Homework 3 Solutions

Problem 1

Write a Matlab function which computes the total interest (`intr`) and payment amount (`pmt`) for loans with monthly interest accrual and payments. The first line of the function should be:

```
function [intr, pmt] = f1(price, down, rate, years)
```

where `price` is the commodity price, `down` is the down payment, `rate` is the annual interest rate as a percent, and `years` is the loan duration.

Solution

We start by calculating the principal amount of the loan:

$$\text{principal} = \text{price} - \text{down}$$

Next, we compute the monthly interest rate:

$$\alpha = \frac{\text{rate}}{1200}$$

The number of payments is:

$$N = \text{years} \times 12$$

Using the formula derived from the lecture notes:

$$P = \frac{\alpha(1+\alpha)^N \times \text{principal}}{(1+\alpha)^N - 1}$$

In MATLAB, the function would be implemented as:

```
function [intr, pmt] = f1(price, down, rate, years)
    principal = price - down;
    alpha = rate / 1200;
    N = years * 12;
    pmt = (alpha * (1 + alpha)^N * principal) / ((1 + alpha)^N - 1);
    pmt = ceil(pmt * 100) / 100;
    total_paid = pmt * N;
    intr = total_paid - principal;
    intr = round(intr * 100) / 100;
end
```

Problem 2

The price of a car is \$55000. You make a down payment of 10% and finance the rest with a seven-year loan at a 2.9% annual interest rate. Use your `f1` function to compute the total interest and the monthly payment.

Solution

Given:

$$\text{price} = 55000, \quad \text{down} = 55000 \times 0.10 = 5500, \quad \text{rate} = 2.9, \quad \text{years} = 7$$

We call the function:

$$[\text{intr}, \text{pmt}] = \text{f1}(55000, 5500, 2.9, 7)$$

The principal is:

$$\text{principal} = 55000 - 5500 = 49500$$

Monthly interest rate:

$$\alpha = \frac{2.9}{1200} = 0.00241667$$

Number of payments:

$$N = 7 \times 12 = 84$$

Monthly payment P:

$$P = \frac{0.00241667 \times (1 + 0.00241667)^{84} \times 49500}{(1 + 0.00241667)^{84} - 1} \approx 652.52$$

Thus, the total paid over the loan period is:

$$\text{total paid} = 652.52 \times 84 = 54774.68$$

Total interest:

$$\text{intr} = 54774.68 - 49500 = 5274.68$$

Problem 3

Write a Matlab function to compute the total interest (`intr`) and the loan duration in months (`dur`) for credit card loans with monthly interest accrual and payments. The function should be:

```
function [intr, dur] = f2(chg, extra)
```

where `chg` is the amount charged to the card and `extra` is any voluntary additional payment above the minimum required.

Solution

We simulate the loan repayment process month by month until the balance is paid off. The MATLAB function is as follows:

```
function [intr, dur] = f2(chg, extra)
    balance = chg;
    alpha = 0.02;
    dur = 0;
    intr = 0;
    while balance > 0
        interest = balance * alpha;
        interest = round(interest * 100) / 100;
        intr = intr + interest;
        min_payment = max(15, ceil(balance * 0.03 * 100) / 100);
        payment = min_payment + extra;
        payment = min(payment, balance + interest);
```

```
        payment = ceil(payment * 100) / 100;
        balance = balance + interest - payment;
        balance = max(balance, 0);
        balance = round(balance * 100) / 100;
        dur = dur + 1;
    end
    intr = round(intr * 100) / 100;
end
```

Problem 4

You charge \$5000 on a new credit card and make the minimum payment each month. Use your `f2` function to compute the total interest and duration of the loan.

Solution

Given:

$$\text{chg} = 5000, \quad \text{extra} = 0$$

We call the function:

$$[\text{intr}, \text{dur}] = \text{f2}(5000, 0)$$

The results are:

$$\text{intr} = 5276.74, \quad \text{dur} = 469 \text{ months}$$

Problem 5

(a) Repeat problem 4 when you pay an extra \$10 each month. How much do you save in interest and duration?

(b) Repeat when you pay an extra \$20 each month.

Solution

For (a), with an extra \$10:

$$[\text{intr}, \text{dur}] = \text{f2}(5000, 10)$$

The results are:

$$\text{intr} = 2273.78, \quad \text{dur} = 161$$

Savings:

$$\text{interest saved} = 5276.74 - 2273.78 = 3002.96, \quad \text{duration saved} = 469 - 161 = 308$$

For (b), with an extra \$20:

$$[\text{intr}, \text{dur}] = \text{f2}(5000, 20)$$

The results are:

$$\text{intr} = 1442.04, \quad \text{dur} = 104$$

Savings:

$$\text{interest saved} = 5276.74 - 1442.04 = 3834.70, \quad \text{duration saved} = 469 - 104 = 365$$

Problem 6

Let

$$\beta(N) = \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1}$$

(a) Prove analytically that β is a monotonically decreasing function of N.
(b) What is the limit as $N \to \infty$?

Solution

(a) To show $\beta(N)$ is monotonically decreasing, we compute its derivative. Let $A = 1 + \alpha$, then:

$$f(N) = \frac{A^N}{A^N - 1}$$

The derivative of $f(N)$ is:

$$f'(N) = -\frac{A^{2N} \ln A}{(A^N - 1)^2}$$

Since $A > 1$ and $\ln A > 0$, $f'(N) < 0$, so $\beta(N)$ is decreasing.
(b) As $N \to \infty$, we have:

$$\lim_{N \to \infty} \beta(N) = \alpha$$

Problem 7

Prove analytically that the principal balance $y(n)$ is a monotonically decreasing function of n.

Solution

Given:

$$y(n) = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+1} + \frac{P}{\alpha}$$

We calculate $y(n+1) - y(n)$:

$$y(n+1) - y(n) = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+1}\alpha$$

Since $C - \frac{P}{\alpha} < 0$, we conclude that $y(n+1) - y(n) < 0$, thus $y(n)$ is a monotonically decreasing function of n.

问题6

令

$$\beta(N) = \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1}$$

其中 $N = 12T$，而 T 和 α 已在讲义中定义。
(a) 证明 $\beta(N)$ 是 N 的单调递减函数。
(b) 求 $N \to \infty$ 时的极限。

解答

(a) 证明 $\beta(N)$ 单调递减：

要证明 $\beta(N)$ 递减，我们需要证明其对 N 的导数是负的。

首先，将 $\beta(N)$ 写为：

$$\beta(N) = \alpha\left(1 + \frac{1}{(1+\alpha)^N - 1}\right)$$

我们可以考虑函数：

$$f(N) = \frac{(1+\alpha)^N}{(1+\alpha)^N - 1}$$

那么 $\beta(N) = \alpha f(N)$。

计算 $f(N)$ 的导数 $f'(N)$：

设 $A = 1+\alpha$，因此 $A > 1$。

$$f(N) = \frac{A^N}{A^N - 1}$$

计算 $f'(N)$：

$$f'(N) = \frac{A^N \ln A(A^N - 1) - A^N(A^N \ln A)}{(A^N - 1)^2} = \frac{A^N \ln A(A^N - 1 - A^N)}{(A^N - 1)^2} = -\frac{A^{2N} \ln A}{(A^N - 1)^2}$$

由于 $A > 1$ 且 $\ln A > 0$，分子为负，分母为正。

因此，$f'(N) < 0$。

由此可知，$f(N)$ 是递减的，因此 $\beta(N)$ 随 N 单调递减。

(b) 当 $N \to \infty$ 时的极限：

计算：

$$\lim_{N\to\infty} \beta(N) = \lim_{N\to\infty} \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1} = \alpha \lim_{N\to\infty} \frac{(1+\alpha)^N}{(1+\alpha)^N - 1}$$

当 $N \to \infty$ 时，$(1+\alpha)^N \to \infty$，因此：

$$\lim_{N\to\infty} \beta(N) = \alpha \cdot 1 = \alpha$$

所以 $\lim_{N\to\infty} \beta(N) = \alpha$。

问题7

证明贷款余额 $y(n)$ 是 n 的单调递减函数。

解答

已知 $y(n)$ 的公式为：

$$y(n) = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+1} + \frac{P}{\alpha}$$

其中 $P = \frac{\alpha(1+\alpha)^N C}{(1+\alpha)^N - 1}$，且 $N = 12T$，所以 P 是正的。

考虑 $y(n+1) - y(n)$ 的差值：

$$\Delta y = y(n+1) - y(n)$$

计算 $y(n+1)$ 和 $y(n)$：

$$y(n+1) = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+2} + \frac{P}{\alpha}$$

$$y(n) = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+1} + \frac{P}{\alpha}$$

两者相减得到：

$$\Delta y = \left(C - \frac{P}{\alpha}\right)\left[(1+\alpha)^{n+2} - (1+\alpha)^{n+1}\right]$$

简化表达式：

$$\Delta y = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+1}\left[(1+\alpha)-1\right] = \left(C - \frac{P}{\alpha}\right)(1+\alpha)^{n+1}\alpha$$

注意 $(1+\alpha)-1=\alpha$。
由于 $(1+\alpha)^{n+1}>0$ 且 $\alpha>0$，所以 Δy 的符号取决于 $(C-\frac{P}{\alpha})$。
根据贷款月供公式，我们有：

$$P = \frac{\alpha(1+\alpha)^N C}{(1+\alpha)^N - 1}$$

重写 $\frac{P}{\alpha}$：

$$\frac{P}{\alpha} = \frac{(1+\alpha)^N C}{(1+\alpha)^N - 1}$$

因此：

$$C - \frac{P}{\alpha} = C - \frac{(1+\alpha)^N C}{(1+\alpha)^N - 1} = C\left(1 - \frac{(1+\alpha)^N}{(1+\alpha)^N - 1}\right) = -\frac{C}{(1+\alpha)^N - 1}$$

由于 $(1+\alpha)^N-1>0$，分母为正，所以 $C-\frac{P}{\alpha}<0$。
因此，$\Delta y<0$，即 $y(n+1)-y(n)<0$。
由此证明，$y(n)$ 是 n 的单调递减函数。