PROBABILITY THEORY CLASS 2

- Agenda for today

  - Take home Q ✓
  - Independence
  - Conditional probability
  - Conditional Independence
  - Simpson's paradox
  - Bayes Rule

- Independence

  Two events A, B are independent iff.

  $$P(AB) = P(A \cap B) = P(A) \cdot P(B)$$

  Proving independence → Assumption : 2 coin tosses

  ↳ Derived : $\underline{P(AB)} = P(A) \cdot P(B)$

Ex: $\Omega = \{1,2,3,4,5,6\}$

$A = \{1,2,3,4\}$, $P(A) = \frac{4}{6} = \frac{2}{3}$

$B = \{2,4,6\}$, $P(B) = \frac{3}{6} = \frac{1}{2}$

$AB = \{2,4\}$, $P(AB) = \frac{2}{6} = \frac{1}{3}$

clearly $P(AB) = \frac{1}{3} = P(A) \cdot P(B)$

Occurance of B has no effect on A

Say B has happened, then

$\underline{2 \in A}$, $\underline{4 \in A}$, $\underline{6 \notin A}$

$\Rightarrow P(A|B) = \frac{2}{3}$ (same as above)

Ex: $\Omega = \{1,2,3,4,5,6\}$

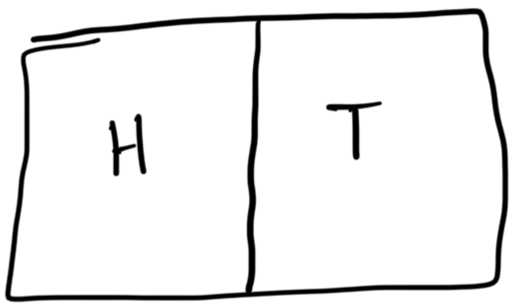
Dependent $\left\{ \begin{array}{l} A = \{2,4,6\}, \quad P(A) = 1/2 \\ B = \{4,5,6\}, \quad P(B) = 1/2 \end{array} \right.$

$AB = \{4,6\}$, $P(AB) = 1/3$

- Disjoint : { H/T , Odd/even }

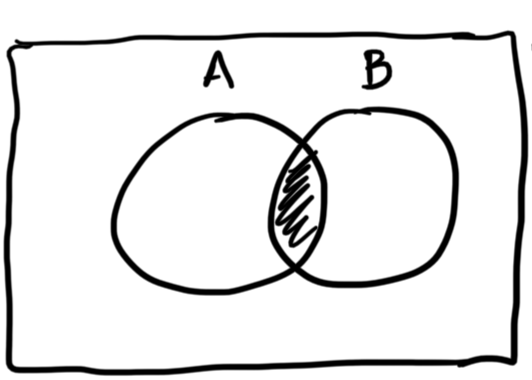
vs

Independence : { 2 tosses, 2 days of stock ret }



- Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

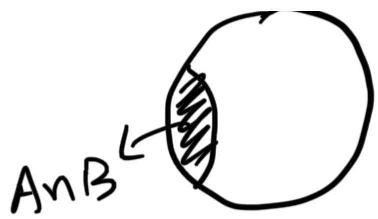


S

$\downarrow$

B

$$\frac{n_{A \cap B}/N}{\ldots} = \frac{P(AB)}{\ldots}$$



$\overline{}\; n_B / N$ $\quad \underbrace{P(B)}_{>0}$

Normalize

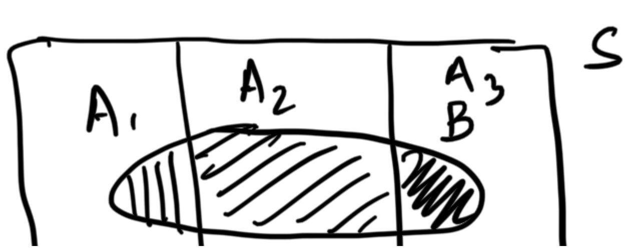
$$P(A|B) = P(A)$$

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$P(B|A) = \frac{P(AB)}{P(A)}$$

$$P(A|B) = P(B|A) \cdot \left.\frac{P(A)}{P(B)}\right\}$$ Bayes Rule

- Law of total Prob.

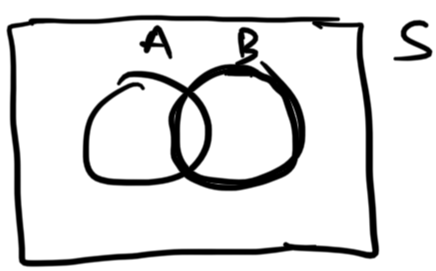


$$B = (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3)$$

$$P(B) = P(BA_1) + P(BA_2) + P(BA_3)$$

$$P(B \mid A_1) = \frac{P(BA_1)}{P(A_1)}$$

$$P(B) = P(B \mid A_1) \cdot P(A_1) + P(B \mid A_2) \cdot P(A_2) + P(B \mid A_3) \cdot P(A_3)$$



B

B^c

$$P(A) = P(A \mid B) \cdot P(B) + P(A \mid B^c) \cdot P(B^c)$$

car alarm

Notes

- Notes

(a) $P(A|B) \neq P(B|A)$

(b) $P(A|B) \neq 1 - P(A|B^c)$

(c) $P(A|B) = 1 - P(A^c|B)$

(d) $$\underbrace{P(A|B)}_{\text{Posterior}} = \frac{\overbrace{P(B|A)}^{\text{Likelihood}} \cdot \overbrace{P(A)}^{\text{Prior}}}{P(B)}$$

- Conditional Independence

A and B are conditionally independent given C iff

$$P(A \cap B \mid C) = P(A|C) \cdot P(B|C)$$

Ex: Unconditional independence $\neq$ conditional Independence

$$\Omega = \{1,2,3,4,5,6\}$$

$$A = \{1,2,3,4\}, P(A) = 2/3$$

$$B = \{2,4,6\}, P(B) = 1/2$$

$$AB = \{2,4\}, P(AB) = 1/3$$

$$C = \{2,3,4,5\}$$

$$A \cap C = \{2,3,4\} = P(AC) = 1/2$$

$$B \cap C = \{2,4\} = P(BC) = 1/3$$

$$AB \cap C = \{2,4\} \quad P(AB \cap C) = 1/3$$

$$\underline{P(A|C)} = \frac{P(A \cap C)}{P(C)} = \frac{3/6}{4/6} = \frac{3}{4}$$

$$P(B|C) = \frac{2/6}{4/6} = \frac{1}{2}$$

$$P(AB|C) = \frac{2/6}{\quad} = \frac{1}{2}$$

$4/6$

[condt. dep.] $P(AB|C) \neq P(A|C) \cdot P(B|C)$

[Uncondt. Indep] $P(AB) = P(A) \cdot P(B)$

Two events that are unconditionally independent but conditionally dependent

Ex: Two events that are dependent but conditionally independent

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{2, 4, 6\}, \; P(A) = 1/2$$

$$B = \{4, 5, 6\}, \; P(B) = 1/2$$

$$AB = \{4, 6\}, \; P(AB) = 1/3$$

$$C = \{2, 3, 4, 5\}$$

$P(AC)$ $2/6$

$$\underline{AC} = \{2, 4\}, \; P(A|C) = \frac{P(AC)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}$$

$$\underline{BC} = \{4, 5\} = P(B|C) = \frac{P(BC)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2}$$

$$AB \cap C = \{4\}, \; P(A \cap B|C) = \frac{P(AB \cap C)}{P(C)} = \frac{1/6}{4/6} = 1/4$$

$$P(AB|C) = \frac{1}{4} = P(A|C) \cdot P(B|C) = \frac{1}{2} \times \frac{1}{2}$$

- Simpson's Paradox

Dr. Dave (Good)

	Heart	Bandaid	
Success	70	10	$= \frac{80}{20}$
Fail	20	0	

$\frac{70}{}$ = 76%, $\frac{10}{}$ = 100%

Dr. Nick (Bad)

	Heart	Bandaid	
Success	2	81	$= \frac{83}{17}$
Fail	8	9	

$\frac{2}{10}$ = 20%, $\frac{81}{90}$ = 90%

90 $\underbrace{\quad}_{10}$

= 80%

$\underbrace{\quad}$ $\underbrace{\quad}$

= 83%

- confounding variables

$$y \sim x$$

$$y \sim \underbrace{x_1}_{\text{Heart}} + \underbrace{x_2}_{\text{Bandaid}}$$

- Bayes Theorem

$$\underbrace{P(A|B)} = \frac{\underbrace{P(B|A)} \cdot P(A)}{P(B)}$$

$$P(\text{Hypothesis} \mid \underset{\text{Data}}{\overset{\text{Evidence}}{\shortparallel}})$$

Ex: P(Car Theft) = 1%

A B

$$P(\text{Car Alarm} \mid \text{Theft}) = 90\%$$

$$P(\underset{A}{\text{Car Alarm}} \mid \underset{B^c}{\text{No theft}}) = 20\%$$

$$P(\text{Theft} \mid \text{Car Alarm})$$

$$P(\text{Theft} \mid \text{Car Alarm}) = \underbrace{P(\text{Car Alarm} \mid \text{Theft})}_{0.9} \times \frac{\overbrace{P(\text{Theft})}^{0.01}}{\underbrace{P(\text{Car Alarm})}}$$

$$P(\text{Car Alarm}) = \overbrace{P(\text{Car Alarm} \mid \text{Theft})}^{=0.9} \times \overbrace{P(\text{Theft})}^{0.01} + \underbrace{P(\text{Car Alarm} \mid \text{No theft})}_{0.2} \times \underbrace{P(\text{No theft})}_{0.99}$$

$$P(\text{Car Alarm}) = 0.207$$

$$P(\text{Theft} \mid \text{Car Alarm}) = \underbrace{P(CA \mid T)} \times \frac{P(T)}{P(CA)}$$

$$= 0.9 \times \frac{0.01}{0.207} = \boxed{0.043}$$

$$P(H_1 \mid \underbrace{Data}) = P(D \mid H) \cdot \frac{P(H)}{P(D)}$$

$H_2 \mid D$