II. Bilateral Z Transform

Digital signal processing has a time-domain component and a frequency domain component. In the frequency domain we can use several different transforms, each having its own applications.

The bilateral *z* transform of a sequence *x* is defined as follows:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The frequency variable *z* is complex and therefore the transform is complex. The set of *z* values for which the transform has a finite magnitude is called the region of convergence (ROC). Since

$$|X(z)| \leq \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n}$$

only the magnitude of *z* is involved and thus the ROC is a disk centered at the origin. The *z* transforms of interest to us will be rational functions:

$$X(z) = \frac{P(z)}{R(z)}$$

where *P* and *R* are polynomials with no common factors. The roots of *P* are called zeros and the roots of *R* are called poles. Clearly, no pole can be in the ROC. If there are *N* poles with *M* distinct magnitudes, there are $M + 1$ possible ROCs for $X(z)$, and $M + 1$ distinct sequences $x(n)$ which will have the same $X(z)$.

As an example, consider the right-sided sequence $x(n) = a^n u(n)$ for which

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \quad \text{iff} \quad |z| > |a|$$

The transform has a pole at $z = a$ and the ROC is the region outside the circle of radius $|a|$. Now let $x(n) = -a^n u(-n-1)$ which is left-sided. Then

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} \quad \text{iff } |z| < |a|$$

The transform has a pole at $z = a$ and the ROC is the region inside the circle of radius $|a|$. Both sequences produce the same $X(z)$; only the ROC is different. Since $X(z)$ has one distinct pole magnitude, we would expect that there are two possible ROCs. It is important to realize that given a rational $X(z)$, we cannot determine the inverse transform unless we know the ROC. If there were two poles with distinct magnitudes, say $z = 2$ and $z = -1/3$, there would be three possible regions of convergence: $|z| < 1/3,\ 1/3 < |z| < 2,\ |z| > 2$, corresponding to left-sided, two-sided, and right-sided $x(n)$.

When $X(z)$ is rational, we can find the inverse transform using a partial fraction expansion. In the case where the numerator degree is less than the denominator degree, and if there are N distinct poles d_k, the transform can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

where

$$A_k = (1-d_k z^{-1})\, X(z)\big|_{z=d_k}$$

For example, let

$$X(z) = \frac{1}{(1-.25z^{-1})(1-.5z^{-1})}$$

with ROC: $|z| > .5$. Then

$$X(z) = \frac{2}{1-.5z^{-1}} - \frac{1}{1-.25z^{-1}}$$

and $x(n) = 2(.5)^n u(n) - (.25)^n u(n)$. Note that the other two possible ROCs would lead to different inverse transforms. Repeated poles can be handled with some additional algebra and the use of readily available transform tables.

The z transform has many useful properties which make it possible to perform certain time-domain operations in the frequency domain. For example, if $x(n) \leftrightarrow X(z)$ and $y(n) = x(n-n_0)$, then $Y(z) = z^{-n_0} X(z)$. If $y(n) = z_0^n x(n)$, then $Y(z) = X(z/z_0)$. If $y(n) = x_1(n) * x_2(n)$, then $Y(z) = X_1(z)X_2(z)$. Note that the convolution, which is generally an infinite sum, will not produce a finite result unless the ROCs of X_1 and X_2 have a nontrivial intersection.

The relationship between a sequence and its z transform is very simple when the sequence has finite length. In that case, the transform is just a polynomial whose coefficients are the elements of the sequence. Consider $x(n) = \delta(n)$. The transform is $X(z) = 1$. Now let $x(n) = \delta(n+2) - .5\delta(n+1) - \delta(n) + .5\delta(n-1)$. Using the shift property, $X(z) = z^2 - .5z - 1 + .5z^{-1}$.

The convolution property allows us to apply the z transform to the analysis of LTI systems. If the impulse response $h(n)$ has z transform $H(z)$, the input-output equation $y = x*h$ is equivalent to $Y(z) = X(z)H(z)$. $H(z)$ is called the <u>transfer function</u> of the LTI system. The tests for causality and stability can be applied to the transfer function. The system is causal iff the ROC of $H(z)$ is outside the circle whose radius is the largest pole magnitude. The system is stable iff the ROC of $H(z)$ includes the unit circle. To have both causality and stability, it is necessary (but not sufficient) that all the poles of the transfer function lie inside the unit circle.

Most mathematical models for LTI systems are difference equations, the general form of which is

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Taking z transforms of both sides and using the shift property, we obtain the rational transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

As an example, consider:

$$y(n) - 2.5y(n-1) + y(n-2) = x(n)$$

The transfer function is

$$H(z) = \frac{1}{1 - 2.5z^{-1} + z^{-2}}$$

The poles are located at $z = .5$ and $z = 2$. The system is causal iff the ROC is the region: $|z| > 2$, and it is stable iff the ROC is the region: $.5 < |z| < 2$. Since there is a pole outside the unit circle, the system cannot have both properties.

The concept of inverse systems was introduced in Chapter I. Two systems with impulse responses h_1 and h_2 are inverses of each other if $h_1 * h_2 = \delta$. The equivalent frequency domain criterion is $H_1H_2 = 1$. The poles of H_1 are the zeros of H_2. Of course, the ROCs must overlap; otherwise, the convolution is not finite. As an example, consider the system modeled by the difference equation:

$$y(n) - 0.9y(n-1) = x(n) - 0.5x(n-1)$$

If we assume that the system is causal, the transfer function is

$$H_1(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}}, \quad |z| > 0.9$$

so the system is also stable. Let

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1-0.9z^{-1}}{1-0.5z^{-1}}$$

Of the two possible ROCs that we could choose for H_2, only the region $|z| > 0.5$ overlaps the ROC of H_1. With that choice the two systems are inverses of each other. The difference equation model for the second system is

$$y(n) - 0.5y(n-1) = x(n) - 0.9x(n-1)$$

and it is also causal and stable. In general, a causal and stable LTI system will have a causal and stable inverse iff all its poles and zeros lie inside the unit circle.

Homework 2 (due 19 Sept.)

1. (a) Show that the LTI system modeled by the fourth-order difference equation

$$y(n) - y(n-4) = x(n) - x(n-1) + x(n-2) - x(n-3)$$

can be modeled by a first-order difference equation, where x and y are the input and output, respectively. (b) Locate the poles and determine whether the system is stable or causal.

2. A LTI system with a <u>two-sided</u> impulse response has the difference equation model

$$y(n-2) - 4y(n-1) + (\lambda + 6)y(n) - 4y(n+1) + y(n+2) = \lambda x(n)$$

where x and y are the input and output, respectively, and $\lambda > 0$. (a) Prove that this system is stable. (b) Does this system have a stable inverse?

3. A causal system is modeled by

$$y(n) - 0.25y(n-2) = 2x(n) + 0.5x(n-2)$$

where x and y are the input and output, respectively. (a) Determine the difference equation model for the inverse system. (b) Let x = [1 2 3 2 1] and use Matlab's **filter** function to verify that the inverse system works the way it should.

4. When the input to a certain LTI system is $x(n) = \delta(n) - \delta(n-1) + \delta(n-2)$, the output is $y(n) = \delta(n) + \delta(n-1) - 2\delta(n-2) + 3\delta(n-3) - \delta(n-4)$. Determine the output of the same system when the input is $x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$. Use your **mconv** function and Matlab's **deconv** function.

5. The input and output of a certain LTI system are, respectively,

$$x(n) = \delta(n) + 2\delta(n-1) - \delta(n-2)$$
$$y(n) = \delta(n-1) + 2\delta(n-2) - 2\delta(n-3) - 2\delta(n-4) + \delta(n-5)$$

If this system is a cascade (series) interconnection of a backward difference system and another LTI system, determine the impulse response of the other system. Use Matlab's **deconv** function.

6. Consider the boundary value difference equation

$$y(n+1) - 10.1y(n) + y(n-1) = -1, \quad y(0) = 0, \quad y(101) = 0$$

If y is the column vector containing $y(1)$, $y(2)$, … , $y(100)$, this boundary value problem can be reformulated as a matrix equation $Ay = g$ where A is a positive definite matrix. (a) Determine A and g. (b) Use **linsolve** with the two appropriate options to create a stem plot of y.