Chapter 9

Show S^2 is an unbiased estimator of σ^2

$$\sum_{i=1}^{n}(X_i-\bar{X})^2 = \sum_{i=1}^{n}[(X_i-\mu)-(\bar{X}-\mu)]^2$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 - 2(\bar{X}-\mu)\sum_{i=1}^{n}(X_i-\mu) + n(\bar{X}-\mu)^2$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2$$

now, $E[S^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right]$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}E(X_i-\mu)^2 - nE(\bar{X}-\mu)^2\right]$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n}\sigma_{X_i}^2 - n\sigma_{\bar{X}}^2\right)$$

however, $\sigma_{X_i}^2 = \sigma^2$ for $i=1,2,\ldots,n$ &

$$\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$$

Therefore, $E[S^2] = \frac{1}{n-1}\left(n\sigma^2 - n.\frac{\sigma^2}{n}\right)$

$$= \sigma^2 \quad \blacksquare$$

95% CI Sleep example

$\bar{x} = 7.85$

$n = 20$

$\sigma = 1.7$

$\alpha = 0.05$ (95% CI)

$$CI = \bar{x} \pm z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} = 7.85 \pm z_{\frac{0.05}{2}} \frac{(1.7)}{\sqrt{20}}$$

$$= 7.85 \pm 1.96 \frac{(1.7)}{\sqrt{20}}$$

$$= (7.10, 8.59)$$

① 95% CI for $\bar{x} = 2491.43$

$\sigma = 842.62$

$n = 45$

$$\bar{x} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \rightarrow 2491.43 \pm Z_{\frac{0.05}{2}} \frac{(842.62)}{\sqrt{45}}$$

$$\rightarrow 2491.43 \pm 1.96 \frac{(842.62)}{\sqrt{45}}$$

$$= (2245.24, 2737.626)$$

② find 99% CI for $\bar{x} = 71.8$

$\sigma = 8.9$

$n = 100$

$$71.8 \pm Z_{\frac{0.01}{2}} \frac{(8.9)}{\sqrt{100}}$$

$$= 71.8 \pm 2.58 \frac{(8.9)}{\sqrt{100}}$$

$$= (69.5, 74.1)$$

for ② above find

① bound of error for $\alpha = 0.01$

$$Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = 2.58 \frac{(8.9)}{100} = 2.296$$

② sample size necessary to make error no bigger than 1.5

$$n = \left(\frac{Z_{\alpha/2}\,\sigma}{e}\right)^2 = \left(\frac{2.58 \times 8.9}{1.5}\right)^2 = 234.85 \Rightarrow 235$$

Vacuum

$\bar{X} = 42$
$S = 11.9$
$n = 12$

90% CI: $\bar{X} \pm t_{\frac{\alpha}{2}} \cdot \frac{s}{\sqrt{n}}$

$= \bar{X} \pm t_{\frac{0.10}{2}} \cdot \frac{s}{\sqrt{n}}$

$= 42 \pm 1.7959 \frac{(11.9)}{\sqrt{12}}$

$= (35.83, 48.17)$

case 1: 99% CI for $\mu_1 - \mu_2$

pop 1	pop 2
$\bar{X}_1 = 81$	$\bar{X}_2 = 76$
$\sigma_1 = 5.2$	$\sigma_2 = 3.4$
$n_1 = 25$	$n_2 = 36$

$\bar{X}_1 - \bar{X}_2 \pm Z_{\frac{0.01}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

$= 81 - 76 \pm 2.58 \sqrt{\frac{5.2^2}{25} + \frac{3.4^2}{36}}$

$= (1.94, 8.05)$

case 2 : 95% CI for $\mu_1 - \mu_2$

lab	no lab
$\bar{x}_1 = 85$	$\bar{x}_2 = 79$
$s_1 = 4.7$	$s_2 = 6.1$
$n_1 = 11$	$n_2 = 17$

$$95\%\ CI = \bar{x}_1 - \bar{x}_2 \pm t_{\frac{0.05}{2}}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$= 85 - 79 \pm 2.056 \sqrt{31.395} \sqrt{\frac{1}{11} + \frac{1}{17}}$$

$$= (1.543, 10.468)$$

case 3: 90% CI for $\mu_1 - \mu_2$

F	M
$\bar{X}_1 = 54.05$	$\bar{X}_2 = 22.42$
$S_1 = 41.61$	$S_2 = 23.02$
$n_1 = 37$	$n_2 = 12$

$$90\%\ CI: (\bar{X}_1 - \bar{X}_2) \pm t_{\frac{0.10}{2}} \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

$$= 54.05 - 22.42 \pm 1.677 \sqrt{\frac{41.61^2}{37} + \frac{23.02^2}{12}}$$

$$= (15.64, 47.61)$$