GlashowSalamWeinberg

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0.1 Glashow-Salam-Weinberg Model

0.1.1 Introduction

In 1934, Enrico Fermi proposed the following 4-fermion model

$$\frac{G_F}{\sqrt{2}}(\overline{\psi}_e\gamma^\mu\psi_{\overline{\nu}})(\overline{\psi}_p\gamma_\mu\psi_n), \tag{1}$$

to describe β-decay processes such as $n \to p + e^- + \overline{\nu}_e$. The quantity $\overline{\psi} \equiv \psi^\dagger\gamma^0$ is called the Dirac conjugate of the spinor ψ, where the operator † is the Hermitian conjugate, i.e., the transpose and complex conjugate and γ^μ are 4×4 matrices defined by the anti-commutation relations $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, where $g^{\mu\nu}$ is the metric tensor of special relativity. G_F is called the Fermi constant.

Fermi assumed that β-decay involved vector currents, $\overline{\psi}\gamma^\mu\psi$, but in 1956 T. D. Lee and C. N. Yang noted that no experiment had been performed to test the assumption of parity invariance, that is, the invariance of particle interactions with respect to a reflection in a mirror. They suggested that certain experimental puzzles could be explained if parity was not conserved in the weak interactions. Furthermore, Lee and Yang suggested experiments to test this hypothesis. Shortly thereafter, Chien-Shiung Wu performed the experiment (see https://en.wikipedia.org/wiki/Wu_experiment) and confirmed the violation of parity invariance. The following year, Lee and Yang won the Nobel Prize in Physics. A year later, in 1958, Robert Marshak and his student George Sudarshan developed the vector axial-vector theory (V-A) of the weak interactions, which Richard Feynman and Murray Gell-Mann developed further.

We do not know why the weak interactions violate parity conservation, in particular, why they involve both vector $\overline{\psi}\gamma^\mu\psi$ and axial-vector $\overline{\psi}\gamma^\mu\gamma^5\psi$ currents ($\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$). However, now that we have a better understanding of the weak interactions, it would be simpler to say that the weak interactions treat left and right-handed fields differently. But, the V-A jargon has stuck.

The Glashow-Salam-Weinberg (GSW) Langrangian (really a Lagrangian density) is built using the fields below for the 1st generation of leptons and quarks:

$$L = \begin{pmatrix} \nu_L \\ \\ e_L \end{pmatrix}, \quad e_R,$$

$$Q = \begin{pmatrix} u_L \\ \\ d_L \end{pmatrix}, \quad u_R, \quad d_R,$$

where

$$P_L = \frac{1}{2}(1 - \gamma^5),$$

$$P_R = \frac{1}{2}(1 + \gamma^5)$$

are projection operators that split a

fermion field into its left and right components

$$\psi_{L,\,R} = P_{L,\,R}\,\psi, \tag{2}$$

such that $\psi = \psi_L + \psi_R$. Note the absence, in this model, of the field ν_R.

The GSW model, inspired by the highly successful quantum field theory quantum electrodynamics (QED) and the quantum field theories developed by Yang and Mills is defined by the Langrangian,

$$\begin{aligned}
\mathcal{L} = {} & \overline{L} i\gamma^\mu(\mathbf{1}\partial_\mu + ig_1 y_L^L \mathbf{1} B_\mu + ig_2 \mathbf{T}\cdot\mathbf{W}_\mu)L \\
& + \overline{Q} i\gamma^\mu(\mathbf{1}\partial_\mu + ig_1 y_L^Q \mathbf{1} B_\mu + ig_2 \mathbf{T}\cdot\mathbf{W}_\mu)Q \\
& + \overline{e}_R i\gamma^\mu(\partial_\mu + ig_1 y_R^e B_\mu)e_R \\
& + \overline{u}_R i\gamma^\mu(\partial_\mu + ig_1 y_R^u B_\mu)u_R \\
& + \overline{d}_R i\gamma^\mu(\partial_\mu + ig_1 y_R^d B_\mu)d_R \\
& - \frac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} \\
& - \frac{1}{4}B_{\mu\nu}B^{\mu\nu},
\end{aligned} \tag{3}$$

which explicitly treats left-handed and right-handed fields differently. The y symbols, related to the constants $Y = 2y$ called *hypercharges*, are chosen to obtain agreement with observations. We use y instead of $Y/2$, as is the usual convention, to make the expressions a bit tidier. The other symbols are:

$\mathbf{1}$ a 2×2 unit matrix,

$T = \boldsymbol{\tau/2}$ where τ are the Pauli matrices,

and the tensors $W_{\mu\nu}$ and $B_{\mu\nu}$ are given by

$$\begin{aligned}
\mathbf{W}_{\mu\nu} &= \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g_2\,\mathbf{W}_\mu \times \mathbf{W}_\nu, \\
(\mathbf{W}_{\mu\nu})_i &= (\partial_\mu\mathbf{W}_\nu)_i - (\partial_\nu\mathbf{W}_\mu)_i - g_2\,\epsilon_{ijk}W_\mu^j W_\nu^k, \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \text{ and} \\
\mathbf{W}_\mu &= (W_\mu^1,\, W_\mu^2,\, W_\mu^3).
\end{aligned}$$

The quantities enclosed in the first and second pair of parentheses in the Lagrangian are 2×2 matrices that act on the lepton and quark doublet fields L and Q, respectively, while the parentheses in the remaining fermionic terms enclose 1×1 matrices, that is, scalars, which act on the singlet fields e_R, u_R, and d_R, respectively. However, each field is intrinsically a 4-component Lorentz object: a 4-component Dirac spinor in the case of the fermionic fields and a 4-vector in the case of the bosonic fields.

In the GSW Lagrangian, the hypercharges are assigned specific numerical values. Therefore, the Lagrangian has only two free parameters, namely, the couplings g_1 and g_2.

0.1.2 $SU(2) \otimes U(1)$ Gauge Invariance

The GSW Lagrangian is invariant under the group $SU(2) \otimes U(1)$. The elements U_2 of $SU(2)$ can be represented as the 2×2 matrices

$$
\begin{aligned}
U &= \exp(i\mathbf{T} \cdot \boldsymbol{\theta}(x)), \text{ while} \\
U &= \exp(iy\,\alpha(x)),
\end{aligned}
$$

represent the elements of $U(1)$. The $SU(2)$ elements are defined by three spacetime-dependent functions, $\theta(x) = (\theta_1(x), \theta_2(x), \theta_3(x))$, each associated with group generators $T_1 = \tau_1/2$, $T_2 = \tau_2/2$, and $T_3 = \tau_3/2$, respectively, while the $U(1)$ elements are defined by a single function $\alpha(x)$ and a single generator $y = Y\,/\,2$.

Invariance under the group $SU(2) \otimes U(1)$ means that the following replacements leave the Lagrangian unchanged,

$$
\begin{aligned}
U \in \boldsymbol{SU(2)} & \\
& L \to UL, \quad Q \to UQ, \\
& e_R \to e_R, \quad u_R \to u_R, \quad d_R \to d_R \\
\mathbf{T} \cdot \mathbf{W}_\mu & \to U(\mathbf{T} \cdot \mathbf{W}_\mu)U^{-1} - \frac{1}{ig_2}(\partial_\mu U)U^{-1}, \\
& B_\mu \to B_\mu, \\
U \in \boldsymbol{U(1)} & \\
& L \to UL, \quad Q \to UQ \\
& e_R \to Ue_R, \quad u_R \to Uu_R, \quad d_R \to Ud_R, \\
& \mathbf{W}_\mu \to \mathbf{W}_\mu, \\
& B_\mu \to UB_\mu U^{-1} - \frac{1}{ig_1}(\partial_\mu U)U^{-1}.
\end{aligned}
$$

Notice the absence of mass terms such as $m\psi\psi$ and $M_B^2 B_\mu B^\mu$ in the Lagrangian. This is necessary because mass terms violate the imposed $SU(2) \otimes U(1)$ invariance.

The model above was developed by Glashow in 1961. However, because it predicted zero mass for all bosonic fields (not just for the photon), it clearly did not accord with the absence of experimental evidence for a long-range weak nuclear force. But in 1964 a mechanism for introducing mass without violating gauge invariance was developed independently by Englert & Brout, by Higgs, and by Guralnik, Hagen & Kibble in a series of papers in that year. These papers were inspired by the work of the condensed matter theorist Phil Anderson. Much to the chagrin of the other theorists, the mechanism was referred to, by Steven Weinberg, as the Higgs mechanism and the name stuck. A more accurate name, though a bit of a mouthful, would be the *Anderson-Englert-Brout-Higgs-Guralnik-Hagen-Kibble* mechanism.

In 1967, Salam and Weinberg independently showed how the Higgs mechanism can be used to correct the mass problem of the Glashow model. In addition, Weinberg introduced gauge invariant interactions between the Higgs field and the fermion fields that induced mass terms for the latter. Thus was born the **electroweak model**.

It is usually argued that **local gauge** invariance means that the phases of the quantum fields can be set to different values at different events in spacetime and that this makes sense because there is no reason why, for example, Andromedans, ten million years from now, should adopt the same phase convention as Earthlings do now. Personally, I find this argument unconvincing. There exist, after all, other symmetries with phases that remain constant over the whole of spacetime. For example, the replacements (see https://arxiv.org/pdf/hep-ph/0410370.pdf, p. 9)

$$\begin{aligned}
Q &\to e^{i\theta/3}Q, \\
u_R &\to e^{i\theta/3}u_R, \\
d_R &\to e^{i\theta/3}d_R, \\
\text{and} & \\
L &\to e^{i\phi}L, \\
e_R &\to e^{i\phi}e_R,
\end{aligned}$$

which correspond to baryon number and lepton number conservation, respectively, are **global symmetries** that leave the Lagrangian invariant. Admittedly, these global symmetries were not imposed but are inherent in the structure of the theory and are, in that sense, *accidental*. My question is: why is it fine to have global symmetries where certain quantities, such as θ and ϕ are the same everywhere and everywhen, that is, Andromedans and Earthlings must use the same values of these phases even though they have never met and yet it is not fine to have the same gauge field phases everywhere? There is no good reason that I can discern. I think it is better simply to accept the discovery that if one imposes local gauge invariance we are led to theories, which for reasons not yet clear, are spectacularly successful.

0.1.3 Hypercharges

We need a name for $y \equiv Y/2$; let's refer to y as the *reduced* hypercharge. The reduced hypercharge y, electric charge Q (in units of the proton charge and not to be confused with the quark doublet), and the third component of the (weak) isospin $I_3 = \pm 1/2$ are related by the Gell-Mann Nishijima relation $y = Q - I_3$. For the left-handed neutrino field, $Q = 0$ and $I_3 = +1/2$, therefore, the reduced hypercharge is $y_L^L = -1/2$. For the left-handed electron field, $Q = -1$ and $I_3 = -1/2$ and again $y_L^L = -1/2$. On the other hand, since the right-handed electron field is an $SU(2)$ singlet $I_3 = 0$ and, therefore, $y_R^e = -1$. We conclude that the reduced hypercharge is a property of the doublet or singlet.

In the quark model, up quarks have electric charge $Q = +2/3$ (in units of the proton charge), while down quarks have electric charge $Q = -1/3$. Therefore, in order to satisfy the Gell-Mann Nishijima relation we must make the assignments $y_L^Q = 1/6$, $y_R^u = 2/3$, and $y_R^d = -1/3$. The hypercharge assignments are collected below.

Reduced Hypercharge	Value
y_L^Q	$+1/6$
y_L^L	$-1/2$
y_R^u	$+2/3$
y_R^d	$-1/3$
y_R^e	-1

0.1.4 Bosons

If we define $X_\mu \equiv \mathbf{T} \cdot \mathbf{W}_\mu$ and $X_{\mu\nu} \equiv \mathbf{T} \cdot \mathbf{W}_{\mu\nu}$, and noting that $T^a T^b = i\frac{1}{2}\epsilon_{abc}T^c + \frac{1}{4}\delta_{ab}\mathbf{1}$ and, therefore, $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$, we can write

$$X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu + ig_2[X_\mu, X_\nu],$$

and, therefore,

$$\mathbf{W}_{\mu\nu} \cdot \mathbf{W}^{\mu\nu} = \frac{1}{2}\mathrm{Tr}(X_{\mu\nu}X^{\mu\nu}).$$

In the low energy limit ($q^2 << M_W$, where the W boson mass $M_W \sim 80\,\mathrm{GeV}$), the GSW model reproduces the Fermi model for β-decay provided that we set

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}.$$

Tip: To disable a cell use **esc r**; use **esc y** to reactivate it and **esc m** to go to markdown mode.

[1]:
```python
# load symbolic algebra module
import sympy as sm
import sympy.functions.special.tensor_functions as t
from sympy.matrices import Matrix
from sympy.physics.matrices import mgamma, msigma
from sympy import I, trigsimp, radsimp

# enable pretty printing of equations
sm.init_printing()

levi = t.LeviCivita
kron = t.KroneckerDelta
```

0.1.5 Define fields

Alas, the particle physics literature is littered with differing conventions for defining the fields. In the textbook by Mark Thomson, $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp W_\mu^2)$, while in the textbook by Gordon Kane (Modern

Elementary Particle Physics, 2nd Edition) these fields are defined by $W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(-W^1_\mu \mp W^2_\mu)$. And in the book by Donnelly, Formaggio, Holstein, Milner, and Surrow (Foundations of Nuclear and Particle Physics), the convention is $W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(W^1_\mu \pm W^2_\mu)$. Here, we'll follow the convention in Thomson's book.

[2]:
```
# make the following symbols non-commutative so that the order in which
# they are written is preserved.
gamma   = sm.Symbol('\gamma^{\mu}', commutative=False)
nubarL = sm.Symbol('\overline{\\nu}_L', commutative=False)
nuL     = sm.Symbol('\\nu_L', commutative=False)
ebarL   = sm.Symbol('\overline{e}_L', commutative=False)
eL      = sm.Symbol('e_L', commutative=False)
ebarR   = sm.Symbol('\overline{e}_R', commutative=False)
eR      = sm.Symbol('e_R', commutative=False)

ubarL   = sm.Symbol('\overline{u}_L', commutative=False)
uL      = sm.Symbol('u_L', commutative=False)
ubarR   = sm.Symbol('\overline{u}_R', commutative=False)
uR      = sm.Symbol('u_R', commutative=False)

dbarL   = sm.Symbol('\overline{d}_L', commutative=False)
dL      = sm.Symbol('d_L', commutative=False)
dbarR   = sm.Symbol('\overline{d}_R', commutative=False)
dR      = sm.Symbol('d_R', commutative=False)

sbarL   = sm.Symbol('\overline{s}_L', commutative=False)
sL      = sm.Symbol('s_L', commutative=False)
sbarR   = sm.Symbol('\overline{s}_R', commutative=False)
sR      = sm.Symbol('s_R', commutative=False)

cbarL   = sm.Symbol('\overline{c}_L', commutative=False)
cL      = sm.Symbol('c_L', commutative=False)
cbarR   = sm.Symbol('\overline{c}_R', commutative=False)
cR      = sm.Symbol('c_R', commutative=False)

# these do not have to be non-commutative
B, W1, W2, W3          = sm.symbols("B_\mu, W^1_\mu, W^2_\mu, W^3_\mu")
Bnu, W1nu, W2nu, W3nu = sm.symbols("B_\\nu, W^1_\\nu, W^2_\\nu, W^3_\\nu")
Wplus, Wminus          = sm.symbols("W^+_\mu, W^-_\mu")
Wplusup, Wminusup      = sm.symbols("W^{+\mu}, W^{-\mu}")
Wplusnu,Wminusnu,W3nu = sm.symbols("W^+_\\nu, W^-_\\nu, W^3_\\nu")

Z, A      = sm.symbols('Z_\mu, A_\mu')
Zup, Aup = sm.symbols('Z^\mu, A^\mu')
Znu, Anu = sm.symbols('Z_\\nu, A_\\nu')

nuL, eL, eR, uL, uR, dL, dR, B, W1, W2, W3
```

[2]:

$$\left(\nu_L,\ e_L,\ e_R,\ u_L,\ u_R,\ d_L,\ d_R,\ B_\mu,\ W^1_\mu,\ W^2_\mu,\ W^3_\mu\right)$$

[3]:
```
Wplus, Wminus, Z, A
```

[3]:

$$\left(W^+_\mu,\ W^-_\mu,\ Z_\mu,\ A_\mu\right)$$

0.1.6 Define couplings, charges, hypercharges, weak isospin, and weak mixing angle θ_W

[4]:
```
g1, g2, q, thetaW  = sm.symbols('g_1 g_2 q \\theta_W')
YQL, YdL, YuR, YdR = sm.symbols('y^Q_L y^d_L y^u_R y^d_R')
YLL, YeR, YphiL      = sm.symbols('y^L_L y^e_R y_L^\phi')
g1, g2, q, YLL, YeR, YQL, YuR, YdR, YphiL, thetaW
```

[4]:

$$\left(g_1,\ g_2,\ q,\ y^L_L,\ y^e_R,\ y^Q_L,\ y^u_R,\ y^d_R,\ y^\phi_L,\ \theta_W\right)$$

[5]:
```
Qf, Qu, Qd, Qe, Qn = sm.symbols('Q^f, Q^u, Q^d, Q^e, Q^\\nu')
If3, Id3, Iu3, Ie3, In3 = sm.symbols('I^f_3, I^d_3, I^u_3, I^e_3, I^\\nu_3')
q, Qf, Qu, Qd, Qe, Qn, If3, Id3, Iu3, Ie3, In3
```

[5]:

$$\left(q,\ Q^f,\ Q^u,\ Q^d,\ Q^e,\ Q^\nu,\ I^f_3,\ I^d_3,\ I^u_3,\ I^e_3,\ I^\nu_3\right)$$

0.1.7 Define $SU(2)$ lepton and quark doublets L and Q

[6]:
```
L = Matrix([nuL, eL])
Lbar = Matrix([nubarL, ebarL]).T

Q = Matrix([uL, dL])
Qbar = Matrix([ubarL, dbarL]).T

Lbar, L, Qbar, Q
```

[6]:

$$\left(\begin{bmatrix}\overline{\nu}_L & \overline{e}_L\end{bmatrix},\ \begin{bmatrix}\nu_L\\ e_L\end{bmatrix},\ \begin{bmatrix}\overline{u}_L & \overline{d}_L\end{bmatrix},\ \begin{bmatrix}u_L\\ d_L\end{bmatrix}\right)$$

0.1.8 Define $\mathbf{T}\cdot W$ matrix

[7]:
```
tau1 = msigma(1)
tau2 = msigma(2)
tau3 = msigma(3)

U    = tau1**2

T1   = tau1/2
T2   = tau2/2
T3   = tau3/2
w    = T1*W1 + T2*W2 + T3*W3
w
```

[7]:

$$\left[\begin{matrix}\frac{W^{3}_{\mu}}{2} & \frac{W^{1}_{\mu}}{2} - \frac{i W^{2}_{\mu}}{2}\\\frac{W^{1}_{\mu}}{2} + \frac{i W^{2}_{\mu}}{2} & - \frac{W^{3}_{\mu}}{2}\end{matrix}\right]$$

The fields $W^1_\mu \mp i W^2_\mu$, which we write as $(W^1_\mu \mp i W^2_\mu)\,/\,\sqrt{2} \to W^\pm_\mu$, are identified as the fields of the charged weak bosons.

[8]:
```
w = w.subs({W1/2 - I*W2/2: Wplus / sm.sqrt(2),
            W1/2 + I*W2/2: Wminus / sm.sqrt(2)})
w
```

[8]:

$$\left[\begin{matrix}\frac{W^{3}_{\mu}}{2} & \frac{\sqrt{2} W^{+}_{\mu}}{2}\\\frac{\sqrt{2} W^{-}_{\mu}}{2} & - \frac{W^{3}_{\mu}}{2}\end{matrix}\right]$$

0.1.9 Define matrix b

[9]:
```
b = U*B
b
```

[9]:

$$\left[\begin{matrix}B_{\mu} & 0\\0 & B_{\mu}\end{matrix}\right]$$

0.1.10 Compute matrices

$$M_L = g_1 y_L^L B_\mu + g_2 \mathbf{T} \cdot \mathbf{W}_\mu \tag{4}$$

and

$$M_Q = g_1 y_L^Q B_\mu + g_2 \mathbf{T} \cdot \mathbf{W}_\mu \tag{5}$$

[10]:
```
ML = g1*YLL*b + g2*w

MQ = g1*YQL*b + g2*w

ML, MQ
```

[10]:

$$\left(\left[\begin{matrix}B_{\mu} g_{1} y^{L}_{L} + \frac{W^{3}_{\mu} g_{2}}{2} & \frac{\sqrt{2} W^{+}_{\mu} g_{2}}{2}\\\frac{\sqrt{2} W^{-}_{\mu} g_{2}}{2} & B_{\mu} g_{1} y^{L}_{L} - \frac{W^{3}_{\mu} g_{2}}{2}\end{matrix}\right], \ \left[\begin{matrix}B_{\mu} g_{1} y^{Q}_{L} + \frac{W^{3}_{\mu} g_{2}}{2} & \frac{\sqrt{2} W^{+}_{\mu} g_{2}}{2}\\\frac{\sqrt{2} W^{-}_{\mu} g_{2}}{2} & B_{\mu} g_{1} y^{Q}_{L} - \frac{W^{3}_{\mu} g_{2}}{2}\end{matrix}\right]\right)$$

0.1.11 Expand

$-\bar{L}\gamma^\mu M_L L - \bar{Q}\gamma^\mu M_Q Q$

[11]:
```
l = sm.expand(-Lbar*gamma*ML*L)[0] + sm.expand(-Qbar*gamma*MQ*Q)[0]
l
```

[11]:

$$-B_\mu g_1 y_L^L \overline{\nu}_L \gamma^\mu \nu_L - B_\mu g_1 y_L^L \overline{e}_L \gamma^\mu e_L - B_\mu g_1 y_L^Q \overline{d}_L \gamma^\mu d_L - B_\mu g_1 y_L^Q \overline{u}_L \gamma^\mu u_L - \frac{\sqrt{2} W_\mu^+ g_2 \overline{\nu}_L \gamma^\mu e_L}{2} - \frac{\sqrt{2} W_\mu^+ g_2 \overline{u}_L \gamma^\mu d_L}{2} - \frac{\sqrt{2} W_\mu^- g_2 \overline{d}_L \gamma^\mu u_L}{2} - \frac{\sqrt{2} W_\mu^- g_2 \overline{e}_L \gamma^\mu \nu_L}{2} - \frac{W_\mu^3 g_2 \overline{\nu}_L \gamma^\mu \nu_L}{2} + \frac{W_\mu^3 g_2 \overline{d}_L \gamma^\mu d_L}{2} + \frac{W_\mu^3 g_2 \overline{e}_L \gamma^\mu e_L}{2} - \frac{W_\mu^3 g_2 \overline{u}_L \gamma^\mu u_L}{2}$$

0.1.12 Compute the singlet terms

$$-g_1 y_R^e B_\mu \bar{e}_R \gamma^\mu e_R - g_1 y_R^u B_\mu \bar{u}_R \gamma^\mu u_R - g_1 y_R^d B_\mu \bar{d}_R \gamma^\mu d_R$$

[12]:

```
r  = g1 * YeR*B*ebarR*gamma*eR
r += g1 * YuR*ubarR*gamma*B*uR
r += g1 * YdR*dbarR*gamma*B*dR
r  = -r
r
```

[12]:

$$-B_\mu g_1 y_R^d \overline{d}_R \gamma^\mu d_R - B_\mu g_1 y_R^e \overline{e}_R \gamma^\mu e_R - B_\mu g_1 y_R^u \overline{u}_R \gamma^\mu u_R$$

0.1.13 The unification of the electromagnetic and weak nuclear forces

The 1961 Glashow hypothesis is that the electromagnetic and weak nuclear forces are related and, therefore, unified in the following sense:

$$\begin{pmatrix} B_\mu \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}. \tag{6}$$

Above a certain critical temperature, the symmetry between the two forces (called the electroweak symmetry) is restored and the two forces merge into a single **electroweak force**. Below the critical temperature, the electroweak symmetry is said to be *spontaneously broken* by the change in shape of the potential energy of the Higgs field, which we discuss below. Because of the changed shape (which, to date, is introduced by hand), the theory yields infinitely many vacuum state solutions. While this infinite *ensemble* of vacuum state solutions collectively respect the electroweak symmetry, the particular vacuum state into which the universe settles does not.

This feature is analogous to that of an infinite ferromagnet. Above a certain critical temperature, called the Curie temperature, the magnetic moments of the ferromagnet's domains are randomly oriented. Therefore, when averaged over the whole ferromagnet there is no net magnetization and the ferromagnet exhibits an $O(3)$ symmetry, that is, it is invariant with respect to rotations in 3-D space. However, well below the Curie temperatue, the magnetic moments line up in the same randomly selected direction, which causes the system to lose its $O(3)$ symmetry. Thus, the ground state of the ferromagnetic with its net magnetization does not share the symmetry of the equations that describe the system. However, collectively, the infinite ensemble of possible ground states still preserves the symmetry in the sense that a simultaneous rotation of all of the ground states in the same way doesn't change the ensemble.

We should be thankful for broken symmetry because without it the universe would be far less diverse and we would not exist.

0.1.14 Replace fields B_μ and W_μ^3 by A_μ and Z_μ

[13]:
```
AZv = Matrix([A, Z])
rot = Matrix([[sm.cos(thetaW), -sm.sin(thetaW)],
              [sm.sin(thetaW),  sm.cos(thetaW)]])
BW  = rot * AZv
BW
```

[13]:

$$\begin{bmatrix} A_\mu \cos(\theta_W) - Z_\mu \sin(\theta_W) \\ A_\mu \sin(\theta_W) + Z_\mu \cos(\theta_W) \end{bmatrix}$$

[15]:
```
a = l + r
a = a.subs(B,  BW[0])
a = a.subs(W3, BW[1])
a = a.expand()
a
```

[15]:

$$-A_\mu g_1 y_L^L \cos(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L - A_\mu g_1 y_L^L \cos(\theta_W)\overline{e}_L\gamma^\mu e_L - A_\mu g_1 y_L^Q \cos(\theta_W)\overline{d}_L\gamma^\mu d_L - A_\mu g_1 y_L^Q \cos(\theta_W)\overline{u}_L\gamma^\mu u_L - A_\mu g_1 y_R^d \cos(\theta_W)\overline{d}_R\gamma^\mu d_R - A_\mu g_1 y_R^e \cos(\theta_W)\overline{e}_R\gamma^\mu e_R - A_\mu g_1 y_R^u \cos(\theta_W)\overline{u}_R\gamma^\mu u_R - \frac{A_\mu g_2 \sin(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L}{2} + \frac{A_\mu g_2 \sin(\theta_W)\overline{d}_L\gamma^\mu d_L}{2} + \frac{A_\mu g_2 \sin(\theta_W)\overline{e}_L\gamma^\mu e_L}{2} - \frac{A_\mu g_2 \sin(\theta_W)\overline{u}_L\gamma^\mu u_L}{2} - \frac{\sqrt{2}W_\mu^+ g_2\overline{\nu}_L\gamma^\mu e_L}{2} - \frac{\sqrt{2}W_\mu^+ g_2\overline{u}_L\gamma^\mu d_L}{2} - \frac{\sqrt{2}W_\mu^- g_2\overline{d}_L\gamma^\mu u_L}{2} - \frac{\sqrt{2}W_\mu^- g_2\overline{e}_L\gamma^\mu \nu_L}{2} + Z_\mu g_1 y_L^L \sin(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L + Z_\mu g_1 y_L^L \sin(\theta_W)\overline{e}_L\gamma^\mu e_L + Z_\mu g_1 y_L^Q \sin(\theta_W)\overline{d}_L\gamma^\mu d_L + Z_\mu g_1 y_L^Q \sin(\theta_W)\overline{u}_L\gamma^\mu u_L + Z_\mu g_1 y_R^d \sin(\theta_W)\overline{d}_R\gamma^\mu d_R + Z_\mu g_1 y_R^e \sin(\theta_W)\overline{e}_R\gamma^\mu e_R + Z_\mu g_1 y_R^u \sin(\theta_W)\overline{u}_R\gamma^\mu u_R - \frac{Z_\mu g_2 \cos(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L}{2} + \frac{Z_\mu g_2 \cos(\theta_W)\overline{d}_L\gamma^\mu d_L}{2} + \frac{Z_\mu g_2 \cos(\theta_W)\overline{e}_L\gamma^\mu e_L}{2} - \frac{Z_\mu g_2 \cos(\theta_W)\overline{u}_L\gamma^\mu u_L}{2}$$

In the above expression there are interactions between the electromagnetic field A_μ and the neutrino current $\bar{\nu}_L\gamma^\mu\nu_L$. However, we have no evidence that such interactions exists. The simplest way to get rid of them is to set

$$-y_L^L\, g_1 \cos\theta_W = \frac{g_2}{2}\sin\theta_W .$$

[16]:
```
a = a.subs(-YLL*g1*sm.cos(thetaW), g2*sm.sin(thetaW)/2)
a
```

[16]:

$$-A_\mu g_1 y_L^Q \cos(\theta_W)\overline{d}_L\gamma^\mu d_L - A_\mu g_1 y_L^Q \cos(\theta_W)\overline{u}_L\gamma^\mu u_L - A_\mu g_1 y_R^d \cos(\theta_W)\overline{d}_R\gamma^\mu d_R - A_\mu g_1 y_R^e \cos(\theta_W)\overline{e}_R\gamma^\mu e_R - A_\mu g_1 y_R^u \cos(\theta_W)\overline{u}_R\gamma^\mu u_R + \frac{A_\mu g_2 \sin(\theta_W)\overline{d}_L\gamma^\mu d_L}{2} + A_\mu g_2 \sin(\theta_W)\overline{e}_L\gamma^\mu e_L - \frac{A_\mu g_2 \sin(\theta_W)\overline{u}_L\gamma^\mu u_L}{2} - \frac{\sqrt{2}W_\mu^+ g_2\overline{\nu}_L\gamma^\mu e_L}{2} - \frac{\sqrt{2}W_\mu^+ g_2\overline{u}_L\gamma^\mu d_L}{2} - \frac{\sqrt{2}W_\mu^- g_2\overline{d}_L\gamma^\mu u_L}{2} - \frac{\sqrt{2}W_\mu^- g_2\overline{e}_L\gamma^\mu \nu_L}{2} + Z_\mu g_1 y_L^L \sin(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L + Z_\mu g_1 y_L^L \sin(\theta_W)\overline{e}_L\gamma^\mu e_L + Z_\mu g_1 y_L^Q \sin(\theta_W)\overline{d}_L\gamma^\mu d_L + Z_\mu g_1 y_L^Q \sin(\theta_W)\overline{u}_L\gamma^\mu u_L + Z_\mu g_1 y_R^d \sin(\theta_W)\overline{d}_R\gamma^\mu d_R + Z_\mu g_1 y_R^e \sin(\theta_W)\overline{e}_R\gamma^\mu e_R + Z_\mu g_1 y_R^u \sin(\theta_W)\overline{u}_R\gamma^\mu u_R - \frac{Z_\mu g_2 \cos(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L}{2} + \frac{Z_\mu g_2 \cos(\theta_W)\overline{d}_L\gamma^\mu d_L}{2} + \frac{Z_\mu g_2 \cos(\theta_W)\overline{e}_L\gamma^\mu e_L}{2} -$$

$$\frac{Z_\mu g_2 \cos(\theta_W)\overline{u}_L\gamma^\mu u_L}{2}$$

So far, so good. Now, if this theory has something to do with reality, we need to recover the known form of the electromagnetic interaction, which for electrons has the form $qA_\mu(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R)$, where q is the magnitude of the electic charge of the electron. We can recover the correct form of the QED interactions if we impose the conditions

$$-y_R^e g_1 \cos\theta_W = g_2 \sin\theta_W = q,$$

which, recalling that $y_R^e = -1$, yields

$$\tan\theta_W = \frac{g_1}{g_2},$$

and,

$$q = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}.$$

[17]:
```
a = a.subs({-YeR*g1*sm.cos(thetaW): q, g2*sm.sin(thetaW): q })
a = a.subs({g1*sm.cos(thetaW): q})
a = a.subs({g1 : q / sm.cos(thetaW), g2: q / sm.sin(thetaW)})
a = a.trigsimp().expand()
a
```

[17]:

$$-A_\mu q y_L^Q \overline{d}_L\gamma^\mu d_L - A_\mu q y_L^Q \overline{u}_L\gamma^\mu u_L - A_\mu q y_R^d \overline{d}_R\gamma^\mu d_R - A_\mu q y_R^u \overline{u}_R\gamma^\mu u_R + \frac{A_\mu q\overline{d}_L\gamma^\mu d_L}{2} + A_\mu q\overline{e}_L\gamma^\mu e_L + A_\mu q\overline{e}_R\gamma^\mu e_R - \frac{A_\mu q\overline{u}_L\gamma^\mu u_L}{2} - \frac{\sqrt{2}W_\mu^+ q\overline{\nu}_L\gamma^\mu e_L}{2\sin(\theta_W)} - \frac{\sqrt{2}W_\mu^+ q\overline{u}_L\gamma^\mu d_L}{2\sin(\theta_W)} - \frac{\sqrt{2}W_\mu^- q\overline{d}_L\gamma^\mu u_L}{2\sin(\theta_W)} - \frac{\sqrt{2}W_\mu^- q\overline{e}_L\gamma^\mu \nu_L}{2\sin(\theta_W)} + Z_\mu q y_L^L \tan(\theta_W)\overline{\nu}_L\gamma^\mu\nu_L + Z_\mu q y_L^L \tan(\theta_W)\overline{e}_L\gamma^\mu e_L + Z_\mu q y_L^Q \tan(\theta_W)\overline{d}_L\gamma^\mu d_L + Z_\mu q y_L^Q \tan(\theta_W)\overline{u}_L\gamma^\mu u_L + Z_\mu q y_R^d \tan(\theta_W)\overline{d}_R\gamma^\mu d_R + Z_\mu q y_R^e \tan(\theta_W)\overline{e}_R\gamma^\mu e_R + Z_\mu q y_R^u \tan(\theta_W)\overline{u}_R\gamma^\mu u_R - \frac{Z_\mu q\overline{\nu}_L\gamma^\mu\nu_L}{2\tan(\theta_W)} + \frac{Z_\mu q\overline{d}_L\gamma^\mu d_L}{2\tan(\theta_W)} + \frac{Z_\mu q\overline{e}_L\gamma^\mu e_L}{2\tan(\theta_W)} - \frac{Z_\mu q\overline{u}_L\gamma^\mu u_L}{2\tan(\theta_W)}$$

0.1.15 Simplify expressions

We extract the coefficients of the bosonic fields, that is, the currents, and manipulate them into the forms in which they are typically expressed.

0.1.16 Get currents associated with W_μ^+

[18]:
```
a = a.collect(Wplus)
Wpluscoeff  = a.coeff(Wplus)
Wpluscoeff  = Wpluscoeff.subs(g2, q/sm.sin(thetaW)).factor(q).simplify()
Wpluscoeff
```

[18]:

$$-\frac{\sqrt{2}q\left(\overline{\nu}_L\gamma^\mu e_L + \overline{u}_L\gamma^\mu d_L\right)}{2\sin(\theta_W)}$$

0.1.17 Get currents associated with W_μ^-

[19]:
```
a = a.collect(Wminus)
Wminuscoeff = a.coeff(Wminus)
Wminuscoeff = Wminuscoeff.subs(g2, q/sm.sin(thetaW)).factor(q).simplify()
Wminuscoeff
```

[19]:

$$-\frac{\sqrt{2}q\left(\bar{d}_L\gamma^\mu u_L+\bar{e}_L\gamma^\mu\nu_L\right)}{2\sin\left(\theta_W\right)}$$

0.1.18 Get currents associated with A_μ

Sympy is currently unable to collect noncommutative variables, so we hack it! Replace noncommutative variables by commutative ones, collect, and go back to the original variables.

[20]:
```
N, E, U, D = sm.symbols('N, E, U, D')

a = a.collect(A)

Acoeff = a.coeff(A).expand()

Acoeff = Acoeff.subs({ebarL*gamma*eL: E,
                      ubarL*gamma*uL: U,
                      dbarL*gamma*dL: D})
Acoeff = Acoeff.collect(E).collect(U).collect(D)

Acoeff = Acoeff.subs({E:ebarL*gamma*eL,
                      U: ubarL*gamma*uL,
                      D: dbarL*gamma*dL})

Acoeff = Acoeff.subs(g1, g2*sm.sin(thetaW)/sm.cos(thetaW))
Acoeff = Acoeff.subs({g2*sm.sin(thetaW): q})

Acoeff
```

[20]:

$$-qy_R^d\bar{d}_R\gamma^\mu d_R-qy_R^u\bar{u}_R\gamma^\mu u_R+q\bar{e}_L\gamma^\mu e_L+q\bar{e}_R\gamma^\mu e_R+\left(-qy_L^Q-\frac{q}{2}\right)\bar{u}_L\gamma^\mu u_L+\left(-qy_L^Q+\frac{q}{2}\right)\bar{d}_L\gamma^\mu d_L$$

Notice that the coefficients of the left-handed quark currents can be written as $-q(y_L^Q\pm 1/2)=-q(y_L^Q+I_3^f)=-qQ^f$, where $f=u,d$. Also, since the 3rd component of weak isospin for singlets is $I_3=0$, we have that $y_R^f=Q^f$.

[21]:
```
Acoeff = Acoeff.subs({-q*YQL + q/2: -q*Qd,
                      -q*YQL - q/2: -q*Qu,
                      YuR: Qu,
                      YdR: Qd})
Acoeff
```

[21]:

$$-Q^dq\bar{d}_L\gamma^\mu d_L-Q^dq\bar{d}_R\gamma^\mu d_R-Q^uq\bar{u}_L\gamma^\mu u_L-Q^uq\bar{u}_R\gamma^\mu u_R+q\bar{e}_L\gamma^\mu e_L+q\bar{e}_R\gamma^\mu e_R$$

Likewise, we notice that the coefficients of the electron currents can be written as $-Q^eq$:

[22]:
```python
# left electron current
Acoeff = Acoeff.subs({q*ebarL*gamma*eL: -Qe*q*ebarL*gamma*eL})

# right electron current
Acoeff = Acoeff.subs({q*ebarR*gamma*eR: -Qe*q*ebarR*gamma*eR})

Acoeff
```

[22]:

$$-Q^d q\bar{d}_L\gamma^\mu d_L - Q^d q\bar{d}_R\gamma^\mu d_R - Q^e q\bar{e}_L\gamma^\mu e_L - Q^e q\bar{e}_R\gamma^\mu e_R - Q^u q\bar{u}_L\gamma^\mu u_L - Q^u q\bar{u}_R\gamma^\mu u_R$$

We now collect terms with the same charge.

[23]:
```python
Acoeff = Acoeff.\
collect(-Qd*q).\
collect(-Qu*q).\
collect(-Qe*q).\
collect(q).\
collect(-1)

Acoeff
```

[23]:

$$-q\left(Q^d\left(\bar{d}_L\gamma^\mu d_L + \bar{d}_R\gamma^\mu d_R\right) + Q^e\left(\bar{e}_L\gamma^\mu e_L + \bar{e}_R\gamma^\mu e_R\right) + Q^u\left(\bar{u}_L\gamma^\mu u_L + \bar{u}_R\gamma^\mu u_R\right)\right)$$

We see that the electromagnetic current J^μ_{em} can be written as

$$J^\mu_{\text{em}} = -q\sum_{f=u,d,e} Q^f\overline{\psi}^f\gamma^\mu\psi^f,$$

where $\psi = \psi_L + \psi_R$, noting that $P_{L,R}\gamma^\mu P_{L,R} = 0$, where Q^f is the electic charge in units of the proton charge for the quantum field of flavor f.

0.1.19 Get currents associated with Z_μ

Do some algebraic manipulations to simplify expression. Unfortunately, collect does not work with noncommutative symbols. So, again, use our hack.

[24]:
```python
a = a.collect(Z)

Zcoeff = a.coeff(Z)

Zcoeff = Zcoeff.subs({nubarL*gamma*nuL: N,
                      ebarL*gamma*eL: E,
                      ubarL*gamma*uL: U,
                      dbarL*gamma*dL: D})
Zcoeff = Zcoeff.collect(E).collect(N).collect(U).collect(D)

Zcoeff = Zcoeff.subs({N:nubarL*gamma*nuL,
                      E:ebarL*gamma*eL,
                      U: ubarL*gamma*uL,
                      D: dbarL*gamma*dL})
```

```
Zcoeff = Zcoeff.subs(g1, g2*sm.sin(thetaW)/sm.cos(thetaW))

Zcoeff = Zcoeff.trigsimp()

Zcoeff
```

[24]:

$$qy_R^d \tan(\theta_W)\overline{d}_R\gamma^\mu d_R + qy_R^e \tan(\theta_W)\overline{e}_R\gamma^\mu e_R + qy_R^u \tan(\theta_W)\overline{u}_R\gamma^\mu u_R + q\left(y_L^L \tan(\theta_W) - \frac{1}{2\tan(\theta_W)}\right)\overline{\nu}_L\gamma^\mu\nu_L + q\left(y_L^L \tan(\theta_W) + \frac{1}{2\tan(\theta_W)}\right)\overline{e}_L\gamma^\mu e_L + q\left(y_L^Q \tan(\theta_W) - \frac{1}{2\tan(\theta_W)}\right)\overline{u}_L\gamma^\mu u_L + q\left(y_L^Q \tan(\theta_W) + \frac{1}{2\tan(\theta_W)}\right)\overline{d}_L\gamma^\mu d_L$$

0.1.20 Weak neutral currents

The weak neutral current J^μ_{nc} can be written as

$$\begin{aligned} J^\mu_{\text{nc}} &= q\tan(\theta_W)\sum_{f=u,d,e} y_R^f \overline{f}_R\gamma^\mu f_R \\ &+ q\tan(\theta_W)\sum_{f=u,d}\left(y_L^Q - \frac{I_3^f}{\tan^2(\theta_W)}\right)\overline{f}_L\gamma^\mu f_L \\ &+ q\tan(\theta_W)\sum_{f=\nu,e}\left(y_L^L - \frac{I_3^f}{\tan^2(\theta_W)}\right)\overline{f}_L\gamma^\mu f_L \end{aligned}$$

The more traditional way of writing the weak neutral current J^μ_{nc} is

$$J^\mu_{\text{nc}} = \frac{2q}{\sin(2\theta_W)}\left(\sum_{f=d,e,u} Q^f\sin^2\theta_W\overline{f}_R\gamma^\mu f_R + \sum_{f=d,e,u,\nu}(-I_3^f + Q^f\sin^2\theta_W)\,\overline{f}_L\gamma^\mu f_L\right).$$

0.1.21 Final form of GSW lagrangian for 1st generation fermions

Electromagnetic interactions

[25]:
```
Acoeff*A
```

[25]:

$$-A_\mu q\left(Q^d\left(\overline{d}_L\gamma^\mu d_L + \overline{d}_R\gamma^\mu d_R\right) + Q^e\left(\overline{e}_L\gamma^\mu e_L + \overline{e}_R\gamma^\mu e_R\right) + Q^u\left(\overline{u}_L\gamma^\mu u_L + \overline{u}_R\gamma^\mu u_R\right)\right)$$

Weak charged current interactions

[26]:
```
Wpm = Wpluscoeff*Wplus + Wminuscoeff*Wminus
Wpm = Wpm.collect(sm.sqrt(2)*q / (2*sm.sin(thetaW))).collect(-1)
Wpm
```

[26]:

$$-\frac{\sqrt{2}q\left(W_\mu^+\left(\overline{\nu}_L\gamma^\mu e_L + \overline{u}_L\gamma^\mu d_L\right) + W_\mu^-\left(\overline{d}_L\gamma^\mu u_L + \overline{e}_L\gamma^\mu\nu_L\right)\right)}{2\sin(\theta_W)}$$

Weak neutral current interactions

[27]:
```
Zcoeff*Z
```

[27]:

$$Z_\mu \left(q y_R^d \tan(\theta_W) \overline{d}_R \gamma^\mu d_R + q y_R^e \tan(\theta_W) \overline{e}_R \gamma^\mu e_R + q y_R^u \tan(\theta_W) \overline{u}_R \gamma^\mu u_R + q \left(y_L^L \tan(\theta_W) - \frac{1}{2\tan(\theta_W)} \right) \overline{\nu}_L \gamma^\mu \nu_L -$$

The GSW model incorporates quantum electrodynamics (QED), predicts weak charged and neutral current interactions and incorporates the observation that neither the electromagnetic nor the weak neutral currrent interactions change flavor, while the weak charged current interactions always do. And, of course, the model incorporates the maximal parity violation of the weak interactions. But, at this stage, all particles are massless.

0.2 The Higgs Sector

The Lagrangian of the Higgs field is given by

$$\mathcal{L}_{\text{Higgs}} = (D^\mu \phi)^\dagger (D_\mu \phi) - V(\phi^\dagger \phi), \tag{7}$$

where

$$D_\mu = \partial_\mu + i g_1 y_L^\phi B_\mu + i g_2 \mathbf{T} \cdot \mathbf{W}_\mu, \tag{8}$$

$$V(\phi^\dagger \phi) = \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2, \quad \text{and the 4-component Higgs field is given by} \tag{9}$$

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}. \tag{10}$$

Qualitative description of the Higgs mechanism The stability of the vacuum requires $\lambda > 0$. In the very early universe, we presume that $\mu^2 > 0$, in which case $\langle 0|\phi|0\rangle = 0$, that is, the expectation (i.e., average) value of the Higgs field in the vacuum is zero. But, at some critical temperature, T_{crit}, there is a phase transition in which it is presumed that μ^2 changes sign thereby causing the Higgs potential to develop infinitely many degenerate vacua in each of which the Higgs field is non-zero. Above T_{crit}, there is a unique vacuum state and this state respects the symmetry of the Lagrangian. Below T_{crit}, the *ensemble* of vacua respects the symmetry, but, the universe cools into one of the infinitely many potential vacua, which on its own breaks the symmetry. This is an example of *spontaneous symmetry breaking*, which we touched upon above. The specific instance is called electroweak symmetry breaking (EWSB).

The vacuum state of the universe is one in which the Higgs field behaves like a superconductor that attenuates the propagation of the W_μ and Z_μ fields, thereby rendering these fields massive, but permits the free propagation of the A_μ field. (The theory is, of course, made to do this!) Moreover, the interactions between the Higgs field and the fermions, which also are put in by hand, cause a rapid flipping between the left-handed and right-handed fermion fields, which causes energy to build up within them that we interpret as the mass of the associated field quanta, that is, particles. The results from the LHC experiment are, so far, consistent with this picture.

Since there is no right-handed neutrino the flipping mechanism cannot occur and therefore the neutrino remains massless in the GSW model. However, neutrinos are now known to have (very small) masses, but, how their masses arise remains an open question.

Quantum numbers We need to specify the value of the reduced hypercharge y_L^ϕ associated with the doublet ϕ. As above, the reduced hypercharge is chosen so that the Gell-Mann Nishijima relation $y = Q - I_3$ holds, where the third component of the weak isospin $I_3 = +1/2$ for the upper component of the doublet and $I_3 = -1/2$ for the lower component. In the GSW model, it is assumed that the vacuum expectation of ϕ is

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}.$$

Since the vacuum has a net electric charge of zero, we must set $Q = 0$ for the lower component $\phi_3 + i\phi_4$. From $y_L^\phi = Q - I_3$ it follows that $y_L^\phi = +1/2$. This also shows that the upper component, $\phi_1 + i\phi_2$ must be assigned $Q = +1$, which of course is why it is assumed to be zero in the vacuum otherwise the entire universe would have a huge net positive electric charge!

Higgs field at low energies At energies not too far from the vacuum state, we assume that the Higgs field can be approximated as follows

$$\phi \approx H = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h \end{pmatrix},$$

where h represents the fluctuations of the Higgs field about its average value $v \approx 246\,\text{GeV}$ in the vacuum state (the vacuum expectation value). The quanta of the field h are identified with the Higgs boson discovered at CERN in 2012. There may be absolutely no significance whatsoever to the numerical fact that $v/\sqrt{2} \approx m_t$, where m_t is the mass of the top quark; or it could be deeply significant. Whatever conclusion is to be drawn, the value of v implies (as we shall see below) that the coupling of the top quark to the Higgs boson is close to (and may even be) unity.

In the following, we shall also need the conjugate field defined by

$$H_c \equiv i\tau_2 H = \frac{1}{\sqrt{2}}\begin{pmatrix} v + h \\ 0 \end{pmatrix}.$$

0.2.1 Vector boson - Higgs boson interactions

In order to see explicitly what interactions are implied by the Higgs field Langrangian, we first compute

$$F = g_1 y_L^\phi B_\mu + g_2 \mathbf{T} \cdot \mathbf{W}_\mu. \tag{11}$$

Note The matrix F is Hermitian, that is, $F^\dagger = F$.

[34]:
```
v = sm.symbols('v')
h = sm.symbols('h')
H = Matrix([0, (v+h)/sm.sqrt(2)])
F = g1*YphiL*b + g2*w
Hc=  sm.I*tau2*H
F, H, Hc
```

[34]:

$$\left(\begin{bmatrix} B_\mu g_1 y_L^\phi + \frac{W_\mu^3 g_2}{2} & \frac{\sqrt{2} W_\mu^+ g_2}{2} \\ \frac{\sqrt{2} W_\mu^- g_2}{2} & B_\mu g_1 y_L^\phi - \frac{W_\mu^3 g_2}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{\sqrt{2}(h+v)}{2} \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}(h+v)}{2} \\ 0 \end{bmatrix}\right)$$

Then, we compute

$$x = FH, \tag{12}$$

$$\overline{x} = H^T F^\dagger, \tag{13}$$

$$= H^T F, \tag{14}$$

$$f = \overline{x} x. \tag{15}$$

[36]:
```
x    = F*H
xbar = H.T*F
f = xbar*x
f = f[0]
f = f.subs({YphiL: 1})
f = f.subs({Wminus: Wminusup})
f
```

[36]:

$$\frac{W_\mu^+ W^{-\mu} g_2^2 (h+v)^2}{4} + \frac{(h+v)^2 \left(B_\mu g_1 - \frac{W_\mu^3 g_2}{2}\right)^2}{2}$$

Map to Z field.

[37]:
```
f = f.subs(g1*B - g2*W3/2, Z * sm.sqrt(g1**2 + g2**2)/2)
f = f.expand()
f = f.subs({Z**2: Z*Zup})
f = f.collect(Z*Zup*v**2).collect(Z*Zup*h*v).collect(Z*Zup*h**2)
f
```

[37]:

$$\frac{W_\mu^+ W^{-\mu} g_2^2 h^2}{4} + \frac{W_\mu^+ W^{-\mu} g_2^2 hv}{2} + \frac{W_\mu^+ W^{-\mu} g_2^2 v^2}{4} + Z^\mu Z_\mu h^2 \left(\frac{g_1^2}{8} + \frac{g_2^2}{8}\right) + Z^\mu Z_\mu hv \left(\frac{g_1^2}{4} + \frac{g_2^2}{4}\right) + Z^\mu Z_\mu v^2 \left(\frac{g_1^2}{8} + \frac{g_2^2}{8}\right)$$

The terms involving the Higgs boson field h in the low-energy limit are the single and di-Higgs boson interactions, while the terms $\propto v^2$ are the mass terms. For a massive vector boson field, V_μ, the mass term has the form $(m_V^2\,/\,2)V_\mu V^\mu$. The term $W_\mu^+ W^{-\mu}$ is the sum of two terms quadratic in the fields W^1 and W^2, with the same mass, so the mass term has the form $m_W^2\, W_\mu^+ W^{-\mu}$. Therefore, in the GSW model the W boson mass is predicted to be $m_W = v g_2\,/\,2$, while for the Z boson the prediction is $m_Z = v\sqrt{g_1^2 + g_2^2}\,/\,2$ and, therefore, $m_W/m_Z = g_2/\sqrt{g_1^2 + g_2^2} = \cos\theta_W$, which modulo higher order corrections is in spectacular agreement with measurements as shown below.

[38]:
```
sw2 = 0.2397              # sin^2(theta_W)
sw  = sm.sqrt(sw2)
cw  = sm.sqrt(1-sw2)
print('cos(theta_W) = %8.3f' % cw)
```

```
cos(theta_W) =    0.872
```

[39]:
```
MW  = 80.385 # GeV
MZ  = 91.188 # GeV
cwp = MW/MZ
```

```python
print('M_W / M_Z    = %8.3f\t%8.3f' % (cwp, cw / cwp))
```

```
M_W / M_Z    =    0.882    0.989
```

0.2.2 The Higgs potential and Higgs boson self-interactions

From

$$\phi \approx H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix},$$

we can write the Higgs potential as $V(H) = \mu^2 H^\dagger H - \lambda(H^\dagger H)^2$, which when expanded becomes

[40]:
```python
mu, lm = sm.symbols('\mu, \lambda')
V = mu**2*H.T*H - lm*(H.T*H)**2
V = V.expand()[0]
V
```

[40]:

$$-\frac{\lambda h^4}{4} - \lambda h^3 v - \frac{3\lambda h^2 v^2}{2} - \lambda h v^3 - \frac{\lambda v^4}{4} + \frac{\mu^2 h^2}{2} + \mu^2 h v + \frac{\mu^2 v^2}{2}$$

0.2.3 h-h interactions

At the minimum of the potential, $\mu^2 = v^2 \lambda$.

[41]:
```python
V = V.subs(mu**2, v**2*lm)
V
```

[41]:

$$-\frac{\lambda h^4}{4} - \lambda h^3 v - \lambda h^2 v^2 + \frac{\lambda v^4}{4}$$

The GSW model predicts the existence of triple and quartic Higgs boson self-interactions and a Higgs boson mass given by $m_H = v\sqrt{2\lambda}$. Using the measured value $m_H = 125.10 \pm 0.14\,\text{GeV}$, this implies the following values for Higgs potential parameters

$$\lambda = \frac{1}{2}\left(\frac{m_H}{v}\right)^2 = 0.129,$$
$$\mu = v\sqrt{\lambda} = 88.4\,\text{GeV}.$$

Note also, the intriguing fact that $v/\sqrt{2} = 173.9\,\text{GeV}$ matches the measured top quark mass of $m_t = 172.8 \pm 0.3\,\text{GeV}$ to within about 0.6%. Nature seems to be telling us something!

0.2.4 Yukawa interactions between fermions and the Higgs field

The interactions between the fermions and the Higgs field are presumed to be Yukawa interactions of the form

$$-\sqrt{2}g_e(\overline{L}He_R + \overline{e}_R H^\dagger L) - \sqrt{2}g_u(\overline{Q}H_c u_R + \overline{u}_R H_c^\dagger Q) - \sqrt{2}g_d(\overline{Q}Hd_R + \overline{d}_R H^\dagger Q)$$

Let us pause for a minute to consider the dimensions of the above interactions. In natural units, the Lagrangian, which recall is really a Lagrangian density, has units of $[M]^4$ since the measure over spacetime is d^4x, which has dimensions of $[M]^{-4}$. Since a fermion mass term looks like

$$m\overline{\psi}\psi$$

and $\dim(m) = [M]$, it follows that a fermion field has dimensions $[M]^{3/2}$. Since the Higgs boson is a scalar field, it has dimensions $[M]$. Consequently, the Yukawa interactions between fermions and the Higgs boson must be a gauge invariant combination of two fermion fields with a combined dimension of $[M]^3$ and the Higgs field with dimension $[M]$. Moreover, since the Higgs field is a doublet, it must be paired with the Dirac conjugate of a *doublet* fermion field giving a term of dimension $[M]^{5/2}$ which must be paired with a *singlet* fermion field, of dimension $[M]^{3/2}$, to yield an interaction term with the correct dimension $[M]^4$; hence terms of the form $\overline{L}He_R$ and its Hermitian conjugate $\overline{e}_R H^\dagger L$. However, there is a subtlety.

In order to avoid a flavor changing Yukawa interaction, we need to fudge the interaction of the up quark field with the Higgs field by using the conjugate field H_c rather than H. By any measure, the GSW model is an amazing intellectual achievement, but it is clear there is room for improvement.

Define coupling parameters and Higgs conjugate fields

[42]:
```
ge, me = sm.symbols('g_e, m_e')
gu, mu, gd, md = sm.symbols('g_u, m_u, g_d, m_d')
gu, mu, gd, md
ge, me, gu, mu, gd, md
```

[42]:

$$(g_e,\ m_e,\ g_u,\ m_u,\ g_d,\ m_d)$$

[43]:
```
Hc = sm.I*tau2*H
Qbar, Q, Hc, Hc.T
```

[43]:

$$\left(\begin{bmatrix}\overline{u}_L & \overline{d}_L\end{bmatrix},\ \begin{bmatrix}u_L\\ d_L\end{bmatrix},\ \begin{bmatrix}\frac{\sqrt{2}(h+v)}{2}\\ 0\end{bmatrix},\ \begin{bmatrix}\frac{\sqrt{2}(h+v)}{2} & 0\end{bmatrix}\right)$$

Expand

$$-\sqrt{2}g_e(\overline{L}He_R + \overline{e}_R H^\dagger L)$$

[44]:
```
Le = -ge*(Lbar*H*eR + ebarR*H.T*L)[0]*sm.sqrt(2)
Le = Le.expand()
Le = Le.collect(-ge*v).collect(-ge*h)
Le = Le.subs(ge*v, me)
Le = Le.subs(ge, me/v)


Le
```

[44]:

$$-\frac{hm_e\left(\overline{e}_L e_R + \overline{e}_R e_L\right)}{v} - m_e\left(\overline{e}_L e_R + \overline{e}_R e_L\right)$$

Let's now do the same thing for the quarks.

[45]:
```
Lq = -gd*(Qbar*H*dR  + dbarR*H.T*Q)[0]*sm.sqrt(2) \
     -gu*(Qbar*Hc*uR + ubarR*Hc.T*Q)[0]*sm.sqrt(2)
Lq = Lq.expand()
Lq = Lq.collect(-gu*v).collect(-gu*h)
```

```
Lq = Lq.collect(-gd*v).collect(-gd*h)
Lq = Lq.subs(gu*v, mu)
Lq = Lq.subs(gd*v, md)
Lq = Lq.subs(gu, mu/v)
Lq = Lq.subs(gd, md/v)

Lq
```

[45]:

$$-\frac{hm_d\left(\overline{d}_L d_R + \overline{d}_R d_L\right)}{v} - \frac{hm_u\left(\overline{u}_L u_R + \overline{u}_R u_L\right)}{v} - m_d\left(\overline{d}_L d_R + \overline{d}_R d_L\right) - m_u\left(\overline{u}_L u_R + \overline{u}_R u_L\right)$$

We see that the Yukawa couplings, together with the particular choice of the value of the Higgs field in the vacuum state, yields two important consequences as alluded to above: the first is mass terms for the fermions and the second is interactions between the Higgs boson and fermions that are proportional to the fermion mass and inversely proportional to the vacuum expectation value v. This implies that the top quark coupling to the Higgs boson is close to, and may be, unity. Why that is, is yet another mystery in a long list of them.

We conclude that the Higgs mechanism generates mass terms for the weak vector bosons as well as the fermions while preserving gauge invariance.

0.2.5 The Cabibbo Hypothesis

So far we have considered only the 1st generation of particles. But three generations have been discovered so far and we have implicitly assumed that there is no "crosstalk" between them. There are, however, decays that can be explained if "crosstalk" exists between generations. For example, the existence of decays such as

$$\Lambda(d, u, s) \to p(d, u, u)\ e^-\ \bar{\nu}_e$$

are readily explained by supposing that the W boson can convert an s-quark to a u-quark, that is, a 2nd generation quark to a 1st generation quark. Further clues are provided by the rates of decays such as

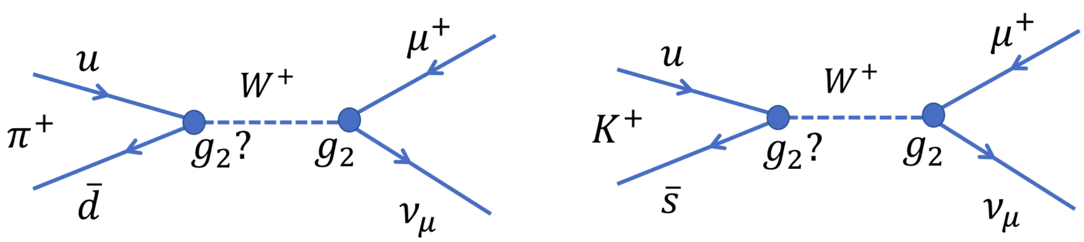
$$\pi^+ \to \mu^+\,\nu_\mu, \tag{16}$$

$$K^+ \to \mu^+\,\nu_\mu, \tag{17}$$

which are measured to be in the ratio

$$\frac{\Gamma(\pi^+ \to \mu^+\,\nu_\mu)}{\Gamma(K^+ \to \mu^+\,\nu_\mu)} \approx \frac{1}{20}. \tag{18}$$

The above decays can be described with the following Feynman diagrams,



in which the couplings g_2? differ from g_2 by a factor chosen to match the measured ratio. The pictorial explanation of these decays, just like the decay of positronium (an electron-positron bound system), is that the particles comprising the π^+ annihilate, in this case to a W boson that subsequently decays leptonically. The observed ratio is readily explained if the W boson can interact with quarks of differing generations, but at a reduced rate compared with that of within-generation interactions. These observations and many others were explained by the Italian physicist Cabibbo who suggested that the lower component of the quark doublet, that is, the down quark field, be replaced with the combination

$$d_L \to d'_L = \cos\theta_C\, d_L + \sin\theta_C\, s_L,$$

where θ_C is called the Cabibbo angle. With a suitable choice of the Cabibbo angle (about 12°), the Cabibbo hypothesis worked well. Unfortunately, however, the replacement

$$\overline{d}_L \gamma^\mu d_L \to \overline{d'}_L \gamma^\mu d'_L, \tag{19}$$

predicts the existence of flavor-changing neutral currents (FCNC) at a rate in sharp disagreement with observations.

[46]:
```
thetaC = sm.symbols('theta_C')
xbarL, xL = sm.symbols('\overline{d^\prime}_L, '\
                       'd^\prime_L', commutative=False)
Nc = xbarL * gamma * xL
Nc = Nc.subs({xL: sm.cos(thetaC)*dL + sm.sin(thetaC)*sL,
              xbarL: sm.cos(thetaC)*dbarL + sm.sin(thetaC)*sbarL})
Nc = Nc.expand()
Nc
```

[46]: $\sin^2(\theta_C)\overline{s}_L\gamma^\mu s_L + \sin(\theta_C)\cos(\theta_C)\overline{d}_L\gamma^\mu s_L + \sin(\theta_C)\cos(\theta_C)\overline{s}_L\gamma^\mu d_L + \cos^2(\theta_C)\overline{d}_L\gamma^\mu d_L$

0.2.6 The GIM Mechanism

In 1970, Sheldon Glashow, John Iliopoulos and Luciano Maiani (GIM) noted that if the strange quark were the lower component of another doublet,

$$\begin{pmatrix} c_L \\ s_L \end{pmatrix},$$

whose upper component would be a new quark field, dubbed "charm", and if one generalized Cabibbo's idea and assumed that the lower components of the two quark doublets were mixed as follows

$$\begin{pmatrix} d'_L \\ s'_L \end{pmatrix} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \begin{pmatrix} d_L \\ s_L \end{pmatrix},$$

then the replacement

$$s_L \to s'_L = -\sin\theta_C\, d_L + \cos\theta_C\, s_L$$

would yield exact cancellation of the unwanted FCNC interactions at the price of introducing yet another quark field, as demonstrated below.

0.2.7 Implemention of GIM Mechanism

[47]:
```
ybarL, yL = sm.symbols('\overline{s^\prime}_L, '\
                       's^\prime_L',
```

```
                                        commutative=False)
# add the GIM terms to the Cabbibo terms
yy = ybarL * gamma * yL
yy = yy.subs({yL:-sm.sin(thetaC)*dL + sm.cos(thetaC)*sL,
              ybarL:-sm.sin(thetaC)*dbarL + sm.cos(thetaC)*sbarL})
yy = yy.expand()
Nc = (Id3 - Qd * sm.sin(thetaW)**2) * Nc
Nc += (Id3 - Qd * sm.sin(thetaW)**2) * yy

# and simplify
Nc = Nc.simplify()
Nc
```

[47]: $\left(I_3^d - Q^d \sin^2\left(\theta_W\right)\right)\left(\bar{d}_L\gamma^\mu d_L + \bar{s}_L\gamma^\mu s_L\right)$

By introducing a 2nd doublet, the GIM mechanism neatly renders the weak neutral currents diagonal again, that is, it elimintes flavor-changing neutral currents and predicts the existence of a 4th quark, the charm quark. This prediction was confirmed in stunning fashion in 1974 with the indepedent discoveries of the J/ψ at the Brookhaven National Laboratory (BNL) and the Stanford Linear Accelerator Center (SLAC).

Today, it is known that mixing of the down quark fields occurs across all three generations. The mixing is described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Unfortunately, however, no one understands why mixing across the (three) generations should exist, nor why it takes the particular form that it does, nor what determines the strength of the mixing.

0.2.8 Vector boson self-interactions

The pure vector boson Lagrangian is given by

$$\mathcal{L}_{VB} = -\frac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu},$$

where

$$\begin{aligned}\mathbf{W}_{\mu\nu} &= \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g_2\,\mathbf{W}_\mu\times\mathbf{W}_\nu,\\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu.\end{aligned}$$

From

$$\begin{pmatrix}B_\mu\\ W_\mu^3\end{pmatrix} = \begin{pmatrix}\cos\theta_W & -\sin\theta_W\\ \sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}A_\mu\\ Z_\mu\end{pmatrix} \tag{20}$$

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp W_\mu^2) \tag{21}$$

we obtain

$$\begin{aligned}W^1 &= (W^- + W^+)/\sqrt{2}\\ W^2 &= (W^- - W^+)/\sqrt{(2)}\\ W^3 &= \sin\theta_W A + \cos\theta_W Z\\ B &= \cos\theta_W A - \sin\theta_W Z,\end{aligned}$$

which we use below to rewrite the vector boson Lagrangian in terms of the usual fields.

[48]:
```python
Wplusmu, Wminusmu = sm.symbols("W^+_\mu, W^-_\mu", commutative=False)
Wplusnu, Wminusnu = sm.symbols("W^+_\\nu, W^-_\\nu", commutative=False)

Zmu, Amu          = sm.symbols('Z_\mu, A_\mu',  commutative=False)
Znu, Anu          = sm.symbols('Z_\\nu, A_\\nu',  commutative=False)

dmu, dnu = sm.symbols('\partial_{\mu}, \partial_{\\nu}',
                      commutative=False)

W1mu, W2mu, W3mu  = sm.symbols('W^1_\mu W^2_\mu W^3_\mu',
                               commutative=False)
W1nu, W2nu, W3nu  = sm.symbols('W^1_\\nu W^2_\\nu W^3_\\nu',
                               commutative=False)
Bmu, Bnu          = sm.symbols('B_\mu B_\\nu',  commutative=False)

Vmu = Matrix([W1mu, W2mu, W3mu])
Vnu = Matrix([W1nu, W2nu, W3nu])

dnuVmu = dnu*Vmu
dmuVnu = dmu*Vnu
dnuBmu = dnu*Bmu
dmuBnu = dmu*Bnu

Vmu, Vnu, Bmu, Bnu, dnuVmu, dmuVnu, dnu*Bmu, dmu*Bnu
```

[48]:

$$\left(\begin{bmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \end{bmatrix}, \begin{bmatrix} W^1_\nu \\ W^2_\nu \\ W^3_\nu \end{bmatrix}, B_\mu, B_\nu, \begin{bmatrix} \partial_\nu W^1_\mu \\ \partial_\nu W^2_\mu \\ \partial_\nu W^3_\mu \end{bmatrix}, \begin{bmatrix} \partial_\mu W^1_\nu \\ \partial_\mu W^2_\nu \\ \partial_\mu W^3_\nu \end{bmatrix}, \partial_\nu B_\mu, \partial_\mu B_\nu \right)$$

[49]:
```python
VmuVnu  = Vmu.cross(Vnu)
VmuVnu
```

[49]:

$$\begin{bmatrix} W^2_\mu W^3_\nu - W^3_\mu W^2_\nu \\ -W^1_\mu W^3_\nu + W^3_\mu W^1_\nu \\ W^1_\mu W^2_\nu - W^2_\mu W^1_\nu \end{bmatrix}$$

[50]:
```python
WW  = dmuVnu - dnuVmu - q*VmuVnu / sm.sin(thetaW)
BB  = dmuBnu - dnuBmu
WW, BB
```

[50]:

$$\left(\begin{bmatrix} -\frac{q\left(W^2_\mu W^3_\nu - W^3_\mu W^2_\nu\right)}{\sin\left(\theta_W\right)} + \partial_\mu W^1_\nu - \partial_\nu W^1_\mu \\ -\frac{q\left(-W^1_\mu W^3_\nu + W^3_\mu W^1_\nu\right)}{\sin\left(\theta_W\right)} + \partial_\mu W^2_\nu - \partial_\nu W^2_\mu \\ -\frac{q\left(W^1_\mu W^2_\nu - W^2_\mu W^1_\nu\right)}{\sin\left(\theta_W\right)} + \partial_\mu W^3_\nu - \partial_\nu W^3_\mu \end{bmatrix}, \partial_\mu B_\nu - \partial_\nu B_\mu \right)$$

[51]:
```python
Vsubs = {W1mu: (Wminusmu + Wplusmu)/sm.sqrt(2),
         W2mu: (Wminusmu - Wplusmu)/sm.sqrt(2),
```

```
         W3mu: sm.sin(thetaW)*Amu + sm.cos(thetaW)*Zmu,
         W1nu: (Wminusnu + Wplusnu)/sm.sqrt(2),
         W2nu: (Wminusnu - Wplusnu)/sm.sqrt(2),
         W3nu: sm.sin(thetaW)*Anu + sm.cos(thetaW)*Znu,
         Bmu:  sm.cos(thetaW)*Amu - sm.sin(thetaW)*Zmu,
         Bnu:  sm.cos(thetaW)*Anu - sm.sin(thetaW)*Znu}
```

[52]:
```
Wplus_nu = sm.Symbol('W^+_{\\nu}')
Wplus_mu = sm.Symbol('W^+_{\mu}')

Wminus_nu = sm.Symbol('W^-_{\\nu}')
Wminus_mu = sm.Symbol('W^-_{\mu}')

A_nu = sm.Symbol('A_{\\nu}')
A_mu = sm.Symbol('A_{\mu}')

Z_nu = sm.Symbol('Z_{\\nu}')
Z_mu = sm.Symbol('Z_{\mu}')

d_muWplus_nu = sm.Symbol('\partial_{\mu}W^+_{\\nu}')
d_nuWplus_mu = sm.Symbol('\partial_{\\nu}W^+_{\mu}')

d_muWminus_nu = sm.Symbol('\partial_{\mu}W^-_{\\nu}')
d_nuWminus_mu = sm.Symbol('\partial_{\\nu}W^-_{\mu}')

d_muA_nu = sm.Symbol('\partial_{\mu}A_{\\nu}')
d_nuA_mu = sm.Symbol('\partial_{\\nu}A_{\mu}')

d_muZ_nu = sm.Symbol('\partial_{\mu}Z_{\\nu}')
d_nuZ_mu = sm.Symbol('\partial_{\\nu}Z_{\mu}')

noncom = {Wplusnu: Wplus_nu, Wplusmu: Wplus_mu,
          Wminusnu: Wminus_nu, Wminusmu: Wminus_mu,
          Znu: Z_nu, Zmu: Z_mu,
          Anu: A_nu, Amu: A_mu,
          dmu*Wplusnu: d_muWplus_nu, dnu*Wplusmu: d_nuWplus_mu,
          dmu*Wminusnu: d_muWminus_nu, dnu*Wminusmu: d_nuWminus_mu,
          dmu*Znu: d_muZ_nu, dnu*Zmu: d_nuZ_mu,
          dmu*Anu: d_muA_nu, dnu*Amu: d_nuA_mu}
```

[53]:
```
for i in range(3):
    WW[i] = WW[i].subs(Vsubs).trigsimp().expand()
WW
```

[53]:

$$\left[\begin{matrix}- \frac{\sqrt{2} q A_{\mu} W^+_{\nu}}{2} + \frac{\sqrt{2} q A_{\mu} W^-_{\nu}}{2} + \frac{\sqrt{2} q W^+_{\mu} A_{\nu}}{2} - \frac{\sqrt{2} q W^-_{\mu} A_{\nu}}{2} + \frac{\sqrt{2} q W^+_{\mu} Z_{\nu}}{2 \tan{\left(\theta_W \right)}} - \frac{\sqrt{2} q W^-_{\mu} Z_{\nu}}{2 \tan{\left(\theta_W \right)}} - \frac{\sqrt{2} q Z_{\mu} W^+_{\nu}}{2 \tan{\left(\theta_W \right)}} + \frac{\sqrt{2} q Z_{\mu} W^-_{\nu}}{2 \tan{\left(\theta_W \right)}} + \frac{\sqrt{2} \partial_{\mu} W^+_{\nu}}{2} + \\ - \frac{\sqrt{2} q A_{\mu} W^+_{\nu}}{2} - \frac{\sqrt{2} q A_{\mu} W^-_{\nu}}{2} + \frac{\sqrt{2} q W^+_{\mu} A_{\nu}}{2} + \frac{\sqrt{2} q W^-_{\mu} A_{\nu}}{2} + \frac{\sqrt{2} q W^+_{\mu} Z_{\nu}}{2 \tan{\left(\theta_W \right)}} + \frac{\sqrt{2} q W^-_{\mu} Z_{\nu}}{2 \tan{\left(\theta_W \right)}} - \frac{\sqrt{2} q Z_{\mu} W^+_{\nu}}{2 \tan{\left(\theta_W \right)}} - \frac{\sqrt{2} q Z_{\mu} W^-_{\nu}}{2 \tan{\left(\theta_W \right)}} - \frac{\sqrt{2} \partial_{\mu} W^+_{\nu}}{2} + \\ - \frac{q W^+_{\mu} W^-_{\nu}}{\sin{\left(\theta_W \right)}} + \frac{q W^-_{\mu} W^+_{\nu}}{\sin{\left(\theta_W \right)}} + \sin{\left(\theta_W \right)} \partial_{\mu} A_{\nu} - \sin{\left(\theta_W \right)} \partial_{\nu} A_{\mu} + \cos{\left(\theta_W \right)} \partial_{\mu} Z_{\nu} - \cos{\left(\theta_W \right)} \partial_{\nu} Z_{\mu}\end{matrix}\right.$$

[54]:
```python
for i in range(3):
    WW[i] = WW[i].subs(noncom).expand()
WW
```

[54]:

$$\left[\begin{array}{l} -\frac{\sqrt{2}A_{\mu}W_{\nu}^{+}q}{2}+\frac{\sqrt{2}A_{\mu}W_{\nu}^{-}q}{2}+\frac{\sqrt{2}A_{\nu}W_{\mu}^{+}q}{2}-\frac{\sqrt{2}A_{\nu}W_{\mu}^{-}q}{2}+\frac{\sqrt{2}W_{\mu}^{+}Z_{\nu}q}{2\tan{(\theta_W)}}-\frac{\sqrt{2}W_{\nu}^{+}Z_{\mu}q}{2\tan{(\theta_W)}}-\frac{\sqrt{2}W_{\mu}^{-}Z_{\nu}q}{2\tan{(\theta_W)}}+\frac{\sqrt{2}W_{\nu}^{-}Z_{\mu}q}{2\tan{(\theta_W)}}+\frac{\sqrt{2}\partial_{\mu}W_{\nu}^{+}}{2}+\sqrt{2}\partial \\ -\frac{\sqrt{2}A_{\mu}W_{\nu}^{+}q}{2}-\frac{\sqrt{2}A_{\mu}W_{\nu}^{-}q}{2}+\frac{\sqrt{2}A_{\nu}W_{\mu}^{+}q}{2}+\frac{\sqrt{2}A_{\nu}W_{\mu}^{-}q}{2}+\frac{\sqrt{2}W_{\mu}^{+}Z_{\nu}q}{2\tan{(\theta_W)}}-\frac{\sqrt{2}W_{\nu}^{+}Z_{\mu}q}{2\tan{(\theta_W)}}+\frac{\sqrt{2}W_{\mu}^{-}Z_{\nu}q}{2\tan{(\theta_W)}}-\frac{\sqrt{2}W_{\nu}^{-}Z_{\mu}q}{2\tan{(\theta_W)}}-\frac{\sqrt{2}\partial_{\mu}W_{\nu}^{+}}{2}+\sqrt{2}\partial \\ -\frac{W_{\mu}^{+}W_{\nu}^{-}q}{\sin{(\theta_W)}}+\frac{W_{\nu}^{+}W_{\mu}^{-}q}{\sin{(\theta_W)}}+\partial_{\mu}A_{\nu}\sin{(\theta_W)}+\partial_{\mu}Z_{\nu}\cos{(\theta_W)}-\partial_{\nu}A_{\mu}\sin{(\theta_W)}-\partial_{\nu}Z_{\mu}\cos{(\theta_W} \end{array}\right.$$

[55]:
```python
BB = BB.subs(Vsubs).expand()
BB
```

[55]:

$$-\sin{(\theta_W)}\partial_{\mu}Z_{\nu}+\sin{(\theta_W)}\partial_{\nu}Z_{\mu}+\cos{(\theta_W)}\partial_{\mu}A_{\nu}-\cos{(\theta_W)}\partial_{\nu}A_{\mu}$$

[56]:
```python
BB = BB.subs(noncom)
BB
```

[56]:

$$\partial_{\mu}A_{\nu}\cos{(\theta_W)}-\partial_{\mu}Z_{\nu}\sin{(\theta_W)}-\partial_{\nu}A_{\mu}\cos{(\theta_W)}+\partial_{\nu}Z_{\mu}\sin{(\theta_W)}$$

[57]:
```python
VV =-sm.Rational(1, 4) * (BB*BB +  (WW.T*WW)[0])
VV = VV.expand()
VV
```

[57]:

$$-\frac{A_{\mu}^{2}\left(W_{\nu}^{+}\right)^{2}q^{2}}{4}-\frac{A_{\mu}^{2}\left(W_{\nu}^{-}\right)^{2}q^{2}}{4}+\frac{A_{\mu}A_{\nu}W_{\mu}^{+}W_{\nu}^{+}q^{2}}{2}+\frac{A_{\mu}A_{\nu}W_{\mu}^{-}W_{\nu}^{-}q^{2}}{2}+\frac{A_{\mu}W_{\mu}^{+}W_{\nu}^{+}Z_{\nu}q^{2}}{2\tan{(\theta_W)}}-$$
$$\frac{A_{\mu}\left(W_{\nu}^{+}\right)^{2}Z_{\mu}q^{2}}{2\tan{(\theta_W)}}+\frac{A_{\mu}W_{\nu}^{+}\partial_{\mu}W_{\nu}^{-}q}{2}-\frac{A_{\mu}W_{\nu}^{+}\partial_{\nu}W_{\mu}^{-}q}{2}+\frac{A_{\mu}W_{\mu}^{-}W_{\nu}^{-}Z_{\nu}q^{2}}{2\tan{(\theta_W)}}-\frac{A_{\mu}\left(W_{\nu}^{-}\right)^{2}Z_{\mu}q^{2}}{2\tan{(\theta_W)}}-$$
$$\frac{A_{\mu}W_{\nu}^{-}\partial_{\mu}W_{\nu}^{+}q}{2}+\frac{A_{\mu}W_{\nu}^{-}\partial_{\nu}W_{\mu}^{+}q}{2}-\frac{A_{\nu}^{2}\left(W_{\mu}^{+}\right)^{2}q^{2}}{4}-\frac{A_{\nu}^{2}\left(W_{\mu}^{-}\right)^{2}q^{2}}{4}-\frac{A_{\nu}\left(W_{\mu}^{+}\right)^{2}Z_{\nu}q^{2}}{2\tan{(\theta_W)}}+$$
$$\frac{A_{\nu}W_{\mu}^{+}W_{\nu}^{+}Z_{\mu}q^{2}}{2\tan{(\theta_W)}}-\frac{A_{\nu}W_{\mu}^{+}\partial_{\mu}W_{\nu}^{-}q}{2}+\frac{A_{\nu}W_{\mu}^{+}\partial_{\nu}W_{\mu}^{-}q}{2}-\frac{A_{\nu}\left(W_{\mu}^{-}\right)^{2}Z_{\nu}q^{2}}{2\tan{(\theta_W)}}+\frac{A_{\nu}W_{\mu}^{-}W_{\nu}^{-}Z_{\mu}q^{2}}{2\tan{(\theta_W)}}+$$
$$\frac{A_{\nu}W_{\mu}^{-}\partial_{\mu}W_{\nu}^{+}q}{2}-\frac{A_{\nu}W_{\mu}^{-}\partial_{\nu}W_{\mu}^{+}q}{2}-\frac{\left(W_{\mu}^{+}\right)^{2}\left(W_{\nu}^{-}\right)^{2}q^{2}}{4\sin^{2}{(\theta_W)}}-\frac{\left(W_{\mu}^{+}\right)^{2}Z_{\nu}^{2}q^{2}}{4\tan^{2}{(\theta_W)}}+\frac{W_{\mu}^{+}W_{\nu}^{+}W_{\mu}^{-}W_{\nu}^{-}q^{2}}{2\sin^{2}{(\theta_W)}}+$$
$$\frac{W_{\mu}^{+}W_{\nu}^{+}Z_{\mu}Z_{\nu}q^{2}}{2\tan^{2}{(\theta_W)}}+\frac{W_{\mu}^{+}W_{\nu}^{-}\partial_{\mu}A_{\nu}q}{2}+\frac{W_{\mu}^{+}W_{\nu}^{-}\partial_{\mu}Z_{\nu}q\cos{(\theta_W)}}{2\sin{(\theta_W)}}-\frac{W_{\mu}^{+}W_{\nu}^{-}\partial_{\nu}A_{\mu}q}{2}-$$
$$\frac{W_{\mu}^{+}W_{\nu}^{-}\partial_{\nu}Z_{\mu}q\cos{(\theta_W)}}{2\sin{(\theta_W)}}-\frac{W_{\mu}^{+}Z_{\nu}\partial_{\mu}W_{\nu}^{-}q}{2\tan{(\theta_W)}}+\frac{W_{\mu}^{+}Z_{\nu}\partial_{\nu}W_{\mu}^{-}q}{2\tan{(\theta_W)}}-\frac{\left(W_{\nu}^{+}\right)^{2}\left(W_{\mu}^{-}\right)^{2}q^{2}}{4\sin^{2}{(\theta_W)}}-\frac{\left(W_{\nu}^{+}\right)^{2}Z_{\mu}^{2}q^{2}}{4\tan^{2}{(\theta_W)}}-$$
$$\frac{W_{\nu}^{+}W_{\mu}^{-}\partial_{\mu}A_{\nu}q}{2}-\frac{W_{\nu}^{+}W_{\mu}^{-}\partial_{\mu}Z_{\nu}q\cos{(\theta_W)}}{2\sin{(\theta_W)}}+\frac{W_{\nu}^{+}W_{\mu}^{-}\partial_{\nu}A_{\mu}q}{2}+\frac{W_{\nu}^{+}W_{\mu}^{-}\partial_{\nu}Z_{\mu}q\cos{(\theta_W)}}{2\sin{(\theta_W)}}+$$
$$\frac{W_{\nu}^{+}Z_{\mu}\partial_{\mu}W_{\nu}^{-}q}{2\tan{(\theta_W)}}-\frac{W_{\nu}^{+}Z_{\mu}\partial_{\nu}W_{\mu}^{-}q}{2\tan{(\theta_W)}}-\frac{\left(W_{\mu}^{-}\right)^{2}Z_{\nu}^{2}q^{2}}{4\tan^{2}{(\theta_W)}}+\frac{W_{\mu}^{-}W_{\nu}^{-}Z_{\mu}Z_{\nu}q^{2}}{2\tan^{2}{(\theta_W)}}+\frac{W_{\mu}^{-}Z_{\nu}\partial_{\mu}W_{\nu}^{+}q}{2\tan{(\theta_W)}}-$$
$$\frac{W_{\mu}^{-}Z_{\nu}\partial_{\nu}W_{\mu}^{+}q}{2\tan{(\theta_W)}}-\frac{\left(W_{\nu}^{-}\right)^{2}Z_{\mu}^{2}q^{2}}{4\tan^{2}{(\theta_W)}}-\frac{W_{\nu}^{-}Z_{\mu}\partial_{\mu}W_{\nu}^{+}q}{2\tan{(\theta_W)}}+\frac{W_{\nu}^{-}Z_{\mu}\partial_{\nu}W_{\mu}^{+}q}{2\tan{(\theta_W)}}-\frac{\partial_{\mu}A_{\nu}^{2}\sin^{2}{(\theta_W)}}{4}-\frac{\partial_{\mu}A_{\nu}^{2}\cos^{2}{(\theta_W)}}{4}+$$
$$\frac{\partial_{\mu}A_{\nu}\partial_{\nu}A_{\mu}\sin^{2}{(\theta_W)}}{2}+\frac{\partial_{\mu}A_{\nu}\partial_{\nu}A_{\mu}\cos^{2}{(\theta_W)}}{2}-\frac{\left(\partial_{\mu}W_{\nu}^{+}\right)^{2}}{4}+\frac{\partial_{\mu}W_{\nu}^{+}\partial_{\nu}W_{\mu}^{+}}{2}-\frac{\left(\partial_{\mu}W_{\nu}^{-}\right)^{2}}{4}+\frac{\partial_{\mu}W_{\nu}^{-}\partial_{\nu}W_{\mu}^{-}}{2}-$$
$$\frac{\partial_{\mu}Z_{\nu}^{2}\sin^{2}{(\theta_W)}}{4}-\frac{\partial_{\mu}Z_{\nu}^{2}\cos^{2}{(\theta_W)}}{4}+\frac{\partial_{\mu}Z_{\nu}\partial_{\nu}Z_{\mu}\sin^{2}{(\theta_W)}}{2}+\frac{\partial_{\mu}Z_{\nu}\partial_{\nu}Z_{\mu}\cos^{2}{(\theta_W)}}{2}-\frac{\partial_{\nu}A_{\mu}^{2}\sin^{2}{(\theta_W)}}{4}-$$

$$\frac{\partial_{\nu} A_{\mu}^{2} \cos^{2}(\theta_{W})}{4} - \frac{\left(\partial_{\nu} W_{\mu}^{+}\right)^{2}}{4} - \frac{\left(\partial_{\nu} W_{\mu}^{-}\right)^{2}}{4} - \frac{\partial_{\nu} Z_{\mu}^{2} \sin^{2}(\theta_{W})}{4} - \frac{\partial_{\nu} Z_{\mu}^{2} \cos^{2}(\theta_{W})}{4}$$

This is a complicated expression, but as expected from the form of the vector boson Lagrangian, the theory predicts triple and quartic boson interactions.

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