STAT 530 Homework 4

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2/25/2022

(1) Let $X_1, \dots, X_n \overset{iid}{\sim} \text{Unif}(\theta, 2\theta)$, where $\theta > 0$. Find the MLE of θ. Is it an unbiased estimator of θ? If not, make some adjustments to get an unbiased estimator of θ.

The likelihood function for θ is

$$\mathcal{L}(\theta|\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{\theta} I(\theta < x_i) I(x_i < 2\theta) = \left(\frac{1}{\theta}\right)^n I(\theta < x_{(1)}) I(x_{(n)}/2 < \theta).$$

Note that if $x_{(n)}/2 > x_{(1)}$, then this likelihood is zero. However, if $x_{(1)} \geq x_{(n)}/2$, then $\mathcal{L}(\theta|\boldsymbol{x})$ is monotonically decreasing in θ, so the MLE $\hat{\theta}$ of θ is $X_{(n)}/2 = \frac{1}{2}\max_i(X_i)$.

To determine whether it is unbiased, we first find the distribution of $T = \max_i(X_i)$:

$$F_T(t) = \prod_{i=1}^{n} F_{X_i}(2t) = (F_{X_1}(2t))^n \, I(\theta/2 < t < \theta)$$

and

$$\begin{aligned} f_T(t) &= \frac{d}{dt} (F_{X_1}(2t))^n = 2n (F_{X_1}(2t))^{n-1} f_X(2t) I(\theta/2 < t < \theta) \\ &= \frac{2n(2t-\theta)^{n-1}}{\theta^n} I(\theta/2 < t < \theta) \end{aligned}$$

Then finding the expectation, we have

$$E(T) = \int_{\theta/2}^{\theta} \frac{2nt(2t-\theta)^{n-1}}{\theta^n} dt$$

and setting $u = 2t - \theta$,

$$\begin{aligned} &= \frac{n}{2\theta^n} \int_0^{\theta} u^n + \theta u^{n-1} du \\ &= \frac{n}{2\theta^n} \left(\frac{u^{n+1}}{n+1} + \frac{\theta u^n}{n} \right) \Big|_0^{\theta} \\ &= \frac{n}{2\theta^n} \left(\frac{n\theta^{n+1} + (n+1)\theta^{n+1}}{n(n+1)} \right) \\ &= \frac{\theta(2n+1)}{2n(n+1)} \end{aligned}$$

So $\hat{\theta} = T(\boldsymbol{X}) = \max_i(X_i)$ is a biased estimator of θ, but if we multiply $T(\boldsymbol{X})$ by $\frac{2n(n+1)}{2n+1}$, we will have an unbiased estimator.

(2) Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$. Show that there is no unbiased estimator of $\tau(\theta) = |\theta|$.

Assume that an unbiased estimator $W(\boldsymbol{X})$ exists for $\tau(\theta) = |\theta|$. Then for all θ,

$$E(W(\boldsymbol{X})) = |\theta|.$$

$E(W)$ is a function of θ, so taking partial derivatives, we have

$$\frac{dE(W)}{d\theta} = \frac{d}{d\theta}|\theta| \quad \text{for all } \theta.$$

However, the right-hand side is not defined at $\theta = 0$, while the left-hand side is presumably continuous, so there must not be an estimator $W(\boldsymbol{X})$ which is unbiased for $|\theta|$.

(3) Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Show that the sample mean and variance, $\bar{X}$ and S^2, are respectively UMVUEs of μ and σ^2 by showing that they are uncorrelated with all unbiased estimators.

$T_1 = \bar{X}$ and $T_2 = S^2$ are independent random variables with respective pdfs

$$f(t_1 \mid \mu, \sigma^2) = \left(\frac{2\pi\sigma^2}{n}\right)^{-1/2} \exp\left(\frac{n}{2\sigma^2}(t_1 - \mu)^2\right)$$

and

$$f(t_2 \mid \mu, \sigma^2) = \frac{(n-1)^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\sigma^{n-1}} t_2^{\frac{n-3}{2}} \exp\left(\frac{-t_2(n-1)}{2\sigma^2}\right)$$

and joint distribution

$$f(t_1, t_2 \mid \mu, \sigma^2) = f(t_1 \mid \mu, \sigma^2) f(t_2 \mid \mu, \sigma^2) I(t_2 \geq 0).$$

Wet $U(T_1, T_2)$ be an arbitrary unbiased estimator of 0. Then

$$E(U) = \int_{-\infty}^{\infty} \int_{0}^{\infty} U(t_1, t_2) f(t_1, t_2 \mid \mu, \sigma^2) dt_2 dt_1 = 0. \tag{1}$$

Taking partial derivatives of the second and third expression above with respect to μ, we have

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} U(t_1, t_2) \frac{-n(t_1 - \mu)}{\sigma^2} f(t_1, t_2 \mid \mu, \sigma^2) dt_2 dt_1 = 0$$

Which becomes

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} U(t_1, t_2) t_1 f(t_1, t_2 \mid \mu, \sigma^2) dt_2 dt_1 = \mu \int_{-\infty}^{\infty} \int_{0}^{\infty} U(t_1, t_2) f(t_1, t_2 \mid \mu, \sigma^2) dt_2 dt_1.$$

The right-hand side is zero though, so we have that the left-hand side, $E(UT_1) = 0$ which establishes that $\mathrm{Cov}(U, T_1) = 0$.

Now, taking the partial derivative of both sides of (1) with respect to σ, we get a really complex expression on the left hand side, but it can essentially be simplified down to

$$C_1 \int\int u t_1^2 f(t_1, t_2) + C_2 \int\int u t_1 f(t_1, t_2) + C_3 \int\int u f(t_1, t_2) + C_4 \int\int u t_2 f(t_1, t_2) = 0.$$

Where C_1, C_2, C_3, C_4 are constants. We have already established that the second and third quantities are zero. To show that the first quantity, $E(UT_1^2)$, is zero, we can simply take the partial derivative of $E(UT_1)$ with respect to μ and solve for $E(UT_1^2)$. Therefore, $E(UT_2)$ must also be zero, and T_2 and U must be uncorrelated.