STAT 530 Homework 5

2022-03-19

(1) Suppose $X_1, \dots, X_n \overset{iid}{\sim} \text{Bernoulli}(p), 0 < p < 1$.

  (a) Calculate the Fisher information number $\mathcal{I}_n(p)$
  (b) Find the value of $p \in (0,1)$ for which $\mathcal{I}_n(p)$ is minimal. This value of p, where the fisher information $\mathcal{I}_n(p)$ is minimal, corresponds to the most difficult case for estimating p. That is, when data are generated under this value of p from the model, the variance of an unbiased estimator of p is potentially largest.
  (c) Show that $\bar{X} = \sum_{i=1}^n X_i/n$ is the UMVUE of p using the Cramer-Rao inequality.
  (d) For $n \geq 4$, show that the product $X_1X_2X_3X_4$ is an unbiased estimator of p^4, and use this fact to find the UMVUE of p^4.

(2) Suppose that the random variables $Y_1, \dots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \quad i = 1, \dots, n$$

where $\{x_i\}_{i=1}^n$ are fixed constants and $\{\epsilon_i\}_{i=1}^n$ are i.i.d. $N(0, \sigma^2)$ with known $\sigma^2 > 0$.

  (a) Find the MLE of β.
  (b) Find the distribution of the MLE.
  (c) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of β.
  (d) Show that the MLE is the UMVUE of β using the Cramer-Rao inequality.

(3) Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 1)$ where $\theta \in \mathbb{R}$. It turns out that $T = \bar{X}^2 - n^{-1}$ is the UMVUE of $\tau(\theta) = \theta^2$ (because it's a function of a complete sufficient statistic).

  (a) Show that T is an unviased estimator of $\tau(\theta)$ and find the variance $\text{Var}_\theta(T)$.
  (b) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of $\tau(\theta)$.
  (c) Show that $\text{Var}_\theta(T)$ is strictly larger than the Cramer-Rao lower bound for all values of $\theta \in \mathbb{R}$.