1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

$$-u''(x) = f(x),\ 0 < x < 1,$$

$$u(0) = u(1) = 0.$$

(pf) By (12.3), we have $u(x) = x\int_0^1 (1-s)f(s)\,ds - \int_0^x (x-s)f(s)\,ds$

for $f(x) = \frac{1}{x}$ $\quad u(x) = x\int_0^1 (1-s)\cdot\frac{1}{s}\,ds - \int_0^x (x-s)\cdot\frac{1}{s}\,ds$

$$= x\left(\ln s - s\Big|_0^1\right) - \left(x\ln s - s\Big|_0^x\right)$$

$$= x\left(-1 - \lim_{s\to 0}\ln s\right) - \left(x\ln x - x - \lim_{s\to 0} x\ln s\right)$$

$$= -x - x\ln x + x = -x\ln x$$

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).
[*Solution*: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \ldots, k-1$ and $l = k+1, \ldots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

$$G(x,s) = \begin{cases} s(1-x) \text{ if } 0 \le s \le x, \\ x(1-s) \text{ if } x \le s \le 1. \end{cases} \qquad (12.4)$$

We have defined $\boxed{L_h G^k = e^k}$

For any fixed $s = x_k$, $x \mapsto G(x,s)$ is linear on $[0,s]$ and $[s,1]$

$\Rightarrow (L_h G)_j = 0$ for $j \neq k$

For $j = k$: $(L_h G)_k = \dfrac{-G(x_{k-1}, s) + 2G(x_k, s) - G(x_{k+1}, s)}{h^2}$

$$= \frac{-x_{k-1}(1-s) + 2s(1-s) - s(1-x_{k+1})}{h^2}$$

$$= \frac{-(k-1)h(1-kh) + 2kh(1-kh) - kh(1-(k+1)h)}{h^2}$$

$$= \frac{1}{h}$$

$\Rightarrow (L_h G)_l = \begin{cases} \frac{1}{h}, & \text{if } l = k \\ 0, & \text{if } l \neq k \end{cases} \quad \Rightarrow (L_h G) = \frac{1}{h} e^k$

$$\Rightarrow \boxed{L_h(hG) = e^k}$$

$\Rightarrow hG = G^k$ $\quad (hG(x_j, x_k) = G^k(x_j))$ □