Statistical Models and Inference - Part I

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Data Modeling

- we perform experiments and make observations to learn about a phenomenon
- to interpret `data`, we have to model them

Inference

- make general statements about a phenomenon through a model, using noisy and incomplete data
- must describe both the Phenomenon (i.e. Model) and the Measurement Process

▷ Key to Data Modeling: use data together with generative model (theory) and measurement model (experimental practice) to derive consistent probabilistic inferences

Data Modeling

- given some `data`, `D`, we want to perform three actions:

▷ parameter estimation:
for a specific Model `M`, with parameters θ, infere the values of model parameters, i.e. $P\left(\theta \middle| D\ M\right)$, the parameter posterior pdf

▷ model comparison:
given a set of models $\{M_j\}$, find out which one is best supported by data. This means finding $P\left(M_j \middle| D\right)$, the model posterior probability

▷ prediction:
given a model `M`, inferred from the data, predict new data at some new location (in the parameter space or time)

Bayesian Model Comparison

- we start by looking at model comparison for the simple case of models with no parameters

▷ using our data `D`, we look for $P\left(M \middle| D\right)$

- since $\mathrm{M} \cdot \overline{\mathrm{M}} = 0$ and $\mathrm{M} + \overline{\mathrm{M}} = \Omega$, we can write

$$
\begin{aligned}
P(D) &= P(DM) + P(D\overline{M}) \\
&= P(D \mid M)\ P(M) + P(D \mid \overline{M})\ P(\overline{M})
\end{aligned}
$$

- our quantity of interest, $P\left(M \middle| D\right)$, is related to Bayes' theorem by

$$
\begin{aligned}
P(M \mid D) &= \frac{P(D \mid M)\ P(M)}{P(D)} = \frac{P(D \mid M)\ P(M)}{P(D \mid M)\ P(M) + P(D \mid \overline{M})\ P(\overline{M})} \\
&= \frac{1}{1 + \dfrac{P(D \mid \overline{M})\ P(\overline{M})}{P(D \mid M)\ P(M)}} = \frac{1}{1 + \dfrac{1}{R}}
\end{aligned}
$$

- with $R = \dfrac{P(D \mid M)\ P(M)}{P(D \mid \overline{M})\ P(\overline{M})}$ the posterior odd ratio of the models

Bayesian Model Comparison

- it is easy to demonstrate that

$$\frac{P(M \mid D)}{P(\overline{M} \mid D)} = R = \frac{P(D \mid M)\ P(M)}{P(D \mid \overline{M})\ P(\overline{M})}$$

- in order to determine $P(M \mid D)$, we need three quantities:

▷ $P(D \mid M)$: the probability of measuring D when M is true

▷ $P(D \mid \overline{M})$: the probability of measuring D when M is not true (i.e. false)

▷ $P(M)$: the probability that M is true, independently of the data (and, of course, $P(\overline{M}) = 1 - P(M)$) $\Rightarrow P(M)$ tells us how probable the model is

- but, shouldn't we have information to tell us that M is more likely than $\overline{M}$, we could set

$$P(M) = P(\overline{M})$$

- and R becomes the Bayes factor

$$BF = \frac{P(D \mid M)}{P(D \mid \overline{M})}$$

- i.e. the ratio of the probability of the data under each model

Bayesian Model Comparison

- should we have more models, $\{M_j\}$, with $\sum P(M_j) = 1$, the probability of data becomes

$$P(D) = \sum_j P\left(D \mid M_j\right)\ P\left(M_j\right)$$

- and the posterior probability of model # 1, M_1, becomes

$$P\left(M_1 \mid D\right) = \frac{P\left(D \mid M_1\right)\ P(M_1)}{P(D)}$$

- if we do not have a complete set of models, we cannot compute the posterior probabilities, but we can still compute the odds ratio or Bayes factor between any two models

$$BF = \frac{P(D \mid M_1)}{P(D \mid M_2)} \quad \text{and} \quad R = \frac{P(D \mid M_1)\ P(M_1)}{P(D \mid M_2)\ P(M_2)}$$

Example

Problem

- a test for a disease is 90% reliable
- the probability of testing positive, in absence of the disease, is 0.07
- we know that among people aged 40 to 50 with no symptoms 8 in 1000 have the disease

Q: if a person in his/her 40 tests positive, what is the probability that he/she has the disease ?

Background information

- we build the following propositions:
  - `D`: a person is tested positive
  - `M`: a person has the disease
- and probabilities
  - $P(D \mid M) = 0.9$
  - $P(D \mid \overline{M}) = 0.07$
  - $P(M) = 0.008$

Example - analytical solution

- we build

$$R = \frac{P(D \mid M)\ P(M)}{P(D \mid \overline{M})\ P(\overline{M})} = \frac{9 \cdot 10^{-1} \times 8 \cdot 10^{-3}}{7 \cdot 10^{-2} \times (1 - 8 \cdot 10^{-3})} = 0.1035$$

- therefore

$$P(M \mid D) = \frac{1}{1 + 1/R} = 0.094$$

- even though a positive test result is quite probable (assuming the person has the disease), it is very unlikely that he/she has the disease
- what is decisive in the computation of $P(M \mid D)$ is the ratio between

$$P(D\ M) = P(D \mid M)\ P(M) = 7.2 \cdot 10^{-3}$$

(positive result, assuming the disease is present)

- and

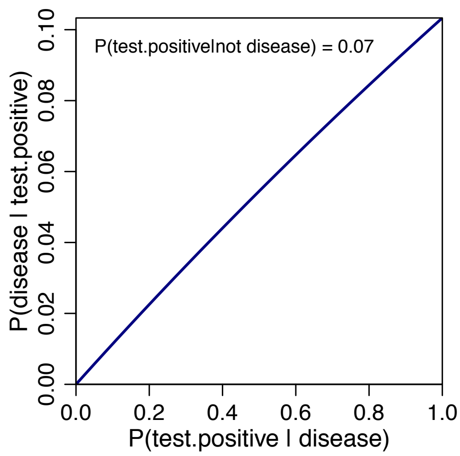
$$P(D\ \overline{M}) = P\left(D \mid \overline{M}\right)\ P\left(\overline{M}\right) = 7 \cdot 10^{-2}$$

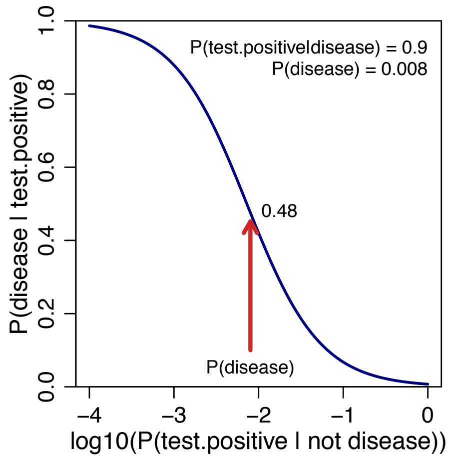
(positive result, assuming the disease is absent)

Example - R solution

```
post <- function(p.d.m, p.d.notm, p.m) {
   p.notm <- 1 - p.m
   odds.ratio <- (p.d.m * p.m) /
                 (p.d.notm * p.notm)
   p.m.d <- 1/(1 + 1/odds.ratio)
}

p.d.m <- seq(0, 1, 0.01) # True positive
p.d.notm <- 0.07         # False positive
p.m <- 0.008             # Disease Prior

p.m.d <- post(p.d.m, p.d.notm, p.m)
plot(p.d.m, p.m.d, type='l', lwd=2, col='navy')

p.d.m <- 0.9                   # True positive
p.d.notm <- 10^seq(-4,0, 0.02) # False positive
p.m <- 0.008                   # Disease Prior

p.m.d <- post(p.d.m, p.d.notm, p.m)
plot(log10(p.d.notm), p.m.d, type='l', col='navy')
```

- only once the false positive rate drops below the base rate ($P(M)$) does the test starts to be useful





Data Modeling with Parametric Models

- generative model : theory predicting observable data from model parameters
  - the model just studied did not have any parameter: it was either `true` or `false`
- the simplest generative model is a straight line

$$f(x; a, b) = a + b \cdot x$$

- but our measurements will differ from the model due to noise

$$y = f(x; a, b) + \epsilon$$

- and the noise model - we call it the measurement model - has also parameters
  - given our set of data $\mathrm{D} = \{y_j\}$ at specified values $\{x_j\}$, we want to infer the values of the parameters for the generative model
  - in some cases we want to find the best set of parameters that predicts the data
  - but data are noisy → there is no unique solution
- we look for the probability distributions of the parameters, $P\left(\theta \,\middle|\, D\ M\right)$, also called parameter posterior pdf. Thanks to Bayes' theorem

$$P\left(\theta \,\middle|\, D\ M\right) = \frac{P\left(D \,\middle|\, \theta\ M\right)\ P\left(\theta \,\middle|\, M\right)}{P\left(D \,\middle|\, M\right)}$$

The Likelihood

- $P(D \mid \theta\, M)$ is the Likelihood probability
  - it is a key function since it describes both the phenomenon and the data
  - it tells us the probability of getting the data we measured, given some value of the parameters

- `M` specifies:
  - a generative model ← the equation for the straight line $f(x; a, b)$
  - a measurement model ← how the measurement of y at a given x differs from $f(x; a, b)$ due to noise

- the measurement model describes ϵ in $y = f(x; a, b) + \epsilon$
  - example: Gaussian distribution with variance σ^2. The Likelihood for any measurement is

$$P(y \mid \theta\, M) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y - f(x; a, b))^2}{2\sigma^2}\right)$$

- telling us that the measurement has a Gaussian distribution about the true value
- $\theta = \theta(a, b; \sigma)$ is the union of the generative and measurements models

The Prior

- $P(\theta \mid M)$ is the Prior probability
  - it encapsulates all the information we have, independent of the data
- it is called Prior because is the background information we have before obtaining the Data
- different people may have different information, or different opinion on what prior information is important
- this is not a weakness of inference
- it just reflects reality: we do not only use our immediate measurements to reach scientific conclusions

The Posterior

- $P\left(\theta \mid D\, M\right)$ is the Posterior probability

- it is the pdf over the model parameters, given `data` and background information

- from Bayes' theorem

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

- the proportionality is through $P\left(D \mid M\right)$, a normalization factor which is independent of θ. Therefore:

$$P\left(\theta \mid D\, M\right) = \frac{1}{Z} P\left(D \mid \theta\, M\right) P\left(\theta \mid M\right)$$

- with $Z = P\left(D \mid M\right)$
- from a conceptual point of view, inference is really that strightforward
- Bayesian inference is the process of improving our knowledge of the model paramaters by using the data

▷ we update the Prior using the Likelihood to obtain the Posterior

The Evidence

- $P\left(D \mid M\right)$ is the Evidence

- is the denominator of Bayes's equation and it gives the probability of observing the Data `D`, assuming the model `M` to be true, for any values of θ

$$P\left(D \mid M\right) = \int P\left(D \mid \theta M\right) P\left(\theta \mid M\right) d\theta$$

- evidence plays a key role in model comparison

- as a normalization constant, it is very important if we want to compute certain quantities from the posterior

- sometimes the integral can be calculated analytically, but for many real-world problems, we have to resort to numerical integration → Markov Chain Monte Carlo