STAT 620 Homework 2

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1. Read Theorem 2.5.1

Theorem 2.5.1: Every nonempty subset of $\widehat{\mathbb{R}}$ has an infimum and supremum. If $\{a_i\}_{i=1}^{\infty}$ is a sequence in $\widehat{\mathbb{R}}$, then $\limsup a_i$ and $\liminf a_i$ exist in $\widehat{\mathbb{R}}$.

(a) Explain why every nonempty subset of $\widehat{\mathbb{R}}$ has a supremum. Is this true for $\mathbb{R}$?

A nonempty subset A can be either bounded from above or not bounded from above.

By definition 2.5.4, if A is a subset in $\widehat{\mathbb{R}}$ that is not bounded from above, $\sup A = \infty$ which is a point in $\widehat{\mathbb{R}}$.

If A is bounded from above by a point M in $\mathbb{R}$, then M is the supremum of A in $\mathbb{R}$. Since $\mathbb{R} \subset \widehat{\mathbb{R}}$, M is also a point in $\widehat{\mathbb{R}}$ and hence A has a supremum in $\widehat{\mathbb{R}}$.

(b) Show that if $\{a_i\}_{i=1}^{\infty}$ is a sequence in $\widehat{\mathbb{R}}$ then $\limsup a_i$ exists in $\widehat{\mathbb{R}}$.

Consider a sequence $\{a_i\}_{i=1}^{\infty}$. The limit supremum is defined as $\limsup a_i := \inf\left\{\sup\{a_j\}_{j=i}^{\infty}\right\}_{i=1}^{\infty}$. There are several different cases to consider, each with subcases:

- $\{a_i\}_{i=1}^{\infty}$ is bounded above and below

Let M be the least upper bound (supremum) and L be the greatest lower bound (infimum) of $\{a_i\}_{i=1}^{\infty}$.

If $\{a_i\}_{i=1}^{\infty}$ is increasing, then $\sup a_i = M$ always, so $\limsup a_i = \inf\{M, M, \dots\} = M$.

If $\{a_i\}_{i=1}^{\infty}$ is decreasing, then $\sup {a_j}_{j=1}^{\infty}$ is also decreasing and will also be bounded below by L, so $\limsup a_i = L$.

If $\{a_i\}_{i=1}^{\infty}$ oscillates between M and L but approaches some value k, $L < k < M$, then $\{\sup a_j\}_{i=j}^{\infty}$ will be a decreasing sequence bounded below by k, so $\limsup a_i = k$.

In any of these cases, the $\limsup a_i$ is an point of the extended reals $\widehat{\mathbb{R}}$ and therefore always exists in this space.

- $\{a_i\}_{i=1}^{\infty}$ is bounded above but not below

If $\{a_i\}_{i=1}^{\infty}$ is increasing, then $\limsup a_i$ will converge to M by the same argument as above.

If $\{a_i\}_{i=1}^{\infty}$ is decreasing, $\{\sup a_j\}$ will also be decreasing and not bounded from below. Therefore $\limsup a_i = \inf\{\sup a_j\}_{i=1}^{\infty} = -\infty$.

If $\{a_i\}_{i=1}^{\infty}$ is oscillating but settling down to a value $k < M$, then $\limsup a_i = k$ again by the same argument as above.

- $\{a_i\}_{i=1}^{\infty}$ is bounded below but not above

If $\{a_i\}_{i=1}^{\infty}$ is decreasing, then $\{\sup a_j\}$ will also be decreasing and bounded below by L, so $\limsup a_i = L$.

If $\{a_i\}_{i=1}^{\infty}$ is increasing, then $\{\sup a_j\}$ is constant and $\limsup a_i = \infty$.

If $\{a_i\}_{i=1}^{\infty}$ oscillates while settling down to a value k, $L < k < \infty$, then $\limsup a_i = k$.

- $\{a_i\}_{i=1}^{\infty}$ is unbounded

If $\{a_i\}_{i=1}^{\infty}$ is increasing, then $\limsup a_i = \infty$.

If $\{a_i\}_{i=1}^{\infty}$ is decreasing, then $\limsup a_i = -\infty$.

For all of the above cases, $\limsup a_i$ is an element of $\widehat{\mathbb{R}}$, so $\limsup a_i$ always exists in $\widehat{\mathbb{R}}$ for any sequence $\{a_i\}_{i=1}^{\infty}$ in $\widehat{\mathbb{R}}$ of the afformentioned cases.

Note: I think the case when the sequence oscillates without settling down to a specific value makes it difficult to explicitly define what $\limsup a_i$ is. However, since the supremum (and infimum) of any sequence in $\widehat{\mathbb{R}}$ is an element $\widehat{\mathbb{R}}$, then the infimum of the sequence of supremums should also be an element of $\widehat{\mathbb{R}}$.