STAT 620 Homework 11

2022-04-29

7.9 Let $(\Omega, \mathscr{F}, P)$ be a probability space.

1. If $\{A_i\}_{i=1}^m$ is a collection of independent events, show

$$P\left(\bigcup_{i=1}^m A_i\right) = 1 - \prod_{i=1}^m (1 - P(A_i)).$$

$$\begin{aligned} P\left(\bigcup_{i=1}^m A_i\right) &= 1 - P\left(\left(\bigcup_{i=1}^m A_i\right)^C\right) \\ &= 1 - P\left(\bigcap_{i=1}^m A_i^C\right) \\ &= 1 - \prod_{i=1}^m P(A_i^C) && \textit{(Theorem 7.4.4 \& Definition 7.4.3)} \\ &= 1 - \prod_{i=1}^m (1 - P(A_i)) && \square \end{aligned}$$

2. If $\{A_i\}_{i=1}^\infty$ is a collection of independent events, show

$$P\left(\bigcap_{i=1}^\infty A_i\right) = \prod_{i=1}^\infty P(A_i)$$

I am going to attempt to prove this by induction. Consider two sets in the collection A_{i_1} and A_{i_2}. By Definition 7.4.2, $P(A_{i_1} \cap A_{i_2}) = P(A_{i_1})P(A_{i_2})$. We also have that by definition, for $m \in \mathbb{N}$, $P\left(\bigcap_{j=1}^m A_{i_j}\right) = \prod_{j=1}^m P(A_{i_j})$.

Now consider $m+1$ sets. We want to show that $P\left(\bigcap_{j=1}^{m+1} A_{i_j}\right) = \prod_{j=1}^{m+1} P(A_{i_j})$. We have

$$\begin{aligned} P\left(\bigcap_{j=1}^{m+1} A_{i_j}\right) &= P\left(\left(\bigcap_{j=1}^{m} A_{i_j}\right) \cap A_{i_{m+1}}\right) \\ &= P\left(\bigcap_{j=1}^{m+1} A_{i_j}\right) \times P(A_{i_{m+1}}) \\ &= \prod_{j=1}^{m} P(A_{i_j}) \times P(A_{i_{m+1}}) \\ &= \prod_{j=1}^{m+1} P(A_{i_j}) && \square \end{aligned}$$

Therefore, the probability of a countable intersection of independent events can be expressed as the countable product of the probabilities of the individual events.

8.2 Let $(\mathbb{X}, \mathscr{M}, \mu)$ be a measure space and $\mathbb{X} = A \cup B$ with $A, B \in \mathscr{M}$. Prove that a real valued function f on $\mathbb{X}$ is measurable if and only if f is measurable on A and on B.

(f is measurable on $\mathbb{X} \implies f$ is measurable on A and B)

If f is measurable on $\mathbb{X} = A \cup B$, then $f^{-1}(C) \in \mathscr{M}$ for all C in the corresponding Borel σ-algebra. Then by definition 8.2.3, and the closure of a σ-algebra under finite intersection, $f^{-1}(C) \cap A \in \mathscr{M}$ and $f^{-1}(C) \cap B \in \mathscr{M}$ for all $C \in \mathscr{N}$ (the Borel σ-algebra), so f is measurable on A and B.

(f is measurable on A and $B \implies f$ is measurable on $\mathbb{X}$)

Starting from the end and going to the beginning in the above proof should be sufficient for this direction, I think. If f is measurable on A and B, then $f^{-1}(C) \cap A \in \mathscr{M}$ and $f^{-1}(C) \cap B \in \mathscr{M}$ for all $C \in \mathscr{N}$. This implies by the properties of σ-algebras that $f^{-1}(C) \in \mathscr{M}$ for all $C \in \mathscr{N}$, so by definition 8.2.1, f is $\mathscr{M}$-measurable.

8.7 Let X be an integer valued random variable on a probability space $(\Omega, \mathscr{F}, P)$ and m a positive integer. Show

$$\sum_{i=-\infty}^{\infty} P(\{\omega : i < X(\omega) \leq i + m\}) = m.$$

$$\begin{aligned}
\sum_{i=-\infty}^{\infty} P(\{\omega : i < X(\omega) \leq i + m\}) &= \sum_{i=-\infty}^{\infty} P(\{i, \dots, i+m\}) \\
&= \sum_{i=-\infty}^{\infty} P\left(\bigcup_{j=i}^{i+m} \{j\}\right) \\
&= \sum_{i=-\infty}^{\infty} \sum_{j=i}^{i+m} P(\{j\}) \\
&= \cdots \sum_{i=k}^{k+m} P(\{i\}) + \sum_{i=k+1}^{k+1+m} P(\{i\}) + \dots \\
&= \sum_{i=-\infty}^{\infty} mP(\{i\}) \\
&= m \sum_{i=-\infty}^{\infty} P(\{i\}) = m \times 1 = m \qquad \square
\end{aligned}$$