

Now expand in h about $h = 0$:

$$K(s, u) = \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{i(s-u)y} (1 + h[isf(y) - |sg(y)|^\alpha] + O(h^2)) dy$$

We can go a bit further:

$$K(s, u) = \delta(s - u) + \frac{his}{2\pi} \int_{y=-\infty}^{\infty} e^{i(s-u)y} f(y) dy - \frac{h|s|^\alpha}{2\pi} \int_{y=-\infty}^{\infty} e^{i(s-u)y} |g(y)|^\alpha dy + O(h^2)$$

If we let \widehat{f} and $\widehat{|g|^\alpha}$ denote Fourier transforms of the respective functions, then we have

$$K(s, u) = \delta(s - u) + his\widehat{f}(s - u) - h|s|^\alpha \widehat{|g|^\alpha}(s - u) + O(h^2)$$

Now we insert this into the integral (9) and obtain

$$\psi_{n+1}(s) = \psi_n(s) + his \int_{u=-\infty}^{\infty} \widehat{f}(s - u) \psi_n(u) du - h|s|^\alpha \int_{u=-\infty}^{\infty} \widehat{|g|^\alpha}(s - u) \psi_n(u) du + O(h^2)$$

Now we move $\psi_n(s)$ to the left-hand side and write

$$\psi(s, n\Delta t) \approx \psi_n(s)$$

where the ψ on the left-hand side is the continuum limit in time of the discrete snapshots $\psi_n(s)$. Then dividing through by h and sending $h \rightarrow 0$, we derive

$$\frac{\partial \psi}{\partial t}(s, t) = is \int_{u=-\infty}^{\infty} \widehat{f}(s - u) \psi(u, t) du - |s|^\alpha \int_{u=-\infty}^{\infty} \widehat{|g|^\alpha}(s - u) \psi(u, t) du \quad (15)$$

Maybe this integrodifferential equation is at the heart of what we are doing? For the particular case of $\alpha = 1$, $f(y) = \tan^{-1} y$ and $g(y) = \sqrt{1 + y^2}$, we obtain

$$\frac{\partial \psi}{\partial t}(s, t) = -\frac{s}{2} \int_{-\infty}^{\infty} \frac{e^{-|s-u|}}{s-u} \psi_n(u) du - \frac{|s|}{\pi} \int_{-\infty}^{\infty} \frac{K_1(|s-u|)}{|s-u|} \psi_n(u) du$$

where $K_1(z)$ is the modified Bessel function of the second kind. It seems like there are unavoidable singularities here.

Fractional Fokker-Planck! If we write the inversion formula

$$p(y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuy} \psi(u, t) du$$

and then differentiate with respect to y on both sides, we will derive

$$\frac{\partial}{\partial y} p(y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuy} (-iu) \psi(u, t) du.$$

More generally, we can *define* a fractional derivative of p via the formula

$$\frac{\partial^\alpha}{\partial|y|^\alpha}p(y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuy} (-|u|^\alpha) \psi(u,t) du.$$

If we use this formula with $\alpha = 2$, we recover a familiar statement:

$$\frac{\partial^2}{\partial y^2}p(y,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuy} (-u^2) \psi(u,t) du.$$

In other words, the choice of signs in the fractional derivative definition matches $(-iu)^2 = -u^2 = -|u|^2$. Now using the above to take the inverse Fourier transform of (15), and using the convolution theorem, we derive

$$\frac{\partial p}{\partial t}(y,t) = -\frac{\partial}{\partial y}[f(y)p(y,t)] + \frac{\partial^\alpha}{\partial|y|^\alpha}[|g(y)|^\alpha p(y,t)]. \quad (16)$$

When $\alpha = 2$, this is the classical Fokker-Planck PDE, modulo a factor of $1/2$ on the diffusion term. This missing factor of $1/2$ is due to the fact that when $\alpha = 2$, the characteristic function of the Lévy increment—defined in (4)—becomes $\exp(-hs^2)$, whereas for SDE driven by Brownian motion we (canonically) define the characteristic function of the increment to be $\exp(-hs^2/2)$.

Back to Physical Space? My original idea was to evolve characteristic functions because they decay exponentially even for fat-tailed distributions like the Cauchy. But what if we try to evolve the PDF instead? Suppose we write the time-advanced version of (8):

$$\tilde{p}(x, t_{n+1}) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} e^{-iux} \psi_{n+1}(u) du$$

We can then combine this with (7) to derive

$$\tilde{p}(x, t_{n+1}) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} e^{-iux} \int_{y=-\infty}^{\infty} e^{iu(y+f(y)h)} \exp(-h|ug(y)|^\alpha) \tilde{p}(y, t_n) dy du.$$

Let us define

$$\mathcal{K}(x, y) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} e^{-iux} e^{iu(y+f(y)h)} \exp(-h|ug(y)|^\alpha) du.$$

Then the update equation becomes

$$\tilde{p}(x, t_{n+1}) = \int_{y=-\infty}^{\infty} \mathcal{K}(x, y) \tilde{p}(y, t_n) dy.$$

Maybe this will avoid singularities?