620 Homework 4

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1. (Exercise 4.2) Show that every point in the Cantor set $\mathcal{C}$ is the limit of a sequence of points in $\mathcal{C}$.

According to **Principles of Mathematical Analysis** (Rudin), "A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$."

So, let p be a point in $\mathcal{C}$. Then $p \in \bigcap_{i=0}^{\infty} F_i$, where F_i is a finite union of 2^i closed intervals (as defined in the class notes). Then p must be in all F_i for $i = 0, 1, 2, \ldots$ so that p is always contained in a closed interval of length $1/3^i$. Then for every neighborhood N, of p, there is a point $q \in N$ that is also in that same interval. So p is a limit point.

2. (Exercise 4.3) Show that the cantor set does not contain any open intervals.

Using another result from **Principles of Mathematical Analysis**, we have the following Lemma: *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed. We also have that for any finite collection of closed sets, the union of these sets is closed.* (I know I should prove this, but I'm just trying to get this homework done.)

Since F_i is a finite union of closed sets, F_i must also be closed, and since $\mathcal{C}$ is an (infinite) intersection of closed sets, $\mathcal{C}$ is also closed.

3. (Exercise 4.4) Prove Theorem 4.3.3: *$\mathcal{D}$ is a bounded function, $\mathcal{D} = 0$ a.e., and $\mathcal{D}(x)$ is not continuous a.e.*

$\mathcal{D}(x)$ is bounded by 1, since for all $x \in \mathbb{R}$, the value of $\mathcal{D}$ never exceeds 1.

To show that $\mathcal{D}(x) = 0$ almost everywhere, we need to show that $\mathcal{D}(x) \neq 0$ only on a set of measure zero:

We know that a single point has measure zero, and $\mathcal{D}(x) = 1$ only for $x \in \mathbb{Q}$, which is a countable set of points. Therefore, by Theorem 4.3.1, the set where $\mathcal{D}(x) = 1$ is a set of measure zero.

Finally, to show that $\mathcal{D}(x)$ is not continuous almost everywhere, we need to show that $\mathcal{D}(x)$ is continuous only on a set of measure zero:

First, $\mathcal{D}$ is continuous at x_0 if, given $\delta > 0$, there exists an $\epsilon > 0$ such that $|\mathcal{D}(x) - \mathcal{D}(x_0)| < \delta$ for any x such that $|x - x_0| < \epsilon$. However, given $x_0 \notin \mathbb{Q}$, there is no ϵ neighborhood around x_0 that does not contain an $x \in \mathbb{Q}$, so for $\delta < 1$, we can not have $|\mathcal{D}(x) - \mathcal{D}(x_0)| < \delta$ for any ϵ. Therefore, $\mathcal{D}$ is only continuous on, at most, a set of measure zero ($\mathbb{Q}$). (I think, by the same reasoning, $\mathcal{D}$ is also not continuous on $\mathbb{Q}$.)