Confinement in SU(2) gauge theory on a lattice

A numerical study

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In this very short review, we present (i) the Metropolis algorithm which is a method of approximately solving path integrals and (ii) the Wilson formalism, a formulation of (in our case) an SU(2) gauge theory on a lattice. By applying the former to the latter, we study the distance dependence of the static potential of a quark antiquark pair. Confinement is demonstrated with this.

1 Introduction

Confinement is a very interesting feature of non-abelian gauge theories: E. g. in Quantum Chromodynamics (an SU(3) gauge theory), there cannot be any free quarks, they only appear in bound states. By studying the potential of a static quark antiquark pair, one may try to find an indication of this: If the potential displays a *linear* dependence on the distance between quark and antiquark, this indicates confinement.

Unfortunately, QCDs running coupling with large values of the coupling for low momenta makes it impossible to study this problem pertubatively. Therefore, one has to resort to non-analytical methods. In our case, the lattice formulation of a non-abelian gauge theory (SU(2), for simplicity, not QCD) was chosen.

This is a short summary of a longer thesis on the topic [1], it is organised as follows: In section 2, the numerical algorithm used to attack the problem is presented. Then, in section 3, we present the formulation of the gauge theory that is suited for simulations. Finally, in section 4, we present the results of the numerical studies and, in sec. 5, we discuss possible further topics of study.

2 Markov

Markov Chain Monte Carlo (MCMC) algorithms allow to numerically approximate path integrals such as [2]

$$\langle O \rangle = \frac{1}{Z} \int \mathscr{D}[x] O[x] e^{-S[x]},\ Z = \int \mathscr{D}[x] e^{-S[x]}.$$

The interpretation is thus: x represents a single trajectory, the measure $\mathscr{D}[x]$ means that all possible trajectories are integrated over. O is an observable whose expectation value one is interested in. It is calculated by evaluating it at the different x where the trajectories are weighted by $\frac{1}{Z} e^{-S[x]}$. S is the action and the path integral is evaluated at $\hbar = 1$ and Euclidian time. Thus just the minus sign in front of it and not the imaginary unit. [1] Here, Z accounts for the normalisation of the integral. It is suggestively named: As one may see, MCMC algorithms may be used equally to numerically compute canonical partition sums from statistical mechanics.

The basic idea of these algorithm is to draw many samples $\sim \frac{1}{Z} e^{-S[x]}$ and then to approximate the expectation value $\langle O \rangle$ by the arithmetic mean $\overline{O}$ of the observable evaluated at the different samples. For MCMC, this is realised by the generation of multiple generations of configurations $\{x\}$ where $W(x, x')$ describes the probability of passing from configuration x to x'. When applied repeatedly, the samples will eventually follow the right distribution if, and only if, the *detailed balance* condition is met: [2]

$$\frac{W(x, x')}{W(x', x)} = \frac{e^{-S[x']}}{e^{-S[x]}} =: e^{-\Delta S[x', x]}.$$

A very simple algorithm that meets this criterion is that of *Metropolis* which works as follows: Given a configuration x, a new configuration x' is generated at random. This new configuration will be accepted with probability [2]

$$W(x, x') = \begin{cases} 1, & \text{if } \Delta S < 0, \\ e^{-\Delta S}, & \text{else.} \end{cases} \tag{1}$$

3 Wilson

Having established a tool to calculate path integrals given the classical action, we now turn to the question of finding a formulation of an action for an SU(2) gauge theory that is appropriate for simulations. In the continuous case, the action describing the gauge field $A(x)$ is the one of Yang and Mills: [3]

$$S_{\mathrm{YM}}[A] = \frac{1}{2g^2} \int \mathrm{d}^4 x\ \operatorname{tr}[F_{\mu\nu}(x) F^{\mu\nu}(x)],\ F_{\mu\nu} = \frac{1}{i}[D_\mu, D_\nu]\ = \partial_\mu A_\nu - \partial_\nu A + i[A_\mu, A_\nu],$$

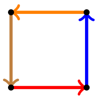
where the trace is taking over the colour indices.

[1] This has the positive side effect that the numerical calculations become much easier given that one has to deal with exponentially supressed factors insted of oscillating phases.

Due to finite memory and computing time, one can only simulate a discretised version of the gauge field. Instead of (Euclidian) space time $\mathbb{R}^4$, a *lattice* Λ of discrete points is studied. [2] On the lattice, the gauge field "lives" on the links between the points. [3] The variables are named $U_\mu(x) \in \mathrm{SU}(2)$ [4] where μ denotes the direction towards which the link points starting at the point x. With these, one may formulate the *Wilson* action [4]

$$S_\mathrm{W} = \frac{\beta}{2} \sum_{x\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\ \mathrm{tr}\ [\mathbb{1} - U_{\mu\nu}(x)],\ \beta = \frac{4}{g^2}. \tag{2}$$

The *plaquette* describes a square of the gauge links:

$$U_{\mu\nu}(x) =$$

$$= U_\mu(x) U_\nu(x + a_\mu) U_\mu(x + a_\nu)^\dagger U_\nu(x)^\dagger.$$

Here, $\hat{\mu}$ is a basis vector in direction μ and a is the lattice spacing. (So, $U_\mu(x)$ points from lattice site x in direction μ, $U_\nu(x + a\hat{\mu})$ points from there in direction ν etc. as illustrated above.) One may demonstrate that, in the continuum limit, the Wilson action approaches the one of Yang and Mills, $S_\mathrm{W} \xrightarrow{a\to 0} S_\mathrm{YM}$. [3] Opposed to a naive discretisation, the Wilsonian approach has the advantage of retaining gauge invariance. A nice way of looking at the two actions is as follows: The fact that the field strength tensor may be calculated as the commutator of the covariant derivatives signifies that the Yang Mills action measures *curvature*. The Wilson action does the equivalent on the lattice by summing all those squares of link variables. Indeed, with the definition of the link variables as the path ordered integral of the gauge field, one may recover the limit mentioned above.

As an observable of this system, we consider the perhaps most simple possible composition of the lattice variables: rectangles. These are generalisations of the plaquettes $U_{\mu\nu}$ and could be written like $W(r,t) = \mathrm{tr}\ \prod_{x\in\mathcal{C}} U(x)$. Here, r denotes the extend of the rectangle $\mathcal{C}$ in time direction and r the extend in one of the spacial directions of the lattice. One may show that the expectation value of these *Wilson loops* approximately scale like [5]

$$\langle W(r,t)\rangle \sim e^{-tV(r)} \tag{3}$$

for large values of t. Here, $V(r)$ is the potential of a static quark antiquark pair. Thus, by studying the system described above, one could indeed draw conclusions about how such a quark antiquark potential V scales with the distance r.

4 Confinement

The simulation carried out was in principle quite simple: A lattice is set up, where on each of the lattice links an SU(2) matrix is saved. Using the Wilson action (2), the Metropolis

[2] Technically, the lattice is actually a *torus* with periodic boundary conditions.

[3] This is contrary to e. g. a bosonic field which will "live" on the lattice points.

[4] The links are elements are elements of the gauge group contrary to the gauge field, which are elements of the algebra.

algorithm (1) is employed to generate lattice configurations. The action is measured after each iteration. Once its value stabilises, the measurements can start. Now, Wilson loops of different sizes are measured with breaks of several iterations (several to reduce correlation between the measurements).

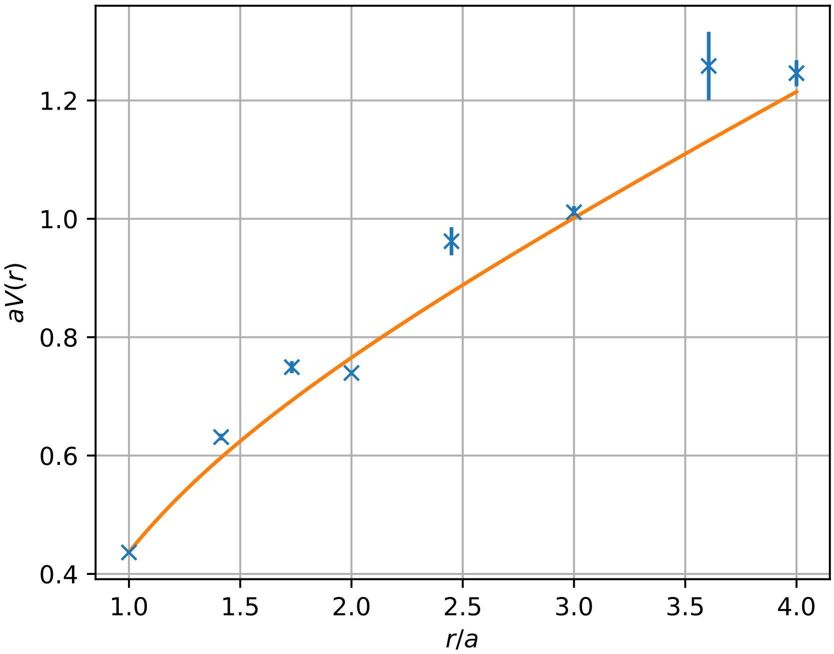


Figure 1: The static potential dep. on the distance r between quark and antiquark (measured in units of lattice spacing a). The non-integer values of r stem from measurements of non planar Wilson loops. A linear dependence is clearly visible.

At fixed r, several measurements of $\overline{W(r,t)}$ allow to extract the value of $V(r)$ by using (3). The results are plotted in figure 1. The most prominent feature of this graph is the fact that there is a linear component of the potential. This indicates confinement: In order to pull apart a quark antiquark pair, one needs to invest more and more as the distance between the two grows. This is a very prominent feature of non-abelian gauge theories. If we had studied an abelian gauge theory, the behaviour of the potential would be purely Coulomb like, without the linear component.

5 Conclusion

We have succesfully demonstrated the appearance of confinement in an SU(2) lattice gauge theory. There are several possible further points of interest, both on the technical and physical side: One might want to improve the estimation of the uncertainty of the estimator of the expectation values. Furthermore, one could study a larger lattice to get more data points and more samples to reduce the uncertainty.

More interestingly, one might want to study SU(3) which bears actual physical relevance for it is realised in nature. However, this would be much more complicated than the SU(2) case since one would actually need to exponentiate the generators to get elements of the group. (For SU(2), this could be circumvented by exploiting that SU(2) $\cong S^3$.) On the other hand, it would be feasible to study U(1), an *abelian* gauge theory. Here, one would expect no confinement.

Finally, it should again be noted that this is but a brief summary of a short thesis on this topic. For more details concerning the Metropolis algorithm, we refer to [2] and for the Wilson formalism [3] and [5] may be recommended.

References

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