ROOTS OF PALINDROMIC QUARTIC POLYNOMIALS

A polynomial with real coefficients is palindromic if its coefficient sequence is even (symmetric). The general form of a palindromic quartic polynomial is

$$x^4 + bx^3 + ax^2 + bx + 1.$$

Note that the roots of this polynomial are the same as the roots of

$$x^2 + bx + a + bx^{-1} + x^{-2}$$

If

$$y = x + x^{-1}$$

then

$$y^2 = x^2 + x^{-2} + 2$$

and the original polynomial becomes

$$y^2 + by + a - 2$$

with roots

$$y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a + 8}}{2}$$

Therefore, the roots of the original quartic are the four roots of the quadratics:

$$x^2 - y_1 x + 1 \quad \text{and} \quad x^2 - y_2 x + 1$$

Let d be the positive square root of $\left|b^2 - 4a + 8\right|$.

Case 1: $b^2 - 4a + 8 < 0$

The roots will be distinct, complex, and off the unit circle. If $p = \left(-b + id\right)/2$, the roots are

r, r^{-1}, r^*, r^{*-1} where $r = \left(p + \sqrt{p^2 - 4}\right)/2$.

Case 2: $b^2 - 4a + 8 \geq 0$

(a) If $\left|d - b\right| \leq 4$ and $\left|d + b\right| \leq 4$, the roots are $e^{\pm i\theta_1}, e^{\pm i\theta_2}$ where

$\cos(\theta_1) = (d - b)/4, \quad \cos(\theta_2) = -(d + b)/4$.

(b) If $\left|d - b\right| \leq 4$ and $\left|d + b\right| > 4$, the roots are $e^{\pm i\theta}, r, r^{-1}$ where

$\cos(\theta) = (d - b)/4, \quad r = \left(-\left(d + b\right) + \sqrt{\left(d + b\right)^2 - 16}\right)/4$.

(c) If $\left|d + b\right| \leq 4$ and $\left|d - b\right| > 4$, the roots are $e^{\pm i\theta}, r, r^{-1}$ where

$\cos(\theta) = -(d + b)/4, \quad r = \left(d - b - \sqrt{\left(d - b\right)^2 - 16)}\right)/4$.

(d) If $\left|d + b\right| > 4$ and $\left|d - b\right| > 4$, the roots will be real and off the unit circle. They are

$r_1, r_1^{-1}, r_2, r_2^{-1}$ where $r_1 = \left(-\left(d + b\right) + \sqrt{\left(d + b\right)^2 - 16}\right)/4, \quad r_2 = \left(d - b - \sqrt{\left(d - b\right)^2 - 16}\right)/4$. The

roots will be distinct iff $d > 0$; otherwise $r_2 = r_1^{-1}$.