Introduction to Running Time Analysis of Algorithms

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CS 251 Data Structures
UIC

Outline

1. Definitions
2. Examples

Definitions

Basics

We want to analyze the running time behavior $T(N)$ of algorithms.

- N is a positive integer capturing the input size of a particular instance where the algorithm is applied.
  (e.g., N is the size of an array to be sorted.)

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  (e.g., N is the size of an array to be sorted.)

In other words, $T(N)$ is a function,

$$T(N) : \mathbb{N}^* \mapsto \mathbb{N}^*$$

- The domain reflects the different values that the problem size can take.
- The range is the running time (in discrete units of time).

Upper Bounds

Definition (Big O - Upper Bound)

$T(N) = O(f(N))$ if there exist positive constants c and n_0 such that

$$T(N) \leq cf(N)$$

for every $N \geq n_0$.

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Example

$3N^2 = O(N^3)$

($c = 1$ and $n_0 = 3$)
($c = 2$ and $n_0 = 2$)
($c = 3$ and $n_0 = 1$)
(one is enough; constants do not matter!)

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We say,

- $3N^2$ is big-oh N to the third, or that,
- $3N^2$ is order of N to the third.

Upper Bounds

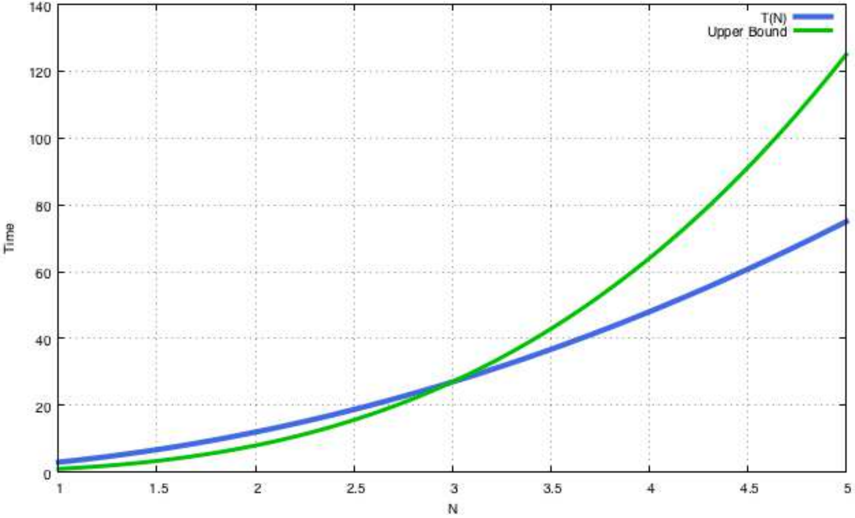
Definition (Set Definition of Big O)

$$O(f(N)) = \{g(N) \quad : \quad \text{there exist } c, n_0 > 0 \text{ such that} \\ 0 \leq g(N) \leq cf(N) \text{ for all } N \geq n_0\}$$

Example

$3N^2 \in O(N^3)$

An Example for the O Notation



$$T(N) = 3N^2 \leq N^3$$

Lower Bounds

Definition (Big Ω - Lower Bound)

$T(N) = \Omega(f(N))$ if there are positive constants c and n_0 such that

$$cf(N) \leq T(N)$$

for every $N \geq n_0$.

Example

$\sqrt{N} = \Omega\left(\log_2 N\right)$ $\qquad$ ($c = 1$ and $n_0 = 16$)

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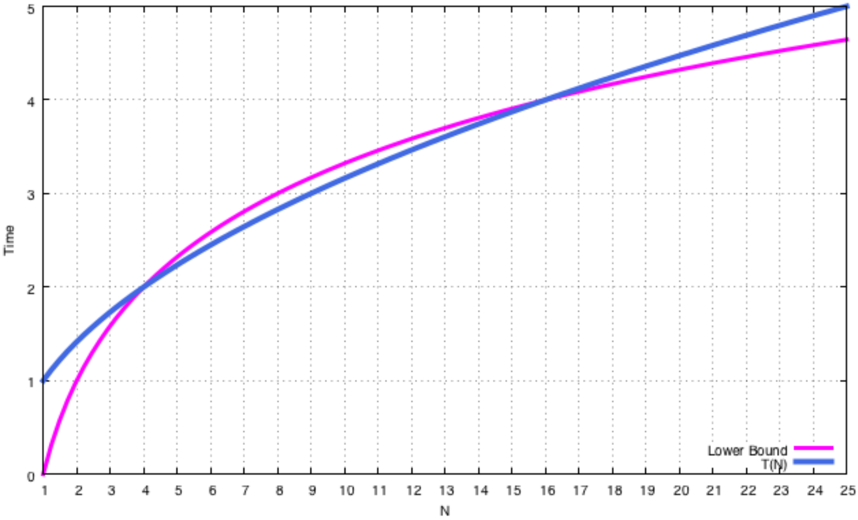
The Example for the Ω Notation



$$\log_2 N \leq T(N) = \sqrt{N}$$

Tight Bounds

Definition

$T(N) = \Theta(f(N))$ if and only if

$$\begin{cases} T(N) = \Omega(f(N)) & \text{and} \\ T(N) = O(f(N)) \end{cases}$$

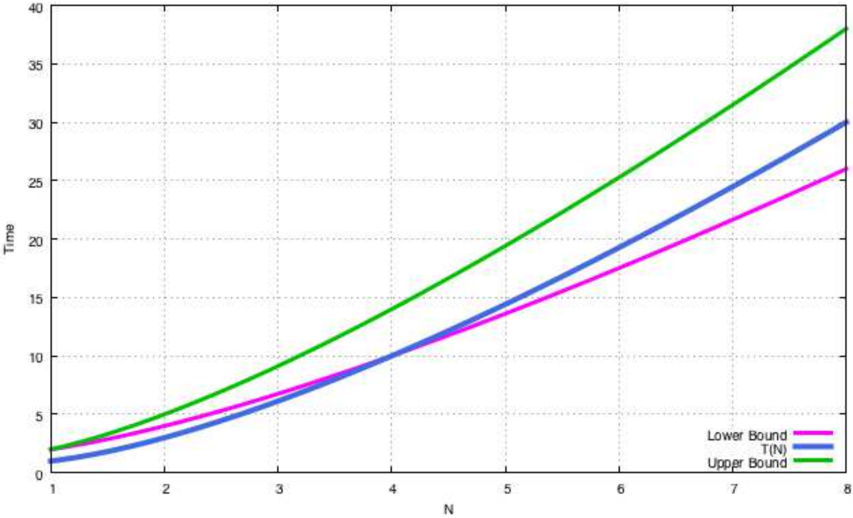
Example

$\frac{3}{2}N\log_2 N - N + 2 = \Theta\left(N\log_2 N\right)$

$$N\log_2 N + 2 \leq \frac{3}{2}N\log_2 N - N + 2 \leq \frac{3}{2}N\log_2 N + 2$$

for every $N \geq 4$.

The Example for the Θ Notation



$$N\log_2 N + 2 \leq T(N) = \frac{3}{2}N\log_2(N) - N + 2 \leq \frac{3}{2}N\log_2(N)$$

Some General Rules

Rule 1 Let $T_1(N) = O(f(N))$ and $T_2(N) = O(g(N))$. Then,

- $T_1(N) + T_2(N) = O(f(N) + g(N))$
  (intuitively, $T_1 + T_2 = O(\max(f(N), g(N)))$)
- $T_1(N) \cdot T_2(N) = O(f(N) \cdot g(N))$

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Rule 2 Let $T(N)$ be a polynomial of degree k. Then, $T(N) = \Theta(n^k)$.

Rule 3 $\log^k(N) = O(N)$ for any constant k.

Typical Growth Rates

Function	Name
c	Constant
$\log N$	Logarithmic
$\log^2 N$	Log-squared
N	Linear
$N \log N$	
N^2	Quadratic
N^3	Cubic
2^N	Exponential

Examples

Constant Time Operations

Definition (Execution in Constant Time)

An operation (more generally an algorithm) is said to be constant time, written as $O(1)$ time, if the **time** required **to execute** the operation (or the algorithm) is upper bounded by a value that **does not depend on the size of the input**.

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- Swap two elements in an array if necessary.
- Assign a value to a pointer.
- Find the minimum value in a sorted array; $a[0]$.

*** numbers stored in registers.

Insert a Node in a Doubly Linked List

```
LIST-INSERT (List L, Listnode x)
x.next = L.head
if (L.head != NULL)
    (L.head).prev = x
L.head = x
x.prev = NULL
```

Insert a Node in a Doubly Linked List

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LIST-INSERT (List L, Listnode x)
1  x.next = L.head
2  if (L.head != NULL)
3      (L.head).prev = x
4  L.head = x
5  x.prev = NULL
```

- Line 3: 1 time step (1 assignment)
- Line 2: 1 time step (1 comparison)
- Lines 2-3: At most 2 time steps
- Lines 1, 4, and 5: 1 time step each

Total time is at most 5 time steps; so $O(1)$.

Find a Node with a Particular Value in a List

```
LIST-FIND (List L, Listvalue v)
 current = L.head
while ((current != NULL) and (current.val != v))
    current = current.next
 return current
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Worst case scenario (v not in the list)

- Line 3: 1 time step $\Rightarrow$ Total: N
- Line 2: Total $2N + 1$ time steps
- Lines 1 and 4: 1 time step each $\Rightarrow$ Total: 2

Total time is at most $3N + 3$ time steps; so $O(N)$.

Compute the Sum $\sum_{i=1}^{N} i^3$

```
int sum( int n )
{
    int partialSum;

    partialSum = 0;
    for( int i = 1; i <= n; ++i )
        partialSum += i * i * i;
    return partialSum;
}
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2     for( int i = 1; i <= n; ++i )
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  }
```

- Lines 1 and 4: 1 unit of time each $\Rightarrow$ Total: 2
- Line 3: 4 units of time (2 multiplications, 1 addition, 1 assignment) $\Rightarrow$ Total: $4N$
- Line 2: $\underbrace{1}_{init} + \underbrace{(N+1)}_{comparisons} + \underbrace{N}_{increments} \Rightarrow$ Total: $2N + 2$

Therefore, the overall total is $6N + 4$ time steps. (O(N))

General Rules for Obtaining Upper Bounds

Rule 1 – FOR Loops The running time of a for loop is at most the running time of the statements inside the for loop (including tests) times the number of iterations.

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Rule 4 – If/Else In a situation as below:

```
if ( condition )
    S1
else
    S2
```

the running time is upper bounded by the running time of the test plus the larger of the running times of S1 and S2.

Insertion Sort

```
INSERTION-SORT(A)
  for j ← 2 to length[A]
       do key ← A[j]
          ▹ Insert A[j] into the sorted sequence A[1 □ j - 1].
          i ← j - 1
          while i > 0 and A[i] > key
              do A[i + 1] ← A[i]
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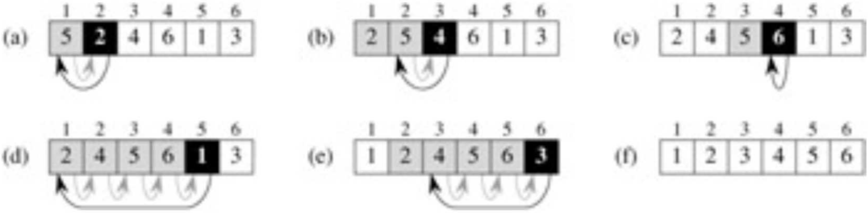

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By the **rule of FOR loops**, since the body of the FOR loop will be executed $N-1$ times, we have that the total running time is $O(N)$.
(We also applied the **rule of nested loops** - but it was trivial here.)

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$$\begin{aligned}\text{Total time } T(N) &\leq c' + \sum_{j=2}^{N}(c' + cj) = c' + c'(N-1) + c\sum_{j=2}^{N} j \\ &= c'N + c\left(\frac{N(N+1)}{2} - 1\right) = O\left(N^2\right).\end{aligned}$$

Binary Search

What is the idea of binary search?

Binary Search

```
/**
 * Performs the standard binary search using two comparisons per level.
 * Returns index where item is found or -1 if not found.
 */
template <typename Comparable>
int binarySearch( const vector<Comparable> & a, const Comparable & x )
{
    int low = 0, high = a.size( ) - 1;

    while( low <= high )
    {
        int mid = ( low + high ) / 2;

        if( a[ mid ] < x )
            low = mid + 1;
        else if( a[ mid ] > x )
            high = mid - 1;
        else
            return mid;   // Found
    }
    return NOT_FOUND;     // NOT_FOUND is defined as -1
}
```

Binary Search - Worst Case

Assumption: Let the array size be $N = 2^k$ for some $k \in \mathbb{N}^*$.

- Worst case: the number we are looking for is not in the array!
- In each iteration, we are left with an array that has size at most half of the previous array.
- For example,

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- Stop when $N/2^q < 1 \Rightarrow 2^q > N \Rightarrow q > \log_2(N) \Rightarrow q = 1 + k$.

Total time $O(\log_2(N))$

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Binary search splits the array ($N \geq 2$) into

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Let $2^{k-1} < N < 2^k$. Then, $2^{k-2} < \lceil N/2 \rceil \leq 2^{k-1}$.

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Queries	Array Size	Array Size Upper Bound
0	N	$< 2^k$
1	$\lceil N/2 \rceil$	$\leq 2^{k-1}$
2	$\lceil \lceil N/2 \rceil /2 \rceil$	$\leq 2^{k-2}$
$\vdots$	$\vdots$	$\vdots$
q	$\lceil \cdots \lceil \lceil N/2 \rceil /2 \rceil \cdots \rceil /2 \rceil$	$\leq 2^{k-q}$

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2	$\lceil \lceil N/2 \rceil /2 \rceil$	$\leq 2^{k-2}$
$\vdots$	$\vdots$	$\vdots$
q	$\lceil \cdots \lceil \lceil N/2 \rceil /2 \rceil \cdots \rceil /2 \rceil$	$\leq 2^{k-q}$

- Stop when $2^{k-q} < 1 \Rightarrow q = 1 + k = 1 + \lceil \log_2(N) \rceil$.

Total time $O\left(\log_2(N)\right)$

Appendix: A Few More Comments

Little Oh

Definition (Little Oh)

$T(N) = o(f(N))$ if, for all positive constants c, there exists an $n_0 > 0$ such that

$$T(N) < cf(N)$$

when $N \geq n_0$.

(Informal: $T(N) = o(f(N))$ if $T(N) = O(f(N))$ and $T(N) \neq \Theta(f(N))$.)

Example

$2N^2 = o(N^3)$ $\left(n_0 = \frac{2}{c}\right)$

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$2N^2 = o(N^3)$ $\left(n_0 = \frac{2}{c}\right)$

Definition (Set Definition of Little Oh)

$$o(f(n)) = \{g(n) \quad : \quad \text{for any constant } c > 0, \text{ there is a constant } n_0 > 0 \text{ such that } 0 \leq g(n) < cf(n) \text{ for all } n \geq n_0\}$$

Little Omega

Definition (Little Omega)

$T(N) = \omega(f(N))$ if, for all positive constants c, there exists an $n_0 > 0$ such that

$$T(N) > cf(N)$$

when $N \geq n_0$.

(Informal: $T(N) = \omega(f(N))$ if $T(N) = \Omega(f(N))$ and $T(N) \neq \Theta(f(N))$.)

Example

$2N^2 = o(N^3)$ $\left(n_0 = \frac{2}{c}\right)$

Little Omega

Definition (Little Omega)

$T(N) = \omega(f(N))$ if, for all positive constants c, there exists an $n_0 > 0$ such that

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when $N \geq n_0$.

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Example

$2N^2 = o(N^3)$ $\left(n_0 = \frac{2}{c}\right)$

Definition (Set Definition of Little Omega)

$$\omega(f(n)) = \{g(n) \quad : \quad \begin{array}{l} \text{for any constant } c > 0, \\ \text{there is a constant } n_0 > 0 \text{ such that} \\ 0 \leq cf(n) < g(n) \text{ for all } n \geq n_0\} \end{array}$$

Using Limits to Compute Relationships

For two functions f and g we can compute the limit,

$$L = \lim_{N \to \infty} \frac{f(N)}{g(N)}$$

using L'Hôpital's rule.

- $L = 0 \Rightarrow f(N) = o(g(N))$.
- $L = c \neq 0 \Rightarrow f(N) = \Theta(g(N))$.
- $L = \infty \Rightarrow g(N) = o(f(N))$.
- L does not exist $\Rightarrow$ There is no relation (will never happen in our context)

Macro Substitution

Convention: A set in a formula represents a function from the set.

Example

$$N^2 + O(N) = O(N^2)$$

means that for any $f(N) \in O(N)$:

$$N^2 + f(N) = g(N)$$

for some $g(N) \in O(N^2)$.