STAT 530 Homework 6

2022-03-18

(1) Problem 7.55 (a) and (b) Casella & Berger: For each of the following pdfs, let $X_1, \ldots, X_n$ be a sample from that distribution. In each case, find the best unbiased estimator of θ^r.

(a) $f(x \mid \theta) = \frac{1}{\theta}, \quad 0 < x < \theta, r < n$
(b) $f(x \mid \theta) = e^{-(x-\theta)}, \quad x > \theta$

(2) Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$, where

$$P(X = x \mid \lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \quad x = 0, 1, \ldots; \quad 0 \leq \lambda < \infty.$$

Find the UMVUE of $\tau(\lambda) = \lambda^r$ for some positive integer r.

(3) Prove the following claims:

(a) Suppose $\hat{\theta}$ is the unique Bayes estimator, then $\hat{\theta}$ is admissible.
(b) Suppose θ^* is the unique minimax estimator, then θ^* is admissible.

(4) For this question, we will study the *breakdown value* in greater depth. The textbook definition of a breakdown value is given below:

Definition 10.2.2 Let $X_{(1)} < \cdots < X_(n)$ be an ordered sample of size n, and let T_n be a statistic based on this sample. T_n has *breakdown value* $b, 0 \leq b \leq 1$, if, for every $\epsilon > 0$,

$$\lim_{X_{(\{(1-b)n\})} \to \infty} T_n < \infty \quad \text{and} \quad \lim_{X_{(\{(1-(b+\epsilon))n\})} \to \infty} T_n = \infty$$

(Recall Definition 5.4.2 on percentile notation)

Where $\{b\}$ is the number b rounded to the nearest integer. That is, if i is an integer and $i - 0.5 \leq b < i + 0.5$, then $\{b\} = i$. The textbook also claims that the sample median M_n has a breakdown value of 50%. These do not make sense. For example, consider $n = 10$, $b = 50\%$, and $\epsilon = 0.01$, then $\{(1-b)n\} = \{(1-(b+\epsilon))n\} = 5$. Obviously we cannot have

$$\lim_{X_{(\{(1-b)n\})} \to \infty} M_n = \lim_{X_{(5)} \to \infty} M_n < \infty \quad \text{and} \quad \lim_{X_{(\{(1-(b+\epsilon))n\})} \to \infty} T_n = \lim_{X_{(5)} \to \infty} M_n = \infty$$

at the same time.

Now consider replacing the equations in Definition 10.2.2 by

$$\lim_{X_{(\lfloor(1-b)n\rfloor)} \to \infty} T_n < \infty \quad \text{and} \quad \lim_{X_{(\lfloor(1-(b+\epsilon))n\rfloor) \to \infty}} T_n = \infty,$$

where $\lfloor b \rfloor$ is the greatest integer less than or equal to b, that is, the floor function of b. Show that, under the new definition, the sample median M_n has a breakdown value of $\frac{\lfloor \frac{n-3}{2} \rfloor}{n}$ (assume $n \geq 3$). Obviously, this converges to 50% as $nto\infty$.