Probability Take Home Questions 3

1. Show that sum of two independent binomial distributions is also binomial, i.e. $X \sim \text{Bin}(n,p), Y \sim \text{Bin}(m,p)$ then $X+Y \sim \text{Bin}(n+m,p)$. *Hint*: You might want to use the Vandermonde's identity, which states $\sum_{i=0}^{k} {}^{n}C_i \; {}^{m}C_{k-i} = {}^{n+m}C_k$.

Solution

$$\text{For } 0 \leq k \leq n+m, P(X+Y=k) = \sum_{i=0}^{k} P(Y=k-i|X=i) \cdot P(X=i)$$

$$P(X+Y=k) = \sum_{i=0}^{k} P(Y=k-i) \cdot P(X=i) = \sum_{i=0}^{k} \underbrace{{}^{n}C_i p^i q^{n-i}}_{\text{Bin}(n,p)} \cdot \underbrace{{}^{m}C_{k-i} p^{k-i} q^{m-k+i}}_{\text{Bin}(m,p)}$$

$$\implies P(X+Y=k) = p^k q^{m+n-k} \sum_{i=0}^{k} {}^{n}C_i \; {}^{m}C_{k-i} = {}^{m+n}C_k p^k q^{m+n-k}$$

2. Say $X \sim \text{Pois}(3)$ and $Y \sim \text{Pois}(2)$ are two independent poisson distributions. Compute the probability that $X+Y=5$. Additionally, compute the probability that $X=4, Y=1$ given $X+Y=5$. *Hint*: Sum of two poisson distributions is also a poisson distribution. For the second part, independence assumption is key!

Solution

$$X \sim \text{Pois}(3), Y \sim \text{Pois}(2)$$
$$X+Y \sim \text{Pois}(2+3=5)$$
$$P(X+Y=5) = e^{-5}5^5/5!$$

$$P(X=4, Y=1|X+Y=5) = \frac{P(X=4, Y=1)}{P(X+Y=5)} = \frac{P(X=4) \cdot P(Y=1)}{P(X+Y=5)} = \frac{e^{-3}3^4/4! \cdot e^{-2}2^1/1!}{e^{-5}5^5/5!} = 0.26$$

3. The arrival of buses at the bus stop can be modeled as a poisson process. Say buses arrive at a rate of 5 per hour ($\lambda = 5$). You just arrived at the bus stop. What is the probability that the next buss will arrive at the stop in the next 10 minutes. Additionally, find the probability that there will be a bus arriving in the next 5 minutes if there was no bus in the last 10 minutes.

Solution

$$P(1 \text{ Bus in } 10 \text{ min}) = \frac{e^{-5/6}(5/6)}{1!} = 0.36$$

$$P(1 \text{ Bus in 5 min after 10 min wait}) = P(1 \text{ Bus in 5 min}) = \frac{e^{-5/12}(5/12)}{1!} = 0.27$$

4. Suppose you are given the PMF function of a random variable X such that $P(X = -2) = P(X = 2) = 1/8, P(X = -1) = P(X = 1) = 1/8, P(X = 0) = 1/2$. Lets define $Y = X^2$. Compute the PMF and the CDF of the random variable Y.

Solution

$$P(X = x) = \begin{cases} 1/8, & x = -2 \\ 1/8, & x = -1 \\ 1/2, & x = 0 \\ 1/8, & x = 1 \\ 1/8, & x = 2 \end{cases}$$

$$P(Y = 1) = P(X = -1) + P(X = 1) = 1/8 + 1/8 = 1/4$$
$$P(Y = 4) = P(X = -2) + P(X = 2) = 1/8 + 1/8 = 1/4$$

$$P(Y = X^2 = y) = \begin{cases} 1/2, & y = 0 \\ 1/4, & y = 1 \\ 1/4, & y = 4 \end{cases}$$