Introduction to Scientific Computing

Assignment 2

許晉　314653002

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Question 1 (Exercise 8.5)

Prove that

$$(n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \leq |\omega_{n+1}(x)| \leq n!h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = \dfrac{2}{n}$.

Solution

Considering

$$\begin{aligned}
\frac{|\omega_{n+1}(x)|}{n!h^{n-1}} &= \frac{1}{n!} \cdot \frac{|x-x_0|}{h} \times \frac{|x-x_1|}{h} \times \cdots \times \frac{|x-x_{n-2}|}{h} |(x-x_{n-1})(x-x_n)| \\
&= \frac{|x-x_0|}{nh} \times \frac{|x-x_1|}{(n-1)h} \times \cdots \times \frac{|x-x_{n-2}|}{2h} |(x-x_{n-1})(x-x_n)| \\
&\overset{(*)}{\leq} |(x-x_{n-1})(x-x_n)|
\end{aligned}$$

(*) Since $|x-x_j| \leq (n-j)h$ for $j = 0, 1, \cdots, n-1$

Similarly, we have

$$\begin{aligned}
\frac{|\omega_{n+1}(x)|}{(n-1)!h^{n-1}} &= \frac{1}{(n-1)!} \cdot \frac{|x-x_0|}{h} \times \frac{|x-x_1|}{h} \times \cdots \times \frac{|x-x_{n-2}|}{h} |(x-x_{n-1})(x-x_n)| \\
&= \frac{|x-x_0|}{(n-1)h} \times \frac{|x-x_1|}{(n-2)h} \times \cdots \times \frac{|x-x_{n-2}|}{2h} |(x-x_{n-1})(x-x_n)| \\
&\overset{(**)}{\geq} |(x-x_{n-1})(x-x_n)|
\end{aligned}$$

(**) Since $|x - x_j| \geq (n - j - 1)h$ for $j = 0, 1, \cdots, n-1$ By above, we have

$$(n-1)!h^{n-1}|(x - x_{n-1})(x - x_n)| \leq |\omega_{n+1}(x)| \leq n!h^{n-1}|(x - x_{n-1})(x - x_n)|.$$

$\square$

Question 2 (Exercise 8.6)

Under the assumption of previous question, show that $|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$.

Solution

First, note that $|\omega_{n+1}(x)|$ is symmetric about y-axis, so it is an even function. That is, we only need to consider the interval $x \in (0, x_n)$.

Claim: $\left|\dfrac{\omega_{n+1}(x+h)}{\omega_{n+1}(x)}\right| > 1$ for $x \in (0, x_{n-1})$, then $|\omega_{n+1}(x)|$ is increasing strictly hence $|\omega_{n+1}|$ reaches its maximum for x in the last interval i.e. $x \in (x_{n-1}, x_n)$.

Let $N = \dfrac{1}{h} = \dfrac{n}{2}$, then we can rewrite $\omega_{n+1}(x)$ as

$$(x + Nh)(x + (N-1)h) \cdots (x + h)(x)(x - h) \cdots (x - (N-1)h)(x - Nh).$$

It follows that

$$\begin{aligned}
\left|\frac{\omega_{n+1}(x+h)}{\omega_{n+1}(x)}\right| &= \left|\frac{(x + h + Nh)(x + h + (N-1)h) \cdots (x+h)x \cdots (x + h - (N-1)h)(x + h - Nh)}{(x + Nh)(x + (N-1)h) \cdots (x + h)(x)(x - h) \cdots (x - (N-1)h)(x - Nh)}\right| \\
&= \left|\frac{x + (N+1)h}{x - Nh}\right|
\end{aligned}$$

Since $x \in (0, x_{n-1})$, then $x + (N+1)h > Nh - x \Rightarrow \left|\dfrac{x + (N+1)h}{x - Nh}\right| > \left|\dfrac{Nh - x}{x - Nh}\right| = 1.$

Thus, we prove the claim. $\square$

Question 3 (Exercise 8.8)

Determine an interpolation polynomial $Hf \in \mathbb{P}_n$ such that

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, \cdots, n,$$

and check that

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j,$$

that is, the Hermite interpolating polynomial on one node coincides the *Taylor Polynomial*.

Solution

By definition, the Hermite polynomial has the form

$$H_{N-1}(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i} y_i^{(k)} L_{ik}(x),$$

where $y_i^{(k)} = f^{(k)}(x_i), i = 0, \cdots, n, k = 0, \cdots, m$

Now we only have on node, so the form becomes

$$Hf(x) = \sum_{k=0}^{n} y_0^{(k)} L_{0k}(x),$$

where $L_{0k}(x) = l_{0k}(x) = \dfrac{(x - x_0)^k}{k!}$

By combining above, we have

$$Hf(x) = \sum_{k=0}^{n} y_0^{(k)} \frac{(x - x_0)^k}{k!} = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$$

which is the desired result. $\square$