III. Unilateral Z Transform

Difference equation models for LTI systems often include specific initial conditions. To analyze such models in the frequency domain, we need a transform which can account for initial conditions. Unfortunately, the bilateral z transform cannot. As an alternative, consider the unilateral z transform, which is defined as follows:

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The properties of the unilateral transform are those of the bilateral transform, except the shift property must be revised. If $x \leftrightarrow X^+$ and $y(n) = x(n\text{-}n_0)$, then

$$Y^+(z) = z^{-n_0}\left(X^+(z) + \sum_{n=1}^{n_0} x(-n) z^{n-1}\right), \quad n_0 \ge 1$$

For example,

$$Y^+(z) = z^{-1}X^+(z) + x(-1), \quad n_0 = 1$$
$$Y^+(z) = z^{-2}X^+(z) + x(-1)z^{-1} + x(-2), \quad n_0 = 2$$

The unilateral transform is useful in bank loan calculations. Suppose you borrow an amount C at an annual interest rate R to be repaid monthly in the amount P over a period of T years. Assume that interest accrues monthly. Let $y(n\text{-}1)$ be the loan balance after the nth payment. Then y satisfies the difference equation:

$$y(n) = (1+\alpha)y(n-1) - Pu(n), \quad 0 \le n \le 12T - 1$$
$$y(-1) = C$$

where $\alpha = R/1200$. The sequence $y(n)$ is called the loan amortization table. If we transform both sides:

$$Y^{+}(z) = (1+\alpha)\left(z^{-1}Y^{+}(z) + C\right) - \frac{P}{1-z^{-1}}$$

Solving for Y^{+}:

$$\begin{aligned} Y^{+}(z) &= \frac{(1+\alpha)C}{1-(1+\alpha)z^{-1}} - \frac{P}{(1-z^{-1})(1-(1+\alpha)z^{-1})} \\ &= (1+\alpha)\frac{(C - P/\alpha)}{1-(1+\alpha)z^{-1}} + \frac{P/\alpha}{1-z^{-1}} \end{aligned}$$

After inverse transforming:

$$y(n) = (C - P/\alpha)(1+\alpha)^{n+1} + P/\alpha, \quad -1 \le n \le 12T - 1$$

It is important for a borrower to know the installment amount for a given loan period and interest rate. Since the loan balance after the last payment should be zero,

$$P = \frac{\alpha(1+\alpha)^{12T}C}{(1+\alpha)^{12T} - 1}$$

As an example, suppose you borrow \$5000 at an 11% annual interest rate and must make monthly payments for 3 years. Then α = 11/1200 and P = \$163.70, rounded up to the next cent. Because of that rounding, the final principal balance y(35) = -.2721 instead of zero, which means the final payment must be reduced by 27 cents to \$163.43. So the total cost of the loan is 35P + \$163.43 = \$5892.93. Note that money owed to the lender is rounded <u>up</u> to the next cent, while money owed to the borrower is rounded <u>down</u> to the next cent.

Credit card loans work differently. The loan duration depends on the payment amounts. The bank specifies a minimum monthly payment. Paying the minimum will maximize the loan duration and the total interest. We will assume a minimum payment of 3% of the loan balance (rounded up to the nearest cent) or $15, whichever is greater, and an annual interest rate of 24%.

Homework 3 (due 26 Sept.)

*1. Write a Matlab function which computes the total interest (**intr**) and payment amount (**pmt**) for loans with monthly interest accrual and payments. The first line of the function should be

function [intr, pmt] = f1(price, down, rate, years)

where **price** is the commodity price, **down** is the down payment, **rate** is the annual interest rate as a percent, and **years** is the loan duration. Make sure to round properly. All monetary amounts must be in dollars and (whole) cents.

2. The price of a car is $55,000. You make a down payment of 10% and finance the rest with a seven-year loan at a 2.9% annual interest rate. Use your f1 function to compute the total interest and the monthly payment.

*3. Write a Matlab function to compute the total interest (**intr**) and the loan duration in months (**dur**) for credit card loans with monthly interest accrual and payments, using the parameters specified in the lecture notes. The first line of the function should be

function [intr, dur] = f2(chg, extra)

where **chg** is the amount charged to the card, and **extra** is any voluntary additional payment above the minimum required. All monetary amounts must be in dollars and (whole) cents. Make sure to round properly.

4. You charge \$5000 on a new credit card and make the minimum payment each month. Use your f2 function to compute the total interest and duration of the loan.

5. (a) Repeat problem 4 when you pay an extra \$10 each month. How much do you save in interest and duration? (b) Repeat when you pay an extra \$20 each month.

6. Let

$$\beta(N) = \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1}$$

where $N = 12T$, and T and α are defined in the lecture notes. (a) Prove analytically that β is a monotonically decreasing function of N. (b) What is the limit as $N \to \infty$?

7. Prove analytically that the principal balance $y(n)$ is a monotonically decreasing function of n.