STAT 620 Homework 1

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1. Do exercise 2.1 from the textbook (page 21): Show that the set of polynomials with rational coefficients is countable.

Consider a polynomial of degree $\leq N \in \mathbb{N}$ defined as

$$\sum_{i=0}^{N} a_i x^i \quad \text{for } a_i \in \mathbb{Q}.$$

Then the polynomial can be defined by an ordered tuple of N coefficients: $(a_0, a_1, \ldots, a_n)$ for $a_i \in \mathbb{Q}$. The set of ordered N-tuples can be defined by the Cartesian product

$$\mathbb{Q}^N = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q},$$

ie, the Cartesian product of a finite number of $\mathbb{Q}$, which is itself countable, by theorem 2.3.4.

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2. Prove the following, which is one of the many relationships in Theorem 2.1.1.

$$\left(\bigcap_{\alpha \in \mathcal{B}} A_\alpha\right)^C = \bigcup_{\alpha \in \mathcal{B}} A_\alpha^C$$

Let $a \in \left(\bigcap_{\alpha \in \mathcal{B}} A_\alpha\right)^C$. Then $a \notin \bigcap_{\alpha \in \mathcal{B}} A_\alpha$, so a not in at least one A_α.

This implies that $a \in A_\alpha^C$, so $a \in \bigcup_{\alpha \in \mathcal{B}} A_\alpha^C$. Therefore, $\left(\bigcap_{\alpha \in \mathcal{B}} A_\alpha\right)^C \subset \bigcup_{\alpha \in \mathcal{B}} A_\alpha^C$.

Reversing these steps, we can show that $\bigcup_{\alpha \in \mathcal{B}} A_\alpha^C \subset \left(\bigcap_{\alpha \in \mathcal{B}} A_\alpha\right)^C$,

so $\left(\bigcap_{\alpha \in \mathcal{B}} A_\alpha\right)^C = \bigcup_{\alpha \in \mathcal{B}} A_\alpha^C$. $\square$

3. Do exercise 2.2: Define $A_i = \{a_i x^i : a_i \in \mathbb{R}\}$ for $i = 0, 1, 2, \ldots$. Construct an increasing sequence of sets that produces $\bigcup_{i=1}^{\infty} A_i$.

A_i is the set of all monomials of degree i, so $\bigcup_{i=1}^{\infty} A_i$ is the set of all monomials. Define $B_i = \{a_i x^j : a_i \in \mathbb{R} \text{ and } j \leq i\}$. Ie, B_i is the set of all monomials with real number coefficients of degree i or less. This constitutes an increasing sequence of sets such that $\lim_{i \to \infty} B_i = \bigcup_{i=1}^{\infty} A_i$.
