Lecture #3
Recursion (2)

Algorithm

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In Previous Lecture

Concept of recursion

- When it is defined in terms of itself, it is called recursion
  - Base case(s) and recursive step
- Recursion can simply describe an algorithm into several terms

How to design and analyze recursion

- Divide and conqure
  - Divide the problem into several (smaller) sub-problems
  - Conquer them separately & aggregate the results if necessary
- Mathematical induction
  - If $k-1$-th domino falls, then k-th domino falls surely
  - Prove base cases and inductive step

In This Lecture

How to analyze a recursive complexity function

- Repeated substitution
- Mathmetical induction
- Master theorem

Outline

Recursive complexity ☜

Repeated substitution

Mathematical induction

Master theorem

Complexity of Recursive Alg.

Problem: Exponentiation (or power)

- **Input**: base number a and exponent n (non-negative int.)
- **Output**: to calculate a^n

```
def power(a, n):
    if n == 0:
        return 1
    else:
        return power(a, n-1) * a
```

- **Q: What is the time complexity of the above algorithm?**
  - It's also represented **recursively** as recurrence relation
  - We need to solve the recurrence relation to obtain its closed form

Recursive Complexity

Let $T(n)$ denote the time complexity of power(a, n)

- $T(n)$: # of operations to solve the problem which size is n
  - **If n is a base case,**
    - It just returns 1 requiring constant time, i.e., $T(n) = C$
  - **If n is in a recursive step,**
    - 1) power(a, n) calls power(a, n-1) which requires $T(n-1)$
    - 2) After then, the result is multiplied by a requiring constant time C
    - i.e., $T(n) = T(n-1) + C$

problem of size n

```
def power(a, n):
    if n == 0:
        return 1
    else:
        return power(a, n-1) * a
```

sub-problem of size n-1

$$T(n) = \begin{cases} C & n = 0 \\ T(n-1) + C & n > 0 \end{cases}$$

Q. What is its closed form?

Another Recursive Algorithm

Let's divide the problem as follows:

If n is even: $$a^n = (a\times a)\times(a\times a)\times\cdots\times(a\times a) = (a^2)^{\frac{n}{2}}$$

If n is odd: $$a^n = a\times(a\times a)\times(a\times a)\times\cdots\times(a\times a) = a\times(a^2)^{\frac{n-1}{2}}$$

```
def power(a, n):
    if n == 0:
        return 1
    else if n is even:
        return power(a * a, n/2)
    else:
        return a * power(a * a, (n-1)/2)
```

$$T(n) = \begin{cases} C & n == 0 \\ T\left(\frac{n}{2}\right) + C & n \text{ is even} \\ T\left(\frac{n-1}{2}\right) + C & n \text{ is odd} \end{cases}$$

Q. What is its closed form?

Fibonacci Number

Problem: Fibonacci number

- **Input**: n indicates n-th Fibonacci number, i.e., F_n
- **Output**: to calculate n-th Fibonacci number F_n defined by

$$F_n = F_{n-2} + F_{n-1}$$

  - The sum of two consecutive Fibonacci numbers
  - $F_0 = 0$ and $F_1 = 1$ by its definition (base cases)
- Let $T(n)$ be the time complexity of its recursive algorithm

```
def fib(n):
    if n <= 1:
        return n
    else:
        return fib(n-2) + fib(n-1)
```

$$\Rightarrow \quad T(n) = \begin{cases} C & n \leq 1 \\ T(n-2) + T(n-1) + C & n > 1 \end{cases}$$

Q. What is its closed form?

Outline

Recursive complexity

Repeated Substitution ☜

Mathematical induction

Master theorem

Repeated Substitution (1)

Basic idea of repeated substitution

- Repeatedly substitute the complexity function whose input size decreases toward a base case

Example: $T(n) = T(n-1) + C$

- $\mathrm{T}(1) \leq \mathrm{C}$ as its base case

substitute
$T(n-1) = T(n-2) + C$

$$
\begin{aligned}
T(n) &= T(n-1) + C \\
&= T(n-2) + 2C \\
&= T(n-3) + 3C \\
&= \cdots \\
&= T(1) + (n-1)C \leq Cn = O(n)
\end{aligned}
$$

Repeated Substitution (2)

Example: $T(n) \le 2T\left(\frac{n}{2}\right) + n$

- $\mathrm{T}(1) \le \mathrm{C}$ as its base case

$$
\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + n \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n = 2^2T\left(\frac{n}{2^2}\right) + 2n \\
&\le 2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + 2n = 2^3T\left(\frac{n}{2^3}\right) + 3n \\
&= \cdots
\end{aligned}
$$

Assume $n = 2^k \rightarrow$

$$
\begin{aligned}
&\le 2^kT\left(\frac{n}{2^k}\right) + kn = nT(1) + n\log n \\
&\le nC + n\log n = O(n\log n)
\end{aligned}
$$

Assumption on $n = 2^k$

Details

$n = 2^k$ is often assumed for computational ease

- **Fact:** There is always a power of 2 between n & $2n$
  - i.e., $n \le 2^k \le 2n$ (see the appendix)
- If $T(n) = O(n^r)$, $T(2n) = O\big((2n)^r\big) = O(2^r n^r) = O(n^r)$
  - $T(n)$ and $T(2n)$ have the same asymptotic behavior, i.e., as $n \to \infty$, $T(n) = T(2n)$
- If $T(n)$ is polynomial & its leading coefficient is positive, $T(n)$ is monotonically increasing as $n \to \infty$
  - $\Rightarrow n \le 2^k \le 2n \Leftrightarrow T(n) \le T\left(2^k\right) \le T(2n)$
  - $\Rightarrow T(n) = T\left(2^k\right) = T(2n)$ because $T(n) = T(2n)$
- Thus, it's okay if we assume the size n of input is 2^k since they have the same Big-O bound

Repeated Substitution

Pros

- Intuitive and easy to calculate
- Effective for a recursive complexity in a simple form

Cons

- Prone to make mistakes
- Require a lot of efforts when we apply the method to a complicated complexity function
  - Try to solve the following problem using repeated substitution

$$T(n) = 3T\left(\frac{n}{4}\right) + \sqrt{n} \times \log n$$

Outline

Recursive complexity

Repeated substitution

Mathematical induction

Master theorem

Mathematical Induction (1)

Basic idea of mathematical induction

- Estimate the closed form of a recursive complexity, and then prove it by induction

Example

- Claim: $T(n) \le 2T(n/2) + n$ and its closed form is $T(n) \le cn \log n$ for positive c and large n
- Base case
  - If $n = 2$, there is always positive c such that $T(2) \le c2 \log 2$
- Inductive step
  - **Previous case**: assume the claim holds for $n = k/2$
  - **Next case**: does the claim hold for $n = k$?

Mathematical Induction (2)

- Claim: $T(n) \leq 2T(n/2) + n$ and its closed form is $T(n) \leq cn\log n$ for positive c and large n
- Inductive step
  - **Previous case**: assume the claim holds for $\frac{k}{2} \Rightarrow T\left(\frac{k}{2}\right) \leq c\left(\frac{k}{2}\right)\log\frac{k}{2}$
  - **Next case**: does the claim hold for $n = k$?

$$T(k) \leq 2T\left(\frac{k}{2}\right) + k$$

Use the assumption → $\leq 2c\left(\frac{k}{2}\right)\log\frac{k}{2} + k = ck\log k - ck\log 2 + k$

$$= ck\log k + \underbrace{(-c\log 2 + 1)}_{\text{Set } c \text{ such that } -c\log 2 + 1 < 0 \Leftrightarrow c > \frac{1}{\log 2} = 1}k$$

Also true for k → $\leq ck\log k$

  - Thus, there is always c satisfying $T(n) \leq cn\log n$ for any n
  - Meaning $T(n) = O(n\log n)$

No Consideration of Base Case

Don’t need to consider the part of base case

- When solving a recursive complexity by induction
- Why?
  - Suppose we should show $T(n) = f(n)$
  - Then, we show $T(a) \leq cf(a)$ for constant a as a base case
  - Note that $T(a)$ is also constant and normally, & $f(n)$ returns a positive number (imagine it’s a time complexity)
  - Thus, there is always c satisfying $T(a) \leq cf(a)$

- **Thus, we only consider the inductive step for such proofs**

Other Example – Wrong Version

Claim: $T(n) \le 2T(n/2) + 1$ and it's $O(n)$

- Estimate $T(n) \le cn$
- Inductive step
  - **Previous case**: assume the claim holds for $n = \frac{k}{2}$

$$T\left(\frac{k}{2}\right) \le c\,\frac{k}{2}$$

  - **Next case**: does the claim hold for $n = k$?

$$\begin{aligned} T(k) &\le 2T\left(\frac{k}{2}\right) + 1 \\ &\le 2c\,\frac{k}{2} + 1 \\ &= ck + 1 \end{aligned}$$

  - Note that we cannot say $ck + 1 \le ck$; the proving fails

Other Example – Correct Version

Claim: $T(n) \leq 2T(n/2) + 1$ and it's $O(n)$

- Estimate $T(n) \leq cn - 2$
- Inductive step
  - **Previous case**: assume the claim holds for $n = \frac{k}{2}$

$$T\left(\frac{k}{2}\right) \leq c\frac{k}{2} - 2$$

  - **Next case**: does the claim hold for $n = k$?

$$\begin{aligned} T(k) &\leq 2T\left(\frac{k}{2}\right) + 1 \\ &\leq 2c\frac{k}{2} - 4 + 1 \\ &= ck - 3 \leq ck - 2 \end{aligned}$$

  - Now the proving is correct since the result has the same form

Mathematical Induction

Pros

- For a complicated function, it's easier than the repeated substitution method
  - If an effective bound is provided…

Cons

- Need to estimate "effective" bound
  - Loose bound is meaningless
  - Excessively tight bound will not be proved
- Intuition for such estimate is from experience
  - Need to solve a lot of problems in this way

Outline

Recursive complexity

Repeated substitution

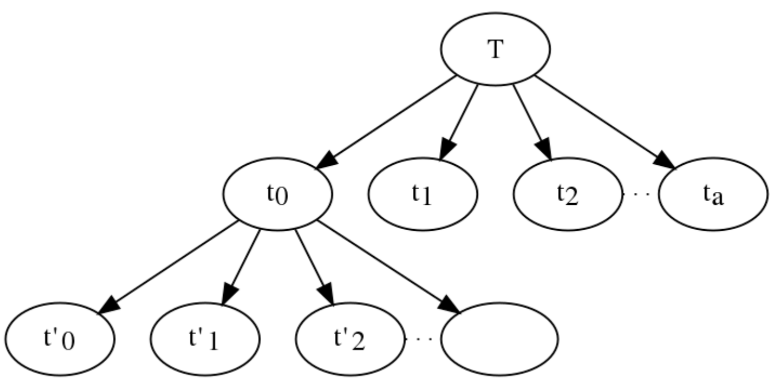
Mathematical induction

Master theorem

Generalization of Recursive Alg.

Most recursive algorithms are based on Divide & Conquer as follows

```
def procedure(n):
    if n <= some constant k like 1
        solve the input directly without recursion
    else:
        create a subproblems, each having size n/b
        call procedure p recursively on each subproblem
        aggregate the results from the subproblems
```



Solution tree

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$f(n)$: remaining cost to split the problem and recombining the results

Master Theorem

Master theorem allows many recurrence relations of the following form to be converted Θ directly!

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- n: input size for problem
- $a \geq 1$: number of sub-problems
- n/b: size of input for each sub-problem where $b \geq 1$
- $f(n)$: remaining cost (overhead)
  - to split the problem and recombining the results at the top level

There is an exact version of Master Theorem

- But not discussed in this lecture since it's complicated
- Instead, let's check its approximate version

Master Theorem

Master Theorem [approximate version]

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

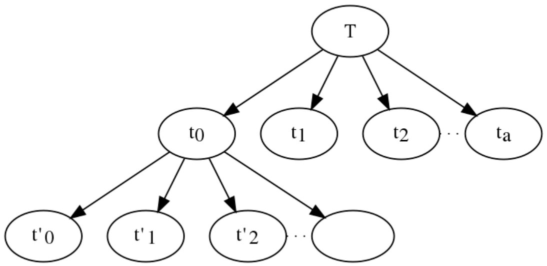
- Suppose $h(n) = n^{\log_b a}$
- **Case 1)** $\lim_{n \to \infty} \frac{f(n)}{h(n)} = 0 \Rightarrow T(n) = \Theta\big(h(n)\big)$
- **Case 2)** $\lim_{n \to \infty} \frac{f(n)}{h(n)} = \infty \,\&\, af\left(\frac{n}{b}\right) < f(n) \Rightarrow T(n) = \Theta\big(f(n)\big)$
- **Case 3)** $\frac{f(n)}{h(n)} = \Theta(1) \Rightarrow T(n) = \Theta(h(n) \log n)$

Proving the Master theorem is beyond this course (it's graduate level)

Interpretation (1)

Master Theorem [approximate version]

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$



Solution tree

- Suppose $h(n) = n^{\log_b a}$
  - ⇒ # of leaf nodes in the solution tree
  - ⇒ # of problems where the input size is 1
  - ⇒ cost for solving all problems where the input size is 1
- The depth of the solution tree is $k = \log_b n$
  - Size changes as $n \rightarrow \frac{n}{b} \rightarrow \cdots \rightarrow \frac{n}{b^k}$; when $\frac{n}{b^k} = 1$, it reaches at a leaf
- The number of leaf nodes at level k is $a^k = a^{\log_b n} = n^{\log_b a}$
  - $a^{\log_b n} = x \Leftrightarrow \log_b n \times \log_b a = \log_b x \Leftrightarrow \log_b n^{\log_b a} = \log_b x$

Interpretation (2)

Master Theorem [approximate version]

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- **Case 1)** $\lim_{n\to\infty} \frac{f(n)}{h(n)} = 0 \Rightarrow T(n) = \Theta\big(h(n)\big)$
  - Condition 1) $\lim_{n\to\infty} \frac{f(n)}{h(n)} = 0$ meaning $h(n)$ overwhelms $f(n)$
    - $h(n)$: cost for solving all problems where the input size is 1
    - $f(n)$: remaining cost (overhead) to split the problem & recombining the results at the top level
  - Then, $h(n)$ determines the complexity $T(n)$
    - The condition is called “the solution tree is **leaf-heavy**”

Interpretation (3)

Master Theorem [approximate version]

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- **Case 2)** $\lim_{n\to\infty}\frac{f(n)}{h(n)} = \infty$ & $af\left(\frac{n}{b}\right) < f(n) \Rightarrow T(n) = \Theta\big(f(n)\big)$
  - Condition 1) $\lim_{n\to\infty}\frac{f(n)}{h(n)} = \infty$ meaning if $f(n)$ overwhelms $h(n)$
  - Condition 2) $af\left(\frac{n}{b}\right) < f(n)$
    - $af\left(\frac{n}{b}\right)$: the sum of remaining costs of all sub-problems at the children level
    - $f(n)$: remaining cost (overhead) of problem at the root level
    - When the recursion goes to the below level, the overhead cost should decrease!
  - Then, $f(n)$ determines the complexity $T(n)$
    - These conditions are called “the solution tree is **root-heavy**”

Interpretation (4)

Master Theorem [approximate version]

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- **Case 3)** $\frac{f(n)}{h(n)} = \Theta(1) \Rightarrow T(n) = \Theta(h(n)\log n)$
  - Condition 1) $\frac{f(n)}{h(n)} = \Theta(1)$ meaning if their weights are comparable by a constant
    - Work to split/recombine a problem is comparable to sub-problems
  - Then, $h(n)\log n$ determines the complexity $T(n)$

Examples With Master Theorem

Get the closed form of the following complexities

- **Case 1**: $T(n) = 2T\left(\frac{n}{3}\right) + c$
  - $a = 2,\ b = 3 \Rightarrow h(n) = n^{\log_3 2}$ and $f(n) = c$
  - $\lim_{n\to\infty} \frac{f(n)}{h(n)} = \lim_{n\to\infty} \frac{c}{n^{\log_3 2}} = \infty$ thus, $T(n) = \Theta\left(n^{\log_3 2}\right)$
- **Case 2**: $T(n) = 2T\left(\frac{n}{4}\right) + n$
  - $a = 2,\ b = 4 \Rightarrow h(n) = n^{\log_4 2} = \sqrt{n}$ and $f(n) = n$
  - $\lim_{n\to\infty} \frac{f(n)}{h(n)} = \lim_{n\to\infty} \frac{n}{\sqrt{n}} = 0$ & $af\left(\frac{n}{b}\right) < f(n) \Rightarrow 2\frac{n}{4} = \frac{n}{2} < n$; thus, $T(n) = \Theta(n)$
- **Case 3**: $T(n) = 2T\left(\frac{n}{2}\right) + n$
  - $a = 2,\ b = 2 \Rightarrow h(n) = n^{\log_2 2} = n$ and $f(n) = n$
  - $\frac{f(n)}{h(n)} = 1 = \Theta(1)$; thus, $T(n) = \Theta(n \log n)$

Other Technique

Changing variables makes an equation simple

- $T(n) = 2T(\sqrt{n}) + \log_2 n$
  - Let $m = \log_2 n \Rightarrow 2^m = n$
- $\Rightarrow T(2^m) = 2T\left(2^{\frac{m}{2}}\right) + m$
  - Let $P(m)$ denote $T(2^m)$
- $\Rightarrow P(m) = 2P\left(\frac{m}{2}\right) + m$
  - By Master theorem, $P(m) = \Theta(m \log m)$

- Thus, $T(n) = P(m) = \Theta(m \log m) = \Theta(\log n \log(\log n))$

Master Theorem [Approx.]

Pros

- Very convenient and can apply it to an arbitrary complexity function in form of $T(n) = aT(n/b) + f(n)$
  - Do not need to calculate or prove something

Cons

- For some cases, this approximate version cannot be applied
  - In these cases, need to use the exact version (see the textbook)
- Note that not all recurrence relations can be solved with this theorem

What You Need To Know

Recursive complexity

- Complexity of a recursive algorithm is also represented recursively as recurrence relation

How to analyze a recursive complexity function

- **Repeated substitution**
  - Repeatedly substitute the complexity function whose input size decreases toward a base case
- **Mathmetical induction**
  - Estimate the closed form of a recursive complexity, and then prove it by induction
- **Master theorem**
  - Can solve any function in form of $T(n) = aT(n/b) + f(n)$
- For some cases, changing variables makes an equation simple

In Next Lecture

Sorting problem

Basic sorting algorithms

- Selection Sort
- Bubble Sort
- Insertion Sort

Thank You

Appendix: Proof for $n \le 2^k \le 2n$

Claim: there exists positive integer k s.t. $n \le 2^k \le 2n$

Proof

- Note that $n = 2^{\log_2 n}$ for a natural number n

$$2^{\lfloor \log_2 n \rfloor} \le n \le 2^{\lceil \log_2 n \rceil}$$

$$\Rightarrow 2^{\lfloor \log_2 n \rfloor + 1} \le 2n \le 2^{\lceil \log_2 n \rceil + 1}$$

- Note that $2^{\lceil \log_2 n \rceil} \le 2^{\lfloor \log_2 n \rfloor + 1}$
  - $\lceil \log_2 n \rceil - \lfloor \log_2 n \rfloor - 1 = \begin{cases} -1, & \log_2 n \text{ is integer} \\ 0, & \log_2 n \text{ isn't integer} \end{cases} \le 0$

- Thus, $n \le 2^{\lceil \log_2 n \rceil} \le 2^{\lfloor \log_2 n \rfloor + 1} \le 2n$
- In other words, $k = \lceil \log_2 n \rceil$ or $\lfloor \log_2 n \rfloor + 1$
  - If $\log_2 n$ isn't integer, $\lceil \log_2 n \rceil = \lfloor \log_2 n \rfloor + 1$