

Minimum Variance calculations

Solver Horse Race

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by

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1 Box/Group Constrained Minimum Variance Portfolio

The most natural mathematical formulation of the box & group-constrained minimum variance problem looks as following:

$$\begin{aligned}
 & \min_w \quad w' \Sigma w \\
 & s.t. \\
 & \quad 1'w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad b \leq Aw \leq b + r \\
 & \quad l \leq w \leq u
 \end{aligned}$$

The corresponding AMPL model file of the problem formulation looks as following:

```

> modelMVGROUPOBOX <- c(
+
+   #Markowitzmodel
+   "param N ;",
+   "param Sigma{1..N,1..N} ;",
+   "var w{1..N}>=0, default 1/N ;",
+   "minimize Objective: sum{i in 1..N} sum{j in 1..N} w[i] * Sigma[i,j] * w[j] ;",
+   "subject to Budget: sum{i in 1..N} w[i] = 1 ;",
+
+ )

```

```

+ #Box Constraints
+ "param lower{1..N} ;",
+ "param upper{1..N} ;",
+ "subject to Lower1 {i in 1..N}: w[i] >= lower[i] ;",
+ "subject to Upper1 {i in 1..N}: w[i] <= upper[i] ;",
+
+ #Group Constraints
+ "param G ;",
+ "param B{1..G} ;",
+ "param R{1..G} ;",
+ "param AA{1..G,1..N} ;",
+ "subject to Lower2{i in 1..G}: sum{j in 1..N} AA[i,j]*w[j]>=B[i] ;",
+ "subject to Upper2{i in 1..G}: sum{j in 1..N} AA[i,j]*w[j]<=B[i]+R[i] ;")
> amplModelFile(model=modelMVGROUPBOX, project="myPortfolio")

```

The problem formulation is quadratic with linear constraints. If the covariance matrix is positive definite ($w'\Sigma w > 0 \quad \forall w$), the mathematical problem is convex and therefore has a locatable solution.

The solvers that are suitable for this task are specialized linear/quadratic solvers. The following solvers are supposed to be efficient on quadratic problems:

- CPLEX (commercial)
- XPRESS (commercial)
- Gurobi (commercial)
- KNITRO (commercial)
- LOQO (commercial)
- SNOPT (commercial)
- MINOS (commercial)
- CONOPT (commercial)
- CBC (open-source)

The solvers *LOQO*, *SNOPT*, *MINOS*, and *CONOPT* is not capable of mixed-integer programming, therefore we exclude it from all integer problems.

In practice, there are some problems regarding large covariance matrices. Large covariance matrices tend to have very small eigenvalues. Therefore, even if the covariance matrix is positive definite, the finite numerical precision of numerical solvers might lead to the possibility that $w'\Sigma w$ is evaluated as smaller than zero for some w . The solver then thinks that the matrix is not positive semi-definite, and therefore gives an error message since the problem is not convex anymore. For example, CPLEX gives the following error message:

"CPLEX 12.6.3.0: QP Hessian is not positive semi-definite. No basis."

For more information see <http://www-01.ibm.com/support/docview.wss?uid=swg21399951>. We will now present two methods of dealing with this problem.

1.1 Method 1: Adjusting covariance matrix

The easiest way to solve this problem is to adjust the covariance matrix by adding a small constant to all diagonal elements. This will increase the eigenvalues of Σ , and if the constant is chosen appropriately large, $w'\Sigma w > 0$ even with finite machine precision. For our data, this constant is given by $\delta = 10^{-6}$.

1.2 Cholesky-factorization approach

Another approach we can pursue is the following:

When the covariance matrix Σ is positive-definite, we can find a matrix A such that $\Sigma = A'A$. Since Σ is symmetric, we can use Cholesky factorization to efficiently compute (in our case in under 1 second) such a matrix A . We can then introduce new variables y and reformulate the problem as

$$\begin{aligned}
 & \min_{w,y} \quad y'y \\
 s.t. \quad & y - Cw = 0 \\
 & 1'w = 1 \\
 & w_i \geq 0 \\
 & b \leq Aw \leq b + r \\
 & l \leq w \leq u
 \end{aligned}$$

This problem is always positive definite, even under finite computing precision.

Apparently, the first method is under normal circumstances significantly faster than the Cholesky-approach. The solver results of these two approaches differ only on a very small scale, the achieved covariance is not significantly bigger for the first approach. Therefore we will use the first approach when comparing different solvers. Some results for the Cholesky-based approach are attached in the end together with a comparison of the results.

1.3 Unconstained Minimum Variance Portfolio

$$\begin{aligned}
& \min_w w' \Sigma w \\
& s.t. \\
& \sum_i^N w_i = 1 \\
& w_i \geq 0
\end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	11	16	9.416e-05	3.5e-06	9.1e-09	5.4e-14
XPRESS	6.7	10	9.416e-05	5.6e-05	1.4e-07	5.8e-11
GUROBI	9.3	15	9.416e-05	3.7e-06	9.4e-09	5.4e-14
SNOPT	12	1462	9.416e-05	1.1e-05	2.7e-08	1.4e-13
MINOS	190	1516	9.416e-05	1.1e-05	2.7e-08	1.4e-13
CONOPT	110	168	9.416e-05	1.1e-05	2.7e-08	1.4e-13
KNITRO	24	11	9.416e-05	2e-04	7.1e-07	2.1e-09
LOQO	120	19	9.416e-05	3.5e-06	8.9e-09	1.2e-12
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO	LOQO
CPLEX	0.00E+00	3.14E-05	2.97E-09	2.87E-04	2.87E-04	2.87E-04	3.16E-04	2.71E-06
XPRESS	5.68E-05	0.00E+00	5.84E-05	1.45E-03	1.45E-03	1.45E-03	2.19E-04	9.27E-05
GUROBI	1.73E-09	3.12E-05	0.00E+00	2.90E-04	2.90E-04	2.90E-04	3.16E-04	2.80E-06
SNOPT	1.43E-05	6.77E-05	1.44E-05	0.00E+00	1.16E-12	1.79E-14	3.70E-04	7.23E-06
MINOS	1.43E-05	6.77E-05	1.44E-05	1.16E-12	0.00E+00	1.13E-12	3.70E-04	7.23E-06
CONOPT	1.43E-05	6.77E-05	1.44E-05	2.08E-14	1.14E-12	0.00E+00	3.70E-04	7.23E-06
KNITRO	2.00E-03	8.46E-04	2.34E-03	8.08E-03	8.08E-03	8.08E-03	0.00E+00	1.82E-03
LOQO	2.17E-06	4.44E-05	2.26E-06	1.40E-04	1.40E-04	1.40E-04	3.38E-04	0.00E+00

Table 1: Kullback-Leibler-divergence

	PF
CPLEX	TRUE
XPRESS	TRUE
GUROBI	TRUE
SNOPT	TRUE
MINOS	TRUE
CONOPT	TRUE
KNITRO	TRUE
LOQO	TRUE
CBC	TRUE

Table 2: Constraint-Fulfilment

1.4 Box Constraints

$$\begin{aligned}
& \min_w \quad w' \Sigma w \\
& s.t. \\
& \quad 1'w = 1 \\
& \quad w_i \geq 0 \\
& \quad l \leq w \leq u
\end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower box constraint: OLZ

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	11	16	0.0001373	8.1e-08	1.3e-10	1.5e-14
XPRESS	6.6	10	0.0001373	4.5e-05	1.1e-07	6.7e-12
GUROBI	8.8	14	0.0001373	8.1e-08	1.6e-10	1.5e-14
SNOPT	12	1473	0.0001373	8.1e-08	1.3e-10	1.6e-14
MINOS	200	1564	0.0001373	8.1e-08	1.3e-10	1.6e-14
CONOPT	44	50	0.0001373	8.1e-08	1.3e-10	1.6e-14
KNITRO	24	11	0.0001373	0.00015	6.3e-07	7.1e-10
LOQO	120	21	0.0001373	2.2e-07	4.9e-10	4.6e-12
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO	LOQO
CPLEX	0.00E+00	6.73E-05	1.33E-07	6.66E-13	6.67E-13	6.67E-13	3.14E-04	2.95E-07
XPRESS	4.78E-04	0.00E+00	3.47E-04	4.78E-04	4.78E-04	4.78E-04	1.52E-04	2.87E-04
GUROBI	3.26E-07	6.63E-05	0.00E+00	3.27E-07	3.27E-07	3.27E-07	3.13E-04	6.66E-08
SNOPT	6.69E-13	6.73E-05	1.34E-07	0.00E+00	1.24E-17	3.50E-16	3.14E-04	2.96E-07
MINOS	6.69E-13	6.73E-05	1.34E-07	-6.13E-18	0.00E+00	3.47E-16	3.14E-04	2.96E-07
CONOPT	6.69E-13	6.73E-05	1.34E-07	2.08E-16	2.25E-16	0.00E+00	3.14E-04	2.96E-07
KNITRO	2.54E-03	3.02E-04	2.11E-03	2.55E-03	2.55E-03	2.55E-03	0.00E+00	1.82E-03
LOQO	8.09E-07	6.52E-05	9.49E-08	8.12E-07	8.12E-07	8.12E-07	3.11E-04	0.00E+00

Table 3: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev
CPLEX	TRUE	FALSE	11	4.6e-08
XPRESS	TRUE	FALSE	11	4.6e-08
GUROBI	TRUE	FALSE	11	4.6e-08
SNOPT	TRUE	FALSE	11	4.6e-08
MINOS	TRUE	FALSE	11	4.6e-08
CONOPT	TRUE	FALSE	11	4.6e-08
KNITRO	TRUE	FALSE	10	4.3e-08
LOQO	TRUE	FALSE	11	4.6e-08

Table 4: Constraint-Fulfilment

1.5 Group Constraints

$$\begin{aligned}
& \min_w \quad w' \Sigma w \\
& s.t. \\
& \quad 1'w = 1 \\
& \quad w_i \geq 0 \\
& \quad b \leq Aw \leq b + r
\end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower group constraint: OLZ

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	12	17	0.0001713	3.1e-06	1.2e-08	6.9e-15
XPRESS	6.7	11	0.0001713	1.3e-05	5.4e-08	2.3e-10
GUROBI	10	15	0.0001713	1.1e-06	4.9e-09	6.9e-15
SNOPT	10	1466	0.0001713	2e-06	9.8e-09	2.7e-14
MINOS	150	1496	0.0001713	2e-06	9.8e-09	2.7e-14
CONOPT	46	392	0.0001713	2e-06	9.8e-09	2.7e-14
KNITRO	25	13	0.0001713	6.8e-05	3.6e-07	6.5e-11
LOQO	120	20	0.0001713	2.9e-06	1.1e-08	3.2e-12
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO	LOQO
CPLEX	0.00E+00	3.94E-06	1.08E-07	3.25E-06	3.25E-06	3.25E-06	9.20E-05	1.47E-07
XPRESS	1.28E-05	0.00E+00	1.49E-05	7.33E-05	7.33E-05	7.33E-05	8.46E-05	6.26E-06
GUROBI	1.02E-07	4.01E-06	0.00E+00	1.67E-06	1.67E-06	1.67E-06	9.51E-05	3.12E-07
SNOPT	5.72E-07	4.48E-06	2.21E-07	0.00E+00	5.24E-14	1.80E-14	9.91E-05	8.67E-07
MINOS	5.73E-07	4.48E-06	2.21E-07	5.19E-14	0.00E+00	7.40E-15	9.91E-05	8.67E-07
CONOPT	5.72E-07	4.48E-06	2.21E-07	3.39E-14	2.38E-14	0.00E+00	9.91E-05	8.67E-07
KNITRO	3.52E-04	1.74E-04	3.95E-04	1.40E-03	1.40E-03	1.40E-03	0.00E+00	2.87E-04
LOQO	2.61E-07	3.10E-06	5.64E-07	8.62E-06	8.62E-06	8.62E-06	9.06E-05	0.00E+00

Table 5: Kullback-Leibler-divergence

	PF	Group	G_ViolN	G_MaxDev
CPLEX	TRUE	FALSE	14	4.9e-06
XPRESS	TRUE	FALSE	14	4.9e-06
GUROBI	TRUE	FALSE	14	4.9e-06
SNOPT	TRUE	FALSE	14	4.9e-06
MINOS	TRUE	FALSE	14	4.9e-06
CONOPT	TRUE	FALSE	14	4.9e-06
KNITRO	TRUE	FALSE	13	4.9e-06
LOQO	TRUE	FALSE	14	4.9e-06

Table 6: Constraint-Fulfilment

1.6 Box & Group Constraints

$$\begin{aligned}
& \min_w w' \Sigma w \\
& s.t. \\
& \quad 1'w = 1 \\
& \quad w_i \geq 0 \\
& \quad b \leq Aw \leq b + r \\
& \quad l \leq w \leq u
\end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	13	23	0.0002238	6.6e-07	2.2e-09	4.3e-14
XPRESS	6.5	12	0.0002238	8.7e-05	4.4e-07	6.7e-11
GUROBI	10	15	0.0002238	1.4e-06	7.6e-09	4.3e-14
SNOPT	11	1588	0.0002238	1.8e-07	1.3e-09	9.2e-14
MINOS	150	1578	0.0002238	1.5e-07	1.2e-09	9.2e-14
CONOPT	53	513	0.0002238	1.8e-07	1.3e-09	9.2e-14
KNITRO	26	14	0.0002238	2.3e-05	1.5e-07	2.8e-11
LOQO	120	23	0.0002238	1.4e-07	8.2e-10	5.3e-12
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO	LOQO
CPLEX	0.00E+00	4.42E-05	6.78E-07	3.12E-08	3.12E-08	3.12E-08	1.65E-05	8.50E-08
XPRESS	1.84E-04	0.00E+00	1.04E-04	2.23E-04	2.23E-04	2.23E-04	1.52E-05	1.42E-04
GUROBI	1.67E-06	4.01E-05	0.00E+00	2.85E-06	2.85E-06	2.85E-06	1.33E-05	5.92E-07
SNOPT	1.93E-08	4.45E-05	7.96E-07	0.00E+00	5.56E-11	5.86E-15	1.68E-05	1.54E-07
MINOS	1.93E-08	4.45E-05	7.94E-07	5.57E-11	0.00E+00	5.56E-11	1.67E-05	1.54E-07
CONOPT	1.93E-08	4.45E-05	7.96E-07	-5.81E-15	5.56E-11	0.00E+00	1.68E-05	1.54E-07
KNITRO	6.78E-05	1.16E-05	2.92E-05	8.73E-05	8.73E-05	8.73E-05	0.00E+00	4.81E-05
LOQO	1.37E-07	4.33E-05	3.74E-07	3.85E-07	3.85E-07	3.85E-07	1.57E-05	0.00E+00

Table 7: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev
CPLEX	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
XPRESS	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
GUROBI	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
SNOPT	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
MINOS	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
CONOPT	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
KNITRO	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
LOQO	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06

Table 8: Constraint-Fulfilment

1.7 Buy-In constraints

If we want to add Buy-In constraints to our model, we have to add binary variables $\delta \in \{0, 1\}$. The mathematical formulation changes to

$$\begin{aligned}
 & \min_{w, \delta \in \{0, 1\}} w' \Sigma w \\
 & s.t. \\
 & \quad 1' w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad B_i * \delta_i \leq w_i \leq \delta_i
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Buy-In constraint: 0.001

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	33	16	0	9.416e-05	0.0021	4.9e-06	0
XPRESS	11	-	0	9.416e-05	5e-04	1.5e-06	7.8e-11
GUROBI	6.5	141 plus 55	-	9.416e-05	5e-04	1.1e-06	0
KNITRO	43	-	2	9.416e-05	6e-04	1.3e-06	0
CBC	DNF						

	CPLEX	XPRESS	GUROBI	KNITRO
CPLEX	0.00E+00	2.05E-03	3.19E-03	-1.04E+00
XPRESS	5.03E-02	0.00E+00	1.03E-03	-1.04E+00
GUROBI	7.70E-02	2.44E-02	0.00E+00	-1.05E+00
KNITRO	3.86E+03	2.69E+03	3.86E+03	0.00E+00

Table 9: Kullback-Leibler-divergence

	PF	BuyIn	BL_ViolN	BL_MaxDev
CPLEX	TRUE	TRUE		
XPRESS	TRUE	FALSE	1413	2.3e-07
GUROBI	TRUE	TRUE		
KNITRO	TRUE	TRUE		

Table 10: Constraint-Fulfilment

1.8 Buy-In & Box & Group constraints

$$\begin{aligned}
& \min_{w, \delta \in \{0,1\}} w' \Sigma w \\
& s.t. \\
& \quad 1'w = 1 \\
& \quad w_i \geq 0 \\
& \quad b \leq Aw \leq b + r \\
& \quad l \leq w \leq u \\
& \quad B_i * \delta_i \leq w_i \leq \delta_i
\end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ
- Buy-In constraint: 0.001

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	34	1855	422	0.0002241	0.003	1.5e-05	0
XPRESS	7.3	-	1	0.0002241	2e-04	9e-07	0
GUROBI	7	828 plus 416	-	0.0002241	6.5e-05	2.4e-07	0
KNITRO	59	-	3	0.0002241	6.4e-05	2.4e-07	0
CBC	DNF						

	CPLEX	XPRESS	GUROBI	KNITRO
CPLEX	0.00E+00	3.33E-02	3.33E-02	3.33E-02
XPRESS	1.84E-01	0.00E+00	2.90E-05	2.82E-05
GUROBI	1.82E-01	2.90E-05	0.00E+00	6.05E-09
KNITRO	1.82E-01	2.82E-05	6.05E-09	0.00E+00

Table 11: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev	BuyIn
CPLEX	TRUE	FALSE	31	4.6e-08	FALSE	14	4.9e-06	TRUE
XPRESS	TRUE	FALSE	7	3.2e-08	FALSE	12	4.9e-06	TRUE
GUROBI	TRUE	FALSE	76	4.6e-08	FALSE	15	4.9e-06	TRUE
KNITRO	TRUE	FALSE	20	4.6e-08	FALSE	14	4.9e-06	TRUE

Table 12: Constraint-Fulfilment

1.9 Turnover Constraints

Existing vector of assets v .

$$\begin{aligned}
 & \min_{w, \delta} \quad w' \Sigma w \\
 & s.t. \\
 & \quad 1' w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad w_i - v_i \leq \delta_i \\
 & \quad v_i - w_i \leq \delta_i \\
 & \quad \sum_{i=1}^N \delta_i \leq T
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Turnover constraint: 0.1
- Existing portfolio vector v : First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	16	20	0.0005772	3.9e-08	5.9e-11	1.4e-13
XPRESS	6.6	9	0.0005772	6.4e-06	1.5e-08	7.9e-12
GUROBI	10	18	0.0005772	6.3e-08	1.8e-10	6.8e-13
SNOPT	4.9	1473	0.0005772	8.6e-08	7.1e-11	1.4e-13
MINOS	8.2	1475	0.0005772	8.4e-08	7.1e-11	1.4e-13
CONOPT	29	1416	0.0005772	8.6e-08	7.1e-11	1.4e-13
KNITRO	29	19	0.0005772	8.5e-08	3e-10	1.1e-10
LOQO	410	38	0.0005772	8.1e-08	7e-11	1.5e-13
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO	LOQO
CPLEX	0.00E+00	8.13E-06	5.63E-08	3.97E-07	3.97E-07	3.97E-07	2.04E-07	3.20E-07
XPRESS	3.85E-05	0.00E+00	2.65E-05	1.55E-04	1.55E-04	1.55E-04	4.48E-05	1.34E-04
GUROBI	9.28E-08	7.66E-06	0.00E+00	1.88E-06	1.88E-06	1.88E-06	3.67E-07	1.55E-06
SNOPT	3.11E-08	8.33E-06	1.31E-07	0.00E+00	3.85E-13	3.93E-15	2.18E-07	6.17E-13
MINOS	3.11E-08	8.33E-06	1.31E-07	3.97E-13	0.00E+00	4.01E-13	2.18E-07	1.03E-12
CONOPT	3.11E-08	8.33E-06	1.31E-07	-3.93E-15	3.81E-13	0.00E+00	2.18E-07	6.14E-13
KNITRO	9.26E-07	8.41E-06	7.92E-07	2.00E-06	2.00E-06	2.00E-06	0.00E+00	1.95E-06
LOQO	3.11E-08	8.33E-06	1.31E-07	2.37E-13	6.42E-13	2.41E-13	2.18E-07	0.00E+00

Table 13: Kullback-Leibler-divergence

	PF	Turnover	Overs.
CPLEX	TRUE	TRUE	
XPRESS	TRUE	TRUE	
GUROBI	TRUE	TRUE	
SNOPT	TRUE	FALSE	5.6e-17
MINOS	TRUE	FALSE	6.7e-15
CONOPT	TRUE	TRUE	
KNITRO	TRUE	TRUE	
LOQO	TRUE	TRUE	

Table 14: Constraint-Fulfilment

1.10 Linear Transaction Costs Constraints

Existing vector of assets v .

$$\begin{aligned}
 & \min_{w, \delta} \quad w' \Sigma w \\
 & s.t. \\
 & \quad 1' w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad w_i - v_i \leq \delta_i \\
 & \quad v_i - w_i \leq \delta_i \\
 & \quad \sum_{i=1}^N c_i \cdot \delta_i \leq T
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Transaction Costs constraint T: 0.0002
- Transaction Costs c: First 730 assets 0.002, last 731 assets 0.001
- Existing portfolio vector v: First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	17	22	0.0005468	5.7e-10	9.7e-13	2.6e-14
XPRESS	6.9	10	0.0005468	4.8e-07	1.4e-09	2.8e-11
GUROBI	9.9	17	0.0005468	2.9e-10	7.3e-13	2.4e-13
SNOPT	5.2	1477	0.0005468	1.5e-10	2.4e-13	2.6e-14
MINOS	8.3	1480	0.0005468	1.5e-10	2.4e-13	2.6e-14
CONOPT	22	1409	0.0005468	1.5e-10	2.4e-13	2.6e-14
KNITRO	35	29	0.0005468	8.7e-08	1.3e-10	1.7e-11
LOQO	400	37	0.0005468	6.4e-09	9.1e-12	2.7e-14
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO	LOQO
CPLEX	0.00E+00	9.95E-07	1.15E-09	5.56E-09	5.56E-09	5.56E-09	3.58E-08	5.56E-09
XPRESS	6.50E-06	0.00E+00	8.10E-06	1.54E-05	1.54E-05	1.54E-05	4.05E-06	1.54E-05
GUROBI	7.92E-10	9.97E-07	0.00E+00	2.45E-09	2.45E-09	2.45E-09	3.53E-08	2.45E-09
SNOPT	7.68E-10	1.00E-06	6.37E-10	0.00E+00	-6.21E-16	-1.90E-15	3.85E-08	6.49E-15
MINOS	7.68E-10	1.00E-06	6.37E-10	6.45E-16	0.00E+00	-1.25E-15	3.85E-08	7.16E-15
CONOPT	7.68E-10	1.00E-06	6.37E-10	1.91E-15	1.25E-15	0.00E+00	3.85E-08	8.40E-15
KNITRO	1.59E-07	9.08E-07	1.19E-07	3.01E-07	3.01E-07	3.01E-07	0.00E+00	3.01E-07
LOQO	7.68E-10	1.00E-06	6.37E-10	6.86E-15	6.21E-15	4.96E-15	3.85E-08	0.00E+00

Table 15: Kullback-Leibler-divergence

	PF	Trans_Lin	Overs.
CPLEX	TRUE	TRUE	
XPRESS	TRUE	TRUE	
GUROBI	TRUE	TRUE	
SNOPT	TRUE	TRUE	
MINOS	TRUE	FALSE	8.6e-17
CONOPT	TRUE	FALSE	3.2e-18
KNITRO	TRUE	FALSE	2.7e-10
LOQO	TRUE	TRUE	

Table 16: Constraint-Fulfilment

1.11 Fixed Transaction Costs Constraints

Existing vector of assets v .

$$\begin{aligned}
 & \min_{w, \delta \in \{0,1\}} w' \Sigma w \\
 & s.t. \\
 & 1'w = 1 \\
 & w_i \geq 0 \\
 & w_i - v_i \leq \delta_i \\
 & v_i - w_i \leq \delta_i \\
 & \sum_{i=1}^N c_i \cdot \delta_i \leq T
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Transaction Costs constraint T: 0.0002
- Transaction Costs c: 6e-7
- Existing portfolio vector v: First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	17	23	0	9.416e-05	8e-16	1.1e-18	0
XPRESS	5.1	-	0	0.0006833	0.09	0.0013	0.00068
GUROBI	5.5	524 plus 53	-	9.416e-05	1.4e-16	9.4e-19	0
KNITRO	DNF						
CBC	DNF						

Note that the objective is the same as for the unconstrained portfolio up to the fourth digit. The constraint acts here as Buy-In constraint that prevents unnecessarily small trades without having a real affect on the portfolio.

	CPLEX	XPRESS	GUROBI
CPLEX	0.00E+00	3.84E+00	3.01E-16
XPRESS	2.41E+01	0.00E+00	2.41E+01
GUROBI	-3.08E-16	3.84E+00	0.00E+00

Table 17: Kullback-Leibler-divergence

	PF	Trans.Fixed	Overs.
CPLEX	TRUE	TRUE	
XPRESS	TRUE	FALSE	0.00068
GUROBI	TRUE	TRUE	

Table 18: Constraint-Fulfilment

1.12 Fixed Transaction Costs Constraints with a higher cost per trade

Existing vector of assets v .

$$\begin{aligned}
 & \min_{w, \delta \in \{0,1\}} w' \Sigma w \\
 & s.t. \\
 & \quad 1'w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad w_i - v_i \leq \delta_i \\
 & \quad v_i - w_i \leq \delta_i \\
 & \quad \sum_{i=1}^N c_i \cdot \delta_i \leq T
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Transaction Costs constraint T: 0.0002
- Transaction Costs c: 6e-6
- Existing portfolio vector v: First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	76	52258	5861	0.0006725	0	0	0
XPRESS	DNF						
GUROBI	DNF						
KNITRO	DNF						
CBC	DNF						

	PF	Trans_Fixed	Overs.
CPLEX	TRUE	FALSE	0.00021

Table 19: Constraint-Fulfilment

1.13 Linear+Fixed Transaction Costs Constraints

Existing vector of assets v .

$$\begin{aligned}
 & \min_{w, \delta, \gamma \in \{0,1\}} w' \Sigma w \\
 & s.t. \\
 & \quad 1'w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad w_i - v_i \leq \delta_i \\
 & \quad v_i - w_i \leq \delta_i \\
 & \quad w_i - v_i \leq \gamma_i \\
 & \quad v_i - w_i \leq \gamma_i \\
 & \quad \sum_{i=1}^N d_i \cdot \gamma_i + \sum_{i=1}^N c_i \cdot \delta_i \leq T_1
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Transaction costs constraint T: 0.0002
- Linear transaction costs c: First 728 assets 0.002, last 729 assets 0.001
- Fixed transaction costs d: 6e-7
- Existing portfolio vector v: First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	69	121	26	0.0005521	6.9e-12	4.7e-15	0
XPRESS	10	-	0	0.0006833	0.063	0.0012	0.00068
GUROBI	22	4410 plus 4756	-	0.0005521	6.9e-12	4.7e-15	0
KNITRO	DNF						
CBC	DNF						

	CPLEX	XPRESS	GUROBI
CPLEX	0.00E+00	2.15E+00	-6.00E-16
XPRESS	2.15E+01	0.00E+00	2.15E+01
GUROBI	6.09E-16	2.15E+00	0.00E+00

Table 20: Kullback-Leibler-divergence

	PF	TransLinFixed	Overs.
CPLEX	TRUE	FALSE	2e-05
XPRESS	TRUE	FALSE	0.0036
GUROBI	TRUE	FALSE	0.00011

Table 21: Constraint-Fulfilment

1.14 Linear+Fixed Transaction Costs Objective

Existing vector of assets v .

$$\begin{aligned} \min_{w, \delta \in \{0,1\}} \quad & w' \Sigma w + \lambda \left(\sum_{i=1}^N c_i \cdot \delta_i + \sum_{i=1}^N d_i \cdot \gamma_i \right) \\ \text{s.t.} \quad & \\ & 1'w = 1 \\ & w_i \geq 0 \\ & w_i - v_i \leq \delta_i \\ & v_i - w_i \leq \delta_i \\ & w_i - v_i \leq \gamma_i \\ & v_i - w_i \leq \gamma_i \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- λ : 200
- Linear transaction costs c : First 728 assets 0.002, last 729 assets 0.001
- Fixed transaction costs d : 6e-7
- Existing portfolio vector v : First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	25	368	150		4.7e-16	9.4e-19	0
XPRESS	23	-	1		0.0058	2.3e-05	0
GUROBI	20	1511 plus 3335	119		1.1e-16	5.1e-19	0
KNITRO	DNF						
CBC	DNF						

	CPLEX	XPRESS	GUROBI
CPLEX	0.00E+00	2.43E-01	2.29E-16
XPRESS	1.61E-01	0.00E+00	1.61E-01
GUROBI	-2.27E-16	2.43E-01	0.00E+00

Table 22: Kullback-Leibler-divergence

	PF
CPLEX	TRUE
XPRESS	TRUE
GUROBI	TRUE

Table 23: Constraint-Fulfilment

1.15 Box Group Linear+Fixed Transaction Costs Objective

Existing vector of assets v .

$$\begin{aligned}
 \min_{w, \delta \in \{0,1\}} \quad & w' \Sigma w + \lambda \left(\sum_{i=1}^N c_i \cdot \delta_i + \sum_{i=1}^N d_i \cdot \gamma_i \right) \\
 \text{s.t.} \quad & \\
 & 1'w = 1 \\
 & w_i \geq 0 \\
 & l \leq w \leq u \\
 & b \leq Aw \leq b + r \\
 & w_i - v_i \leq \delta_i \\
 & v_i - w_i \leq \delta_i \\
 & w_i - v_i \leq \gamma_i \\
 & v_i - w_i \leq \gamma_i
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ
- λ : 200
- Linear transaction costs c: First 728 assets 0.002, last 729 assets 0.001
- Fixed transaction costs d: 6e-7
- Existing portfolio vector v: First 200 assets $\frac{1}{200}$, all other assets are 0

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	41	6987	2199		0.00075	3.5e-06	0
XPRESS	37	-	1		0.0016	1.2e-05	0
GUROBI	31	6378 plus 3771	795		0.00075	3.3e-06	0
KNITRO	DNF						
CBC	DNF						

	CPLEX	XPRESS	GUROBI
CPLEX	0.00E+00	9.72E-03	2.78E-02
XPRESS	1.78E-01	0.00E+00	1.73E-01
GUROBI	1.96E-02	9.59E-03	0.00E+00

Table 24: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev
CPLEX	TRUE	FALSE	35	4.6e-08	FALSE	16	4.9e-06
XPRESS	TRUE	FALSE	17	4.6e-08	FALSE	16	4.9e-06
GUROBI	TRUE	FALSE	17	4.6e-08	FALSE	16	4.9e-06

Table 25: Constraint-Fulfilment

1.16 Tail Dependency Constraints

Existing vector of assets v .

$$\begin{aligned}
 & \min_w w' \Sigma w \\
 & s.t. \\
 & 1'w = 1 \\
 & w_i \geq 0 \\
 & w' P w \leq T
 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Tail dependence matrix: gldDependencyFit from fPortfolio + 10e-4 on the diagonal for numerical stability of the solvers (see Markowitz tests)
- Tail dependency constraint T: 0.3

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	49	22	9.417e-05	9.5e-05	3e-07	2.8e-08
XPRESS	15	No	9709	1100	2.1	0.52
GUROBI	47	20	9.416e-05	3.5e-06	6.6e-09	0
SNOPT	29	1583	9.415e-05	1.6e-05	1e-07	4.8e-12
MINOS	850	1541	9.416e-05	3.6e-06	8.6e-09	4.8e-12
CONOPT	380	154	9.416e-05	3.6e-06	8.6e-09	4.8e-12
KNITRO	76	12	9.416e-05	0.00012	3.3e-07	6.6e-10
LOQO	DNF					
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO
CPLEX	0.00E+00	-3.36E+00	9.78E-04	3.40E-03	3.40E-03	3.40E-03	1.96E-04
XPRESS	6.52E+04	0.00E+00	9.10E+04	1.09E+05	1.09E+05	1.09E+05	7.64E+04
GUROBI	1.28E-04	-3.36E+00	0.00E+00	2.90E-05	2.89E-05	2.89E-05	1.33E-04
SNOPT	1.36E-04	-3.36E+00	1.65E-06	0.00E+00	4.54E-08	4.54E-08	1.42E-04
MINOS	1.36E-04	-3.36E+00	1.61E-06	4.53E-08	0.00E+00	9.67E-15	1.42E-04
CONOPT	1.36E-04	-3.36E+00	1.61E-06	4.53E-08	6.94E-15	0.00E+00	1.42E-04
KNITRO	1.66E-05	-3.36E+00	5.79E-04	3.07E-03	3.07E-03	3.07E-03	0.00E+00

Table 26: Kullback-Leibler-divergence

	PF	PF_ViolN	PFMaxDev	Tail	Oversh.
CPLEX	TRUE			TRUE	
XPRESS	FALSE	0	3000	FALSE	5500000
GUROBI	TRUE			TRUE	
SNOPT	TRUE			TRUE	
MINOS	TRUE			TRUE	
CONOPT	TRUE			TRUE	
KNITRO	TRUE			TRUE	

Table 27: Constraint-Fulfilment

1.17 Box Group Tail Dependency Constraints

Existing vector of assets v .

$$\begin{aligned}
 & \min_w \quad w' \Sigma w \\
 & s.t. \\
 & \quad 1'w = 1 \\
 & \quad w_i \geq 0 \\
 & \quad l \leq w \leq u \\
 & \quad b \leq Aw \leq b + r \\
 & \quad w' Pw \leq T
 \end{aligned} \tag{1}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ
- Tail dependence matrix: gldDependencyFit from fPortfolio + 10e-4 on the diagonal for numerical stability of the solvers (see Markowitz tests)
- Tail dependency constraint T: 0.3

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	55	20	0.0002238	9.7e-06	4.9e-08	1.1e-09
XPRESS	16	34	0.0002238	1.6e-05	7.4e-08	0
GUROBI	58	30	0.0002239	0.00013	5.6e-07	3.6e-09
SNOPT	17	2062	0.000224	0.0045	4.8e-05	2.9e-10
MINOS	620	1614	0.0002238	2e-05	1.1e-07	2.8e-10
CONOPT	83	497	0.0002241	0.0059	5.5e-05	2.9e-10
KNITRO	95	17	0.0002238	4.4e-05	2.8e-07	5.3e-10
LOQO	DNF					
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO
CPLEX	0.00E+00	1.23E-05	-1.29E-05	3.17E-02	2.56E-04	3.80E-01	2.88E-05
XPRESS	4.55E-06	0.00E+00	-8.60E-06	3.16E-02	1.37E-04	3.80E-01	4.57E-05
GUROBI	6.39E-05	8.80E-05	0.00E+00	3.18E-02	6.00E-04	3.80E-01	9.49E-05
SNOPT	1.48E-01	1.45E-01	1.37E-01	0.00E+00	3.94E-01	6.72E-01	1.56E-01
MINOS	1.26E-05	8.35E-06	3.48E-06	3.15E-02	0.00E+00	3.80E-01	6.11E-05
CONOPT	1.66E-01	1.55E-01	1.58E-01	1.83E-01	3.70E-01	0.00E+00	1.60E-01
KNITRO	3.40E-05	7.65E-05	1.88E-05	3.19E-02	9.40E-04	3.81E-01	0.00E+00

Table 28: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev	Tail
CPLEX	TRUE	FALSE	42	4.8e-08	FALSE	17	5.2e-06	TRUE
XPRESS	TRUE	FALSE	17	4.5e-08	FALSE	14	4.9e-06	TRUE
GUROBI	TRUE	FALSE	76	8.9e-08	FALSE	27	3.7e-05	TRUE
SNOPT	TRUE	FALSE	19	4.6e-08	FALSE	15	4.9e-06	TRUE
MINOS	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06	TRUE
CONOPT	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06	TRUE
KNITRO	TRUE	FALSE	15	4.4e-08	FALSE	13	4.9e-06	TRUE

Table 29: Constraint-Fulfilment

1.18 Buy-In Tail Dependence Constraint

$$\begin{aligned}
& \min_{w, \delta} \quad w' \Sigma w \\
& s.t. \\
& \quad 1' w = 1 \\
& \quad w_i \geq 0 \\
& \quad l \leq w \leq u \\
& \quad w' P w \leq T \\
& \quad B_i * \delta_i \leq w_i \leq \delta_i
\end{aligned} \tag{2}$$

- Σ : OLZ + 10e-6 on the diagonal
- Tail dependence matrix: gldDependencyFit from fPortfolio + 10e-4 on the diagonal for numerical stability of the solvers (see Markowitz tests)
- Tail dependency constraint T: 0.3
- Buy-In constraint B: 0.001

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	43	22	-	9.417e-05	0.00065	2e-06	1.4e-08
XPRESS	22	-	0	0.0006833	0.09	0.0013	0.00068
GUROBI	11	141	-	9.416e-05	0.00042	1.8e-06	1.4e-08
KNITRO	350	-	5	9.416e-05	0.001	3.7e-06	1.4e-08
CBC	DNF						

	CPLEX	XPRESS	GUROBI	KNITRO
CPLEX	0.00E+00	3.84E+00	2.76E-02	-1.05E+00
XPRESS	9.60E+00	0.00E+00	2.41E+01	-4.89E+00
GUROBI	1.15E-03	3.85E+00	0.00E+00	-1.05E+00
KNITRO	1.93E+03	6.53E+02	3.87E+03	0.00E+00

Table 30: Kullback-Leibler-divergence

	PF	BuyIn	BL_ViolN	BL_MaxDev	Tail
CPLEX	TRUE	FALSE	1419	0.00099	TRUE
XPRESS	TRUE	FALSE	1461	0.00068	TRUE
GUROBI	TRUE	TRUE			TRUE
KNITRO	TRUE	TRUE			TRUE

Table 31: Constraint-Fulfilment

1.19 Tail Dependency Objective

Existing vector of assets v .

$$\begin{aligned} \min_{w, \delta} \quad & w' \Sigma w + \lambda \cdot w' P w \\ \text{s.t.} \quad & 1' w = 1 \\ & w_i \geq 0 \end{aligned}$$

- Σ : OLZ + 10e-6 on the diagonal
- Tail dependence matrix: gldDependencyFit from fPortfolio + 10e-4 on the diagonal for numerical stability of the solvers (see Markowitz tests)
- λ : 1

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	22	15	1.074	6.9e-08	1.1e-10	6.9e-14
XPRESS	15	10	1.074	5e-05	1.6e-07	5.5e-12
GUROBI	16	13	1.074	2.3e-06	7.2e-09	0
SNOPT	32	1461	1.074	6.4e-07	1.9e-09	1.2e-13
MINOS	620	1519	1.074	6.4e-07	1.8e-09	1.2e-13
CONOPT	350	176	1.074	6.4e-07	1.9e-09	1.2e-13
KNITRO	71	11	1.074	0.00016	9e-07	3.3e-09
LOQO	DNF					
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO
CPLEX	0.00E+00	5.50E-05	1.35E-06	1.09E-05	1.09E-05	1.09E-05	4.13E-04
XPRESS	2.18E-04	0.00E+00	1.51E-04	1.22E-03	1.22E-03	1.22E-03	2.74E-04
GUROBI	2.22E-06	4.71E-05	0.00E+00	5.47E-05	5.47E-05	5.47E-05	4.03E-04
SNOPT	6.45E-07	5.85E-05	2.96E-06	0.00E+00	1.76E-13	4.74E-14	4.18E-04
MINOS	6.45E-07	5.85E-05	2.96E-06	1.75E-13	0.00E+00	2.13E-13	4.18E-04
CONOPT	6.45E-07	5.85E-05	2.96E-06	4.78E-14	2.15E-13	0.00E+00	4.18E-04
KNITRO	3.13E-03	8.08E-04	3.16E-03	9.08E-03	9.08E-03	9.08E-03	0.00E+00

Table 32: Kullback-Leibler-divergence

PF	
CPLEX	TRUE
XPRESS	TRUE
GUROBI	TRUE
SNOPT	TRUE
MINOS	TRUE
CONOPT	TRUE
KNITRO	TRUE

Table 33: Constraint-Fulfilment

1.20 Box Group Tail Dependency Objective

Existing vector of assets v .

$$\begin{aligned}
 \min_w \quad & w' \Sigma w + \lambda \cdot w' P w \\
 s.t. \quad & \\
 & 1' w = 1 \\
 & w_i \geq 0 \\
 & l \leq w \leq u \\
 & b \leq A w \leq b + r
 \end{aligned} \tag{3}$$

- Σ : OLZ + 10e-6 on the diagonal
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ
- Tail dependence matrix: gldDependencyFit from fPortfolio + 10e-4 on the diagonal for numerical stability of the solvers (see Markowitz tests)
- λ : 1

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	20	23	2.465	3.3e-09	1.3e-11	3.7e-14
XPRESS	13	11	2.465	0.00016	1.1e-06	1.2e-09
GUROBI	17	15	2.465	9.4e-08	2.5e-10	0
SNOPT	19	1575	2.465	8.4e-09	2.4e-11	2.3e-15
MINOS	390	1603	2.465	8.5e-09	2.4e-11	2.3e-15
CONOPT	110	497	2.465	8.2e-09	2.8e-11	2.3e-15
KNITRO	64	14	2.465	7.4e-06	3.8e-08	6.3e-11
LOQO	DNF					
CBC	Mem. over.					

	CPLEX	XPRESS	GUROBI	SNOPT	MINOS	CONOPT	KNITRO
CPLEX	0.00E+00	3.38E-04	5.06E-09	6.34E-08	6.34E-08	6.34E-08	7.18E-06
XPRESS	3.00E-03	0.00E+00	2.81E-03	5.81E-03	5.81E-03	5.81E-03	9.01E-04
GUROBI	5.07E-09	3.38E-04	0.00E+00	6.49E-08	6.49E-08	6.49E-08	7.17E-06
SNOPT	5.56E-09	3.38E-04	5.66E-09	0.00E+00	2.25E-15	-1.65E-15	7.22E-06
MINOS	5.56E-09	3.38E-04	5.66E-09	-1.36E-15	0.00E+00	-3.33E-15	7.22E-06
CONOPT	5.56E-09	3.38E-04	5.66E-09	9.21E-15	1.11E-14	0.00E+00	7.22E-06
KNITRO	4.93E-05	3.02E-04	4.38E-05	1.30E-04	1.30E-04	1.30E-04	0.00E+00

Table 34: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev
CPLEX	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06
XPRESS	TRUE	FALSE	16	4.3e-08	FALSE	15	4.9e-06
GUROBI	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06
SNOPT	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06
MINOS	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06
CONOPT	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06
KNITRO	TRUE	FALSE	20	4.6e-08	FALSE	16	4.9e-06

Table 35: Constraint-Fulfilment

1.21 Buy-In Tail Dependency Objective

$$\begin{aligned}
& \min_{w, \delta} \quad w' \Sigma w + \lambda \cdot w' P w \\
& s.t. \quad 1' w = 1 \\
& \quad \quad w_i \geq 0 \\
& \quad \quad l \leq w \leq u \\
& \quad \quad B_i * \delta_i \leq w_i \leq \delta_i
\end{aligned} \tag{4}$$

- Σ : OLZ + 10e-6 on the diagonal
- Tail dependence matrix: gldDependencyFit from fPortfolio + 10e-4 on the diagonal for numerical stability of the solvers (see Markowitz tests)
- λ : 1
- Buy-In constraint B: 0.001

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	26	16	0	1.074	1.7e-07	7.2e-10	0
XPRESS	15	-	0	1.074	1e-06	3.4e-09	2.2e-10
GUROBI	12	157 plus 65	-	1.074	1.7e-07	7.2e-10	0
KNITRO	78	-	2	1.074	4.1e-06	1.1e-08	0
CBC	DNF						

	CPLEX	XPRESS	GUROBI	KNITRO
CPLEX	0.00E+00	1.16E-06	2.91E-16	-1.12E+00
XPRESS	1.52E-05	0.00E+00	1.52E-05	-1.12E+00
GUROBI	-2.93E-16	1.16E-06	0.00E+00	-1.12E+00
KNITRO	3.86E+03	2.55E+03	3.86E+03	0.00E+00

Table 36: Kullback-Leibler-divergence

	PF	BuyIn	BLViolN	BLMaxDev
CPLEX	TRUE	TRUE		
XPRESS	TRUE	FALSE	1411	2.6e-07
GUROBI	TRUE	TRUE		
KNITRO	TRUE	TRUE		

Table 37: Constraint-Fulfilment

2 Appendix: Cholesky Approaches

2.1 No Constraints

$$\begin{aligned}
& \min_{w,y} \quad y/y \\
& s.t. \\
& y - Cw = 0 \\
& 1'w = 1 \\
& w_i \geq 0
\end{aligned}$$

- Cholesky matrix C: R-function *chol*

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	9.1	12	9.415e-05	7.4e-08	2.8e-10	3.9e-15
XPRESS	6.3	8	9.415e-05	2e-04	6.1e-07	1.2e-10
GUROBI	8.4	17	9.415e-05	7.4e-08	2.3e-10	3.9e-15
SNOPT	DNF					
MINOS	160	1585	9.415e-05	1.4e-07	6.3e-10	4.4e-15
CONOPT	21	201	9.415e-05	1.4e-07	3.6e-10	4.4e-15
KNITRO	21	45	9.422e-05	0.00012	1.5e-06	2.9e-07
LOQO	82	27	9.415e-05	1.3e-07	3.5e-10	5.4e-12
CBC	10	-	0.0006833	0.094	0.0013	0.00068

	CPLEX	XPRESS	GUROBI	MINOS	CONOPT	KNITRO	LOQO	CBC
CPLEX	0.00E+00	2.70E-04	1.03E-07	3.43E-06	3.43E-06	-1.07E+00	5.18E-07	3.86E+00
XPRESS	2.11E-03	0.00E+00	2.60E-03	5.98E-03	5.98E-03	-1.07E+00	2.56E-03	3.86E+00
GUROBI	6.99E-08	2.71E-04	0.00E+00	9.95E-07	9.95E-07	-1.07E+00	8.90E-08	3.86E+00
MINOS	2.16E-07	2.71E-04	6.75E-08	0.00E+00	1.83E-12	-1.07E+00	7.70E-09	3.86E+00
CONOPT	2.16E-07	2.71E-04	6.75E-08	1.84E-12	0.00E+00	-1.07E+00	7.70E-09	3.86E+00
KNITRO	3.91E+03	2.71E+03	4.00E+03	4.01E+03	4.01E+03	0.00E+00	3.85E+03	6.80E+02
LOQO	2.20E-07	2.71E-04	9.12E-08	1.48E-07	1.48E-07	-1.07E+00	0.00E+00	3.86E+00
CBC	2.34E+01	1.47E+01	2.40E+01	2.41E+01	2.41E+01	-4.93E+00	2.29E+01	0.00E+00

Table 38: Kullback-Leibler-divergence

	PF	PF_ViolN	PFMaxDev
CPLEX	TRUE		
XPRESS	TRUE		
GUROBI	TRUE		
MINOS	TRUE		
CONOPT	TRUE		
KNITRO	TRUE		
LOQO	FALSE	1147	8.1e-12
CBC	TRUE		

Table 39: Constraint-Fulfilment

2.2 Box Constraints

$$\begin{aligned}
& \min_{w,y} \quad y/y \\
& s.t. \\
& y - Cw = 0 \\
& 1'w = 1 \\
& w_i \geq 0 \\
& l \leq w \leq u
\end{aligned}$$

- Cholesky matrix C: R-function *chol*
- Upper & lower box constraint: OLZ

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	8.8	12	0.0001373	2e-06	3.8e-09	1.2e-12
XPRESS	6.7	14	0.0001373	3e-05	9.9e-08	1.3e-09
GUROBI	9.1	17	0.0001373	4.8e-09	1.9e-11	1e-16
SNOPT	DNF					
MINOS	200	1762	0.0001373	7.3e-09	2.9e-11	9.7e-17
CONOPT	11	47	0.0001373	7.3e-09	2.9e-11	9.7e-17
KNITRO	42	118	0.0001373	4.4e-09	5e-11	2e-11
LOQO	97	32	0.0001373	3.4e-09	2.9e-11	1.3e-11
CBC	29	-	0.0006833	0.0077	0.0012	0.00068

	CPLEX	XPRESS	GUROBI	MINOS	CONOPT	KNITRO	LOQO	CBC
CPLEX	0.00E+00	4.03E-05	7.90E-06	9.07E-06	9.07E-06	-1.42E-01	8.01E-06	2.37E+00
XPRESS	1.35E-04	0.00E+00	3.18E-04	3.36E-04	3.36E-04	-1.43E-01	3.17E-04	2.37E+00
GUROBI	2.03E-06	4.83E-05	0.00E+00	2.39E-09	2.39E-09	-1.42E-01	6.34E-10	2.37E+00
MINOS	2.07E-06	4.84E-05	1.98E-09	0.00E+00	-9.52E-17	-1.42E-01	2.19E-09	2.37E+00
CONOPT	2.07E-06	4.84E-05	1.98E-09	3.18E-16	0.00E+00	-1.42E-01	2.19E-09	2.37E+00
KNITRO	1.52E+02	1.50E+02	1.52E+02	1.52E+02	1.52E+02	0.00E+00	1.52E+02	3.09E+01
LOQO	2.03E-06	4.83E-05	6.42E-10	2.49E-09	2.49E-09	-1.42E-01	0.00E+00	2.37E+00
CBC	9.83E+00	9.68E+00	9.84E+00	9.84E+00	9.84E+00	-2.51E+00	9.84E+00	0.00E+00

Table 40: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev
CPLEX	TRUE	FALSE	38	4.6e-08
XPRESS	TRUE	FALSE	10	4.3e-08
GUROBI	TRUE	FALSE	11	4.6e-08
MINOS	TRUE	FALSE	25	4.6e-08
CONOPT	TRUE	FALSE	11	4.6e-08
KNITRO	TRUE	FALSE	47	4.6e-08
LOQO	TRUE	FALSE	41	4.6e-08
CBC	TRUE	TRUE		

Table 41: Constraint-Fulfilment

2.3 Group Constraints

$$\begin{aligned}
& \min_{w,y} \quad y/y \\
& s.t. \\
& y - Cw = 0 \\
& 1'w = 1 \\
& w_i \geq 0 \\
& b \leq Aw \leq b + r
\end{aligned}$$

- Cholesky matrix C: R-function *chol*
- Upper & lower group constraint: OLZ

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	9.2	13	0.0001713	9.5e-08	2e-10	2.5e-12
XPRESS	7.8	22	0.0001713	6.4e-07	9.4e-09	3.5e-09
GUROBI	9	17	0.0001713	8.4e-09	4.4e-11	3.8e-16
SNOPT	DNF					
MINOS	150	1562	0.0001713	9e-09	5e-11	3.8e-16
CONOPT	15	388	0.0001713	9e-09	5e-11	3.8e-16
KNITRO	30	54	0.0001713	7.4e-09	5.5e-11	3.8e-12
LOQO	96	30	0.0001713	6.3e-07	2e-09	8.6e-12
CBC	8.6	-	0.0006833	0.079	0.0013	0.00068

	CPLEX	XPRESS	GUROBI	MINOS	CONOPT	KNITRO	LOQO	CBC
CPLEX	0.00E+00	2.32E-06	2.28E-07	1.60E-06	1.60E-06	-1.05E+00	3.01E-06	3.71E+00
XPRESS	5.53E-05	0.00E+00	1.02E-04	1.20E-04	1.20E-04	-1.05E+00	8.50E-05	3.71E+00
GUROBI	7.91E-08	2.60E-06	0.00E+00	1.34E-07	1.34E-07	-1.05E+00	2.57E-06	3.71E+00
MINOS	1.06E-07	2.66E-06	1.09E-08	0.00E+00	-5.94E-15	-1.05E+00	2.54E-06	3.71E+00
CONOPT	1.06E-07	2.66E-06	1.09E-08	7.18E-15	0.00E+00	-1.05E+00	2.54E-06	3.71E+00
KNITRO	2.70E+03	1.90E+03	3.33E+03	3.34E+03	3.34E+03	0.00E+00	3.22E+03	5.54E+02
LOQO	-2.43E-06	7.35E-08	-2.49E-06	-2.48E-06	-2.48E-06	-1.05E+00	0.00E+00	3.71E+00
CBC	1.84E+01	1.15E+01	2.38E+01	2.39E+01	2.39E+01	-4.76E+00	2.29E+01	0.00E+00

Table 42: Kullback-Leibler-divergence

	PF	PF_ViolN	PFMaxDev	Group	G_ViolN	G_MaxDev
CPLEX	TRUE			FALSE	14	4.9e-06
XPRESS	TRUE			FALSE	21	6e-06
GUROBI	TRUE			FALSE	14	4.9e-06
MINOS	TRUE			FALSE	14	4.9e-06
CONOPT	TRUE			FALSE	14	4.9e-06
KNITRO	FALSE	1164	6.2e-12	FALSE	18	4.9e-06
LOQO	FALSE	1189	1.3e-11	FALSE	17	4.3e-06
CBC	TRUE			FALSE	23	0.1

Table 43: Constraint-Fulfilment

2.4 Box and Group Constraints

$$\begin{aligned}
& \min_{w,y} \quad y/y \\
& s.t. \\
& y - Cw = 0 \\
& 1'w = 1 \\
& w_i \geq 0 \\
& b \leq Aw \leq b + r \\
& l \leq w \leq u
\end{aligned}$$

- Cholesky matrix C: R-function *chol*
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ

	Time [s]	It. steps	Objective	Max-Dev.	Mean-Dev.	Median-Dev.
CPLEX	9.5	14	0.0002238	5e-07	3.1e-09	1.8e-13
XPRESS	6.8	12	0.0002238	2.9e-05	1.9e-07	9.3e-12
GUROBI	8.9	15	0.0002238	1.9e-07	1.2e-09	1.9e-13
SNOPT	DNF					
MINOS	250	1966	0.0002238	2.4e-07	8.7e-10	1.9e-13
CONOPT	21	504	0.0002238	2.4e-07	8.7e-10	1.9e-13
KNITRO	61	136	0.0002238	1.4e-07	5.1e-10	2.5e-13
LOQO	110	34	0.0002238	1.9e-07	6.5e-10	4e-12
CBC	26	-	0.0006833	0.0085	0.0012	0.00068

	CPLEX	XPRESS	GUROBI	MINOS	CONOPT	KNITRO	LOQO	CBC
CPLEX	0.00E+00	1.86E-05	1.14E-07	7.92E-07	7.92E-07	-2.70E-01	4.51E-07	2.32E+00
XPRESS	4.94E-05	0.00E+00	6.35E-05	9.88E-05	9.88E-05	-2.70E-01	8.24E-05	2.32E+00
GUROBI	8.16E-08	1.96E-05	0.00E+00	1.87E-07	1.87E-07	-2.70E-01	7.24E-08	2.32E+00
MINOS	2.77E-07	2.04E-05	8.59E-08	0.00E+00	4.38E-14	-2.70E-01	1.00E-08	2.32E+00
CONOPT	2.77E-07	2.04E-05	8.59E-08	6.91E-14	0.00E+00	-2.70E-01	1.00E-08	2.32E+00
KNITRO	1.65E+02	1.65E+02	1.65E+02	1.65E+02	1.65E+02	0.00E+00	1.65E+02	3.48E+01
LOQO	2.17E-07	2.02E-05	4.73E-08	1.45E-08	1.45E-08	-2.70E-01	0.00E+00	2.32E+00
CBC	9.73E+00	9.72E+00	9.73E+00	9.73E+00	9.73E+00	-2.59E+00	9.73E+00	0.00E+00

Table 44: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev
CPLEX	TRUE	FALSE	39	4.6e-08	FALSE	15	4.9e-06
XPRESS	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
GUROBI	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
MINOS	TRUE	FALSE	29	4.6e-08	FALSE	15	4.9e-06
CONOPT	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
KNITRO	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06
LOQO	TRUE	FALSE	34	4.6e-08	FALSE	15	4.9e-06
CBC	TRUE	TRUE			FALSE	23	0.1

Table 45: Constraint-Fulfilment

2.5 Buy-In constraints

$$\begin{aligned}
& \min_{w, y, \delta \in \{0,1\}} y/y \\
& s.t. \\
& y - Cw = 0 \\
& 1'w = 1 \\
& w_i \geq 0 \\
& B_i * \delta_i \leq w_i \leq \delta_i
\end{aligned}$$

- Cholesky matrix C: R-function *chol*
- Buy-In constraint B: 0.001

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	57	635	279	9.415e-05	4.6e-08	1.5e-10	0
XPRESS	81	-	1	9.415e-05	4.3e-06	1.2e-08	0
GUROBI	220	3368 plus 9214	-	9.415e-05	7.9e-08	5.7e-10	0
KNITRO	290	-	40	9.415e-05	4.6e-08	1.5e-10	0
CBC	DNF						

	CPLEX	XPRESS	GUROBI	KNITRO
CPLEX	0.00E+00	9.49E-09	2.42E-12	-3.58E-16
XPRESS	9.50E-09	0.00E+00	9.51E-09	9.50E-09
GUROBI	2.43E-12	9.49E-09	0.00E+00	2.43E-12
KNITRO	3.42E-16	9.49E-09	2.42E-12	0.00E+00

Table 46: Kullback-Leibler-divergence

	PF	BuyIn
CPLEX	TRUE	TRUE
XPRESS	TRUE	TRUE
GUROBI	TRUE	TRUE
KNITRO	TRUE	TRUE

Table 47: Constraint-Fulfilment

2.6 Buy-In & Box & Group constraints

$$\begin{aligned}
& \min_{w, y, \delta \in \{0,1\}} y/y \\
& s.t. \\
& y - Cw = 0 \\
& 1'w = 1 \\
& w_i \geq 0 \\
& b \leq Aw \leq b + r \\
& l \leq w \leq u \\
& B_i * \delta_i \leq w_i \leq \delta_i
\end{aligned}$$

- Cholesky matrix C: R-function *chol*
- Buy-In constraint B: 0.001
- Upper & lower box constraint: OLZ
- Upper & lower group constraint: OLZ

	Time [s]	It. steps	Nodes	Objective	Max-Dev.	Mean-Dev.	Median-Dev
CPLEX	59	1199	218	0.0002241	5e-04	2.4e-06	0
XPRESS	86	-	0	0.0002241	5e-04	2.4e-06	0
GUROBI	130	30920 plus 4075	-	0.0002241	5e-04	2.3e-06	0
KNITRO	820	-	15	0.0002241	5e-04	2.3e-06	0
CBC	DNF						

	CPLEX	XPRESS	GUROBI	KNITRO
CPLEX	0.00E+00	1.77E-06	2.48E-02	2.48E-02
XPRESS	1.78E-06	0.00E+00	2.48E-02	2.48E-02
GUROBI	1.49E-03	1.48E-03	0.00E+00	1.87E-09
KNITRO	1.50E-03	1.48E-03	1.87E-09	0.00E+00

Table 48: Kullback-Leibler-divergence

	PF	Box	B_ViolN	B_MaxDev	Group	G_ViolN	G_MaxDev	BuyIn
CPLEX	TRUE	FALSE	20	4.6e-08	FALSE	15	4.9e-06	TRUE
XPRESS	TRUE	FALSE	15	4.3e-08	FALSE	13	4.9e-06	TRUE
GUROBI	TRUE	FALSE	52	4.6e-08	FALSE	15	4.9e-06	TRUE
KNITRO	TRUE	FALSE	20	4.6e-08	FALSE	14	4.9e-06	TRUE

Table 49: Constraint-Fulfilment