1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\left\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\right\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

Heun's method: $y_{n+1} = y_n + h\Phi(t_n, y_n, h) + h\tau_{n+1}$

$$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n, h)$$

$$= y_{n+1} - y_n - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

$$= \underbrace{\int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]}_{E_1} + \underbrace{\frac{h}{2}[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]}_{E_2}$$

$$:= E_1 + E_2$$

where E_1 is the error given by trapezoidal rule, which is $-\frac{h^3}{12} f''(\xi_n, y(\xi_n))$ for $\xi_n \in (t_n, t_{n+1})$

$\Rightarrow E_1 = O(h^3)$

For E_2, consider Taylor expansion:

$$f(t_{n+1}, y_n + hf(t_n, y_n)) = f(t_{n+1}, y_{n+1}) + \frac{\partial f}{\partial y}(t_{n+1}, y_{n+1})\underline{(y_n + hf(t_n, y_n) - y_{n+1})} + O(h^2)$$

where $y_n + hf(t_n, y_n) - y_{n+1} = O(h^2)$ is given by Forward Euler method

$\Rightarrow E_2 := hO(h^2) = O(h^3)$

Thus we have $h\tau_{n+1} = E_1 + E_2 = O(h^3) \Rightarrow \tau_{n+1} = O(h^2)$ □

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

CN method: $y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] + h\tau_{n+1}$

$$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - \frac{h}{2}[f(t_n, y_n) - f(t_{n+1}, y_{n+1})]$$

$$= \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) - f(t_{n+1}, y_{n+1})]$$

Let $g(s) = f(s, y(s)) \Rightarrow h\tau_{n+1} = \int_{t_n}^{t_{n+1}} g(s)ds - \frac{h}{2}[g(t_n) - g(t_{n+1})] = -\frac{h^3}{12} g''(\xi_n) = O(h^3)$

for $\xi_n \in (t_n, t_{n+1})$ from trapezoidal rule of integration

$\Rightarrow \tau_{n+1} = O(h^2)$ □