Week 2 Assignment

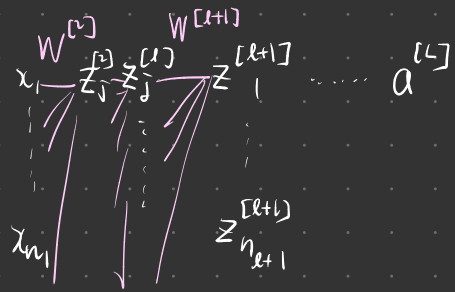
1. Read Deep Learning: An Introduction for Applied Mathematicians. Consider a network as defined in (3.1) and (3.2). Assume that $n_L = 1$, find an algorithm to calculate $\nabla a^{[L]}(x)$.

Consider $\begin{cases} a^{[1]} = x \in \mathbb{R}^{n_1} \\ a^{[\ell]} = \sigma(W^{[\ell]} a^{[\ell-1]} + b^{[\ell]}) \in \mathbb{R}^{n_\ell} \end{cases}$, for $\ell = 2, \dots, L$

Define $z^{[\ell]} = W^{[\ell]} a^{[\ell-1]} + b^{[\ell]} \in \mathbb{R}^{n_\ell}$, then $a^{[\ell]} = \sigma(z^{[\ell]})$

$$g^{[\ell]} = (g_1^{[\ell]}, g_2^{[\ell]}, \dots, g_{n_\ell}^{[\ell]}) = \left(\frac{\partial a^{[L]}}{\partial z_1^{[\ell]}}, \frac{\partial a^{[L]}}{\partial z_2^{[\ell]}}, \dots, \frac{\partial a^{[L]}}{\partial z_{n_\ell}^{[\ell]}} \right)$$

Fixed j-th neuron of ℓ-th layer : ($\ell = L-1, \dots, 2$)



$$g_j^{[\ell]} = \frac{\partial a^{[L]}}{\partial z_j^{[\ell]}} = \sum_{k=1}^{n_{\ell+1}} \frac{\partial a^{[L]}}{\partial z_k^{[\ell+1]}} \cdot \frac{\partial z_k^{[\ell+1]}}{\partial z_j^{[\ell]}}$$

$$= \sum_{k=1}^{n_{\ell+1}} g_k^{[\ell+1]} \left(\frac{\partial}{\partial z_j^{[\ell]}} \left(\sum_{r=1}^{n_\ell} W_{kr}^{[\ell+1]} a_r^{[\ell]} + b_k^{[\ell+1]} \right) \right)$$

$$= \sum_{k=1}^{n_{\ell+1}} g_k^{[\ell+1]} \left(\sum_{r=1}^{n_\ell} \frac{\partial z_k^{[\ell+1]}}{\partial a_r^{[\ell]}} \cdot \frac{\partial \boxed{a_r^{[\ell]}}}{\partial z_j^{[\ell]}} \right) \qquad = \sigma(z_r^{[\ell]})$$

$r \neq j$時，$a_r^{[\ell]}$ 和 $z_j^{[\ell]}$ 無關，偏導數為0

$$= \sum_{k=1}^{n_{\ell+1}} g_k^{[\ell+1]} \left(\sum_{r=1}^{n_\ell} W_{kr}^{[\ell+1]} \cdot (\sigma'(z_r^{[\ell]}) \cdot \delta_{rj}) \right)$$, where δ_{rj} is Kronecker delta

$$= \sum_{k=1}^{n_{\ell+1}} g_k^{[\ell+1]} \cdot \left(W_{kj}^{[\ell+1]} \sigma'(z_j^{[\ell]}) \right) = \sigma'(z_j^{[\ell]}) \sum_{k=1}^{n_{\ell+1}} W_{kj}^{[\ell+1]} g_k^{[\ell+1]}$$

$$= \left((W_j^{[\ell+1]})^T g^{[\ell+1]} \right) \circ \sigma'(z^{[\ell]})$$, where $x \circ y \in \mathbb{R}^n$ is defined by $(x \circ y)_i = x_i y_i$, the pairwise component multiplication.

$$\Rightarrow \underbrace{g^{[\ell]}}_{n_\ell} = \left(\underbrace{(W^{[\ell+1]})^T}_{n_{\ell+1} \times n_\ell} \underbrace{g^{[\ell+1]}}_{n_{\ell+1}} \right) \circ \underbrace{\sigma'(z^{[\ell]})}_{n_\ell} \qquad (*)$$

Note that $g^{[L]} = \frac{\partial a^{[L]}}{\partial z^{[L]}} = \sigma'(z^{[L]}) \in \mathbb{R}$

Next, focus on 2^{nd} layer: $z_j^{[2]} = \sum_{i=1}^{n_1} W_{ji} x_i + b_j^{[2]} \Rightarrow \frac{\partial z^{[2]}}{\partial x_i} = W_{ji}^{[2]}$

$$\therefore \frac{\partial a^{[L]}}{\partial x_i} = \sum_{j=1}^{n_2} \frac{\partial a^{[L]}}{\partial z_j^{[2]}} \cdot \frac{\partial z_j^{[2]}}{\partial x_i} = \sum_{j=1}^{n_2} g_j^{[2]} W_{ji}^{[2]}$$

$$\Rightarrow \nabla_x a^{[L]}(x) = (g^{[2]})^T W^{[2]} \qquad (**)$$

By $(*)$ & $(**)$, we can formulate an algorithm for calculating $\nabla_x a^{[L]}(x)$.