Solution

Recall that Chebyshev points of the second kind is $x_i = \cos\left(\dfrac{i}{n}\pi\right), \quad i = 0, \cdots, n$

By previous assignment, we have the relation

$$w_i = \frac{1}{\omega'_{n+1}(x_i)}$$

Here, $\omega_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$

First, we calculate $w_0 = \dfrac{1}{\omega'_{n+1}(x_0)}$.

$$\begin{aligned}
\omega'_{n+1}(x_0) &= \prod_{j=1}^{n}(x_0 - x_j) = \prod_{j=1}^{n}\left(1 - \cos\left(\frac{j\pi}{n}\right)\right) \\
&\overset{(1)}{=} \prod_{j=1}^{n}\left(2\sin^2\left(\frac{j\pi}{2n}\right)\right) \\
&= 2^n\left(\prod_{j=1}^{n}\sin\left(\frac{j\pi}{2n}\right)\right)^2 \\
&\overset{(2)}{=} 2^n \cdot \frac{2n}{2^{2n-1}} = n \cdot 2^{2-n}
\end{aligned}$$

(1) $1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}, \quad 1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$

(2) Let $\alpha = e^{\frac{2\pi i}{m}}$ be the root of the equation $z^m = 1$. Thus, $\alpha^2, \alpha^3, \cdots, \alpha^{m-1}$ are also the roots of $z^m = 1$

By factorizing we have

$$z^m - 1 = (z-\alpha)(z-\alpha^2)\cdots(z-\alpha^{m-1})(z-1) \Rightarrow \prod_{j=1}^{m-1}(z-\alpha^j) = \frac{z^m - 1}{z - 1} = 1 + z + z^2 + \cdots + z^{m-1}.$$

If $z = 1$, we have the relation

$$\prod_{j=1}^{m-1}(1 - \alpha^j) = m$$

Note that for every θ, we have $|1 - e^{i\theta}|^2 = |1 - (\cos\theta + i\sin\theta)|^2 = (1 - \cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta = 4\sin^2\dfrac{\theta}{2}$, following that $|1 - e^{i\theta}| = 2\left|\sin\dfrac{\theta}{2}\right|$.

Let $\theta = \dfrac{2j\pi}{m}$, then $|1 - e^{\frac{2j\pi}{m}\cdot i}| = 2\left|\sin\left(\dfrac{j\pi}{m}\right)\right| \Rightarrow |1 - \alpha^j| = 2\left|\sin\left(\dfrac{j\pi}{m}\right)\right| = 2\sin\left(\dfrac{j\pi}{m}\right)$ for $j = 1, 2, \cdots, m$

From the relation above, we have

$$\begin{aligned}|m| &= \prod_{j=1}^{m-1} |1-\alpha^j| \\ &= \prod_{j=1}^{m-1} 2\sin\left(\frac{j\pi}{m}\right) \\ &= 2^{m-1}\prod_{j=1}^{m-1} \sin\left(\frac{j\pi}{m}\right)\end{aligned}$$

Then we have the identity

$$\prod_{j=1}^{m-1} \sin\left(\frac{j\pi}{m}\right) = \frac{m}{2^{m-1}}$$

Let $m = 2n \Rightarrow \dfrac{2n}{2^{2n-1}} = \displaystyle\prod_{j=1}^{2n-1} \sin\left(\frac{j\pi}{2n}\right)$

Moreover, notice that when $j > n$, $\sin\left(\dfrac{j\pi}{2n}\right) = \sin\left(\pi - \dfrac{j\pi}{2n}\right) = \sin\left(\dfrac{(2n-j)\pi}{2n}\right) \Rightarrow$ $\displaystyle\prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{2n}\right) = \prod_{j=n+1}^{2n-1} \sin\left(\frac{j\pi}{2n}\right)$

Therefore,

$$\begin{aligned}\prod_{j=1}^{2n-1} \sin\left(\frac{j\pi}{2n}\right) &= \prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{2n}\right) \cdot \sin\left(\frac{n\pi}{2n}\right) \cdot \prod_{j=n+1}^{2n-1} \sin\left(\frac{j\pi}{2n}\right) \\ &= \left[\prod_{j=1}^{n-1} \sin\left(\frac{j\pi}{2n}\right)\right]^2 = \left[\prod_{j=1}^{n} \sin\left(\frac{j\pi}{2n}\right)\right]^2 = \frac{2n}{2^{2n-1}}\end{aligned}$$

Similarly, we can calculate $w_n = \dfrac{1}{\omega'_{n+1}(x_n)}$

$$\begin{aligned}
\omega'_{n+1}(x_n) &= \prod_{j=0}^{n-1}(x_n - x_j) = \prod_{j=0}^{n-1}\left(-1-\cos\left(\frac{j\pi}{n}\right)\right)\\
&\overset{(1)}{=} (-1)^n \prod_{j=0}^{n-1}\left(2\cos^2\left(\frac{j\pi}{2n}\right)\right)\\
&= (-1)^n \cdot 2^n \left(\prod_{j=0}^{n-1}\cos\left(\frac{j\pi}{2n}\right)\right)^2\\
&= (-1)^n \cdot 2^n \left(\prod_{j=0}^{n-1}\sin\left(\frac{(n-j)\pi}{2n}\right)\right)^2\\
&\overset{(j'=n-j)}{=} (-1)^n \cdot 2^n \left(\prod_{j'=1}^{n}\sin\left(\frac{(j')\pi}{2n}\right)\right)^2\\
&\overset{(2)}{=} (-1)^n \cdot 2^n \cdot \frac{2n}{2^{2n-1}} = n\cdot(-1)^n\cdot 2^{2-n}
\end{aligned}$$

Next, consider $1 \le k \le n-1$, $w_k = \dfrac{1}{\omega'_{n+1}(x_k)}$.

Note that Chebyshev polynomials of second kind $U_n(x)$ satisfy

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad n = 0, 1, 2, \cdots$$

For the nodes x_k, we have $U_{n-1}(x_k) = U_{n-1}\left(\cos\dfrac{k\pi}{n}\right) = \dfrac{\sin\left(n\cdot\dfrac{k\pi}{n}\right)}{\sin\dfrac{k\pi}{n}} = 0$

Then we can say that $x_1, x_2, \cdots, x_{n-1}$ are $(n-1)$ roots of $U_{n-1}(x)$. If we write $U_{n-1}(x)$ as $\prod_{k=1}^{n-1}(x-x_k)$, then $\omega_{n+1}(x) = \alpha(x^2-1)U_{n-1}(x)$, where $\alpha \in \mathbb{R}$.

Recall that the leading coefficient of $U_{n-1}(x)$ is 2^{n-1} and $\omega_{n+1}(x)$ is monic. It implies that $\alpha = 2^{1-n} \Rightarrow \omega_{n+1}(x) = 2^{1-n}(x^2-1)U_{n-1}(x)$.

Next, do the derivative then we have

$$\omega'_{n+1}(x) = 2^{1-n}\left(2xU_{n-1}(x) + (x^2-1)U'_{n-1}(x)\right)$$

$$\omega'_{n+1}(x_k) = 2^{1-n}\underbrace{(x_k^2-1)}_{(3)}\cdot\underbrace{U'_{n-1}(x_k)}_{(4)} = (-1)^n\cdot n\cdot 2^{1-n}$$

(3) $x_k^2 - 1 = \cos^2\left(\frac{k\pi}{n}\right) - 1 = -\sin^2\left(\frac{k\pi}{n}\right)$

(4) Recall that Chebyshev polynomials of second kind is $U_{n-1}(x) = \dfrac{\sin\left(n\cos^{-1}(x)\right)}{\sin\left(\cos^{-1}(x)\right)}$.

Do the derivative we will get

$$U'_{n-1}(x) = \frac{\cos(n\cos^{-1}(x)) \cdot \left(n \cdot \frac{-1}{\sqrt{1-x^2}}\right) \cdot \sin(\cos^{-1}(x)) + x \cdot \frac{1}{\sqrt{1-x^2}} \cdot \sin(n\cos^{-1}(x))}{\left[\sin\left(\cos^{-1}(x)\right)\right]^2}$$

$$U'_{n-1}(x_k) = U'_{n-1}\left(\cos\frac{k\pi}{n}\right) = \frac{(-1)^{k+1} \cdot n}{\left(\sin\frac{k\pi}{n}\right)^2}$$

Combining the result above, we have

$$\omega'_{n+1}(x_k) = \begin{cases} n \cdot 2^{2-n} & , k = 0 \\ (-1)^k \cdot n \cdot 2^{1-n} & , 1 \le k \le n-1 \\ (-1)^n \cdot n \cdot 2^{2-n} & , k = n \end{cases}$$

Then the Barycentric weight is

$$w_k = \frac{1}{\omega'_{n+1}(x_k)} = \begin{cases} \frac{2^{n-2}}{n} & , k = 0 \\ (-1)^k \cdot \frac{2^{n-1}}{n} & , 1 \le k \le n-1 \\ (-1)^n \cdot \frac{2^{n-2}}{n} & , k = n \end{cases}$$

By rescaling the weight w_k $(1 \le k \le n-1)$ to $(-1)^k$, then we have the Barycentric weight after rescaling is

$$w_k = \frac{1}{\omega'_{n+1}(x_k)} = \begin{cases} \frac{1}{2} & , k = 0 \\ (-1)^k & , 1 \le k \le n-1 \\ \frac{(-1)^n}{2} & , k = n \end{cases},$$

which is the desired result. $\square$