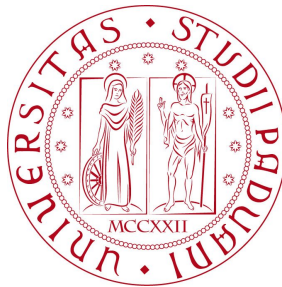


Statistical Models and Inference - Part I

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Data Modeling

- we perform **experiments** and make **observations** to **learn about a phenomenon**
- to interpret data, we have to model them

Inference

- make general statements about a phenomenon **through a model**, using **noisy and incomplete data**
 - must **describe** both the **Phenomenon** (i.e. Model) and the **Measurement Process**
- ▷ Key to Data Modeling: use data together with **generative model (theory)** and **measurement model (experimental practice)** to derive consistent probabilistic inferences

- given some data, D , we want to perform three actions:

▷ parameter estimation:

for a specific Model M , with parameters θ , infer the values of model parameters, i.e. $P(\theta | D, M)$, the **parameter posterior pdf**

▷ model comparison:

given a set of models $\{M_j\}$, find out which one is best supported by data. This means finding $P(M_j | D)$, the **model posterior probability**

▷ prediction:

given a model M , inferred from the data, **predict new data at some new location** (in the parameter space or time)

Bayesian Model Comparison

- we start by looking at model comparison for the simple case of models with no parameters

▷ using our data D , we look for $P(M | D)$

- since $M \cdot \bar{M} = 0$ and $M + \bar{M} = \Omega$, we can write

$$\begin{aligned} P(D) &= P(DM) + P(D\bar{M}) \\ &= P(D | M) P(M) + P(D | \bar{M}) P(\bar{M}) \end{aligned}$$

- our quantity of interest, $P(M | D)$, is related to Bayes' theorem by

$$\begin{aligned} P(M | D) &= \frac{P(D | M) P(M)}{P(D)} = \frac{P(D | M) P(M)}{P(D | M) P(M) + P(D | \bar{M}) P(\bar{M})} \\ &= \frac{1}{1 + \frac{P(D | \bar{M}) P(\bar{M})}{P(D | M) P(M)}} = \frac{1}{1 + \frac{1}{R}} \end{aligned}$$

- with $R = \frac{P(D | M) P(M)}{P(D | \bar{M}) P(\bar{M})}$ the **posterior odd ratio** of the models

Bayesian Model Comparison

- it is easy to demonstrate that

$$\frac{P(M | D)}{P(\bar{M} | D)} = R = \frac{P(D | M) P(M)}{P(D | \bar{M}) P(\bar{M})}$$

- in order to determine $P(M | D)$, we need three quantities:

▷ $P(D | M)$: the probability of measuring D when M is true

▷ $P(D | \bar{M})$: the probability of measuring D when M is not true (i.e. false)

▷ $P(M)$: the probability that M is true, independently of the data (and, of course, $P(\bar{M}) = 1 - P(M) \Rightarrow P(M)$ tells us how probable the model is

- but, shouldn't we have information to tell us that M is more likely than \bar{M} , we could set

$$P(M) = P(\bar{M})$$

- and R becomes the Bayes factor

$$BF = \frac{P(D | M)}{P(D | \bar{M})}$$

- i.e. the ratio of the probability of the data under each model

Bayesian Model Comparison

- should we have more models, $\{M_j\}$, with $\sum P(M_j) = 1$, the probability of data becomes

$$P(D) = \sum_j P(D | M_j) P(M_j)$$

- and the posterior probability of model # 1, M_1 , becomes

$$P(M_1 | D) = \frac{P(D | M_1) P(M_1)}{P(D)}$$

- if we do not have a complete set of models, we cannot compute the posterior probabilities, but we can still compute the odds ratio or Bayes factor between any two models

$$BF = \frac{P(D | M_1)}{P(D | M_2)} \quad \text{and} \quad R = \frac{P(D | M_1) P(M_1)}{P(D | M_2) P(M_2)}$$

Example

Problem

- a test for a disease is 90% reliable
- the probability of testing positive, in absence of the disease, is 0.07
- we know that among people aged 40 to 50 with no symptoms 8 in 1000 have the disease

Q: if a person in his/her 40 tests positive, what is the probability that he/she has the disease ?

Background information

- we build the following propositions:
 - D : a person is tested positive
 - M : a person has the disease
- and probabilities
 - $P(D | M) = 0.9$
 - $P(D | \bar{M}) = 0.07$
 - $P(M) = 0.008$

Example - analytical solution

- we build

$$R = \frac{P(D | M) P(M)}{P(D | \bar{M}) P(\bar{M})} = \frac{9 \cdot 10^{-1} \times 8 \cdot 10^{-3}}{7 \cdot 10^{-2} \times (1 - 8 \cdot 10^{-3})} = 0.1035$$

- therefore

$$P(M | D) = \frac{1}{1 + 1/R} = 0.094$$

- even though a positive test result is quite probable (assuming the person has the disease), it is very unlikely that he/she has the disease
- what is decisive in the computation of $P(M | D)$ is the ratio between

$$P(D | M) = P(D | M) P(M) = 7.2 \cdot 10^{-3}$$

(positive result, assuming the disease is present)

- and

$$P(D | \bar{M}) = P(D | \bar{M}) P(\bar{M}) = 7 \cdot 10^{-2}$$

(positive result, assuming the disease is absent)

Example - R solution

```
post <- function(p.d.m, p.d.notm, p.m) {
  p.notm <- 1 - p.m
  odds.ratio <- (p.d.m * p.m) /
                (p.d.notm * p.notm)
  p.m.d <- 1/(1 + 1/odds.ratio)
}

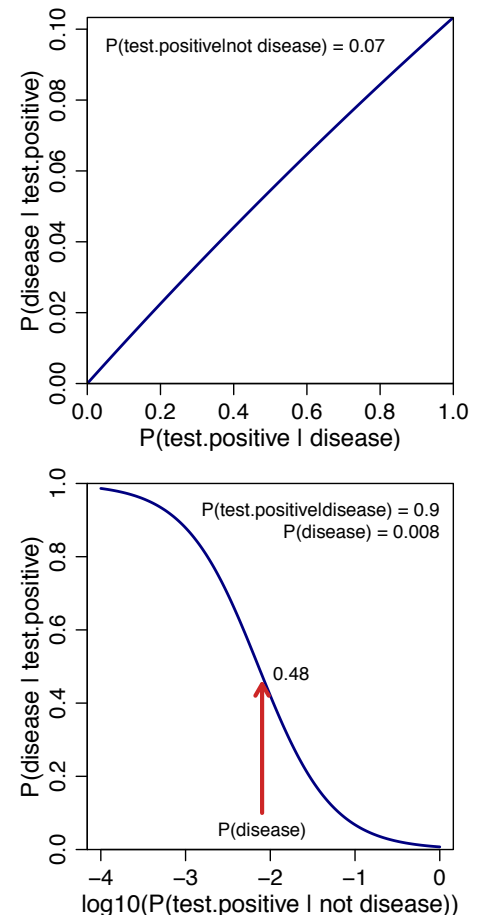
p.d.m <- seq(0, 1, 0.01) # True positive
p.d.notm <- 0.07         # False positive
p.m <- 0.008             # Disease Prior

p.m.d <- post(p.d.m, p.d.notm, p.m)
plot(p.d.m, p.m.d, type='l', lwd=2, col='navy')

p.d.m <- 0.9             # True positive
p.d.notm <- 10^seq(-4,0, 0.02) # False positive
p.m <- 0.008             # Disease Prior

p.m.d <- post(p.d.m, p.d.notm, p.m)
plot(log10(p.d.notm), p.m.d, type='l', col='navy')
```

- only once the false positive rate drops below the base rate ($P(M)$) does the test starts to be useful



Data Modeling with Parametric Models

- generative model** : theory predicting observable data from model parameters
 - the model just studied did not have any parameter: it was either true or false
- the simplest generative model is a straight line

$$f(x; a, b) = a + b \cdot x$$



- but our measurements will differ from the model due to noise

$$y = f(x; a, b) + \epsilon$$

- and the noise model - we call it the **measurement model** - has also parameters
 - given our set of data $D = \{y_j\}$ at specified values $\{x_j\}$, we want to infer the values of the parameters for the generative model
 - in some cases we want to find the best set of parameters that predicts the data
 - but data are noisy \rightarrow there is no unique solution
- we look for the probability distributions of the parameters, $P(\theta | D M)$, also called **parameter posterior pdf**. Thanks to Bayes' theorem

$$P(\theta | D M) = \frac{P(D | \theta M) P(\theta | M)}{P(D | M)}$$

The Likelihood

- $P(D | \theta M)$ is the Likelihood probability
 - it is a key function since it describes both the phenomenon and the data
 - it tells us the probability of getting the data we measured, given some value of the parameters
- M specifies:
 - a generative model  the equation for the straight line $f(x; a, b)$
 - a measurement model  how the measurement of y at a given x differs from $f(x; a, b)$ due to noise
- the measurement model describes ϵ in $y = f(x; a, b) + \epsilon$
 - example: Gaussian distribution with variance σ^2 . The Likelihood for any measurement is

$$P(y | \theta M) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(y - f(x; a, b))^2}{2\sigma^2}\right)$$

- telling us that the measurement has a Gaussian distribution about the true value
- $\theta = \theta(a, b; \sigma)$ is the union of the generative and measurements models

The Prior

- $P(\theta | M)$ is the Prior probability
 - it encapsulates all the information we have, independent of the data
- it is called Prior because is the background information we have before obtaining the Data
- different people may have different information, or different opinion on what prior information is important
- this is not a weakness of inference
- it just reflects reality: we do not only use our immediate measurements to reach scientific conclusions

The Posterior

- $P(\theta | D M)$ is the Posterior probability
- it is the pdf over the model parameters, given data and background information
- from Bayes' theorem

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

- the proportionality is through $P(D | M)$, a normalization factor which is independent of θ . Therefore:

$$P(\theta | D M) = \frac{1}{Z} P(D | \theta M) P(\theta | M)$$

- with $Z = P(D | M)$
 - from a conceptual point of view, inference is really that straightforward
 - Bayesian inference is the process of improving our knowledge of the model parameters by using the data
- we update the Prior using the Likelihood to obtain the Posterior

The Evidence

- $P(D | M)$ is the Evidence
- is the denominator of Bayes's equation and it gives the probability of observing the Data D , assuming the model M to be true, for any values of θ

$$P(D | M) = \int P(D | \theta M) P(\theta | M) d\theta$$

- evidence plays a key role in model comparison
- as a normalization constant, it is very important if we want to compute certain quantities from the posterior
- sometimes the integral can be calculated analytically, but for many real-world problems, we have to resort to numerical integration → Markov Chain Monte Carlo