STAT 530 Homework 2

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(1) (10 pts) Find the method of moment estimators of the unknown parameters based on a random sample $X_1, \dots, X_n$ of size n from the following distributions. *(See "Table of Common Distributions" in Casella & Berger for definitions and properties of these distributions)*:

(a) (5 pts) Negative-binomial$(3, p)$, unknown p;

Note about notation: I am using $\widehat{E(X^k)}$ to denote the kth population moment for the distribution with the method of moments estimator $\hat{\theta}$. Also, I will use M_1 to denote $\frac{1}{n}\sum_{i=1}^n X_i$ and M_2 to denote $\frac{1}{n}\sum_{i=1}^n X_i^2$.

Set $M_1 = \widehat{E(X)}$, where $\widehat{E(X)} = \frac{3(1-\hat{p})}{\hat{p}}$ (the mean of the negative-binomial distribution). Then solving for the method of moments estimator, $\hat{p}$, we have

$$\hat{p} = \frac{3}{M_1 + 3} = \frac{3}{\bar{X} + 3}.$$

(b) (5 pts) Double-exponential(μ, σ), unknown μ and σ.

To find the method of moments estimators for parameters μ and σ, we begin with the system of equations:

$$\begin{aligned} M_1 &= \widehat{E(X)} = \hat{\mu} \\ M_2 &= \widehat{E(X^2)} = \widehat{\mathrm{Var}(X)} + \widehat{E(X)}^2 = 2\hat{\sigma}^2 + \hat{\mu}^2 \end{aligned}$$

The M.o.M. estimator for μ is $\hat{\mu} = \bar{X}$, and substituting this into the second equation and solving for $\hat{\sigma}$ we get

$$\hat{\sigma} = \sqrt{\frac{M_2 - M_1^2}{2}}$$

As the M.o.M. estimator for σ.

(2) (14 pts) Suppose we have a random sample of size n, $X_1, \dots, X_n \sim f(x|\theta)$, where

$$f(x|\theta) = 2\sqrt{\frac{\theta}{\pi}} \exp(-\theta x^2) \cdot \mathbb{I}(x > 0).$$

(a) (7 pts) Find the method of moment estimator of θ by matching the 1st sample moment with the first population moment.

We first equate the 1st sample moment, M_1, with the estimated 1st population moment. We can then integrate f to compute the estimated population moment.

$$
\begin{aligned}
M_1 = \widehat{E(X)} &= \sqrt{\frac{\hat{\theta}}{\pi}} \int_0^\infty 2xe^{-\hat{\theta}x^2} dx \\
&= \sqrt{\frac{\hat{\theta}}{\pi}} \int_0^\infty e^{-\hat{\theta}u} du && (u - substitution) \\
&= \sqrt{\frac{\hat{\theta}}{\pi}} \left(-\frac{1}{\hat{\theta}} e^{-\hat{\theta}u} \right)_0^\infty \\
&= \frac{1}{\sqrt{\hat{\theta}\pi}}
\end{aligned}
$$

Solving for $\hat{\theta}$, we get

$$\hat{\theta} = \frac{1}{\pi M_1^2}$$

(b) (7 pts) Now, suppose we instead match the 2nd sample moment $\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)$ with the 2nd population moment moment $(E(X_1^2))$. We can obtain another estimator of θ. What is this estimator? Is it the same as your answer in (a)?

(3) (8 pts) Problem 7.1, Casella & Berger:

(4) (8 pts) Let $\mathbb{I}(A)$ denote the indicator function of an event A, where $\mathbb{I}(A) = 1$ if event A holds true and $\mathbb{I}(A) = 0$ otherwise. Suppose that $A_1, \ldots, A_n$ are n separate events, and B is the event that "events $A_1, \ldots, A_n$ hold true at the same time." Use your knowledge in STAT 520, rigorously show that

$$\prod_{i=1}^n \mathbb{I}(A_i) = \mathbb{I}(B).$$

(5) (18 pts) Given a random sample $X_1, \ldots, X_n$ from a pdf/pmf $f(x|\theta)$, $\theta \in \Theta \subset \mathbb{R}$, we know that the likelihood function is

$$L(\theta|x) = \prod_{i=1}^n f(x_i|\theta), \quad \theta \in \Theta,$$

but there exists one subtle point to highlight about how to exactly write the likelihood expression depending on the support of $f(x|\theta)$.

- Recall the support of $f(x|\theta)$ is $S_\theta = \{x \in \mathbb{R} : f(x|\theta) > 0\}$, which could possibly depend on $\theta \in \Theta$. For example, an exponential distribution has a pdf whose support is free from the parameter θ, while a uniform distribution may have a pdf whose support depends on θ.
- It is always true that $f(x|\theta) = f(x|\theta) \cdot \mathbb{I}(x \in S_\theta)$ for all $x \in \mathbb{R}$ and so always true that