STAT 620 Homework 6

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1. (5.12) Let $\mathbb{X} = \{1, 2, 3\}$ let ν be a set function on $\mathscr{P}_{\mathbb{X}}$. with values $\nu(\emptyset) = 0$ and $\nu(\{1, 2, 3\}) = 1$. Set $x = \nu(\{1, 2\})$, $y = \nu(\{2, 3\})$, and $z = \nu(\{1, 3\})$. Find necessary and sufficient conditions on x, y, and z such that it is possible for ν to be additive.

We must have that

$$\begin{aligned}\nu(\{1\}) + \nu(\{2\}) &= x, \\ \nu(\{2\}) + \nu(\{3\}) &= y, \text{ and} \\ \nu(\{1\}) + \nu(\{3\}) &= z\end{aligned}$$

also, we must have

$$\nu(\{1\}) + \nu(\{2\}) + \nu(\{3\}) = \nu(\{1, 2, 3\}) = 1.$$

Solving for $\nu(\{1\})$, $\nu(\{2\})$, $\nu(\{3\})$, respectively, we have

$$\begin{aligned}\nu(\{1\}) &= x - \nu(\{2\}), \\ \nu(\{2\}) &= y - \nu(\{3\}), \\ \nu(\{3\}) &= z - \nu(\{1\}).\end{aligned}$$

Plugging these in, we see that

$$\begin{aligned}1 &= (x - \nu(\{2\})) + (y - \nu(\{3\})) + (z - \nu(\{1\})) \\ &= x + y + z - (\nu(\{1\}) + \nu(\{2\}) + \nu(\{3\})) \\ &= x + y + z - 1\end{aligned}$$

So we have that $x + y + z = 2$ as a necessary and sufficient condition for ν to be an additive measure. It is sufficient, since this condition was found on the basis that all subsets of 2 and 3 are additive.

2. (5.14) Let $\mathbb{X}$ be a nonempty set and fix $x \in \mathbb{X}$. Define

$$\mu^*(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

for $A \subset \mathbb{X}$. Prove that μ^* is an outer measure, and describe the collection $\mathcal{M}$ of μ^*-measurable sets.

To show that μ^* is an outer measure, we must show that $\mu^*(\emptyset) = 0$, monotonicity, and sub-additivity.

Since $\emptyset$ is in fact, empty, $x \notin \emptyset$. So $\mu^*(\emptyset) = 0$.

To show monotonicity, let $A \subset B \in \mathscr{P}_{\mathbb{X}}$. If $x \in A$, then $x \in B$ and $1 = \mu^*(A) = \mu^*(B)$. Similarly, if $x \notin A$, then x may or may not be in B. if $x \notin B$, then $0 = \mu * (A) = \mu^*(B)$. If $x \in B$, then $\mu^*(A) < \mu^*(B) = 1$. In any case $\mu^*(A) \leq \mu^*(B)$.

Finally, to show subadditivity, let $\{A_i\}_{i=1}^{\infty}$ be a collection of sets in $\mathscr{P}_{\mathbb{X}}$ and let x be an element of $\bigcup_{i=1}^{\infty} A_i$. Then $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 1$. However, if $x \in \bigcup_{i=1}^{\infty} A_i$, then x is an element of at least one of the A_i, so

$1 = \mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$. If x is not an element of $\bigcup_{i=1}^{\infty} A_i$, then $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$. Additionally, x can not an element of any A_i, so $\sum_{i=1}^{\infty} \mu^*(A_i) = 0$ as well. This still satisfies sub-additivity, so μ^* is a sub-additive measure.

The set of μ^*-measurable sets are those sets A that satisfy

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

For all sets E in $\mathscr{P}_{\mathbb{X}}$. For any set in $\mathscr{P}_{\mathbb{X}}$, x is either an element of the set or an element of the compliment. If x is not in a set E, then $\mu^*(E) = \mu^*(E \cap A) = \mu^*(E \cap A^c) = 0$ and $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Otherwise, if x is in E, then $\mu^*(E) = 1$. In this case x is in exactly one of $E \cap A$ or $E \cap A^c$, so $\mu^*(E \cap A) + \mu^*(E \cap A^c) = 1$ as well. This holds for all sets in $\mathscr{P}_{\mathbb{X}}$, so all subsets of $\mathbb{X}$ are μ^*-outer measurable.

3. (5.17) on $\mathbb{X} = \mathbb{N}$, define the set function $\nu : \mathscr{P}_{\mathbb{X}} \to [0, \infty]$ by

$$\nu(A) = \begin{cases} \sum_{i \in A} \frac{1}{2^i} & A \text{ finite} \\ \infty & A \text{ infinite.} \end{cases}$$

   for $A \in \mathscr{P}_{\mathbb{X}}$.

- Prove that ν is finitely additive but not countably additive.
- Compute μ_ν^* and the collection $\mathcal{M}$ of μ_ν^*- measurable sets
- Show that $\nu \neq \mu_\nu^*$.

Let $\{A_i\}_{i=1}^{n}$ be a finite collection of disjoint sets in $\mathscr{P}_{\mathbb{X}}$. If any of the A_i are infinite, then $\bigcup_{i=1}^{n} A_i$ is infinite and $\nu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} A_i = \infty$, since adding any positive value to ∞ is still ∞. If every set in the collection is finite, then $\nu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i \in \cup A_i} \frac{1}{2^i} = \sum_{i \in A_1} \frac{1}{2^i} + \sum_{i \in A_2} \frac{1}{2^i} + \cdots + \sum_{i \in A_n} \frac{1}{2^i} = \sum_{i=1}^{n} \nu(A_i)$, since the A_i are disjoint.

Consider, on the other hand, the countable collection of set $\{A_i\}_{i=1}^{\infty}$ where $A_i = \{i\}$. Then $\nu(\bigcup A_i) = \infty$, since $\bigcup A_i$ is infinite. but $\sum_{i=1}^{\infty} \nu(A_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1$. So ν is not countably additive.

Using Theorem 5.4.3, we can construct an outer measure using the set function ν:

$$\mu_\nu^*(A) = \inf\left\{ \sum_{i=1}^{\infty} \nu(A_i) : \{A_i\} \subset \mathscr{P}_{\mathbb{X}} \text{ and } A \subset \bigcup_{i=1}^{\infty} A_i \right\}.$$