9. Given the following set of data

$$\{f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1\},$$

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

$$H(-1) = -a_3 + a_2 - a_1 + a_0 = 1$$
$$H'(-1) = 3a_3 - 2a_2 + a_1 = 1$$
$$H'(1) = 3a_3 + 2a_2 + a_1 = 2$$
$$H'(2) = 8a_3 + 4a_2 + 2a_1 + a_0 = 1$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -1 & 1 & 1 \\ 3 & -2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 & 2 \\ 8 & 4 & 2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & -2 & 3 & 4 \\ 0 & 5 & -2 & 3 & 5 \\ 0 & 12 & -6 & 9 & 9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & -2 & 3 & 4 \\ 0 & 0 & 8 & -12 & -15 \\ 0 & 0 & 18 & -27 & -39 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 0 & 1 & -2 & 3 & 4 \\ 0 & 0 & 8 & -12 & -15 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \Rightarrow$$ The system has no solution.

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12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}, \qquad (8.75)$$

called the *Padé approximation*. Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10} x^{10} + \ldots.$$

[*Solution*: $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

Taylor expansion: $\frac{1}{1+b_2x^2} = 1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + b_2^4x^8 - \cdots$

$$\Rightarrow r(x) = (a_0 + a_2x^2 + a_4x^4)(1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + b_2^4x^8 - \cdots)$$
$$= a_0 + (a_2 - a_0b_2)x^2 + (a_4 - a_2b_2 + a_0b_2^2)x^4 - (a_4b_2 + a_2b_2^2 + a_0b_2^3)x^6 + O(x^8)$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)$$

$$\Rightarrow \begin{cases} a_0 = 1 \\ a_2 - b_2 = -\frac{1}{2} \\ a_4 - a_2b_2 + b_2^2 = \frac{1}{24} \\ -a_4b_2 - a_2b_2^2 - b_2^3 = -\frac{1}{720} \end{cases} \Rightarrow \begin{cases} a_0 = 1 \\ a_2 = -\frac{7}{15} \\ a_4 = \frac{1}{40} \\ b_2 = \frac{1}{30} \end{cases}$$