STAT 640: Homework 3

Due **Wednesday, February 9, 11:59pm MT** on the course Canvas webpage. Please follow the homework guidelines on the syllabus.

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Problem 1

Show Proposition 2.10: if $\boldsymbol{A}$ is nonsingular, then the unique generalized inverse of $\boldsymbol{A}$ is $\boldsymbol{A}^{-1}$. (In other words, if $\boldsymbol{G}$ is any generalized inverse of $\boldsymbol{A}$, then $\boldsymbol{G} = \boldsymbol{A}^{-1}$.)

Answer: If $\boldsymbol{A}$ is nonsingular, then by definition 2.6, there exists a matrix $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$. Therefore, to show that a generalized inverse $\boldsymbol{G}$ of $\boldsymbol{A}$ is the unique inverse $\boldsymbol{A}^{-1}$, we must show that $\boldsymbol{G}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{G} = \boldsymbol{I}$.

$\boldsymbol{I} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{G}\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{G}$.

Similarly, $\boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{G}\boldsymbol{A} = \boldsymbol{G}\boldsymbol{A}$.

Therefore, $\boldsymbol{G}$ is a matrix which satisfies the properties of $\boldsymbol{A}^{-1}$. To Show that $\boldsymbol{G}$ is the unique inverse of $\boldsymbol{A}$, consider the difference $\boldsymbol{A}^{-1} - \boldsymbol{G}$. (Note: $\boldsymbol{A}^{-1}$ and $\boldsymbol{G}$ must be the same dimension if they are conformable with $\boldsymbol{A}$ on either side). Then

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{A}^{-1} - \boldsymbol{G})\boldsymbol{A} &= \boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{G}\boldsymbol{A} \\
&= \boldsymbol{A} - \boldsymbol{A} \\
&= \boldsymbol{0}
\end{aligned}$$

Since $\boldsymbol{A} \neq \boldsymbol{0}$ (because it is nonsingular), then we must conclude that $\boldsymbol{G} = A^{-1}$ is the only matrix which is an inverse of $\boldsymbol{A}$.

Problem 2

Let $\boldsymbol{A}$ be a positive definite matrix.

a. Show that all diagonal elements of $\boldsymbol{A}$ are positive.

Answer: Since $\boldsymbol{A}$ is positive definite, there exists an $n \times n$ matrix $\boldsymbol{R}$ such that $\boldsymbol{A} = \boldsymbol{R}\boldsymbol{R}^T$. We have

$$\boldsymbol{A} = \boldsymbol{R}\boldsymbol{R}^T = \begin{bmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \dots & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & \dots & r_{n1} \\ \vdots & \ddots & \vdots \\ r_{1n} & \dots & r_{nn} \end{bmatrix}$$

The first diagonal element of this matrix will be the sum of squares of the first row of $\boldsymbol{R}$, the second will be the sum of squares of the second row of $\boldsymbol{R}$ and so on. Sums of squares are always non-negative values, so every diagonal element of $\boldsymbol{A}$ will be non-negative. Furthermore, the determinant of the diagonal elements of $\boldsymbol{A}$ can be computed as their product. Since $\boldsymbol{A}$ is non-singular, $\det(\mathsf{diag}(a_1, \dots, a_n)) \neq 0$. This implies that no diagonal elements of $\boldsymbol{A}$ are zero, so they must all be positive.

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b. Show that $\det(\boldsymbol{A}) > 0$.

Answer: A positive-definite matrix is symmetric, so we can define the eigendecomposition $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T$. Then $\det(\boldsymbol{A}) = \det(\boldsymbol{Q}\boldsymbol{\Lambda}\boldsymbol{Q}^T) = \det(\boldsymbol{Q}\boldsymbol{Q}^T)\det(\boldsymbol{\Lambda}) = \det(\boldsymbol{\Lambda}) = \prod_{i=1}^{n} \lambda_i$. Because $\boldsymbol{A}$ is positive-definite, all eigenvalues of $\boldsymbol{A}$, $\lambda_i > 0$, so $\det(\boldsymbol{A}) > 0$.

Problem 3

Let $\boldsymbol{X}$ be any $n \times p$ matrix. Show that $\boldsymbol{X}^T\boldsymbol{X}$ and $\boldsymbol{X}\boldsymbol{X}^T$ are nonnegative definite.

Answer: Define $\boldsymbol{R} = \boldsymbol{X}^T$. Then $\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{R}\boldsymbol{R}^T$. By proposition 2.14, $\boldsymbol{X}^T\boldsymbol{X}$ is non-negative definite. Similarly, define $\boldsymbol{R} = \boldsymbol{X}$. Then $\boldsymbol{X}\boldsymbol{X}^T = \boldsymbol{R}\boldsymbol{R}^T$, so $\boldsymbol{X}\boldsymbol{X}^T$ is also non-negative definite.

Problem 4

Let $\boldsymbol{X} \in \mathbb{R}^{n\times p}$ have rank p, with $n > p$. Denote the SVD of $\boldsymbol{X}$ as $\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^\top$. Show that $\boldsymbol{P} = \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top$ can be written $\boldsymbol{U}\boldsymbol{A}\boldsymbol{U}^\top$ for some matrix $\boldsymbol{A}$, *which does not involve the values of* $\boldsymbol{X}$, $\boldsymbol{V}$, *or* $\boldsymbol{D}$. Write out the values of $\boldsymbol{A}$ (in terms of numbers you can compute, not an algebraic expression).

Answer:

$$
\begin{aligned}
\boldsymbol{P} &= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \\
&= \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T\left((\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)^T(\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)\right)^{-1}(\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)^T \\
&= \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T(\boldsymbol{V}\boldsymbol{D}^T\boldsymbol{U}^T\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)^{-1}(\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)^T \\
&= \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T(\boldsymbol{V}\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{V}^T)^{-1}(\boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T)^T \\
&= \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T(\boldsymbol{V}\mathrm{diag}(\sigma_1^2,\ldots,\sigma_p^2)\boldsymbol{V}^T)^{-1}\boldsymbol{V}\boldsymbol{D}\boldsymbol{U}^T \\
&= \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T(\boldsymbol{V}^T)^{-1}\mathrm{diag}(\sigma_1^2,\ldots,\sigma_p^2)\boldsymbol{V}^{-1}\boldsymbol{V}\boldsymbol{D}\boldsymbol{U}^T \\
&= \boldsymbol{U}\boldsymbol{D}\mathrm{diag}(\sigma_1^2,\ldots,\sigma_p^2)\boldsymbol{D}\boldsymbol{U}^T \\
&= \boldsymbol{U}\begin{bmatrix} \boldsymbol{I}_{p\times p} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{U}^T
\end{aligned}
$$

The values of $\boldsymbol{A}$ are 0's and 1's.

Problem 5

Consider vectors in $\mathbb{R}^3$ and the subspaces $\mathcal{V} = \mathsf{span}\left(\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right)$ and $\mathcal{W} = \mathsf{span}\left(\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}\right\}\right)$

a. In words, describe the geometry of the spaces $\mathcal{V}$ and $\mathcal{W}$. You can reference the standard x-, y-, and z-axes canonically used for representing $\mathbb{R}^3$.

Answer:

$\mathcal{V}$ is the xz-plane in $\mathbb{R}^3$. $\mathcal{W}$ is a line embedded in the xz-plane. Specifically, it is the line $z = 2x$, with y fixed at 0.

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b. Compute $\boldsymbol{P}_{\mathcal{V}}$.

Answer:

The projection matrix $\boldsymbol{P}_\mathcal{V}$ can be computed as $\boldsymbol{P}_\mathcal{V} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}$. However, since both columns of $\boldsymbol{X}$ are already normalized, we can forgo the normalization bit $(\boldsymbol{X}^T\boldsymbol{X})^{-1}$ and just compute $\boldsymbol{P}_\mathcal{V} = \boldsymbol{X}\boldsymbol{X}^T$:

$$\boldsymbol{P}_\mathcal{V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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c. Compute $\boldsymbol{P}_\mathcal{W}$.

Answer: We can first normalize the vector $\boldsymbol{w} = [1, 0, 2]^T$, then compute the projection matrix $P_\mathcal{W} = \hat{\boldsymbol{w}}\hat{\boldsymbol{w}}^T$, where $\hat{\boldsymbol{w}}$ is the vector in the direction of $\boldsymbol{w}$ with unit length.

First, $\hat{\boldsymbol{w}} = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}$. Then

$$\boldsymbol{P}_\mathcal{W} = \left[\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right] \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

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d. Verify that $\boldsymbol{P}_\mathcal{V}\boldsymbol{P}_\mathcal{W} = \boldsymbol{P}_\mathcal{W}\boldsymbol{P}_\mathcal{V} = \boldsymbol{P}_\mathcal{W}$.

Answer:

$$\boldsymbol{P}_\mathcal{V}\boldsymbol{P}_\mathcal{W} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} = \boldsymbol{P}_\mathcal{W}$$

and

$$\boldsymbol{P}_\mathcal{W}\boldsymbol{P}_\mathcal{V} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} = \boldsymbol{P}_\mathcal{W}$$

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e. In words, describe the geometry of the spaces $\mathcal{V} \cap \mathcal{W}^\perp$ and $\mathcal{W}^\perp$.

Answer: Since $\mathcal{W}$ is a line in the xz-plane, then $\mathcal{V} \cap \mathcal{W}^\perp$ must be the perpendicular line in the xz-plane. Namely, the line $z = -\frac{1}{2}x$ with y fixed at 0. in $\mathbb{R}^3$, $\mathcal{W}^\perp$ is the plane $z = -\frac{1}{2}x$ with varying y, or the plane in $\mathbb{R}^3$ defined with the normal vector $[1, 0, 2]^T$.

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f. Compute $\boldsymbol{P}_{\mathcal{V}\cap\mathcal{W}^\perp}$.

Answer: $\mathcal{V} \cap \mathcal{W}^\perp$ is spanned by the vector $\boldsymbol{u} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, so we can compute the projection matrix $\boldsymbol{P}_{\mathcal{V}\cap\mathcal{W}^\perp}$ by normalizing and finding the outer product.

The norm of $\boldsymbol{u} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ is $\sqrt{5}$, so $\hat{\boldsymbol{u}} = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix}$. The outer product is then

$$\boldsymbol{u}\boldsymbol{u}^T = \begin{bmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

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g. Compute $\boldsymbol{P}_{\mathcal{W}^\perp}$.

Answer: A basis for $\mathcal{W}^\perp$ can be found by identifying a vector in $\mathbb{R}^3$ that is perpendicular to every element in $\mathcal{W} = \left\{ \begin{bmatrix} a \\ 0 \\ 2a \end{bmatrix} : a \in \mathbb{R} \right\}$. Using the inner product, we have

$$0 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 2a \end{bmatrix} = a(x + 2z)$$

For $a \neq 0$, we must have that $x = -2z$. y is a free parameter and can be equal to anything. So a normalized basis for $\mathcal{W}^\perp$ is $\left\{ \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Now we can compute a projection matrix onto this space by defining the column space $\boldsymbol{W}^\perp = \begin{bmatrix} -\frac{2}{\sqrt{5}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{5}} & 0 \end{bmatrix}$ and finding $\boldsymbol{P}_{\mathcal{W}^\perp}$ as

$$\boldsymbol{P}_{\mathcal{W}^\perp} = \boldsymbol{W}^\perp(\boldsymbol{W}^\perp)^T = \begin{bmatrix} -\frac{2}{\sqrt{5}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{2}{5} \\ 0 & 1 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

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h. For the vector $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, compute $\boldsymbol{P}_{\mathcal{V}}\boldsymbol{a}$, $\boldsymbol{P}_{\mathcal{W}}\boldsymbol{a}$, $\boldsymbol{P}_{\mathcal{V}\cap\mathcal{W}^\perp}\boldsymbol{a}$, and $\boldsymbol{P}_{\mathcal{W}^\perp}\boldsymbol{a}$.

Answer:

$$\boldsymbol{P}_{\mathcal{V}}\boldsymbol{a} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{W}}\boldsymbol{a} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ 0 \\ \frac{1}{5} \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{V}\cap\mathcal{W}^\perp}\boldsymbol{a} = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ 0 \\ \frac{1}{5} \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{W}^\perp}\boldsymbol{a} = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{2}{5} \\ 0 & 1 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ 2 \\ \frac{1}{5} \end{bmatrix}$$

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i. For the vector $\boldsymbol{b} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$, compute $\boldsymbol{P}_{\mathcal{V}}\boldsymbol{b}$, $\boldsymbol{P}_{\mathcal{W}}\boldsymbol{b}$, $\boldsymbol{P}_{\mathcal{V}\cap\mathcal{W}^\perp}\boldsymbol{b}$, and $\boldsymbol{P}_{\mathcal{W}^\perp}\boldsymbol{b}$.

Answer:

$$\boldsymbol{P}_{\mathcal{V}}\boldsymbol{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{W}}\boldsymbol{b} = \begin{bmatrix} \frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 \\ \frac{2}{5} & 0 & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{V}\cap\mathcal{W}^\perp}\boldsymbol{b} = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{2}{5} \\ 0 & 0 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{P}_{\mathcal{W}^\perp}\boldsymbol{b} = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{2}{5} \\ 0 & 1 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Problem 6

Consider the matrix $\boldsymbol{J}_n = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$.

a. Write $\boldsymbol{J}_n$ as the product of a column vector and row vector. That is, find $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{J}_n = \boldsymbol{u}\boldsymbol{v}^\mathsf{T}$.

Answer: We can write $\boldsymbol{J}_n$ as the outer product $\hat{\boldsymbol{j}}\hat{\boldsymbol{j}}^T$, where $\hat{\boldsymbol{j}} = (1/\sqrt{n}, \dots 1/\sqrt{n})^T$. (In other words, it is a vector $\boldsymbol{j}$ of all ones that has been normalized).

$$\hat{\boldsymbol{j}}\hat{\boldsymbol{j}}^T = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n \frac{1}{n} & \cdots & \sum_{i=1}^n \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \frac{1}{n} & \cdots & \sum_{i=1}^n \frac{1}{n} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} = \boldsymbol{J}_n$$

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b. Show that $\boldsymbol{J}_n$ is idempotent.

Answer:

$$\boldsymbol{J}_n\boldsymbol{J}_n = \frac{1}{5}\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} = \frac{1}{n^2}\begin{bmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & & \vdots \\ n & n & \dots & n \end{bmatrix} = \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix} = \boldsymbol{J}_n$$

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c. Since it is both symmetric and idempotent, $\boldsymbol{J}_n$ is a projection matrix. Describe the subspace of $\mathbb{R}^n$ onto which it projects (either a formal definition of the elements of the subspace, or a geometric description).

Answer: $\boldsymbol{J}_n$ has only one linearly independent vector, so $\boldsymbol{J}_n$ projects onto a line in $\mathbb{R}^n$ where all components are the same. Given an arbitrary vector $\boldsymbol{a}$ in $\mathbb{R}^n$, the projection $\boldsymbol{J}_n\boldsymbol{a} = \left[\frac{1}{n}\sum_{i=1}^n a_i, \quad \dots, \quad \frac{1}{n}\sum_{i=1}^n a_i\right]^T$. The components of the projection $\boldsymbol{J}_n\boldsymbol{a}$ will all be the sample average of the components in $\boldsymbol{a}$.

Problem 7

Consider the matrix $\boldsymbol{A} = \begin{bmatrix} 8 & 4 & 0 \\ 4 & 18 & 4 \\ 0 & 4 & 1 \end{bmatrix}$.

a. Find a square matrix $\boldsymbol{R}$ such that $\boldsymbol{A} = \boldsymbol{R}\boldsymbol{R}^\top$. You can use base R.

Answer:

```
A <- cbind(c(8,4,0), c(4,18,4), c(0,4,1))

# Find eigenvalues and eigenvectors
eigs <- eigen(A)
Q <- eigs$vectors
Lambda <- diag(eigs$values)

R <- Q %*% sqrt(Lambda) %*% t(Q)
print(R)
```

```
##            [,1]      [,2]       [,3]
## [1,]  2.7550234 0.6108256 -0.1916715
## [2,]  0.6108256 4.0904158  0.9462507
## [3,] -0.1916715 0.9462507  0.2605218
```

```
# Check that RR^T = A
round(R %*% t(R), 5)
```

```
##      [,1] [,2] [,3]
## [1,]    8    4    0
## [2,]    4   18    4
## [3,]    0    4    1
```

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b. We know that if $\text{Var}(\boldsymbol{Z}) = \boldsymbol{I}$, then $\text{Var}(\boldsymbol{R}\boldsymbol{Z}) = \boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^\top = \boldsymbol{A}$. Use the following code (set `eval=TRUE`), and the $\boldsymbol{R}$ from part (a), to simulate 10,000 random 3-vectors. Is their empirical covariance matrix close to $\boldsymbol{A}$?

```
set.seed(2021640)
N <- 10000
Z <- matrix(rnorm(N*3), 3, N) # N 3x1 vectors in 3xN matrix
Y <- R %*% Z
cov(t(Y))
```

```
##             [,1]      [,2]        [,3]
## [1,]  8.18153633  4.054482 -0.00907157
## [2,]  4.05448191 17.823035  3.94894843
## [3,] -0.00907157  3.948948  0.98837105
```

Answer: Yeah it's pretty close

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c. Find a vector $\boldsymbol{a}$ such that $\text{Var}(\boldsymbol{a}^\top \boldsymbol{Y}) = 0$ for $\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{Z}$ as in (b). Check this empirically using your sample from (b).

Answer: The matrix $\boldsymbol{A}$ has rank 2 and is non-negative definite (see Proposition 2.14). This means that all of the eigenvalues are greater than or equal to zero. In fact, there are as many positive eigenvalues as the $\mathsf{rank}A$. This means that the third eigenvalue λ_3 must be zero with eigenvalue $\boldsymbol{q}_3$. So we have $\boldsymbol{A}\boldsymbol{q}_3 = \lambda_3 \boldsymbol{q}_3 = \boldsymbol{0}$.

So

$$\mathrm{Var}(\boldsymbol{a}^T \boldsymbol{Y}) = \mathrm{Var}(\boldsymbol{a}^T \boldsymbol{R}\boldsymbol{Z}) = \boldsymbol{a}^T \boldsymbol{A}\boldsymbol{a}.$$

If we let $\boldsymbol{a} = \boldsymbol{q}_3$, then we have

$$\boldsymbol{q}_3^T \boldsymbol{A}\boldsymbol{q}_3 = \boldsymbol{q}_3^T \lambda_3 \boldsymbol{q}_3 = \boldsymbol{q}_3^T \boldsymbol{0} = 0$$

```r
# eigenvector corresponding to zero eigenvalue
a <- eigs$vectors[,3]
# Pretty close to zero
cov(t(Y) %*% a)
```

```
##              [,1]
## [1,] 3.211886e-14
```

This is pretty close to zero.
