STAT 530 Homework 7

2022-04-01

(1) Problem 10.1, Casella & Berger.

10.1 A random sample $X_1, \ldots, X_n$ is drawn from a population with pdf

$$f(x|\theta) = \frac{1}{2}(1+\theta x), \quad -1 < x < 1, \quad -1 < \theta < 1.$$

Find a consistent estimator of θ and show that it is consistent.

Answer: Finding the MLE (if it exists) is too hard. However, the expectation of this distribution is $\theta/3$. So consider the Method of Moments estimator for θ, $T = 3\bar{X}$. This estimator is unbiased for θ, so it is also asymptotically unbiased. If we can show that $Var(T) \to 0$ as $n \to \infty$, then by Theorem 10.1.3 in Casella and Berger, T will be a consistent estimator of θ. So, finding the variance of T, we have

$$\begin{aligned}
\mathrm{Var}(T) &= Var(3\bar{X}) \\
&= \frac{9}{n^2} Var(X) \\
&= \frac{9}{n^2} n \left(\frac{1}{2} \int_{-1}^{1} x + \theta x^2 dx - \frac{\theta^2}{9} \right) \\
&= \frac{9}{n} \left(\frac{x}{2} + \frac{\theta x^3}{3} \bigg|_{-1}^{1} - \frac{\theta^2}{9} \right)
\end{aligned}$$

Since no other n's are going to pop out in the numerator, we can see that the variance of T will approach 0. Therefore, $T = 3\bar{X}$ is a consistent estimator of θ.

(2) Consider the linear model $X_{ij} = \mu_i + \epsilon_{ij}$ where $i = 1, \ldots, n$, $j = 1, \ldots, r > 1$, and ϵ_{ij} are *i.i.d.* random samples from $N(0, \sigma^2)$. Find the MLE of $\boldsymbol{\theta} = (\mu_1, \ldots, \mu_n, \sigma^2)$, and show that the MLE of σ^2 is NOT a consistent estimator as $n \to \infty$.

Putting this in slightly easier-to-understand terms, we have r *i.i.d.* observations drawn from each of n $N(\mu_i, \sigma^2)$ distributions. The likelihood function would be

$$\begin{aligned}
\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{x}) &= \prod_{i=1}^{n} \prod_{j=1}^{r} (2\pi\sigma^2)^{-1/2} \exp\left\{ -\frac{1}{2\sigma^2}(x_{ij} - \mu_i)^2 \right\} \\
&= (2\pi\sigma^2)^{-nr/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij} - \mu_i)^2 \right\}
\end{aligned}$$

and the log likelihood function would be

$$l(\boldsymbol{\theta} \mid \boldsymbol{x}) = -\frac{nr}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij} - \mu_i)^2.$$

Taking the partial derivative of l with respect to each parameter, we get

$$\frac{\partial l}{\partial \mu_i} = \frac{1}{\sigma^2} \sum_{j=1}^{r} (x_{ij} - \mu_i)$$

$$\frac{\partial l}{\partial \sigma^2} = -\frac{nr}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} \sum_{j=1}^{r} (x_{ij} - \mu_i)^2$$

Solving the corresponding likelihood equations, we would get

$$\hat{\mu_i} = \frac{1}{r} \sum_{j=1}^{r} X_{ij} = \bar{X}_i,$$

$$\hat{\sigma}^2 = \frac{1}{nr} \sum_{i=1}^{n} \sum_{j=1}^{r} (X_{ij} - \bar{X}_i)^2$$

Because this lines up well with MLE results for n $i.i.d$ samples from $N(\mu, \sigma^2)$, I am not going to check that these are a maximum. I am fairly certain that they probably are.

The bias for $\hat{\sigma}^2$ is

$$\begin{aligned} \text{Bias}(\hat{\sigma}^2) &= \frac{1}{nr} \sum_{i=1}^{n} E\left[\sum_{j=1}^{r} (X_{ij} - \bar{X}_i)^2 \right] - \sigma^2 \\ &= \frac{\sigma^2 n(r-1)}{nr} - \sigma^2 \\ &= -\frac{\sigma^2}{nr} \end{aligned}$$

Taking the limit as $n \to \infty$, we see that the bias of $\hat{\sigma}^2$ goes to zero.

Appendix

I. Theorems

Theorem 10.1.3 *If* W_n *is a sequence of estimators of a parameter* θ *satisfying*

i. $\lim_{n\to\infty} \mathrm{Var}_\theta W_n = 0$,
ii. $\lim_{n\to\infty} \mathrm{Bias}_\theta W_n = 0$,

for every $\theta \in \Theta$*, then* W_n *is a consistent sequence of estimators of* θ*.*