Characteristic Function Evolution for Lévy SDE
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Problem Statement. Let L_t^α denote an α-stable Lévy processes, i.e., a process such that:

1. $L_0^\alpha = 0$ almost surely,
2. L_t^α has independent increments, and
3. For $t > s \geq 0$, $L_t^\alpha - L_s^\alpha \sim S_\alpha((t-s)^{1/\alpha}, 0, 0)$. That is, the increment over a time interval of length $t-s$ has an α-stable distribution with scale parameter $\sigma = (t-s)^{1/\alpha}$, skewness parameter $\beta = 0$, and location parameter $\mu = 0$. The characteristic function of this increment is:

$$E[\exp(is(L_t^\alpha - L_s^\alpha))] = \exp(-(t-s)|s|^\alpha). \tag{1}$$

Now consider the stochastic differential equation (SDE)

$$dX_t = f(X_t)dt + g(X_t)dL_t^\alpha. \tag{2}$$

Let $p(x,t)$ denote the probability density function (PDF) of X_t—note that p is the exact PDF of the exact solution of the SDE.

Suppose that $p(x,0)$ is given. Our goal is to compute $p(x,t)$ for $t > 0$.

Brief Review of Characteristic Functions. Given any random variable X with density $p(x)$, we can define the characteristic function as the Fourier transform:

$$\psi(s) = \int_{x=-\infty}^{\infty} e^{isx} p(x)\, dx.$$

Note that

$$\psi(0) = \int_{x=-\infty}^{\infty} p(x)\, dx = 1.$$

Using $|e^{isx}| = 1$ and $p(x) \geq 0$, we have

$$\|\psi(s)\| = \left| \int_{x=-\infty}^{\infty} e^{isx} p(x)\, dx. \right| \leq \int_{x=-\infty}^{\infty} p(x)\, dx = 1,$$

Because $\psi(0) = 1$, we see that $\|\psi\|_\infty = 1$.

Derivation of Method (Temporal Discretization). To derive our method, we first discretize (2) in time via Euler-Maruyama with step $h > 0$:

$$x_{n+1} = x_n + f(x_n)h + g(x_n)\Delta L_{n+1}^\alpha, \tag{3}$$

where ΔL_{n+1}^α is independent of x_n and has characteristic function

$$\psi_{\Delta L_{n+1}^\alpha}(s) = \exp(-h|s|^\alpha). \tag{4}$$

The drift f and diffusion g functions can be assumed to be smooth. We can also assume that g is bounded away from zero, i.e., that there exists $\delta > 0$ such that $|g(x)| \geq \delta$ for all x. In fact, it is of interest to solve this problem (well) in the case where g is a positive constant.

We let $\widetilde{p}(x, t_n)$ denote the exact PDF of x_n, itself an approximation to the exact solution at time t_n, $X(t_n)$.

Let us denote the conditional density of x_{n+1} given $x_n = y$ by $p_{n+1,n}(x|y)$. Applying this to (3), we obtain the following evolution equation for the marginal density of x_n:

$$\widetilde{p}(x, t_{n+1}) = \int_{-\infty}^{\infty} p_{n+1,n}(x|y)\widetilde{p}(y, t_n)\; dy. \tag{5}$$

From (3), we can show that the characteristic function of the conditional density $p_{n+1,n}(x|y)$ is

$$e^{is(y+f(y)h)} \exp(-h|sg(y)|^{\alpha}).$$

Therefore, we can compute the characteristic function using

$$\psi_{n+1}(s) = \int_{x=-\infty}^{\infty} e^{isx}\widetilde{p}(x, t_{n+1})dx. \tag{6}$$

The characteristic function is given by

$$\psi_{n+1}(s) = \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp\left(-h|sg(y)|^{\alpha}\right)\widetilde{p}(y, t_n)\, dy. \tag{7}$$

Since

$$\widetilde{p}(y, t_n) = \frac{1}{2\pi} \int_{u=-\infty}^{\infty} e^{-iuy}\psi_n(u)\, du \tag{8}$$

from (7) we get

$$\psi_{n+1}(s) = \int_{u=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp\left(-h|sg(y)|^{\alpha}\right)e^{-iuy}\, dy\right] \psi_n(u)\, du.$$

Defining

$$\widetilde{K}(s, u) = \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp\left(-h|sg(y)|^{\alpha}\right)e^{-iuy}\, dy,$$

we get

$$\psi_{n+1}(s) = \int_{u=-\infty}^{\infty} \widetilde{K}(s, u)\psi_n(u)\, du. \tag{9}$$

Up to quadrature, (9) is the algorithm. By repeatedly applying (9) we evolve the characteristic function forward in time. If, at any point, we want to retrieve the PDF from the characteristic function, we use (8).

Collocation. Any method to evaluate (9) will require spatial discretization, i.e., a finite-dimensional approximation of ψ_n. We have investigated thoroughly a trapezoidal discretization of the integral, using both equispaced and non-equispaced grids. Such approaches suffer from the problem that for $|u| \neq 0$ sufficiently small, $|\psi_n(u)| > 1$, a fatal issue.

Let us consider a collocation method in which we approximate ψ_n using a mixture of Gaussians:

$$\psi_n(u) \approx \sum_{m=-M}^{m=M} \gamma_m^n \exp\left(\frac{-(u-u_m)^2}{\zeta}\right) \tag{10}$$

Here we take $u_m = m\Delta u$ for some $\Delta u > 0$. There is a compatibility condition between Δu and the parameter $\zeta > 0$—the Gaussians must overlap not too little and not too much. (TODO: make this precise!) Note that the normalization condition $\psi_n(0) = 1$ turns into

$$\sum_{m=-M}^{m=M} \gamma_m^n \exp\left(\frac{-u_m^2}{\zeta}\right) = 1 \tag{11}$$

We now use (10) in (9) to obtain

$$\sum_{\ell=-M}^{\ell=M} \gamma_\ell^{n+1} \exp\left(\frac{-(s-u_\ell)^2}{\zeta}\right) = \int_{u=-\infty}^{\infty} \widetilde{K}(s,u) \sum_{m=-M}^{m=M} \gamma_m^n \exp\left(\frac{-(u-u_m)^2}{\zeta}\right) du. \tag{12}$$

Using the definition of $\widetilde{K}$, we see that

$$\begin{aligned} K(s,u_m) &= \int_{u=-\infty}^{\infty} \widetilde{K}(s,u) \exp\left(\frac{-(u-u_m)^2}{\zeta}\right) du \\ &= \frac{1}{2\pi} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha) \int_{u=-\infty}^{\infty} e^{-iuy} \left(\frac{-(u-u_m)^2}{\zeta}\right) du\, dy \\ &= \frac{\sqrt{\zeta}}{2\sqrt{\pi}} \int_{y=-\infty}^{\infty} e^{is(y+f(y)h)} \exp(-h|sg(y)|^\alpha - y^2\zeta/4) e^{-iu_m y}\, dy \end{aligned} \tag{13}$$

We have carried out the integral over u exactly. Using this in (12), we have

$$\sum_{m'=-M}^{m'=M} \gamma_{m'}^{n+1} \exp\left(\frac{-(s-u_{m'})^2}{\zeta}\right) = \sum_{m=-M}^{m=M} K(s,u_m)\gamma_m^n \tag{14}$$

We enforce this equality at the $2M+1$ points $s = u_\ell$ for $-M \leq \ell \leq M$. Let

$$C_{\ell,m} = \exp\left(\frac{-(u_\ell-u_m)^2}{\zeta}\right).$$

Similarly, let us write

$$K_{\ell,m} = K(u_\ell, u_m).$$

Then C and K are $(2M+1)\times(2M+1)$ matrices. The update equation (14) becomes

$$\gamma^{n+1} = C^{-1}K\gamma^n. \tag{15}$$

Numerical Analysis (Spatial Discretization). The idea now is to compute $\widetilde{K}(s,u)$. We split the domain of integration:

$$K(s,u) = \frac{1}{2\pi}\int_{y=-\infty}^{-L/2} e^{is(y+f(y)h)} \exp\big(-h|sg(y)|^\alpha - y^2\zeta/4\big)e^{-iuy}\,dy$$
$$+\frac{1}{2\pi}\int_{y=-L/2}^{L/2} e^{is(y+f(y)h)} \exp\big(-h|sg(y)|^\alpha - y^2\zeta/4\big)e^{-iuy}\,dy$$
$$+\frac{1}{2\pi}\int_{y=L/2}^{\infty} e^{is(y+f(y)h)} \exp\big(-h|sg(y)|^\alpha - y^2\zeta/4\big)e^{-iuy}\,dy$$

The inner integral over the finite domain $[-L/2, L/2]$ is the one we will compute using a quadrature rule. We set up an equispaced grid with $N > 0$ grid points. Then $\Delta y = L/N$ and $y_j = -L/2 + (\Delta y)j$ for $j = 0, 1, 2, \ldots, N-1$. The inner integral then becomes a sum of terms of the form

$$\int_{y_j}^{y_{j+1}} e^{iky}\phi(y)\,dy.$$

We now begin the derivation of the quadrature rule. Assume ϕ is sufficiently smooth so that we can approximate it well using the Lagrange interpolant:

$$\phi(y) \approx \frac{y-y_j}{\Delta y}\phi(y_j+\Delta y) - \frac{y-y_{j+1}}{\Delta y}\phi(y_j).$$

Using this approximation, we derive the quadrature rule:

$$\begin{aligned}\int_{y_j}^{y_{j+1}} e^{iky}\phi(y)\,dy &\approx \int_{y_j}^{y_{j+1}} e^{iky}\frac{y-y_j}{\Delta y}\phi(y_j+\Delta y)\,dy - \int_{y_j}^{y_{j+1}} e^{iky}\frac{y-y_{j+1}}{\Delta y}\phi(y_j)\,dy \\ &= \frac{m_1(y_j)-y_jm_0(y_j)}{\Delta y}\phi(y_j+\Delta y) - \frac{m_1(y_j)-y_{j+1}m_0(y_j)}{\Delta y}\phi(y_j) \qquad (16)\end{aligned}$$

In this derivation, we have

$$m_0(y_j) = \int_{y_j}^{y_{j+1}} e^{iky}\,dy = \begin{cases}\Delta y & k=0\\ (ik)^{-1}(e^{iky_{j+1}}-e^{iky_j}) & k\neq 0\end{cases}$$

and

$$m_1(y_j) = \int_{y_j}^{y_{j+1}} ye^{iky}\,dy = \begin{cases}\frac{1}{2}(y_{j+1}^2-y_j^2) & k=0\\ (ik)^{-1}(y_{j+1}e^{iky_{j+1}}-y_je^{iky_j}) + k^{-2}(e^{iky_{j+1}}-e^{iky_j}) & k\neq 0.\end{cases}$$

Equipped with the quadrature rule, we set

$$\phi(x) = \exp\left[h(isf(x) - |sg(x)|^\alpha) - x^2\zeta/4\right]$$

and complete the calculation:

$$
\begin{aligned}
\frac{1}{2\pi}\int_{y=-L/2}^{L/2} & e^{is(y+f(y)h)}\exp\left(-h|sg(y)|^{\alpha}-y^2\zeta/4\right)e^{-iuy}\,dy \\
&= \frac{1}{2\pi}\sum_{j=0}^{N-1}\int_{y_j}^{y_{j+1}} e^{i(s-u)y}\phi(y)\,dy \\
&\approx \frac{1}{2\pi}\sum_{j=0}^{N-1}\frac{m_1(y_j)-y_j m_0(y_j)}{\Delta y}\phi(y_j+\Delta y)-\frac{m_1(y_j)-y_{j+1}m_0(y_j)}{\Delta y}\phi(y_j). \quad (17)
\end{aligned}
$$

Asymptotics. We now turn to the integrals over the unbounded domains $(-\infty,-L/2)$ and $(L/2,+\infty)$. Our strategy will be to use asymptotic approximations of f and g to compute the integrals. We will explain this strategy by example.

Suppose we have a potential $V(x)$—here we think of harmonic or anharmonic, single- or multiple-well potentials. Then we can define

$$f(x) = -\frac{dV}{dx}.$$

Suppose that the potential is eventually linear, i.e., for $|x| \geq L/2$,

$$V(x) = C|x| + D.$$

This implies that for $|x| \geq L/2$,

$$f(x) = -C\,\mathrm{sgn}(x).$$

Further assume that the diffusion coefficient is constant, i.e.,

$$g(x) = g > 0.$$

Let erfc be the complementary error function defined by

$$\mathrm{erfc}(z) = 1 - \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt.$$

Then the asymptotic contributions will be:

$$
\begin{aligned}
\frac{1}{2\pi}\int_{y=-\infty}^{-L/2} & e^{is(y+f(y)h)}\exp\left(-h|sg(y)|^{\alpha}-y^2\zeta/4\right)e^{-iuy}\,dy \\
&\approx \frac{1}{2\pi}\int_{y=-\infty}^{-L/2} e^{i(s-u)y+isCh}e^{-h|s|^{\alpha}g^{\alpha}-y^2\zeta/4}\,dy \\
&= (2\sqrt{\pi\zeta})^{-1}\exp\left(-h|s|^{\alpha}g^{\alpha}-\frac{(s-u)^2}{\zeta}+isCh\right)\mathrm{erfc}\left(\frac{\zeta L-4i(s-u)}{4\sqrt{\zeta}}\right) \quad (18)
\end{aligned}
$$

and

$$\begin{aligned}
\frac{1}{2\pi}\int_{y=L/2}^{\infty} & e^{is(y+f(y)h)}\exp\left(-h|sg(y)|^{\alpha}-y^{2}\zeta/4\right)e^{-iuy}\,dy \\
&\approx \frac{1}{2\pi}\int_{y=L/2}^{\infty} e^{i(s-u)y-isCh}e^{-h|s|^{\alpha}g^{\alpha}-y^{2}\zeta/4}\,dy \\
&= (2\sqrt{\pi\zeta})^{-1}\exp\left(-h|s|^{\alpha}g^{\alpha}-\frac{(s-u)^{2}}{\zeta}-isCh\right)\operatorname{erfc}\left(\frac{\zeta L+4i(s-u)}{4\sqrt{\zeta}}\right) \qquad (19)
\end{aligned}$$