STAT 620 Homework 7

2022-03-30

Problem 1

(5.19) Prove that if $\{\mu_i\}_{i=1}^{\infty}$ is a collection of measure on a measurable space $(\mathbb{X}, \mathscr{M})$ and $\{a_i\}_{i=1}^{\infty}$ is a collection of numbers with $a_i \geq 0$ for all i, then $\sum_{i=1}^{\infty} a_i \mu_i$ is a measure on $(\mathbb{X}, \mathscr{M})$

Answer: Define $M(A) = \sum_{i=1}^{\infty} a_i \mu_i(A)$ for $A \in \mathscr{M}$. Then $M(\emptyset) = \sum_{i=1}^{\infty} a_i \mu_i(\emptyset) = \sum_{i=1}^{\infty} 0 = 0$. Let $\bigcup_{j=1}^{\infty} A_j$ be a sequence of disjoint sets in $\mathscr{M}$. Then

$$\begin{aligned} M\left(\bigcup_{j=1}^{\infty} A_j\right) &= \sum_{i=1}^{\infty} a_i \mu_i \left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \sum_{i=1}^{\infty} a_i \sum_{j=1}^{\infty} \mu_i(A_j) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_i \mu_i(A_j) \\ &= \sum_{j=1}^{\infty} M(A_j). \end{aligned}$$

So M satisfies the requirements for a countable measure on $(\mathbb{X}, \mathscr{M})$.

Problem 2

a. (5.20) Prove that if $(\mathbb{X}, \mathscr{M}, \mu)$ is a measure space, $B \in \mathscr{M}$, and we define $\nu(A) = \mu(A \cap B)$ for $A \in \mathscr{M}$, then ν is a measure on $(\mathbb{X}, \mathscr{M})$.

Answer: First, $\nu(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$. Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint collection of sets in $\mathscr{M}$. Then

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i \cap B\right) = \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \nu(A_i).$$

So ν satisfies the requirements to be a countable additive measure on $(\mathbb{X}, \mathscr{M})$.

b. How does this problem relate to theorem 5.1.3? What is different? Suppose $A \in \mathscr{M}$ and $A \cap B = \emptyset$. Is $\nu(A)$ defined?

Answer: This problem shows that all the set of the form $A \cap B$, given $B \in \mathscr{M}$ and A is any set in $\mathscr{M}$ make up a new σ-algebra. However, in Theorem 5.1.3, B need not be in the original σ-algebra, $\mathscr{M}$. B can be any subset of the sample space $\mathbb{X}$. By this theorem, $\nu(A)$ where $A \cap B = \emptyset$ should be defined.