LAB 3

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Section 1

Manually calculate the Tensor Product of the following two vectors in two ways: $\begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Task 1.1 Conduct calculation by their vector forms.

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \times \frac{1}{\sqrt{2}} \\ \frac{1}{2} \times -\frac{1}{\sqrt{2}} \\ 0 \times \frac{1}{\sqrt{2}} \\ 0 \times -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} \times \frac{1}{\sqrt{2}} \\ -\frac{1}{2} \times -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$
$$= \frac{1}{2\sqrt{2}}|000\rangle - \frac{1}{2\sqrt{2}}|001\rangle - \frac{1}{2\sqrt{2}}|100\rangle + \frac{1}{2\sqrt{2}}|101\rangle + \frac{1}{2}|110\rangle - \frac{1}{2}|111\rangle$$

Task 1.2 Convert these two vectors into their basis forms and then conduct calculation by their basis forms.

$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}|00\rangle - \frac{1}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$$
$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$
$$\begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{2}|00\rangle - \frac{1}{2}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$
$$= \frac{1}{2\sqrt{2}}|000\rangle - \frac{1}{2\sqrt{2}}|001\rangle - \frac{1}{2\sqrt{2}}|100\rangle + \frac{1}{2\sqrt{2}}|101\rangle + \frac{1}{2}|110\rangle - \frac{1}{2}|111\rangle$$

As we can see this is the same output in task 1.1.

Section 2

Factor the joint state $\frac{4}{5\sqrt{2}}|00\rangle - \frac{3}{5\sqrt{2}}|01\rangle - \frac{4}{5\sqrt{2}}|10\rangle + \frac{3}{5\sqrt{2}}|11\rangle$ into two independent qubit states.

Task 2.1 Verify this joint state is not entangled.

Given a joint state $r|00\rangle + s|01\rangle + t|10\rangle + u|11\rangle$. For the state to not be entangled ru = st.

We have: $ru = \frac{4}{5\sqrt{2}} \times \frac{3}{5\sqrt{2}} = \frac{12}{50}$ and $st = -\frac{3}{5\sqrt{2}} \times -\frac{4}{5\sqrt{2}} = \frac{12}{50}$, so $ru = st$ holds and is not entangled. Thus, this joint state can be factored into two independent states.

Task 2.2 Include the detailed steps for the factoring.

$$\begin{aligned} \frac{4}{5\sqrt{2}}|00\rangle - \frac{3}{5\sqrt{2}}|01\rangle - \frac{4}{5\sqrt{2}}|10\rangle + \frac{3}{5\sqrt{2}}|11\rangle &= \frac{4}{5\sqrt{2}}|0\rangle \otimes |0\rangle - \frac{3}{5\sqrt{2}}|0\rangle \otimes |1\rangle - \frac{4}{5\sqrt{2}}|1\rangle \otimes |0\rangle + \frac{3}{5\sqrt{2}}|1\rangle \otimes |1\rangle \\ &= |0\rangle \otimes (\frac{4}{5\sqrt{2}}|0\rangle - \frac{3}{5\sqrt{2}}|1\rangle) - |1\rangle \otimes (\frac{4}{5\sqrt{2}}|0\rangle - \frac{3}{5\sqrt{2}}|1\rangle) \\ &= \frac{1}{\sqrt{2}}|0\rangle \otimes (\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle) - \frac{1}{\sqrt{2}}|1\rangle \otimes (\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle) \\ &= (\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle) \otimes (\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle) \end{aligned}$$

So the two individual qubits are $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ and $\frac{4}{5}|0\rangle - \frac{3}{5}|1\rangle$. Both qubits satisfy that the sum of their amplitudes squared equals 1.

Section 3

You are given three vectors: $|x\rangle$ = $\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$ and $|y\rangle$ = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $|z\rangle$ = $\begin{bmatrix} 1 \\ 8 \end{bmatrix}$ use these to demonstrate the following properties using python in the next sections:

Task 3.1 Distributivity:

We need to test if the following holds: $|x\rangle(|y\rangle + |z\rangle) = |x\rangle|y\rangle + |x\rangle|z\rangle$

In [28]:
```
import numpy as np

x = np.array([5,6,7])
x
```
Out[28]:
```
array([5, 6, 7])
```

In [27]:
```
y = np.array([2,3])
y
```
Out[27]:
```
array([2, 3])
```

In [26]:
```
z = np.array([1,8])
z
```
Out[26]:
```
array([1, 8])
```

Let's first calculate $|x\rangle \otimes (|y\rangle + |z\rangle)$:

In [29]:
```
yaz = y+z
yaz
prod_x_yaz = np.kron(x, yaz)
prod_x_yaz
```
Out[29]:
```
array([15, 55, 18, 66, 21, 77])
```

Then, let's calculate $(|x\rangle \otimes |y\rangle) + (|x\rangle \otimes |z\rangle)$:

In [30]:
```
prod_xy = np.kron(x, y)
prod_xz = np.kron(x, z)
sum_xy_xz = prod_xy + prod_xz
sum_xy_xz
```
Out[30]:
```
array([15, 55, 18, 66, 21, 77])
```

In [31]:
```
prod_x_yaz == sum_xy_xz
```
Out[31]:
```
array([ True,  True,  True,  True,  True,  True])
```

We notice in the above that both calculations produce the same array and thus satisfies Distributivity. Since $|x\rangle(|y\rangle + |z\rangle) = |x\rangle|y\rangle + |x\rangle|z\rangle$

Task 3.2 Associativity:

We need to test if the following holds: $(|x\rangle \otimes |y\rangle) \otimes |z\rangle = |x\rangle \otimes (|y\rangle \otimes |z\rangle)$

In [32]:
```
prod_xy_z = np.kron(prod_xy, z)
prod_xy_z
```
Out[32]:
```
array([ 10,  80,  15, 120,  12,  96,  18, 144,  14, 112,  21, 168])
```

In [33]:
```
prod_yz = np.kron(y,z)
prod_x_yz = np.kron(x, prod_yz)
prod_x_yz
```
Out[33]:
```
array([ 10,  80,  15, 120,  12,  96,  18, 144,  14, 112,  21, 168])
```

In [34]:
```
prod_xy_z == prod_x_yz
```
Out[34]:
```
array([ True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True])
```

We notice in the above that both calculations produce the same array and thus satisfies Associativity. Since $(|x\rangle \otimes |y\rangle) \otimes |z\rangle = |x\rangle \otimes (|y\rangle \otimes |z\rangle)$

Task 3.3 Commutativity:

To test commutativity, we will test on the vectors $|x\rangle$, $|y\rangle$ and therefore, we must show that $|x\rangle \otimes |y\rangle \neq |y\rangle \otimes |x\rangle$.

In [22]:
```
prod_yx = np.kron(y,x)
prod_yx
```
Out[22]:
```
array([10, 12, 14, 15, 18, 21])
```

In [23]:
```
prod_xy
```
Out[23]:
```
array([10, 15, 12, 18, 14, 21])
```

In [24]:
```
prod_xy == prod_yx
```
Out[24]:
```
array([ True, False, False, False, False,  True])
```

We can see in the above that the output of the vectors are different and thus $|x\rangle \otimes |y\rangle \neq |y\rangle \otimes |x\rangle$ and not commutative.

Section 4

You are given two independent qubits: $q_0 = \frac{1}{3}|0\rangle - \frac{2\sqrt{2}}{3}|1\rangle$ and $q_1 = \frac{5}{13}|0\rangle - \frac{12}{13}|1\rangle$. Use Qiskit package to perform the following.

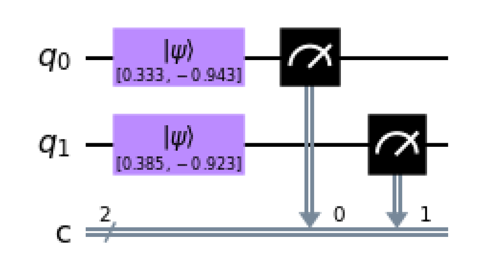
Task 4.1:

Construct a circuit with 2-qubit input and 2-bit output. Initialize q_0 as $\frac{1}{3}|0\rangle - \frac{2\sqrt{2}}{3}|1\rangle$ and q_1 as $\frac{5}{13}|0\rangle - \frac{12}{13}|1\rangle$. Measure these two qubits. Draw this circuit.

In [6]:
```
from qiskit import *
from qiskit.providers.aer import QasmSimulator
from qiskit.visualization import plot_histogram
from math import sqrt

# Create a quantum circuit with 2 input qubits and 2 output bits
qc = QuantumCircuit(2,2)
# Initialize qubit 0 as |q0>
initial_state = [1/3, -(2*sqrt(2))/3]
qc.initialize(initial_state, 0)
# Initialize qubit 1 as |q1>
initial_state = [5/13, -12/13]
qc.initialize(initial_state, 1)
# Store the measurement results of qubits 0 and 1 into bits 0 and 1 respectively
qc.measure([0,1], [0,1])

# Visualize this circuit
%matplotlib inline
qc.draw()
```
Out[6]:

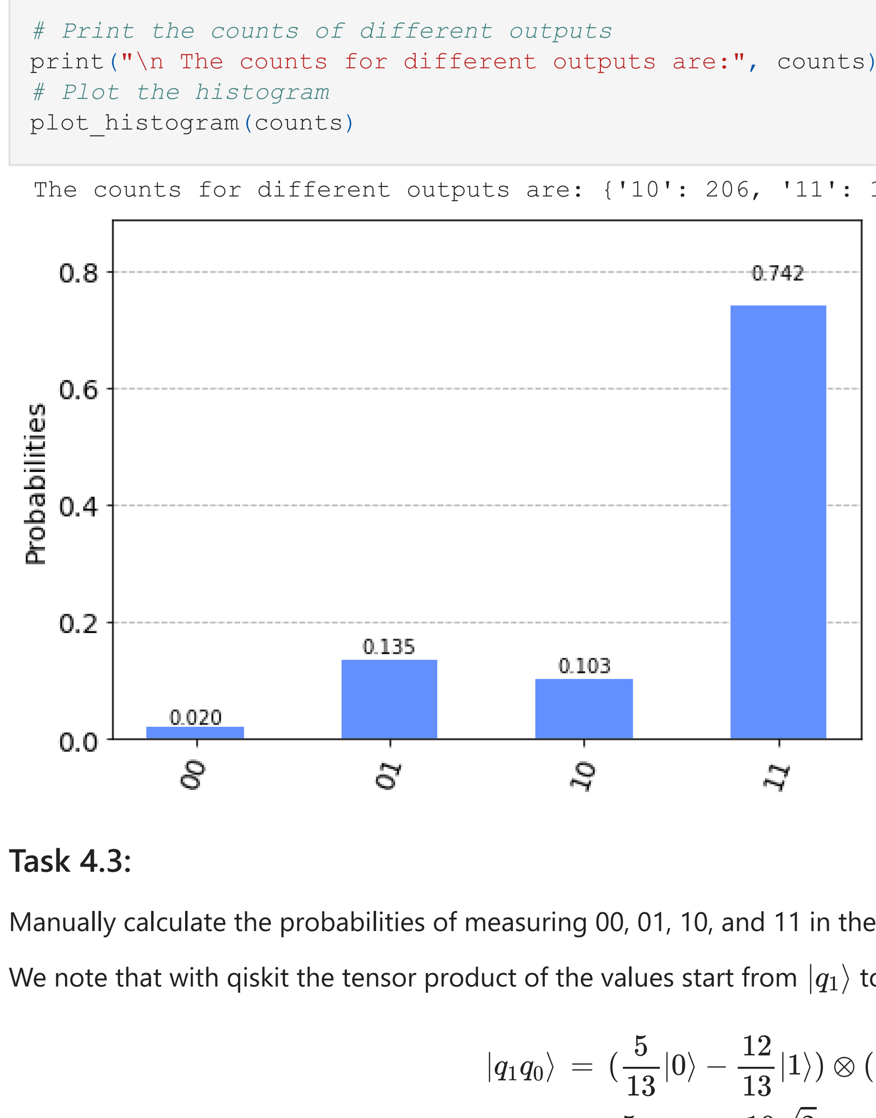


Task 4.2:

Use a QasmSimulator to run the above circuit with 2000 shots, and then plot the measurement results.

In [38]:
```
simulator = QasmSimulator()
job = execute(qc, backend=simulator, shots=2000)
# Call result() to obtain the result
result = job.result()
# Call get_counts() to obtain the counts of different outputs
counts = result.get_counts(qc)

# Print the counts of different outputs
print("\n The counts for different outputs are:", counts)
# Plot the histogram
plot_histogram(counts)
```
```
 The counts for different outputs are: {'10': 206, '11': 1484, '01': 270, '00': 40}
```
Out[38]:



Task 4.3:

Manually calculate the probabilities of measuring 00, 01, 10, and 11 in theory.

We note that with qiskit the tensor product of the values start from $|q_1\rangle$ to $|q_0\rangle$ thus we calculate the tensor product in the following way:

$$\begin{aligned} |q_1 q_0\rangle &= (\frac{5}{13}|0\rangle - \frac{12}{13}|1\rangle) \otimes (\frac{1}{3}|0\rangle - \frac{2\sqrt{2}}{3}|1\rangle) \\ &= \frac{5}{39}|00\rangle - \frac{10\sqrt{2}}{39}|01\rangle - \frac{12}{39}|10\rangle + \frac{24\sqrt{2}}{39}|11\rangle \end{aligned}$$

So we have:

$P(|q_1 q_0\rangle \text{ measured as } |00\rangle) = (\frac{5}{39})^2 = \frac{25}{1521} \approx 0.016,$

$P(|q_1 q_0\rangle \text{ measured as } |01\rangle) = (-\frac{10\sqrt{2}}{39})^2 = \frac{200}{1521} \approx 0.131,$

$P(|q_1 q_0\rangle \text{ measured as } |10\rangle) = (-\frac{12}{39})^2 = \frac{16}{169} \approx 0.095$

$P(|q_1 q_0\rangle \text{ measured as } |11\rangle) = (\frac{24\sqrt{2}}{39})^2 = \frac{128}{169} \approx 0.757.$

Task 4.4:

Show by calculation that the probabilities obtained in 4.2 roughly match the probabilities obtained in 4.3.

We can see in task 4.3 the approximate probabilities are given in the calculations. With the graph we take the values for the bit with index 0 is for $|q_0\rangle$, and the bit with index 1 is for $|q_1\rangle$.For the four outcomes in this plot, if we index them by subscripts, they should look like $0_1 0_0$, $0_1 1_0$, $1_1 0_0$, and $1_1 1_0$. So the bit with index `1` is for $|q_1\rangle$, and the bit with index `0` is for $|q_0\rangle$. We read this off the graph from right to left.

The probability calculated in the previous section was:

$P(|q_1 q_0\rangle \text{ measured as } |00\rangle) = (\frac{5}{39})^2 = \frac{25}{1521} \approx 0.016,$

$P(|q_1 q_0\rangle \text{ measured as } |01\rangle) = (-\frac{10\sqrt{2}}{39})^2 = \frac{200}{1521} \approx 0.131,$

$P(|q_1 q_0\rangle \text{ measured as } |10\rangle) = (-\frac{12}{39})^2 = \frac{16}{169} \approx 0.095$

$P(|q_1 q_0\rangle \text{ measured as } |11\rangle) = (\frac{24\sqrt{2}}{39})^2 = \frac{128}{169} \approx 0.757.$

The probability for $|11\rangle$ is 0.757 and 0.742 for the simulator, for $|00\rangle$ is 0.016 and 0.02 for the simulator, for $|01\rangle$ is 0.131 and 0.135 for the simulator, and for $|10\rangle$ 0.095 and 0.103 for the simulator.

This shows that the calculation roughly matches the values in the simulator obtained in 4.2.