STAT430 Homework #7: Due Friday, April 15, 2022.

Name: **KEY**

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Question 1

Let $Y_1, \ldots, Y_n$ be i.i.d. Uniform$(0, \theta)$. Show that $Y_{(n)} = \max(Y_1, \ldots, Y_n)$ is sufficient for θ. Hint: The Uniform$(0, \theta)$ density can be written $f(y) = \theta^{-1} \cdot \mathbb{1}_{\{0<y<\theta\}}$, where $\mathbb{1}_{\{A\}}$ is an indicator function which equals 1 if A is true and equals 0 otherwise.

Answer:

The likelihood function is

$$\mathcal{L}(\theta \mid \boldsymbol{y}) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}_{\{0<y_i<\theta\}} = \frac{1}{\theta^n} \mathbb{1}_{\{y_{(n)}<\theta\}} \mathbb{1}_{\{y_{(1)}>0\}}$$

We see that $\mathcal{L}(\theta \mid \boldsymbol{y})$ can be written as a product of $g(\theta, Y_{(n)}) = \frac{1}{\theta^n} \mathbb{1}_{\{y_{(n)}<\theta\}}$ and $h(\boldsymbol{y}) = \mathbb{1}_{\{y_{(1)}>0\}}$, So by the factorization theorem, $Y_{(n)}$ is a sufficient statistic for θ.

Question 2

Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with density

$$f(y) = \frac{\alpha y^{\alpha-1}}{\beta^\alpha} \qquad 0 < y < \beta.$$

If β is known, find a one-dimensional sufficient statistic for α.

Answer:

The likelihood function is

$$\mathcal{L}(\alpha, \beta \mid \boldsymbol{y_i}) = \prod_{i=1}^{\infty} \frac{\alpha y^{\alpha-1}}{\beta^\alpha} \mathbb{1}_{\{0<y_i<\beta\}} = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} \mathbb{1}_{\{y_{(1)}>0\}} \mathbb{1}_{\{y_{(n)}<\beta\}}.$$

which can be factored into a product of $g(\alpha, T(\boldsymbol{y})) = \left(\frac{\alpha}{\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} y_i\right)^{\alpha-1}$ and $h(\boldsymbol{y}) = \mathbb{1}_{\{y_{(1)}>0\}} \mathbb{1}_{\{y_{(n)}<\beta\}}$, so $T(\boldsymbol{Y}) = \prod_{i=1}^{n} Y_i$ is a sufficient statistic for α.

Question 3

Let $X_1, \ldots X_n$ be an observation from the pdf

$$P(X = x; \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \qquad x = -1, 0, 1 \qquad 0 \leq \theta \leq 1.$$

a) Show $T_1 = T(X_1)$ is an unbiased estimator of θ, where $T(X)$ is defined

$$T(X) = \begin{cases} 2 & \text{if x = 1} \\ 0 & \text{otherwise} \end{cases}$$

Show $T_n = \frac{1}{n} \sum_{i=1}^{n} T(X_i)$ is unbiased for θ.

Answer:

$$E(T_1) = T_1(-1)P(X_1 = -1) + T_1(0)P(X_1 = 0) + T_1(1)P(X_1 = 1) = 2\left(\frac{\theta}{2}\right) = \theta$$

Since $ET_1 = \theta$, T_1 is an unbiased estimator for θ. Similarly, we can show that T_n is an unbiased estimator of θ

$$ET_n = \frac{1}{n} \sum_{i=1}^{n} ET(X_i) = \frac{n\theta}{n} = \theta$$

Since $ET_n = \theta$, T_n is an unbiased estimator for θ.

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b) Find the MVUE of θ and show this estimator is better than T_n.

Answer:

This was fun to figure out, but definitely a bit tricky. The UMVUE for θ can be found by taking the expectation of an unbiased estimator of θ conditioned on a sufficient (and complete) statistic. A sufficient statistic for θ is $S = \sum_{i=1}^n |X_i|$. I am just going to assume it is complete as well, so the UMVUE for θ is

$$E(T_n \mid S = s) = \frac{1}{n}\sum_{i=1}^n E(T(X_i)|S = s)$$

The expectation $E(T(X_i)|S)$ is

$$\begin{aligned}
E(T(X_i)|S = s) &= T(-1)P(X_i = -1 \mid S = s) + T(0)P(X_i = 0 \mid S = s) + T(1)P(X_i = 1 \mid S = s) \\
&= 2P(X_i = 1 \mid S = s) \\
&= \frac{2P(X_i = 1, S = s)}{P(S = s)} \\
&= \frac{2P(X_i = 1, \sum_{j\neq i}|X_j| = s-1)}{P(S = s)} \\
&= \frac{2P(X_i = 1)P(\sum_{j\neq i}|X_j| = s-1)}{P(S = s)} \\
&= \frac{2(\theta/2)\binom{n-1}{s-1}\theta^{s-1}(1-\theta)^{n-s}}{\binom{n}{s}\theta^s(1-\theta)^{n-s}} \\
&= \frac{\binom{n-1}{s-1}}{\binom{n}{s}} = \frac{(n-1)!}{(s-1)!(n-s)!}\frac{s!(n-s)!}{n!} = \frac{s}{n}
\end{aligned}$$

Try figuring my reasoning out on that.

So the expectation $E(T_n|S) = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^n |X_i|$ is the UMVUE for θ. To show that this is a "better" estimator that T_n, we have to show that the variance of the UMVUE is smaller than the variance of T_n. The variance of T_n is

$$\begin{aligned}
VarT_n &= \frac{1}{n^2}\sum_{i=1}^n VarT(X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^n (ET(X_i)^2 - (ET(X_i))^2) \\
&= \frac{n(4(\theta/2) - (\theta)^2)}{n^2} \\
&= \frac{\theta(2-\theta)}{n}
\end{aligned}$$

and the variance of $\frac{1}{n}S$ is

$$\begin{aligned} Var\frac{1}{n}S &= \frac{1}{n^2}\sum_{i=1}^{n} Var|X_i| \\ &= \frac{1}{n^2}\sum_{i=1}^{n}(E|X_i|^2 - (E|X_i|)^2) \\ &= \frac{n(\theta/2+\theta/2-(\theta/2+\theta/2)^2)}{n^2} \\ &= \frac{\theta(1-\theta)}{n} \end{aligned}$$

and it should be easy to see that $\frac{\theta(2-\theta)}{n} > \frac{\theta(1-\theta)}{n}$, so S/n is better than T_n, as it should be.

Question 4

Let $X_1,\ldots,X_n$ be i.i.d. Geometric(p).

a) Find the MoM estimator of p.

b) Prove that your estimator in part (a) is consistent for p.

Answer:

Question 5

Suppose $Y_1, ..., Y_n$ are i.i.d. according to density

$$f(y) = e^{-(y-\theta)}, \qquad y \geq \theta, \qquad \theta > 0.$$

a) Find the MoM estimator of θ.

b) Is the MoM estimator of θ unbiased? If no, compute the bias.

c) Find the variance of the MoM estimator of θ.

d) Find a sufficient statistic for θ.

e) Find the MVUE of θ.

f) Compare the mean squared error of the MoM estimator and the MVUE. Which one has the smallest MSE?

g) Find the MoM estimator of $(log(\theta))^{1/4}$.

Answer:

Question 6

Let $X_1, \ldots, X_n$ be i.i.d. Uniform$(-\theta,\theta)$.

a) Find the MoM estimator of θ.

b) Find a one-dimensional sufficient statistic for θ.

c) Is the MoM estimator you found in part (a) the MVUE? Explain.

Answer:

Question 7

For each of the following, state whether the statement is TRUE or FALSE. (*Note: Although I do not ask you to explain WHY each statement is true/false, for an exam you should understand the material well enough that you can WHY.*)

1) Suppose $Y_1,..,Y_n$ are i.i.d. from distribution with parameter θ, which is a function of $\mu = E[Y_i]$, and $V[Y_i] < \infty$. The MoM estimator of $\theta^{-2/5}$ is a consistent estimator.

2) Suppose X is a random variable and g is a function. Then $E[g(X)] = g(E[X])$.

3) The MVUE and MLE are always functions of a minimal sufficient statistic.

4) MoM estimators are always unbiased.

5) MLEs are always unbiased.

6) MVUEs are always unbiased.

Answer:
