On Adaptive Wavelet-based Methods for the Maxwell Equations

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Summary

On Adaptive Wavelet-based Methods for the Maxwell Equations

The subject of the thesis is the applicability of recently developed wavelet-based methods for efficient numerical solution of the Maxwell equations, which constitute a model for electromagnetic phenomena. An introduction to wavelet bases in Hilbert spaces, and the adaptive wavelet-based methods is given, followed by an introduction to computational electromagnetics. Then, variational formulations for the Maxwell equations are developed, and results of well-posedness are proven. We put particular attention to the existence of sigularities in the solution to Maxwells equations, and the danger that they may not be resolved by certain numerical schemes. The formulations belong to the class of Fictitious Domain formulations. Among the advantages of such formulations are that they allow the use of simple, for instance periodic, functions for their discretization, and boundary and other conditions need not be incorporated into the employed basis functions. Finally, being an important issue for the adaptive wavelet-based methods, formulations of the Maxwell equations in an infinitely-dimensional Euclidean space setting are considered, and initial results of well-posedness are stated.

Resumé

Om Adaptive Wavelet-baserede Metoder for Maxwells Ligninger

Emnet for denne afhandling er anvendeligheden af adaptive, wavelet-baserede metoder til numerisk løsning af Maxwells ligninger. Disse udgør en model for elektromagnetiske fænomener i naturen. Der indledes med en introduktion til wavelet baser i Hilbertrum, samt til de førnævnte metoder, og derefter gives en introduktion til matematisk og numerisk analyse af elektromagnetiske problemer. Dernæst udvikles nye, velstillede variationsformuleringer for Maxwells ligninger, under særlig hensyntagen til at variationsproblemerne kan udgøre fundamentet for numeriske metoder, som kan opløse eventuelle singulariteter i løsningen til Maxwells ligninger. Variationsformuleringerne er såkaldte Fictitious Domains formuleringer. Blandt fordelene ved disse er blandt andet at de retfærdiggør brugen af periodiske basisfunktioner til diskretisering af problemet, samt at rand- og andre betingelser ikke nødvendigvis skal indbygges i de anvendte basisfunktioner. Slutteligt, som et vigtigt aspekt for adaptive wavelet-baserede metoder, gives indledende resultater om velstillethed i Euklidisk metrik af diskrete formuleringer for Maxwells ligninger.

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Chapter 1

Introduction

Why wavelets and Partial Differential Equations ?

In recent years, wavelet techniques have gained popularity in the applied sciences, mainly because of the ability of wavelet bases to represent and describe complicated objects efficiently. In image and signal processing, wavelet techniques have been applied successfully over the last decade, so it is interesting to examine if the properties of wavelets can be exploited in other scientific areas as well. The subject of this work is the application of wavelet techniques to the numerical solution of operator equations, or more specifically, partial differential equations (PDEs). Contrary to the case of applications of wavelets in signal and image processing, the object to be analyzed with wavelets (i.e. the solution) is not given beforehand.

Several problems in science are variations over the following theme: A function f is given, representing an object to be analyzed. Expand it as a linear combination of basis functions $\{f_k\}$

$$f = \sum_k a_k f_k$$

and hope that the relevant properties of f can be described by a few coefficients $\{a_k\}$ in the expansion.

Depending on the basis f_k, various kinds of information about f can be extracted from the expansion coefficients $\{a_k\}$. In Fourier analysis, f could be a continuous, differentiable periodic function, and f_k could be sines and cosines oscillating at integer multiples of the angular frequency of f. For this case, a_k reflects the amount of energy of f residing at the frequency corresponding to the basis function f_k. These basis functions are *global*, in the sense that they take on nonzero values on the entire domain on which f is defined.

Wavelet bases have a different structure than the Fourier bases. They can be *locally* supported, hence the coefficients in an expansion relative to a wavelet basis reflect, to a larger extent than the Fourier expansion coefficients, the local behaviour of f, such as oscillations, discontinuitites, and steep gradients.

The main theme in applications of wavelet-based techniques is the ability of concentrating the computational effort where it is needed the most. In the context of numerical solution of PDEs, this leads naturally to the concept of *adaptivity*. We shall present two intuitive ideas of adaptivity:

- *Given an accuracy, find an approximation to the solution to the operator equation within this accuracy, with the minimal number of degrees of freedom.*

- *Given a number of degrees of freedom, find an approximation to the solution of the operator equation, with this number of degrees of freedom, that realizes the best possible accuracy.*

Solutions to PDEs may exhibit irregular behavior locally. An adaptive algorithm attempts to spend the computational effort mainly on localizing and resolving the irregular behaviour of the solution. The unique properties of wavelets facilitates this procedure.

Adaptive wavelet-based methods

In the papers [24],[25], A. Cohen, W. Dahmen and R. DeVore developed adaptive wavelet-based algorithms for the numerical solution of operator equations. We shall in this thesis focus on the algorithms of the latter paper. These schemes essentially rely on three properties. First, the recasting of the operator equation into a well-posed variational formulation in a Hilbert space, which we shall denote H. In other words, there should exist a bilinear form $a(\cdot,\cdot)$, mapping $H \times H$ into some scalar field, such that for a given element $f \in H'$, being the dual space of H, there holds

$$a(u,v) = \langle f, v\rangle, \quad v \in H \tag{1.1}$$

for some $u \in H$. Well-posedness means here the existence and uniqueness of u, and the continuous dependence of u on the given data. A well-posed variational formulation allows an operator $A : H \mapsto H'$ to be defined from $a(\cdot,\cdot)$, such that A is an isomorphism.

The second ingredient is the availability of a *wavelet basis* $\Psi_H := \{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ of H, inducing a norm equivalence with the structure

$$\hat{c}\|\boldsymbol{D}\boldsymbol{c}\|_{\ell_2(\mathcal{J})} \leq \left\|\sum_\lambda c_\lambda\psi_\lambda\right\|_H \leq \hat{C}\|\boldsymbol{D}\boldsymbol{c}\|_{\ell_2(\mathcal{J})}, \tag{1.2}$$

where $\boldsymbol{c} := \{c_\lambda\}_{\lambda\in\mathcal{J}}$ is the array of expansion coefficients, $\boldsymbol{D}$ is a bi-infinite, diagonal matrix with positive entries, and $0 < \hat{c} \leq \hat{C} < \infty$ are constants that do not depend on the choice of $\boldsymbol{c}$. The well-posed variational problem and the wavelet basis can be employed to define a (correctly scaled) wavelet representation of the operator A, let us call it $\boldsymbol{A}$, which is an isomorphism on $\ell_2(\mathcal{J})$.

The third ingredient is the *compressibility* of the involved operators, and the vectors on which they act. The compressibility of a vector is to be understood in the sense that an infinite vector $\boldsymbol{c}$ can be approximated well, in an appropriate norm, by a few of its elements. By *wavelet characterizations of function spaces*, this property is related to smoothness properties of the solution to the underlying operator equation. The compressibility, or *quasi-sparsity* of a bi-infinite operator such as $\boldsymbol{A}$ means that it can be approximated well by appropriately chosen finite sections of $\boldsymbol{A}$, with a low number of nonzero entries. In summary, compressibility properties of wavelet realizations of operators, and vectors of expansion coefficients with respect to wavelet bases, rely on the unique approximation and cancellation properties of wavelet bases.

A classical approach for solving a variational problem numerically is to restrict the problem to a fixed finite-dimensional subspace, so that one discretizes the bilinear form using a basis for the subspace. Thereby, a linear system of equations is obtainined, which is often solved most efficiently by iterative methods. The paper [25] introduced a new paradigm, in which a

convergent iterative method is devised in the infinitely-dimensional setting, using the $\ell_2(\mathcal{J})$ isomorphism $\boldsymbol{A}$. In order to do numerical computations, one needs to approximate the action of $\boldsymbol{A}$ on finite vectors up to any given accuracy, and this is where the *adaptivity* enters the picture. Balancing approximation errors give a convergent algorithm, and compressibility properties of the operator $\boldsymbol{A}$, and smoothness properties of the solution to the operator equation, determine work estimates and convergence rates of the algorithm.

The Maxwell equations

Electromagnetics as a physical discipline plays a very important role in the sciences, and in the modern society in general, since several important technological innovations of the recent decades rely on an understanding of electromagnetic phenomena, and the ability to solve problems from electromagnetics. For instance, understanding electromagnetic waves and their propagation is important for cell phones, optical fibres, and for radars and satellite communication. Turbines generating electricity at power plants, induction heaters, electric engines, loudspeakers and microphones all rely on the interaction between electricity and magnetism.

For applications in the engineering sciences, such as the ones mentioned above, the Maxwell equations have, since their derivation in the late 19th century, fully explained electromagnetic phenomena. They read as follows:

$$\begin{aligned} \mathbf{curl}\ \frac{\boldsymbol{B}}{\mu} &= \varepsilon \frac{\partial}{\partial t}\boldsymbol{E} + \boldsymbol{J} \\ \mathbf{curl}\ \boldsymbol{E} &= -\frac{\partial}{\partial t}\boldsymbol{B}, \end{aligned}$$

where $\boldsymbol{J}$ is the current density, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic flux density. ε and μ are material parameters describing electric and magnetic properties.

Since analytical solution methods exist only for very simple and idealized physical configurations, the ability to solve the Maxwell equations numerically is highly appreciated. The approach taken in this thesis consists of eliminating $\boldsymbol{B}$ from the Maxwell equations, to obtain a variational formulation involving the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\mathbf{curl}\boldsymbol{u}, \mathbf{curl}\boldsymbol{v}) + \kappa(\boldsymbol{u}, \boldsymbol{v}),$$

for $\boldsymbol{u}, \boldsymbol{v}$ in (a subspace of) an appropriately defined function space $\mathbf{H}(\mathbf{curl}; \Omega)$. κ is a parameter depending on the material parameters μ, ε, as well as other quantities such as employed time stepping lengths. κ may be positive, negative or complex valued, which in turn has an influence on the analysis leading to results of well-posedness of variational formulations.

An important issue in the developments presented in this thesis will be the existence of singularities in the electric field $\boldsymbol{E}$. This refers to the fact that $\boldsymbol{E}$ belongs to (subspaces of) $\mathbf{H}(\mathbf{curl}; \Omega)$, of which in general the Sobolev space $\mathbf{H}^1(\Omega)$ is a proper subspace. If one is not careful, one may prescribe numerical algorithms that do converge, but to the wrong solution, namely the 'smooth component of $\boldsymbol{E}$', belonging to $\mathbf{H}^1(\Omega)$.

Today, electromagnetics is a source of many interesting and challenging problems for many areas of science, ranging from pure mathematics, over numerics and to engineering science. To the knowledge of the author, no research has been carried out on the adaptive wavelet-based solution to the Maxwell equations. Therefore, the combination of the rapid

development of efficient numerical schemes, and the interesting and challenging problems posed by electromagnetics and the Maxwell equations, comprise an open and interesting field of research.

Outline

The new work presented in this thesis is based on results from several sources. In order to motivate the problems we consider, as well as to put the new developments into perspective, we choose to give a coherent exposition of the results that constitute the foundation for the work in this thesis.

Chapter 2 gives a coherent outline of wavelet theory, in the terminology of numerical analysis. In particular, we focus on construction principles of wavelet bases applied in connection with the adaptive schemes of [25].

Chapter 3 treats the new paradigm for numerical solution of PDEs, and the adaptive wavelet methods from [25], along with the concepts and results from modern approximation theory used for their construction and analysis.

Chapter 4 focuses on saddle point formulations, of which the Fictitious Domain formulation is a special case, which will be of particular importance for later developments. We also treat the transformation of a saddle point problem into a well-posed problem in the infinite-dimensional Euclidean metric, which is important for the adaptive wavelet-based solution of such problems.

Chapter 5 introduces some fundamental concepts for the mathematical and numerical analysis of the Maxwell equations. The relevant function spaces are introduced, along with simple variational formulations, and corresponding results of well-posedness. An overview of some contemporary methods is also given. This chapter is to some extent based on the paper [37].

Chapter 6 treats Fictitious Domain formulations for the Maxwell equations, and it is based on the paper [37]. Results of well-posedness are given, and special attention is paid to the issue of resolving non-$\mathbf{H}^1(\Omega)$ singularities.

Chapter 7 considers the transformation of the variational formulations of Chapter 6 into a well-posed formulation in Euclidean metric, using appropriately chosen, and recently developed wavelet bases for the function spaces occuring in the analysis of the Maxwell equations. Initial results of well-posedness are given.

Chapter 2

Elements of wavelet theory

The purpose of this chapter is to give a coherent exposition of the construction of wavelet bases for the numerical solution of PDEs. For good reasons, most introductory texts on wavelet theory choose the path and terminology of signal and image processing [43], [54], [77], [73], [61]. The community considering wavelet-based techniques for numerical solution of PDEs and other operator equations often has slightly different preferences, both with respect to terminology and specific requirements of the wavelet bases. The present exposition is written with this in mind, hoping that it may provide an easy transition to the operator equation language of wavelet analysis, for those who may have studied wavelets previously in a signal or image processing context.

The outline of the material is as follows. In Section 2.1, we present the fundamental structural properties of wavelet bases that are common for all constructions in the sequel, and that are crucial for their application to the numerical solution of operator equations. Section 2.2 treats the now classical construction of biorthogonal wavelets in the space $L_2(\mathbb{R})$, carried out in the late 1980's by Cohen, Daubechies and Feauveau [26], and it constitutes the foundation for the constructions that follow. Section 2.3 introduces a framework that generalizes the constructions of Section 2.2, and in particular, it facilitates the construction of wavelets on bounded subdomains of $\mathbb{R}$. In Section 2.4, this generalized framework is applied to outline the construction of wavelet bases for the space $L_2(0,1)$, as carried out by Dahmen, Kunoth and Urban [40].

After the introduction of tensor product wavelets in Section 2.5, which gives us wavelet bases for $L_2(\mathbb{R}^n)$, or $L_2((0,1)^n)$, we explain briefly in Section 2.6 some wavelet constructions on general domains. The last section is devoted to a property of wavelets that is of great importance for applications in numerical analysis, namely wavelet characterizations of function spaces. We shall focus on wavelet characterizations of Sobolev and Besov spaces, which are known from many different sources. We choose [34] and [45], but others are [62], [54], [79], [43].

2.1 The basic requirements

In this section, we introduce four properties of wavelet bases, shared by all constructions in this thesis, and which are important for their application to the numerical analysis of PDEs. For simplicity, we present them in the univariate case, but the properties extend immediately to the multivariate case. Two of these properties of wavelet bases, namely the *translation*

and dilation property and *locality*, are in most cases taken as a point of departure, and are (almost) fulfilled by construction. The two other properties of wavelet bases are the *Riesz basis property*, and the *vanishing moment property*, which one must strive to fulfill in the process of construction.

We shall introduce the following notation: $A \lesssim B$ means that A can be bounded by a constant times B, where the constant is independent of parameters that A and B may depend on. We write $A \sim B$ whenever $A \lesssim B$ and $B \lesssim A$.

In this chapter, we shall work in the space $L_2(\Omega)$, equipped with the usual inner product, which we shall designate $(\cdot,\cdot)_{0,\Omega}$. Ω is some (Borel) subset of $\mathbb{R}$.

The translation and dilation property

The elements in the basis for $L_2(\Omega)$ are mainly on the following form:

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \tag{2.1}$$

for some function $\psi \in L_2(\Omega)$. The *level* j refers to the scale on which the function is living, and the *translation parameter* k reflects its position on the domain. For wavelets on $\mathbb{R}$, all elements in the basis are of the above form, and j and k range through all of $\mathbb{Z}$. For the case of a bounded domain, every element in the basis cannot have the above form, as we shall see in Section 2.4, and j and k then range through appropriate subsets of $\mathbb{Z}$.

Locality

The basis functions must have *local support*, in the sense that

$$\operatorname{diam}(\operatorname{supp}(\psi_{j,k})) \sim 2^{-j}.$$

In particular, the basis functions should be compactly supported.

The Riesz basis property

A Riesz basis for a separable Hilbert space $(X,(\cdot,\cdot))$ is an at most countable sequence $\{x_\lambda\}_{\lambda\in\Lambda}$ of vectors in X with the property that the vector space of finite linear combinations is dense in X, and that there exist constants $0 < A \le B < \infty$ such that for any sequence $c = \{c_\lambda\}_{\lambda\in\Lambda} \in \ell_2(\Lambda)$, the norm equivalence

$$A\|c\|^2_{\ell_2(\Lambda)} \le \left\| \sum_{\lambda\in\Lambda} c_\lambda x_\lambda \right\|^2_X \le B\|c\|^2_{\ell_2(\Lambda)}$$

holds. The constants A, B are called the *Riesz bounds*.

An orthonormal basis is a special case of a Riesz basis, where $A = B = 1$. A Riesz basis has a biorthogonal sequence $\{\tilde{x}_\lambda\}_{\lambda\in\Lambda}$, i.e. a sequence that satisfies $(x_{\lambda_i}, \tilde{x}_{\lambda_j}) = \delta_{i,j}$, and $\{\tilde{x}_\lambda\}_{\lambda\in\Lambda}$ is also a Riesz basis. Any $x \in X$ has the representations

$$x = \sum_{\lambda\in\Lambda}(x,\tilde{x}_\lambda)x_\lambda = \sum_{\lambda\in\Lambda}(x,x_\lambda)\tilde{x}_\lambda.$$

For the construction of Riesz bases in, say, $L_2(\mathbb{R})$ having the structure (2.1), we shall introduce a hierarchy of stability notions of bases. A family of vectors which is a Riesz

basis for its closed linear span is called *stable*[1]. A family $\{\psi_{j,k}\}_{j,k}$ is *uniformly stable* if the families $\{\psi_{j,k}\}_k$ are stable with constants independent of j. Finally, the Riesz basis property of $\{\psi_{j,k}\}_{j,k}$ in $L_2(\mathbb{R})$ is referred to as *stability over all levels*.

For a Riesz basis generated by the compactly supported function ψ, called the *wavelet*, the dual basis has the structure $2^{j/2}\tilde{\psi}(2^j x - k)$, where $\tilde{\psi} \in L_2(\mathbb{R})$ is a compactly supported function called the *dual wavelet*.

The vanishing moment property

We say that the function ψ resp. $\tilde{\psi}$ has m resp. $\tilde{m}$ *vanishing moments* if

$$\begin{aligned} ((\cdot)^l, \psi)_{0,\Omega} &= 0, \quad l = 0, \ldots, m, \quad \text{resp.} \\ ((\cdot)^l, \tilde{\psi})_{0,\Omega} &= 0, \quad l = 0, \ldots, \tilde{m}. \end{aligned}$$

For the case of compactly supported and continuous ψ and $\tilde{\psi}$, existence of the integrals above is obvious. Later in the chapter, it will be described how additional interesting and desirable properties of wavelets can be inferred from the four properties just mentioned.

Perspective

Orthonormal bases of compactly supported wavelets do indeed exist, but it turns out that the requirements for their construction are too restrictive for solving PDEs numerically [35], [19]. We shall therefore treat biorthogonal bases exclusively.

Wavelets are desirable for many applications in signal processing, in that they can provide good localization (i.e. exponential decay) in the time as well as in the frequency domain. This excludes compact support, since it leads, by Fourier transformation, to poor localization in the frequency domain.

Many advantages of wavelet-based techniques rely on the ability of wavelets to represent functions to high accuracy by an expansion with a low number of significant terms, compared to other basis functions. This is due to the vanishing moment property, as a local Taylor-expansion argument shows. The combination of good localization and vanishing moments explains in many cases the successful application of wavelet-based techniques.

2.2 Biorthogonal wavelet bases in $L_2(\mathbb{R})$

In this section, we introduce some important concepts of wavelet theory, namely those of a *multiresolution sequence*, *refinability*, and the *scaling function*. The multiresolution sequence quantifies what is meant by approximation of $L_2(\mathbb{R})$, initially by scaling function expansions, but later on by wavelet expansions. The scaling function and the refinability arise in a natural way from the multiresolution sequence, and eventually paves the way for the definition and construction of wavelets. Being of great importance for later developments, we shall finally show a concrete example of a family of wavelet bases, namely those consisting of *B-splines*. The material in this subsection is borrowed from [26], [43].

[1] In the literature, it is also known as a Riesz sequence.

Biorthogonal multiresolution, scaling functions, refinability

A *biorthogonal multiresolution sequence* in $L_2(\mathbb{R})$ is a pair of sequences of subspaces $\{\mathcal{S}_j\}_{j\in\mathbb{Z}}$, $\{\tilde{\mathcal{S}}_j\}_{j\in\mathbb{Z}}$ of $L_2(\mathbb{R})$, called the *primal* and the *dual* multiresolution sequence, respectively, which satisfy the conditions:

(i) The spaces are *nested and dense in* $L_2(\mathbb{R})$, in the sense that

$$\mathcal{S}_j \subset \mathcal{S}_{j+1} \subset \ldots, \quad \mathrm{clos}_{L_2(\mathbb{R})}\left(\bigcup_{j\in\mathbb{Z}} \mathcal{S}_j\right) = L_2(\mathbb{R}), \tag{2.2}$$

$$\tilde{\mathcal{S}}_j \subset \tilde{\mathcal{S}}_{j+1} \subset \ldots, \quad \mathrm{clos}_{L_2(\mathbb{R})}\left(\bigcup_{j\in\mathbb{Z}} \tilde{\mathcal{S}}_j\right) = L_2(\mathbb{R}), \tag{2.3}$$

and the spaces are *separating*:

$$\bigcap_{j\in\mathbb{Z}} \mathcal{S}_j = \{0\}, \quad \bigcap_{j\in\mathbb{Z}} \tilde{\mathcal{S}}_j = \{0\},$$

(ii) The spaces in (2.2), (2.3) are generated by, respectively, $L_2(\mathbb{R})$-functions ξ and $\tilde{\xi}$, called the *scaling function*, and the *dual scaling function*, forming a *dual pair*, i.e.

$$\forall j,k,k' \in \mathbb{Z}: \quad (\xi_{j,k}, \tilde{\xi}_{j,k'})_{0,\mathbb{R}} = \delta_{k,k'},$$

such that

$$\Xi_j := \{2^{j/2}\xi(2^j x - k) := \xi_{j,k}(x),\ k \in \mathbb{Z}\}$$

is a Riesz basis for $\mathcal{S}_j := S(\Xi_j) := \mathrm{clos}_{L_2(\mathbb{R})}\mathrm{span}(\Xi_j)$. The same holds for the dual scaling function: $\tilde{\mathcal{S}}_j = S(\tilde{\Xi}_j)$.

The nestedness of $\mathcal{S}_j$ and $\tilde{\mathcal{S}}_j$ expresses the fact that these spaces capture an increasing amount of details in the space $L_2(\mathbb{R})$, and the density of the spaces ensures that any element in $L_2(\mathbb{R})$ can be approximated arbitrarily well in $L_2(\mathbb{R})$ by choosing an element from a sufficiently large space $\mathcal{S}_j$.

The factor $2^{j/2}$ is chosen such that the transformation $\xi(\cdot) \mapsto 2^{j/2}\xi(2^j\cdot)$ becomes an isomorphism on $L_2(\mathbb{R})$. The spaces $\mathcal{S}_j, \tilde{\mathcal{S}}_j$ are *translation invariant*:

$$\forall j \in \mathbb{Z}: \quad f = f_{j,0} \in \mathcal{S}_j \ \Leftrightarrow f_{j,k} \in \mathcal{S}_j,\ k \in \mathbb{Z}.$$

The nestedness of the spaces $\mathcal{S}_j, \tilde{\mathcal{S}}_j$ is equivalent to the scaling function and the dual being *refinable*:

$$\xi(x) = \sum_{k\in\mathbb{Z}} a_k \xi(2x-k), \quad \tilde{\xi}(x) = \sum_{k\in\mathbb{Z}} \tilde{a}_k \tilde{\xi}(2x-k), \tag{2.4}$$

where the coefficients $\{a_k\}_{k\in\mathbb{Z}}, \{\tilde{a}_k\}_{k\in\mathbb{Z}}$, are called the *masks* of the scaling functions ξ and $\tilde{\xi}$. (2.4) are called *refinement equations*. Compact support of ξ is equivalent to its mask being finite, i.e. only finitely many a_k are nonzero. [2]

[2] We adopt the notation of [76] and designate the univariate scaling function by ξ, since the more standard symbol ϕ will be reserved for *tensor product scaling functions* in Section 2.5.

Complement spaces, wavelets, and the wavelet transform

We define complements $\mathcal{U}_j, \tilde{\mathcal{U}}_j$ of $\mathcal{S}_j,\ \tilde{\mathcal{S}}_j$ in $\mathcal{S}_{j+1},\ \tilde{\mathcal{S}}_{j+1}$ so that the conditions (iii)-(v) below hold true:

(iii) The spaces decompose as

$$\mathcal{S}_{j+1} = \mathcal{S}_j + \mathcal{U}_j, \quad \tilde{\mathcal{S}}_{j+1} = \tilde{\mathcal{S}}_j + \tilde{\mathcal{U}}_j, \tag{2.5}$$

in the sense that for each $f \in \mathcal{S}_{j+1}$, there exist unique $g \in \mathcal{S}_j,\ h \in \mathcal{U}_j$ so that $f = g + h$.

(iv) The orthogonality conditions

$$\tilde{\mathcal{U}}_j \perp \mathcal{S}_j, \quad \mathcal{U}_j \perp \tilde{\mathcal{S}}_j \quad \forall j \in \mathbb{Z}, \quad \text{and } \tilde{\mathcal{U}}_j \perp \mathcal{U}_r \quad \forall j, r \in \mathbb{Z}, j \neq r \tag{2.6}$$

hold.

(v) There exist functions $\eta \in L_2(\mathbb{R})$ and $\tilde{\eta} \in L_2(\mathbb{R})$ such that $\Upsilon_j := \{\eta_{j,k}(x) :\ k \in \mathbb{Z}\}$ is a Riesz basis for $\mathcal{U}_j = S(\Upsilon_j)$, and similarly for the dual wavelet: $\tilde{\mathcal{U}}_j = S(\tilde{\Upsilon}_j)$.

Stability of Υ_j implies uniform stability, but this is in general not sufficient in order to realize stability over all levels (see Theorem 2.3.2). If, however, η_j and $\tilde{\eta}_j$ are stable over all levels, we shall call the families

$$\Upsilon := \bigcup_{j \in \mathbb{Z}} \Upsilon_j, \quad \tilde{\Upsilon} := \bigcup_{j \in \mathbb{Z}} \tilde{\Upsilon}_j$$

biorthogonal wavelet bases for $L_2(\mathbb{R})$, and we shall denote η as the *wavelet*, and $\tilde{\eta}$ as the *dual wavelet*.

The interpretation of the decomposition in (iii) is that the function g is a *coarse approximation*, or an *average* of f, and the function g contains the *details* in f that g does not contain. In the literature, the spaces $\mathcal{S}_j$ are sometimes referred to as *average spaces*, and $\mathcal{U}_j$ as *detail spaces*.

Examples of scaling functions and wavelets do indeed exist. A recipe for constructing wavelets is given in [26], and takes as point of departure the masks $\{a_k\}_k$, $\{\tilde{a}_k\}_k$. From this, one can construct (initially, the Fourier transform of) the scaling functions, so that they are automatically refinable. On certain additional decay requirements on the Fourier transforms of ξ and $\tilde{\xi}$, and $(\xi_{j,0}, \tilde{\xi}_{j,k}) = \delta_{0,k}$ for any $j \in \mathbb{Z}$, one ends up with a biorthogonal basis of wavelets, which can be constructed as follows:

$$b_k = (-1)^{k+1} \tilde{a}_{1-k}, \quad \tilde{b}_k = (-1)^{k+1} a_{1-k} \tag{2.7}$$

$$\eta(x) = \sum_{k \in \mathbb{Z}} b_k \xi(2x - k), \quad \tilde{\eta}(x) = \sum_{k \in \mathbb{Z}} \tilde{b}_k \tilde{\xi}(2x - k). \tag{2.8}$$

It should be noted that in these (early) developments, translation invariance and the Fourier transform play a significant role in the constructions.

Wavelet and multiscale transforms

An element

$$\sum_k c_{j+1,k}\xi_{j+1,k} \in \mathcal{S}_{j+1}$$

also has a representation

$$\sum_k c_{j,k}\xi_{j,k} + \sum_k d_{j,k}\eta_{j,k} \in \mathcal{S}_j + \mathcal{U}_j.$$

The change of basis operation that maps $\{c_{j+1,k}\}$ to $\{c_{j,k}\} \cup \{d_{j,k}\}$ is called the (one-step) *wavelet transform*, and

$$c_{j,k} = \sum_n a_{n-2k}c_{j+1,n}, \quad d_{j,k} = \sum_n b_{n-2k}c_{j+1,n}.$$

The inverse mapping $\{c_{j,k}\}_k \cup \{d_{j,k}\}_k \mapsto \{c_{j+1,k}\}_k$ is the *inverse (one-step) wavelet transform*, and

$$c_{j+1,k} = \sum_n \left(\tilde{a}_{k-2n}c_{j,n} + \tilde{b}_{k-2n}d_{j,n}\right).$$

Iterating the transforms, one obtains a *multiscale transform*, i.e. a change of coordinates over several levels.

2.2.1 Example: Biorthogonal B-spline wavelets on $\mathbb{R}$

We conclude this section with an example of biorthogonal wavelet bases in $L_2(\mathbb{R})$, which is of fundamental importance for the later developments, namely that of [26], where the primal multiresolution consists of *B-splines*, see also [40]. A B-spline of order $d \geq 1$ on an interval $I \subseteq \mathbb{R}$ is a function $f \in C^{d-2}(I)$ (where $C^{-1}(I)$ are discontinuous functions), which is a piecewise polynomial of order $d-1$. The first few examples of B-splines on $\mathbb{R}$ are

$${}_1\xi(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases},$$

$${}_2\xi(x) = \begin{cases} 1+x, & -1 \leq x < 0 \\ 1-x, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases},$$

$${}_3\xi(x) = \begin{cases} \frac{(x+1)^2}{2}, & -1 \leq x < 0 \\ -\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}, & 0 \leq x < 1 \\ \frac{(x-2)^2}{2}, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Denoting by $f * g$ the convolution of $f, g \in L_2(\mathbb{R})$, the ${}_d\xi$ can be generated (up to translation) by convolution as follows: ${}_1\xi = 1_{[0,1]}$, and for $d \geq 2$, ${}_d\xi = {}_{d-1}\xi * {}_1\xi$, see [54, chapter 4] for details. Defining $\mu(d) = d \mod 2$, the ${}_d\xi$ can be constructed to be symmetric around $\frac{\mu(d)}{2}$, and has support

$$\operatorname{supp}{}_d\xi = \left[\frac{1}{2}(-d+\mu(d)), \frac{1}{2}(d+\mu(d))\right] = \left[-\left\lfloor\frac{d}{2}\right\rfloor, \left\lceil\frac{d}{2}\right\rceil\right] =: [l_1, l_2], \tag{2.9}$$

where $\lfloor x \rfloor$ resp. $\lceil x \rceil$ is the largest integer smaller than or equal to x, resp. the smallest integer larger than or equal to x. Moreover, ${}_d\xi$ is refinable:

$$ {}_d\xi(x) = \sum_{k=l_1}^{l_2} 2^{1-d} \binom{d}{k + \lfloor \frac{d}{2} \rfloor} \; {}_d\xi(2x-k) := \sum_{k=l_1}^{l_2} a_k \; {}_d\xi(2x-k), \quad x \in \mathbb{R}. $$

It can be proved [26] that for $\tilde{d} \in \mathbb{N}$ such that $\tilde{d} \geq d$ and $d + \tilde{d}$ is even, there exists a refinable function ${}_{d,\tilde{d}}\tilde{\xi} \in L_2(\mathbb{R})$, with the same symmetry properties as ${}_d\xi$, such that it forms a dual pair with ${}_d\xi$ and has compact support

$$ \begin{aligned} \operatorname{supp} {}_{d,\tilde{d}}\tilde{\xi} &= \left[-\frac{1}{2}d - \tilde{d} + 1 + \frac{1}{2}\mu(d), \frac{1}{2}d + \tilde{d} - 1 + \frac{1}{2}\mu(d) \right] \\ &= [l_1 - \tilde{d} + 1, l_2 + \tilde{d} - 1] =: [\tilde{l}_1, \tilde{l}_2]. \end{aligned} \tag{2.10} $$

Furthermore, in a pointwise sense,

$$ x^l = \sum_k c_k \; {}_{d,\tilde{d}}\tilde{\xi}(x-k), \quad x \in \mathbb{R}, \quad l = 0, \ldots, \tilde{d}. \tag{2.11} $$

We denote this property as *exactness of order* $\tilde{d}$. By (2.6), it implies that the wavelet η has $\tilde{d}$ vanishing moments. It can be proved that the wavelets ${}_d\eta$ and ${}_{d,\tilde{d}}\tilde{\eta}$ arising from the scaling functions, as explained by (2.8), provide Riesz bases $\eta_{j,k}, \tilde{\eta}_{j,k}$ of $L_2(\mathbb{R})$.

Figure 2.1 show two examples of scaling functions, wavelets, and their duals. In the left column, the family $d = 2, \tilde{d} = 2$ is depicted, from top down the primal scaling function, the dual scaling function, the primal wavelet and the dual wavelet. Similarly, in the right column, the family corresponding to $d = 3, \tilde{d} = 5$ are shown. All depicted functions are translations of the respective functions on level $j = 5$.

Note that both the primal and dual scaling functions have a symmetry axis, whereas the primal and dual wavelets have a symmetry or an antisymmetry axis, depending on the spline order. For orthonormal bases of compactly supported wavelets, this symmetry is not present, since the only real-valued wavelet, generating an orthonormal basis for $L_2(\mathbb{R})$, which is compactly supported and has a symmetry or an antisymmetry axis, is the so-called *Haar wavelet* [43, thm 8.1.4], which is the wavelet generated from ${}_1\xi$ defined as above.

Note also that the regularity of the scaling function and wavelet increases with the spline order, whereas the regularity of the dual scaling function and wavelet decreases.

Finally we mention that the B-spline scaling functions and wavelets enjoy simple properties of differentiation which will be of use later. One has, for $d \geq 2$ and $\mu(d) = d \bmod 2$,

$$ \frac{d}{dx} \; {}_d\xi(x) = \; {}_{d-1}\xi(x + \mu(d-1)) - \; {}_{d-1}\xi(x - \mu(d)), \tag{2.12} $$

$$ \frac{d}{dx} \; {}_{d-1,\tilde{d}+1}\tilde{\xi}(x) = \; {}_{d,\tilde{d}}\tilde{\xi}(x + 1 - \mu(d-1)) - \; {}_{d,\tilde{d}}\tilde{\xi}(x - \mu(d-1)), \tag{2.13} $$

and for the wavelets,

$$ \frac{d}{dx} \; {}_d\eta(x) = 4 \; {}_{d-1}\eta(x), \quad \frac{d}{dx} \; {}_{d-1,\tilde{d}+1}\tilde{\eta}(x) = -4 \; {}_{d,\tilde{d}}\tilde{\eta}(x). \tag{2.14} $$

see for instance [76], and the references therein.

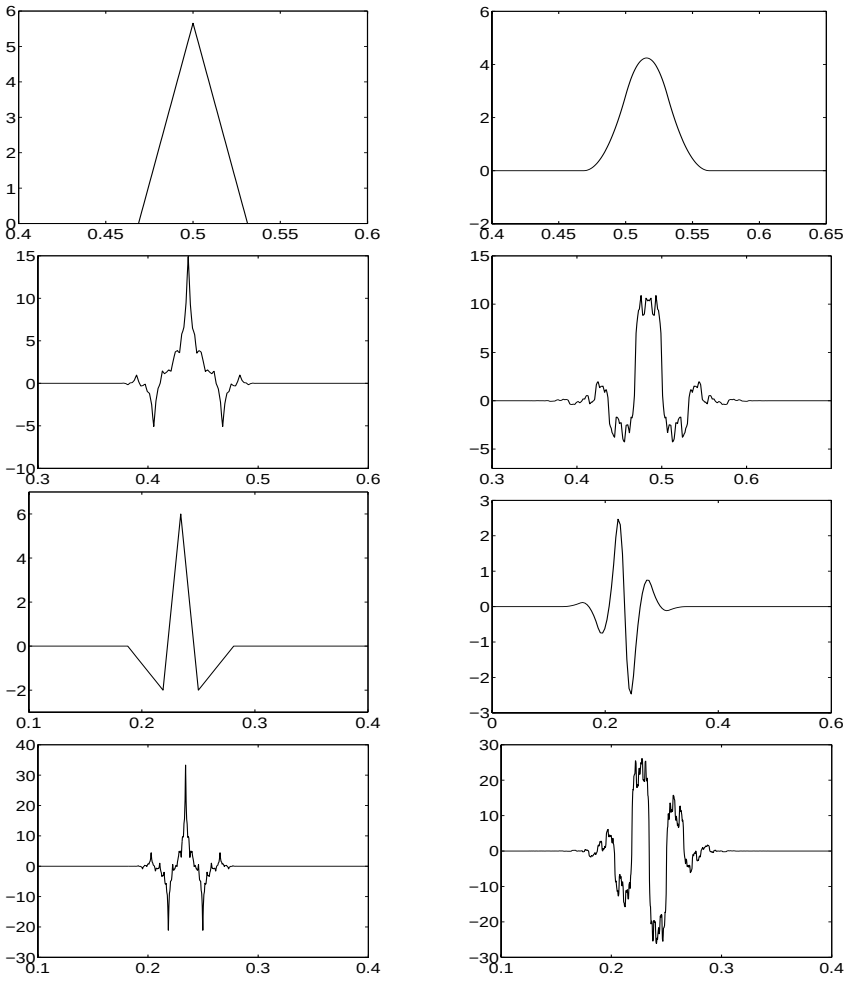

Figure 2.1: Examples of B-spline scaling functions and wavelets

2.3 Generalizing the multiresolution concept

The construction outlined in the previous section is of little use for the construction of bases of spaces of functions defined on a bounded subset of $\mathbb{R}$, or more generally, $\mathbb{R}^n$. In the transition to these cases of a bounded domain, difficulties are caused by the fact that translation invariance does not make sense, so that techniques based on the Fourier transform no longer apply. The aim of this section is to carry out the program from the previous sections, in a framework from [19] that does *not* rely on Fourier transform techniques.

Of particular importance is the formulation of refinability of the scaling function in terms of a *refinement operator*, and the introduction of the wavelet in terms of a so-called *stable completion*. Together, these operators describe the wavelet transforms in a simple and elegant fashion.

Throughout this section, we shall work in a separable Hilbert space $(\mathcal{H}, (\cdot, \cdot))$. The contents of this section is borrowed from [19].

Notation

For $j \in \mathbb{Z}$, we consider families of refinable functions

$$\Phi_j := \{\phi_{j,k} : \ k \in \Delta_j\} \subset \mathcal{H}$$

with closed linear span $\mathcal{S}_j$, where Δ_j is an index set. $\mathcal{S}_j$ and $\tilde{\mathcal{S}}_j$ are biorthogonal multiresolution sequences in $\mathcal{H}$. We shall think of the basis functions in Φ_j and a sequence of coefficients with respect to a basis: $\boldsymbol{c} = \{c_k\}_{k\in\Delta_j}$ as column vectors. For $v \in \mathcal{H}$, (v, Φ_j) is the column vector of inner products between v and the elements of Φ_j. We write

$$\boldsymbol{c}^T\Phi := \sum_{k\in\Delta} c_k\phi_k$$

where Δ is some index set, and provided that the sum is meaningful. Assuming the existence of biorthogonal bases Φ and $\tilde{\Phi}$ for $\mathcal{H}$, we shall write, for $v \in \mathcal{H}$,

$$v = (v, \tilde{\Phi})^T\Phi := \sum_k (v, \tilde{\phi}_{j,k})\phi_{j,k}.$$

By (Φ_j, Φ_j), we understand the matrix consisting of inner products between elements from Φ_j in $\mathcal{H}$, also called the *Gram matrix*, or the *Gramian* of Φ.

Refinability, wavelets, and uniform stability

Assuming that for all $j \in \mathfrak{J}$ for $\mathfrak{J} = \mathbb{Z}$ or $\mathfrak{J} = \{j_0, j_0 + 1, \ldots\}$, Φ_j and Φ_{j+1} are stable bases with nested spans $\mathcal{S}_j \subset \mathcal{S}_{j+1}$, one can derive the existence of an operator $\mathbf{M}_{j,0} \in \mathcal{L}(\ell_2(\Delta_{j+1}), \ell_2(\Delta_j))$ such that

$$\Phi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,0}.$$

This is the refinement equation, and $\mathbf{M}_{j,0}$ is called the refinement operator. For the case of scaling functions on the real line, $\mathbf{M}_{j,0}$ is a bi-infinite matrix with entries consisting of the mask of the scaling function, see (2.4).

As in the previous section, we look for complement spaces $\mathcal{U}_j$ of $\mathcal{S}_j$ in $\mathcal{S}_{j+1}$, and wavelet bases that span these spaces. The task of wavelet construction can be considered as a *matrix completion problem* in the following way: Given the refinement operator $\mathbf{M}_{j,0}$, find $\mathbf{M}_{j,1} \in \mathcal{L}(\ell_2(\Delta_{j+1}), \ell_2(\nabla_j))$ such that

$$\mathbf{M}_j := [\mathbf{M}_{j,0}, \mathbf{M}_{j,1}] : \ \ell_2(\Delta_{j+1}) \mapsto \ell_2(\Delta_j \cup \nabla_j)$$

is bounded and boundedly invertible. For this reason, $\mathbf{M}_{j,1}$ is called a *stable completion of* $\mathbf{M}_{j,0}$. With a stable completion, we associate the functions $\Psi_j = \{\psi_{j,k} : \ k \in \nabla_j\}$ defined by

$$\Psi_j^T = \Phi_{j+1}^T\mathbf{M}_{j,1}$$

as candidates for the wavelets. Assuming that Φ_j is stable, then Φ_{j+1} and $\Phi_j \cup \Psi_j$ as defined above are stable bases for $\mathcal{S}_{j+1}$ if and only if $\mathbf{M}_j$ is boundedly invertible. Furthermore, one can show that the family $\Phi_j \cup \Psi_j$ is uniformly stable if and only if the norms of $\mathbf{M}_j$ and its inverse are uniformly bounded in j.

Ideally, the dependence of $\mathbf{M}_{j,0}, \mathbf{M}_{j,1}$ on j should be minimal, in the sense that once they are known for one value of j, their assembly for other values of j is done without any computational work.

Given a refinement operator, a stable completion is not uniquely determined, so a characterization of all possible stable completions is desirable. Also, a criterion for when the wavelet bases are biorthogonal Riesz bases for $\mathcal{H}$ is desirable. This is not guaranteed by uniform stability of Φ_j alone, it turns out that an additional requirement is needed to ensure stability over all levels (Theorem 2.3.2 below). First, we shall define the wavelet transforms in the new terminology.

Wavelet transforms

Assume Φ_j and Ψ_j are uniformly stable bases of scaling functions and wavelets, respectively, and let $\mathbf{M}_{j,0},\ \mathbf{M}_{j,1}$ be the corresponding refinement operator and stable completion. Then, by definition, there exist operators $\mathbf{B}_{j,0} \in \mathcal{L}(\ell_2(\Delta_{j+1}), \ell_2(\Delta_j))$ and $\mathbf{B}_{j,1} \in \mathcal{L}(\ell_2(\Delta_{j+1}), \ell_2(\nabla_j))$ such that

$$\begin{bmatrix}\mathbf{B}_{j,0}\\ \mathbf{B}_{j,1}\end{bmatrix}\begin{bmatrix}\mathbf{M}_{j,0} & \mathbf{M}_{j,1}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \\ & \mathbf{I}\end{bmatrix}, \quad \begin{bmatrix}\mathbf{M}_{j,0} & \mathbf{M}_{j,1}\end{bmatrix}\begin{bmatrix}\mathbf{B}_{j,0}\\ \mathbf{B}_{j,1}\end{bmatrix} = \mathbf{I}.$$

We say that $\mathbf{M}_j$ is *uniformly banded* if the rows and columns contain only a uniformly bounded number of nonzero entries, and that only a uniformly bounded finite number of columns have a nonempty intersection. The stable completion $\mathbf{M}_{j,1}$ will be called *local* if both $\mathbf{M}_j$ and

$$\mathbf{B}_j = \begin{bmatrix}\mathbf{B}_{j,0}\\ \mathbf{B}_{j,1}\end{bmatrix}$$

are uniformly banded.

For $\boldsymbol{c}_j \in \ell_2(\Delta_j)$ and $\boldsymbol{d}_j \in \ell_2(\nabla_j)$, we can write

$$\boldsymbol{c}_j\Phi_j + \boldsymbol{d}_j\Psi_j = (\mathbf{M}_{j,0}\boldsymbol{c}_j)\Phi_{j+1} + (\mathbf{M}_{j,1}\boldsymbol{d}_j)\Phi_{j+1} = \left(\mathbf{M}_j\begin{bmatrix}\boldsymbol{c}_j\\ \boldsymbol{d}_j\end{bmatrix}\right)\Phi_{j+1}.$$

In the other direction, given $\boldsymbol{c}_{j+1} \in \ell_2(\Delta_{j+1})$, we can write

$$\boldsymbol{c}_{j+1}\Phi_{j+1} = (\mathbf{B}_{j,0}\boldsymbol{c}_{j+1})\Phi_j + (\mathbf{B}_{j,1}\boldsymbol{c}_{j+1})\Psi_j$$

The mapping

$$\ell_2(\Delta_{j+1}) \ni \boldsymbol{c}_{j+1} \mapsto \begin{bmatrix}\mathbf{B}_{j,0}\boldsymbol{c}_{j+1}\\ \mathbf{B}_{j,1}\boldsymbol{c}_{j+1}\end{bmatrix} \in \ell_2(\Delta_j) \times \ell_2(\nabla_j)$$

is the (one-step) wavelet transform, and the mapping

$$\ell_2(\Delta_j) \times \ell_2(\nabla_j) \ni \begin{bmatrix}\boldsymbol{c}_j\\ \boldsymbol{d}_j\end{bmatrix} \mapsto \mathbf{M}_{j,0}\boldsymbol{c}_j + \mathbf{M}_{j,1}\boldsymbol{d}_j = \mathbf{M}_j\begin{bmatrix}\boldsymbol{c}_j\\ \boldsymbol{d}_j\end{bmatrix} \in \ell_2(\Delta_{j+1})$$

is the (one-step) inverse wavelet transform. For the case of finite-dimensional spaces $\mathcal{S}_j$, we mention that if the stable completion is local, then the execution of a multiscale transform from $\mathcal{S}_J$ to $\mathcal{S}_{j_0} \oplus \mathcal{U}_{j_0} \oplus \cdots \oplus \mathcal{U}_{J-1}$ and its inverse can be carried out at the optimal expense of $\mathcal{O}(\#\Delta_J)$ floating point operations.

Characterization of stable completions.

We turn now to the theorem giving the characterization of all local stable completions of a given refinement operator $\mathbf{M}_{j,0}$, or stated equivalently, characterizing all possible wavelets, given a refinable scaling function. For notational simplicity, we drop in this subsection reference to a particular level j, and we understand that all properties of operators, such as boundedness and bandedness, will have to hold uniformly in j. We shall consider bounded and banded operators

$$L \in \mathcal{L}(\ell_2(\nabla), \ell_2(\Delta)), \quad K \in \mathcal{L}(\ell_2(\nabla), \ell_2(\nabla)), \tag{2.15}$$

and assume furthermore that K has a bounded and banded inverse. From these operators, we build $\mathbf{H}_{L,K} \in \mathcal{L}(\ell_2(\Delta) \times \ell_2(\nabla), \ell_2(\Delta) \times \ell_2(\nabla))$ as

$$\mathbf{H}_{L,K} := \begin{bmatrix} I & L \\ \mathbf{0} & K \end{bmatrix}, \tag{2.16}$$

which has the bounded inverse

$$\mathbf{H}_{L,K}^{-1} = \begin{bmatrix} I & -LK^{-1} \\ \mathbf{0} & K^{-1} \end{bmatrix}. \tag{2.17}$$

THEOREM 2.3.1 *[19] If $\check{\mathbf{M}}_1$ is one local stable completion of $\mathbf{M}_0$ with associated inverse*

$$\check{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_{j,0} \\ \mathbf{B}_{j,1} \end{bmatrix},$$

then another local stable completion $\mathbf{M}_1$ is achieved by the relation

$$\begin{bmatrix} \mathbf{M}_0 & \mathbf{M}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_0 & \check{\mathbf{M}}_1 \end{bmatrix} \mathbf{H}_{L,K}, \tag{2.18}$$

and its associated inverse is given by

$$\mathbf{B} = \mathbf{H}_{L,K}^{-1} \check{\mathbf{B}}. \tag{2.19}$$

On the other hand, if $\mathbf{M}_1$ and $\check{\mathbf{M}}_1$ are any two local stable completions of $\mathbf{M}_0$ with associated inverses $\mathbf{B}$ and $\check{\mathbf{B}}$, then there exists operators L and K as in (2.15), (2.16), and (2.17), such that (2.18) and (2.19) hold.

The theorem states that once an *initial local stable completion* is available, all other stable completions are parametrized. The theorem also holds if the locality is removed. Recall that locality of the stable completion reflects locality of the wavelet bases and optimal complexity of multiscale transforms.

Stability over all levels

We turn now to the Riesz basis property of the wavelet bases. Denote by $\|\cdot\|_{s,\Omega}$ the norm on the Sobolev space $H^s(\Omega)$, where for $s < 0$, $H^s(\Omega) = (H^{-s}(\Omega))'$. The following theorem states a number of conditions that suffice in order to ensure stability over all levels.

THEOREM 2.3.2 *[40] Let $\mathcal{H} = L_2(\Omega)$ for some bounded, Lipschitz domain Ω. Assume that*

(i) Φ_j *is uniformly stable for* $j \geq j_0$.

(ii) There exists a dual family $\tilde{\Phi}_j$ *which is refinable and satisfies* $(\phi_{j,k}, \tilde{\phi}_{j,k'}) = \delta_{k,k'}$.

(iii) The operators $Q_j : L_2 \mapsto \mathcal{S}_j$ *defined by* $Q_j f = (f, \tilde{\Phi}_j)\Phi_j$ *are uniformly bounded in* j.

(iv) $\mathcal{S}_j$ *and the ranges* $\tilde{\mathcal{S}}_j$ *of the adjoints* Q_j' *of* Q_j *satisfy the Jackson inequality*

$$\inf_{v_j \in V_j} \|v - v_j\|_{0,\Omega} \lesssim 2^{-sj} \|v\|_{s,\Omega}, \quad s \leq \sigma,$$

and the Bernstein inequality

$$\|v_j\|_{s,\Omega} \lesssim 2^{sj} \|v_j\|_{0,\Omega} \quad \forall v_j \in V_j, \ s \leq \gamma$$

for

$$V_j = \begin{Bmatrix} \mathcal{S}_j \\ \tilde{\mathcal{S}}_j \end{Bmatrix}, \quad \sigma = \begin{Bmatrix} d \\ \tilde{d} \end{Bmatrix}, \quad \gamma = \begin{Bmatrix} t \leq d \\ \tilde{t} \leq \tilde{d} \end{Bmatrix}.$$

Then, for $Q_{j_0-1} = 0$,

$$\|v\|_{s,\Omega}^2 \sim \sum_{j=j_0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_{0,\Omega}^2, \quad -t < s < t. \tag{2.20}$$

The Jackson and Bernstein inequalities have an origin in interpolation and approximation theory, which we describe in Section 3.2. Later in this chapter (Lemma 2.7.1), we shall have a closer look how to fulfill the Jackson and Bernstein inequalities.

Setting $s = 0$ in (2.20) we infer that we can establish the Riesz basis property of

$$\Psi := \bigcup_{j \in \mathfrak{J}} \Psi_j$$

by fulfilling the assumptions of Theorem 2.3.2, and by constructing uniformly stable bases Ψ_j resp. $\tilde{\Psi}_j$ for the complement spaces $\mathcal{U}_j = (Q_{j+1} - Q_j)\mathcal{S}_{j+1}$ resp. $\tilde{\mathcal{U}}_j = (Q_{j+1}' - Q_j')\tilde{\mathcal{S}}_{j+1}$ such that the biorthogonality (2.6) holds. This amounts to projecting the initial stable completion onto the right subspace. The interesting result is the following, which is a special case of [19, cor. 3.1]:

PROPOSITION 2.3.3 *Let* Φ_j *and* $\tilde{\Phi}_j$ *be uniformly stable, biorthogonal bases of scaling functions with* $\mathcal{S}_j = S(\Phi_j)$ *and* $\tilde{\mathcal{S}}_j = S(\tilde{\Phi}_j)$. *Denote by* $\mathbf{M}_{j,0}$ *and* $\tilde{\mathbf{M}}_{j,0}$ *the corresponding refinement operators. Let* $\hat{\mathbf{M}}_{j,1}$ *be any stable completion of* $\mathbf{M}_{j,0}$ *with* $\hat{\mathbf{G}}_j^{-1} = \hat{\mathbf{M}}_j^{-1}$. *Then*

$$\mathbf{M}_{j,1} = (\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T)\hat{\mathbf{M}}_{j,1}$$

is also a stable completion, and

$$\mathbf{G}_j = \mathbf{M}_j^{-1} = \begin{bmatrix} \tilde{\mathbf{M}}_{j,0}^{-T} \\ \hat{\mathbf{G}}_{j,1}^{-T} \end{bmatrix}.$$

Furthermore, the families

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \hat{\mathbf{G}}_{j,1} \tilde{\Phi}_{j+1}$$

satisfy the orthogonality conditions

$$(\Psi_j, \tilde{\Psi}_j) = \mathbf{I}, \quad (\Psi_j, \tilde{\Phi}_j) = \mathbf{0}, \quad (\tilde{\Psi}_j, \Phi_j) = \mathbf{0},$$

which is equivalent to (2.6).

2.4 Biorthogonal wavelet bases on the interval

As an example of an application of the framework described in the previous section, we shall in this section give an account of the construction of biorthogonal B-spline wavelets on the interval, first published in [40]. This construction is a cornerstone for the wavelet-based solution of PDEs, since it, by the tensor product structure presented later, facilitates the construction of wavelet bases on unit cubes in arbitrary dimensions, and eventually by mapping and parametrization techniques, wavelet bases on more general domains.

The construction is based on the biorthogonal B-spline scaling functions and wavelets on the real line, treated in Subsection 2.2.1. The approach we shall take is the following: Given a B-spline scaling function ξ, one retains a number of translates of this function whose support does not overlap the boundaries. These translates are called *interior functions*. A scaling function basis is then obtained by adding a number of appropriately defined scaling functions near the left and right boundary points. Finally, we shall introduce the concept of boundary adapted wavelet bases. This allows one to construct wavelet bases for the interval for which the basis functions satisfy homogeneous boundary conditions.

Throughout this section, we shall work in the space $L_2(0,1)$, equipped with the inner product $(\cdot,\cdot)_{0,[0,1]}$.

2.4.1 Biorthogonal B-spline multiresolution on the interval

The aim in this subsection is to outline the construction of biorthogonal multiresolution sequences

$$\mathcal{S}_j = S(\Xi_j) \subset L_2(0,1), \quad \tilde{\mathcal{S}}_j = S(\tilde{\Xi}_j) \subset L_2(0,1),\ j \in \mathfrak{J} = \{j_0, j_0+1, \ldots\}$$

with the following properties:

(i) Ξ_j consists of B-splines of degree $d-1$, and for fixed d and any $\tilde{d} \geq d$ with $\tilde{d}+d$ even, $\tilde{\Xi}_j$ has exactness $\tilde{d}-1$.

(ii) Ξ_j and $\tilde{\Xi}_j$ are refinable, with finitely supported masks, and biorthogonal for each $j \geq j_0$, where j_0 is a *minimal level*, defined below.

(iii) Ξ_j and $\tilde{\Xi}_j$ have the same cardinality.

(iv) The basis functions are locally supported, in the sense that

$$\operatorname{diam}(\operatorname{supp} \xi_{j,k}),\ \operatorname{diam}(\operatorname{supp} \tilde{\xi}_{j,k}) \sim 2^{-j}, \quad j \geq j_0.$$

(v) The bases Ξ_j and $\tilde{\Xi}_j$ are uniformly stable.

(vi) The projectors

$$Q_j v := (v, \tilde{\Xi}_j)^T_{0,[0,1]} \Xi_j, \quad Q'_j v = (v, \Xi_j)^T_{0,[0,1]} \tilde{\Xi}_j$$

are uniformly bounded.

The point of departure is the biorthogonal B-spline multiresolution scaling functions and wavelets from Subsection 2.2.1. Choose $d, \tilde{d}$ as in Subsection 2.2.1. For notational simplicity, we shall write simply $\xi = {}_d\xi$ and $\tilde{\xi} = {}_{d,\tilde{d}}\tilde{\xi}$. Following the notation of [40], we shall denote by

$\xi_{[j,k]}$ the translates and dilates from Section 2.2, in order to distinguish these 'pure' translates from other elements in the bases we are about to construct, namely the boundary functions.

First, the dual multiresolution sequence is treated, since the supports of the dual functions are at least as large as those of the primal functions (see [40]). Fix an integer $\tilde{l} \geq \tilde{l}_2$ (recall (2.10)) and choose $j \in \mathbb{Z}$ so that the set of indices

$$\tilde{I}_j^0 := \{\tilde{l}, \ldots, 2^j - \tilde{l} - \mu(d)\}$$

where $\mu(d) = d \mod 2$ corresponds to translates ${}_{d,\tilde{d}}\tilde{\xi}_{[j,k]}$, $k \in \tilde{I}_j^0$ whose supports are strictly contained in $[0,1]$. To realize the exactness of order $\tilde{d}-1$, $\tilde{d}$ boundary functions are provided near the left and right endpoints. The *boundary functions* near the left endpoint are, for $r = 0, \ldots, \tilde{d}-1$ defined as[3]

$$\tilde{\xi}^L_{j,\tilde{l}-\tilde{d}+r} := \sum_{m=-\tilde{l}_2+1}^{\tilde{l}-1} \alpha_{m,r}\tilde{\xi}_{[j,m]}|_{\mathbb{R}_+}, \text{ with } \alpha_{m,r} = \int_{\mathbb{R}} x^r \xi(x-m)dx. \tag{2.21}$$

The index set corresponding to the left boundary functions is defined as

$$\tilde{I}_j^L := \{\tilde{l} - \tilde{d}, \ldots, \tilde{l} - 1\}.$$

Similarly, $\tilde{d}$ boundary functions are provided near the right endpoint: For $r = 0, \ldots, \tilde{d}-1$

$$\tilde{\xi}^R_{j,2^j-\tilde{l}+\tilde{d}-\mu(d)-r} := \sum_{m=2^j-\tilde{l}-\mu(d)+1}^{2^j-\tilde{l}_1-1} \alpha^R_{j,m,r}\tilde{\xi}_{[j,m]}|_{[0,1]} \tag{2.22}$$

where

$$\alpha^R_{j,m,r} = \int_{\mathbb{R}} (2^j - x)^r \xi(x-m)dx,$$

thus the index set for the right boundary functions is

$$\tilde{I}_j^R := \{2^j - \tilde{l} + 1 - \mu(d), 2^j - \tilde{l} + \tilde{d} - \mu(d)\}.$$

We define the union of the index sets

$$\tilde{I}_j := \tilde{I}_j^L \cup \tilde{I}_j^0 \cup \tilde{I}_j^R.$$

The left and right boundary functions are refinable, so that alltogether we infer the existence of a refinement operator $\tilde{\boldsymbol{M}}_{j,0}$ and write $\tilde{\Xi}_j := \{\tilde{\xi}_{j,k} : k \in \tilde{I}_j\}$,

$$\tilde{\Xi}_j^T = \tilde{\Xi}_{j+1}^T \tilde{\boldsymbol{M}}_{j,0}. \tag{2.23}$$

We assume that the left and right boundary functions do not overlap. This corresponds to requiring

$$j \geq \left\lceil \log_2(\tilde{l} + \tilde{l}_2 - 1) + 1 \right\rceil =: j_0,$$

where j_0 is the *minimal level*.

[3] This indexing has been chosen because of convenience in implementations of the bases

The same procedure is now carried out for the primal multiresolution sequence. Requiring the same cardinality of the primal basis Ξ_j and defining

$$l := \tilde{l} - (\tilde{d} - d),$$

one arrives at index sets for the functions at the left boundary, the interior and the right boundary:

$$I_j^L = \{l - d, \ldots, l - 1\}, \tag{2.24}$$
$$I_j^0 = \{l, \ldots, 2^j - l - \mu(d)\}, \tag{2.25}$$
$$I_j^R = \{2^j - l + 1 - \mu(d), \ldots, 2^j - l + d - \mu(d)\}. \tag{2.26}$$

As before, we define the union of these index sets

$$I_j := I_j^L \cup I_j^0 \cup I_j^R.$$

With

$$\tilde{\alpha}_{m,r}^L = \int_{\mathbb{R}} x^r \tilde{\xi}(x - m)dx, \quad \text{and} \quad \tilde{\alpha}_{j,m,r}^R = \int_{\mathbb{R}} (2^j - x)^r \tilde{\xi}(x - m)dx,$$

the left and right boundary functions are, for $r = 0, \ldots, d - 1$,

$$\xi_{j,l-d+r}^L := \sum_{m=-l_2+1}^{l-1} \tilde{\alpha}_{m,r}^L \xi_{[j,m]}|_{[0,1]}, \tag{2.27}$$

$$\xi_{j,2^j-l+d-\mu(d)-r}^R := \sum_{m=2^j-l-\mu(d)+1}^{2^j-l_1-1} \tilde{\alpha}_{j,m,r}^R \xi_{[j,m]}|_{[0,1]}. \tag{2.28}$$

These functions satisfy the same refinability and symmetry properties as their dual counterparts, so that in particular we infer the existence of a refinement operator $\boldsymbol{M}_{j,0}$ so that for $\Xi := \{\xi_{j,k} : \ k \in I_j\}$,

$$\Xi_j^T = \Xi_{j+1}^T \boldsymbol{M}_{j,0}. \tag{2.29}$$

At this point, one can prove that the sets

$$\Xi'_j := \{\xi_{j,k}^L : \ k \in I_j^L\} \cup \{\xi_{[j,k]} : \ k \in I_j^0\} \cup \{\xi_{j,k}^R : \ k \in I_j^R\} \tag{2.30}$$
$$\tilde{\Xi}'_j := \{\tilde{\xi}_{j,k}^L : \ k \in \tilde{I}_j^L\} \cup \{\tilde{\xi}_{[j,k]} : \ k \in \tilde{I}_j^0\} \cup \{\tilde{\xi}_{j,k}^R : \ k \in \tilde{I}_j^R\} \tag{2.31}$$

satisfy the points in the program on page 17, except for the biorthogonality in point (ii). They cannot be expected to be biorthogonal, since only the interior functions, which are of course biorthogonal to each other, have been retained from the biorthogonal bases on $\mathbb{R}$. The idea is to perform a change of basis operation, in that one chooses regular matrices C_j and $\tilde{C}_j$ and defines

$$\Xi_j := C_j \Xi'_j, \quad \tilde{\Xi}_j := \tilde{C}_j \tilde{\Xi}'_j. \tag{2.32}$$

Biorthogonality of Ξ_j and $\tilde{\Xi}_j$ is equivalent to

$$\boldsymbol{I} = (\Xi_j, \tilde{\Xi}_j)_{0,[0,1]} = C_j (\Xi'_j, \tilde{\Xi}'_j)_{0,[0,1]} \tilde{C}_j^T, \tag{2.33}$$

which is possible if and only if the Gramian $(\Xi'_j, \tilde{\Xi}'_j)_{0,[0,1]}$ is nonsingular for each $j \geq j_0$ and each choice of d and $\tilde{d}$ as in Subsection 2.2.1. This is indeed the case, and it should be noted that the proof of this fact relies crucially on the properties of B-splines, see [40].

There are several possible choices of the change-of-basis matrices in (2.33), and these choices influence, for instance, the Riesz bounds of the basis. However, since these issues are not within our scope, we shall not treat this any further, only refer to [9] for more information.

Figure 2.2 shows two examples of primal and dual scaling functions. In the left column, the primal and dual scaling functions for $d = 2, \tilde{d} = 2$ are shown, and in the right column, the primal and dual scaling functions for $d = 3,\ \tilde{d} = 5$ are depicted.

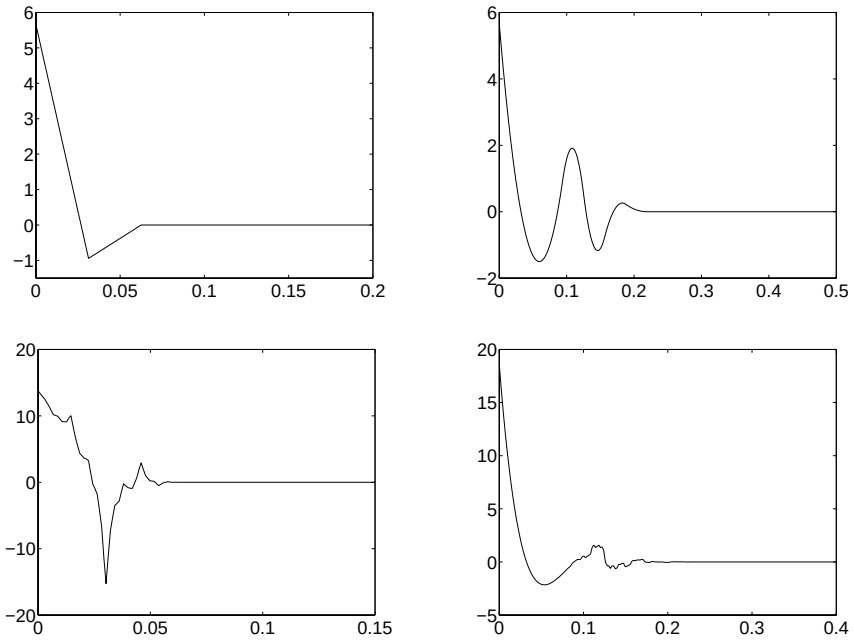

Figure 2.2: Primal and dual scaling functions at the boundary.

REMARK 2.4.1 *An important property of these bases is that the boundary corrections are the same for each level. The only index sets that change their cardinality as the level grows are those corresponding to the interior functions I_j^0 and $\tilde{I}_j^0$. This ensures the minimal dependence of $\mathbf{M}_{j,0},\ \mathbf{M}_{j,1}$ on j as noted on page 13.*

2.4.2 Biorthogonal wavelet bases

Let $\mathbf{M}_{j,1}$ be any stable completion of $\mathbf{M}_{j,0}$. Then, as previously mentioned, $\Upsilon_j := \mathbf{M}_{j,1}\Xi_{j+1}$, is a Riesz basis for the complement $\mathcal{U}_j$ of $\mathcal{S}_j$ in $\mathcal{S}_{j+1}$. Requirements similar to (2.6) for the interval are not automatically fulfilled, but can be realized by employing Proposition 2.3.3 from Section 2.3.

The question of how to find such an initial stable completion remains. In this case, it can be found by a Gaussian elimination procedure applied to the refinement matrix $\mathbf{M}_{j,0}$. We refer to [40],[76] for details.

Note that the Riesz basis property, i.e. the stability over all levels of the basis

$$\Upsilon := \Xi_{j_0} \cup \bigcup_{j \geq j_0} \Upsilon_j$$

follows again by fulfilling the assumptions of Theorem 2.3.2. We shall see later (Lemma 2.7.1) that these assumptions are included in the program from the beginning of Subsection 2.4.1, so that Υ above is indeed a Riesz basis for $L_2(0,1)$.

2.4.3 Boundary adapted wavelets

In this subsection, we describe a procedure from [17], but also found in [76], on how homogeneous boundary conditions can be incorporated in a multiresolution sequence. For a finite set $I \subset \mathbb{Z}$, we denote by $\mathfrak{f}(I)$ the minimal element, and by $\mathfrak{l}(I)$ the maximal element. For a given scale j, a family of scaling functions Ξ_j and wavelets Υ_j are called *boundary adapted* if at the points $\{0,1\}$:

(i) Only one basis function is nonvanishing at the points $\{0,1\}$, i.e.

$$\begin{aligned} \xi_{j,k}(0) \neq 0 \Leftrightarrow k = \mathfrak{f}(I_j), \quad \xi_{j,k}(1) \neq 0 \Leftrightarrow k = \mathfrak{l}(I_j), \\ \eta_{j,k}(0) \neq 0 \Leftrightarrow k = \mathfrak{f}(J_j), \quad \eta_{j,k}(1) \neq 0 \Leftrightarrow k = \mathfrak{l}(J_j). \end{aligned}$$

(ii) The nonvanishing scaling functions and wavelets take the same value at the boundary:

$$\begin{aligned} \xi_{j,\mathfrak{f}(I_j)}(0) = \eta_{j,\mathfrak{f}(J_j)}(0) = c_0 2^{j/2}, \\ \xi_{j,\mathfrak{l}(I_j)}(1) = \eta_{j,\mathfrak{l}(J_j)}(1) = c_1 2^{j/2}, \end{aligned}$$

where the constants c_0, c_1 are independent of j.

In [17], boundary adapted biorthogonal wavelet bases based on B-splines are constructed.

Given a boundary adapted system Ξ_j, incorporating a homogeneous boundary condition at 0, say, simply amounts to discarding the only scaling function taking a nonzero value there. We introduce a parameter $\beta = (\beta_0, \beta_1)$, where $\beta_i \in \{0,1\}$, $i = 0,1$, and $\beta_i = 0$ amounts to a homogeneous boundary condition at 0 if $i = 0$, and a homogeneous boundary condition at 1 if $i = 1$. Similarly, $\beta_i = 1$ denotes that no boundary condition is imposed. This leads to the following definition of index sets:

$$I_j^\beta = \begin{cases} I_j & \beta = (1,1) \\ I_j / \mathfrak{f}(I_j) & \beta = (0,1) \\ I_j / \mathfrak{l}(I_j) & \beta = (1,0) \\ I_j / (\mathfrak{f}(I_j), \mathfrak{l}(I_j)) & \beta = (0,0) \end{cases} \tag{2.34}$$

Accordingly we define the biorthogonal multiresolution sequences

$$\mathcal{S}_j^\beta := \operatorname{span}(\Xi_j^\beta) := \operatorname{span}\{\xi_{j,k} : \ k \in I_j^\beta\}, \tag{2.35}$$

$$\tilde{\mathcal{S}}_j^\beta := \operatorname{span}(\tilde{\Xi}_j^\beta) := \operatorname{span}\{\tilde{\xi}_{j,k} : \ k \in I_j^\beta\}. \tag{2.36}$$

Using (ii) above, we see that incorporating homogeneous boundary conditions at 0, say, in the complement space $\mathcal{U}_j$ amounts to replacing the first wavelet $\eta_{j,\mathfrak{f}(J_j)}$ and the first dual wavelet $\tilde{\eta}_{j,\mathfrak{f}(J_j)}$ with, respectively,

$$\eta_{j,k} := \frac{1}{\sqrt{2}}(\eta_{j,\mathfrak{f}(J_j)} - \xi_{j,\mathfrak{f}(I_j)}), \;\; \tilde{\eta}_{j,k} := \frac{1}{\sqrt{2}}(\tilde{\eta}_{j,\mathfrak{f}(J_j)} - \tilde{\xi}_{j,\mathfrak{f}(I_j)}),$$

and finally defining $k = \mathfrak{f}(J_j)$. Analogously, boundary conditions can be incorporated at the point 1, and one can define boundary adapted complement spaces, just like in (2.35), (2.36):

$$\mathcal{U}_j^\beta := \mathrm{span}(\Upsilon_j^\beta) := \mathrm{span}\{\eta_{j,k} : \; k \in J_j^\beta\}, \tag{2.37}$$

$$\tilde{\mathcal{U}}_j^\beta := \mathrm{span}(\tilde{\Upsilon}_j^\beta) := \mathrm{span}\{\tilde{\eta}_{j,k} : \; k \in J_j^\beta\}. \tag{2.38}$$

2.4.4 Periodic wavelets

In this subsection, we outline a simple construction of wavelets on the interval, or more specificly, on the *torus* $\mathbb{T} = \mathbb{R}/[0,1[$, namely *periodic wavelets*. For a countinuous and compactly supported function f, one can define its *periodization* $\bar{f}$ as

$$\bar{f}(x) = \sum_{k \in \mathbb{Z}} f(x+k).$$

Obviously, $\bar{f}$ is well-defined, continuous, bounded, and 1-periodic. It turns out that by periodizing the scaling functions and wavelets, living on levels $j \geq 0$, one obtains a biorthogonal Riesz basis for $L_2(\mathbb{T})$. We record:

THEOREM 2.4.2 *Let ϕ, ψ be compactly supported B-spline scaling functions and wavelets from Section 2.2, and $\tilde{\phi}$, $\tilde{\psi}$ their duals. Let them generate a biorthogonal wavelet basis for $L_2(\mathbb{R})$. Define for $j \geq 0$*

$$\bar{\Phi}_j := \{\bar{\phi}_{j,k} : \; k = 0, \ldots, 2^j - 1\}, \quad \bar{\Psi}_j := \{\bar{\psi}_{j,k} : \; k = 0, \ldots, 2^j - 1\}$$
$$\bar{\tilde{\Phi}}_j := \{\bar{\tilde{\phi}}_{j,k} : \; k = 0, \ldots, 2^j - 1\}, \quad \bar{\tilde{\Psi}}_j := \{\bar{\tilde{\psi}}_{j,k} : \; k = 0, \ldots, 2^j - 1\}$$

as well as

$$\bar{\mathcal{S}}_j := \mathit{span}(\bar{\Phi}_j), \quad \bar{\tilde{\mathcal{S}}}_j := \mathit{span}(\bar{\tilde{\Phi}}_j).$$

The following properties hold:

(i) For $j \geq 0$, $\bar{\mathcal{S}}_j \subset \bar{\mathcal{S}}_{j+1}$ and $\bar{\tilde{\mathcal{S}}}_j \subset \bar{\tilde{\mathcal{S}}}_j$.

(ii) For $j \geq 0$, $\bar{\Psi}_j$ is a uniformly stable Riesz basis for $\bar{\mathcal{S}}_j$, and similarly for the dual family.

(iii) For $j_0 \geq 0$, the family

$$\bar{\Phi}_{j_0} \cup \bigcup_{s \geq j_0}^{j-1} \bar{\Psi}_s$$

is a Riesz basis for $\mathcal{S}_j$, and similarly for the dual family.

(iv) The sequences $\{\bar{\mathcal{S}}_j\}_{j\geq 0}$ *and* $\{\bar{\bar{\mathcal{S}}}_j\}_{j\geq 0}$ *are dense in* $L_2(\mathbb{T})$*, so that the families*

$$\bar{\Psi} := \bar{\Phi}_{j_0} \cup \bigcup_{j\geq j_0} \bar{\Psi}_j, \quad \bar{\bar{\Psi}} := \bar{\bar{\Phi}}_{j_0} \cup \bigcup_{j\geq j_0} \bar{\bar{\Psi}}_j$$

are biorthogonal wavelet bases for $L_2(\mathbb{T})$*.*

The minimal level j_0 is often chosen such that the involved scaling functions and wavelets do not 'wrap around themselves' by the periodization.

For information on periodic wavelets, see for instance, [78],[43],[54], where the orthogonal case is considered, but an inspection of the proof shows that it carries over to the biorthogonal case without problems.

2.4.5 Differentiation

Following [75], we mention that the properties of differentiation treated in Subsection 2.2.1 carry over to the present situation.

THEOREM 2.4.3 *Let*

$$\Xi_j^{(1)} = \{\xi_{j,k}^{(1)} : \ k \in I_j\} \quad \text{and} \quad \Upsilon_j^{(1)} = \{\eta_{j,k}^{(1)} : \ k \in J_j\}, \ i = 0, 1, \tag{2.39}$$

be, respectively, systems of univariate scaling functions and wavelets on $(0, 1)$*, and define the dual families similarly. Note that the index sets for primal and dual functions are identical. Assume furthermore that* $\Upsilon^{(1)} \subset H^1(0, 1)$*. Then there exists a second system of scaling functions and wavelets*

$$\Xi_j^{(0)} = \{\xi_{j,k}^{(0)} : \ k \in I_j\} \quad \text{and} \quad \Upsilon_j^{(0)} = \{\eta_{j,k}^{(0)} : \ k \in J_j\}, \ i = 0, 1, \tag{2.40}$$

and similarly for the dual families, such that there exist uniformly sparse matrices

$$D_{j,0} : \mathbb{R}^{I_j \times I_j}, \quad D_{j,1} : \mathbb{R}^{J_j \times J_j}$$

with the property that

$$\frac{d}{dx}\Xi_j^{(1)} = D_{j,0}\Xi_j^{(0)}, \quad \frac{d}{dx}\tilde{\Xi}_j^{(0)} = -D_{j,0}^T\tilde{\Xi}_j^{(0)}, \tag{2.41}$$

$$\frac{d}{dx}\Upsilon_j^{(1)} = D_{j,1}\Upsilon_j^{(0)}, \quad \frac{d}{dx}\tilde{\Upsilon}_j^{(0)} = -D_{j,1}^T\tilde{\Upsilon}_j^{(1)}. \tag{2.42}$$

By uniformly sparse matrices is meant that the bandwidth is bounded uniformly in j. Explicitly,

$$D_{j,1} = (d_{mn}), \quad d_{mn} = 0 \text{ if } |n - m| > k,$$

where k is the bandwidth, being independent of j. From the construction of bases on the interval, one infers that the matrices $D_{j,0}, \ D_{j,1}$ consist of three blocks, as depicted in Figure 2.3: a block D^L describing the boundary corrected functions near the left endpoint, a block D^I describing the interior functions, and a block D^R describing the boundary corrected functions near the right endpoint. The matrices $D_{j,0}, D_{j,1}$ exhibit the level independence described in Remark 2.4.1, since the boundary corrections are the same on all levels, and only the number

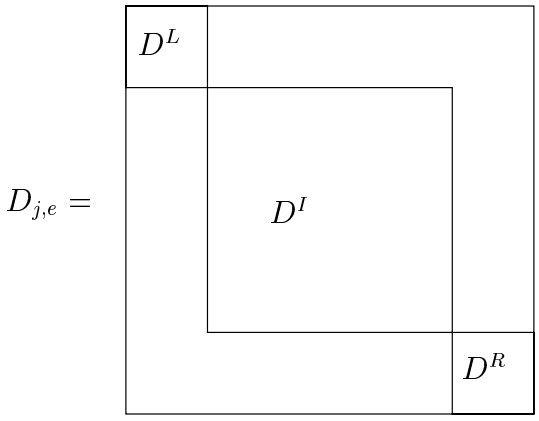

Figure 2.3: Structure of $D_{j,e}$, $e = 0, 1$.

of interior functions change with the level. From (2.12) and (2.14) we infer that the interior blocks have the structure

$$(D_{j,0})_{k,k'} = 2^j(\delta_{k,k'} - \delta_{k,k'-1}), \quad (D_{j,1})_{k,k'} = 2^{j+2}\delta_{k,k'}. \tag{2.43}$$

The uppercase indices $(0), (1)$ will be referred to as *smoothness indices*, and will be of convenience later. Note that the index sets for the primal scaling functions $\Xi_j^{(1)}$ are the same as those for the primal scaling functions $\Xi_j^{(0)}$. Suppose that the systems with smoothness index (1) are generated by the B-spline scaling functions from Subsection 2.2.1 as follows:

$$\xi^{(1)}(x) = \xi_d(x), \quad \tilde{\xi}^{(1)}(x) = {}_{d-1,\tilde{d}+1}\tilde{\xi}(x) \tag{2.44}$$

$$\eta^{(1)}(x) = {}_d\eta(x), \quad \tilde{\eta}^{(1)}(x) = {}_{d-1,\tilde{d}+1}\tilde{\eta}(x) \tag{2.45}$$

then from (2.12) and (2.14) it can be inferred that the systems with smoothness index (0) are generated by

$$\xi^{(0)}(x) = {}_{d-1}\xi(x), \quad \tilde{\xi}^{(0)}(x) = {}_{d,\tilde{d}}\tilde{\xi}(x), \tag{2.46}$$

$$\eta^{(0)}(x) = {}_{d-1}\eta(x), \quad \tilde{\eta}^{(0)}(x) = {}_{d,\tilde{d}}\tilde{\eta}(x) \tag{2.47}$$

see [76].

2.5 Tensor product wavelets

From the previously discussed univariate wavelets on $\Omega \in \{(0,1), \mathbb{R}, \mathbb{T}\}$, one can construct wavelet bases for function spaces over higher dimensional domains, of which we shall now consider $\Omega^n \in \{(0,1)^n, \mathbb{R}^n, \mathbb{T}^n\}$ in a *tensor product fashion*. Apart from understanding (2.48), the contents of this subsection is mainly notational conventions.

We suppose that we have given a biorthogonal multiresolution sequence in $L_2(\Omega)$, with corresponding families of *univariate* scaling functions

$$\Xi_j = \{\xi_{j,k} : \quad k \in I_j\}, \quad \tilde{\Xi}_j = \{\tilde{\xi}_{j,k} : \quad k \in I_j\}$$

and wavelets

$$\Upsilon_j = \{\eta_{j,k} : \quad k \in J_j\}, \quad \tilde{\Upsilon}_j = \{\tilde{\eta}_{j,k} : \quad k \in J_j\}.$$

A multiresolution sequence in $L_2(\Omega^n)$ is generated by scaling functions on the form

$$\phi_{j,k}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \xi_{j,k_i}(x_i),$$

where $k = (k_1, \ldots, k_n)$ and $k_i \in I_j$, and $i = 1, \ldots, n$. Introducing $E := \{0,1\}^n$ and $E^* := E/\{\mathbf{0}\}$, the wavelets take for $e = (e_1, \ldots, e_n) \in E^*$ the form

$$\psi_{j,e,k} = \prod_{k=1}^{n} \zeta_{j,e_i,k_i}(x_i), \quad \zeta_{j,e_i,k_i}(x_i) = \begin{cases} \xi_{j,k_i}(x_i), & k_i \in I_j \quad e_i = 0 \\ \eta_{j,k_i}(x_i), & k_i \in J_j \quad e_i = 1 \end{cases}. \tag{2.48}$$

e is called the *type parameter*. For notational simplicity, we define

$$J_{j,e_i} = \begin{cases} I_j, & e_i = 0 \\ J_j, & e_i = 1 \end{cases}, \text{ and } \mathcal{J}_{j,e} = J_{j,e_1} \times \ldots \times J_{j,e_n}.$$

The dual tensor product scaling functions and wavelets are defined analogously. The various types of basis functions are collected in sets

$$e \in \{0,1\}^n, \quad \Psi_{j,e} = \{\psi_{j,e,k} : \ k \in \mathcal{J}_{j,e}\}.$$

Note that $\Psi_{j,\mathbf{0}}$ corresponds to a pure scaling function basis, and that in dimension n, there are $2^n - 1$ types of wavelets. The scaling function or wavelet basis on level j is

$$\Psi_j = \bigcup_{e \in E} \Psi_{j,e}.$$

Later, it will be convenient to consider tensor product bases with factors having different smoothness indices. The smoothness parameter is then a vector $s \in E$, and for some type parameter $e \in E$,

$$\mathcal{J}_{j,e}^{(s)} = J_{j,e_1}^{(s_1)} \times \ldots \times J_{j,e_n}^{(s_n)}. \tag{2.49}$$

We shall write

$$\Psi_j^{(s)} := \{\psi_{j,e,k}^{(s)} : \ k \in \mathcal{J}_{j,e}^{(s)}\},$$

where, following the notation of (2.48) and Theorem 2.4.3,

$$\psi_{j,k,e}^{(s)} = \prod_{i=1}^{n} \eta_{j,e_i,k_i}^{(s_i)}.$$

Similarly, we shall indicate tensor products of boundary adapted wavelets with a vector $\beta = (\beta_1, \ldots, \beta_n) \in E$ (recall the notation in (2.34),

$$\mathcal{J}_{j,e}^{\beta} = J_{j,e_1}^{\beta_1} \times \ldots \times J_{j,e_n}^{\beta_n}.$$

For notational simplicity, we shall often concatenate indices in the following way:

$$\lambda = (j, e, k) \in \mathfrak{J} \times E \times \mathcal{J}_{j,e} =: \mathcal{J}_j, \quad |\lambda| = j, \tag{2.50}$$

where $\mathfrak{J} \subset \mathbb{Z}$ is the range of levels. We make the general convention that the omission of the level refers to a union over all levels:

$$\mathcal{J} := \bigcup_{j \in \mathfrak{J}} \mathcal{J}_j. \tag{2.51}$$

Thus we understand that for tensor product wavelets living on a domain $\Omega \subset \mathbb{R}^n$, the translation parameter, boundary parameter, smoothness index and type index are all n-dimensional vectors of the corresponding parameters for the univariate case.

It is straightforward to see that the properties described in section 2.1 are preserved. Furthermore, from Theorem 2.4.3, we infer the following

REMARK 2.5.1 *For $i \in \{1, \dots, n\}$ and a smoothness index $s \in E$ such that $s - \delta_i \in E$, we can write*

$$\partial_i \Psi_j^{(s)} = D_j^{(i)} \Psi_j^{(s-\delta_i)}$$

for a uniformly sparse matrix $D_j^{(i)} : \mathbb{R}^{\mathcal{J}_j \times \mathcal{J}_j}$ appropriately composed of the $D_{j,0}$ and $D_{j,1}$ from Theorem 2.4.3. For each level j, D_j has a non-diagonal block, due to univariate scaling functions being factors of multivariate wavelets.

2.6 Wavelets on general domains

In this section, we consider the construction of wavelet bases on domains that can be expressed as a union of 'patches', i.e. images of the n-dimensional square under appropriate parametric mappings,

$$\overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega_i}, \quad \Omega_i \cap \Omega_j = \emptyset, \ i \neq j, \quad \kappa_i : \square \mapsto \Omega_i, \ i = 1, \dots, m$$

see figure 2.4. Since the constructions mentioned below are very technical, we shall not go into the somewhat intricate mathematical details, but only sketch some of the underlying ideas, and point out some of their differences, regarding construction principles and applicability.

The idea, followed in [17],[18],[42],[41], consists in taking for each $i = 1, \dots, m$ a wavelet basis on the open n-dimensional unit cube $\square$, and map the basis onto each patch $\kappa_i(\square)$. Homogeneous boundary conditions can be inforced by identifying which part of the patch $\kappa_i(\square)$ (if any) that lies on the boundary of Ω, and identifies on which part of the boundary $\square$ the homogeneous boundary condition must be incorporated in the corresponding wavelet basis. Global continuity requirements are met by stitching together the wavelet bases across interpatch boundaries. Specifically, one identifies two patches $\kappa_i(\square)$ and $\kappa_j(\square)$ with an adjacent edge (or face in 3D), and makes sure that the corresponding wavelet bases on $\square$ are unaltered along the relevant edges (faces) of $\square$, thus making it possible to stitch together the functions across the interpatch boundary.

Certain regularity requirements on the mappings must be assumed, and for each case below, we refer the reader to the corresponding reference. We introduce the non-standard parametrization-dependent inner products on $L_2(\Omega)$:

$$\langle f, g \rangle_{0,\Omega} := \sum_{i=1}^{m} (f \circ \kappa_i, g \circ \kappa_i)_{0,\square}. \tag{2.52}$$

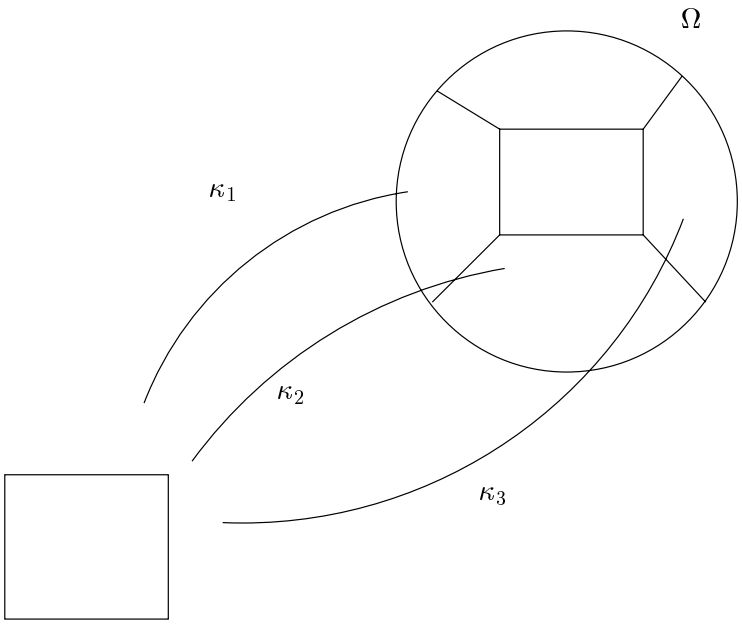

Figure 2.4: The domain Ω as a union of patches

We would like to point out some differences between the approaches of the before mentioned papers. First, the *composite wavelet bases* approach of [41], where a stable completion approach is employed to arrive at a wavelet basis on Ω with a desired degree of global smoothness. Depending on details of the construction, it applies to $H^s(\Omega)$ for $-1/2 < s \leq 3/2$, but in some cases, it works for even lower values of s. Secondly, the *wavelet element approach* of [17], [18], where explicit matching conditions across interpatch boundaries are derived for the scaling functions and wavelets. This approach relies crucially on a norm equivalence of the form

$$\|f\|_{H^s}^2 \sim \sum_i \|f|_{\Omega_i}\|_{H^s(\Omega_i)}^2, \quad -\frac{1}{2} < s \leq \frac{3}{2}.$$

so that this approach also covers a limited range of global smoothness, still, however, governing a wide range of applications. Both of these methods arrive at biorthogonal wavelet bases with respect to the inner product (2.52). Finally, the approach of [42] covers arbitrary s, and gives biorthogonality with respect to the canonical inner product on $L_2(\Omega)$, but the construction and analysis is considerably more involved.

It should also be noted that the case of Neumann boundary conditions on all of or a part of the boundary can also be covered, and we refer to the above mentioned papers for details.

2.7 Wavelet-induced norm equivalences

A very useful property of wavelets in connection with efficient numerical solution of PDEs is their ability to *characterize* normed function spaces, in the sense that one can estimate the norm of a function from above and below by considering the coefficients in the wavelet expansion of the function. We wish in this section to consider wavelet-induced norm equivalences in two ranges of function spaces, namely Sobolev and Besov spaces. For the rest of this chapter,

we assume that $\Omega \subset \mathbb{R}^n$ is some domain, for which we have a biorthogonal multiresolution sequence with corresponding biorthogonal scaling function and wavelet bases, all in the space $L_2(\Omega)$.

Recall the Jackson and Bernstein inequalities from Theorem 2.3.2. The lemma below relates the Jackson-Bernstein inequalities to smoothness, polynomial exactness, and local support. Again, we postpone the discussion of the Jackson and Bernstein inequalities, referring to Section 3.2.

LEMMA 2.7.1 *[40] For $s < \sup\{t : \ \Phi_j \subset H^t(\Omega)\}$, the Bernstein inequality holds, and similarly for the dual family. If the biorthogonal families Φ_j and $\tilde{\Phi}_j$ satisfy*

(i) $\|\phi_{j,k}\|_{0,\Omega}, \|\tilde{\phi}_{j,k}\|_{0,\Omega} \lesssim 1, \quad k \in \mathcal{J}_j,$

(ii) Φ_j *and* $\tilde{\Phi}_j$ *are locally finite, in the sense that with*

$$\sigma_{j,k} = supp(\phi_{j,k}), \quad \tilde{\sigma}_{j,k} = supp(\tilde{\phi}_{j,k}), \quad k \in \mathcal{J}_j$$

one has

$$\#\{k' \in \mathcal{J}_j : \sigma_{j,k} \cap \sigma_{j,k'} \neq \emptyset\}, \#\{k' \in \mathcal{J}_j : \ \tilde{\sigma}_{j,k} \cap \tilde{\sigma}_{j,k'} \neq \emptyset\} \lesssim 1, \ \forall k \in \mathcal{J}_j,$$

then the families Φ_j and $\tilde{\Phi}_j$ are uniformly stable. If in addition, Ω is a bounded Lipschitz domain and $\Pi_l(\Omega) \subset \mathcal{S}_j$, where $\Pi_l(\Omega)$ is the space of polynomials on Ω of degree at most $l-1$, then

$$\inf_{v_j \in \mathcal{S}_j} \|v - v_j\|_{0,\Omega} \lesssim h_j^l \|v\|_{l,\Omega}, \quad v \in H^l(\Omega)$$

where $h_j = \sup_{k \in \mathcal{J}_j}\{diam(\sigma_{j,k}), diam(\tilde{\sigma}_{j,k})\}$.

Then, Theorem 2.3.2 gives us the desired wavelet characterization of the Sobolev space. Define the matrix $\boldsymbol{D} : \mathbb{R}^{\mathcal{J} \times \mathcal{J}}$ as

$$\boldsymbol{D} = \text{diag } (2^{|\lambda|}),$$

so that $\boldsymbol{D}^s = \text{diag}(2^{|\lambda| s})$, and assume that we have wavelet bases Ψ_j for the complement spaces $\mathcal{U}_j = (Q_{j+1} - Q_j)\mathcal{S}_{j+1}$, such that the assumptions of Theorem 2.3.2 are fulfilled. Then the norm equivalence (2.20) can be written as

$$\|\boldsymbol{c}^T \Psi\|_{s,\Omega}^2 \sim \|\boldsymbol{D}^s \boldsymbol{c}\|_{\ell_2(\mathcal{J})}^2 \tag{2.53}$$

where $\boldsymbol{D}^s \boldsymbol{c} \in \ell_2(\mathcal{J})$. A duality argument [36],[62] shows that the dual family $\tilde{\Psi}$ is a basis for $H^{-s}(\Omega) = (H^s(\Omega))'$, and induces a norm equivalence

$$\|\boldsymbol{c}^T \tilde{\Psi}\|_{-s,\Omega}^2 \sim \|\boldsymbol{D}^{-s} \boldsymbol{c}\|_{\ell_2}^2, \tag{2.54}$$

where we note that $\boldsymbol{D}^s$, being a diagonal transformation, is invertible, when regarded as a bi-infinite matrix. These equivalences show that by a appropriate scaling of the L_2-normalized wavelets, one can obtain a Riesz basis for $H^s(\Omega)$:

$$\|\boldsymbol{c}^T \boldsymbol{D}^{-s} \Psi\|_{s,\Omega}^2 \sim \|\boldsymbol{c}\|_{\ell_2}^2,$$

where we write

$$\boldsymbol{c}^T \boldsymbol{D}^{-s} \Psi = \sum_{\lambda} c_\lambda d_\lambda^{-s} \psi_\lambda$$

for the entries $d_\lambda = (\boldsymbol{D})_{\lambda,\lambda}$.

Another way of measuring smoothness of functions is in terms of Besov spaces, see appendix A for their definition. We quote [45]:

THEOREM 2.7.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, on which $\Psi = \{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ is a wavelet basis for $L_2(\Omega)$. For $0 < \tau < 2$, let*

$$\frac{1}{\tau} = \frac{s}{n} + \frac{1}{2}.$$

Assume that $\psi \in B^t_\tau(L_2(\Omega))$ for some $t > s$, and that ψ has r vanishing moments, with $r > s$. If

$$f = \sum_{\lambda\in\mathcal{J}} c_\lambda \psi_\lambda \in L_2(\Omega)$$

and $f \in B^s_\tau(L_\tau(\Omega))$, then $\{c_\lambda\}_{\lambda\in\mathcal{J}} \in \ell_\tau(\mathcal{J})$, and

$$|f|_{B^s_\tau(L_\tau(\Omega))} \sim \left(\sum_\lambda |c_\lambda|^\tau\right)^{1/\tau}.$$

Wavelet characterizations are available for a wide range of function spaces, such as Hardy-, Lipschitz-, Triebel-Lizorkin spaces, and the space BMO, see for instance [54], [62], [78].

Chapter 3

Adaptive wavelet methods

As in most applications of wavelets, the efficiency of numerical methods for solution of PDEs, and many other operator equations, rely on the ability of wavelets to represent, or approximate in some sense, functions and operators, in an economical fashion. We focus in this chapter on the adaptive wavelet-based algorithms presented in [25]. These methods exploit the fact that in many cases, wavelet expansions of functions, and wavelet representations of differential operators, are in an appropriate sense, sparse, or compressible. We describe in this chapter the adaptive wavelet-based methods of [25], and the mathematical framework used for the construction and analysis of the adaptive algorithms.

In [25], the authors introduced a *new paradigm* for the numerical solution of PDEs, which differs significantly from previous, what we shall call *classical* approaches. Eventually, the introduction of the new paradigm removes some difficulties which can be present when employing the classical approaches.

Essentially, the applicability and efficiency of the methods in [25] rely on three properties. Firstly, the *mapping property*, which means that a Hilbert space H exists, such that the operator in question is an isomorphism of H onto its dual H'. Secondly, one must have at hand a wavelet basis inducing a norm equivalence in H. Thirdly, there is the *compressibility*, meaning that wavelet representation of the operator in question is sufficiently sparse.

The outline of the chapter is as follows. In Section 3.1, we introduce the classical approach to the numerical solution of PDEs, including the concept of a variational formulation of a PDE, the Galerkin method, and its error estimation and convergence analysis. This is done in order to emphasize the main ideas behind the new paradigm presented in [25]. For the sake of perspective and comparison, we outline some fundamental properties of the *finite element method*, which is widely used in engineering applications, and to which wavelet-based methods are often compared. In Section 3.2, we present some approximation theoretical concepts, such as linear and nonlinear approximation, aproximation spaces and their relationships with interpolation and smoothness spaces, in particular Sobolev and Besov spaces. In Section 3.3 we present the adaptive algorithm of [25], along with its convergence rates and work estimates. Section 3.4 employs the theory of Section 3.2 to relate some of the parameters in the efficiency analysis of the adaptive algorithm to tractable properties of the solution to the PDE (regularity) and the operator (compressibility of wavelet representations).

3.1 The classical approach

We treat in this section the classical approach to the numerical solution of PDEs, for which the following four issues are fundamental:

(i) Formulation of a well-posed variational problem in a Hilbert space H.

(ii) Discretization of the variational problem using a basis for a finite-dimensional subspace of H.

(iii) Well-posedness of the discrete problems, and error analysis.

(iv) Solution of the linear system of equations from (ii).

In this section, we treat these issues in turn. First, we need some preliminaries, which we shall use throughout the chapter. $\Omega \subset \mathbb{R}^n$ is a bounded, Lipschitz domain, unless noted. Let H be a Hilbert spaces over the scalar field $\mathbb{K}$, such that $H \hookrightarrow L_2(\Omega)$. Each $v \in L_2(\Omega)$ induces a functional on H, as follows:

$$u \in H, \ u \mapsto (u, v)_{0,\Omega}. \tag{3.1}$$

One can define the norm

$$\|v\|_{H'} := \sup_{\|u\|_H \leq 1} |(u, v)_{0,\Omega}|,$$

and we can define the space H' as the closure of $L_2(\Omega)$ in the above norm. H' is called the dual space of H with $L_2(\Omega)$ as pivot space, or shortly, the dual of H. Note that $L_2(\Omega) \hookrightarrow H'$. Such a triple $H, L_2(\Omega), H'$ is often called a *Gelfand triple* in the literature. An example is $H = H_0^1(\Omega), \ H' = H^{-1}(\Omega)$. We denote by $\langle \cdot, \cdot \rangle_{H' \times H}$ the duality form on H', but when there is no risk of confusion, we shall just write $\langle , \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle$.

Note here that the Riesz representation theorem considers functionals of the form $u \in H$, $H \ni v \mapsto (u, v)_H$, i.e. functionals represented by inner product in H, and not in $L_2(\Omega)$, as in (3.1).

3.1.1 Variational formulation, well-posedness

Let $a(\cdot, \cdot) : H \times H \mapsto \mathbb{K}$ be a bilinear form which is continuous and coercive over H, meaning, respectively, that there exist constants $c, \ C$ such that for any $u, v \in H$,

$$|a(u, v)| \leq C\|u\|_H \|v\|_H, \tag{3.2}$$

$$|a(u, u)| \geq c\|u\|_H^2. \tag{3.3}$$

By a *variational problem* we understand a problem of the following form

PROBLEM 3.1.1 *Given $f \in H'$, find $u \in H$ such that*

$$a(u, v) = \langle f, v \rangle \quad \forall v \in H.$$

The fundamental result on existence and uniqueness, or more generally well-posedness, is the following:

THEOREM 3.1.2 (LAX-MILGRAM) *If $a(\cdot, \cdot)$ is continuous and coercive on H, then Problem 3.1.1 is well-posed, in the sense that for any $f \in H'$ there exists a unique solution $u \in H$, and a constant C independent of u such that $\|u\|_H \leq C\|f\|_{H'}$.*

This result states that the operator $A : H \mapsto H'$, defined by

$$\langle Au, v\rangle := a(u, v) \tag{3.4}$$

fulfills

$$\|Au\|_{H'} \sim \|u\|_H. \tag{3.5}$$

Whenever an operator satisfies (3.5), we say that it satisfies the *mapping property*. The space $(H, a(\cdot,\cdot))$ is traditionally denoted as the *energy space*, and the norm induced by $a(\cdot,\cdot)$ the *energy norm*. For more details, see for instance [68],[12], [11], [21].

3.1.2 Discretization, the Galerkin method

A commonly employed method for the numerical approximation of the solution is the *Galerkin method*, where one picks a finitely dimensional space $H_h \subset H$, spanned by a basis $\{v_h\}$, and considers the following problem.

PROBLEM 3.1.3 *Given $f \in H'$, find $u_h \in H_h$ such that*

$$a(u_h, v) = f(v) \quad \forall v \in H_h. \tag{3.6}$$

The solution u to Problem 3.1.1 is approximated by an expansion u_h in terms of the basis $\{v_h\}$. Testing with every element in this basis, i.e. letting v in (3.6) sweep through the basis $\{v_h\}$, gives a linear system of equations, yielding the coefficients in the expansion of u_h. The system matrix is usually denoted as the *stiffness matrix*. One can prove that under the same assumptions as in Theorem 3.1.2, there exists a unique solution $u_h \in H_h$.

3.1.3 Convergence, error analysis

Estimates of the error $u - u_h$ are important for proving convergence and estimating efficiency of Galerkin methods. The method is said to converge if, as the space H_h gets larger, the error tends to zero, when measured in some prescribed norm. The following result, called the Cea lemma, is useful in this respect, since it relates the estimation of the error to the so-called *approximation power* of the subspace H_h.

THEOREM 3.1.4 *Assume that $a(\cdot,\cdot)$ is continuous and coercive on the space H, and let $u \in H$ and $u_h \in H_h$ be, respectively, the solutions of problems 3.1.1 and 3.1.3. Then*

$$\|u - u_h\|_H \leq \frac{C}{c} \inf_{v \in H_h} \|u - v\|_H,$$

where the constants c, C are those from (3.2) and (3.3).

For a variational problem such as 3.1.1, the well-posedness and the error analysis of the discrete problem follow from the same conditions as the well-posedness of the continuous problem. This is not the case in general, as we shall see in the next chapter on mixed formulations.

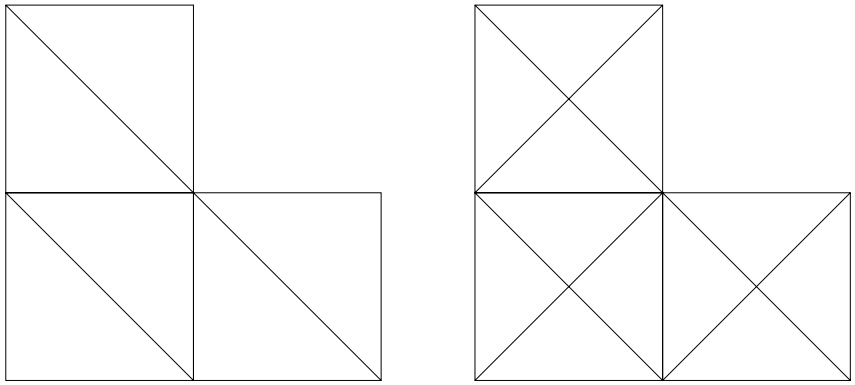

Figure 3.1: Examples of meshes

3.1.4 Solution of the linear system

Employing the Galerkin method, one must finally solve a linear system of equations, which is in many cases sparsely populated. The systems are often so large that use of direct methods such as Gaussian elimination, is not recommendable, instead one uses iterative methods, such as for instance conjugate gradients (CG), or other so-called Krylov subspace methods [67], [2], [72], [70]. Unfortunately, one often encounters the problem that as one increases the resolution, i.e. as the subspaces H_h gets larger, the condition number of the stiffness matrix increases. This has a negative impact on the performance of the iterative methods, in that the number of iterations that are necessary in order to produce an approximation up to a prescribed tolerance increases. Several *preconditioning* techniques exist in order to remedy this situation, see for instance [67], and [35], and the references cited therein.

3.1.5 An example

In the following example, we introduce a well-studied variational problem and discuss some fundamental issues concerning the numerical approximation. We describe some aspects of the *finite element method*, since it is widely used in engineering problems, and often compared to wavelet-based methods in terms of efficiency.

EXAMPLE 3.1.5 *Suppose* $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, *is bounded and Lipschitz. For* $f \in L_2(\Omega)$, *consider the problem*

$$-\Delta u = f \text{ on } \Omega, \ u = 0 \text{ on } \Gamma = \partial\Omega.$$

Multiplying by any $v \in H_0^1(\Omega)$ *and integrating by parts gives the formulation from Problem 3.1.1 with* $H = H_0^1(\Omega)$, $a(u, v) = (\nabla u, \nabla v)_{0,\Omega}$. *Continuity is obviously fulfilled, and the Poincare inequality (page 107) ensures coercivity, and hence well-posedness.*

The (conforming) finite element method consists in subdividing the domain Ω *in subdomains, such as triangles, and together the subdomains comprise a mesh. In standard notation, the mesh size parameter* $h > 0$ *gives an estimate of the size of the triangles. On each triangle, one defines in appropriate ways polynomial basis functions, spanning the subspace* $H_h \subset H$. *The goal is that as* $h \to 0$, *the approximations from* H_h *should converge to the exact solution, measured in an appropriate norm. Usually, the mesh is assumed to be uniform, meaning*

roughly that the ratio between the sizes of largest and smallest triangle is bounded from above and below, independently of h. Such methods will be denoted as uniform methods, and will stand in contrast to adaptive methods, treated later.

Assuming $u \in H^k(\Omega)$ for some $k > 1$, and denoting by u_h the approximation obtained from a Galerkin approach, Bramble-Hilbert type arguments [12, section 4.3] produce error estimates for finite element approximation of the form

$$|u - u_h|_{m,\Omega} \leq Ch^{k-m}|u|_{k,\Omega}, \quad 0 \leq m \leq k. \tag{3.7}$$

The convergence rate, i.e. the decay rate of the error as $h \to 0$, depends on the regularity of the solution, measured in terms of Sobolev space, and the degree of polynomials chosen for the construction of the basis functions. There are many excellent accounts of the Finite Element Method, among these are [11],[12], [21].

3.2 Linear and nonlinear approximation

In this section, we shall follow [45] and formalize the concept of approximation in a Hilbert space setting. We shall distinguish between linear approximation, which is performed in uniform numerical methods, and nonlinear approximation, which is performed in adaptive numerical methods. We shall put particular focus on one type of nonlinear approximation, namely that of *N-term approximation with wavelets*. We introduce the concept of approximation spaces, being spaces of functions which can be approximated, either linearly or nonlinearly, to a certain accuracy. Within the scope of this thesis, these approximation spaces have, via Jackson-Bernstein inequalities and interpolation spaces, a characterization in terms of smoothness, measured in terms either Sobolev space (linear approximation), or Besov space (nonlinear approximation). Eventually, this will furnish the relation between efficiency of adaptive wavelet-based numerical methods, and regularity of the solution to the PDE.

Approximation in Hilbert space

We define the concept of approximation in a Hilbert space setting as follows. Let X be a Hilbert space over the scalar field $\mathbb{K}$. We consider families of subsets $X_n \subset X$. The parameter n usually describes a characteristic quantity of the subset X_n. If X_n is a subspace, n could be related to its dimension.

We shall assume the following properties of the sets X_n:

(i) $X_0 = \{0\}$.

(ii) $X_n \subset X_{n+1}$, $n \in \mathbb{N}$.

(iii) $aX_n = X_n$, $a \in \mathbb{R}$, $a \neq 0$, $n \in \mathbb{N}$.

(iv) $X_n + X_n \subset X_{cn}$ for some integer $c \geq 1$ independent of n.

(v) Each $f \in X$ has a *best approximation* from X_n, i.e. an element $f_n \in X_n$ such that $\|f_n - f\|_X$ equals

$$E_n(f)_X := \inf_{g \in X_n} \|f - g\|_X.$$

(vi) For any $f \in X$, $E_n(f)_X \to 0$ as $n \to \infty$.

Note that this definition does not require X_n to be linear spaces. Suppose that we wish to approximate a given function $f \in X$ called the *target function*. In the case of *linear approximation*, the sets X_n are linear spaces and are fixed in advance. In example 3.1.5 of Subsection 3.1, an example of linear approximation was mentioned. In the case of *nonlinear approximation*, the sets X_n are not linear spaces, and they may *depend on the target function.*

Let now Ψ be a wavelet basis of X, and consider spaces of the form

$$\Sigma_N = \{\sum_{\lambda \in \mathcal{I}} a_\lambda \psi_\lambda : a_\lambda \in \mathbb{K},\ \#\mathcal{I} \leq N\},\ N \in \mathbb{N} \tag{3.8}$$

i.e. the space of linear combinations of at most N wavelets. Approximation with functions from Σ_N is called *N-term approximation with wavelets.* It is a special case of nonlinear approximation. The important aspect of this particular type of approximation is, as noted above, that *the selection of wavelet basis functions may depend on the target function.* In what follows, we shall denote N as the *number of degrees of freedom*.

The approximation spaces $\mathcal{A}_q^\alpha(X)$

Letting $\alpha > 0$, we introduce the *approximation spaces* $\mathcal{A}_q^\alpha = \mathcal{A}_q^\alpha(X)$ as the subspace of X for which

$$|f|_{\mathcal{A}_q^\alpha} := \begin{cases} \left(\sum_{n=1}^\infty \left(n^\alpha E_n(f)_X\right)^q \frac{1}{n}\right)^{1/q}, & 0 < q < \infty \\ \sup_{n \geq 1} n^\alpha E_n(f)_X, & q = \infty \end{cases}$$

is finite. As we shall see later, given some sequence X_n from which approximations are chosen, in a linear or nonlinear fashion, the corresponding approximation space of interest to us has a characterization in terms of *smoothness*, in the sense of either Sobolev or Besov spaces. It should come as no surprise that the approximation spaces related to nonlinear approximation are significantly larger than the ones related to linear approximation.

REMARK 3.2.1 *Note that the term approximation space suits both $\mathcal{A}_q^\alpha$, and the sets X_n from which approximands to the target function are chosen. In order to avoid confusion, we speak only of the spaces $\mathcal{A}_q^\alpha$ as approximation spaces.*

Jackson-Bernstein inequalities, interpolation and approximation spaces

In order to relate the connection between the Jackson and Bernstein inequalities and approximation theory, we need some fundamental concepts from interpolation theory.

Let $X, (\|\cdot\|_X)$ and $(Y, \|\cdot\|)$ be normed linear spaces, with $Y \hookrightarrow X$. For any $t > 0$, one can define the *K-functional* as

$$K(f,t) := \inf_{g \in Y} \|f - g\|_X + t|g|_Y$$

where $|\cdot|_Y$ is a seminorm on Y. Hence, one approximates elements from X with elements from Y, but with a penalty term depending on the approximating element from Y. For $0 < \theta < 1$, $0 < q \leq \infty$, the *interpolation space* $(X,Y)_{\theta,q}$ is defined as the subspace of X for which

$$|f|_{(X,Y)_{\theta,q}} := \begin{cases} \left(\int_0^\infty \left(t^{-\theta} K(f,t)\right)^q \frac{1}{t} dt\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\theta} K(f,t), & q = \infty \end{cases}$$

is finite. We can now state the following theorem, borrowed from [45]:

THEOREM 3.2.2 *If the Jackson inequality*

$$E_n(f)_X \lesssim n^{-r}|f|_Y, \quad f \in Y, \quad n = 1, 2, \ldots \tag{3.9}$$

and the Bernstein inequality

$$|g|_Y \lesssim n^r \|g\|_X, \quad g \in X_n, \quad n = 1, 2, \ldots \tag{3.10}$$

hold, then for each $0 < \gamma < r$ *and* $0 < q \leq \infty$, *the relation*

$$\mathcal{A}_q^\gamma(X) = (X, Y)_{\gamma/r,q}$$

holds, with equivalent norms.

Interpolation spaces is a well studied area, and hence the characterization of an approximation space is carried out by identifying the two spaces X, Y for which the Jackson and Bernstein inequalities (3.9) and (3.10) hold, and characterizing the corresponding interpolation space. For more information on interpolation spaces, we refer to [10], [60], [74]. In particular, we shall focus on Besov spaces as interpolation spaces.

Often, the parameters γ and r can be thought of as the parameters reflecting the smoothness properties of the spaces, wereas the q is often employed for 'fine tuning', whenever it is necessary. In (3.13) below, we shall see such an example.

Linear approximation with wavelets

Let $\Omega \subset \mathbb{R}^n$ be an open connected Lipschitz domain. The following characterization of Besov spaces is known [45]:

$$(L_p(\Omega), W^{r,p}(\Omega))_{\theta,q} = B_q^{\theta r}(L_p(\Omega)), \quad 0 < \theta < 1, \ 0 < q \leq \infty, \tag{3.11}$$

with equivalent norms. Consider the case $X_j = \mathcal{S}_j$, where the latter is some multiresolution sequence, for which the Jackson and Bernstein inequalities hold as in Theorem 2.3.2 and Lemma 2.7.1. This is a case of linear approximation in the space $L_2(\Omega)$, hence $p = 2$, and

$$E_j(f)_{L_2} = \inf_{g \in \mathcal{S}_j} \|f - g\|_{0,\Omega}.$$

From (3.11) we see that the corresponding approximation spaces are then the Besov spaces $B_q^{\theta r}(L_2(\Omega))$, for a range of θr depending on the validity ranges of the Jackson and Bernstein inequalities. There is no restriction on q, other than $0 < q \leq \infty$.

Nonlinear approximation with wavelets

Still, $\Omega \subset \mathbb{R}^n$ is an open connected Lipschitz domain. Let us assume that Ψ is a wavelet basis for $L_2(\Omega)$, and define Σ_N as in (3.8). Hence we consider nonlinear approximation with wavelets, with

$$E_N(f) = \inf_{g \in \Sigma_N} \|f - g\|_{0,\Omega}.$$

An element $g \in \Sigma_N$ realizing the infimum is called a *best* N-term approximation. Under the assumptions and with the terminology and use of Theorem 2.7.2, one can prove [45] the Jackson inequality

$$E_N(f) \lesssim |f|_{B_\tau^s(L_\tau(\Omega))} N^{-s/n}$$

and the Bernstein inequality

$$f \in \Sigma_N, \quad \|f\|_{B^s_\tau(L_\tau(\Omega))} \lesssim N^{s/n} \|f\|_{0,\Omega}$$

for a range of s depending as in Theorem 2.7.2 on the regularity and vanishing moments of the involved wavelets. The corresponding approximation spaces are

$$\mathcal{A}^{\gamma/n}_q(L_2(\Omega)) = (L_2(\Omega), B^s_\tau(L_\tau(\Omega)))_{\gamma/s,q}. \tag{3.12}$$

For one value of q, namely the one that satisfies

$$\frac{1}{q} = \frac{\gamma}{n} + \frac{1}{2} \tag{3.13}$$

this approximation/interpolation space is a Besov space, namely

$$\mathcal{A}^{\gamma/n}_q(L_2(\Omega)) = B^\gamma_q(L_q(\Omega)). \tag{3.14}$$

Characterizing functions that can be approximated to a given accuracy, either in a linear or nonlinear fashion, these results state nothing about *how* to find, say, a best N-term approximation of a given target function. Indeed, this is the role of the numerical algorithms. These characterizations of approximation spaces provide, along with regularity theory for PDEs, *bench marks*, to which the performance of numerical algorithms should be compared. The role of regularity theory for PDEs is to give estimates of smoothness of solutions to the PDE, in terms of membership of smoothness spaces, given information about the differential operator, the domain, and the given data.

The smoothness space diagram: advantage of nonlinear approximation

In this subsection, we present a way of graphically visualizing the smoothness spaces just introduced. It is borrowed from the survey by DeVore, [45]. Consider figure 3.2, which consists of the upper right quadrant of a coordinate system. Along the ordinate axis, we depict $\frac{1}{p}$ for $0 < p < \infty$. The corresponding interpretation is to measure smoothness related quantities in the $L_p(\Omega)$ metric, i.e. partial derivatives in the case of Sobolev space, and differences in the case of Besov space, see appendix A. Along the abscissa, we depict s, interpreted as a parameter measuring smoothness. Each point $(\frac{1}{p}, s)$ represents one or more smoothness spaces.

Take the case of linear approximation, and the corresponding interpolation space (3.11), and let $p = 2$. $L_2(\Omega)$ and $W^{2,r}(\Omega) = H^r(\Omega)$ are associated to the points $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, r)$, respectively. For each $0 < \theta < 1$, we associate each family of Besov spaces $B^{\theta r}_q(L_2(\Omega))$, $0 < q \leq \infty$, to the point $(\frac{1}{2}, \theta r)$. These families of smoothness spaces are in this way identified with the vertical line in figure 3.2 designated L, and so we have depicted the approximation spaces governing linear approximation.

For the case of nonlinear approximation, recall (3.12) and associate $B^s_\tau(L_\tau(\Omega))$ to the point $(\frac{1}{\tau}, s)$. To each $0 < \gamma < s$, we associate to the point $(\frac{1}{q}, \gamma)$ satisfying (3.13) the Besov space $B^\gamma_q(L_q(\Omega))$, thus forming the line designated NL, and representing the approximation spaces governing nonlinear approximation with wavelets.

The Sobolev imbedding theorem gives information on the size of the approximation spaces. In fact, spaces corresponding to points on and to the left of the line designated NL are imbedded into L_2, the spaces corresponding to the points to the right of the line are not.

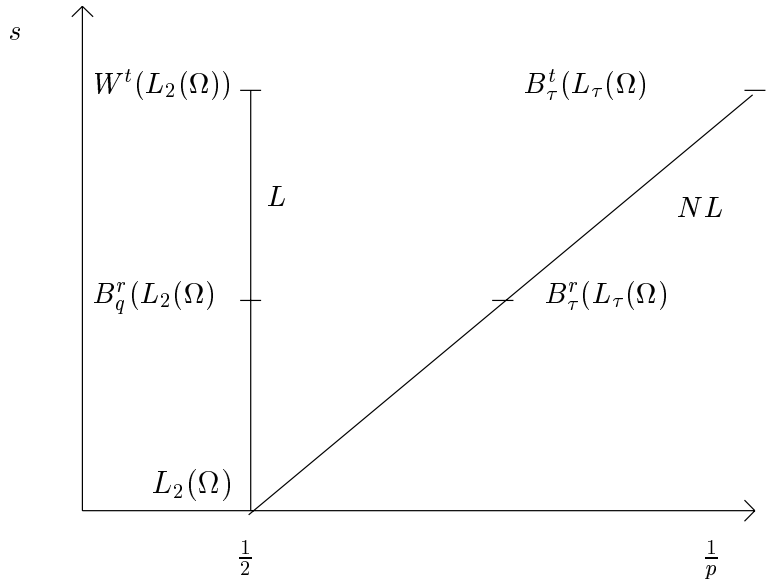

Figure 3.2: Smoothness space diagram

The sequence space $\ell^w_\tau(\mathcal{J})$

For the analysis of the performance of N-term approximation in various settings, we shall introduce a family of important sequence spaces. Suppose that $v \in \ell_2(\mathcal{J})$. Let $v^* = \{v^*_n\}_{n\in\mathbb{N}}$ denote a *nondecreasing rearrangement* of v, i.e. $|v^*_n| \geq |v^*_{n+1}|$, $n \in \mathbb{N}$. For $0 < \tau < 2$ we define the seminorm

$$|v|_{\ell^w_\tau(\mathcal{J})} := \sup_{n\in\mathbb{N}} n^{1/\tau}|v^*_n|,$$

and the space $\ell^w_\tau(\mathcal{J})$ as the subspace of $\ell_2(\mathcal{J})$ for which the norm

$$\|v\|_{\ell^w_\tau(\mathcal{J})} := \|v\|_{\ell_2(\mathcal{J})} + |v|_{\ell^w_\tau(\mathcal{J})}$$

is finite. We denote by v_N the first N terms of v^*, also denoted as a *best N-term approximation* of v. Such an approximation is not unique. For $0 < \tau < 2$, $\|v\|_{\ell^w_\tau(\mathcal{J})} \leq 2\|v\|_{\ell_\tau(\mathcal{J})}$. A reverse inclusion does not hold, a counterexample is provided by considering a properly 'normalized' harmonic series. As previously noted, vectors of wavelet coefficients are often sparsely populated. We shall use the $\ell^w_\tau(\mathcal{J})$ to measure and quantify the sparsity of such vectors. A characterization of the space $\ell^w_\tau(\mathcal{J})$ is:

PROPOSITION 3.2.3 *Fix τ and s such that*

$$\frac{1}{\tau} = s + \frac{1}{2}. \tag{3.15}$$

Then $v \in \ell^w_\tau(\mathcal{J})$ if and only if $\|v - v_N\|_{\ell_2(\mathcal{J})} = \mathcal{O}(N^{-s})$ and one has the estimate

$$\|v - v_N\|_{\ell_2(\mathcal{J})} \lesssim N^{-s}\|v\|_{\ell^w_\tau(\mathcal{J})}.$$

3.3 The new paradigm

The paper [25] introduced a new paradigm for the numerical solution of operator equations. In this section, we give an outline of the basic underlying ideas, which also provide motivation for the developments later in the thesis.

We consider the variational problem 3.1.1. Both the 'old' and the new paradigm applies to a larger class of problems, some of which will be treated later, but for the moment, and to keep the presentation simple, we explain the ideas of [25] using Problem 3.1.1.

The new paradigm is concerned with the following issues:

(i) Formulation of a well-posedness variational problem.

(ii) Transformation the original problem into an equivalent, well-posed problem in the sequence space ℓ_2.

(iii) Formulate a convergent iterative scheme in the infinitely dimensional ℓ_2 setting.

(iv) Approximate the action of the involved operators on finitely supported vectors in an adaptive fashion, balancing the dynamically updated approximation errors in order to ensure convergence.

The first point is identical to that of the classical approach. Now we treat the remaining points in turn.

3.3.1 Transformation to the sequence space ℓ_2

We explain now how to transform the variational problem 3.1.1 into a problem posed in Euclidean space, and we adopt the terminology and notation of Section 3.1. We want to emphasize that the scope of problems, which can be treated by the methods of [25], is much larger than variational formulations for elliptic PDEs.

Let $A : H \mapsto H'$ be generated from a bilinear form $a(\cdot,\cdot)$ on $H \times H$, as in (3.4), and let it fulfill the mapping property (3.5). Suppose that we have to our disposal a wavelet basis $\Psi = \{\psi_\lambda\}_{\lambda\in\mathcal{J}}$, inducing a norm equivalence of the form

$$\|\boldsymbol{c}^T\Psi\|_H \sim \|\boldsymbol{D}\boldsymbol{c}\|_{\ell_2(\mathcal{J})}, \tag{3.16}$$

where D is a diagonal, bi-infinite matrix. Then, by a duality argument [35], [36] we also have that $\tilde{\Psi}$ characterizes H', as in (2.54). Given $f \in H'$, the equation

$$Au = f \text{ in } H' \tag{3.17}$$

has a unique solution $u \in H$. Define now the bi-infinite matrix

$$\check{\boldsymbol{A}} = a(\Psi,\Psi) = (a(\psi_\lambda,\psi_\nu))_{\lambda,\nu\in\mathcal{J}},$$

which we shall call the *wavelet representation of the operator* A. Define the vector $\check{\boldsymbol{F}} = (\langle\psi_\lambda, f\rangle)_{\lambda\in\mathcal{J}}$. Represent the solution in the given wavelet basis as $u = \sum_\lambda c_\lambda\psi_\lambda$, and let $\check{\boldsymbol{U}} = (c_\lambda)_{\lambda\in\mathcal{J}}$. Solving now the equation

$$\check{\boldsymbol{A}}\check{\boldsymbol{U}} = \check{\boldsymbol{F}}$$

would, in general, lead to instability, since $\check{\boldsymbol{A}}$ is unbounded as an operator on $\ell_2(\mathcal{J})$. Therefore, we perform the substitutions

$$\boldsymbol{A} := \boldsymbol{D}^{-1}\check{\boldsymbol{A}}\boldsymbol{D}^{-1}, \quad \boldsymbol{U} := \boldsymbol{D}\check{\boldsymbol{U}}, \quad \boldsymbol{F} := \boldsymbol{D}^{-1}\check{\boldsymbol{F}},$$

and consider instead the equation

$$\boldsymbol{A}\boldsymbol{U} = \boldsymbol{F}. \tag{3.18}$$

This strategy gives us the desired ℓ_2 isomorphism, as the following theorem from [36] shows.

THEOREM 3.3.1 *Under the assumption (3.16), $\boldsymbol{F} \in \ell_2(\mathcal{J})$. The function $u = \sum_\lambda c_\lambda \psi_\lambda$ is a solution to (3.17) if and only if $\boldsymbol{U}$ solves (3.18). Furthermore, if A satisfies the mapping property, and a wavelet characterization as in (3.16) of $\boldsymbol{U}$ holds, then $\boldsymbol{A} : \ell_2(\mathcal{J}) \mapsto \ell_2(\mathcal{J})$ is an isomorphism, i.e.*

$$\|\boldsymbol{A}\boldsymbol{U}\|_{\ell_2(\mathcal{J})} \sim \|\boldsymbol{U}\|_{\ell_2(\mathcal{J})}.$$

In the terminology of the classical approach and Example 3.1.5, the matrix $\boldsymbol{D}^{-1}$ plays the role of a *preconditioner* of the matrix $\check{\boldsymbol{A}}$. This preconditioner is optimal, in the sense that condition numbers of restrictions of $\check{\boldsymbol{A}}$ to finite levels, preconditioned with corresponding sections of the matrix $\boldsymbol{D}^{-1}$, are uniformly bounded with respect to the level.

The transformation of the original problem into sequence space has the immediate benefit that the task of approximation becomes conceptually simple: Finding the best N-term approximation of the solution $\boldsymbol{U} \in \ell_2(\mathcal{J})$ with a fixed number of terms N amounts to picking the N largest elements (in magnitude) of $\boldsymbol{U}$. However, this requires knowledge of *all* entries in $\boldsymbol{U}$, which is, of course, not available. The challenge is to construct an algorithm that nevertheless produces the optimal decay in the error of N term approximation. As noted previously, approximation theory and regularity theory for PDEs provides a *bench mark* for the convergence rate of the adaptive algorithm, and one of the remarkable accomplishments of [25] is the construction of an algorithm producing the optimal error decay.

3.3.2 Iteration in the infinitely-dimensional setting

We assume the existence of some norm $\|\cdot\|$ equivalent to the ℓ_2 norm:

$$\hat{c}\|\boldsymbol{U}\|_{\ell_2} \leq \|\boldsymbol{U}\| \leq \hat{C}\|\boldsymbol{U}\|_{\ell_2}, \tag{3.19}$$

and an $\alpha > 0$ such that

$$\|\boldsymbol{I} - \alpha\boldsymbol{A}\| < \rho < 1. \tag{3.20}$$

The norm in (3.20) is the operator norm defined in the usual way. Then it is immediately verified that the *Richardson iteration* $\boldsymbol{U}^0 = 0$,

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \alpha(\boldsymbol{F} - \boldsymbol{A}\boldsymbol{U}^n), \quad n+1 \to n$$

converges to $\boldsymbol{U}$ in ℓ_2.

The Richardson iteration is chosen in order to keep the presentation simple. A more efficient performance in practical realizations would be realized by choosing gradient or conjugate gradient iterations, where the parameter α changes its value for each iteration.

3.3.3 Adaptive application of operators

The involved quantities in the iteration above are infinite dimensional, and must be approximated by finite dimensional quantitites in order to do numerical computations. This task is accomplished, in an adaptive fashion, by the following routine:
APPLY$[\eta, \boldsymbol{A}, \boldsymbol{V}] \to (\boldsymbol{W}, \Lambda)$
determines for a given, finitely supported vector $\boldsymbol{V}$ a vector $\boldsymbol{W}$, supported on some finite index set $\Lambda \subset \mathcal{J}$ such that

$$\|\boldsymbol{A}\boldsymbol{V} - \boldsymbol{W}\| \leq \eta. \tag{3.21}$$

For estimates of computational work required for the execution of **APPLY**, the concept of *compressible matrices* is important. Similar to the way in which the sequence space ℓ_τ^w quantifies the sparsity of vectors of wavelet coefficients of functions, the compressible matrices quantify the sparsity of matrix representations of differential operators, relative to wavelet bases.

DEFINITION 3.3.2 *We shall say that a matrix $\boldsymbol{A} \in \mathcal{L}(\ell_2(\mathcal{J}))$ is s^*-compressible if for every $0 < s < s^*$ and for some summable sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ there exists for every $j \in \mathbb{N}_0$ a matrix $\boldsymbol{A}_j$ having at most $\alpha_j 2^j$ nonzero entries per row and column such that*

$$\|\boldsymbol{A} - \boldsymbol{A}_j\| \leq \alpha_j 2^{-sj},\ j \in \mathbb{N}_0.$$

The class of s^* compressible matrices is denoted by $\mathcal{C}_{s^*}$.

In [24], it is proven that when $s < s^*$ and s, τ are related by (3.15), then $\mathcal{C}_s^*$ are bounded operators on $\ell_\tau^w(\mathcal{J})$.

Because of its important role for the adaptive algorithm, and since the role of the N-term approximation to the adaptive method is clearly displayed, we show below a realization of the algorithm **APPLY** . For a finite vector $\boldsymbol{V} \in \ell_2(\mathcal{J})$, we denote by $\boldsymbol{V}_{2^j}$ the best 2^j-approximation of $\boldsymbol{V}$. For later convenience, write

$$\boldsymbol{V}_{[j]} := \boldsymbol{V}_{2^j} - \boldsymbol{V}_{2^{j-1}},\ \text{and}\ \boldsymbol{V}_{[0]} := \boldsymbol{V}_{2^0}.$$

The idea is to approximate the infinitely supported $\boldsymbol{A}\boldsymbol{V} \in \ell_2(\mathcal{J})$ by an expression of the form

$$\boldsymbol{W}_j := \boldsymbol{A}_j \boldsymbol{V}_{[0]} + \boldsymbol{A}_{j-1}\boldsymbol{V}_{[1]} + \cdots + \boldsymbol{A}_0 \boldsymbol{V}_{[j]}, \tag{3.22}$$

with the matrices $\boldsymbol{A}_j$ chosen in accordance with Definition 3.3.2. The following error estimate can then be proven:

$$\|\boldsymbol{A}\boldsymbol{V} - \boldsymbol{W}_k\|_{\ell_2(\mathcal{J})} \leq c\|\boldsymbol{V} - \boldsymbol{V}_{2^k}\|_{\ell_2(\mathcal{J})} + \sum_{j=0}^{k} a_j \|\boldsymbol{V}_{[k-j]}\|_{\ell_2(\mathcal{J})} \tag{3.23}$$

where the a_j are upper bounds for $\|\boldsymbol{A} - \boldsymbol{A}_j\| \leq \alpha_j 2^{-sj}$. Recall from Lemma 3.2.3 that if $\boldsymbol{V} \in \ell_\tau^w(\mathcal{J})$, then $\|\boldsymbol{V} - \boldsymbol{V}_{2^k}\| \lesssim 2^{-k}$.

ALGORITHM 3.3.3
APPLY$[\eta, \boldsymbol{A}, \boldsymbol{V}] \mapsto (\boldsymbol{W}, \Lambda)$*:*

(i) Sort the nonzero entries of the finitely supported vector $\boldsymbol{V}$ *and form the vectors* $\boldsymbol{V}_{[0]}, V_{[j]}$ *for* $j = 0, \ldots, \lfloor \log N \rfloor$, $N := \#supp\boldsymbol{V}$. *Define for* $j > \log N$ $\boldsymbol{V}_{[j]} = 0$.

(ii) Compute $\|\boldsymbol{V}_{[j]}\|_{\ell_2(\mathcal{J})}$, $j = 0, \ldots, \lfloor \log N \rfloor + 1$, *and*

$$\|\boldsymbol{V}\|^2_{\ell_2(\mathcal{J})} = \sum_{j=0}^{N} \|\boldsymbol{V}_{[j]}\|^2_{\ell_2(\mathcal{J})}.$$

(iii) Set $k = 0$.

(a) Compute the right hand side R_k *of (3.23), for the given* k.

(b) If $R_k < \eta$, *stop and output* k, *otherwise* $k := k + 1$, *and return to (a).*

(iv) For the output k *of (iii) and for* $j = 0, \ldots, k$, *compute nonzero entries in the matrices* $\boldsymbol{A}_{k-j}$ *which have a column index in common with one of the nonzero entries of* $\boldsymbol{V}_{[j]}$.

(v) For the output k *of (iii), compute* $\boldsymbol{W}_k$ *as in (3.22), and let* $\boldsymbol{W} := \boldsymbol{W}_k$ *and* $\Lambda = supp\boldsymbol{W}$.

On the efficiency of the **APPLY** algorithm, we have the following result [25]:

REMARK 3.3.4 *If* $\boldsymbol{A}$ *is* s^* *compressible, then for* $s < s^*$ *and* s, τ *related by (3.15), the following holds. For any given tolerance* $\eta > 0$ *and any vector* $\boldsymbol{V}$ *with finite support, the output* $\boldsymbol{W}_\eta, \Lambda_\eta$ *of* $\mathbf{APPLY}[\eta, \boldsymbol{A}, \boldsymbol{V}]$ *has the following properties: There exists a constant* $C > 0$ *(depending on* s *as it tends to* ∞*) such that*

(i) The size of the output Λ_η *is bounded by* $\#(\Lambda_\eta) \leq C\|\boldsymbol{V}\|^{1/s}_{\ell^w_\tau(\mathcal{J})}\eta^{-1/s}$.

(ii) The number of arithmetic operations needed to compute $\boldsymbol{W}_\eta$ *does not exceed*

$$C\left(\eta^{-1/s}\|\boldsymbol{V}\|^{1/s}_{\ell^w_\tau(\mathcal{J})} + N\right),$$

where $N = \#supp\boldsymbol{V}$.

(iii) The number of sorting operations needed to compute $\boldsymbol{W}_\eta$ *does not exceed* $CN \log N$.

(iv) The output vector $\boldsymbol{W}_\eta$ *satisfies* $\|\boldsymbol{W}\|_{\ell^w_\tau(\mathcal{J})} \leq C\|\boldsymbol{V}\|_{\ell^w_\tau(\mathcal{J})}$.

An operator $\boldsymbol{A}$ for which an **APPLY** routine can be devised, with the properties of the above remark, is said to be s^*-*admissible*. Hence s^*-compressible operators are clearly s^*-admissible.

3.3.4 The adaptive algorithm

We assume that the following routines are at our disposal:
$\mathbf{RHS}[\eta, \boldsymbol{F}] \to (\boldsymbol{F}_\eta, \Lambda)$
determines for a positive tolerance η a finitely supported sequence $\boldsymbol{F}_\eta$ with support $\Lambda \subset \mathcal{J}$ such that

$$\|\boldsymbol{F} - \boldsymbol{F}_\eta\| \leq \eta. \tag{3.24}$$

$\mathbf{COARSE}[\eta, \boldsymbol{W}] \to (\overline{\boldsymbol{W}}, \Lambda)$
determines for a finitely supported $\boldsymbol{W}$ an approximation $\overline{\boldsymbol{W}}$ such that

$$\|\boldsymbol{W} - \overline{\boldsymbol{W}}\| \leq \eta.$$

Without displaying the details of **COARSE**, its role is to approximate a given finitely supported vector by discarding insignificant (small in magnitude) entries, eventually ensuring better complexity estimates.

Let us now display the adaptive algorithm [1].

ALGORITHM 3.3.5

(i) Initialization: Fix the target accuracy ε and set

$$\overline{\boldsymbol{U}}^0 = 0,\ \Lambda_0 = \emptyset,\ \varepsilon_0 = c_M^{-1}\hat{C}\|\boldsymbol{F}\|_{\ell_2},\ j = 0.$$

(ii) if $\varepsilon_j \leq \hat{c}\varepsilon$, stop and accept $\overline{\boldsymbol{U}}^j$ as solution. Otherwise, set $\boldsymbol{V}^0 = \overline{\boldsymbol{U}}^j$.

(ii.1) For $l = 0, \ldots, K-1$, compute

$$\mathbf{RHS}[\rho^l\varepsilon_j, \boldsymbol{F}] \to (\boldsymbol{F}_l, \Lambda_{l,\boldsymbol{F}})$$
$$\mathbf{APPLY}[\rho^l\varepsilon_j, \boldsymbol{A}, \boldsymbol{V}^l] \to (\boldsymbol{W}^l, \Lambda_{l,\boldsymbol{A}})$$

as well as

$$\boldsymbol{V}^{l+1} = \boldsymbol{V}^l + \alpha(\boldsymbol{F}_l - \boldsymbol{W}^l).$$

(ii.2) Apply $\mathbf{COARSE}[\varepsilon_j, \boldsymbol{V}^K] \to (\overline{\boldsymbol{U}}^{j+1}, \Lambda_{j+1})$, set $\varepsilon_{j+1} = \varepsilon_j/2$, $j+1 \to j$, and go to (ii).

Under the assumptions (3.19),(3.20),(3.24), (3.21) the algorithm is proven to converge [25, prop. 4.2], in that the error $\|\boldsymbol{U} - \overline{\boldsymbol{U}}\|$ decreases with a fraction $\frac{1}{2}$ for each iteration over j.

3.3.5 Convergence rates and work estimates

In this subsection, we present the concepts used for the analysis of the adaptive wavelet-based algorithm. Assumptions on certain parameters are mentioned, but their relation to wavelet bases and other concepts is postponed until the next section.

The following properties will be *assumed* on **RHS**.

ASSUMPTION 3.3.6 *There exists some $s^* > 0$ such that, whenever for some positive $s < s^*$ the solution $\boldsymbol{U}$ to equation (3.18) belongs to $\ell_\tau^w(\mathcal{J})$ with τ according to (3.15), then, for any $\eta > 0$, the output $(\boldsymbol{F}_\eta, \Lambda_\eta)$ of $\mathbf{RHS}[\eta, \boldsymbol{F}]$ satisfies*

$$\|\boldsymbol{F}_\eta\|_{\ell_\tau^w(\mathcal{J})} \leq C\|\boldsymbol{F}\|_{\ell_\tau^w(\mathcal{J})},$$

and

$$\#(\Lambda_\eta) \leq C\|\boldsymbol{F}\|^{1/s}_{\ell_\tau^w(\mathcal{J})}\eta^{-1/s},$$

where $\boldsymbol{A}$ depends only on s when s tends to infinity. Moreover, the number of arithmetic operations needed to compute $\boldsymbol{F}_\eta$ remains proportional to $\#(\Lambda_\eta)$, and the number of sort operations stays bounded by some fixed multiple of $\#(\Lambda_\eta)\log(\#(\Lambda_\eta))$.

On the calculation of entries in the matrix $\boldsymbol{A}$, we have

[1] Recall the constants in (3.19)

ASSUMPTION 3.3.7 *Each entry in* $\boldsymbol{A}$ *is computable at unit cost.*

The main result of [25] is the following:

THEOREM 3.3.8 *Let* $\boldsymbol{A}$ *be an isomorphism on* $\ell_2(\mathcal{J})$*, let* $\boldsymbol{F} \in \ell_2(\mathcal{J})$*, and consider the equation*

$$\boldsymbol{A}\boldsymbol{U} = \boldsymbol{F}.$$

Assume (3.19),(3.20), and the properties of **RHS** *around equation (3.24), and make moreover the Assumptions 3.3.6, 3.3.7. Then Algorithm 3.3.5 converges, that is, given any accuracy* $\varepsilon > 0$ *to the solution* $\boldsymbol{U}$ *of (3.18), the algorithm produces after a finite number of steps an approximation* $\overline{\boldsymbol{U}}$ *such that* $\|\boldsymbol{U} - \overline{\boldsymbol{U}}\|_{\ell_2} \leq \varepsilon$*. Moreover, if, for some* s^**,* $\boldsymbol{A}$ *is* s^**-compressible, assumption 3.3.6 hold for* s^**, and if* $\boldsymbol{U}$ *has an error of best* N*-term approximation of* $\mathcal{O}(N^{-s})$ *for some* $s < s^*$*, then*

(i) The support size of $\overline{\boldsymbol{U}}$ *is bounded by*

$$\#supp\overline{\boldsymbol{U}}(\varepsilon) = \#\Lambda(\varepsilon) \leq C\|\boldsymbol{U}\|^{1/s}_{\ell^w_\tau(\mathcal{J})}\varepsilon^{-1/s},$$

where C *depends on* s*, and on the involving constants.*

(ii) One has

$$\|\overline{\boldsymbol{U}}\|_{\ell^w_\tau(\mathcal{J})} \leq C\|\boldsymbol{U}\|_{\ell^w_\tau(\mathcal{J})},$$

with C *independent on* ε*.*

(iii) The number of arithmetic operations needed to compute $\overline{\boldsymbol{U}}$ *is bounded by* $C\varepsilon^{-1/s}$ *arithmetic operations and* $C\varepsilon^{-1/s}|\log\varepsilon|$ *sort operations with* C *as above.*

This theorem displays the work balance of the algorithm, i.e. how the decay of error and the amount of computational work depends on $N = \#\Lambda$, where Λ is the support of the produced approximation $\overline{\boldsymbol{U}}$, i.e. the set of involved degrees of freedom. Firstly, as N increases, the error decays as N^{-s}, where s depends on the compressibility of the operator $\boldsymbol{A}$, the s^* in assumption 3.3.6, and the *sparsity* of $\boldsymbol{U}$, i.e. to which ℓ^w_τ space it belongs. The latter will in the next section be related to a regularity measure of the solution of the underlying PDE. Note also what is called the *asymptotical optimality* of the algorithm, i.e. it realizes the optimal convergence rate as $N \to \infty$, while requiring the lowest possible ($\mathcal{O}(N)$) floating point operations.

3.4 Wavelets, efficiency and regularity

In this section, we investigate further the properties of wavelets that relate to the approximation theoretical concepts and assumptions discussed in the previous section.

Regularity of solution, decay rate of error

The role of regularity theory of PDEs is, given information on the differential operator, the given data, and the domain, to state smoothness properties of solutions, in terms of appropriate smoothness spaces, in particular parameters describing the smoothness in an appropriate sense. When this is known, one can relate it to numerical methods, and if these

are efficient, one can prescribe an optimal convergence rate. Recall, for instance, the discussion in Example 3.1.5.

We shall now relate the convergence rate of Algorithm 3.3.5 to the smoothness of the solution of the PDE in question. The assumption in Theorem 3.3.8 that '$\boldsymbol{U}$ has an error of best N-term approximation of $\mathcal{O}(N^{-s})$ for some $s < s^*$', i.e. $\boldsymbol{U} \in \ell_\tau^w(\mathcal{J})$, relates to *smoothness properties* of the solution u of Problem 3.1.1, measured in terms of Besov space. The connection is Proposition 3.2.3 stating that $\boldsymbol{U} \in \ell_\tau^w(\mathcal{J}) \subset \ell_\tau(\mathcal{J})$, and Theorem 2.7.2, connecting $\ell_\tau(\mathcal{J})$ to the scale of Besov spaces $B_\tau^s(L_\tau(\Omega))$.

To be more specific, consider Problem 3.1.1 posed in the energy space $H = H^t(\Omega)$. We can express the solution

$$u = \check{\boldsymbol{c}}^T \Psi,$$

where we assume that Ψ fulfills the assumptions of Theorems 2.3.2 and 2.7.2. From Theorem 3.3.1, the output of the algorithm is finite approximations to the *scaled* vector $\boldsymbol{U} = \boldsymbol{D}\check{\boldsymbol{c}}$, with $\boldsymbol{D} = \mathrm{diag}(2^{t|\lambda|})$. Via Theorem 2.7.2, it can be proven that if $\boldsymbol{U} \in \ell_\tau(\mathcal{J})$, which by Proposition 3.2.3 follows from the above assumption, then

$$u = \check{\boldsymbol{c}}^T \Psi \in B_\tau^{sn+t}(L_\tau(\Omega)). \tag{3.25}$$

Hence, functions belonging to $B_\tau^{sn+t}(L_\tau(\Omega))$ have *scaled* wavelet coefficients in $\ell_\tau(\mathcal{J})$, with the assumed decay $\mathcal{O}(N^{-s})$ in $\ell_2(\mathcal{J})$ of best N-term approximation, thus fulfilling the above mentioned assumption of Theorem 3.3.8. This corresponds to an error decay of $\mathcal{O}(N^{-s})$ of $u - \boldsymbol{D}^{-1}\overline{\boldsymbol{U}}_N^T \Psi$, measured in the energy norm of $H^t(\Omega)$, where $\overline{\boldsymbol{U}}_N$ is an approximation to $\boldsymbol{U}$ with N nonzero entries. If one measured the error in the weaker norm $L_2(\Omega)$, the decay would, of course, be considerably faster.

Whether the use of an adaptive method is recommendable, can now (in principle) be determined by regularity theory of PDEs. Thereby, one (ideally) obtains information about the maximal Sobolev smoothness parameter s_S such that the solution $u \in H^{s_S}(\Omega)$, and the maximal Besov smoothness parameter s_B such that $u \in B_\tau^{s_B}(L_\tau(\Omega))$. If $s_B >> s_S$, then the use of an adaptive method is suggested, since it would produce higher convergence rates than a linear method.

Compressibility

On compressibility of wavelet representations of operators, one has the following result, which can be found in [24]:

PROPOSITION 3.4.1 *Let $\boldsymbol{A}$ be a wavelet representation of an operator, so that*

$$|\boldsymbol{A}_{\lambda,\lambda'}| \lesssim \frac{2^{-||\lambda|-|\lambda'||\sigma}}{\left(1 + 2^{min(|\lambda|,|\lambda'|)}\, dist(\sigma_\lambda, \sigma_{\lambda'})\right)^\beta} \tag{3.26}$$

where $\sigma_\lambda = supp\psi_\lambda$. For

$$s^* := \min\left\{\frac{\sigma}{n} - \frac{1}{2}, \frac{\beta}{n} - 1\right\},$$

the matrix $\boldsymbol{A}$ is s^*-compressible.

In the literature, wavelet realizations that satisfy (3.26) for some $\sigma > n/2$ and $\beta > n$ are said to be *quasi-sparse*. Wavelet realizations of a wide range of operators are known to be quasi-sparse, see [39], [35], and the references therein. In general, σ depends on the regularity of the wavelet basis, and β depends on the order of the underlying operator, the spatial dimension n of the domain Ω, and the order of vanishing moments of the wavelet basis.

As an example [23],[39], take a wavelet basis Ψ for $H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^n$, and consider the wavelet representation of the Laplacian: $\check{\boldsymbol{A}} = (\nabla\Psi, \nabla\Psi)$. With the matrix $\boldsymbol{D}^{-1}$ from (2.54), define $\boldsymbol{A} := \boldsymbol{D}^{-1}\check{\boldsymbol{A}}\boldsymbol{D}^{-1}$. Suppose that $\nabla\psi_\lambda$ has m' vanishing moments, such that for any vector valued function P with components being polynomials of order m', $(P, \nabla\psi_\lambda)_{0,\Omega} = 0$. Then, for $|\lambda| \geq |\lambda'|$,

$$(\nabla\psi_\lambda, \nabla\psi_{\lambda'})_{0,\Omega} = ((\nabla\psi_{\lambda'} - P), \nabla\psi_\lambda)_{0,\Omega} \leq \|\nabla\psi_\lambda\|_{1,\Omega}\|\nabla\psi_{\lambda'} - P\|_{\infty,\sigma_\lambda}.$$

where $\sigma_\lambda = \mathrm{supp}(\psi_\lambda)$. If $\sigma' > 0$ is the Hölder continuity exponent [31] of $\nabla\psi_{\lambda'}$, one can estimate

$$\|\nabla\psi_{\lambda'} - P\|_{\infty,\sigma_\lambda} \lesssim 2^{-|\lambda|\sigma'}2^{|\lambda'|(1+\sigma')}2^{\frac{n}{2}|\lambda'|}, \quad \|\nabla\psi_\lambda\|_{1,\Omega} \lesssim 2^{-n|\lambda|}2^{\frac{n}{2}|\lambda}2^{|\lambda|},$$

and from this the estimate

$$2^{-(|\lambda|+|\lambda'|)}|(\nabla\psi_\lambda, \nabla\psi_{\lambda'})| \lesssim \begin{cases} 2^{(\sigma'+n/2)(|\lambda|-|\lambda'|)}, & \text{if } \sigma_\lambda \cap \sigma_{\lambda'} \neq \emptyset \\ 0, & \text{if } \sigma_\lambda \cap \sigma_{\lambda'} = \emptyset \end{cases} \tag{3.27}$$

follows, where $\sigma_{\lambda'} = \mathrm{supp}(\psi_{\lambda'})$. Consequently, the matrix $\boldsymbol{A}$ satisfies Proposition 3.4.1 for $\sigma = \sigma' + n/2$ and *any* β. We emphasize that the *scaled* matrix $\boldsymbol{A}$ is compressible, whereas the unscaled matrix $\check{\boldsymbol{A}}$ is not.

3.5 Final remarks

We conclude by some comments on the methods described above, compared to other methods. As mentioned earlier, in an adaptive finite-element contexts, a significant ingredient is that of an *a-priori error estimator*, the role of which is to estimate the approximation error locally, in order to obtain information about which basis functions are needed in order to improve the current approximation. The methods described above do not make use of a-posteriori error estimators. The reason is the employment of the new paradigm, where the adaptivity does not enter through a-posteriori error estimates, but in the adaptive approximation of the action of an operator on an infinitely-dimensional space to a finitely supported vector, taking place in the routine **APPLY**.

Also, the connection between on one hand the performance of the algorithm, i.e. convergence speed and work balance, and on the other hand tractable properties of the underlying problem, namely the compressibility of wavelet realizations of the operator in question, and the regularity of the wavelet bases employed, and the regularity of the solution, measured in terms of a scale of Besov spaces, deserves to be emphasized.

It should be mentioned that regularity theory of PDEs that measure smoothness in the range of Besov spaces relevant for the adaptive wavelet-based methods is not that well studied. We can refer to regularity studies of the Laplace problem [56], and the Stokes problem [33].

Once more, we would like to stress that the applicability of the adaptive wavelet-based method of [25] is not restricted to variational problems of the type 3.1.1, but also applies to systems of variational problems, such as saddle point problems, a wide range of integral equations, and transport problems, see [25].

Chapter 4

Saddle point formulations of boundary value problems

A saddle point formulation of a boundary value problem is a variational formulation of a PDE that employs two or more bilinear forms, defined on appropriately chosen Hilbert spaces. One of the bilinear forms represents the operator in question, such as for instance the Laplacian in example 3.1.5. The other bilinear form(s) can be employed in order to impose additional constraints, which could be necessary or convenient for a given situation.

This chapter introduces the concept of a saddle point formulation, and outlines the theory governing well-posedness and discretizations of such formulations, which can be found in detail in [13], [11]. In particular, we focus on one particular formulation, namely the Fictitious Domain formulation. In short, the Fictitious Domain formulation offers a simple and attractive alternative in situations where one faces a difficult task in constructing bases over geometrically complicated and/or moving domains, or incorporating complicated boundary conditions in basis functions. Recall for instance the discussion in Section 2.6.

4.1 Saddle point formulations

We adopt in this section the framework and terminology of Section 3.1. Let H and M be Hilbert spaces, subspaces of L_2, with L_2 duals H' resp. M'. Let $a(\cdot,\cdot) : H \times H \mapsto \mathbb{K}$ and $b(\cdot,\cdot) : H \times M \mapsto \mathbb{K}$ be continuous bilinear forms, where in this section, $\mathbb{K} = \mathbb{R}$. Define the three operators $A \in \mathcal{L}(H, H')$, $B \in \mathcal{L}(H, M')$ and $B' \in \mathcal{L}(M, H')$ as follows

$$\begin{aligned} \langle Au, v\rangle_{H'\times H} &:= a(u, v), \\ \langle Bu, p\rangle_{M'\times M} &:= b(u, p), \\ \langle u, B'p\rangle_{H\times H'} &:= b(v, p). \end{aligned}$$

A saddle point problem is a variational problem of the following form.

PROBLEM 4.1.1 *Given $f \in H'$ and $g \in M'$ in the range of B, find $(u, p) \in H \times M$, such that*

$$a(u, v) + b(v, p) = \langle f, v\rangle \ \forall v \in H, \tag{4.1}$$

$$b(u, q) = \langle g, q\rangle \ \forall q \in M. \tag{4.2}$$

Written in operator form, it reads as follows:

$$\mathcal{L}U := \begin{pmatrix} A & B' \\ B & \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \text{ in } H' \times M'. \tag{4.3}$$

4.1.1 Well-posedness

The fundamental result on well-posedness of mixed formulations is the following theorem, due to Brezzi [13]:

THEOREM 4.1.2 *If the inf-sup conditions*

$$\inf_{v \in ker(B)} \sup_{u \in ker(B)} \frac{|a(u,v)|}{\|u\|_H \ \|v\|_H} \geq \alpha > 0 \ , \tag{4.4}$$

$$\inf_{u \in ker(B)} \sup_{v \in ker(B)} \frac{|a(u,v)|}{\|u\|_H \ \|v\|_H} \geq \alpha > 0 \ , \tag{4.5}$$

$$\inf_{p \in M} \sup_{u \in H} \frac{|b(u,p)|}{\|u\|_H \ \|p\|_{M/ker(B')}} \geq \beta > 0 \tag{4.6}$$

hold, then Problem 4.1.1 is well-posed in the sense that the operator $\mathcal{L}$ induces an isomorphism of $H \times M/ker(B)$ onto its dual $H' \times (M/ker(B))'$.

REMARK 4.1.3 *It should be noted that coercivity of $a(\cdot,\cdot)$ on H is not required for the inf-sup conditions (4.4),(4.5) to hold; coercivity on the kernel of B suffices.*

The inf-sup conditions (4.4), (4.5), (4.6) are equivalent to, respectively, the restriction of A to $\ker(B)$ being onto, being one-to-one, and $\mathrm{Ran}(B)$ being closed in M'.

Note that well-posedness in this sense only implies uniqueness of the solution up to addition of elements in $\ker(B')$, which is nontrivial if and only if B is not onto. If B is onto, then the solution of Problem 4.1.1 is unique in $H \times M$. Hence the claim of Theorem 4.1.2 can be stated as

$$\|U\|_{H \times M} \sim \|\mathcal{L}U\|_{H' \times M'}. \tag{4.7}$$

This is the *mapping property* for saddle point problems.

Well-posedness over the complex scalar field

In [13], well-posedness is proven in the case where the scalar field $\mathbb{K} = \mathbb{R}$. Since we shall later consider the case $\mathbb{K} = \mathbb{C}$, we present the ingredients necessary to prove well-posedness in this case as well. An inspection of the proof in [13, p. 26-42] reveals that the claim follows from the Riesz representation theorem, the Lax-Milgram theorem (both theorems hold for $\mathbb{K} = \mathbb{C}$), and the following

PROPOSITION 4.1.4 *Let H, Q be Hilbert spaces, and let $B \in \mathcal{L}(H, Q')$ and $B' \in \mathcal{L}(Q, H')$. The following assertions are equivalent:*

(i) $Im(B)$ is closed in Q'.

(ii) $Im(B')$ is closed in H'.

(iii) $(Ker(B))^{\perp} = Im(B')$.

(iv) ${}^{\perp}(Ker(B')) = Im(B)$.

(v) $\exists k > 0$ *such that for any* $g \in Im(B)$ *there is a* $v_g \in H$ *such that* $Bv_g = g$ *and* $\|v_g\|_V \leq k\|g\|_{Q'}$.

(vi) $\exists k > 0$ *such that for any* $g \in Im(B')$ *there is a* $v_g \in Q$ *such that* $Bv_g = g$ *and* $\|v_g\|_Q \leq k\|g\|_{H'}$.

To see that this proposition is equally valid for $\mathbb{K} = \mathbb{C}$, we cite four classical theorems from functional analysis, valid for the complex case, and from which the proposition can easily be inferred. For proofs, see for instance [69]. Still supposing that H and Q are Hilbert spaces, hence reflexive, we let R and S be subspaces of H and H', respectively, where H' is the dual of H. Define

$$R^{\perp} := \{v' \in H' : \ \langle v, v' \rangle = 0 \ \forall v \in R\}, \tag{4.8}$$

$${}^{\perp}S := \{v \in H : \langle v, v' \rangle = 0 \ \forall v' \in S\}. \tag{4.9}$$

THEOREM 4.1.5 *$R^{\perp}$ is weak-star closed in H', and ${}^{\perp}S$ is norm closed in H. ${}^{\perp}(R^{\perp})$ is the norm closure of R in H, and $({}^{\perp}S)^{\perp}$ is the weak-star closure of S in H'.*

THEOREM 4.1.6 *If X, Y are Banach spaces, and $T \in \mathcal{L}(X, Y)$, then the following assertions are equivalent:*

(i) *$Im(T)$ is closed in Y.*

(ii) *$Im(T')$ is weak-star closed in X'.*

(iii) *$Im(T')$ is norm closed in X'.*

An operator T mapping a Banach space X into another Banach space Y is called *open* if for each open $U \subset X$, $T(U)$ is open in Y.

THEOREM 4.1.7 *If X, Y are Banach spaces, and $T \in \mathcal{L}(X, Y)$ is onto, then T is open, and there exists a $k > 0$ such that for each $f \in Y$ there exists a $v \in X$ with $Tv = f$ and $\|v\|_X \leq k\|f\|_Y$.*

THEOREM 4.1.8 *If X, Y are Banach spaces and $T \in \mathcal{L}(X, Y)$, then*

$$Ker(T') = (Im(T))^{\perp}, \quad ker(T) = {}^{\perp}Im(T') \tag{4.10}$$

Employing (4.8), (4.9), and the four theorems just mentioned, one easily proves Proposition 4.1.4.

By definition of the dual norm, the absolute value of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ in the conditions (4.4)-(4.6) is necessary in the complex case, whereas it is obviously not necessary in the real valued case.

4.2 Discretization, well-posedness and error analysis

For the numerical approximation of the solution of a saddle-point problem, we introduce

PROBLEM 4.2.1 *Let H_h and M_h be finite dimensional subspaces of H and M, respectively. Given $(f, g) \in H' \times M'$, find $(u_h, p_h) \in H_h \times M_h$ such that*

$$a(u_h, v) + b(v, p_h) = \langle f, v \rangle \quad \forall v \in H_h, \tag{4.11}$$
$$b(u_h, q) = \langle g, q \rangle \quad \forall q \in M_h. \tag{4.12}$$

On well-posedness, we have the following proposition [13, p.52]:

PROPOSITION 4.2.2 *Assume that*

$$\{v_h \in H_h : \ b(v_h, q_h) = \langle g, q_h \rangle_M, \ \forall q_h \in Q_h\} \neq \emptyset,$$

and that (4.4) holds with $ker(B)$ replaced by

$$ker(B_h) := \{v_h \in H_h : \ b(v_h, q_h) = 0 \ \forall q_h \in M_h\}.$$

The Problem 4.2.1 has a unique solution $(u_h, p_h) \in H_h \times Q_h / ker(B_h)$.

A Cea-type result holds for the error estimates in this case as well [13].

THEOREM 4.2.3 *Suppose that the so-called discrete LBB condition holds, i.e. that (4.4), (4.5), (4.6) hold with H, M, , $ker(B)$, $ker(B')$ replaced by, respectively, H_h, M_h, $ker(B_h)$ defined above, and*

$$ker(B_h') := \{q_h \in M_h : \ b(v_h, q_h) = 0 \ \forall v_h \in H_h\}.$$

Then

$$\|u - u_h\|_H + \|p - p_h\|_M \leq C \left(\inf_{v \in H_h} \|u - v\|_H + \inf_{q \in M_h} \|p - q\|_M \right).$$

REMARK 4.2.4 *Bearing in mind the issues of the classical approach to the numerical solution of PDEs, see Subsection 3.1, we see that for saddle point formulations, the conditions that ensure well-posedness of the variational problem do not suffice in order to ensure well-posedness in the discrete setting, i.e. convergence.*

It should also be noted that for the solution of the system of equations arising from a discretization of a saddle-point formulation, methods such as the conjugate gradients (CG) algorithm cannot be used directly, since such systems are only indefinite. Instead, one can apply the so-called *Uzawa algorithm*, which essentially performs a conjugate gradient method on the so-called *Schur complement* of the system matrix. We get back to this later, in Section 4.4.

4.3 Fictitious Domain-Lagrange Multiplier formulation

The Fictitious Domain-Lagrange Multiplier formulation is an example of a saddle point formulation of a boundary value problem, in which the physical domain $\Omega \subset \mathbb{R}^n$ is imbedded into a larger, simpler domain $\Box$, which we choose as an n-dimensional cube. Boundary constraints are imposed weakly, using the bilinear form $b(\cdot,\cdot)$. Taking this approach, one avoids both incorporating boundary constraints in the test- and trial spaces, and adapting basis functions for test- and trial spaces to the underlying domain. Furthermore, the treatment of the operator and the boundary constraints are in a convenient way 'separated', since, for instance, the cases of moving boundaries and/or changing boundary conditions only affects $b(\cdot,\cdot)$ and/or the right hand side denoted by g in Problem 4.1.1, and neither the underlying spaces H, M, nor subspaces for the discretization.

We explain the Fictitious Domain formulation by considering a boundary value problem posed in $H^1(\Omega)$. Let Δ be the Laplacian, and let us consider the following problem:

PROBLEM 4.3.1 *Given $f \in L_2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, solve*

$$\Delta u = f \text{ on } \Omega, \tag{4.13}$$

$$u = g \text{ on } \Gamma. \tag{4.14}$$

Integration by parts applied to $\int_\Omega \Delta uv dx$ for u, v in, say, $C^2(\Omega)$ yields a boundary term, because of the non-homogeneous boundary condition (4.14). Denote by $B : H^1(\Omega) \mapsto H^{1/2}(\Gamma)$ the trace operator. Following [38], we define

$$a(u,v) := (\nabla u, \nabla v)_{0,\Omega}, \quad u, v \in H^1(\Omega) \tag{4.15}$$

$$b(u,q) := \langle Bu, q\rangle_{H^{1/2}(\Gamma)}, \quad u \in H^1(\Omega), q \in (H^{1/2}(\Gamma))', \tag{4.16}$$

and thereby Problem 4.3.1 is recast into

PROBLEM 4.3.2 *Given $f \in L_2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, find $(u,p) \in H^1(\Omega) \times (H^{1/2}(\Gamma))'$ such that*

$$\begin{aligned} a(u,v) \; + b(v,p) &= (f,v)_{0,\Omega} \qquad \forall v \in H^1(\Omega) \\ b(u,q) \qquad\quad &= \langle g, q\rangle_{H^{1/2}(\Gamma)} \; \forall q \in (H^{1/2}(\Gamma))'. \end{aligned}$$

Now, let Ω be imbedded in $\Box$. Thus, the bilinear form $a(\cdot,\cdot)$ acts on functions in $H^1(\Box)$, which we emphasize by writing $a(\cdot,\cdot)_\Box$, and the right hand side must be extended to $f_\Box \in L_2(\Box)$. The *Fictitious Domain formulation* of Problem 4.3.2 reads

PROBLEM 4.3.3 *Given $f_\Box \in L_2(\Box)$ and $g \in H^{1/2}(\Gamma)$, find $(u,p) \in H^1(\Box) \times (H^{1/2}(\Gamma))'$ such that*

$$\begin{aligned} a(u,v)_\Box \; + \; b(v,p) &= (f_\Box, v)_{0,\Box} \qquad \forall v \in H^1(\Box) \\ b(u,q) \qquad\quad\; &= \langle g, q\rangle_{H^{1/2}(\Gamma)} \; \forall q \in (H^{1/2}(\Gamma))'. \end{aligned}$$

It has been shown in [49], that a solution (u,p) of Problem 4.3.3 provides, if it exists, by restriction of u to Ω a solution to Problem 4.3.2. We emphasize once more the separation of the differential operator from the boundary constraints, which is now apparent from Problem

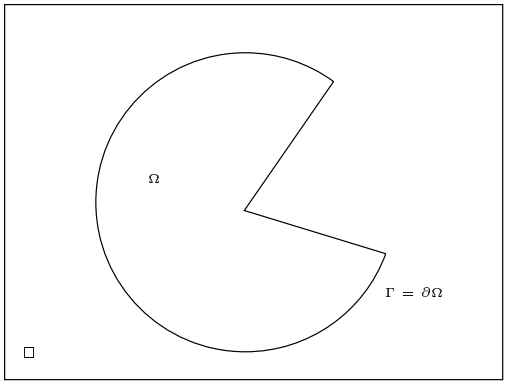

Figure 4.1: The Fictitious Domain-Lagrange Multiplier method

4.3.3. Another remarkable feature of the Fictitious Domain formulation is that it allows the use of *periodic* functions on $\square$.

For a practical realization of the Fictitious Domain formulation, one imposes the boundary constraints in the case of a smooth boundary Γ by taking a parametrization $\kappa : [0,1[\mapsto \Gamma$, and employing periodized wavelets and scaling functions on the real line. For a piecewise smooth boundary, one can take wavelets adapted to each piece, based on the constructions in Section 2.4, and provide the necessary matching conditions across the endpoints, as described in Section 2.6.

Even though we focus on boundary constraints in this section, other constraints can be imposed weakly as well, with obvious modifications. The rest of this section is devoted to reviewing known results of well-posedness of Fictitious Domain Formulations.

4.3.1 Well-posedness of Fictitious Domain Formulations

The material below is borrowed from [57], and is outlined here for the sake of completeness. Let $\Gamma = \partial\Omega$, and let $B : H(\Omega) \mapsto M(\Gamma)$ be a given, bounded trace operator on $H(\Omega)$, where $M(\Gamma)$ is the space of traces of $H(\Omega)$ functions, equipped with the norm $\|\cdot\|_{M(\Gamma)}$. It is often the case that to any trace $h \in M(\Gamma)$, there exists an $f \in H(\Omega)$ with $f|_\Gamma = h$ and $\|f\|_H \lesssim \|h\|_M$ with constants independent of h. This property allows the definition of a bounded right inverse B^{-1} to B, which is called a *continuous lifting*. We shall also need a bounded extension operator $E : H(\Omega) \mapsto H(\mathbb{R}^n)$. The bilinear form $b(\cdot,\cdot) : H(\square) \times M(\Gamma)' \mapsto \mathbb{K}$ is defined as in (4.16). The following lemma from [57] will be useful later when considering well-posedness of the Fictitious Domain-Lagrange Multiplier formulation.

LEMMA 4.3.4 *Suppose that the bounded trace operator $B : H(\Omega) \mapsto M(\Gamma)$ admits a continuous lifting, and that there exists a bounded extension operator $E : H(\Omega) \mapsto H(\mathbb{R}^n)$. Then the inf-sup condition (4.6) holds, i.e. there exists a constant $c > 0$ such that*

$$\inf_{p \in M(\Gamma)'} \sup_{u \in H(\square)} \frac{|b(u,p)|}{\|p\|_{M(\Gamma)'} \|u\|_{H(\square)}} > c.$$

Proof: Let any $p \in M(\Gamma)'$ be given. Using that B is onto, and the continuous lifting and

extension operators, we get

$$\begin{aligned}\|p\|_{M(\Gamma)'} &= \sup_{q\in M, q\neq 0} \frac{|\langle q,p\rangle_{M\times M'}|}{\|q\|_{M(\Gamma)}} \\ &= \sup_{u\in H(\Omega),\ u\neq 0} \frac{|\langle Bu,p\rangle_{M\times M'}|}{\|Bu\|_{M(\Gamma)}} \\ &\lesssim \sup_{u\in H(\Box),\ u\neq 0} \frac{|b(u,p)|}{\|u\|_{H(\Box)}}.\end{aligned}$$

□

We should stress that the choice of extension of the right hand side is of course not unique. Once it is fixed, there is, under appropriate assumptions, a unique solution, as noted in the following theorem that follows from Lemma 4.3.4 and Theorem 4.1.2.

THEOREM 4.3.5 *Assume*

(i) There is a bounded trace operator $B : H(\Omega) \mapsto M(\Gamma)$ with a continuous lifting.

(ii) There is a continuous extension operator $E : H(\Omega) \mapsto H(\mathbb{R}^n)$

(iii) The bilinear form $a(\cdot,\cdot) : H(\Box) \times H(\Box) \mapsto \mathbb{K}$ is bounded and coercive on the kernel of B.

Then Problem 4.3.3 is well-posed, and the mapping property

$$\|U\|_{H\times M'} \sim \|\mathcal{L}U\|_{H'\times M}$$

holds.

For more material on Fictitious Domain methods, we refer to works by Glowinski et.al, for instance [52], [51], [50]. We also note that in the classical setting, the discrete LBB condition can be realized by employing a coarser resolution on Γ than on Ω, see [38].

4.4 Adaptive wavelet methods for saddle point problems

We turn now to the generalization of the adaptive wavelet-based method of sections 3.3 and 3.4 to saddle point problems. The main goal is still to transform the saddle point problem to a well-posed problem in the Euclidean metric, which can be solved adaptively with an algorithm based on the Uzawa algorithm mentioned earlier in this chapter. The main sources of this section are [25] and [32].

4.4.1 Transformation to sequence space

We explain in this section how to transform a saddle point formulation as in Problem 4.1.1 or in (4.3) into a well-posed problem in the Euclidean metric. Let H and M be Hilbert spaces. Suppose that $a(\cdot,\cdot) : H \times H \mapsto \mathbb{R}$ is symmetric and continuous, and let moreover $b(\cdot,\cdot) : M\times H \mapsto \mathbb{R}$ be continuous. Note that in this section, we let, for later convenience, $b(\cdot,\cdot)$ be defined on $M \times H$, and not as in Section 4.1, and in the literature in general, $H \times M$. We

assume that we have to our disposal wavelet bases $\Psi_H = \{\psi_{H,\lambda}\}_{\lambda\in\mathcal{J}_H}$ and $\Psi_M = \{\psi_{M,\lambda}\}_{\lambda\in\mathcal{J}_M}$ for the spaces H and M, inducing the following norm equivalences:

$$\left\|\boldsymbol{c}^T\Psi_H\right\|_H \sim \|\boldsymbol{D}_H\boldsymbol{c}\|_{\ell_2(\mathcal{J}_H)},$$
$$\left\|\boldsymbol{d}^T\Psi_M\right\|_M \sim \|\boldsymbol{D}_M\boldsymbol{d}\|_{\ell_2(\mathcal{J}_M)},$$

for some diagonal transformations $\boldsymbol{D}_H$ and $\boldsymbol{D}_M$. Equivalently, and following the notation of [32], this can be restated as

$$\left\|\boldsymbol{c}^T\boldsymbol{D}_H^{-1}\Psi_H\right\|_H \sim \|\boldsymbol{c}\|_{\ell_2(\mathcal{J}_H)}, \tag{4.17}$$
$$\left\|\boldsymbol{d}^T\boldsymbol{D}_M^{-1}\Psi_M\right\|_M \sim \|\boldsymbol{d}\|_{\ell_2(\mathcal{J}_M)}, \tag{4.18}$$

where we write

$$\boldsymbol{c}^T\boldsymbol{D}_H^{-1}\Psi_H := \sum_{\lambda\in\mathcal{J}_H} c_\lambda d_{H,\lambda}^{-1}\psi_{H,\lambda}$$

where $d_{\lambda,H}$ are the entries in the diagonal matrix $\boldsymbol{D}_H$. Following [32], we can now state

THEOREM 4.4.1 *Consider Problem 4.1.1, and make the following assumptions:*

(i) *Ψ_H and Ψ_M are wavelet bases for H and M, respectively, such that norm equivalences (4.17) and (4.18) hold.*

(ii) *The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ fulfill the inf-sup conditions (4.4), (4.5), and (4.6).*

Then

(i) *The bi-infinite blocks*

$$\boldsymbol{A} := a(\boldsymbol{D}_H^{-1}\Psi_H, \boldsymbol{D}_H^{-1}\Psi_H),$$
$$\boldsymbol{B} := b(\boldsymbol{D}_M^{-1}\Psi_M, \boldsymbol{D}_H^{-1}\Psi_H).$$

generate an operator

$$\boldsymbol{L} := \begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}^T\\ \boldsymbol{B} & \end{pmatrix} : \ell_2(\mathcal{J}) := \ell_2(\mathcal{J}_H)\times\ell_2(\mathcal{J}_M) \mapsto \ell_2(\mathcal{J}) \tag{4.19}$$

which is an isomorphism on the space $\ell_2(\mathcal{J})$.

(ii) *The arrays $\boldsymbol{f} := \boldsymbol{D}_H^{-1}\langle\Psi_H, f\rangle \in \ell_2(\mathcal{J}_H)$ and $\boldsymbol{g} := \boldsymbol{D}_M^{-1}\langle\Psi_M, g\rangle \in \ell_2(\mathcal{J}_M)$.*

(iii) *Problem 4.1.1 is equivalent to the system*

$$\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}^T\\ \boldsymbol{B} & \end{pmatrix}\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{p}\end{pmatrix} = \begin{pmatrix}\boldsymbol{f}\\ \boldsymbol{g}\end{pmatrix}. \tag{4.20}$$

(iv) *If, moreover, $a(\cdot,\cdot)$ is* coercive *on H, then $\boldsymbol{A}$ is an isomorphism on $\ell_2(\mathcal{J}_H)$.*

For an adaptive version of the already mentioned Uzawa algorithm to converge, the block $\boldsymbol{A}$ must be an isomorphism on $\ell_2(\mathcal{J}_H)$. However, as stated in Remark 4.1.3, in order for Problem 4.1.1 to be well-posed, it is not necessary for the bilinear form $a(\cdot,\cdot)$ to be coercive over all of H, so $\boldsymbol{A}$ may not be positive definite. If this is the case, one can employ a so-called *augmented Lagrangian method* and define $\hat{\boldsymbol{A}} = \boldsymbol{A} + c\boldsymbol{B}^T\boldsymbol{B}$ for some $c > 0$, and $\hat{\boldsymbol{f}} = \boldsymbol{f} + c\boldsymbol{B}^T\boldsymbol{g}$, by which one obtains a system

$$\begin{pmatrix} \hat{\boldsymbol{A}} & \boldsymbol{B}^T \\ \boldsymbol{B} & \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{f}} \\ \boldsymbol{g} \end{pmatrix}, \tag{4.21}$$

which is equivalent to the original system (4.20). Under the assumptions of Theorem 4.4.1, $\hat{\boldsymbol{A}}$ is an isomorphism on $\ell_2(\mathcal{I}_H)$ [32], and the block operator in system (4.21 is an ℓ_2 isomorphism as desired.

We assume now that we have established a discrete system on the form of (4.20) where $\boldsymbol{L}$ is an $\ell_2(\mathcal{J})$-isomorphism, and $\boldsymbol{A}$ is an $\ell_2(\mathcal{J}_H)$-isomorphism. Then we can proceed by a block elimination, and define the (infinite dimensional) Schur complement $\boldsymbol{S} := \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}^T$, which is an isomorphism on $\ell_2(\mathcal{I}_M)$ [32]. The system 4.20 is equivalent to

$$\boldsymbol{S}\boldsymbol{p} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{f} - \boldsymbol{g}, \tag{4.22}$$

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{B}^T\boldsymbol{p}. \tag{4.23}$$

An adaptive *Uzawa* algorithm in the spirit of the algorithm in Section 3.3 can be constructed by first letting $\boldsymbol{M} = \boldsymbol{S}$ and solving (4.22) to an appropriate accuracy, and afterwards letting $\boldsymbol{M} = \boldsymbol{A}$, and solving (4.23). We choose not to show here the algorithmic details and the theorems on convergence rates and complexity, instead we refer the reader to [32]. For information on the Uzawa algorithm in general, we refer to [11], [13].

4.4.2 Compressibility

As to the compressibility, we make some remarks on problems posed in sequence space of the form

$$\boldsymbol{M}\boldsymbol{U} = \boldsymbol{F}$$

where $\boldsymbol{F} = (\boldsymbol{F}_1, \ldots, \boldsymbol{F}_m), \boldsymbol{U} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_m) \in \ell_2(\mathcal{I}) := \ell_2(\mathcal{I}_1) \times \ldots \times \ell_2(\mathcal{I}_m)$, and that $\boldsymbol{M}$ consists of blocks denoted $\boldsymbol{M}_{i,j}$, where the $\boldsymbol{M}_{i,j}$ are mappings between $\ell_2(\mathcal{I}_i)$ and $\ell_2(\mathcal{I}_j)$. The space $\ell_2(\mathcal{I})$ is equipped with the norm

$$\|\boldsymbol{U}\|^2_{\ell_2(\mathcal{I})} := \sum_{i=1}^{m} \|\boldsymbol{U}_i\|^2_{\ell_2(\mathcal{I}_i)}. \tag{4.24}$$

We generalize the concept of a best N-term approximation $\boldsymbol{U}_N$ to any $\boldsymbol{U} \in \ell_2(\mathcal{I})$ as follows: Sort the entries in $\boldsymbol{U} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_m)$ in a non-decreasing order (in magnitude) regardless to which $\ell_2(\mathcal{I}_j)$ space they belong, and pick the N largest ones. These are then redistributed out to their components:

$$\boldsymbol{U}_N = (\boldsymbol{U}_{N_1}, \ldots, \boldsymbol{U}_{N_m}), \quad N_1 + \ldots + N_m = N.$$

The definition of $\ell^w_\tau(\mathcal{I})$ and Theorem 3.2.3 carry over without difficulties. The compressibility of the block matrix $\boldsymbol{M}$ follows from the compressibilities of each of the blocks, in that $\boldsymbol{M} \in \mathcal{C}^{s^*}$ for $s^* = \min_{i,j}\{s^*_{i,j} : \ \boldsymbol{M}_{i,j} \in \mathcal{C}^{s^*_{i,j}}\}$.

Closing remarks

We emphasize that by the adaptive wavelet-based approach in [25] and [32], the discrete LBB condition in Theorem 4.2.3 does not enter the picture anywhere. This is due to the fact that the solution of finite-dimensional Galerkin discretizations, as in Problem 4.2.1 is completely avoided, in that one follows the strategy of the new paradigm, which consists in formulating the schemes in the infinite dimensional setting, and approximating the the involved quantities adaptively, to appropriate accuracies.

It should be noted that recently, properties in this direction have been established for finite elements as well [8].

Chapter 5

Foundations of computational electromagnetics

The purpose of this chapter is to present the Maxwell equations as a model for electromagnetic phenomena, and to introduce a mathematical framework, which is useful for the mathematical and numerical analysis of problems from electromagnetics, and which we shall use in the sequel. In Section 5.1, we present the function spaces that are convenient for the mathematical and numerical analysis of the Maxwell equations. In Section 5.2, we present the Maxwell equations as a physical model, and in Section 5.3, we present the mathematical formulations of electromagnetic problems which we shall treat, along with existence, uniqueness and regularity properties. Section 5.4 presents some contemporary methods in the mathematical and numerical analysis of electromagnetic problems. A part of this chapter is based upon [37].

5.1 Function spaces for electromagnetics

In this section, the spaces $\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$ are defined, and the most important properties, needed for our subsequent analysis of electromagnetic problems, are presented. Throughout we shall work with spaces of functions defined on an open domain $\Omega \subset \mathbb{R}^3$, which is in general, except where noted, assumed to be bounded with a Lipschitz continuous boundary Γ. When additional or fewer assumptions are needed, it is stated explicitly.

We consider scalar functions, in general denoted by ordinary letters, as well as n-tuples of scalar functions, called *vector fields*, for $n = 2, 3$. These will be denoted by boldface letters, for instance $\boldsymbol{u} = (u_1, \ldots, u_n)$. Spaces of fields will be denoted by boldface letters as well.

The material in this section is borrowed from [48], except where explicitly noted.

The space $\mathbf{H}(\mathbf{curl};\Omega)$

In the two-dimensional case, for $\Omega \subset \mathbb{R}^2$, the curl operator is defined on $\boldsymbol{\mathcal{D}}(\Omega)$ fields $\boldsymbol{\phi} = (\phi_1, \phi_2) : \Omega \mapsto \mathbb{R}^2$ as

$$\mathbf{curl}\boldsymbol{\phi} := \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2}.$$

In the three-dimensional case, for $\Omega \subset \mathbb{R}^3$, the curl operator is defined on $\boldsymbol{\mathcal{D}}(\Omega)$ fields $\boldsymbol{\phi} := (\phi_1, \phi_2, \phi_3) : \Omega \mapsto \mathbb{R}^3$ as

$$\mathbf{curl}\boldsymbol{\phi} = \left(\frac{\partial \phi_3}{\partial x_2} - \frac{\partial \phi_2}{\partial x_3}, \frac{\partial \phi_1}{\partial x_3} - \frac{\partial \phi_3}{\partial x_1}, \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2} \right).$$

For the rest of this section, we shall treat the case $\Omega \subseteq \mathbb{R}^3$ with 3-dimensional vector fields, and understand that the properties mentioned carry over to the two-dimensional case.

Let $\boldsymbol{f} \in \mathbf{L}_2(\Omega)$. If there exists a function $\boldsymbol{g} \in \mathbf{L}_2(\Omega)$ such that

$$(\boldsymbol{f}, \mathbf{curl}\boldsymbol{\phi})_{0,\Omega} = (\boldsymbol{g}, \boldsymbol{\phi})_{0,\Omega} \quad \forall \boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}(\Omega),$$

then we define $\mathbf{curl}\boldsymbol{f} := \boldsymbol{g}$ and say that $\boldsymbol{f} \in \mathbf{H}(\mathbf{curl}; \Omega)$. Equipped with the inner product

$$(\boldsymbol{f}, \boldsymbol{g})_{\mathbf{H}(\mathbf{curl};\Omega)} := (\boldsymbol{f}, \boldsymbol{g})_{0,\Omega} + (\mathbf{curl}\boldsymbol{f}, \mathbf{curl}\boldsymbol{g})_{0,\Omega},$$

$\mathbf{H}(\mathbf{curl}; \Omega)$ becomes a Hilbert space. We define

$$\mathbf{H}_0(\mathbf{curl}; \Omega) := \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\mathbf{H}(\mathbf{curl};\Omega)}.$$

The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is dense in $\mathbf{H}(\mathbf{curl}; \Omega)$, and the tangential trace mapping

$$\mathcal{D}(\overline{\Omega}) \ni \boldsymbol{v} \mapsto \boldsymbol{\gamma}_\tau(\boldsymbol{v}) := \boldsymbol{v}|_\Gamma \times \boldsymbol{n} \tag{5.1}$$

extends by continuity to a bounded mapping in $\mathcal{L}(\mathbf{H}(\mathbf{curl}; \Omega), \mathbf{H}^{-\frac{1}{2}}(\Gamma))$.

Integration by parts in $\mathbf{H}(\mathbf{curl}; \Omega)$ is a very important tool:

THEOREM 5.1.1

(i) $\forall \boldsymbol{f} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \forall \boldsymbol{g} \in \mathbf{H}(\mathbf{curl}; \Omega)$,

$$(\mathbf{curl}\boldsymbol{f}, \boldsymbol{g})_{0,\Omega} = (\boldsymbol{f}, \mathbf{curl}\boldsymbol{g})_{0,\Omega}. \tag{5.2}$$

(ii) $\forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}; \Omega)$ $\forall \boldsymbol{\phi} \in \mathbf{H}^1(\Omega)$

$$(\mathbf{curl}\boldsymbol{v}, \boldsymbol{\phi})_{0,\Omega} - (\boldsymbol{v}, \mathbf{curl}\boldsymbol{\phi})_{0,\Omega} = \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi} \rangle_\Gamma. \tag{5.3}$$

Note that (5.3) holds for a $\mathbf{H}(\mathbf{curl}; \Omega)$ function and a $\mathbf{H}^1(\Omega)$ function. The formula is not particularly satisfying, since one would, in principle, like to be able to integrate by parts for two $\mathbf{H}(\mathbf{curl}; \Omega)$ functions. However, this turns out to be a delicate matter for general domains [16],[14],[15], essentially due to difficulties in characterizing the space of tangential traces of $\mathbf{H}(\mathbf{curl}; \Omega)$ functions, which is necessary in order to obtain an integration by parts formula where the meaning of the duality form on the boundary is clear.

As in the case of Sobolev spaces, the space $\mathbf{H}_0(\mathbf{curl}; \Omega)$ has an interpretation in terms of boundary conditions, namely

$$\mathbf{H}_0(\mathbf{curl}; \Omega) = \mathbf{ker}(\boldsymbol{\gamma}_\tau) = \{\boldsymbol{f} \in \mathbf{H}(\mathbf{curl}; \Omega) : \ \boldsymbol{f} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\}.$$

The space $\mathbf{H}(\mathrm{div};\Omega)$

The ∇ and div and operators are defined as follows:

$$\phi \in \mathcal{D}(\Omega), \quad \nabla\phi := \left(\frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_n}\right),$$
$$\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}(\Omega), \quad \mathrm{div}\boldsymbol{\phi} := \sum_{i=1}^{n} \frac{\partial\phi_i}{\partial x_i}.$$

Let $\boldsymbol{f} \in \mathbf{L}_2(\Omega)$. If there exists a function $g \in L_2(\Omega)$ such that

$$(\boldsymbol{f}, \nabla\phi)_{0,\Omega} = -(g, \phi)_{0,\Omega} \quad \forall\phi \in \mathcal{D}(\Omega),$$

then we define $\mathrm{div}\boldsymbol{f} := g$ and say that $\boldsymbol{f} \in \mathbf{H}(\mathrm{div};\Omega)$. Equipped with the inner product

$$(\boldsymbol{f}, \boldsymbol{g})_{\mathbf{H}(\mathrm{div};\Omega)} := (\boldsymbol{f}, \boldsymbol{g})_{0,\Omega} + (\mathrm{div}\boldsymbol{f}, \mathrm{div}\boldsymbol{g})_{0,\Omega},$$

the space $\mathbf{H}(\mathrm{div};\Omega)$ becomes a Hilbert space. We shall also need the space

$$\mathbf{V}(\mathrm{div};\Omega) := \{\boldsymbol{f} \in \mathbf{H}(\mathrm{div};\Omega) : \ \mathrm{div}\boldsymbol{f} = 0\}.$$

The space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ is also dense in $\mathbf{H}(\mathrm{div};\Omega)$, and the trace mapping

$$\boldsymbol{\mathcal{D}}(\overline{\Omega}) \ni \boldsymbol{v} \mapsto \gamma_n(\boldsymbol{v}) := \boldsymbol{v} \mapsto \boldsymbol{v}|_\Gamma \cdot \boldsymbol{n}$$

extends by continuity to a bounded surjective mapping in $\mathcal{L}(\mathbf{H}(\mathrm{div};\Omega), H^{-\frac{1}{2}}(\Gamma))$. The following integration by parts formula holds:

THEOREM 5.1.2 $\forall\boldsymbol{v} \in \mathbf{H}(\mathrm{div};\Omega)\ \forall\phi \in H^1(\Omega)$,

$$(\boldsymbol{v}, \nabla\phi)_{0,\Omega} + (\mathrm{div}\boldsymbol{v}, \phi)_{0,\Omega} = \langle \boldsymbol{v}\cdot\boldsymbol{n}, \phi\rangle_\Gamma. \tag{5.4}$$

This formula also yields that for $u \in H^1(\Omega)$ such that $\Delta u \in L_2(\Omega)$, then $\partial u/\partial n \in H^{-\frac{1}{2}}(\Gamma)$ and

$$(\nabla u, \nabla v)_{0,\Omega} = -(\Delta u, v)_{0,\Omega} + \langle(\partial u/\partial\nu), v\rangle_\Gamma. \tag{5.5}$$

We define

$$\mathbf{H}_0(\mathrm{div};\Omega) := \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\|\cdot\|_{\mathbf{H}(\mathrm{div};\Omega)}},$$

and similarly

$$\mathbf{V}_0(\mathrm{div};\Omega) := \{\boldsymbol{f} \in \mathbf{H}_0(\mathrm{div};\Omega) : \ \mathrm{div}\boldsymbol{f} = 0\}.$$

It can be proven that

$$\mathbf{H}_0(\mathrm{div};\Omega) = \mathbf{ker}(\gamma_n) = \{\boldsymbol{f} \in \mathbf{H}(\mathrm{div};\Omega) : \ \boldsymbol{f}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma\}.$$

Hodge decompositions

In the analysis of problems involving the spaces $\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$, it is of great help to have orthogonal decompositions at hand, and such decompositions are the subject of this section.

For the rest of this section we shall, unless otherwise stated, make the following assumptions: We consider the three-dimensional case, where the domain $\Omega \subset \mathbb{R}^3$ is bounded, Lipschitz, connected but not necessarily simply connected. Hence the boundary Γ is allowed to have p components Γ_i, $i = 0, \ldots, p$, where $p = 0$ corresponds to the case where Ω is simply connected. We denote by Γ_0 the exterior boundary. We say that Ω is $\mathcal{C}^{1,1}$ if the boundary locally is the graph of a function whose first derivative is Lipschitz continuous.

First, we present some results that introduce the concept of so-called *potential* functions.

THEOREM 5.1.3 *[48, Thm. 3.6] For every function $\boldsymbol{v} \in \mathbf{L}_2(\Omega)$ that satisfies*

$$\mathrm{div}\boldsymbol{v} = 0 \text{ in } \Omega, \quad \langle \boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ i = 0, \ldots, p,$$

there exists a function $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega)$ so that $\boldsymbol{v} = \mathbf{curl}\boldsymbol{u}$ and

$$\mathrm{div}\boldsymbol{u} = 0 \text{ in } \Omega, \ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \ \text{and } \langle \boldsymbol{u}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ i = 0, \ldots, p. \tag{5.6}$$

If furthermore Ω is simply connected, $\boldsymbol{u}$ is the unique solution of the boundary value problem

$$\begin{cases} -\Delta\boldsymbol{u} = \mathbf{curl}\boldsymbol{v} \quad \text{in } \mathbf{H}^{-1}(\Omega) \\ \mathrm{div}\boldsymbol{u} = 0 \quad \text{in } \Omega \\ \boldsymbol{u} \quad \text{satisfies (5.6).} \end{cases} \tag{5.7}$$

If Ω is a convex polyhedron or has a $\mathcal{C}^{1,1}$ boundary, then $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$.

The function $\boldsymbol{u}$ is called the *vector potential* of $\boldsymbol{v}$.

THEOREM 5.1.4 *If Ω is simply-connected, a function $\boldsymbol{u} \in \mathbf{L}_2(\Omega)$ satisfes $\mathbf{curl}\ \boldsymbol{u} = \boldsymbol{0}$ in Ω if and only if there exists a function $p \in H^1(\Omega)$, unique up to a constant, such that $\boldsymbol{u} = \nabla\ p$.*

The function p is called the *scalar potential* of $\boldsymbol{u}$. Now, one arrives at a so-called Hodge decomposition of $\mathbf{L}_2(\Omega)$, stating that the kernels of the curl and the divergence operators induce an orthogonal decomposition.

COROLLARY 5.1.5 *With*

$$\boldsymbol{\mathcal{H}}_1 = \{\nabla q : \ q \in H^1(\Omega)/\mathbb{R}\} \tag{5.8}$$

$$\boldsymbol{\mathcal{H}}_2 = \{\mathbf{curl}\boldsymbol{f} : \ \boldsymbol{f} \in \mathbf{H}^1(\Omega) : \ \mathrm{div}\boldsymbol{f} = 0, \ (\mathbf{curl}\boldsymbol{f})\cdot\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\} \tag{5.9}$$

one has the orthogonal decomposition

$$\mathbf{L}_2(\Omega) = \boldsymbol{\mathcal{H}}_1 \oplus \boldsymbol{\mathcal{H}}_2. \tag{5.10}$$

For $\boldsymbol{v} \in \mathbf{L}_2(\Omega)$, write $\boldsymbol{v} = \nabla q + \mathbf{curl}\boldsymbol{u}$, then q is the unique solution in $H^1(\Omega)/\mathbb{R}$ of the problem

$$(\nabla q, \nabla\nu)_{0,\Omega} = (\boldsymbol{v}, \nabla\nu)_{0,\Omega} \ \forall\nu \in H^1(\Omega). \tag{5.11}$$

If, moreover, Ω is simply connected, $\boldsymbol{u}$ is the only solution of (5.7).

Imbedding properties

Still considering the three-dimensional case, we shall now describe imbedding properties related to the $\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$. In particular the relation between the spaces $\mathbf{H}^1(\Omega)$ and $\mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$ is of interest in applications. It turns out that the properties of the domain Ω and the boundary Γ are important. The first, simple result, is the following:

REMARK 5.1.6 *[48, p. 35]*

$$\mathbf{H}_0(\Omega)=\mathbf{H}_0(\mathrm{div};\Omega)\cap\mathbf{H}_0(\mathbf{curl};\Omega).$$

Another more interesting result, is the following. Recall that a $\mathcal{C}^{1,1}$ boundary locally lies below a function whose first order derivatives are Lipschitz continuous [48]. If X and Y are normed spaces with $X\subset Y$ and X is imbedded continuously in Y, we write $X\hookrightarrow Y$.

THEOREM 5.1.7 *[48, p. 51] Assume that either Ω is a convex polyhedron, or Γ is a $\mathcal{C}^{1,1}$ boundary. Then the continuous imbedding*

$$\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)\hookrightarrow\mathbf{H}^1_{\mathrm{Tan}}(\Omega):=\{\boldsymbol{u}\in\mathbf{H}^1(\Omega):\ \boldsymbol{u}\times\boldsymbol{n}=\mathbf{0}\ \text{on}\ \Gamma\}$$

holds.

Another imbedding result, which is of importance, is the following, originally due to Weber [78].

THEOREM 5.1.8 *For a bounded Lipschitz domain Ω, the space $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$ is compactly imbedded in $\mathbf{L}_2(\Omega)$.*

From this result, one can prove the following coercivity result [7]:

COROLLARY 5.1.9 *On the space $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega)$, there exists a constant C depending only on the domain Ω and its boundary, such that $\forall\boldsymbol{u}\in\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega)$,*

$$C\|\boldsymbol{u}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}\leq(\mathbf{curl}\boldsymbol{u},\mathbf{curl}\boldsymbol{u})_{0,\Omega}.$$

An integration by parts formula

We mention another integration by parts formula, which is important for the analysis of the Maxwell equations.

THEOREM 5.1.10 ([28]) *If the bounded domain $\Omega\subset\mathbb{R}^3$ is a polyhedron with straight faces, one has for each $\boldsymbol{u},\boldsymbol{v}$ in the space $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$ defined in Theorem 5.1.7, the integration by parts formula*

$$(\mathbf{curl}\boldsymbol{u},\mathbf{curl}\boldsymbol{v})_{0,\Omega}+(\mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v})_{0,\Omega}=(\nabla\boldsymbol{u},\nabla\boldsymbol{v})_{0,\Omega},\tag{5.12}$$

holds, where

$$(\nabla\boldsymbol{u},\nabla\boldsymbol{v})_{0,\Omega}=\int_\Omega\nabla\boldsymbol{u}:\nabla\boldsymbol{v},\quad\nabla\boldsymbol{u}:\nabla\boldsymbol{v}=\sum_{i,j}\partial_iu_j\partial_iv_j.$$

A boundary term enters the formula if the polyhedron is curved, but this will not be important for our purposes. As another important fact, we mention that *for any bounded Liptschitz domain* Ω, the space $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$ is *closed* for the norm of $\mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$ [30] and the references therein. As we shall see later, the solution to the Maxwell equations belong to the space $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$, which, in the general case, is a true subset of $\mathbf{H}^1(\Omega)$. This closedness property of $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$ has for some numerical schemes for the Maxwell equations the consequence that the part of the solution residing outside $\mathbf{H}^1(\Omega)$ is not resolved.

5.2 Maxwells equations

In this subsection, we present the formulations of the Maxwell equations, with which we shall be working. We present the Maxwell equations as a physical model only, and postpone the discussion of mathematical issues of existence, uniqueness and regularity to the next section.

The Time-Dependent Formulation

We consider a domain $\Omega \subset \mathbb{R}^3$, which we assume is occupied by a homogeneous, isotropic and linear [1]dielectric material, characterized by the electric permittivity ε and the magnetic permeability μ, which we assume are functions of spatial coordinates only. A current density $\boldsymbol{J}$ can flow through the medium, and charge with density ρ can be present. Both quantities may depend on space and time. They are related by the fundamental principle of continuity, or charge conservation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, \boldsymbol{J} = 0. \tag{5.13}$$

We denote the electric field by $\boldsymbol{E}$ and the magnetic flux density by $\boldsymbol{B}$. Currents can generate fields and vice versa. Their connections are given by *Maxwell's equations*

$$\mathbf{curl}\ \frac{\boldsymbol{B}}{\mu} = \varepsilon \frac{\partial}{\partial t}\boldsymbol{E} + \boldsymbol{J}, \tag{5.14}$$

$$\mathbf{curl}\ \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}. \tag{5.15}$$

For a perfectly conducting boundary Γ, the following boundary conditions are fulfilled:

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \text{ and } \boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma.$$

Taking the divergence of these equations and using (5.13) yields

$$\mathrm{div}\, \boldsymbol{E} = \frac{\rho}{\varepsilon}, \tag{5.16}$$

$$\mathrm{div}\, \boldsymbol{B} = 0. \tag{5.17}$$

These equations are often included in the set of Maxwell's equations, although they are consequences of (5.14), (5.15) and (5.13).

Another material parameter is the conductivity σ. In the presence of an electric field $\boldsymbol{E}$, the material will conduct a current with density given by Ohm's law:

$$\boldsymbol{J} = \sigma \boldsymbol{E}. \tag{5.18}$$

[1]See for instance [20] for an explanation of these properties

Materials with $\sigma > 0$ resp. $\sigma = 0$ are usually denoted *lossy* resp. *loss free* materials. For the case of positive conductivity, the current density $\boldsymbol{J}$ that enters (5.14), is the total current density, which is composed of (5.18) and perhaps a current density imposed by external sources. As we shall see, the mathematical properties of electromagnetic problems are very different for the two cases $\sigma = 0$ and $\sigma > 0$.

A common procedure for the resolution of Maxwell's equations is the elimination of $\boldsymbol{B}$, [7, 22]. Letting $\boldsymbol{J}$ represent an externally imposed current source, this gives rise to a second order differential equation for determining $\boldsymbol{E}$

$$\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \sigma \frac{\partial \boldsymbol{E}}{\partial t} + \mathbf{curl}\ \frac{1}{\mu} \mathbf{curl}\ \boldsymbol{E} = \frac{\partial \boldsymbol{J}}{\partial t}, \tag{5.19}$$

with initial conditions

$$\begin{aligned} \boldsymbol{E}(\cdot, 0) &= \boldsymbol{E}_0, \qquad \operatorname{div} \boldsymbol{E}_0 = \frac{\rho_0}{\varepsilon}, \\ \frac{\partial \boldsymbol{E}}{\partial t}(\boldsymbol{x}, 0) &= \frac{1}{\varepsilon} \left(-\boldsymbol{J}(\boldsymbol{x}, 0) + \frac{1}{\mu} \mathbf{curl}\ \boldsymbol{B}_0(\boldsymbol{x}) - \sigma \boldsymbol{E}_0(\boldsymbol{x}) \right), \quad \boldsymbol{x} \in \Omega. \end{aligned}$$

In many physics and engineering texts, the relation $\mathbf{curlcurl} = \nabla \operatorname{div} - \boldsymbol{\Delta}$ is often utilized in the common case where $\boldsymbol{E}$ has zero divergence and $\sigma = 0$. Thereby, one can replace the double curl operator by the Laplacian (up to sign), and thus (5.19) gets the form of a wave equation. However, due to regularity properties mentioned on page 70, we shall not employ this approach.

The Time-Harmonic Formulation

Maxwell's equations in the time-harmonic formulation are obtained by assuming that all involved quantities have a sinusoidal time variation with angular frequency ω. If $\boldsymbol{g}(\boldsymbol{x}, t)$ is a field quantity, we describe it as

$$\boldsymbol{g}(\boldsymbol{x}, t) = \operatorname{Re}\left(\tilde{\boldsymbol{g}}(\boldsymbol{x}) \mathrm{e}^{-i\omega t} \right),$$

where $\tilde{\boldsymbol{g}}(\boldsymbol{x})$ is a complex valued amplitude, only depending on spatial variables. For notational simplicity, we shall not write the tilde, but only understand that unless nothing else is said, we work with complex valued fields. Upon inserting $\boldsymbol{E}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ and $\boldsymbol{B}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ into Maxwell's equations in the time-dependent formulation, one obtains

$$\frac{1}{\mu} \mathbf{curl}\, \boldsymbol{B} = -i\varepsilon\omega \boldsymbol{E} + \boldsymbol{J}, \tag{5.20}$$

$$\mathbf{curl}\, \boldsymbol{E} = i\omega \boldsymbol{B}, \tag{5.21}$$

$$\operatorname{div} \boldsymbol{E} = \frac{1}{\varepsilon} \rho, \tag{5.22}$$

$$\operatorname{div} \boldsymbol{B} = 0. \tag{5.23}$$

Once $\boldsymbol{E}$ and $\boldsymbol{B}$ are known, the physical fields are determined by taking the real parts of $\boldsymbol{E}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$ and $\boldsymbol{B}(\boldsymbol{x})\mathrm{e}^{-i\omega t}$.

As in the case of the time-dependent formulation of the Maxwell equations, one can eleminate $\boldsymbol{B}$ by taking the curl of (5.21) and insert into (5.20). This gives the following

problem:

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\left(i\frac{\sigma}{\omega} + \mu\varepsilon\right)\boldsymbol{E} = -i\omega\mu\boldsymbol{J} \tag{5.24}$$

$$\boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{5.25}$$

Employment of the time-harmonic formulation of the Maxwell equations is particularly attractive when fields oscillating at one or a few frequencies are dominant.

5.3 Existence, uniqueness and regularity of solutions

We turn now to treat existence, uniqueness and regularity properties of solutions to the Maxwell equations. We choose in this section to give proofs of some of the claims. This is done in case of statements often used in the literature, but where the author has not found a proof, or in order to emphasize procedures with significant potential outside the realm of electromagnetics.

On existence and uniqueness for the time-dependent formulation, we cite the following result [44, Thm. 2.1]:

THEOREM 5.3.1 *Let $\Omega \subset \mathbb{R}^3$ be a simply-connected, Lipschitz domain. $\sigma = 0$, $\rho \in C^1([0,T], L_2(\Omega))$, $\boldsymbol{J} \in C^1([0,T], \mathbf{L}_2(\Omega)) \cap C([0,T], \mathbf{H}(\mathrm{div};\Omega))$, ρ and $\boldsymbol{J}$ satisfy (5.13), $\boldsymbol{E}_0 := \boldsymbol{E}(\boldsymbol{x},0)$ and $\boldsymbol{B}_0 := \boldsymbol{B}(\boldsymbol{x},0)$ satisfy*

$$\mathrm{div}\boldsymbol{E}_0 = \frac{\rho(\cdot,0)}{\varepsilon},\ \mathrm{div}\boldsymbol{B}_0 = 0\ \in\Omega,\ \boldsymbol{E}_0\times\boldsymbol{n} = \boldsymbol{0} \quad \textit{on } \Gamma.$$

Then the equations (5.14), (5.15), (5.16) and (5.17) along with

$$\boldsymbol{E}\times\boldsymbol{n} = \boldsymbol{0} \quad \textit{on } \Gamma, \frac{\partial}{\partial t}\boldsymbol{B}\cdot\boldsymbol{n} = 0 \quad \textit{on } \Gamma$$

has unique solutions

$$\boldsymbol{E}, \boldsymbol{B} \in C^1([0,T], \mathbf{L}_2(\Omega)) \cap C([0,T], \mathbf{H}(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)).$$

In order to arrive at variational formulations for the Maxwell equations in the time-dependent formulation, one can use the equation (5.19). Let Δt denote some time step. Denote, for a field quantity, $\boldsymbol{u}^n(\cdot) := \boldsymbol{u}(\cdot, n\Delta t)$. Approximating the derivatives with respect to the time variable, using the first, respectively second order backward finite differences:

$$\partial_{\Delta t}\boldsymbol{u}^n := \frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\Delta t}, \quad \partial^2_{\Delta t}\boldsymbol{u}^n := \frac{\partial_{\Delta t}\boldsymbol{u}^n - \partial^{n-1}_{\Delta t}\boldsymbol{u}}{\Delta t}$$

one must for each time instant $n\Delta t$ solve a variational problem of the form: find $\boldsymbol{E}^n \in \mathbf{H}_0(\mathbf{curl};\Omega)$ such that

$$(\mathbf{curl}\,\boldsymbol{E}^n, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + \kappa(\boldsymbol{E}^n, \boldsymbol{v})_{0,\Omega} = (\boldsymbol{f}, \boldsymbol{v})\ \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega) \tag{5.26}$$

where the constant κ is positive, and depends on $\varepsilon, \mu, \sigma, \Delta t$, and $\boldsymbol{f}$ is a field quantity depending on $\boldsymbol{J}$ and values of $\boldsymbol{E}$ at earlier times.

REMARK 5.3.2 *If* $\boldsymbol{J} \in \mathbf{V}(\mathrm{div};\Omega)$, *then any solution* $\boldsymbol{u} \in \mathbf{X}$ *to*

$$(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + \kappa(\boldsymbol{u},\boldsymbol{v})_{0,\Omega} = (\boldsymbol{J},\boldsymbol{v})\ \forall \boldsymbol{v} \in \mathbf{X}$$

where $\boldsymbol{\mathcal{D}}(\Omega) \subset \mathbf{X} \subseteq \mathbf{H}(\mathbf{curl};\Omega)$ *has* $\mathrm{div}\boldsymbol{v} = 0$. *This holds for real valued as well as complex valued fields, and for any* $\kappa \in \mathbb{C}$. *This follows by by choosing* $\boldsymbol{v} = \nabla\phi$ *for any* $\phi \in \mathcal{D}(\Omega)$, *using that gradients have zero curl, and finally the definition of having zero divergence in the weak sense.*

THEOREM 5.3.3 *For each* $\boldsymbol{f} \in (\mathbf{H}_0(\mathbf{curl};\Omega))'$, *the problem of finding* $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ *such that*

$$(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + \kappa(\boldsymbol{u},\boldsymbol{v})_{0,\Omega} = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega) \tag{5.27}$$

is uniquely solvable.

Noting that the bilinear form is obviously coercive on $\mathbf{H}(\mathbf{curl};\Omega)$, the claim follows directly from the Lax-Milgram theorem.

For the time-harmonic problem, we derive a variational formulation of the original problem (5.24) by multiplying with a field $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ and employing (5.2). Note here that from (5.15), $\mathbf{curl}\,\boldsymbol{E} \in \mathbf{H}(\mathbf{curl};\Omega)$. The problem is:

PROBLEM 5.3.4 *Given* $\boldsymbol{J} \in (\mathbf{H}_0(\mathbf{curl};\Omega))'$, *find* $\boldsymbol{E} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ *such that*

$$(\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2\left(i\frac{\sigma}{\omega} + \mu\varepsilon\right)(\boldsymbol{E},\boldsymbol{v})_{0,\Omega} = (\boldsymbol{J},\boldsymbol{v})\ \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega).$$

We split up the treatment in the two cases: $\sigma > 0$ and $\sigma = 0$. First, for $\sigma > 0$, we understand that fields and bilinear forms take on complex values.

Let us now establish the coercivity of the bilinear form in Problem 5.3.4, for a space as large as possible. In [58], it is mentioned without proof that the bilinear form of Problem 5.3.4 is coercive for the space $\mathbf{H}_0(\mathbf{curl};\Omega)$. The author does not know of a proof, so below the coercivity is proven for the, in general, larger space $\mathbf{H}(\mathbf{curl};\Omega)$. We also emphasize that the proof is elementary, it involves essentially only the calculus of polynomials of second degree. Moreover, it does not depend on properties of the domain Ω, so the coercivity holds for *any* domain.

First, we prove the following technical lemma.

LEMMA 5.3.5 *Given* η, $\xi > 0$, *there exists a* $\alpha > 0$ *so that*

$$f(x,y) := x^2 + (\eta^2+\xi^2)y^2 - 2\eta xy - \alpha^2(x^2+y^2+2xy) > 0$$

for all $x, y \in \mathbb{R}$ *such that* $x \geq 0$, $y \geq 0$.

Proof: Restricting f to either of the coordinate axes shows that necessarily

$$\alpha < \min(1, \sqrt{\eta^2+\xi^2}).$$

Moreover, these conditions ensure that f is positive in a neighborhood of any point on either of the coordinate axes. For any fixed $x > 0$, consider the second degree polynomial in y defined by $g_x(y) := f(x,y)$. Rewriting, we get

$$g_x(y) = (\eta^2+\xi^2-\alpha^2)y^2 - 2x(\eta+\alpha^2)y + (1-\alpha^2)x^2.$$

Varying x, we get a family of second degree polynomials, all with a global minimum, whose position depends on x. The claim holds by requiring that for any x, $g_x(y)$ should have no real zeros. This amounts to

$$4x^2(\eta+\alpha^2)^2 - 4x^2(\eta^2+\xi^2-\alpha^2)(1-\alpha^2) < 0,$$

which is equivalent to

$$r(\alpha) := -\alpha^2((\eta^2+1)^2+\xi^2)+\xi^2 > 0.$$

Clearly, $r(\alpha)$ has one positive zero α^*, hence we can choose $0 < \alpha < \min(1, \sqrt{\eta^2+\xi^2}, \alpha^*)$. □

COROLLARY 5.3.6 *For $\sigma > 0$, the bilinear form*

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2\left(i\frac{\sigma}{\omega}+\mu\varepsilon\right)(\boldsymbol{E},\boldsymbol{v})_{0,\Omega}$$

from Problem 5.3.4 is coercive on $\mathbf{H}(\mathbf{curl};\Omega)$, in the sense that there exists a constant C such that for any $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega)$,

$$|a(\boldsymbol{u},\boldsymbol{u})| \geq C\|\boldsymbol{u}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

The coercivity holds for all domains, bounded as well as unbounded.

Proof: For $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega)$, define $x(\boldsymbol{u}) := \|\mathbf{curl}\ \boldsymbol{u}\|^2_{0,\Omega}$, $y(\boldsymbol{u}) := \|\boldsymbol{u}\|^2_{0,\Omega}$ and $\eta := \omega^2\mu\varepsilon$, $\xi := \mu\omega\sigma$. Note that $a(\boldsymbol{u},\boldsymbol{u}) = x(\boldsymbol{u}) - (\eta+i\xi)y(\boldsymbol{u}) = x(\boldsymbol{u}) - \eta y(\boldsymbol{u}) - i\xi y(\boldsymbol{u})$ as well as $\|\boldsymbol{u}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} = x(\boldsymbol{u}) + y(\boldsymbol{u})$. It is then readily seen that the bilinear form $a(\cdot,\cdot)$ is coercive if and only if there exists a constant $\alpha > 0$ such that

$$(x(\boldsymbol{u}) - \eta y(\boldsymbol{u}))^2 + \xi^2 y(\boldsymbol{u})^2 > \alpha^2(x(\boldsymbol{u})^2 + y(\boldsymbol{u})^2 + 2x(\boldsymbol{u})y(\boldsymbol{u})),$$

which follows from Lemma 5.3.5. □

From Corollary 5.3.6, the Lax-Milgram theorem, and Remark 5.3.2, one concludes

THEOREM 5.3.7 *Problem 5.3.4 is uniquely solvable, and if $\boldsymbol{J} \in \mathbf{V}(\mathrm{div};\Omega)$, then $\mathrm{div}\boldsymbol{E} = 0$ on Ω.*

For the case $\sigma = 0$, the linearity of the Maxwell equations allows one to consider real valued fields and bilinear forms only, for which one can pose the following variational problem:

PROBLEM 5.3.8 *Given $\boldsymbol{J} \in \mathbf{V}(\mathrm{div};\Omega)$, find $\boldsymbol{E} \in \mathbf{H}_0(\mathbf{curl};\Omega)$ such that*

$$(\mathbf{curl}\,\boldsymbol{E}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - \omega^2\mu\varepsilon(\boldsymbol{E},\boldsymbol{v})_{0,\Omega} = (\boldsymbol{J},\boldsymbol{v})_{0,\Omega}\ \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega).$$

In the case this problem has a unique solution, one exploits the linearity of the Maxwell equations and multiplies the solution by an appropriate factor of the imaginary unit, thereby obtaining the solution of the original Problem 5.3.4 with $\sigma = 0$, posed for complex valued fields.

We turn now to the solvability of Problem 5.3.8. We state and prove the following theorem, since it gives a canonical approach to proving well-posedness and the mapping property to a given operator equation, such as for instance (3.5).

THEOREM 5.3.9 *Assume* $\boldsymbol{J} \in \mathbf{V}(\mathrm{div};\Omega)$, *and that the problem of finding* $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ *such that*

$$(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} = \omega^2\mu\varepsilon(\boldsymbol{u},\boldsymbol{v})_{0,\Omega} \;\forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega). \tag{5.28}$$

has no nonzero solutions. Then Problem 5.3.8 is uniquely solvable and one has the mapping property

$$\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega)} \sim \|(\mathbf{curl}\,\mathbf{curl} - \omega^2\mu\varepsilon)\boldsymbol{u}\|_{(\mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega))'}, \tag{5.29}$$

$$\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega). \tag{5.30}$$

A number $\omega^2\mu\varepsilon$ such that (5.28) has one or more nonzero solutions is called a *Maxwell eigenvalue.*

Proof: Note that if there is a solution, it must have zero divergence, according to Remark 5.3.2. Consider the operator $C : \mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega) \mapsto (\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega))'$ defined by

$$\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega) \ni \boldsymbol{v} \mapsto (C\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega}.$$

Clearly $C \in \mathcal{L}(\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega), (\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega))')$. From [78], the bilinear form $(\mathbf{curl}\,,\mathbf{curl}\,)_{0,\Omega}$ is coercive on $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$. Since $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ is closed in $\mathbf{H}(\mathbf{curl};\Omega)$, an application of the Lax-Milgram theorem yields that C maps $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ bijectively onto its dual space. This means that C is a Fredholm operator, with Fredholm index zero. The canonical imbedding of $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ into $\mathbf{L}_2(\Omega)$ is compact, hence the imbedding of $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ into its dual is also compact. Hence the operator $C - \lambda I$ also has Fredholm index zero [68, thm 7.103]. If $\omega^2\mu\varepsilon := \lambda$ is not a Maxwell eigenvalue, then $C - \lambda I$ is injective, hence also surjective. Thus there exists constants $0 < d \le D < \infty$ such that for all $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$

$$d||\boldsymbol{u}||_{\mathbf{H}(\mathbf{curl};\Omega)} \le ||(C-\lambda I)\boldsymbol{u}||_{\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega)'} \le D||\boldsymbol{u}||_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

□

Later, we shall establish well-posedness by other means, relying on symmetry of the involved bilinear forms. The interesting aspect of the theorem above is that it generalizes to non-symmetric bilinear forms.

The general formulation

To sum up, we see that the Maxwell equations in the time-dependent as well as the time-harmonic form lead to examinations of a bilinear form

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_{0,\Omega} + \kappa(\boldsymbol{u},\boldsymbol{v})_{0,\Omega} \tag{5.31}$$

where $\boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$, or subspaces hereof. We formulate it as a separate problem, which will serve as a foundation for later developments.

PROBLEM 5.3.10 *Given* $\mathbf{J} \in \mathbf{H}(\mathbf{curl};\Omega)'$, *find* $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega)$ *such that*

$$a(\boldsymbol{u},\boldsymbol{v}) = \langle \mathbf{J},\boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$$

As outlined above, the functional $\mathbf{f}$ is the current density $\boldsymbol{J}$, and perpaps linear combinations of approximations to the electric field $\boldsymbol{E}$ from previous time steps.

On well-posedness of this problem, we make the following

REMARK 5.3.11 *For the time dependent case, fields are real valued, and κ is real and positive. For the time-harmonic case with conductivity $\sigma > 0$, the fields as well as κ are complex valued. For both these cases, the bilinear form is coercive on $\mathbf{H}(\mathbf{curl};\Omega)$, so well-posedness follows immediately from the Lax-Milgram theorem. These conclusions holds as well if one replaces the variational space by $\mathbf{H}_0(\mathbf{curl};\Omega)$. For the time-harmonic, nonconducting ($\sigma = 0$) case, $\kappa < 0$, and the fields can be taken as being real valued. Coercivity of the bilinear form cannot be expected without further assumptions, but this will be treated in more depth below.*

Singularities

From Theorem 5.1.7, one sees that under weak conditions on the domain, the spaces $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$ and $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$ are isomorphic. In the general case, where Ω may be non-convex with a non-smooth boundary, the solution of boundary value problems from electromagnetics belongs to $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$ and *not* to $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$, i.e. electromagnetic fields can have non-$\mathbf{H}^1(\Omega)$ singularities. As we shall see below, when constructing variational formulations for boundary value problems, posed over irregular and non-convex domains, and discretizing them, there is a danger that one may end up approximating only the 'regular' part of the problem, i.e. the part of the solution belonging to $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$. We shall see such examples below.

As already mentioned, when the Maxwell equations are treated in engineering and physics texts, the relation

$$\mathbf{curlcurl}\boldsymbol{u} = \nabla\mathrm{div}\boldsymbol{u} - \boldsymbol{\Delta u}, \tag{5.32}$$

is often used, and for the case of a divergence free vector field in a lossless medium ($\sigma = 0$), the equation (5.19) reduces to the well-known wave equation for the electric field:

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \frac{1}{\mu\varepsilon}\boldsymbol{\Delta E}.$$

However, (5.32) requires sufficient smoothness of the fields. In a variational version, (5.32) reads

$$(\mathbf{curl}\boldsymbol{u}, \mathbf{curl}\boldsymbol{v})_{0,\Omega} + (\mathrm{div}\boldsymbol{u}, \mathrm{div}\boldsymbol{v})_{0,\Omega} = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_{0,\Omega}$$

which by Theorem 5.1.10 holds for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}^1_{\mathrm{Tan}}(\Omega)$ if Ω is a polyhedron with straight faces. Consequently, the relation is not valid when $\boldsymbol{E}$ has non-$\mathbf{H}^1(\Omega)$ singularities, and therefore we do not make use of (5.32) in the equation (5.19).

5.4 Trends in analysis of Maxwells equations

The purpose of this section is to present some current viewpoints and methods for the numerical solution of electromagnetic problems.

5.4.1 Finite difference methods

As in many other areas of physics and engineering science, methods based on finite differences, the most famous one being the *Yee scheme* [80],[71], find applications in computational electromagnetics. The advantages and drawbacks of the method in electromagnetics are also the same as in other areas, most notably the conceptual simplicity and ease of implementation, and similarly the drawbacks, in particular problems of numerical pollution when handling curved boundaries, and rather restrictive stability requirements.

5.4.2 Nedelec finite elements

In [65],[66], Nedelec introduced families of $\mathbf{H}(\mathbf{curl};\Omega)$ conforming finite elements well suited for discretization of bilinear forms arising from the Maxwell equations. They are also commonly referred to as edge elements, since, in the finite element terminology, the degrees of freedom can be line integrals of the shape functions along edges of the triangles (2D) or quadrilaterals or hexahedra (3D). In the case where the unknown is the electric field, the degrees of freedom has a convenient physical interpretation as the potential difference between the endpoints of the edge. The Nedelec elements possess a property, in the sequel denoted as a discrete divergence free property, that makes them particularly attractive in the context of numerical solution of the Maxwell equations. For a uniform triangulation, denote by $\boldsymbol{v}_h \subset \mathbf{H}(\mathbf{curl};\Omega)$ the corresponding Nedelec finite element space. As usual, the parameter h describes the mesh size. There exists a subspace $S_h \subset H_0^1(\Omega)$ such that $\nabla\xi_h \in \boldsymbol{v}_h$ for any $\xi_h \in S_h$. Define

$$\boldsymbol{x}_h = \{\boldsymbol{u}_h \in \boldsymbol{v}_h : \ (\boldsymbol{u}_h, \nabla\xi_h)_{0,\Omega} = 0 \ \forall \xi_h \in S_h\},$$

and write $\boldsymbol{v}_h = \boldsymbol{x}_h \oplus \nabla S_h$. Being perpendicular to a space of gradients, $\boldsymbol{x}_h$ plays the role of discrete divergence free functions. Let $\boldsymbol{f}$ be some linear combination of divergence free $\mathbf{L}_2(\Omega)$ functions and functions in $\boldsymbol{x}_h$. One solves for $\boldsymbol{E}_h \in \boldsymbol{v}_h$ the (uniquely solvable) problem

$$(\mathbf{curl}\boldsymbol{E}_h, \mathbf{curl}\boldsymbol{v}_h) + k(\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{v}_h.$$

Now, testing with $\nabla\xi_h$ for any $\xi_h \in S_h$ and applying (5.4) yields $\boldsymbol{E}_h \in \boldsymbol{x}_h$. Consequently, applying Nedelec elements for the time stepping scheme arising from the time dependent Maxwell equations on a *stationary* triangulation, the sequence $\boldsymbol{E}^n$ of approximations of the electric field at times $n\Delta t$ all belong to $\boldsymbol{X}_h$.

5.4.3 Standard finite elements, weighted regularization

Application of the Nedelec elements has drawbacks, most notably a significantly larger number of degrees of freedom corresponding to a triangulation, compared to the more widely used nodal finite elements, based on, for instance, the piecewise linear hat function, depicted in Figure 2.1, upper left corner. In [30], Costabel and Dauge consider the applicability of nodal finite elements for the solution of the time-harmonic lossless ($\sigma = 0$) Maxwell equations in the form of Problem 5.3.4 with $\mathrm{div}\boldsymbol{J} = 0$ and $\rho = 0$, but with another choice of energy space, which eventually becomes crucial. Choosing $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ yields from Theorem 5.3.9 a well-posed problem, but a conforming discretization requires divergence free elements, excluding the standard nodal elements.

On the space $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{H}(\mathrm{div};\Omega)$, the bilinear form of Problem 5.3.4 is not coercive. However, from Remark 5.3.2, the electric field is divergence free, so a solution of Problem

5.3.4 would also be a solution if the bilinear form in question was replaced by

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\boldsymbol{u},\mathbf{curl}\boldsymbol{v})_{0,\Omega} + (\mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v})_{0,\Omega} - \omega^2\mu\varepsilon(\boldsymbol{u},\boldsymbol{v})_{0,\Omega},$$

for $\boldsymbol{u},\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$. Assuming for the moment that the bilinear form above is coercive on the space $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$, thereby ensuring well-posedness of the problem, and equivalence between the energy norm $\|\cdot\|_a$ and the norm on $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$. Since the hat function has $\mathbf{H}^1(\Omega)$ regularity, approximations using these elements (properly corrected for the boundary condition) belong to the space $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$, defined in Theorem 5.1.7, which is *closed* for the norm of $\mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$, hence also for the energy norm. Consequently, for the mesh size tending to zero, the convergence in the energy norm entails convergence to an element in $\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$. From Theorem 5.1.7, one sees that for convex domains, or domains with a sufficiently smooth boundary, the electric field is in $\mathbf{H}^1_{\mathrm{tan}}(\Omega)$, and one has convergence towards the true solution. However, for nonconvex domains with a non-smooth boundary, the electric field has a non-$\mathbf{H}^1(\Omega)$ singularity, which would *not* be resolved by the approach just outlined.

As a remedy, Costabel and Dauge suggest to employ a weight in the L_2-form involving the divergence term, so that the bilinear form becomes

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\boldsymbol{u},\mathbf{curl}\boldsymbol{v})_{0,\Omega} + \langle\mathrm{div}\boldsymbol{u},\mathrm{div}\boldsymbol{v}\rangle_{Y} - \omega^2\mu\varepsilon(\boldsymbol{u},\boldsymbol{v})_{0,\Omega},$$

where the space Y is larger than $L_2(\Omega)$ but included in $H^{-1}(\Omega)$. The weight is adapted to the domain, for instance, in the case of an L-shaped domain, the weight tends to zero near the re-entrant corner, so that the divergence of the test and trial fields are allowed to have steep gradients there. They construct Y so that the bilinear form becomes coercive, and one obtains convergence to the true solution.

5.4.4 Decomposition in a regular and a singular part

In [7], [6], [5], a method is suggested that relies on a decomposition of the electric field into a regular part, and a singular part. Defining

$$\boldsymbol{V}_R := \{\boldsymbol{v}\in\mathbf{H}^1(\Omega):\ \mathrm{div}\boldsymbol{v}=0,\ \boldsymbol{v}\times\boldsymbol{n}=\mathbf{0}\text{ on }\Gamma\},$$

one can prove a decomposition $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega) := \boldsymbol{V}_R\oplus\boldsymbol{V}_S$, where the decomposition may or may not be orthogonal. The regular part $\boldsymbol{V}_R$ can be approximated numerically, using standard nodal finite elements, whereas the space $\boldsymbol{V}_S$ is subject to considerations of a more analytical nature. For the $2D$ case, it can be proven that $\boldsymbol{V}_S$ is finite-dimensional. In $3D$, it is infinite-dimensional.

Chapter 6

On Fictitious Domain Formulations for Maxwells Equations

The contents of this chapter, and the major part of the previous chapter, appeared in the paper [37], co-authored by prof. dr. Wolfgang Dahmen, and prof., dr. rer nat Karsten Urban.

6.1 Introduction

PDEs arising from computational electromagnetics like Maxwell's equations, still pose very challenging problems in numerical analysis and simulation, partly due to the subtle nature of the relevant function spaces arising in variational formulations. In spite of the significant progress in understanding these equations that has been achieved during the past years, not much appears to be known about *Fictitious Domain Formulations* (FDF) for Maxwell's equations. As described in Section 4.3, in a FDF, the domain of interest $\Omega \subset \mathbb{R}^n$ is embedded into a larger but simpler domain $\square \subset \mathbb{R}^n$. A typical example is $\square = [0,1]^n$. Of course, in general, simple domains support the design of fast numerical methods. Specifically, in the context of Maxwell's equations, simple geometries facilitate a more convenient realization of appropriate discretizations which, due to the above mentioned nature of the relevant function spaces, tend to be quite complex in nature. Isoparametric techniques, known in the finite element context as well as in the wavelet context, recall section 2.6, for adapting trial spaces with incorporated boundary conditions to complex boundaries would typically interfere with the desired structural properties of the trial spaces. In addition, the FDF would help dealing with moving boundaries or treating control problems where boundary values act as a control variable. In such a case, only the discretization of the boundary has to be changed unless complex boundaries require local refinements on $\square$ which, however, would still benefit from the simple geometry of $\square$.

One way of realizing an FDF is to append boundary conditions by *Lagrange multipliers*. Once the saddle point character of the resulting variational problem has been accepted, other constraints could, of course, be treated in the same fashion. In most applications, the electric field $\boldsymbol{E}$ is divergence-free, i.e., an extra condition is automatically imposed on the discretization. This can either be enforced directly by using (at least discretely) divergence-free trial functions (as long as they are available) or implicitly, through a suitable variational formulation (see Section 6.3.1 (6.8), (6.9)), or by incorporating also such constraints with the aid of Lagrange multipliers. In this article, we consider FDFs for variational problems in the space

$\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{H}(\mathrm{div};\Omega)$ arising from Maxwell's equations, where not only the boundary conditions are imposed weakly by using Lagrange Multipliers, but also the divergence constraint. Aside from stability considerations, this allows one to treat also inhomogeneous divergence constraints as in the case of the electric field. One can then use trial functions that do not have to satisfy a divergence constraint and remain independent of (possibly varying) domain geometries. We consider both the time dependent and the time harmonic formulations of Maxwell's equations.

Of course, there is a well-known price to be paid. As mentioned above, FDFs, or more generally, appending any extra condition weakly in terms of Lagrange multipliers, give rise to *saddle point problems*, which are symmetric but no longer positive definite. This has at least two drawbacks, namely efficient solvers for positive definite systems can no longer be used and, secondly, standard discretizations cannot be chosen arbitrarily but have to fulfill the Ladyshenskaja-Babuška-Brezzi (LBB) condition, recall Theorem 4.2.3. Meeting such compatibility conditions, might be a delicate task, in particular, when the extra conditions are non-trivial.

However, recent progress in the analysis of adaptive methods for saddle point problems make the latter issue appear in a somewhat different light. As outlined in Section 4.4, and also in [8] for Finite Elements, convergent adaptive algorithms have been constructed where the involved discretization spaces need *not* meet the LBB condition. All that matters is the well-posedness of the original infinite dimensional problem. Moreover, the methods in [25, 32], which are also outlined in Sections 3.3 and 4.4, are proven to converge at an asymptotically optimal rate, i.e., the error is comparable with the error of the best approximation that is obtained by *any* linear combination of N trial functions, when N is the number of adaptively generated degrees of freedom (best N-term approximation), see Theorem 3.3.8. These developments motivate us to address here the FDF in the context of Maxwell's equations. In particular, well-posedness (in the sense to be explained in Section 2.3) will be a center of focus, as it is an essential prerequisite for the techniques developed in [25, 32].

Aside from the above aspects, our interest in this subject is fueled by an inherent difficulty that arises when applying a FDF to Maxwell's equations, namely the already mentioned fact that the electric field belongs to a space which is, in general, *not* a subspace of $\boldsymbol{H}^1(\Omega)$. However, recalling Theorem 5.1.7, this difference is very small, in the sense that under certain mild conditions on the domain this space is embedded in $\boldsymbol{H}^1(\Omega)$. Since a fictitious domain typically gives rise to such an embedding one has to be careful in formulating a fictitious domain approach so as to capture also possible non-$\boldsymbol{H}^1$-singularities. By this we shall always mean singularities that prevent the underlying function from being in $\boldsymbol{H}^1(\Omega)$.

The outline of the rest of the Chapter is as follows. Section 6.2 introduces some concepts related to the relevant function spaces for the Maxwell equations, which are not mentioned in chapter 5. Section 6.3 is concerned with FDFs for the time dependent Maxwell's equations. We also show an example where a straightforward formulation of a FDF fails since it only captures the smooth $\boldsymbol{H}^1(\Omega)$-part. We derive then a well-posed FDF that indeed preserves the possible non-$\boldsymbol{H}^1(\Omega)$-singularities. Section 6.4 is concerned with FDFs for the time harmonic formulation of Maxwell's equation which usually is significantly harder to treat, because the bilinear form in question is in general non-coercive. We show that the formulation derived for the time dependent case does not immediately carry over to a well-posed formulation for the time harmonic non-conducting case. We present a strategy, though, to obtain a well-posed problem.

6.2 Basic Notation and Facts

We always consider an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with boundary $\Gamma := \partial\Omega$, which consists of m smooth components.

Trace Spaces

With $\boldsymbol{n}$ being the outward unit normal on Γ, one can prove that for $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega)$, the mapping $\gamma_\tau : \mathbf{H}(\mathbf{curl};\Omega) \mapsto H^{-\frac{1}{2}}$ defined by (5.1)

$$\gamma_\tau(\boldsymbol{u}) := \boldsymbol{u} \times \boldsymbol{n}$$

is bounded, and that

$$\mathbf{H}_0(\mathbf{curl};\Omega) = \{\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega) : \ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \ \text{ on } \Gamma\}.$$

Recall the Green's formula (5.3)

$$\int_\Omega \mathbf{curl}\, \boldsymbol{v} \cdot \phi - \int_\Omega \boldsymbol{v} \cdot \mathbf{curl}\, \phi = \langle \boldsymbol{v} \times \boldsymbol{n}, \phi \rangle_\Gamma, \tag{6.1}$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$ and $\phi \in \mathbf{H}^1(\Omega)1$, where $\langle \cdot,\cdot \rangle_\Gamma$ is the usual duality form induced by the $\mathbf{L}_2(\Gamma)$-inner product.

The formula (5.3) is not quite adequate for our purposes, since it does not hold in general when $\boldsymbol{v}$ and ϕ both belong only to $\mathbf{H}(\mathbf{curl};\Omega)$. This problem is addressed in [16], from which we briefly outline the results that are of importance for the present work. The trace mapping γ_τ is *not* surjective on the space $H^{-\frac{1}{2}}$. The range space of this trace mapping is a normed space usually denoted by $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, and the mapping $\gamma_\tau : \mathbf{H}(\mathbf{curl};\Omega) \mapsto \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ is also bounded. In order to formulate a proper duality for the trace spaces one defines another trace mapping π_τ on $\mathbf{H}(\mathbf{curl};\Omega)$:

$$\pi_\tau(\boldsymbol{u}) := \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}).$$

The range of π_τ is a normed space denoted by $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$, and this mapping is bounded as well. Recall from [4] that $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Omega)$ can be decomposed as

$$\boldsymbol{u} = \boldsymbol{\Phi} + \nabla p, \quad \boldsymbol{v} = \boldsymbol{\Psi} + \nabla q,$$

with $\boldsymbol{\Phi}, \boldsymbol{\Psi} \in \mathbf{H}^1(\Omega)$ and $p, q \in H^1(\Omega)$. The following formula can then be proven to be meaningful

$$\begin{aligned} &\int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} - \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{v} = \langle \gamma_\tau(\boldsymbol{u}), \pi_\tau(\boldsymbol{v}) \rangle_\Gamma \\ &:= \int_\Gamma \gamma_\tau(\boldsymbol{\Phi}) \cdot \pi_\tau(\boldsymbol{\Psi}) - \langle \mathrm{div}_\Gamma\, \gamma_\tau(\boldsymbol{\Phi}), q \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &\qquad + \langle \mathbf{curl}_\Gamma\, \pi_\tau(\boldsymbol{\Psi}), p \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}, \end{aligned} \tag{6.2}$$

where div_Γ and $\mathbf{curl}_\Gamma$ are appropriately defined surface differential operators and as usual $\langle \cdot,\cdot \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$ denotes the dual pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Whenever the form $\langle \cdot,\cdot \rangle_\Gamma$ is used in this chapter, in connection with trace mappings on $\mathbf{H}(\mathbf{curl};\Omega)$, we

understand the form defined in (6.2), and it induces a duality between the spaces $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ and $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$ with the space

$$\mathbf{L}_{2,t} = \{\boldsymbol{f} \in \boldsymbol{L}_2(\Gamma) : \ \boldsymbol{f}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$$

as pivot space.

For $\boldsymbol{u} \in \mathbf{H}(\mathrm{div};\Omega)$, one can prove that the normal trace on the boundary, $\boldsymbol{u}\cdot\boldsymbol{n}$ belongs to $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ and that this trace mapping is onto, [13]. Furthermore, recall the Green formula (5.4)

$$\int_\Omega \boldsymbol{v}\cdot\nabla\,\phi + \int_\Omega (\mathrm{div}\,\boldsymbol{v})\phi = \langle \boldsymbol{v}\cdot\boldsymbol{n},\phi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},$$

for all $\boldsymbol{v} \in \mathbf{H}(\mathrm{div};\Omega)$ and $\phi \in H^1(\Omega)$.

6.3 FDFs for the Time-Dependent Formulation

Recall from Section 6.2 that boundary constraints involve the tangential components $\gamma_\tau(\boldsymbol{u}) := \boldsymbol{u}|_\Gamma \times \boldsymbol{n}$ which belong to the space $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$.

6.3.1 An Essential Obstruction

To our knowledge, not much is known so far on FDFs for Maxwell's equations. The issue is somewhat more delicate than in other situations due to a complication caused by the above mentioned subtle difference between the spaces $\mathbf{H}^1_{\mathrm{Tan}}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \gamma_\tau(\boldsymbol{u}) = \mathbf{0} \ \text{ on } \ \Gamma\}$ and $\mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$. We will clarify the consequences of this fact first for the problem (5.31) with $\kappa \in \mathbb{R}^+$ and $\boldsymbol{u},\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl};\Omega)\cap\mathbf{V}(\mathrm{div};\Omega)$. Without loss of generality we can assume that

$$\Omega \subset \square = (0,a)^n, \qquad a \in \mathbb{R}^+,$$

where the 'simple domain' $\square$ will serve as the fictitious domain. Moreover, for simplicity, we will use *periodic* functions on $\square$, see e.g. [53]. Let us denote by

$$\mathcal{C}^\infty_{\mathrm{per}}(\square) := \{\boldsymbol{v} \in \mathcal{C}^\infty : \boldsymbol{v}(\mathbf{x}+a\mathbf{k}) = \boldsymbol{v}(\mathbf{x}),\ \mathbf{k} \in \mathbb{Z}^3\}$$

the a-periodic infinitely differentiable vector fields. Then for any function space $\boldsymbol{X}$ we denote by $\boldsymbol{X}(\square)_{\mathrm{per}}$ the closure of $\mathcal{C}^\infty_{\mathrm{per}}(\square)$ in the $\boldsymbol{X}$-norm: $\boldsymbol{X}(\square)_{\mathrm{per}} := \mathrm{clos}_{\boldsymbol{X}}\,\mathcal{C}^\infty_{\mathrm{per}}(\square)$. Note that also other boundary conditions on $\square$ can be considered. In fact, the subsequent analysis also applies to homogeneous Dirichlet boundary conditions on $\square$.

One would then be tempted to employ the following FDF:

PROBLEM 6.3.1 *Given*

$$\tilde{\boldsymbol{f}} \in \mathbf{X}_{\mathrm{per}}(\square)', \quad \mathbf{X}(\square) := \mathbf{H}(\mathbf{curl};\square)\cap\mathbf{H}(\mathrm{div};\square),$$

determine

$$(\boldsymbol{u},\phi,\lambda) \in \mathbf{X}_{\mathrm{per}}(\square)\times L_2(\square)\times(\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$$

such that

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v})_\square + (\phi,\mathrm{div}\,\boldsymbol{v})_{0,\square} \qquad +\langle\lambda,\gamma_\tau(\boldsymbol{v})\rangle_\Gamma &= (\tilde{\boldsymbol{f}},\boldsymbol{v})_{0,\square}, && \boldsymbol{v} \in \mathbf{X}_{\mathrm{per}}(\square),\\
(\mathrm{div}\,\boldsymbol{u},\psi)_{0,\square} &= 0, && \psi \in L_2(\square),\\
\langle\mu,\gamma_\tau(\boldsymbol{u})\rangle_\Gamma &= 0, && \mu \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'.
\end{aligned}$$

The reason why this approach would in general fail can be explained as follows. On one hand, if Problem 6.3.1 has a solution, its first component $\mathbf{u}$ belongs to $\mathbf{X}_{\rm per}(\square)$. But there is a counterpart to Theorem 5.1.7 for $\mathbf{X}_{\rm per}(\square)$.

LEMMA 6.3.2 *The space* $\mathbf{X}_{\rm per}(\square)$ *is continuously embedded in* $\mathbf{H}^1(\square)$.

Proof: We claim that for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\square)_{\rm per}$

$$\int_\square \nabla \boldsymbol{u} : \nabla \boldsymbol{v} = \int_\square \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} + \int_\square \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}. \tag{6.3}$$

From this formula, the lemma follows by standard density arguments.

For $u, v \in C^\infty(0,a)_{\rm per}$, integration by parts yields

$$\int_0^a u' \, v \, dx = [u\, v]_0^a - \int_0^a u \, v' \, dx = - \int_0^a u \, v' \, dx,$$

because of periodicity. Consequently, for $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\square)_{\rm per}$ and $i, j, k \in \{1,2,3\}$,

$$\int_\square \partial_i u_j v_k d\boldsymbol{x} = - \int_\square u_j \partial_i v_k d\boldsymbol{x}.$$

Note that since the functions are C^∞ on $\square$, the order of integration does not matter. Employing this, one can easily prove the following useful formulas:

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\square)_{\rm per} : \quad \int_\square \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{v} = \int_\square \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{u}, \tag{6.4}$$

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\square)_{\rm per} : \quad \int_\square \nabla \boldsymbol{u} : \nabla \boldsymbol{v} = - \int_\square \Delta \boldsymbol{u} \cdot \boldsymbol{v}, \tag{6.5}$$

$$\forall \boldsymbol{u} \in \mathcal{C}^\infty(\square)_{\rm per}, \forall v \in C^\infty(\square)_{\rm per} : \quad \int_\square \boldsymbol{u} \cdot \nabla v = - \int_\square \operatorname{div} \boldsymbol{u} v. \tag{6.6}$$

For instance, choosing $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}^\infty(\square)_{\rm per}$,

$$\begin{aligned} \int_\square \nabla \boldsymbol{u} : \nabla \boldsymbol{v} &= \int_\square \sum_{i=1}^3 \sum_{j=1}^3 \partial_j u_i \, \partial_j v_i \\ &= - \sum_{i=1}^3 \sum_{j=1}^3 \int_\square \partial_j^2 u_i v_i \\ &= - \int_\square \Delta \boldsymbol{u} \cdot \boldsymbol{v}, \end{aligned}$$

which proves (6.5). The formula (6.3) follows now from the identity

$$\boldsymbol{u} \in \mathcal{C}^\infty(\square)_{\rm per} : \quad \mathbf{curl}\, \mathbf{curl}\, \boldsymbol{u} = \nabla \operatorname{div} \boldsymbol{u} - \Delta \boldsymbol{u}$$

which holds since $\boldsymbol{u}$ is smooth. Multiplying by any $\boldsymbol{v} \in \mathcal{C}^\infty(\square)_{\rm per}$ and using the three relations above, proves the claim. □

Hence also the restriction $\mathbf{u}|_\Omega$ of a solution belongs to $\mathbf{H}^1(\Omega)$. On the other hand, as already noted above, the electric field $\boldsymbol{E}$ solving Maxwell's equations may be an element of $\mathbf{H}(\mathbf{curl};\Omega)\cap\mathbf{H}(\mathrm{div};\Omega)$, but, depending on Ω may have non-$\boldsymbol{H}^1$-singularities, see the remarks following theorem 5.1.7.

The above reasoning, combined with Theorem 5.1.7 actually shows the following somewhat more general fact.

REMARK 6.3.3 *Assume that the physical domain Ω is embedded in a convex polyhedron $\square$. Then FDFs for (5.31) that fall in either one of the following categories:*

- *test and trial functions belong to $\mathbf{X}_{\mathrm{per}}(\square)$ defined above;*
- *test and trial functions have global $\mathbf{H}(\mathbf{curl};\square)\cap\mathbf{H}(\mathrm{div};\square)$ regularity and have vanishing tangential components $\mathbf{u}\times\mathbf{n}_\square$ on $\partial\square$;*

fail in the sense that they may not capture possible non-$\boldsymbol{H}^1$-singularities, i.e. parts of the solution that do not belong to $\mathbf{H}^1(\Omega)$.

REMARK 6.3.4 *Since trivially $(\mathbf{H}(\mathbf{curl};\square)\cap\mathbf{V}(\mathrm{div};\square))_{\mathrm{per}}\hookrightarrow(\mathbf{H}(\mathbf{curl};\square)\cap\mathbf{H}(\mathrm{div};\square))_{\mathrm{per}}$, the availability of divergence-free trial functions on $\square$ (as e.g. divergence-free wavelet bases, see [59, 75]) would be no remedy.*

These results are in line with those of subsection 5.4.3. As immediate consequences of the proof of Lemma 6.3.2, we state some properties of the periodic spaces introduced above, and their relations with other subspaces of $\mathbf{H}(\mathbf{curl};\square)$ and $\mathbf{H}(\mathrm{div};\square)$.

COROLLARY 6.3.5

(i) For each $\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}^1(\square)_{\mathrm{per}}$,

$$\int_\square \nabla\boldsymbol{u}:\nabla\boldsymbol{v}=\int_\square \mathrm{div}\,\boldsymbol{u}\;\mathrm{div}\,\boldsymbol{v}+\int_\square \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}.$$

(ii) For each $\boldsymbol{u}\in\mathbf{H}(\mathrm{div};\square)_{\mathrm{per}}$ and $g\in H^1(\square)_{\mathrm{per}}$,

$$(\mathrm{div}\;\boldsymbol{u},g)_{0,\square}=-(\boldsymbol{u},\nabla\;g)_{0,\square}.$$

(iii) For each $\boldsymbol{u},\boldsymbol{v}\in\mathbf{H}(\mathbf{curl};\square)_{\mathrm{per}}$,

$$(\mathbf{curl}\;\boldsymbol{u},\boldsymbol{v})_{0,\square}=(\boldsymbol{u},\mathbf{curl}\;\boldsymbol{v})_{0,\square}$$

(iv) $\mathbf{L}_2(\square)_{\mathrm{per}}=\mathbf{L}_2(\square)$.

(v)

$$\begin{aligned}\mathbf{H}_0(\mathbf{curl};\square)\subsetneq\mathbf{H}(\mathbf{curl};\square)_{\mathrm{per}}&\subsetneq\mathbf{H}(\mathbf{curl};\square)\\ \mathbf{H}_0(\mathrm{div};\square)\subsetneq\mathbf{H}(\mathrm{div};\square)_{\mathrm{per}}&\subsetneq\mathbf{H}(\mathrm{div};\square).\end{aligned}$$

Proof: (i), (ii) and (iii) follow from (6.3), (6.6) and (6.4) and density arguments. (iv) is clear from standard Fourier analysis. Regarding (v), note that a field whose components are nonzero constants belong to $\mathbf{H}(\mathrm{div};\Box)_{\mathrm{per}}$ but not $\mathbf{H}_0(\mathrm{div};\Box)$. As to the other true inclusion, observe that any $g \in H^1(\Box)_{\mathrm{per}}$ with nonvanishing trace on $\Box$ induces a functional Λ_g on $\mathbf{H}(\mathrm{div};\Box)$

$$\boldsymbol{u} \in \mathbf{H}(\mathrm{div};\Box),\ \Lambda_g(\boldsymbol{u}) = (\mathrm{div}\ \boldsymbol{u}, g)_{0,\Box} + (\boldsymbol{u}, \nabla\ g)_{0,\Box}.$$

Clearly, Λ_g is continuous for the norm of $\mathbf{H}(\mathrm{div};\Box)$, and vanishes on $\mathcal{C}^\infty_{\mathrm{per}}(\Box)$ by (i). Choosing for instance $\boldsymbol{u} \in \mathbf{H}(\mathrm{div};\Box)$ with vanishing normal trace on all but one face of $\Box$ gives by (5.4) $\Lambda_g(\boldsymbol{u}) \neq 0$, showing the nondensity of $\mathcal{C}^\infty_{\mathrm{per}}(\Box)$ in $\mathbf{H}(\mathrm{div};\Box)$. The proof of these properties for the curl operator is completely analogous. ∎

A FDF for a Scattering Problem

The above difficulty has actually been already properly accounted for in [27] in a closely related context, namely for a scattering problem [1], i.e., for an exterior boundary value problem. Defining the bilinear form

$$a(\boldsymbol{u},\boldsymbol{v})_\Box = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v})_\Box + \kappa(\boldsymbol{u},\boldsymbol{v})_\Box, \quad \boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Box), \tag{6.7}$$

the following variational problem is analyzed in [27]:
Given $(\boldsymbol{J},\eta) \in (\mathbf{H}(\mathbf{curl};\Box))' \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, find $(\boldsymbol{u},\lambda) \in \mathbf{H}(\mathbf{curl};\Box) \times (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$ such that

$$a(\boldsymbol{u},\boldsymbol{v})_\Box \quad + \langle \boldsymbol{v}\times\boldsymbol{n}, \lambda\rangle_\Gamma = (\boldsymbol{J}_\Box, \boldsymbol{v}), \quad \boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\Box), \tag{6.8}$$

$$\langle \boldsymbol{u}\times\boldsymbol{n}, \xi\rangle \qquad\qquad = \langle \eta, \xi\rangle_\Gamma, \quad \xi \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))', \tag{6.9}$$

where $\boldsymbol{J}_\Box$ is some extension of $\boldsymbol{J}$ from Ω to $\Box$, and $\langle\cdot,\cdot\rangle_\Gamma$ is the duality form on the boundary. For the definition of div_Γ, we again refer to [14, 15, 16]. In [27], this problem is discretized using Nedelec elements on a fixed grid for $\Box$, imposing zero boundary conditions on the tangential components $\mathbf{u}\times\mathbf{n}|_\Box$ on $\partial\Box$. Choosing $\boldsymbol{v} := \nabla p$ for $p \in H^1_0(\Omega)$, (6.8) says that $a(\boldsymbol{u},\boldsymbol{v})_\Box = \kappa(\boldsymbol{u},\nabla p)_{0,\Box} = (\boldsymbol{J}_\Box, \nabla p)_{0,\Box}$ for all $p \in H^1_0(\Omega)$ and hence

$$\mathrm{div}\,\boldsymbol{u} = \kappa^{-1}\mathrm{div}\boldsymbol{J}_\Box \quad \text{in} \quad H^{-1}(\Omega) \tag{6.10}$$

and, by the same argument also in $H^{-1}(\Box \setminus \Omega)$. Hence $\boldsymbol{u}|_\Omega \in \mathbf{H}(\mathrm{div};\Omega)$ whenever $\boldsymbol{J} \in \mathbf{H}(\mathrm{div};\Omega)$. However, this does not imply $\boldsymbol{u} \in \mathbf{H}(\mathrm{div};\Box)$ because the normal traces of $\boldsymbol{u}$ may jump across Γ. Thus, choosing just $\mathbf{H}(\mathbf{curl};\Box)$ as test and trial space, suffices to identify the correct solution while avoiding the above mentioned trap concerning non-$\mathbf{H}^1$–singularities.

A few comments on the Nedelec elements [65],[66] in the above context are in order, since their role leads us to considering the FDF presented below. The Nedelec elements are $\mathbf{H}(\mathbf{curl};\Omega)$ conforming finite elements and thus well suited for discretizing the bilinear forms arising from the Maxwell equations. They possess a property, in the sequel denoted as a discrete divergence free property, that makes them particularly attractive for the numerical solution of the Maxwell equations. To explain this, denote (for a uniform triangulation, say) by $\boldsymbol{v}_h \subset \mathbf{H}(\mathbf{curl};\Omega)$ the corresponding Nedelec finite element space where, as usual, the

[1] see also page 88f

parameter h describes the mesh size. There exists a subspace $S_h \subset H_0^1(\Omega)$ such that $\nabla\, \xi_h \in \boldsymbol{v}_h$ for any $\xi_h \in S_h$. Defining

$$\boldsymbol{X}_h = \{\boldsymbol{u}_h \in \boldsymbol{v}_h : \ (\boldsymbol{u}_h, \nabla\, \xi_h)_{0,\Omega} = 0 \ \forall \xi_h \in S_h\},$$

we obtain the orthogonal decomposition $\boldsymbol{v}_h = \boldsymbol{X}_h \oplus \nabla\, S_h$. Being perpendicular to a space of gradients, $\boldsymbol{X}_h$ plays the role of discrete divergence free functions. Now consider the problem to find (the uniquely determined) $\boldsymbol{E}_h \in \boldsymbol{v}_h$ satisfying

$$(\mathbf{curl}\, \boldsymbol{E}_h, \mathbf{curl}\, \boldsymbol{v}_h) + k(\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{v}_h,$$

where $\boldsymbol{f}$ is a linear combination of divergence free $\mathbf{L}_2(\Omega)$ functions and functions in $\boldsymbol{X}_h$. Now, testing with $\nabla \xi_h$ for any $\xi_h \in S_h$ and applying (5.4), yields $\boldsymbol{E}_h \in \boldsymbol{X}_h$. Thus, the situation in the finite dimensional setting parallels exactly what happens on the continuous infinite dimensional level. Hence, applying Nedelec elements for the time stepping scheme arising from the time dependent Maxwell equations on a *stationary* triangulation, the sequence $\boldsymbol{E}^n$ of approximations of the electric field at times $n\Delta t$ all belong to $\boldsymbol{X}_h$. Therefore a stable realization of the divergence constraint is warrented by this discretization.

However, when employing adaptive wavelet techniques, for the reasons indicated above, neither stay the spatial discretizations stationary in the course of the time stepping nor will trial functions such as the $\mathbf{H}(\mathbf{curl}; \Omega)$ conforming wavelets of [75] satisfy a dicrete divergence condition in the above sense. Consequently, the approximate solutions at two different time steps may belong to complements of two different spaces of gradients. Hence, imposing the divergence condition weakly, in a saddle point formulation, will enhance stability by controlling those parts of the approximate solutions that belong to the gradient space to which the approximation at any given time step should be perpendicular.

Moreover, appending the divergence constraint explicitly with the aid of Lagrange multipliers would allow us to treat inhomogeneous divergence conditions in the same stable way. Therefore we shall be concerned in the following with such a variational formulation which, however, avoids the unwanted effect of Problem 6.3.1.

6.3.2 Preserving non-$\boldsymbol{H}^1$-Singularities

We present now an alternative FDF which preserves the possible non-$\boldsymbol{H}^1$-singularities of the solutions to Maxwell's equations. The above observations suggest to require only $\mathbf{H}(\mathbf{curl}; \square)$ regularity and impose the divergence constraint only on Ω which is reflected by the following

PROBLEM 6.3.6 *Given*

$$(\boldsymbol{J}, f) \in (\mathbf{H}_0(\mathbf{curl}; \Omega))' \times H^{-1}(\Omega),$$

find a pair

$$(\boldsymbol{u}, p) \in \mathbf{H}_0(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$$

such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + (\boldsymbol{v}, \nabla\, p)_{0,\Omega} &= (\boldsymbol{J}, \boldsymbol{v})_{0,\Omega}, \quad \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega), \\ (\boldsymbol{u}, \nabla\, q)_{0,\Omega} &= (f, q)_{0,\Omega}, \quad q \in H_0^1(\Omega), \end{aligned}$$

where we assume again first that $\kappa > 0$ in (5.31). The next step is to append boundary conditions. Recalling (6.7), we consider

PROBLEM 6.3.7 *Given the triple*

$$(\boldsymbol{J}, f, \eta) \in (\boldsymbol{H}_{\rm per}(\mathbf{curl}; \square))' \times H^{-1}(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma),$$

find a triple

$$(\boldsymbol{u}, p, \lambda) \in \boldsymbol{H}_{\rm per}(\mathbf{curl}; \square) \times H^1_0(\Omega) \times (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma))',$$

so that

$$\begin{aligned}
a(\boldsymbol{u}, \boldsymbol{v})_\square \quad + (\boldsymbol{v}, \nabla p)_{0,\Omega} + \langle \gamma_\tau(\boldsymbol{v}), \lambda \rangle_\Gamma &= (\boldsymbol{J}, \boldsymbol{v})_{0,\square}, \qquad \boldsymbol{v} \in \boldsymbol{H}_{\rm per}(\mathbf{curl}; \square), \\
(\boldsymbol{u}, \nabla q)_{0,\Omega} &= (f, q)_\Omega, \qquad q \in H^1_0(\Omega), \\
\langle \gamma_\tau(\boldsymbol{u}), \mu \rangle_\Gamma &= \langle \eta, \mu \rangle_\Gamma, \qquad \mu \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma))',
\end{aligned}$$

where as above $\gamma_\tau(\boldsymbol{u}) = \boldsymbol{u}|_\Gamma \times \boldsymbol{n}$ *on* Ω.

By the remarks at the end of the previous subsection, Problem 6.3.7 is equivalent to (6.8), (6.9) whenever $f = \mathrm{div}\, \boldsymbol{J}$ on Ω.

Considering Problem 5.3.10 with $\mathbf{H}_0(\mathbf{curl}; \Omega)$ as variational space and, say, $\kappa > 0$ so that it is well-posed, the presence of the second variable p, in the literature sometimes denoted as a *pseudo-pressure*, in the above formulations may seem mysterious. Indeed, if $\boldsymbol{u}$ solves Problem 5.3.10, then it is not a-priori clear that $\boldsymbol{u}$ is the first component of the solution to Problems 6.3.6 and 6.3.7. The answer is provided by imposing additional assumptions on the right hand side, which by no means limit the generality of the framework. We return to this discussion later.

Of course, in the above formulation one solution component still involves functions restricted to the physical domain. So the question arises whether there is still any gain with regard to using wavelets along the lines of chapter 3. The point is that wavelet bases for $H^1_0(\Omega)$ are known also for realistic domain geometries, while $\mathbf{H}(\mathbf{curl}; \Omega)$ stable bases are only available for very simple domain geometries such as the cube used in the above fictitious domain formulation.

The main result of this section can be stated as follows.

THEOREM 6.3.8 *The Problems 6.3.6 and 6.3.7 are well-posed and have unique solutions. The first two components of the solutions to both problems coincide on* Ω *when the boundary conditions in 6.3.7 are homogeneous.*

We begin with some prerequisites that will eventually allow us to apply Theorem 4.1.2. As before, we assume always that Ω is some bounded, simply connected polyhedral Lipschitz domain with boundary $\Gamma = \partial\Omega$. Recall the orthogonal decomposition of $\mathbf{L}_2(\Omega)$ in (5.10). A similar decomposition also holds for $\mathbf{H}(\mathbf{curl}; \Omega)$. In fact, defining

$$\begin{aligned}
\mathbf{X}_1(\Omega) := \big\{ &\mathbf{curl}\, \boldsymbol{u} : \ \boldsymbol{u} \in \boldsymbol{H}^1(\Omega), \ \mathbf{curl}\, \boldsymbol{u} \in \mathbf{H}(\mathbf{curl}; \Omega), \\
&\mathrm{div}\, \boldsymbol{u} = 0, \ (\mathbf{curl}\ \boldsymbol{u}) \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \big\}, \\
\mathbf{X}_2(\Omega) := \big\{ &\nabla \phi : \ \phi \in H^1(\Omega) \big\},
\end{aligned}$$

we record the following observation.

LEMMA 6.3.9 *The spaces* $\mathbf{X}_1(\Omega)$ *and* $\mathbf{X}_2(\Omega)$ *form an orthogonal decomposition*

$$\mathbf{H}(\mathbf{curl}; \Omega) = \mathbf{X}_1(\Omega) \oplus \mathbf{X}_2(\Omega).$$

Proof: Given $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\Omega)$, write $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ according to (5.10). It is clear that $\boldsymbol{u}_i \in \mathbf{X}_i(\Omega)$, $i = 1, 2$. Since $\mathbf{curl}\,\boldsymbol{u}_2 = \mathbf{0}$, $\boldsymbol{u}_1 \perp \boldsymbol{u}_2$ in $\mathbf{H}(\mathbf{curl};\Omega)$. ∎

Now we embed Ω into a larger, simpler domain $\square$. Problems 6.3.6 and 6.3.7 suggest considering the following operators.

LEMMA 6.3.10 *The operators* $\mathrm{Div} : \mathbf{X}_2(\Omega) \to H^{-1}(\Omega)$ *and* $\mathrm{Div}_\Omega : \mathbf{X}_2(\square) \to H^{-1}(\Omega)$ *defined by*

$$\langle \mathrm{Div}\,\boldsymbol{u}, p\rangle := (\boldsymbol{u}, \nabla p)_{0,\Omega}, \quad \langle \mathrm{Div}_\Omega\,\boldsymbol{u}, p\rangle := (\boldsymbol{u}, \nabla p)_{0,\Omega}, \quad p \in H_0^1(\Omega), \tag{6.11}$$

are bounded and onto.

Proof: The boundedness is clear. For $f \in H^{-1}(\Omega)$ the boundary value problem

$$-\Delta\phi = f \quad \text{on } \Omega, \quad \phi = 0 \quad \text{on } \Gamma = \partial\Omega,$$

has a unique solution $\phi \in H_0^1(\Omega)$. This means $\nabla\phi \in \mathbf{X}_2(\Omega)$ and $\mathrm{Div}\,\nabla\phi = f$. To confirm also the surjectivity of Div_Ω, it suffices to extend ϕ by zero from Ω to $\square$. ∎

LEMMA 6.3.11 *The spaces* $\mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ *and*

$$\mathbf{H}_{\mathrm{per}}(\square,\Omega) := \{\boldsymbol{u} \in \mathbf{H}_{\mathrm{per}}(\mathbf{curl};\square) : \mathrm{Div}_\Omega\,\boldsymbol{u} = 0, \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \Gamma\}$$

are closed in $\mathbf{H}(\mathbf{curl};\Omega)$ *and* $\mathbf{H}(\mathbf{curl};\square)$*, respectively.*

Proof: The trace mapping $\gamma_\tau(\boldsymbol{u}) = \boldsymbol{u}|_\Gamma \times \boldsymbol{n}$ is bounded and, by Lemma 6.3.10, so are the operators Div and Div_Ω. The spaces defined above are the kernels of these operators. ∎

The following result can be found in [47, p. 966]. It is based on continuous extensions of trace data, see [3, 16]

LEMMA 6.3.12 *There exists a continuous extension operator*

$$E : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}(\mathbf{curl};\mathbb{R}^3).$$

The last ingredient concerns the inf-sup condition (4.6) in Theorem 4.1.2.

LEMMA 6.3.13 *There exists a positive constant* $\beta > 0$ *such that the inf-sup condition*

$$\inf_{\lambda\in(\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'} \sup_{\boldsymbol{u}\in\mathbf{H}_{\mathrm{per}}(\mathbf{curl};\square)} \frac{\langle\lambda, \boldsymbol{u}\times\boldsymbol{n}\rangle_\Gamma}{\|\lambda\|_{(\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'}\,\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl};\square)}} \geq \beta$$

holds.

Proof: The proof is similar in spirit to [57]. Since the extension in Lemma 6.3.12 is bounded, the above extension from $\mathbf{H}(\mathbf{curl};\Omega)$ to $\mathbf{H}(\mathbf{curl};\mathbb{R}^3)$ also yields a bounded extension to the space $\mathbf{H}_{\rm per}(\mathbf{curl};\square)$ provided that $B \subset \square$. Fix any $\lambda \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$. Using the surjectivity and boundedness of the trace mapping

$$\mathbf{H}(\mathbf{curl};\Omega) \ni \boldsymbol{u} \mapsto \gamma_\tau(\boldsymbol{u}) = \boldsymbol{u}|_\Gamma \times \boldsymbol{n} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma),$$

see [16], and finally the boundedness of the lifting map (see Lemma 6.3.12), we obtain

$$\begin{aligned}\|\lambda\|_{(\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'} &= \sup_{\mu\in\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)} \frac{\langle\lambda,\mu\rangle_\Gamma}{\|\mu\|_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)}} = \sup_{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl};\Omega)} \frac{\langle\lambda,\gamma_\tau(\boldsymbol{u})\rangle_\Gamma}{\|\gamma_\tau(\boldsymbol{u})\|_{\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)}} \\ &\lesssim \sup_{\boldsymbol{u}\in\mathbf{H}(\mathbf{curl};\Omega)} \frac{\langle\lambda,\gamma_\tau(\boldsymbol{u})\rangle_\Gamma}{\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl};\Omega)}} \lesssim \sup_{\boldsymbol{u}\in\mathbf{H}_{\rm per}(\mathbf{curl};\square)} \frac{\langle\lambda,\gamma_\tau(\boldsymbol{u})\rangle_\Gamma}{\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl};\square)}},\end{aligned}$$

which is the desired inf-sup condition. ∎

Proof of Theorem 6.3.8: Let us consider first Problem 6.3.6. Since $\kappa > 0$, the bilinear form $a(\cdot,\cdot)$ in (5.31) is coercive on $\boldsymbol{H}(\mathbf{curl};\Omega)$, hence also on the subspace $\boldsymbol{H}_0(\mathbf{curl};\Omega)$. This implies the validity of (4.4) and (4.5). The inf-sup condition (4.6) corresponding to the divergence constraint in Problem 6.3.6 is equivalent to the closedness of the range of the divergence mapping Div_Ω, which is indeed ensured by Lemma 6.3.10. Moreover, the corresponding mapping is onto. Thus, Problem 6.3.6 is, by Theorem 4.1.2, well-posed and thus has a unique solution.

Now, we investigate Problem 6.3.7. The bilinear form $a(\cdot,\cdot)_\square$ in (6.7) is coercive on $\boldsymbol{H}(\mathbf{curl};\square)$ and hence also on the closed subspace $\mathbf{H}_{\rm per}(\square,\Omega) = \{\boldsymbol{u} \in \mathbf{H}_{\rm per}(\mathbf{curl};\square) : \mathrm{Div}_\Omega \boldsymbol{u} = 0,\ \boldsymbol{u}\times\boldsymbol{n} = 0 \text{ on } \Gamma\}$ which is the kernel of the operator B defined by the Lagrange multipliers in Problem 6.3.7. The inf-sup condition corresponding to the trace mapping in Problem 6.3.7 is fulfilled by Lemma 6.3.13. Moreover, the operator $\mathcal{L}$ induced by the variational problem: Given $(\boldsymbol{J},f) \in (\mathbf{H}_{\rm per}(\square,\Omega))' \times H^{-1}(\Omega)$, find a pair $(\boldsymbol{u},p) \in \mathbf{H}_{\rm per}(\square,\Omega) \times H^1_0(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{u},\boldsymbol{v})_\square + (\boldsymbol{v},\nabla p)_{0,\Omega} &= (\boldsymbol{J},\boldsymbol{v})_{0,\Omega}, \quad \boldsymbol{v} \in \boldsymbol{H}_{\rm per}(\square,\Omega), \\ (\boldsymbol{u},\nabla q)_{0,\Omega} &= (f,q)_{0,\Omega}, \quad q \in H^1_0(\Omega), \end{aligned}$$

is by the preceeding remarks a norm isomorphism from the space $\mathbf{H}_{\rm per}(\square,\Omega) \times H^1_0(\Omega)$ onto its dual. This implies the validity of (4.4) and (4.5) for the bilinear form corresponding to $\mathcal{L}$. Thus, again Theorem 4.1.2 yields that Problem 6.3.7 is well-posed and hence has a unique solution. □

The pseudo-pressure

We return now to a discussion of the role played by the pseudo-pressure p in Problem 6.3.6. Similar arguments are carried out in [29].

REMARK 6.3.14 *Assume that* $\mathbf{J} \in \mathbf{L}_2(\Omega)$ *has* $\mathrm{div}\mathbf{J} = 0$. *Then* $\boldsymbol{u}$ *solves Problem 5.3.10 with* $\mathbf{H}_0(\mathbf{curl};\Omega)$ *as variational space, if and only if* $(\boldsymbol{u},p) = (\boldsymbol{u},0)$ *solves Problem 6.3.7.*

Proof: Both problems are, from the preceding analysis, well-posed. Assume $(\boldsymbol{u}, p)$ solves Problem 6.3.7. For any $\phi \in H_0^1(\Omega)$, $\nabla\,\phi \in \mathbf{H}_0(\mathbf{curl};\Omega)$. From the formula (5.4) we see that a function $\boldsymbol{g} \in \mathbf{L}_2(\Omega)$ has div$\boldsymbol{g} = 0$ if and only if $(\boldsymbol{g}, \nabla\,\phi)_{0,\Omega} = 0$ for every $\phi \in H_0^1(\Omega)$. Testing with $\nabla\,\phi$ for any $\phi \in H_0^1(\Omega)$ in Problem 6.3.7 yields that $p \in H_0^1(\Omega)$ solves $(\nabla\, p, \nabla\phi)_{0,\Omega} = 0$ for any $\phi \in H_0^1(\Omega)$. Since the Laplacian Δ is an isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$, $p = 0$. Then it is clear that $\boldsymbol{u}$ solves Problem 5.3.10.

Similarly, if $\boldsymbol{u}$ solves 5.3.10, testing with gradients yield that $\boldsymbol{u} \perp \nabla\,\phi$ in $\mathbf{L}_2(\Omega)$ for any $\phi \in H_0^1(\Omega)$. Then $(\boldsymbol{u}, 0)$ is the unique solution to problem 6.3.7. □

6.4 FDFs for the Time-Harmonic Formulation

Finally, we discuss properly defined FDFs also for the time-harmonic case. Again, we need some preparations.

DEFINITION 6.4.1 *We say that $\lambda > 0$ is a* Maxwell eigenvalue, *if there exists a non-zero function $\mathbf{u} \in \mathbf{V}_0(\Omega) := \mathbf{H}_0(\mathbf{curl};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$ such that*

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{0,\Omega} = \lambda(\mathbf{u}, \mathbf{v})_{0,\Omega}, \quad \mathbf{v} \in \mathbf{V}_0(\Omega).$$

One can prove that the Maxwell eigenvalues form a discrete subset of $\mathbb{R}$, [30]. Recall from corollary 5.3.6 that the bilinear form $a(\cdot,\cdot) : \mathbf{H}(\mathbf{curl};\Omega) \times \mathbf{H}(\mathbf{curl};\Omega) \mapsto \mathbb{C}$ defined by

$$a(\mathbf{u}, \mathbf{v}) := (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{0,\Omega} - \omega^2\mu\left(\varepsilon + i\frac{\sigma}{\omega}\right)(\mathbf{u}, \mathbf{v})_{0,\Omega} \tag{6.12}$$

is coercive on $\mathbf{H}(\mathbf{curl};G)$ for any domain G.

REMARK 6.4.2 *For $\sigma > 0$, there exists a positive constant $\alpha > 0$ such that we have the inf-sup condition*

$$\inf_{\mathbf{u}\in\mathbf{V}_0(\Omega)} \sup_{\mathbf{v}\in\mathbf{V}_0(\Omega)} \frac{|a(\mathbf{u}, \mathbf{v})|}{||\mathbf{u}||_{\boldsymbol{H}(\mathbf{curl};\Omega)}||\mathbf{v}||_{\boldsymbol{H}(\mathbf{curl};\Omega)}} \geq \alpha.$$

For $\sigma = 0$ it holds as well, provided that $\lambda := \omega^2\mu\varepsilon$ is not a Maxwell eigenvalue.

In fact, for $\sigma > 0$, the claim is a trivial consequence of the coercivity of $a(\cdot,\cdot)$ as a bilinear form on $\mathbf{H}_0(\mathbf{curl};\Omega)$, [58]. For $\sigma = 0$, the operator A induced by $a(\cdot,\cdot)$ in (6.12) is injective on $\mathbf{V}_0(\Omega)$ since λ is assumed not to be a Maxwell eigenvalue. Moreover, A is selfadjoint, so that A is also surjective, hence one-to-one, which is the desired inf-sup-condition (4.5).

We arrive at the following saddle-point formulation realizing the divergence constraint on Ω for the time-harmonic problem.

PROBLEM 6.4.3 *Given*

$$(\boldsymbol{J}, f) \in (\mathbf{H}_0(\mathbf{curl};\Omega))' \times H^{-1}(\Omega),$$

find a pair

$$(\mathbf{u}, p) \in \mathbf{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$$

such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \nabla\, p)_{0,\Omega} &= i\omega\mu\,(\mathbf{J}, \mathbf{v})_{0,\Omega}, \quad \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega), \\ (\mathbf{u}, \nabla\, q)_{0,\Omega} &= \quad (f, q)_{0,\Omega}, \qquad q \in H_0^1(\Omega), \end{aligned}$$

where $a(\cdot,\cdot)$ is defined by (6.12).

THEOREM 6.4.4 *Assume that $\sigma > 0$ or, when $\sigma = 0$, that $\omega^2\mu\varepsilon$ is not a Maxwell eigenvalue. Then, Problem 6.4.3 is well-posed.*

Proof: By Remark 6.4.2 the bilinear form $a(\cdot,\cdot)$ satisfies the required inf-sup condition, and the claim follows from Lemma 6.3.10 and Theorem 4.1.2. ∎

Boundary Conditions

Above we have imposed only the divergence constraint weakly in terms of Lagrange multipliers, which still requires incorporating the boundary conditions into a discretization in $\mathbf{H}(\mathbf{curl};\Omega)$. In order to treat them with the aid of Lagrange multipliers, we extend the bilinear form in (6.12) from Ω to $\square$ as follows: For $\boldsymbol{u},\boldsymbol{v} \in \mathbf{H}(\mathbf{curl};\square)$, let as in (6.7)

$$a(\boldsymbol{u},\boldsymbol{v})_\square := (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_{0,\square} - \omega^2\mu\left(\varepsilon + i\frac{\sigma}{\omega}\right)(\boldsymbol{u},\boldsymbol{v})_{0,\square}. \tag{6.13}$$

The resulting variational problem then reads:

PROBLEM 6.4.5 *Given the triple*

$$(\boldsymbol{J}, f, \eta) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)' \times H^{-1}(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma),$$

find a triple

$$(\boldsymbol{u}, p, \lambda) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square) \times H^1_0(\Omega) \times (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$$

so that

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v})_\square \quad &+ (\boldsymbol{v},\nabla p)_{0,\Omega} + \langle \boldsymbol{v}\times\boldsymbol{n},\lambda\rangle_\Gamma &&= (\boldsymbol{J},\boldsymbol{v})_{0,\square}, && \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square),\\
(\boldsymbol{u},\nabla q)_{0,\Omega} & &&= (f,q)_\Omega, && q \in H^1_0(\Omega),\\
\langle \boldsymbol{u}\times\boldsymbol{n},\mu\rangle_\Gamma & &&= \langle\eta,\mu\rangle_\Gamma, && \mu \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'.
\end{aligned}$$

Let us start with the case $\sigma > 0$.

COROLLARY 6.4.6 *Problem 6.4.5 is well-posed for $\sigma > 0$.*

Proof: Again we wish to apply Theorem 4.1.2. On account of Lemma 6.3.13, it remains to verify the validity of the conditions (4.4) and (4.5) for the form $a(\cdot,\cdot)_\square$. Since by Corollary 5.3.6 the bilinear form in (6.13) is, in particular, coercive on $\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)$, (4.4) and (4.5) obviously hold. ∎

The question of well-posedness of Problem 6.4.5 for the non-conducting case $\sigma = 0$ turns out to be less straightforward since $a(\cdot,\cdot)_\square$ is no longer coercive. We assume throughout this section that that $\lambda := \omega^2\mu\varepsilon$ is not a Maxwell eigenvalue with respect to Ω in the sense of Definition 6.4.1. This is a reasonable assumption, since otherwise the original Maxwell problem would not be well-posed. However, since $\sigma = 0$, it may very well happen that the equation $(\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_\square = \lambda(\boldsymbol{u},\boldsymbol{v})_\square$, $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)$, has a non-trivial solution $\boldsymbol{u} \neq \boldsymbol{0}$ on $\square$. We shall show that this can be avoided by judiceously choosing $\square$.

Let us again assume that the problem is scaled in such a way that $\Omega \subset \square := (0,a)^3$ for a sufficiently large $a \in \mathbb{R}$.

THEOREM 6.4.7 *Suppose that $\lambda := \omega^2\mu\varepsilon$ is not a Maxwell eigenvalue with respect to Ω. Choose $\square := (0,a)^3$ such that*

$$\lambda \neq \frac{4\pi^2}{a^2}(m^2+n^2+p^2) \tag{6.14}$$

for all m, n, $p \in \mathbb{N}$. Then, Problem 6.4.5 is well-posed.

Note that (6.14) can *always* be satisfied. In fact, if $\pi^2/\lambda \in \mathbb{Q}$ choose $a \in \mathbb{R}\setminus\mathbb{Q}$, otherwise some $a \in \mathbb{Q}$ such that $\Omega \subset \square$. In order to prove Theorem 6.4.7, we consider the following auxiliary problem:

PROBLEM 6.4.8 *Find $\boldsymbol{u} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)$ such that*

$$(\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_\square = \lambda(\boldsymbol{u},\boldsymbol{v})_\square, \qquad \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square). \tag{6.15}$$

LEMMA 6.4.9 *Suppose that $\lambda > 0$ is not a Maxwell eigenvalue. Then, Problem 6.4.5 is well-posed if Problem 6.4.8 has only the trivial solution $\boldsymbol{u} \equiv \mathbf{0}$.*

Proof: If Problem 6.4.8 has only the trivial solution $\boldsymbol{u} \equiv \mathbf{0}$, the operator induced by $a(\cdot,\cdot)_\square$ is injective on $\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)$ and thus the inf-sup-condition (4.5) is satisfied. Hence Lemma 6.3.10 and Lemma 6.3.13 show that Problem 6.4.5 is well-posed. ∎

We shall next identify the non-trivial solutions of (6.15), i.e., eigenfunctions.

LEMMA 6.4.10 *There exists a family of eigenfields for the $\mathbf{curl}\,\mathbf{curl}$ operator on $\boldsymbol{C}^\infty_{\mathrm{per}}(\square)$ with eigenvalues*

$$\lambda_{m,n,p} := \frac{4\pi^2}{a^2}(m^2+n^2+p^2), \quad m,n,p \in \mathbb{N}. \tag{6.16}$$

Moreover, this family is orthogonal and complete in $\boldsymbol{L}_2(\square)$, i.e., the set of finite linear combinations of eigenfields is dense in $\boldsymbol{L}_2(\square)$.

Proof: We rely on the identity $\mathbf{curl}\,\mathbf{curl} = \nabla\,\mathrm{div} - \Delta$, which is valid since we consider smooth functions. Define

$$k_m := \frac{2\pi m}{a},$$

where $m \in \mathbb{N}$, and consider

$$u_3^{(1)}(x_1,x_2,x_3)_{m,n,p} := \sin(k_m x_1)\,\sin(k_n x_2)\,\sin(k_p x_3),$$

where $m,n,p \in \mathbb{N}$. In order to enforce

$$\boldsymbol{u}^{(1)}(\boldsymbol{x})_{m,n,p} := (u_1^{(1)}(\boldsymbol{x})_{m,n,p}, u_2^{(1)}(\boldsymbol{x})_{m,n,p}, u_3^{(1)}(\boldsymbol{x})_{m,n,p})$$

to be divergence-free we set

$$u_1^{(1)}(\boldsymbol{x})_{m,n,p} := -\tfrac{mp}{m^2+n^2}\cos(k_m x_1)\,\sin(k_n x_2)\,\cos(k_p x_3), \tag{6.17}$$

$$u_2^{(1)}(\boldsymbol{x})_{m,n,p} := -\tfrac{np}{m^2+n^2}\sin(k_m x_1)\,\cos(k_n x_2)\,\cos(k_p x_3). \tag{6.18}$$

It is easy to see that

$$\Delta u_i^{(1)}(\boldsymbol{x})_{m,n,p} = -\frac{4\pi^2}{a^2}(m^2+n^2+p^2)\; u_i^{(1)}(\boldsymbol{x})_{m,n,p}, \quad i=1,2,3.$$

Hence $\boldsymbol{u}^{(1)}(\boldsymbol{x})_{m,n,p}$ is an eigenfield of **curl curl** with the desired eigenvalue. Define now

$$u_3^{(2)}(\boldsymbol{x})_{m,n,p} := \sin(k_m x_1)\,\sin(k_n x_2)\,\cos(k_p x_3).$$

In the same manner as just outlined, one can choose the remaining components of $\boldsymbol{u}^{(2)}(\boldsymbol{x})_{m,n,p}$ so that it becomes divergence-free and is an eigenfield of the Laplacian, hence also for **curl curl**. Letting

$$\begin{aligned}
u_3^{(3)}(\boldsymbol{x})_{m,n,p} &:= \sin(k_m x_1)\,\cos(k_n x_2)\,\sin(k_p x_3),\\
u_3^{(4)}(\boldsymbol{x})_{m,n,p} &:= \sin(k_m x_1)\,\cos(k_n x_2)\,\cos(k_p x_3),\\
&\;\;\vdots\quad\vdots\\
u_3^{(8)}(\boldsymbol{x})_{m,n,p} &:= \cos(k_m x_1)\,\cos(k_n x_2)\,\cos(k_p x_3),
\end{aligned}$$

one obtains for all components all necessary combinations of sines and cosines to form a basis for $\boldsymbol{L}_2(\square)$. Because of the constants in (6.17,6.18), the basis is orthogonal, but not orthonormal. Since the family

$$(\boldsymbol{u}^{(1)}(\boldsymbol{x})_{m,n,p}, \boldsymbol{u}^{(2)}(\boldsymbol{x})_{m,n,p}, \ldots, \boldsymbol{u}^{(8)}(\boldsymbol{x})_{m,n,p}), \quad m,n,p \in \mathbb{N}, \tag{6.19}$$

is just a scaled version of the canonical Fourier basis of $\boldsymbol{L}_2(\square)$, it is complete in the sense that any function in $\boldsymbol{L}_2(\square)$ can be approximated arbitrarily well by a finite linear combination of elements in (6.19). ∎

PROPOSITION 6.4.11 *All eigenvalues of* (6.15) *are given by* (6.16).

Proof: Let $\boldsymbol{u}_{m,n,p}$ be any of the eigenfunctions $\boldsymbol{u}^{(i)}(\boldsymbol{x})_{m,n,p}$, $i=1,\ldots,8$, arising in the proof of Lemma 6.4.10. Integrating by parts leads to

$$(\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_{m,n,p}, \boldsymbol{v})_\square = (\mathbf{curl}\,\boldsymbol{u}_{m,n,p}, \mathbf{curl}\,\boldsymbol{v})_\square = \lambda_{m,n,p}(\boldsymbol{u}_{m,n,p}, \boldsymbol{v})_\square,$$

for all $\mathbf{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)$ and hence the eigenvalues (6.16) are also eigenvalues of (6.15).

On the other hand, assume now that $(\boldsymbol{u},\lambda)$ is an eigenpair of (6.15), where $\lambda \neq \lambda_{m,n,p}$ for all $m,n,p \in \mathbb{N}$. Then for any element $\boldsymbol{u}_{m,n,p}$ of the basis (6.19) one obtains

$$\lambda(\boldsymbol{u}, \boldsymbol{u}_{m,n,p})_\square = (\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{u}_{m,n,p})_\square = (\boldsymbol{u}, \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}_{m,n,p})_\square = \lambda_{m,n,p}(\boldsymbol{u}, \boldsymbol{u}_{m,n,p})_\square.$$

Since $\lambda \neq \lambda_{m,n,p}$ for every $m,n,p \in \mathbb{N}$, $\boldsymbol{u}$ is perpendicular to the dense set of finite linear combinations of elements from (6.19), hence $\boldsymbol{u} = \mathbf{0}$. ∎

Proof of Theorem 6.4.7: If (6.14) holds, Lemma 6.4.10 and Proposition 6.4.11 imply that λ cannot be an eigenvalue of (6.15), i.e., Problem 6.4.8 only has the trivial solution $\mathbf{u} = \mathbf{0}$. Hence, by Lemma 6.4.9, Problem 6.4.5 is well-posed. □

6.5 Applications

We conclude by mentioning an interesting potential application of the developments of this chapter.

Formulation of a scattering problem

In the following, we consider a scattering problem. $\Omega \subset \mathbb{R}^n$ is the scatterer, on which a given field $\boldsymbol{E}_i$ is incident, see figure 6.1. The space $\mathbb{R}^n/\Omega$, surrounding the scatterer, is assumed to be occupied by vacuum, hence $\sigma = 0$. The incident field is often taken to be varying sinusoidally in time. The presence of the scatterer gives rise to a scattered field $\boldsymbol{E}_s$, which, due to linearity of the Maxwell equations, also varies sinusoidally in time, and consequently the time-harmonic formulation of the Maxwell equations is the common choice for scattering problems. The scattered field $\boldsymbol{E}_s$ is to be calculated in some subdomain of $\mathbb{R}^n/\Omega$. The total electric field $\boldsymbol{E} = \boldsymbol{E}_i + \boldsymbol{E}_s$ must solve the Maxwell equations. Assuming the scatterer to be a perfectly electrically conducting surface gives $\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$ on $\Gamma_i := \partial\Omega$, where $\boldsymbol{n}$ as usual is the outward pointing normal. Assuming source charges and currents reside outside the domain in which $\boldsymbol{E}_s$ is sought, one has $\rho = 0$ and $\boldsymbol{J} = 0$. Inserting $\boldsymbol{E} = \boldsymbol{E}_i + \boldsymbol{E}_s$ in (5.19) gives that the scattered field $\boldsymbol{E}_s$ must solve (5.19), where the source term on the right hand side is replaced by a term depending appropriately on the known incident field $\boldsymbol{E}_i$. The boundary condition is $\boldsymbol{E}_s \times \boldsymbol{n} = -\boldsymbol{E}_i \times \boldsymbol{n}$ on Γ_i.

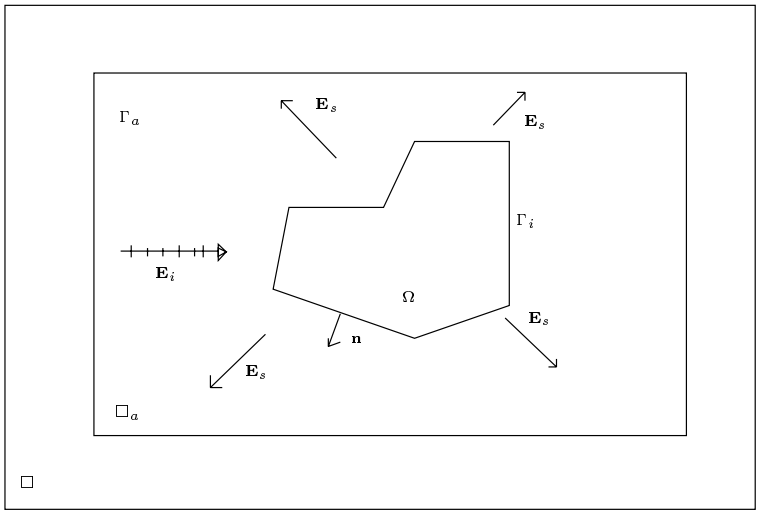

Figure 6.1: The scattering problem.

Imbed now the scatterer Ω in a larger domain $\square_a$, which again is imbedded in $\square$. Absorbing boundary conditions must be imposed on $\Gamma_a = \partial\square_a$, in order to imitate the effect that the scattered field propagates outward without being affected by the artificial boundary Γ_a. The boundary is thus composed by two components: $\Gamma = \Gamma_i \cup \Gamma_a$. We assume that the boundary conditions on the two components is expressed in such a way that

$$\tau := \begin{cases} \boldsymbol{E}_i \times \boldsymbol{n}, & \text{on } \Gamma_i \\ \eta_{ABC}, & \text{on } \Gamma_a \end{cases} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma).$$

The assumption that the absorbing boundary condition on Γ_a makes the tangential trace on Γ belong to $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ is reasonable, since otherwise it would conflict with the overall purpose of absorbing boundary conditions. Indeed, η_{ABC} should ideally be the tangential component of the true solution in all of $\mathbb{R}^3/\Omega$, in particular, a tangential trace of an electric field in $\mathbf{H}(\mathbf{curl};\square/\Omega)$.

Note the sign on the term on Γ_i and the fact that $-\boldsymbol{n}$ is the unit normal pointing outward from $\square/\overline{\Omega}$.

The right hand side is given by

$$\chi(\boldsymbol{E}_i) = \omega^2\mu\varepsilon(\boldsymbol{E}_i - \mathbf{curlcurl}\boldsymbol{E}_i).$$

Without the presence of the scatterer, $\boldsymbol{E}_i$ would be a propatating electromagnetic wave, i.e. a solution to the Maxwell equations. In particular, $\boldsymbol{E}_i$ has sufficient regularity to ensure $\chi(\boldsymbol{E}_i) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)'$.

We arrive at the following problem:

PROBLEM 6.5.1 *Given the triple*

$$(\chi(\boldsymbol{E}_i), 0, \tau) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square)' \times H^{-1}(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma),$$

find a triple

$$(\boldsymbol{u}, p, \lambda) \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square) \times H^1_0(\Omega) \times (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$$

so that

$$\begin{aligned}
a(\boldsymbol{u},\boldsymbol{v})_\square \quad + (\boldsymbol{v}, \nabla p)_{0,\Omega} + \langle \boldsymbol{v}\times\boldsymbol{n}, \lambda\rangle_\Gamma &= (\chi(\boldsymbol{E}_i), \boldsymbol{v})_{0,\square}, \quad && \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square),\\
(\boldsymbol{u}, \nabla q)_{0,\Omega} &= 0, && q \in H^1_0(\Omega),\\
\langle \boldsymbol{u}\times\boldsymbol{n}, \mu\rangle_\Gamma &= \langle \tau, \mu\rangle_\Gamma, && \mu \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'.
\end{aligned}$$

The well-posedness for this problem can be established by theorem 6.4.7, by knowing the Maxwell eigenvalues for Ω, and choosing the dimensions of $\square$ appropriately.

Note that this is an example of a problem with a non-connected boundary, on which one has nonzero boundary conditions. This is a situation which, in the general case, would be very hard to realize if all these properties were to be incorporated into the basis functions employed for the discretization of the problem.

6.6 Closing remarks

In this chapter, we considered Fictitious Domains formulations for the Maxwell equations, with particular focus on preserving non-$\mathbf{H}^1(\Omega)$ singularities in the electric field. The important issue was the combination of imposing weakly the divergence constraint only on the physical domain Ω, and the tangential boundary condition on $\Gamma = \partial\Omega$. This allows the *normal* trace of the electric field to jump across Γ, thereby ensuring non-$\mathbf{H}(\mathrm{div};\square)$ conformity, and in particular, non-$\mathbf{H}^1(\square)$ conformity.

Chapter 7

On sequence space formulations for the Maxwell equations

In the previous chapter, we established well-posed Fictitious Domain formulations for the Maxwell equations. According to the paradigm of Section 3.3, the next step should be the transformation of these variational formulations into well-posed problems in the infinite-dimensional Euclidean ℓ_2 metric, using an appropriately chosen wavelet basis. In the paper [75], wavelet bases in function spaces relevant for the analysis of the Maxwell equations were constructed, whose potential for application to the numerical solution of the Maxwell equations should be investigated. Indeed, this is the main purpose of this chapter.

The outline is as follows. After some preliminary remarks, we introduce the wavelet bases from [75] that are of importance for our purpose. In Section 7.3 we discuss wavelet bases for the space $\mathbf{H}(\mathbf{curl};\Omega)$ in connection with a very important issue for the numerical analysis of the Maxwell equations, namely the influence of the large kernel of the curl operator. Section 7.4 treats a Hodge decomposition approach to the discretization of variational formulations from the previous chapter.

7.1 Preliminaries and objectives

With the road map of Section 4.4 in mind, we consider in this chapter various ways of transforming the variational formulations of Chapter 6 into well-posed problems in the Euclidean metric, for which the method of Section 4.4 applies. We shall take Problem 6.3.7 as point of departure, and consider the bilinear forms

$$a(\boldsymbol{u},\boldsymbol{v}) := (\mathbf{curl}\ \boldsymbol{u},\mathbf{curl}\ \boldsymbol{v})_{0,\square} + (\boldsymbol{u},\boldsymbol{v})_{0,\square}$$

for $\boldsymbol{u},\boldsymbol{v}$ in some subspace $\boldsymbol{X}(\square)$ of $\mathbf{H}(\mathbf{curl};\square)$, and

$$b(\boldsymbol{u},p) := (\boldsymbol{u},\nabla p)_{0,\Omega}$$

for $\boldsymbol{u}\in\boldsymbol{X}(\square)$ and $p\in H^1_0(\Omega)$.

We discuss different choices of wavelet bases $\boldsymbol{\Psi}_X = \{\boldsymbol{\psi}_{\boldsymbol{\lambda},X}\}_{\boldsymbol{\lambda}\in\boldsymbol{\mathcal{J}}_X}$ for the discretization of $a(\cdot,\cdot)$, and $\Psi_M = \{\psi_{\lambda,M}\}_{\lambda\in\mathcal{J}_M}$ for the discretization of $b(\cdot,\cdot)$. The goal is to find wavelet bases having the property that the discretization of $a(\cdot,\cdot)$ with $\boldsymbol{\Psi}_X$, and $b(\cdot,\cdot)$ with $\boldsymbol{\Psi}_X$ and Ψ_M, leads to two operators $\boldsymbol{A}$ and $\boldsymbol{B}$ meeting the requirements of Section 4.4, in particular, the conditions in Theorem 4.4.1.

7.2 Wavelets for the Maxwell equations

Wavelets for the spaces arising in the analysis of the Maxwell equations have been constructed in [75]. Now we outline their theory and construction, and afterwards we discuss their applicability for the solution of the Maxwell equations.

Fundamental assumptions

In this section, we restrict ourselves to domains $\Omega \in \{(0,1)^n, \mathbb{R}^n, \mathbb{T}^n\}$, except where noted. We shall denote by $\square$ an n-dimensional cube. We only consider Hilbert spaces over the real scalar field. From Chapter 2, the following assumptions, which will be the foundation for later developments, hold. First, we define $\boldsymbol{\delta}_i = \{\delta_{1,i}, \ldots, \delta_{n,i}\}$ to be the i-th canonical basis vector in $\mathbb{R}^n$.

ASSUMPTION 7.2.1 *We base the developments on univariate scaling function and wavelet systems of Theorem 2.4.3, and we make the assumptions of this theorem. For all $s = (s_1, \ldots, s_n) \in E$ and all $i \in \{1, \ldots, n\}$ such that $s - \boldsymbol{\delta}_i \in E$, there exist biorthogonal systems $\Psi^{(s)}$, $\tilde{\Psi}^{(s)}$ on $L_2(\Omega)$ with the following properties:*

(i) For any smoothness parameter $s \in E$, $\Psi^{(s)}$ and $\tilde{\Psi}^{(s)}$ are indexed by the same index sets $\mathcal{J} = \{\lambda = (j,k) : \quad j \geq j_0,\ k \in \mathcal{J}_j\}$, where j_0 is a coarsest level.

(ii) The functions in $\Psi^{(s)}$ are compactly supported.

(iii) Regarding $\Psi^{(s)}$ as a column vector of basis functions, one has

$$\partial_i \Psi^{(s)} = D^{(i)} \Psi^{(s-\boldsymbol{\delta}_i)},$$

where $D^{(i)}$ are sparse transformations[1]*.*

(iv) For each $\lambda \in \mathcal{J}$ with $|\lambda| > j_0$ there exists an $i_\lambda \in \{1, \ldots, n\}$ such that the matrix

$$D_{i_\lambda}^{(i_\lambda)} := (D_{\mu,\mu'}^{(i_\lambda)})_{\mu,\mu' \in \mathcal{J}_{|\lambda|}; i_\mu = i_{\mu'} = i_\lambda}$$

is invertible, with

$$\|(D_{i_\lambda}^{(i_\lambda)})^{-1} D_{i_\mu}^{(i_\mu)}\| \lesssim 1$$

for $\mu \in \mathcal{J}_{|\lambda|}$. $\|\cdot\|$ is the matrix norm induced by the Euclidean metric.

(v) An entry $D_{\lambda,\mu}^{(i)} = 0$ if $|\lambda| \neq |\mu|$.

We denote $\ell(\mathcal{J})$ the space of sequences labelled by the index set $\mathcal{J}$. Let $\Psi^{(s)} = \{\psi_\lambda^{(s)}\}_{\lambda \in \mathcal{J}}$ for a smoothness index $s \in E^*$ such that $s - \boldsymbol{\delta}_i \in E^*$. The structure above allows for the definition of a discrete differentiation operator $\mathsf{d}^{(i)} : \ell(\mathcal{J}) \mapsto \ell(\mathcal{J})$, such that for

$$f = c^T \Psi^{(s)} \in H^1(\Omega),$$

then

$$\partial_i f = (\mathsf{d}^{(i)} c)^T \Psi^{(s-\boldsymbol{\delta}_i)}.$$

Specifically,

$$(\mathsf{d}^{(i)} \boldsymbol{c})_\lambda = \sum_{\mu \in \mathcal{J}_{|\lambda|}} D_{\mu,\lambda}^{(i)} c_\mu.$$

[1] see Theorem 2.4.3

Vector wavelets

Wavelet bases for spaces of *vector fields* are defined in a componentwise manner. Recall the indexing of tensor product wavelets in (2.50). We shall provide indexing for the i-th component in the following way:

$$\Psi_j^{[i]} := \{\psi_{j,e,k} : \ e \in E, k \in \mathcal{J}_{j,e}^{[i]}\},$$

and define

$$\boldsymbol{\lambda} := (i,\lambda) \in \{1,\dots,n\} \times \mathcal{J}_j^{[i]}, \text{ and } \boldsymbol{\mathcal{J}}_j := \bigcup_{i=1}^{n} \bigcup_{\lambda \in \mathcal{J}_j^{[i]}} \boldsymbol{\lambda}.$$

A field $\boldsymbol{\psi}_{\boldsymbol{\lambda}}$ is defined componentwise as

$$\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{[i]} = \psi_\lambda \boldsymbol{\delta}_i, \quad \lambda \in \mathcal{J}^{[i]} =: \bigcup_{j \geq j_0} \mathcal{J}_j^{[i]},$$

We reserve boldface letters and symbols for wavelet bases for vector fields. Hence, we write families of vector wavelets as

$$\boldsymbol{\Psi} = \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \ \boldsymbol{\lambda} \in \boldsymbol{\mathcal{J}}\},$$

with a similar notation for dual wavelets.

Discrete differential operators

We can then define the *discrete gradient* $\mathsf{grad} : \ell(\mathcal{J}) \mapsto \ell(\boldsymbol{\mathcal{J}}) := \ell(\mathcal{J}) \times \cdots \times \ell(\mathcal{J})$ as

$$\mathsf{grad}\ \boldsymbol{c} := (\mathsf{d}^{(1)}\boldsymbol{c}, \dots, \mathsf{d}^{(n)}\boldsymbol{c}).$$

We introduce the following short-hand notation for a sequence $\boldsymbol{c} \in \ell(\boldsymbol{\mathcal{J}})$: $\boldsymbol{c} = (c^{(1)}, \dots, c^{(n)})$, where $c^{(i)} = \{c_{i,\lambda}\}_{\lambda \in \mathcal{J}}$. Then, we can define the 2D *discrete curl operator* $\mathsf{curl} : \ell(\boldsymbol{\mathcal{J}}) \mapsto \ell(\mathcal{J})$

$$\boldsymbol{c} = (c^{(1)}, c^{(2)}), \quad \mathsf{curl}\ \boldsymbol{c} := \mathsf{d}^{(1)}c^{(2)} - \mathsf{d}^{(2)}c^{(1)}$$

and the 3D discrete curl operator $\mathsf{curl} : \ell(\boldsymbol{\mathcal{J}}) \mapsto \ell(\boldsymbol{\mathcal{J}})$ as

$$\begin{aligned} &\boldsymbol{c} = (c^{(1)}, c^{(2)}, c^{(3)}), \\ &\mathsf{curl}\ \boldsymbol{c} := (\mathsf{d}^{(2)}c^{(3)} - \mathsf{d}^{(3)}c^{(2)}, \mathsf{d}^{(3)}c^{(1)} - \mathsf{d}^{(1)}c^{(3)}, \mathsf{d}^{(1)}c^{(2)} - \mathsf{d}^{(2)}c^{(1)}). \end{aligned}$$

Recalling Remark 2.5.1, we note that in the multivariate case, the restriction of the discrete differential operator d to any level j has a non-diagonal block.

7.2.1 Wavelets for the space $\mathbf{H}(\mathbf{curl};\Omega)$

DEFINITION 7.2.2 *Given two univariate wavelet families $\Psi_j^{(1)}$ and $\Psi_j^{(0)}$ as in section 2.4.2, we define, in 2D as well as 3D, the basis $\boldsymbol{\Psi}^{curl}$ componentwise, having a tensor product structure, as follows:*

$$\boldsymbol{\psi}_{\boldsymbol{\lambda}}^{[i]} = \psi_\lambda^{(\mathbf{1}-\boldsymbol{\delta}_i)} \boldsymbol{\delta}_i$$

where $\mathbf{1}$ *denotes the* 2 *resp.* 3*-dimensional vector with entries* 1*. Here, we have used the smoothness index convention of (2.49).*

It is immediately verified that all involved basis functions are in $\mathbf{H}(\mathbf{curl};\Omega)$. Let us point out some details of the properties of the $\mathbf{H}(\mathbf{curl};\Omega)$ wavelet basis. They rely on the differentiation calculus described in subsections 2.4.5 and section 2.5, in that they are 'tensorized' such as to facilitate an *exact* evaluation of the curl of an expansion relative to $\mathbf{\Psi}^{\mathrm{curl}}$. This may be clarified by writing the components explicitly, in 2D resp. 3D:

$$2D: \ \mathbf{\Psi}^{\mathrm{curl}} = \begin{pmatrix} \Psi^{(0,1)} \\ \Psi^{(1,0)} \end{pmatrix}, \quad 3D: \ \mathbf{\Psi}^{\mathrm{curl}} = \begin{pmatrix} \Psi^{(0,1,1)} \\ \Psi^{(1,0,1)} \\ \Psi^{(1,1,0)} \end{pmatrix}.$$

Taking, say, in the 2D case a finite expansion $\boldsymbol{f} = \boldsymbol{c}^T \mathbf{\Psi}^{\mathrm{curl}}$ with $\boldsymbol{c} = (c^{[1]}, c^{[2]})$, we can write

$$\mathbf{curl}\ \boldsymbol{f} = (\mathsf{curl}\ \boldsymbol{c})^T \Psi^{(0,0)},$$

and similarly for the 3D case. We define in 2D the space $h(\mathsf{curl})$ as the subspace of $\ell_2(\boldsymbol{\mathcal{J}})$ for which the norm

$$\|\boldsymbol{c}\|^2_{h(\mathsf{curl})} := \sum_{i=1}^{2} \|c^{(i)}\|^2_{\ell_2(\mathcal{J})} + \|\mathsf{curl}\boldsymbol{c}\|^2_{\ell_2(\mathcal{J})}$$

is finite. For the 3D case, we define the space $h(\mathsf{curl})$ as the subspace of $\ell_2(\boldsymbol{\mathcal{J}})$ for which

$$\|\boldsymbol{c}\|^2_{h(\mathsf{curl})} := \sum_{i=1}^{3} \|c^{(i)}\|^2_{\ell_2(\mathcal{J})} + \|\mathsf{curl}\boldsymbol{c}\|^2_{\ell_2(\boldsymbol{\mathcal{J}})}$$

is finite. The following theorem appeared in [75]:

THEOREM 7.2.3 *Under Assumption 7.2.1, the basis $\mathbf{\Psi}^{curl}$ is a basis for $\mathbf{H}(\mathbf{curl};\Omega)$, and one has the estimates*

$$\|\boldsymbol{c}^T \mathbf{\Psi}^{curl}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \sim \|\boldsymbol{c}\|^2_{h(\mathsf{curl})}. \tag{7.1}$$

Although $\mathbf{\Psi}^{\mathrm{curl}}$ is a wavelet basis for $\mathbf{H}(\mathbf{curl};\Omega)$, the naturally induced norm equivalence does not have a form that makes it fit into the framework outlined in Section 3.3. The culprit is the non-diagonality of the curl operator. As well as we can write (3.16) as $\|c^T\Psi\|_H \sim c^T D^2 c$, one can also easily show the existence of bi-infinite matrices $\boldsymbol{C}, \mathbf{G}$ so that (7.1) has the format

$$\|\boldsymbol{c}^T \mathbf{\Psi}^{\mathrm{curl}}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \sim \boldsymbol{c}^T(\boldsymbol{I} + \boldsymbol{C})\boldsymbol{c} =: \boldsymbol{c}^T \mathbf{G} \boldsymbol{c}, \tag{7.2}$$

where $\boldsymbol{C}$ can be thought of as 'the square of curl', and is appropriately composed of the operators $\mathsf{d}^{(i)}$. Since the latter are non-diagonal, so are $\boldsymbol{C}$ and $\mathbf{G}$. This fact is a source of problems, which we shall describe in more detail below.

7.2.2 Divergence free wavelets

From [75], we borrow the following definition.

DEFINITION 7.2.4 *Make the assumption 7.2.1. Define*

$$\boldsymbol{\mathcal{J}}^{df} := \{\boldsymbol{\lambda} = (i,\lambda): \ \lambda \in \mathcal{J}, \ 1 \le i \le n, \ i \ne i_\lambda\}.$$

For $\boldsymbol{\lambda} \in \boldsymbol{\mathcal{J}}^{df}$, define

$$(\boldsymbol{\psi}^{df}_{\boldsymbol{\lambda}})_\nu := \begin{cases} 0, & \nu \notin \{i, i_\lambda\} \\ \psi^{(\boldsymbol{\delta}_i)}_\lambda, & \nu = i \\ -\partial_i (D^{(i_\lambda)}_{i_\lambda})^{-1} \Psi^{(\boldsymbol{\delta}_i + \boldsymbol{\delta}_{i_\lambda})}_{|\lambda|}, & \nu = i_\lambda \end{cases}$$

The system of divergence free wavelets is denoted by $\boldsymbol{\Psi}^{df}$.

The corresponding characterization is given in the following theorem, also borrowed from [75]:

THEOREM 7.2.5 *The basis $\boldsymbol{\Psi}^{df}$ is a basis for $\mathbf{V}_0(\mathrm{div}; \Omega)$, and one has the estimate*

$$\|\boldsymbol{c}^T \boldsymbol{\Psi}^{df}\|^2_{0,\Omega} \sim \|\boldsymbol{c}\|^2_{\ell_2(\boldsymbol{\mathcal{J}}^{df})}$$

The definition above covers divergence free wavelets on $\mathbb{R}^n$, as well as for $(0,1)^n$. In the sequel, we shall employ the *periodic* divergence free wavelets on $\mathbb{R}^n$ (see Section 2.4.4), which we shall denote by $\bar{\boldsymbol{\Psi}}^{df}$.

7.2.3 Curl-free wavelets

Keeping Corollary 5.1.5 in mind, it is natural to define a wavelet basis for the space of functions with vanishing curl as follows: For a wavelet basis $\Psi^{(\mathbf{1})} = \{\psi^{(\mathbf{1})}_\lambda\}_{\lambda \in \mathcal{J}^{cf}}$ for $H^1(\Omega)$, let

$$\boldsymbol{\Psi}^{cf} := \nabla \Psi^{(\mathbf{1})}.$$

Citing [75],

THEOREM 7.2.6 *$\boldsymbol{\Psi}^{cf}$ is a basis for $\mathbf{V}(\mathbf{curl}; \Omega) := \{\boldsymbol{f} \in \mathbf{H}(\mathbf{curl}; \Omega) : \ \mathbf{curl}\, \boldsymbol{f} = \mathbf{0}\}$, and one has the estimate*

$$\|\boldsymbol{c}^T \boldsymbol{\Psi}^{cf}\|^2_{0,\Omega} \sim \|\mathsf{grad}\boldsymbol{c}\|^2_{\ell_2(\mathcal{J}^{cf})} \tag{7.3}$$

In this section, we shall elaborate further on this basis, showing that it allows a slightly different and simpler characterization than that of Theorem 7.2.6. Suppose that the wavelet basis $\Psi^{(\mathbf{1})}$ induces the norm equivalence

$$\|\boldsymbol{c}^T \Psi^{(\mathbf{1})}\|^2_{1,\Omega} \sim \|\boldsymbol{D}\boldsymbol{c}\|^2_{\ell_2(\mathcal{J}^{cf})}. \tag{7.4}$$

holds, with $\boldsymbol{D} = \mathrm{diag}(2^{|\lambda|})$. Regarding the norm equivalence (7.3), we note that on all levels, the univariate scaling functions occur as factors in the tensor product wavelets, hence on each level, the discrete gradient has a non-diagonal block. Below, we shall exploit a well-known norm equivalence in Sobolev space to arrive at a characterization with a diagonal weighting matrix on all levels but the coarsest one. Indeed, it is proven in [65], that for a bounded and connected Lipschitz domain Ω, the norms $\|\cdot\|_{1,\Omega}$ is equivalent to the seminorm $|\cdot|_{1,\Omega}$ on the factor space $H^1(\Omega)/\mathbb{R}$. By definition of the norm on $H^1(\Omega)/\mathbb{R}$, we can write this norm equivalence as

$$\inf_{c \in \mathbb{R}} \|f + c\|_{1,\Omega} \sim |f|_{1,\Omega} = \|\nabla f\|_{0,\Omega}. \tag{7.5}$$

If $f \in H^1(\Omega)$ has zero mean, i.e. if

$$\int_\Omega f dx = 0$$

then straightforward calculations yield $\inf_{c\in\mathbb{R}} \|f+c\|_{1,\Omega} = \|f\|_{1,\Omega}$. With these preparations, we can now prove

PROPOSITION 7.2.7 *Assume that Ω is bounded, connected and Lipschitz. The family $\boldsymbol{\Psi}^{cf}$ is a basis for $\boldsymbol{\mathcal{H}}_2(\Omega)$, for which one has the norm equivalence*

$$q \in H^1(\Omega), \quad q = \boldsymbol{c}^T \Psi^{(\mathbf{1})}, \tag{7.6}$$

$$\|\nabla q\|^2_{0,\Omega} = \left\| \sum_\lambda c_\lambda \boldsymbol{\psi}^{cf}_\lambda \right\|^2_{0,\Omega} \sim \|\mathsf{grad}\boldsymbol{c}\|^2_{\ell_2(\boldsymbol{\mathcal{I}}^{cf}_{j_0})} + \|\boldsymbol{D}\boldsymbol{c}\|^2_{\ell_2(\mathcal{J}^{cf}/\mathcal{J}^{cf}_{j_0})}. \tag{7.7}$$

Proof: Expand any $q \in H^1(\Omega)$ as in (7.6). The estimate for the 'scaling function part' of q takes the form

$$\left\| \sum_{\lambda\in\mathcal{J}_{j_0}} c_\lambda \boldsymbol{\psi}^{cf}_\lambda \right\|^2_{0,\Omega} \sim \|\mathsf{grad}\boldsymbol{c}\|^2_{\ell_2(\boldsymbol{\mathcal{I}}^{cf}_{j_0})},$$

and is the same as that of Theorem 7.2.6. Consider now the 'wavelet part' of q, i.e.

$$q_w = \sum_{\lambda\in\mathcal{J}^{cf}/\mathcal{J}^{cf}_{j_0}} c_\lambda \psi^{(\mathbf{1})}_\lambda.$$

Due to the vanishing moments of the wavelets in $\Psi^{(\mathbf{1})}$, q_w has zero mean. The desired estimate follows now from (7.4) applied to q_w and (7.5). □

7.2.4 A discrete Hodge decomposition

The constructions of the previous two subsections and Corollary 5.1.5 gives, noting that naturally $\boldsymbol{\mathcal{H}}_1(\Omega) \subset \mathbf{V}(\mathrm{div};\Omega)$, a discrete Hodge decomposition as follows [75]:

PROPOSITION 7.2.8 *The decomposition*

$$\mathbf{L}_2(\Omega) = clos_{\mathbf{L}_2(\Omega)} span(\boldsymbol{\Psi}^{df}) \oplus clos_{\mathbf{L}_2(\Omega)} span(\boldsymbol{\Psi}^{cf}) \tag{7.8}$$

is orthogonal in $\mathbf{L}_2(\Omega)$, in $\mathbf{H}(\mathrm{div};\Omega)$ and in $\mathbf{H}(\mathbf{curl};\Omega)$. Furthermore, the decomposition (7.8) is stable, in the sense that for any $\boldsymbol{f} \in \mathbf{L}_2(\Omega)$, the coefficients in the expansion

$$\boldsymbol{f} = \boldsymbol{c}^{df,T}\boldsymbol{\Psi}^{df} + \boldsymbol{c}^{cf,T}\boldsymbol{\Psi}^{cf}$$

satisfy the estimate

$$\|\boldsymbol{f}\|^2_{\mathbf{L}_2(\Omega)} \sim \|\boldsymbol{c}^{df}\|^2_{\ell_2(\boldsymbol{\mathcal{I}}^{df})} + \|\mathsf{grad}\boldsymbol{c}^{cf}\|^2_{\ell_2(\boldsymbol{\mathcal{I}}^{cf}_{j_0})} + \|\boldsymbol{D}\boldsymbol{c}^{cf}\|^2_{\ell_2(\mathcal{J}^{cf}/\mathcal{J}^{cf}_{j_0})}.$$

7.3 The kernel of the curl operator

The variational formulations in Chapter 6 suggest employing wavelet bases for the space $\mathbf{H}(\mathbf{curl};\square)$. We discuss in this section the perspective of employing these wavelet bases for the discretization of variational formulations, such as Problem 6.3.7.

In general, one can say that the progress of fast numerical methods for the Maxwell equations has been slow, in particular because of the large kernel of the curl operator, see Corollary 5.1.5. Along with implications of certain structural properties of the before mentioned wavelet bases, this will be the major topic of discussion in this section. Indeed, this issue will be the main source of difficulty when applying the wavelet bases for the space $\mathbf{H}(\mathbf{curl};\square)$. As outlined in the previous section, the norm equivalence induced naturally by the wavelet bases for $\mathbf{H}(\mathbf{curl};\square)$ involves a *non-diagonal* weighting matrix $\mathbf{G}$ acting on the expansion coefficients. As we shall describe below, the presence of the kernel makes a scaling with a *diagonal* matrix impossible for the wavelet realizations of the involved operators. Then, one has to come to terms with the non-diagonality of the preconditioner, which is a source of difficulty, and places this approach outside the 'natural' framework of the adaptive wavelet-based methods of [25], presented in Chapter 3. The aim of this section is to present some details of this approach, in order to give a clear impression of the nature of the difficulties.

We let $\boldsymbol{\Psi}_X = \boldsymbol{\Psi}^{\mathrm{curl}} = \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\mathcal{J}}_X\}$ be the wavelet basis for $\mathbf{H}(\mathbf{curl};\square)$ from the previous section, and define

$$\check{\boldsymbol{A}}_1 = \big((\mathbf{curl}\boldsymbol{\psi}_{\boldsymbol{\lambda}}, \mathbf{curl}\boldsymbol{\psi}_{\boldsymbol{\lambda}'})_{0,\square}\big)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}, \quad \check{\boldsymbol{A}}_0 := \big((\boldsymbol{\psi}_{\boldsymbol{\lambda}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'})_{0,\square}\big)_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$$

so that the (unscaled) wavelet realization of $a(\cdot,\cdot)$ is

$$\check{\boldsymbol{A}}_{\boldsymbol{\Psi}_X} = (a(\boldsymbol{\psi}_{\boldsymbol{\lambda}}, \boldsymbol{\psi}_{\boldsymbol{\lambda}'}))_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} = \check{\boldsymbol{A}}_1 + \check{\boldsymbol{A}}_0.$$

Taking the approach of Chapter 3, and bearing Theorem 7.2.3 and (7.2) in mind, it is easy to prove that $\mathbf{G}^{-1/2}\check{\boldsymbol{A}}_{\boldsymbol{\Psi}_X}\mathbf{G}^{-1/2}$ is an isomorphism on $\ell_2(\boldsymbol{\mathcal{J}}_X)$. The application of this operator is crucial to Algorithm 3.3.5, and is made difficult by the presence of the inverse square root of the non-diagonal weighting matrix $\mathbf{G}$. For an $N \times N$ (symmetric positive definite) matrix, the extraction of an inverse square root, or any power for that matter, involves a singular value decomposition, which is an $\mathcal{O}(N^3)$ operation, and this would seriously damage the optimal work counts of Algorithm 3.3.5. Therefore, it is tempting to try to employ instead the usual diagonal scaling with $\boldsymbol{D} = \mathrm{diag}(2^{|\boldsymbol{\lambda}|})$.

The operator $\check{\boldsymbol{A}}_1$ is neither bounded on $\ell_2(\boldsymbol{\mathcal{J}}_X)$ [2], nor compressible, cf. Section 3.4, and hence the same holds for $\boldsymbol{A}_{\boldsymbol{\Psi}_X}$. Therefore, we attempt to employ the usual scaling, and consider

$$\boldsymbol{A}_{\boldsymbol{\Psi}_X} := \boldsymbol{D}^{-1}\check{\boldsymbol{A}}_{\boldsymbol{\Psi}_X}\boldsymbol{D}^{-1} = \boldsymbol{D}^{-1}\check{\boldsymbol{A}}_1\boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\check{\boldsymbol{A}}_0\boldsymbol{D}^{-1} =: \boldsymbol{A}_1 + \boldsymbol{A}_0.$$

Now $\boldsymbol{A}_1$ is bounded, and the same arguments in [31] establishing (3.27) can be employed to show that it is also compressible. Consequently, $\boldsymbol{A}_{\boldsymbol{\Psi}_X}$ is bounded and compressible. However, a bound of this operator from below cannot exist. A simple counterexample is provided by considering a sequence $\boldsymbol{u}_k := 2^{-k}\boldsymbol{\psi}^{\mathrm{cf}}_{\lambda_k}$, $k = j_0+1, \ldots$, where $|\lambda_k| = k$. In other words, $\boldsymbol{u}_k$ is a sequence of scaled gradients of wavelets, and belong therefore to the kernel of the curl operator. Since, in particular, $\boldsymbol{u}_k \in \mathbf{H}(\mathbf{curl};\square)$, each $\boldsymbol{u}_k$ has an expansion relative to $\boldsymbol{\Psi}^{\mathrm{curl}}$,

$$\boldsymbol{u}_k = \boldsymbol{c}_k^T\boldsymbol{\Psi}^{\mathrm{curl}}.$$

By the normalization, as showed in (7.3), $\|\boldsymbol{u}_k\|_{\mathbf{H}(\mathbf{curl};\square)} = \|\boldsymbol{u}_k\|_{0,\square} \gtrsim 1$, which by (7.1) shows $\|\boldsymbol{c}_k\|_{\ell_2(\mathcal{J}_X)} \gtrsim 1$. Since $\check{\boldsymbol{A}}_0$ is the Gramian of the Riesz basis $\boldsymbol{\Psi}^{\mathrm{curl}}$ on $\mathbf{L}_2(\Omega)$, it is an isomorphism on $\ell_2(\boldsymbol{\mathcal{J}}_X)$. Consequently, $\|\boldsymbol{A}_{\boldsymbol{\Psi}_X}\boldsymbol{c}_k\| = \|\boldsymbol{A}_0\boldsymbol{c}_k\| \to 0$ as $k \to \infty$, showing the nonexistence of the lower bound.

[2] because of the scaling factors in the operators d

Facing these difficulties, we choose to focus on another approach, based on another wavelet basis for the discretization of the variational problems of Chapter 6.

7.4 Transformation into sequence space via Hodge decomposition

In this section, we still consider the transformation of Problem 6.3.7 into a well-posed problem in the Euclidean metric. We shall consider a strategy, based on the discrete Hodge decomposition of Subsection 7.2.4, in order to circumvent the problem with the kernel of the curl operator, as described above. The 'continuous' Hodge decomposition of Corollary 5.1.5 states that the (first component of the) solution $\boldsymbol{u}$ to Problem 6.3.7 is the orthogonal sum of a divergence free function $\boldsymbol{u}^{df}$ on $\square$ and the gradient of some function $q \in H^1(\square)$. We try to take advantage of this by separating the problem into two subproblems, expanding $\boldsymbol{u}^{df}$ in terms of a (periodic) divergence free wavelet basis, and seek q as a solution to an appropriately defined subproblem, so that when (an approximation to) q is available, its gradient is easily computed and represented in terms of a curl-free wavelet basis.

From now on, we will consider the bilinear form $b(\cdot,\cdot)$ on $H_0^1(\Omega) \times \boldsymbol{X}(\square)$ for $\boldsymbol{X}(\square)$ being a subset of $\mathbf{H}(\mathbf{curl};\square)$, as it was done in Section 4.4.

7.4.1 Motivation and preliminary remarks

We begin by sharpening the Hodge decomposition from Corollary 5.1.5, taking advantage of the simple structure of the domain $\square$. According to [55], for a domain such as $\square$ being *convex, with polyhedral faces*, the space $\boldsymbol{\mathcal{H}}_2(\square)$ has the representation

$$\boldsymbol{\mathcal{H}}_2(\square) = \{\nabla q : \ q \in H_0^1(\square)\}.$$

Obviously, the advantage of this refined decomposition compared to that of Corollary 5.1.5 is that the uniqueness of q up to additive constants is removed.

The wavelet bases

We describe now the choice of wavelet bases we apply for the discretization of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. First, we choose $\boldsymbol{\Psi}_X$ for the discretization of $a(\cdot,\cdot)$ as follows. Let $\Psi^{(\mathbf{1})} = \{\psi_\lambda^{(\mathbf{1})}\}_{\lambda \in \mathcal{J}_X^{cf}}$ be a wavelet basis for $H_0^1(\square)$, boundary adapted along the lines of Subsection 2.4.3, and let it satisfy the characterization

$$\|\boldsymbol{c}\|_{\ell_2(\mathcal{J}_X^{cf})} \sim \|\boldsymbol{c}^T \boldsymbol{D}_{X^{cf}}^{-1} \Psi^{(\mathbf{1})}\|_{1,\square}$$

where $\boldsymbol{D}_{X^{cf}} = \mathrm{diag}(2^{|\lambda|})$. Define as above, $\boldsymbol{\Psi}^{cf} := \nabla \Psi^{(\mathbf{1})}$. Let $\bar{\boldsymbol{\Psi}}^{df} = \{\psi_\lambda^{df}\}_{\lambda \in \mathcal{J}_X^{df}}$ be the periodic divergence free wavelet basis, and we shall assume that they are based upon univariate B-splines of order $d \geq 3$. This ensures that each element in this basis has $\boldsymbol{H}^1(\square)$ regularity, which will be needed later. We define

$$\boldsymbol{\Psi}_X := \bar{\boldsymbol{\Psi}}^{df} \cup \boldsymbol{\Psi}^{cf}.$$

As a basis for the discretization of $b(\cdot,\cdot)$ we choose a basis $\Psi_M = \{\psi_{M,\lambda}\}_{\lambda\in\mathcal{J}_M}$ for $H_0^1(\Omega)$, which induces the norm equivalence

$$\|\boldsymbol{c}\|_{\ell_2(\mathcal{J}_M)} \sim \|\boldsymbol{c}^T \boldsymbol{D}_M^{-1}\Psi_M\|_{1,\Omega}$$

where as above, $\boldsymbol{D}_M = \mathrm{diag}(2^{|\lambda|})$. As indicated in Subsection 2.4.3 and Section 2.6, such bases are known to exist for a large class of domains Ω.

A decomposition

Denote by $\boldsymbol{u}$ the first component of the solution to Problem 6.3.7. Noting first that we are interested in the restriction $\boldsymbol{u}|_\Omega$, we note that according to Corollary 5.1.5, we can write

$$\boldsymbol{u}|_\Omega = \boldsymbol{u}_\Omega^{df} + \nabla q_\Omega,$$

where $\boldsymbol{u}_\Omega^{df} \in \boldsymbol{\mathcal{H}}_1(\Omega)$, $q_\Omega \in \boldsymbol{\mathcal{H}}_2(\Omega)$. With Proposition 7.2.8 and the choice of wavelet basis above in mind, we look for $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl};\square)_{\mathrm{per}}$ on the form

$$\boldsymbol{u} = \boldsymbol{u}^{df} + \nabla q, \tag{7.9}$$

where $\boldsymbol{u}^{df}$ is to be expanded in $\bar{\boldsymbol{\Psi}}^{df}$, and q uniquely determined in $H_0^1(\square)$, so that $\nabla\ q$ has an expansion relative to $\boldsymbol{\Psi}^{cf}$. Somewhat contrary to the spirit of the Fictitious Domain approach, we choose not to employ periodic basis functions on $\square$ for the resolution of ∇q. The reason is that below, we shall eventually solve for q, and the above choice of basis functions brings over to the discrete setting the advantage that uniqueness of q up to constants is removed. This would not be the case if one chose to employ periodic basis functions for the resolution of q.

Define the (for the moment, *unscaled*) operator

$$\check{\boldsymbol{A}}_{\boldsymbol{\Psi}_X} := a(\boldsymbol{\Psi}_X, \boldsymbol{\Psi}_X).$$

The block $\check{\boldsymbol{A}}_{\boldsymbol{\Psi}_X}$ decomposes in a convenient way. From (the proof of) Lemma 6.3.2 and periodicity, we can write

$$(\mathbf{curl}\psi_\lambda^{df}, \mathbf{curl}\psi_{\lambda'}^{df})_{0,\square} = (\nabla\psi_\lambda^{df}, \nabla\psi_{\lambda'}^{df})_{0,\square}$$

for any two $\psi_\lambda^{df}, \psi_{\lambda'}^{df}$ in $\bar{\boldsymbol{\Psi}}^{df}$. Here, we have used the assumption that the elements in $\bar{\boldsymbol{\Psi}}^{df}$ have $\boldsymbol{H}^1(\square)$ smoothness. Then, due to Proposition 7.2.8,

$$\begin{aligned}
\check{\boldsymbol{A}}_{\boldsymbol{\Psi}_X} &:= (\mathbf{curl}\ \boldsymbol{\Psi}, \mathbf{curl}\ \boldsymbol{\Psi})_{0,\square} + (\boldsymbol{\Psi}, \boldsymbol{\Psi})_{0,\square} && (7.10)\\
&= \begin{pmatrix} (\nabla\ \bar{\boldsymbol{\Psi}}^{df}, \nabla\ \bar{\boldsymbol{\Psi}}^{df})_{0,\square} + (\bar{\boldsymbol{\Psi}}^{df}, \bar{\boldsymbol{\Psi}}^{df})_{0,\square} & 0 \\ 0 & (\boldsymbol{\Psi}^{cf}, \boldsymbol{\Psi}^{cf})_{0,\square} \end{pmatrix} && (7.11)\\
&=: \begin{pmatrix} \check{\boldsymbol{A}}^{df} & 0 \\ 0 & \check{\boldsymbol{A}}^{cf} \end{pmatrix}. && (7.12)
\end{aligned}$$

Recalling the remark above, the large kernel of the curl operator is now handled by the block $\boldsymbol{A}^{cf}$. Moreover, employing (5.4), the block $\check{\boldsymbol{B}} := b(\Psi_M, \boldsymbol{\Psi}_X)$ has the following structure:

$$\check{\boldsymbol{B}} = \begin{pmatrix} (\nabla\Psi_M, \bar{\boldsymbol{\Psi}}^{df})_{0,\Omega} \\ (\nabla\Psi_M, \boldsymbol{\Psi}^{cf})_{0,\Omega} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ (\nabla\Psi_M, \boldsymbol{\Psi}^{cf})_{0,\Omega} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \check{\boldsymbol{B}}^{cf} \end{pmatrix}. \tag{7.13}$$

Now we have essentially performed a decoupling of Problem 6.3.7 into a divergence free and a curl free part, and obviously, it suffices to impose the local divergence condition only on the curl free part.

7.4.2 The transformation

The decoupled problems

As a result of the choice of wavelet bases, we can now prescribe a *discrete* formulation of Problem 6.3.7 which decouples into the following two problems. Before we formulate them, we assume the existence of a basis $\Psi_S = \{\psi_{S,\lambda}\}_{\lambda\in\mathcal{J}_S}$ for the space $(\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$. To the knowledge of the author, wavelet bases for the trace space $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ and its dual are still to be constructed. Therefore, as the author is sad to admit, this assumption is somewhat hypothetical.

First, the problem governing the 'divergence free part':

PROBLEM 7.4.1 *For the given* $(\boldsymbol{J}_\square,\eta)\in(\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square))'\times\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$, *define the* (unscaled) *data*

$$\check{\boldsymbol{j}}^{df} := \langle\bar{\boldsymbol{\Psi}}^{df},\boldsymbol{J}_\square\rangle,\quad \check{\boldsymbol{\eta}}^{df} := \langle\Psi_S,\eta\rangle_\Gamma,$$

and the (unscaled) *operators*

$$\check{\boldsymbol{A}}^{df} = a(\bar{\boldsymbol{\Psi}}^{df},\bar{\boldsymbol{\Psi}}^{df})_\square,\quad \check{\boldsymbol{C}}^{df} = \langle\gamma_\tau(\bar{\boldsymbol{\Psi}}^{df}),\Psi_s\rangle_\Gamma.$$

The problem reads

$$\begin{pmatrix}\check{\boldsymbol{A}}^{df} & \check{\boldsymbol{C}}^{df,T}\\ \check{\boldsymbol{C}}^{df} & \end{pmatrix}\begin{pmatrix}\check{\boldsymbol{c}}^{df}\\ \check{\boldsymbol{e}}^{df}\end{pmatrix} = \begin{pmatrix}\check{\boldsymbol{j}}^{df}\\ \check{\boldsymbol{\eta}}^{df}\end{pmatrix}$$

Secondly, the problem governing the 'curl free part':

PROBLEM 7.4.2 *For the given* $(\boldsymbol{J},f,\eta)\in(\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl};\square))'\times H^{-1}(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ *, define the* (unscaled) *data*

$$\check{\boldsymbol{j}}^{cf} := \langle\boldsymbol{\Psi}^{cf},\boldsymbol{J}_\square\rangle,\quad \check{\boldsymbol{f}}^{cf} = \langle\boldsymbol{\Psi}^{cf},f\rangle,\quad \check{\boldsymbol{\eta}}^{cf} := \langle\Psi_S,\eta\rangle_\Gamma,$$

and the (unscaled) *operators*

$$\check{\boldsymbol{A}}^{cf} = a(\boldsymbol{\Psi}^{cf},\boldsymbol{\Psi}^{cf})_\square,\quad \check{\boldsymbol{B}}^{cf} = (\nabla\Psi_M,\boldsymbol{\Psi}^{cf})_{0,\Omega},\quad \check{\boldsymbol{C}}^{df} = \langle\gamma_\tau(\bar{\boldsymbol{\Psi}}^{df}),\Psi_s\rangle_\Gamma.$$

The problem reads

$$\begin{pmatrix}\check{\boldsymbol{A}}^{cf} & \check{\boldsymbol{B}}^{cf,T} & \check{\boldsymbol{C}}^{cf,T}\\ \check{\boldsymbol{B}}^{cf} & & \\ \check{\boldsymbol{C}}^{cf} & & \end{pmatrix}\begin{pmatrix}\check{\boldsymbol{c}}^{df}\\ \check{\boldsymbol{d}}^{cf}\\ \check{\boldsymbol{e}}^{cf}\end{pmatrix} = \begin{pmatrix}\check{\boldsymbol{j}}^{cf}\\ \check{\boldsymbol{f}}^{cf}\\ \check{\boldsymbol{\eta}}^{cf}\end{pmatrix}$$

Clearly, the problems defined above cannot be expected to be well-posed in their corresponding infinte Euclidean metrics. The aim of the next subsections is to consider possibilities of rescaling the problems, in order to turn them into well-posed problems.

The discrete operators

We consider now the two decoupled problems, and formulate conditions that fulfill the assumptions of Theorem 4.4.1, furnishing the transformation of the given variational formulation into sequence space. We focus on the discretization of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, since a closer consideration of the bilinear form on the boundary would be hypothetical, due to the fact that wavelet bases of $(\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma))'$ have, to the knowledge of the author, not been constructed.

First, we make some preparations. In the literature, the following spaces are often considered: For $s > 0$,

$$\boldsymbol{V}^s(\mathrm{div}, \square) := \{\boldsymbol{f} \in \boldsymbol{H}^s(\square) : \ \mathrm{div}\boldsymbol{f} = 0\}.$$

For our purposes, we shall need its periodic version:

$$\boldsymbol{V}^s(\mathrm{div}, \square)_{\mathrm{per}} := \{\boldsymbol{f} \in \boldsymbol{H}^s(\square)_{\mathrm{per}} : \ \mathrm{div}\boldsymbol{f} = 0\}.$$

PROPOSITION 7.4.3

(i) Assume that $\bar{\boldsymbol{\Psi}}^{df}$ *characterizes* $\boldsymbol{V}^1(\mathrm{div}, \square)_{\mathrm{per}}$ *as follows*

$$\|\boldsymbol{c}^T \boldsymbol{D}_{X^{df}}^{-1} \bar{\boldsymbol{\Psi}}^{df}\|_{1,\square}^2 \sim \|\boldsymbol{c}\|_{\ell_2(\boldsymbol{\mathcal{J}}_X^{df})}^2 \tag{7.14}$$

with $\boldsymbol{D}_{X^{df}} = diag 2^{|\lambda|}$. *Then*

$$\boldsymbol{A}^{df} := a(\boldsymbol{D}_{X^{df}}^{-1} \bar{\boldsymbol{\Psi}}^{df}, \boldsymbol{D}_{X^{df}}^{-1} \bar{\boldsymbol{\Psi}}^{df})$$

is an isomorphism on $\ell_2(\mathcal{J}_X^{df})$.

(ii) With $\boldsymbol{D}_{X^{cf}} = diag(2^{|\lambda|})$, *the operator*

$$\boldsymbol{A}^{cf} := a(\boldsymbol{D}_{X^{cf}}^{-1} \boldsymbol{\Psi}^{cf}, \boldsymbol{D}_{X^{cf}}^{-1} \boldsymbol{\Psi}^{cf})$$

is an isomorphism on $\ell_2(\mathcal{J}_X^{cf})$.

(iii) With

$$\boldsymbol{B}^{cf} = (\boldsymbol{D}_M^{-1} \nabla \Psi_M, \boldsymbol{D}_{X^{cf}}^{-1} \boldsymbol{\Psi}^{cf})_{0,\Omega},$$

the operator

$$\begin{pmatrix} \boldsymbol{A}^{cf} & \boldsymbol{B}^{cf,T} \\ \boldsymbol{B}^{cf} & \end{pmatrix} : \ell_2(\mathcal{J}^{cf}) := \ell_2(\mathcal{J}_X^{cf}) \times \ell_2(\mathcal{J}_M) \mapsto \ell_2(\mathcal{J}^{cf})$$

is an isomorphism on $\ell_2(\mathcal{J}^{cf})$.

Proof: We proceed by verifying the conditions of Theorem 4.4.1. For (i), we need only to establish the coercivity of $a(\cdot, \cdot)$ on $\boldsymbol{V}^1(\mathrm{div}, \square)_{\mathrm{per}}$. However, since $\boldsymbol{V}^1(\mathrm{div}, \square)_{\mathrm{per}}$ is equipped with the norm $\|\cdot\|_{1,\square}$, and since Lemma 6.3.2 allows us to write, for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(\mathrm{div}, \square)_{\mathrm{per}}$

$$a^{df}(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_{0,\square} + (\boldsymbol{u}, \boldsymbol{v})_{0,\square},$$

we see that we get coercivity on the space $\boldsymbol{V}^1(\mathrm{div}, \square)_{\mathrm{per}}$. Regarding (ii), we can write

$$\boldsymbol{A}^{cf} = (\boldsymbol{D}_{X^{cf}}^{-1} \nabla \Psi^{(\mathbf{1})}, \boldsymbol{D}_{X^{cf}}^{-1} \nabla \Psi^{(\mathbf{1})})_{0,\square}.$$

Bearing in mind that $\Psi^{(\mathbf{1})}$ is a basis for $H_0^1(\square)$ and that the bilinear form $(\nabla \cdot, \nabla \cdot)_{0,\square}$ is coercive on $H_0^1(\square)$, by the Poincare inequality, we see that $\boldsymbol{A}^{cf}$ is nothing but the (correctly scaled) wavelet realization of the Laplacian on $H_0^1(\square)$, which is, as is well known, an isomorphism on $\ell_2(\mathcal{J}_X^{cf})$. As to (iii), note that the bilinear form $(\nabla \cdot, \nabla \cdot)_{0,\square} : \ H_0^1(\square) \times H_0^1(\square)$ is coercive, and the bilinear form $(\nabla \cdot, \nabla \cdot)_{0,\Omega} : H_0^1(\square) \times H_0^1(\Omega)$ satisfies the inf-sup condition (4.6), since by Lemma 6.3.10, the operator $\mathrm{Div}_\Omega : \ H_0^1(\square) \mapsto H^{-1}(\Omega)$ defined there is onto. ∎

The condition (7.14) can be realized by choosing sufficiently regular univariate scaling functions and wavelets for the construction of the divergence free wavelets.

We note that the operator defined in (iii) is the one arising from the discretization of the following subproblem, which determines q in (7.9):

PROBLEM 7.4.4 *Given* $(\boldsymbol{J}_\Box, f) \in \mathbf{L}_2(\Box) \times H^{-1}(\Omega)$, *find* $(q,p) \in H_0^1(\Box) \times H_0^1(\Omega)$, *such that*

$$(\nabla q, \nabla v)_{0,\Box} + (\nabla v, \nabla p)_{0,\Omega} = (\boldsymbol{J}_\Box, \nabla v)_{0,\Box} \quad \forall v \in H_0^1(\Box), \tag{7.15}$$

$$(\nabla q, \nabla y)_{0,\Omega} = \langle f, y\rangle \qquad \forall y \in H_0^1(\Omega). \tag{7.16}$$

By the remarks in the preceding proof, the inf-sup conditions of the bilinear forms are fulfilled, thereby ensuring well-posedness.

Finally, we note that the compressibility of the involved operators can be established along the lines of the remarks following Proposition 3.4.1.

The data

We make some remarks on the given data. The quantity $\boldsymbol{u}$ in Problem 6.3.7 represents an electric field, and we assume that it is driven by a given current density $\boldsymbol{J}$. As indicated in Theorem 5.3.1, $\boldsymbol{J}$ has regularity in $C^1([0,T],:\mathbf{L}_2(\Omega)) \cap C([0,T],\mathbf{H}(\mathrm{div};\Omega))$. We assume that to any given time instant, we extend $\boldsymbol{J}$ to $\boldsymbol{J}_\Box \in \mathbf{L}_2(\Box)$.

REMARK 7.4.5 *The array defined by*

$$\boldsymbol{j} := \begin{pmatrix} \boldsymbol{j}^{df} \\ \boldsymbol{j}^{cf} \end{pmatrix} = \begin{pmatrix} \boldsymbol{D}^{-1}_{X^{df}}(\bar{\boldsymbol{\Psi}}^{df}, \boldsymbol{J}_\Box)_{0,\Box} \\ \boldsymbol{D}^{-1}_{X^{cf}}(\boldsymbol{\Psi}^{cf}, \boldsymbol{J}_\Box)_{0,\Box} \end{pmatrix}. \tag{7.17}$$

belongs to $\ell_2(\mathcal{J}_X^{df}) \times \ell_2(\mathcal{J}_X^{cf})$. *Indeed, it is clear from the Cauchy-Schwartz inequality that* $(\bar{\boldsymbol{\Psi}}^{df}, \boldsymbol{J}_\Box)_{0,\Box} \in \ell_2(\mathcal{J}_X^{df})$. *Since* $\boldsymbol{J}_\Box \in \mathbf{L}_2(\Box)$, $\mathrm{div}\ \boldsymbol{J}_\Box \in H^{-1}(\Box)$, *and by definition*

$$\boldsymbol{j}^{cf} = \boldsymbol{D}^{-1}_{X^{cf}}(\nabla\Psi^{(\mathbf{1})}, \boldsymbol{J}_\Box)_{0,\Box} = \boldsymbol{D}^{-1}_{X^{cf}}\langle \Psi^{(\mathbf{1})}, \mathrm{div}\ \boldsymbol{J}_\Box\rangle \in \ell_2(\mathcal{J}_X^{cf}).$$

7.5 Closing discussion

We close this chapter with a discussion of the above developments.

Regularity

First, we would like to relate the developments of Section 7.4 to the issues of regularity of solutions to the Maxwell equations, in particular the danger that numerical schemes may not resolve non-$\mathbf{H}^1(\Omega)$ singularities. As described in Chapter 6, for a Fictitious Domains formulation for the Maxwell equations, a way to avoid the trap of not resolving non-$\mathbf{H}^1(\Omega)$ singularities is to impose the divergence condition *weakly* on the physical domain Ω, and imposing the tangential trace condition weakly on Γ as well. As mentioned on Page 79, this allows the *normal* component of the solution $\boldsymbol{u}$ to jump across the boundary Γ, thereby ensuring that $\boldsymbol{u}$ does *not* have a divergence in $\mathbf{L}_2(\Box)$, in particular, $\boldsymbol{u} \notin \mathbf{H}^1(\Box)$.

However, the result stated in (i), Proposition 7.4.3, seems to conflict with these observations. Indeed, in order for the operator $\boldsymbol{A}^{df}$ to be an isomorphism in the Euclidean metric,

the claim (i) suggests the energy space of the first component of a continuous version of Problem 7.4.1 should be $\boldsymbol{V}^1(\mathrm{div}, \square)_{\mathrm{per}}$, which is a subset of $\mathbf{H}^1(\square)_{\mathrm{per}}$. This does not seem to be directly compatible with the fact that the normal trace of the solution to this subproblem shold be allowed to jump across Γ, since this implies non-$\mathbf{H}(\mathrm{div}; \square)$ as well as non-$\mathbf{H}^1(\Omega)$ conformity.

The given data

The 'canonical' approach of Section 3.3 and in particular Theorem 3.3.1 for finding variational problems on appropriate energy spaces, and wavelet bases inducing norm equivalences in the energy spaces, covers automatically the general case of a right hand side in the dual of the energy space. This is not the case of the approach taken above, recall the regularity assumptions on $\boldsymbol{J}$ leading to the estimates of Remark 7.4.5, and as such, this approach is not completely satisfying. The reason for this is that we do not employ a wavelet basis which is tailored for the energy space $\mathbf{H}(\mathbf{curl}; \Omega)$, in the same way as in the case of the Poisson problem, treated with wavelet bases in $H_0^1(\Omega)$. However, relaxing the regularity requirements of the data to $\boldsymbol{J} \in (\mathbf{H}(\mathbf{curl}; \square)_{\mathrm{per}})'$, while obtaining that the discretizations, relative to the union of divergence free and curl free wavelet bases, belong to ℓ_2, does not seem trivial.

Comparison

In summary, we considered the discretization of the bilinear form $a(\cdot, \cdot)$ using two wavelet bases, namely the wavelet basis for $\mathbf{H}(\mathbf{curl}; \square)$, and the union of divergence- and curl free wavelet bases. Both approaches have advantages and disadvantages. The discretization of $a(\cdot, \cdot)$ with the $\mathbf{H}(\mathbf{curl}; \square)$ basis, and employing the usual scaling, leads to an operator, which in principle is easily computable, but which is not an isomorphism in the Euclidean metric, due to the kernel of the curl operator. This problem is less severe when employing the union of divergence- and curl free basis, since this approach leads to a decomposition of the underlying problem into two subproblems, where the kernel of the curl operator only has influence on one of the subproblems. Moreover, this subproblem can be recast as a well-posed problem in the Euclidean metric. The drawbacks of this approach is on one hand the above mentioned regularity issue in the 'divergence free' part of the problem, and the need for imposing additional regularity requirements on the given data.

Periodic wavelets for the 'curl free part'

As described on Page 99, we chose for the 'curl free part' of $\boldsymbol{u}$ to employ a boundary adapted wavelet basis $\Psi^{(\mathbf{1})}$ for the space $H_0^1(\square)$. This has the advantage that q in the (continuous) Hodge decomposition of $\boldsymbol{u}$ is *uniquely* defined, and by the Poincare inequality, the bilinear forms discretized with $\Psi_{X^{cf}}$ immediately fulfill the conditions of 4.4.1. Furthermore, due to the simple structure of the domain $\square$, this basis is very easy to realize, viz. Subsection 2.4.3.

Instead of the basis $\Psi^{(\mathbf{1})}$ of $H_0^1(\square)$ above, one could have used its periodic version $\bar{\Psi}^{(\mathbf{1})}$. Since the span of this basis contains the constants, $\boldsymbol{c}^{cf}$ in the corresponding formulation of Problem (7.4.2) would not be uniquely determined, and one would have to impose an additional requirement on the discretization in order to take care of this nonuniqueness. Due to the remarks leading to Proposition 7.2.7, a suitable norm equivalence induced by this basis is indeed available, and the uniqueness up to constants is reflected by the presence of the (singular) grad operator on the coarsest level in the norm equivalence.

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Chapter 8

Conclusions and outlook

This thesis has concentrated on the applicability of solving the Maxwell equations adaptively, using the wavelet-based methods of [25].

The first major ingredient for the adaptive wavelet-based methods of [25] is well-posed variational formulations. We derived well-posed variational formulations for the Maxwell equations. In particular, we focused on Fictitious Domains methods. It was shown how to ensure convergence to the true solution, i.e. how to resolve non-$\mathbf{H}^1(\Omega)$ singularities. The key point is that imposing the divergence condition only on the physical domain Ω allows, along with the constraint of tangential traces on Γ, the normal trace of the solution to jump across Γ, avoiding in this way global $\mathbf{H}(\mathrm{div};\square)$ regularity for the case of periodic functions on $\square$. Then we considered different approaches for the transformation of the Fictitious Domain formulations into the infintely-dimensional Euclidean space setting. We presented initial results of well-posedness along the lines of [32], unfortunately without reaching fully satisfying results.

Consequently, there is a lot of work to be done before solving the Maxwell equations with the methods of [25] is as well-founded theoretically as, for instance, the Poisson and Stokes problem. Apart from the problems of transforming the variational formulations for the Maxwell equations into well-posed problems in the Euclidean metric, described in the closing discussion of Chapter 7, let us identify some of the most important issues.

The transformation of the Fictitious Domains formulations of Chapter 6 into well-posed problems in the Euclidean metric requires wavelet bases for $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ and its dual. To the knowledge of the author, such wavelet bases have not been constructed. One reason may be that a sound understanding of the properties of these spaces over general Lipschitz domains was not achieved until the recent papers [16],[14],[15]. The results in these papers indicate the need for estimating wavelet expansions in the norm of $H^s(\Gamma)$ for $s \leq -\frac{1}{2}$, which is, in the general case, governed neither by the Wavelet-Element approach of [17], [18], nor the glueing procedure of [41]. Therefore it seems as if one has to employ the work of [42], since it governs trace spaces of a wide range of regularity.

Secondly, there is the question of regularity theory of the Maxwell equations in the scale of Besov spaces $B^s_\tau(L_\tau(\Omega))$, which is important because it provides an upper bound on the error decay of the adaptive algorithm developed in [25]. Regularity theory in this scale of spaces has not been studied very extensively. For the Poisson problem [56] and the Stokes problem [33], such regularity results are known, but to the knowledge of the author, this issue is still open for the Maxwell equations. Valuable information on this subject could very well

be extracted from papers by Mitrea et.al, such as [63], [46].

A third issue, which is important, not in particular for the Maxwell equations, but for the Fictitious Domain formulations in connection with adaptive wavelet-based methods, is the convergence rate when employing a Fictitious Domain formulation. In the course of the work of this thesis, the compressibility potential of the discrete versions of the trace operator was studied closely (by co-researchers of the author), and found to be quite low, which could result in prohibitely slow convergence rates for the adaptive algorithm. Current research is carried out to circumvent this problem, by exploiting the freedom of choosing extensions of the given data to $\square$, in order to obtain a solution on $\square$ which has the optimal smoothness across the boundary Γ of the physical domain Ω.

Of course, an implementation of the algorithm for the solution of the Maxwell equations is important, but as we have just outlined, it is questionable if the time is right for that to be meaningful. Most significantly, this is due to the fact that wavelet bases for $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ are not (to the author's knowledge) known, thus leaving issues of implementation without a sound theoretical foundation.

It should be mentioned that other approaches exist for the numerical solution of the Maxwell equations. So-called *potential formulations* have been considered, in which one does not seek the electric field $\boldsymbol{E}$, but rather the components in the Hodge decomposition of $\boldsymbol{E}$, i.e. the *potential* functions q and $\boldsymbol{u}$ in Corollary 5.1.5. In this approach, there are two unknowns to be resolved before one has an approximation to the electric field, but on the other hand, both of these potentials have $\mathbf{H}^1(\Omega)$ regularity. Since adaptive wavelet-based methods for problems posed in $\mathbf{H}^1(\Omega)$ is a well-studied area, it is plausible that adaptive wavelet-based methods could be prescribed for a potential formulation, in an easier way than for the $\mathbf{H}(\mathbf{curl};\Omega)$ formulations used in this thesis. However, for simplicity, formulations with a minimal number of unknowns were preferred in this work. Furthermore, the interest of the author and collaborators was fueled by the challenges posed by formulations in the space $\mathbf{H}(\mathbf{curl};\Omega)$, such as the existence of non-$\mathbf{H}^1(\Omega)$ singularities, and the fact that wavelet-based methods for problems posed in $\mathbf{H}(\mathbf{curl};\Omega)$ had not been considered previously. Indeed, one can say that the approach of Section 7.4 employing the union of divergence- and curl free wavelet bases, is essentially borrowed from the above mentioned decomposition approach, and suggests an interesting approach for the transformation of the variational formulations into an equivalent, well-posed problem in the Euclidean metric.

In summary, there are still many important and interesting open problems to be dealt with, before efficient adaptive wavelet-based solvers, based upon the work of Cohen, Dahmen and DeVore in [45] are available for the variational formulations of the Maxwell equations, presented in this thesis.

Appendix A

Sobolev and Besov spaces

A.1 Sobolev spaces

We recall some basic properties of Sobolev spaces. There are numerous good sources, of which we mention [1], [68]. $L_2(\Omega)$ is the Hilbert space of square integrable functions over Ω. For $k \in \mathbb{N}$, the Sobolev space $H^k(\Omega)$ is the space of $L_2(\Omega)$ functions having (weak) partial derivatives of order less than or equal to k in $L_2(\Omega)$. Using the multiindex notation, the Sobolev spaces are equipped with the inner product and norm

$$(u,v)_{k,\Omega} := \sum_{|\alpha|\leq k} (\partial^\alpha u, \partial^\alpha v)_{0,\Omega}, \quad \|u\|^2_{k,\Omega} = (u,u)_{k,\Omega}.$$

We denote by $\mathbf{H}^k(\Omega)$ spaces of n-tuples of functions belonging to $H^k(\Omega)$, equipped with the Euclidean inner product.

More generally, for $0 < p \leq \infty$, Sobolev space $W^{k,p}(\Omega)$ is the Banach space of $L_p(\Omega)$ functions with (weak) partial derivatives in $L_p(\Omega)$, equipped with the obvious norm.

The trace mapping in Sobolev spaces is denoted by $\gamma_0 : H^k(\Omega) \mapsto H^{k-\frac{1}{2}}(\Gamma)$, thus the latter space is defined as the space of traces of $H^k(\Omega)$ functions, equipped with the norm

$$||g||_{k-\frac{1}{2},\Gamma} := \inf\{||f||_{k,\Omega} : \ \gamma_0(f) = g\}.$$

We denote by $\mathcal{D}(\Omega)$ the space of C^∞ and compactly supported functions on Ω. We shall also need the space of functions

$$\mathcal{D}(\overline{\Omega}) := \{\phi|_\Omega : \ \phi \in \mathcal{D}(\mathbb{R}^3)\}, \tag{A.1}$$

which is dense in $H^k(\Omega)$ for any $k \geq 0$. We define

$$H^k_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^k(\Omega)}.$$

If Ω is bounded and $k \geq 1$, $H^k_0(\Omega)$ is a closed, genuine subspace of $H^k(\Omega)$. The $H^k_0(\Omega)$ space is also the space of $H^k(\Omega)$ functions with the property that the first $k-1$ normal derivatives vanish on the boundary:

$$u = \frac{\partial u}{\partial n} = \ldots = \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0$$

in the sense of trace. An important result is that when Ω is bounded in at least one direction, the Sobolev seminorm

$$|u|^2_{k,\Omega} := \sum_{|\alpha|=k} \|\partial^\alpha u\|^2_{0,\Omega} \tag{A.2}$$

is equivalent to the Sobolev norm on the space $H_0^k(\Omega)$. The nontrivial inequality in this norm equivalence is known as *Poincares inequality.*

We shall also need Sobolev spaces with negative exponents. Let $k > 0$. Every $L_2(\Omega)$ element u gives rise to a functional on $H_0^k(\Omega)$ as

$$H_0^k(\Omega) \ni v \mapsto (u, v)_{0,\Omega}. \tag{A.3}$$

One can equip the space of such functionals with the usual dual norm, defined as

$$\|u\|_{-k,\Omega} := \sup_{0 \neq v \in H_0^k(\Omega)} \frac{|(u, v)_{0,\Omega}|}{\|v\|_{k,\Omega}}, \tag{A.4}$$

and one defines the space $H^{-k}(\Omega)$ to be the closure of $L_2(\Omega)$ with respect to the $\|\cdot\|_{-k,\Omega}$ norm.

The following integration by parts formulas hold [48]:

(i) For $u, v \in H^1(\Omega)$ and $1 \leq i \leq 3$,

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} v dx + \int_\Gamma \gamma_0(uv) n_i ds \tag{A.5}$$

where n_i is the ith component of the outward unit normal to Γ.

(ii) If also $u \in H^2(\Omega)$,

$$\sum_{i=1}^3 \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = -\sum_{i=1}^3 \int_\Omega \frac{\partial^2 u}{\partial x_i^2} v dx + \sum_{i=1}^3 \int_\Gamma \gamma_0 \left(\frac{\partial u}{\partial x_i} \right) n_i ds. \tag{A.6}$$

We define for $k \in \mathbb{N}$ the negative Sobolev spaces $H^{-k}(\Omega) := (H_0^k(\Omega))'$. The duality pairing is denoted $\langle \cdot, \cdot \rangle_k$.

We shall also need the space $H^1(\Omega)/\mathbb{R}$, which is the space of equivalent classes $\dot{v}$ of $H^1(\Omega)$ modulo the relation

$$f \sim g \;\Leftrightarrow f - g = c, \quad c \in \mathbb{R} \tag{A.7}$$

equipped with the norm

$$\|\dot{v}\|_{H^1(\Omega)/\mathbb{R}} := \inf\{\|v\|_{1,\Omega} : \; v \in \dot{v}\}. \tag{A.8}$$

For the space $H^1(\Omega)/\mathbb{R}$, a Poincare type inequality also holds [64], and hence the Sobolev seminorm (A.2) is equivalent to the Sobolev norm on this space.

A.2 Besov spaces

Secondly, we define for $s > 0$ the Besov space $B_q^s(L_p(\Omega))$, a subspace of $L_p(\Omega)$, as follows. For $f \in L_2(\Omega)$, r an integer and $h > 0$, we define the r-th order difference operator with step h as follows:

$$\Delta_h^r(f, x) := \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f(x + kh). \tag{A.9}$$

The function $\Delta_h^r(f,x)$ is zero by convention when any of the points $x,\ldots,x+kh$ in which f is evaluated falls outside of Ω. The rth order modulus of smoothness in $L_p(\Omega)$ is defined as

$$\omega_r(f,t)_p := \sup_{|h|\le t} \|\Delta_h^r(f,\cdot)\|_{L_p(\Omega)}. \tag{A.10}$$

For $s>0$, we let r denote the smallest integer larger than s, and define f to belong to the Besov space $B_q^s(L_p(\Omega))$ if

$$|f|_{B_q^s(L_p(\Omega))} := \begin{cases} \left(\int_0^\infty (t^{-s}\omega_r(f,t)_p)^q \frac{dt}{t}\right)^{1/q} & 0<q<\infty \\ \sup_{t>0}(t^{-\alpha}\omega_r(f,t)_p) & q=\infty \end{cases} \tag{A.11}$$

is finite. With $\|f\|_{B_q^s(L_p(\Omega))} := |f|_{B_q^s(L_p(\Omega))} + \|f\|_{L_2(\Omega)}$, the Besov space $B_q^s(L_p(\Omega))$ becomes normed.

Recall the terminology from Section 3.2 about interpolation spaces. For an open, connected set Ω, it can be proved [45] that for $X=L_p(\Omega)$ and $Y=W^{r,p}(\Omega)$, one can prove that for $t>0$,

$$K(f,t^r) \sim \omega_r(f,t)_p$$

and consequently

$$(L_p(\Omega), W^{r,p}(\Omega))_{\theta,q} = B_q^{\theta r}(L_p(\Omega)), \quad 0<\theta<1,\ 0<q\le\infty$$

hence Besov spaces arise from interpolation between L_p and Sobolev spaces.

List of symbols

General

$\mathbb{R}$	The real number field.
$\mathbb{C}$	The complex number field.
$\mathbb{N}$	The natural numbers.
$\mathbb{Z}$	The integers.
$\mathbb{Q}$	The rational numbers.
$\mathbb{T}$	The torus $\mathbb{R}/[0,1)$.
$\mathcal{L}(X,Y)$	Bounded operators from the Banach space X to the Banach space Y.
$X \hookrightarrow Y$	The Banach space X is continuously imbedded in the Banach space Y.
$\boldsymbol{c}^T$	Transpose of a vector or a matrix.
$\mathrm{clos}_X(Y)$	Closure of the set Y in the norm of the Banach space X.
$\mathrm{span}(X)$	Space of finite linear combinations of vectors in the space X.
X'	The normed dual of the Banach space X.
T'	The adjoint of an operator T.
$(\cdot,\cdot)_{k,\Omega}$	Inner product on the Hilbert space $H^k(\Omega)$, $k \in \mathbb{N}$.
$\langle\cdot,\cdot\rangle_X$	Duality pairing between the Banach space X and its normed dual X'.
∂_i	Short notation for partial differentiation $\frac{\partial}{\partial x_i}$.
$\bar{f}$	Periodization of a function f.

Wavelets

j	Scale parameter, level.
k	Translation parameter.
ξ	Univariate scaling function.
η	Univariate wavelet.
$\zeta_{j,k}$	For a function $\zeta : \mathbb{R} \mapsto \mathbb{R}$, $\zeta_{j,k} = 2^{j/2}\zeta(2^j \cdot -k)$
${}_d\xi$	Univariate B-spline of order d.
I_j	Index set of translates of the univariate scaling function ξ on level j.
J_j	Index set of translates of the univariate wavelet η on the level j.
Ξ_j	Family $\{\xi_{j,k} : \ k \in I_j\}$ of translates of the scaling function ξ.
Υ_j	Family $\{\eta_{j,k} : \ k \in J_j\}$ of translates of the wavelet η.

${}_{d,\tilde{d}}\xi$	Dual B-spline of order $d, \tilde{d}$.
${}_{d,\tilde{d}}\eta$	Primal B-spline wavelet.
${}_{d,\tilde{d}}\tilde{\eta}$	Dual B-spline wavelet.
$\mathcal{S}_j$	Multiresolution space, average space.
$\mathcal{U}_j$	Wavelet space, detail space.
$\tilde{\mathcal{S}}_j$	Dual multiresolution space.
$\tilde{\mathcal{U}}_j$	Dual wavelet space.
Q_j	Projector $Q_j : L_2 \mapsto \mathcal{S}_j$.
$\boldsymbol{M}_{j,0}$	Refinement operator.
$\boldsymbol{M}_{j,1}$	Completion.
$\boldsymbol{M}_j$	Transformation matrix $\boldsymbol{M}_j = [\boldsymbol{M}_{j,0}, \boldsymbol{M}_{j,1}]$
$\boldsymbol{B}_j$	Inverse transformation matrix.
E	$\{0,1\}^n$, wavelet type parameter.
E^*	$E/\{\mathbf{0}\}$.
e	Type parameter $e \in E$.
ϕ	Tensor product of univariate scaling functions.
ψ	Tensor product of univariate wavelets and scaling functions.
λ	Index for multivariate scaling function or wavelet: $\lambda = (j, e, k_1, \ldots, k_n)$.
$\lvert\lambda\rvert$	j
$\mathcal{I}_j$	Index set of multivariate scaling functions on level j.
$\mathcal{J}_j$	Index set of multivariate wavelets on level j.
Φ_j	Family $\{\phi_\lambda : \lambda \in \mathcal{I}_j\}$ of tensor products of univariate scaling functions.
Ψ_j	Family $\{\psi_\lambda : \lambda \in \mathcal{J}_j\}$ of tensor products of wavelets and scaling functions.
$\boldsymbol{\lambda}$	(i, λ), index for vector valued scaling functions and wavelets.
$\boldsymbol{\phi}$	Vector valued scaling function.
$\boldsymbol{\psi}$	Vector valued wavelet.
$\boldsymbol{\mathcal{I}}_j$	Index set for fields of scaling functions.
$\boldsymbol{\mathcal{J}}_j$	Index set for fields of wavelets.
$\boldsymbol{\Phi}_j$	Family $\{\boldsymbol{\phi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\mathcal{I}}_j\}$ of vector fields of scaling functions.
$\boldsymbol{\Psi}_j$	Family $\{\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \boldsymbol{\mathcal{J}}_j\}$ of vector fields of wavelets.

Function spaces

$L_2(\Omega)$	Square integrable functions f on Ω.
$\mathbf{L}_2(\Omega)$	Square integrable vector fields $\boldsymbol{f}$ on Ω.
$(\cdot,\cdot)_{0,\Omega}$	Inner product on $L_2(\Omega)$.
$H^s(\Omega)$	Sobolev space of functions over Ω.
$\mathbf{H}^1_{\mathrm{Tan}}(\Omega)$	$\{\boldsymbol{f} \in \mathbf{H}^1(\Omega) : \ \boldsymbol{f} \times \boldsymbol{n} = \mathbf{0}\}$.
$(\cdot,\cdot)_{k,\Omega}$	Inner product on $H^k(\Omega)$, $k \in \mathbb{Z}$.
$B^s_q(L_p(\Omega))$	Besov space of smoothness s measured wrt. q and p.
$\mathbf{H}(\mathbf{curl};\Omega)$	$\{\boldsymbol{f} \in \mathbf{L}_2(\Omega) : \ \mathbf{curl}\ \boldsymbol{f} \in \mathbf{L}_2(\Omega)\}$
$\mathbf{H}(\mathrm{div};\Omega)$	$\{\boldsymbol{f} \in \mathbf{L}_2(\Omega) : \mathrm{div}\ \boldsymbol{f} \in L_2(\Omega)\}$
$\mathbf{H}_0(\mathbf{curl};\Omega)$	$\mathrm{clos}_{\mathbf{H}(\mathbf{curl};\Omega)}(\mathcal{D}(\Omega))$.
$\mathbf{H}_0(\mathrm{div};\Omega)$	$\mathrm{clos}_{\mathbf{H}(\mathrm{div};\Omega)}(\mathcal{D}(\Omega))$.
$\mathbf{V}(\mathrm{div};\Omega)$	$\{\boldsymbol{f} \in \mathbf{L}_2(\Omega) : \mathrm{div}\ \boldsymbol{f} = 0\}$

Physical quantities

$\boldsymbol{E}$	Electric field.
$\boldsymbol{B}$	Magnetic flux density.
ε	Electric permeability.
μ	Magnetic permittivity.
σ	Conductivity.
$\boldsymbol{J}$	Current density.
ρ	Charge density.
ω	Angular frequency.

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