STAT 430 HW07 Problem 3b Solution

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The UMVUE for θ can be found by taking the expectation of an unbiased estimator of θ conditioned on a sufficient statistic. A sufficient statistic for θ is $S = \sum_{i=1}^n |X_i|$, so the UMVUE for θ is

$$E(T_n \mid S = s) = \frac{1}{n}\sum_{i=1}^n E(T(X_i)|S = s)$$

The expectation $E(T(X_i)|S)$ is

$$\begin{aligned}
E(T(X_i)|S = s) &= T(-1)P(X_i = -1 \mid S = s) + T(0)P(T(X_i = 0 \mid S = s)) + T(1)P(T(X_i = 1 \mid S = s)) \\
&= 2P(T(X_i = 1 \mid S = s)) \\
&= \frac{2P(X_i = 1, S = s)}{P(S = s)} \\
&= \frac{2P(X_i = 1, \sum_{j \neq i} |X_i| = s - 1)}{P(S = s)} \\
&= \frac{2P(X_i = 1)P(\sum_{j \neq i} |X_i| = s - 1)}{P(S = s)} \\
&= \frac{2(\theta/2)\binom{n-1}{s-1}\theta^{s-1}(1-\theta)^{n-s}}{\binom{n}{s}\theta^s(1-\theta)^{n-s}} \\
&= \frac{\binom{n-1}{s-1}}{\binom{n}{s}} = \frac{(n-1)!}{(s-1)!(n-s)!}\frac{s!(n-s)!}{n!} = \frac{s}{n}
\end{aligned}$$

So the expectation $E(T_n|S) = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^n |X_i|$ is the UMVUE for θ. To show that this is a "better" estimator that T_n, we have to show that the variance of the UMVUE is smaller than the variance of T_n. The variance of T_n is

$$\begin{aligned}
VarT_n &= \frac{1}{n^2}\sum_{i=1}^n VarT(X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^n (ET(X_i)^2 - (ET(X_i))^2) \\
&= \frac{n(4(\theta/2) - (\theta)^2)}{n^2} \\
&= \frac{\theta(2-\theta)}{n}
\end{aligned}$$

and the variance of $\frac{1}{n}S$ is

$$\begin{aligned}
Var\frac{1}{n}S &= \frac{1}{n^2}\sum_{i=1}^{n} Var|X_i| \\
&= \frac{1}{n^2}\sum_{i=1}^{n}(E|X_i|^2 - (E|X_i|)^2) \\
&= \frac{n(\theta/2 + \theta/2 - (\theta/2 + \theta/2)^2)}{n^2} \\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}$$

and it should be easy to see that $\frac{\theta(2-\theta)}{n} > \frac{\theta(1-\theta)}{n}$, so S/n is better than T_n, as it should be.