1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

9.2.2 The Trapezoidal Formula

This formula is obtained by replacing f with $\Pi_1 f$, its Lagrange interpolating polynomial of degree 1, relative to the nodes $x_0 = a$ and $x_1 = b$ (see Figure 9.2, left). The resulting quadrature, having nodes $x_0 = a$, $x_1 = b$ and weights $\alpha_0 = \alpha_1 = (b-a)/2$, is

$$I_1(f) = \frac{b-a}{2}\,[f(a) + f(b)]\,. \qquad (9.11)$$

If $f \in C^2([a,b])$, the quadrature error is given by

$$E_1(f) = -\frac{h^3}{12} f''(\xi),\ h = b-a, \qquad (9.12)$$

where ξ is a point within the integration interval.

9.2.1 The Midpoint or Rectangle Formula

This formula is obtained by replacing f over $[a,b]$ with the constant function equal to the value attained by f at the midpoint of $[a,b]$ (see Figure 9.1, left). This yields

$$I_0(f) = (b-a) f\left(\frac{a+b}{2}\right), \qquad (9.5)$$

with weight $\alpha_0 = b-a$ and node $x_0 = (a+b)/2$. If $f \in C^2([a,b])$, the quadrature error is

$$E_0(f) = \frac{h^3}{3} f''(\xi),\ h = \frac{b-a}{2}, \qquad (9.6)$$

where ξ lies within the interval (a,b).

(9.6) By Taylor expansion:

$$f(x) = f\left(\tfrac{a+b}{2}\right) + f'\left(\tfrac{a+b}{2}\right)\left(x - \tfrac{a+b}{2}\right) + \frac{f''(\xi)}{2}\left(x - \tfrac{a+b}{2}\right)^2$$

$$E_0(f) = \left|\int_a^b f(x)\,dx - (b-a) f\left(\tfrac{a+b}{2}\right)\right| = \left|\int_a^b f(x) - f\left(\tfrac{a+b}{2}\right) dx\right|$$

$$= \left|\int_a^b \cancelto{0}{f'\left(\tfrac{a+b}{2}\right)\left(x - \tfrac{a+b}{2}\right)} + \frac{f''(\xi)}{2}\left(x - \tfrac{a+b}{2}\right)^2 dx\right| \le \frac{\|f''\|_\infty}{2}\int_a^b \left(x - \tfrac{a+b}{2}\right)^2 dx$$

$$= \frac{\|f''\|_\infty}{2}\cdot\frac{1}{12}(b-a)^3 = \frac{h^3}{24}\|f''\|_\infty$$

Then $|E_1(f)| = \dfrac{(b-a)^3}{12}\|f''\|_\infty = 2|E_0(f)|$ #

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae:
(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,
(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.
Which is the order of infinitesimal p for (a) and (b)?
[*Solution*: $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

Besides its degree of exactness, a quadrature formula can also be qualified by its *order of infinitesimal* with respect to the integration stepsize h, which is defined as the maximum integer p such that $|I(f) - I_n(f)| = \mathcal{O}(h^p)$. Regarding this, the following result holds

(a) $f(x)=1$: $\int_{-1}^{1} 1\,dx = 2$; $I_2(1) = \frac{2}{3}(2-1+2) = 2$ ✓

$f(x)=x$: $\int_{-1}^{1} x\,dx = 0$; $I_2(x) = \frac{2}{3}(-1-0+1) = 0$ ✓

$f(x)=x^2$: $\int_{-1}^{1} x^2\,dx = \frac{2}{3}$; $I_2(x^2) = \frac{2}{3}\left(\frac{1}{2} - 0 + \frac{1}{2}\right) = \frac{2}{3}$ ✓

$f(x)=x^3$: $\int_{-1}^{1} x^3\,dx = 0$; $I_2(x^3) = \frac{2}{3}\left(-\frac{1}{4} - 0 + \frac{1}{4}\right) = 0$ ✓

$f(x)=x^4$: $\int_{-1}^{1} x^4\,dx = \frac{2}{5}$; $I_2(x^4) = \frac{2}{3}\left(\frac{1}{8} - 0 + \frac{1}{8}\right) = \frac{1}{6}$ (×) $\Rightarrow$ degree of exactness $= 3$

Then the order of infinitesimal $= 3+2 = 5$

(b) $f(x)=1$: $I_4(1) = 2$ ✓ / $f(x)=x$: $I_4(x) = 0$ ✓ / $f(x)=x^2$: $I_4(x^2) = \frac{2}{3}$ ✓

$f(x)=x^3$: $I_4(x^3) = 0$ ✓ / $f(x)=x^4$: $I_4(x^4) = \frac{14}{27}$ (×)

$\Rightarrow$ degree of exactness $= 3$, order of infinitesimal $= 5$

5. Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find a and x_1 in such a way that Q has maximum degree of exactness r.
[*Solution*: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

For $r=0$, wanted: $\int_0^1 \sqrt{x}\cdot 1\, dx = Q(1) = a \Rightarrow a = 2/3$

$r=1$, wanted: $\int_0^1 \sqrt{x}\cdot x\, dx = Q(x) = a x_1 \Rightarrow \frac{2}{5} = a x_1$ then $(a, x_1) = (\frac{2}{3}, \frac{3}{5})$ ✓

$r=2$, wanted: $\int_0^1 \sqrt{x}\cdot x^2 dx = Q(x^2) = a x_1^2 \Rightarrow \frac{2}{7} \neq a x_1^2$ when $(a, x_1) = (\frac{2}{3}, \frac{3}{5})$

$\Rightarrow$ maximum degree of exactness $= 1$, $a = 2/3$, $x_1 = 3/5$

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1,2,3$ in such a way that Q has degree of exactness $r = 2$.
[*Solution*: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

if $r=2 \Rightarrow I(f) - Q(f) = \int_0^1 f(x)dx - (\alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)) = 0$ ———— $(*)$

for every $f(x)$ of the form ax^2+bx+c, that is, for polynomials $f(x) = 1$ or x or x^2, $(*)$ holds.

$\Rightarrow f(x)=1 : \int_0^1 1\, dx = 1 = \alpha_1 + \alpha_2$

$f(x) = x : \int_0^1 x\, dx = \frac{1}{2} = \alpha_2 + \alpha_3$

$f(x) = x^2 : \int_0^1 x^2 dx = \frac{1}{3} = \alpha_2$

$\Rightarrow \alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{6}$ #