STAT 620 Homework 5

Hannah Butler

2/28/2022

1. (Exercise 5.7) Prove Theorem 5.2.2: $(\mathbb{X}, \mathscr{M}, \mu)$ *is σ-finite if and only if there is a countable disjoint collection of sets $\{A_i\}_{i=1}^{\infty}$ with finite measure such that $\mathbb{X} = \bigcup_{i=1}^{\infty} A_i$.*

$(\mathbb{X}, \mathscr{M}, \mu)$ is σ-finite $\implies$ there is a countable disjoint collection of sets $\{A_i\}_{i=1}^{\infty}$ with finite measure such that $\mathbb{X} = \bigcup_{i=1}^{\infty} A_i$.

If $(\mathbb{X}, \mathscr{M}, \mu)$ is σ-finite, then $\mathscr{M}$ contains an increasing sequence of sets $B_1 \subset B_2 \subset \dots$ such that $\mathbb{X} = \bigcup_{i=1}^{\infty} B_i$ and $\mu(B_i) < \infty$. If we have an increasing sequence of sets, we can construct a disjoint sequence of sets $\{A_i\}_{i=1}^{\infty}$ such that $\mathbb{X} = \bigcup_{i=1}^{\infty} A_i$. We do this by defining $A_1 = A_1$, $A_i = A_i \backslash A_{i-1}$ for $i = 2, 3, \dots$. Additionally, since each A_i is the subset of a set of finite measure, $\mu(A_i) < \infty$ as well.

there is a countable disjoint collection of sets $\{A_i\}_{i=1}^{\infty}$ with finite measure such that $\mathbb{X} = \bigcup_{i=1}^{\infty} A_i \implies (\mathbb{X}, \mathscr{M}, \mu)$ is σ-finite.

By defining $B_i = \cup_{j=1}^{i} A_i$, we can construct an increasing sequence of sets $\{B_i\}_{i=1}^{\infty}$ such that $\mathbb{X} = \bigcup_{i=1}^{\infty} A_i$. Additionally, since $\mu(A_i) < \infty$, and B_i is a finite union of A_i, $\mu(B_i) < \infty$ as well. So $(\mathbb{X}, \mathscr{M}, \mu)$ is σ-finite.

---

2. (Exercise 5.9) Prove that an algebra $\mathscr{A}$ is a σ-algebra if and only if $\mathscr{A}$ is closed under countable unions of increasing sequences of sets, i.e., if $\{E_i\}_{i=1}^{\infty} \subset \mathscr{A}$ and $E_1 \subset E_2 \subset E_3 \dots$, then $\cup_{i=1}^{\infty} E_i \in \mathscr{A}$.

An algebra $\mathscr{A}$ is a σ-algebra $\implies$ $\mathscr{A}$ is closed under countable unions of increasing sequences of sets

This follows by definition.

$\mathscr{A}$ is closed under countable unions of increasing sequences of sets $\implies$ $\mathscr{A}$ is a σ-algebra.

Theorem 5.1.4 states: *An algebra of sets $\mathscr{M}$ on $\mathbb{X}$ is a σ-algebra if and only if $\{A_i\}_{i=1}^{\infty} \subset \mathscr{M}$ is a disjoint collection implies $\bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$.*

Let $E_1 \subset E_2 \subset E_3 \dots$ be an increasing sequence of sets in $\mathscr{A}$ such that $\bigcup_{i=1}^{\infty} E_i \in \mathscr{M}$. Then we can construct a sequence of disjoint sets $\{A_i\}_{i=1}^{\infty}$ with $A_1 = E_1$ and $A_i = E_i \backslash E_{i-1}$ for $i = 2, 3, \dots$. By Theorem 5.1.2, $A_i \in \mathscr{A}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathscr{M}$. So I think that by Theorem 5.1.4, $\mathscr{M}$ is a σ-algebra.

---

3. (Exercise 5.10) Let $\mathbb{X}$ be a nonempty set. A family of sets $\mathscr{R} \subset \mathscr{P}_{\mathbb{X}}$ is called a σ-ring if it is closed under countable unions and differences, i.e., if $\{E_i\}_{i=1}^{\infty} \subset \mathscr{R}$ then $\cup_{i=1}^{\infty} E_i \in \mathscr{R}$ and if $E, F \in \mathscr{R}$ then $E \backslash F \in \mathscr{R}$. Prove the following

a. σ-rings are closed under countable intersections.

Consider two sets A_1 and A_2 in $\mathscr{R}$. Then $A_1 \cup A_2 \in \mathscr{R}$, $A_1 \backslash A_2 \in \mathscr{R}$, $A_2 \backslash A_1 \in \mathscr{R}$, and $(A_1 \backslash A_2) \cup (A_2 \backslash A_1) \in \mathscr{R}$. So $A_1 \cap A_2 = (A_1 \cup A_2) \backslash ((A_1 \backslash A_2) \cup (A_2 \backslash A_1)) \in \mathscr{R}$. This can be generalized to countable intersections, so $\mathscr{R}$ is closed under countable intersections.

b. If $\mathscr{R}$ is a σ-ring, then $\mathscr{R}$ is a σ-algebra if and only if $\mathbb{X} \in \mathscr{R}$.

$\mathscr{R}$ is a σ-algebra $\implies \mathbb{X} \in \mathscr{R}$

This is by definition.

$\mathbb{X} \in \mathscr{R} \implies \mathscr{R}$ is a σ-algebra

We already have that $\mathscr{R}$ is closed under countable unions. If $\mathbb{X} \in \mathscr{R}$, then for any set $A \in mathscrR$, $A^c = \mathbb{X} \backslash A$ is also in $\mathscr{R}$. By definition 5.1.2, this satisfies the requirements for a σ-algebra.