7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \mathrm{Re}z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = -t^z e^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} z \cdot t^{z-1} dt$$

$$= \lim_{b\to\infty} \left(-b^z e^{-b}\right) + z\int_0^\infty e^{-t} t^{z-1} dt$$

Note that $|b^z e^{-b}| = e^{-b}|b^z| = e^{-b}|e^{z\ln b}| = e^{-b}|e^{(\alpha+\beta i)\ln b}|$

$$= e^{-b+\alpha\ln b}|e^{\beta \ln b\, i}| = e^{-b+\alpha\ln b}\cdot 1 = e^{-b}\cdot b^{\alpha} = e^{-b}\cdot b^{\mathrm{Re}\, z}$$

$$\therefore \lim_{b\to\infty}(-b^z e^{-b}) = \lim_{b\to\infty} e^{-b}\cdot b^{\mathrm{Re}\, z} = 0$$

$$\Rightarrow \Gamma(z+1) = z\Gamma(z) \quad \square$$

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1-\frac{\alpha}{2})f(x_n, u_n) + \frac{\alpha}{2}f(x_{n+1}, u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

$\lim_{h\to 0} \tau(h) = 0 \Rightarrow$ consistent

(pf) Let y be real solution

$$y_{n+1} = y_n + h\left[(1-\tfrac{\alpha}{2})\, y'(x_n) + \tfrac{\alpha}{2} y'(x_{n+1})\right] + h\,\tau_{n+1}(h)$$

$$= u_n + h\left[(1-\tfrac{\alpha}{2}) f(x_n, u_n) + \tfrac{\alpha}{2} f(x_{n+1}, u_{n+1})\right]$$

$$\Rightarrow \tau_{n+1}(h) = \frac{y_{n+1} - y_n}{h} - \left[(1-\tfrac{\alpha}{2})\, y'(x_n) + \tfrac{\alpha}{2} y'(x_{n+1})\right]$$

Expanding y at x_n

$$y_{n+1} = y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4)$$

$$\Rightarrow y'(x_{n+1}) = y'(x_n) + h y''(x_n) + \frac{h^2}{2} y'''(x_n) + O(h^3)$$

Therefore, $\tau_{n+1}(h) = \dfrac{h y'(x_n) + \frac{h^2}{2} y''(x_n) + O(h^3)}{h} - \left[y'(x_n) + \dfrac{\alpha h}{2} y''(x_n) + O(h^2)\right]$

$$= \frac{h(1-\alpha)}{2} y''(x_n) + O(h^2)$$

If $\alpha = 1$, $\tau_{n+1} = O(h^2)$
$\alpha \neq 1$, $\tau_{n+1} = O(h)$

Is there possible to have higher order of consistency? If we try to have order 3:

Consider $\tau_{n+1}(h) = \dfrac{h y'(x_n) + \frac{h^2}{2} y''(x_n) + \frac{h^3}{6} y'''(x_n) + O(h^4)}{h}$

$$- \left[y'(x_n) + \frac{\alpha h}{2} y''(x_n) + \frac{\alpha h^2}{4} y'''(x_n) + O(h^3) \right]$$

$$= \frac{h(1-\alpha)}{2} y''(x_n) + \frac{2-3\alpha}{12} h^2 y'''(x_n) + O(h^3)$$

τ_{n+1} has order 3 only if $1-\alpha=0$ and $2-3\alpha=0$. Therefore, we have proved that τ_{n+1} has order at most 3.

Now, take $\alpha=1$, i.e. $u_{n+1} = u_n + \frac{h}{2}[f(x_n, u_n) + f(x_{n+1}, u_{n+1})]$ and consider

$$\begin{cases} y' = -10y, & x>0 \\ y(0)=1 \end{cases}$$

Apply the method we have $u_{n+1} = u_n + \frac{h}{2}(-10u_n + (-10u_{n+1}))$

$$\Rightarrow (1+5h)u_{n+1} = (1-5h)u_n$$

$$\Rightarrow u_{n+1} = \frac{1-5h}{1+5h} u_n$$

Given that $u_0 = 1 \Rightarrow u_n = \left(\frac{1-5h}{1+5h}\right)^n$

$u_n \to 0$ if $\left|\frac{1-5h}{1+5h}\right| < 1 \Rightarrow -1 < \frac{1-5h}{1+5h} < 1 \Rightarrow -1-5h < 1-5h < 1+5h$

$$\Rightarrow h > 0$$

Hence, for the case $\alpha=1$, the method is absolutely stable for any $h>0$.