國立臺灣大學
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CHAPTER 10
LINEAR PROGRAMMING

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Linear Programming

- Course contents:
  - Linear programming
  - Formulation
  - Duality
  - The simplex method
- Reading:
  - Chapter 7 (Dasgupta)
  - Chapter 29 (Cormen)

Linear Programming

- Linear programming describes a broad class of optimization tasks in which both the optimization criterion and the constraints are linear functions.

- Linear programming consists of three parts:
  - A set of decision variables
  - An objective function:
    - maximize or minimize a given linear objective function
  - A set of constraints:
    - satisfy a set of linear inequalities involving these variables

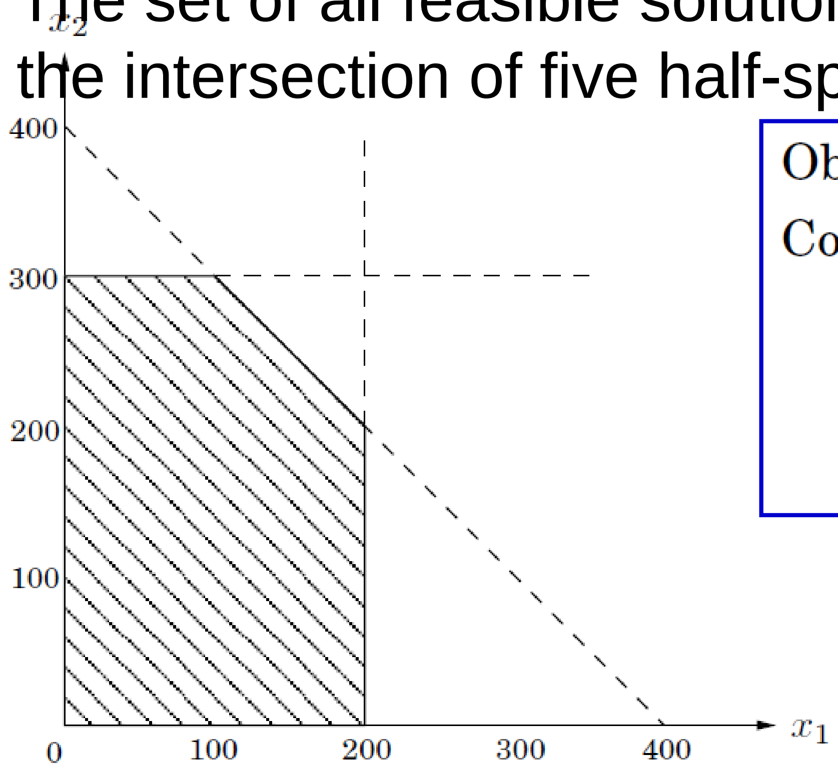
Example: Profit Maximization (1/4)

- A boutique chocolatier has two products:
  - A (Pyramide): profit \$1 per box
  - B (Nuit): profit \$6 per box
- Constraints:
  - The daily demand for these exclusive chocolates is limited to at most 200 boxes of A and 300 boxes of B
  - The current workforce can produce a total of at most 400 boxes of chocolate per day
- Decision variables:
  - x_1 = Boxes of A
  - x_2 = Boxes of B
- Objective Function:
  - Maximize profit

Objective function	$\max x_1 + 6x_2$
Constraints	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$

Example: Profit Maximization (2/4)

- A linear equation in x_1 and x_2 defines a line in the 2D plane
- A linear inequality designates a half-space
- The set of all feasible solutions of this linear program is the intersection of five half-spaces. It is a convex polygon



Objective function	$\max x_1 + 6x_2$
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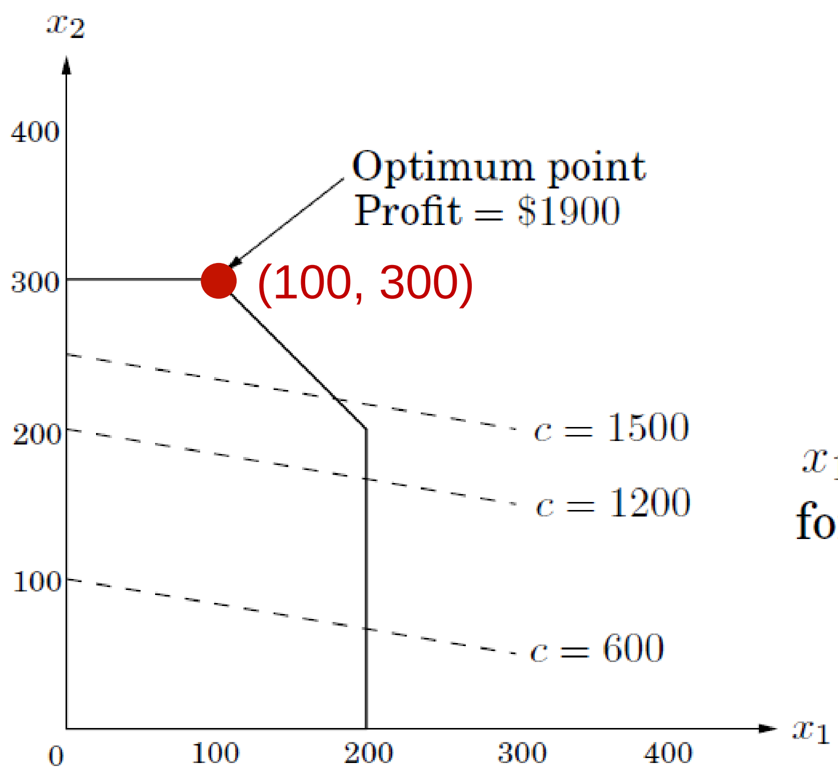
Example: Profit Maximization (3/4)

- Search for the optimal solution
  - It is a general rule of linear programs that the optimum is achieved at a vertex of the feasible region.

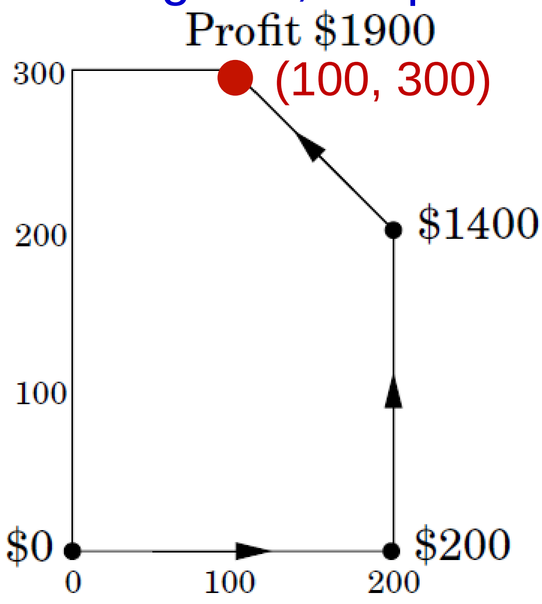


$x_1 + 6x_2 = c$

for different values of the profit c.

Example: Profit Maximization (4/4)

- The Simplex method: hill climbing
  - George Dantzig, 1947
  - Starts at a vertex, say (0, 0)
  - Repeatedly looks for an adjacent vertex (connected by an edge of the feasible region) of better objective value
  - Upon reaching a vertex that has no better neighbor, simplex declares it to be optimal and halts



Multipliers?

$$\begin{aligned} &\max x_1 + 6x_2 \\ &\quad x_1 \le 200 \quad (1) \\ &\quad x_2 \le 300 \quad (2) \\ &x_1 + x_2 \le 400 \quad (3) \\ &x_1, x_2 \ge 0. \end{aligned}$$

- Optimal: $(x_1, x_2) = (100, 300)$; objective value = 1900
- Can this answer be checked somehow?
  - (1) + 6*(2):
    $x_1 + 6x_2 <= 2000$
  - 0*(1) + 5*(2) + (3):
    $x_1 + 6x_2 <= 1900$
  - The multipliers (0, 5, 1) constitute a certificate of optimality
  - How would we systematically find the magic multipliers?

Duality (1/3)

- Multipliers y_i's must be nonnegative

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3.$$

max $x_1 + 6x_2$
$x_1 <= 200$ (1)
$x_2 <= 300$ (2)
$x_1 + x_2 <= 400$ (3)
$x_1, x_2 >= 0.$

- If the left-hand side looks like our objective function, the right-hand side is an upper bound on the optimum solution

$$x_1 + 6x_2 \leq (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

$$\text{if} \quad \left\{ \begin{array}{c} y_1, y_2, y_3 \geq 0 \\ y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \end{array} \right\}.$$

- We want a tight bound!

$$\text{minimize } 200y_1 + 300y_2 + 400y_3$$

Duality (2/3)

- A new LP: finding multipliers that gives the best upper bound on our original LP
  - Primal LP

$$\begin{aligned} \max\ x_1 + 6x_2 & \\ x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

  - Dual LP

$$\begin{aligned} \min\ 200y_1 + 300y_2 + 400y_3 & \\ y_1 + y_3 &\ge 1 \\ y_2 + y_3 &\ge 6 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

  - Any feasible value of dual LP is an upper bound on primal LP
  - If we find a pair of primal and dual feasible values that are equal, they must be both optimal.

$$\text{Primal} : (x_1, x_2) = (100, 300); \quad \text{Dual} : (y_1, y_2, y_3) = (0, 5, 1).$$

Duality (3/3)

- Generic form:

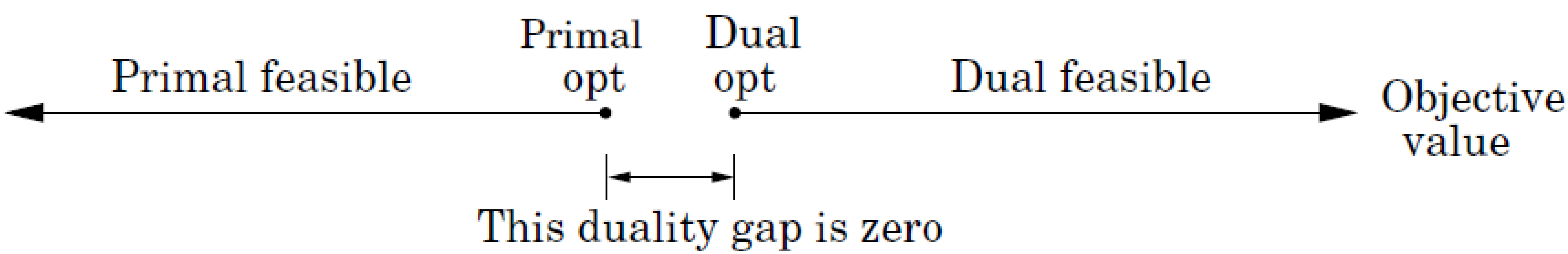
Primal LP:

$$\max \ \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

Dual LP:

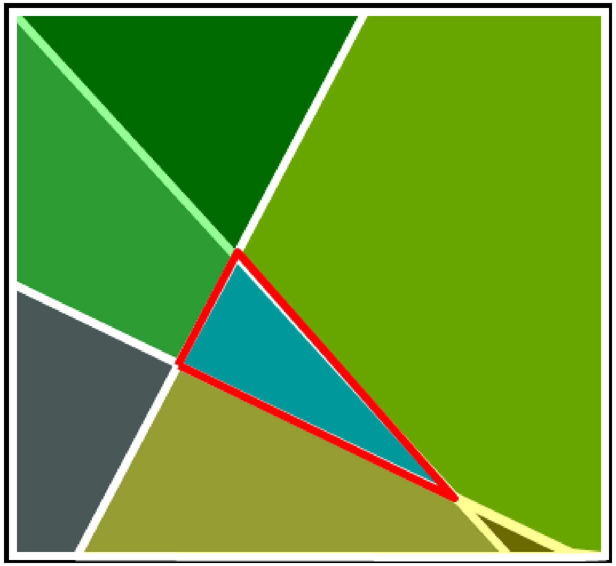
$$\min \ \mathbf{y}^T\mathbf{b}$$
$$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$$
$$\mathbf{y} \geq 0$$

- Dual theorem: If a linear program has a bounded optimum, then so does its dual, and the new optimum

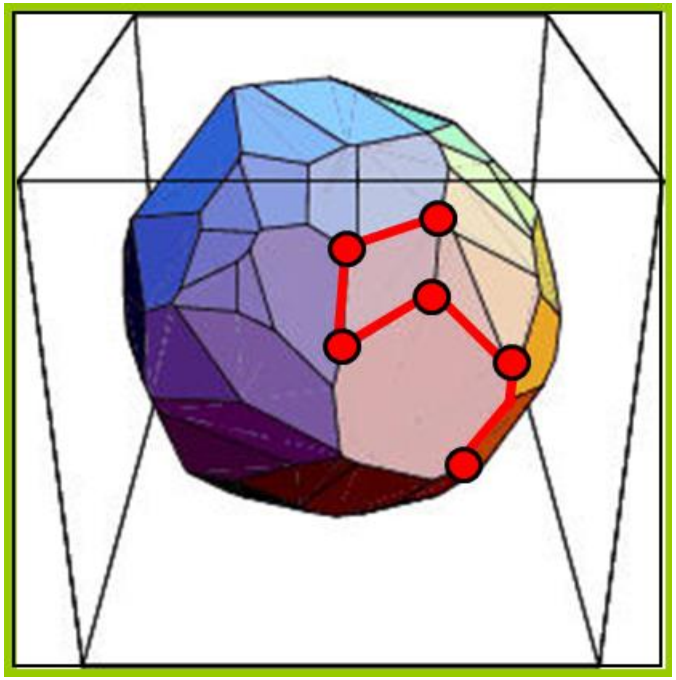


The Simplex Algorithm

```
let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
```



Every constraint specifies an *n*-dimensional half-space

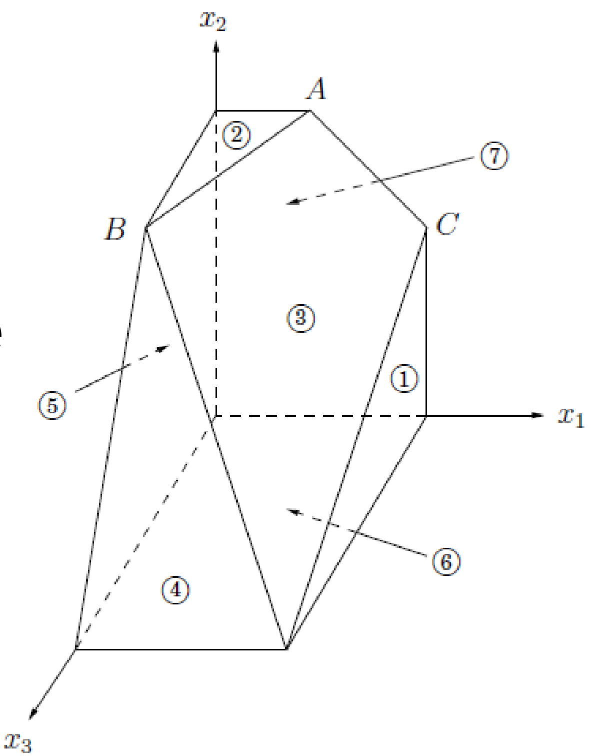


Travel along “edges” until no improvement can be made

Vertex and Neighbors

```
let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
```

- Pick a subset of the inequalities. If there is a unique point that satisfies them with equality, and this point happens to be feasible, then it is a vertex
  - {2, 3, 7} $\rightarrow$ *A*
  - {4, 6} $\rightarrow$ no vertex
- Two vertices are neighbors if they have *n* - 1 defining inequalities in common
  - {2, 3, 7} $\rightarrow$ *A*
  - {1, 3, 7} $\rightarrow$ *C*



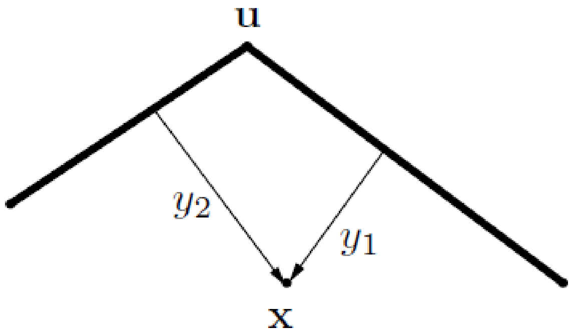
The Simplex Algorithm

- On each iteration, simplex has two tasks:
  - Task 1: Check whether the current vertex is optimal
  - Task 2: Determine where to move next
- Both tasks are easy if the vertex happens to be at the origin
  - Transform the coordinate system to move vertex *u* to the origin

$$\max\ \mathbf{c}^T\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

if one of these enclosing inequalities is $\mathbf{a}_i \cdot \mathbf{x} \leq b_i$,

$$y_i = b_i - \mathbf{a}_i \cdot \mathbf{x}.$$



- Task 1:
  - The origin is optimal if and only if all c_i <= 0
- Task 2:
  - We can move by increasing some x_i for which c_i > 0
  - Until we hit some other constraint

Example (1/3)

Initial LP:	*Current vertex:* {④, ⑤} (origin).

Initial LP:

$$\max\ 2x_1 + 5x_2$$

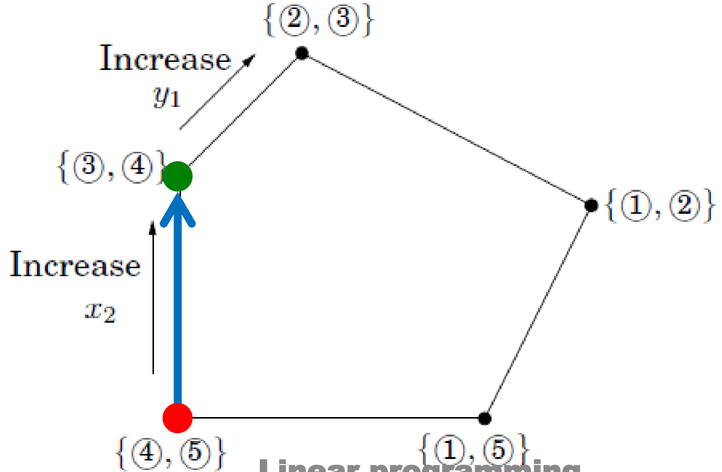
$$\begin{array}{rcll} 2x_1 - x_2 & \leq & 4 & ① \\ x_1 + 2x_2 & \leq & 9 & ② \\ -x_1 + x_2 & \leq & 3 & ③ \\ x_1 & \geq & 0 & ④ \\ x_2 & \geq & 0 & ⑤ \end{array}$$

Current vertex: {④, ⑤} (origin).
Objective value: 0.

Move: increase x_2.
⑤ is released, ③ becomes tight. Stop at $x_2 = 3$.

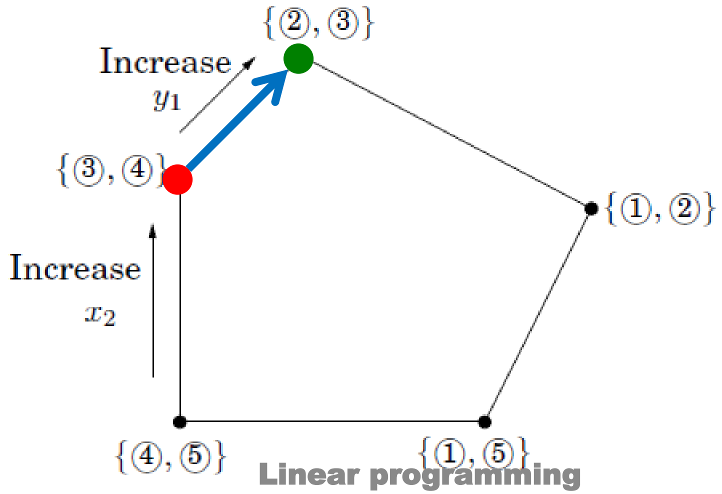
New vertex {④, ③} has local coordinates (y_1, y_2):

$$y_1 = x_1, \quad y_2 = 3 + x_1 - x_2$$



Example (2/3)

Rewritten LP:	*Current vertex:* {④, ③}. *Objective value:* 15.
$$\max\ 15 + 7y_1 - 5y_2$$ $$\begin{array}{rcll} y_1 + y_2 & \leq & 7 & ① \\ 3y_1 - 2y_2 & \leq & 3 & ② \\ y_2 & \geq & 0 & ③ \\ y_1 & \geq & 0 & ④ \\ -y_1 + y_2 & \leq & 3 & ⑤ \end{array}$$	*Move:* increase y_1. ④ is released, ② becomes tight. Stop at $y_1 = 1$. New vertex {②, ③} has local coordinates (z_1, z_2): $$z_1 = 3 - 3y_1 + 2y_2, \quad z_2 = y_2$$



Example (3/3)

Rewritten LP:	*Current vertex:* {②, ③}.

Rewritten LP:

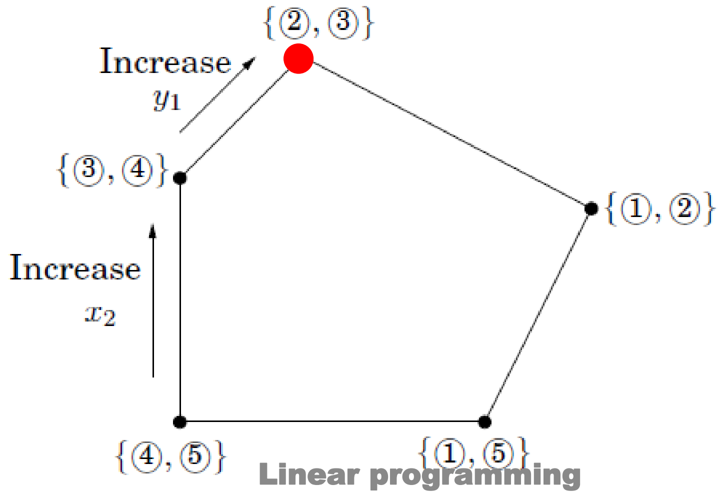
$$\max\ 22 - \tfrac{7}{3}z_1 - \tfrac{1}{3}z_2$$

$$\begin{array}{rcll} -\frac{1}{3}z_1 + \frac{5}{3}z_2 & \leq & 6 & ① \\ z_1 & \geq & 0 & ② \\ z_2 & \geq & 0 & ③ \\ \frac{1}{3}z_1 - \frac{2}{3}z_2 & \leq & 1 & ④ \\ \frac{1}{3}z_1 + \frac{1}{3}z_2 & \leq & 4 & ⑤ \end{array}$$

Current vertex: {②, ③}.
Objective value: 22.

Optimal: all $c_i < 0$.

Solve ②, ③ (in original LP) to get optimal solution $(x_1, x_2) = (1, 4)$.



Standard Form

- Variants
  - Either a maximization or a minimization problem
  - Constraints can be equations and/or inequalities
  - Variables are restricted to be nonnegative or unrestricted in sign
- Standard form
  - Objective function: minimization
  - Constraints: equations
  - Variables: nonnegative

$$\begin{aligned} \max\ & x_1 + 6x_2 \\ x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & -x_1 - 6x_2 \\ x_1 + s_1 &= 200 \\ x_2 + s_2 &= 300 \\ x_1 + x_2 + s_3 &= 400 \\ x_1, x_2, s_1, s_2, s_3 &\ge 0 \end{aligned}$$

Slack variables