Probability Take Home Questions 1 (Solutions)

1. There are 10 people. What is are the number of ways in which you can split them into a team of 6 and a team of 4? Additionally, what are the number of ways in which you can split them into two teams of 5 each?

Solution

- Selecting a team of 6 and a team of 4 out of 10 people : ${}^{10}C_6 = {}^{10}C_4$
- Selecting a two teams of 5 out of 10 people is done ${}^{10}C_5/2$ number of ways
- For the second part, instead if we selected just one team of 5, we would be doing that in ${}^{10}C_5$ number of ways. Since we need to choose two team so 5, when we select one team of 5, the 5 remaining players automatically form another team. So there would be a double count when we form two teams of 5. Hence the division by 2!

2. Prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
Hint: The goal is to prove this mathematically (not using Venn diagrams). Try writing $A \cup B$ as union of three disjoint sets and then apply axiom 3 that we discussed in class.

Solution

- $A \cup B$ can be written as a union of three disjoint sets and then using axiom 3 (probability of union of disjoint sets is the sum of their individual probabilities)

$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B) \implies P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$$

$$P(A \cap B^C) = P(A) - P(A \cap B) \text{ and } P(A^C \cap B) = P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

3. Two people take turns trying to sink a basketball into a net. Person 1 succeeds with probability 1/3 and person 2 succeeds with probability 1/4. What is the probability that person 2 succeeds before person 1. Additionally, compute the probability that person 1 succeeds before person 2.

Solution

- Defining event $A_1 = \{\text{Person 2 shoots in try 1}\}$, $P(A_1) = 1/4$
- Defining event $A_2 = \{\text{Person 2 shoots in try 2 given Person 1 and Person 2 miss in try 1}\}$

$$P(A_2) = \underbrace{3/4}_{\text{P2 miss}} \cdot \underbrace{2/3}_{\text{P1 miss}} \cdot \underbrace{1/4}_{\text{P2 score}} = 1/2 \cdot 1/4$$

$$P(A_3) = (1/2)^2 \cdot 1/4$$

$$P(A_n) = (1/2)^{n-1} \cdot 1/4$$

$$P(\text{P2 succeeding before P1}) = P(A_1) + P(A_2) + .. = \sum_{i=1}^{\infty} (1/2)^{i-1} \cdot 1/4 = 1/4 \cdot \frac{1}{1 - 1/2} = \frac{1}{2}$$

- Similarly we can compute the same probability for P1 as

$$P(\text{P1 succeeding before P2}) = P(A_1) + P(A_2) + .. = \sum_{i=1}^{\infty} (1/2)^{i-1} \cdot 1/3 = 1/3 \cdot \frac{1}{1 - 1/2} = \frac{2}{3}$$

4. Suppose we toss a fair coin until we get exactly two heads. Describe the sample space S. What is the probability that exactly k tosses are required?

- Sample space will look like $\Omega = \{TT..HT..H, THT..TH, ..\}$
- In k-tosses, we know that there are two H and k-2 T and we also know that the last toss is a H. This is equivalent to choosing 1 position from k-1 positions to place the one remaining H and the left over positions are all filled by T.

Solution

$$P(\text{k-tosses for 2 H}) = \underbrace{{}^{k-1}C_1}_{\text{Choosing 1H from k-1 spots}} \cdot \underbrace{(1/2)^{k-2}}_{\text{k-2 Tails}} \cdot \underbrace{(1/2)^2}_{\text{2 Heads}}$$

5. **Birth Month Problem -** This is a variation of the birthday problem. How many people are needed in a room to make it possible that atleast two people have the same birth month with a probability of 50%. There are 12 months in a year and probability of being born in any month is the same. *Hint*: Try expressing the probability of atleast 2 people having the same birth month as a complementary event.

Solution

If we have 13 people, atleast 2 of them will have the same birth month. Say we have n-people in our sample $(n < 12)$. Since each person is different, ordering matters. We can assign unique birth ${}^12P_n$ number of ways -

$$p = P(\text{No one shares birth month}) = \frac{12 \cdot 11..(12 - n + 1)}{12^n}$$

$$P(\text{Atleast 2 people share birth-month}) = 1 - P(\text{No one shares birth-month}) = 1 - p$$

6. **(optional) Birth Month Paradox Advanced Version -** In the above question, we assumed that the probability of being born in any month is equal. Historically, month of August has seen the largest number of births compared to any other month. If we were to incorporate this new information, say probability of being born in August is p and probability of being born in any other month is q. What would be your approach to solving the above problem with this new information? *Hint* : The trick here would still be to write the probabilities in terms of its complementary event. Try enumerating possible combinations and compute the probability using the non-uniform distribution.