

# Heterogeneous beliefs and routes to chaos in a simple asset pricing model

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## Abstract

This paper investigates the dynamics in a simple present discounted value asset pricing model with heterogeneous beliefs. Agents choose from a finite set of predictors of future prices of a risky asset and revise their ‘beliefs’ in each period in a boundedly rational way, according to a ‘fitness measure’ such as past realized profits. Price fluctuations are thus driven by an evolutionary dynamics between different expectation schemes (‘rational animal spirits’). Using a mixture of local bifurcation theory and numerical methods, we investigate possible bifurcation routes to complicated asset price dynamics. In particular, we present numerical evidence of strange, chaotic attractors when the intensity of choice to switch prediction strategies is high. © 1998 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Recently a number of structural asset pricing models have been introduced, emphasizing the role of heterogeneous beliefs in financial markets, with different groups of traders having different expectations about future prices. In most of these heterogeneous agent models two typical investor types arise. The first type are rational ‘smart money’ traders or fundamentalists, believing that the price of an asset is determined solely by its efficient market hypothesis (EMH) fundamental value, as given by the present discounted value of the stream of future dividends. The second typical trader type are ‘noise traders’, sometimes called

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chartists or technical analysts, believing that asset prices are not completely determined by fundamentals, but that they may be predicted by simple technical trading rules, extrapolation of trends and other patterns observed in past prices. An early example of a heterogeneous agent model can be found in Zeeman (1974); more recent examples include Haltiwanger and Waldmann (1985), Frankel and Froot (1988), DeLong et al. (1990), and Dacorogna et al. (1995).

An important question in heterogeneous agents modelling is whether ‘irrational’ traders can survive in the market, or whether they would lose money and be driven out of the market by rational investors, who would trade against them and drive prices back to fundamentals, as argued, e.g. by Friedman (1953). Recently, DeLong et al. (1990) have shown that in a finite horizon financial market model, a constant fraction of ‘noise traders’ may on average earn a higher expected return than ‘rational’ or ‘smart money’ traders, and may survive in the market with positive probability. In related empirical work, Brock et al. (1992) have shown that simple technical trading strategies applied to the Dow Jones index may outperform several popular EMH stochastic finance models such as the random walk or a (G)ARCH-model. In a series of related papers, Kurz (1994a,b,c) introduces heterogeneous, ‘rational beliefs’, where agents may use the ‘wrong’ probability distribution, but the long-run average statistics of realizations coincide with agents’ anticipations. Kurz (1994c) introduces this concept into asset pricing models, and applies it to explain fluctuations in the postwar Dow Jones Index.

A number of recent papers have emphasized that heterogeneity in beliefs may lead to market instability and complicated dynamics, such as cycles or even chaotic fluctuations, in financial markets (e.g. Chiarella, 1992; Day and Huang, 1990; DeGrauwe et al., 1993; Lux, 1995; Sethi, 1996). In these nonlinear models, asset price fluctuations are caused by an endogenous mechanism relating the fraction or the weights of fundamentalists and chartists to the distance between the fundamental and the actual price. A large fraction or weight of the fundamentalists tends to stabilize prices, whereas a large fraction of chartists tends to destabilize prices. Asset price fluctuations are caused by the interaction between these stabilizing and destabilizing forces. Brock and Hommes (1997b) give a more detailed discussion of the role of nonlinear dynamics in modelling financial markets.

In the related ‘artificial economic life’ literature, financial markets are modelled as an evolutionary system, with an ‘ocean’ of traders using different prediction and trading strategies (e.g. Arthur et al. (1996) or the review of LeBaron (1995)). Most of this work is computationally oriented, since the number of different trader types is usually large and therefore the artificial financial market models are not analytically tractable. Brock (1993, 1995) and Brock and LeBaron (1995) have started to build a theoretical framework and analyze simple versions of these adaptive belief systems. Brock and Hommes (1997a) investigate a simple, demand–supply cobweb type adaptive belief

system, with rational versus naive expectations and show how heterogeneous beliefs may lead to market instability and strange attractors, when rational expectations are more costly to obtain than naive expectations. Brock and de Fontnouvelle (1996) investigate an overlapping generations monetary economy with adaptive heterogeneous beliefs and show that different types of bifurcations of the monetary steady state can arise, as the intensity of choice to switch prediction strategies increases. Other work related to this evolutionary approach employs genetic algorithms as a learning and expectation formation device (e.g. Marimon et al., 1989; Arifovich, (1994, 1995); Bullard and Duffy, 1994). Timmermann (1993, 1996) investigates learning dynamics in asset pricing models. Another related work is Krusell and Smith (1996), who investigate the effect of different rules of thumb for savings decisions, which vary in sophistication and effort costs, in a stochastic growth model. See Sargent (1993), Marimon (1997) and Evans and Honkapohja (1997) for extensive surveys of the learning literature.

This paper presents a tractable form of evolutionary dynamics which we call, *Adaptive Belief Systems*, in a simple present discounted value (PDV) asset pricing model. Agents can choose from a finite set of different beliefs or predictors of the future price of a risky asset. Predictor selection is based upon a ‘fitness’ or ‘performance’ measure such as past realized profits. Predictor choice is (boundedly) *rational* in the sense that, at each date, most agents choose the predictor generating the highest past performance. The paper shows how an increase in the ‘intensity of choice’ to switch predictors can lead to market instability and the emergence of complicated dynamics for asset prices and returns. Keynes already argued that stock prices are not only determined by fundamentals, but in addition market psychology and investors’ animal spirits influence financial markets significantly. In our heterogeneous beliefs asset pricing model, when the intensity of choice to switch predictors is high, asset price fluctuations are indeed characterized by an irregular switching between phases where prices are close to the EMH fundamental, phases of optimism (‘castles in the air’), where traders become excited and extrapolate upward trends, and phases of pessimism, where traders become nervous causing a sharp decline in asset prices. A key feature of our adaptive belief systems is that, this irregular switching is triggered by a rational choice between simple prediction strategies. One may thus say that the market is driven by *rational animal spirits*.

The asset pricing model we use is essentially the same as in Lucas (1978). Under homogeneous, rational expectations and the assumption that the dividend process of the risky asset is independently identically distributed (IID), the asset price dynamics is extremely simple: one constant price over time! Our paper shows that, under the hypothesis of heterogeneous expectations among traders the situation changes dramatically, and an extremely rich asset price dynamics emerges, with bifurcation routes to strange attractors, especially if switching to more successful strategies becomes more rapid.

In nonlinear dynamic models such as our asset pricing model, it is in general impossible to obtain explicit analytic expressions for the periodic and chaotic solutions. Therefore, in applied nonlinear dynamics it is common practice to use a mixture of theoretical and numerical methods to analyze the dynamics. Using local bifurcation theory, we are able to detect the primary and the secondary bifurcations in the routes to complexity in asset price fluctuations. In addition, by using numerical tools, such as phase diagrams, bifurcation diagrams and computation of Lyapunov exponents and fractal dimension the occurrence of strange, chaotic attractors is demonstrated.

The plan of the paper is as follows. In Section 2, we present the PDV asset pricing model with heterogeneous, adaptive beliefs. A key parameter in the model is the ‘intensity of choice’, measuring how sensitive the mass of traders is to differences in fitnesses across trading strategies. Section 3 briefly discusses some standard numerical tools for the analysis of nonlinear (discrete time) dynamical systems. Section 4 focusses on the dynamics of equilibrium asset prices with a small number (two, three or four) of simple, linear belief types, such as rational (perfect foresight) agents, fundamentalist, trend extrapolators, contrarians and biased traders. This section studies the change in equilibrium dynamics of asset prices, as parameters of interest such as the choice intensity,  $\beta$ , increase. The purpose of this exercise is to uncover common features such as the character of the primary and secondary generic bifurcations, if any, and possible deviations from fundamentals. Proofs of the results in this section are given in an appendix. Finally, Section 5 concludes and briefly discusses some future work.

## 2. Adaptive beliefs in the PDV asset pricing model

Consider an asset pricing model with one risky asset and one risk free asset. Let  $p_t$  denote the price (ex dividend) per share of the risky asset at time  $t$ , and let  $\{y_t\}$  be the stochastic dividend process of the risky asset. The risk free asset is perfectly elastically supplied at gross return  $R > 1$ . We have

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t, \quad (2.1)$$

for the dynamics of wealth where bold face type denotes random variables and  $z_t$  denotes the number of shares of the asset purchased at date  $t$ . Let  $E_t, V_t$  denote the conditional expectation and conditional variance operators, based on a publically available information set consisting of past prices and dividends. Let  $E_{ht}, V_{ht}$  denote the ‘beliefs’ of investor type  $h$  about the conditional expectation and conditional variance. Note that the conditional variance of wealth  $W_{t+1}$  equals  $z_t^2$  times the conditional variance of *excess return* per share  $p_{t+1} + y_{t+1} - Rp_t$ . We shall assume that beliefs about the conditional variance of excess returns are constant and the same for everyone, i.e.

$V_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t) \equiv \sigma^2$  for all types  $h$ .<sup>1</sup> Assume each investor type is a myopic mean variance maximizer, so for type  $h$  the demand for shares  $z_{ht}$  solves

$$\text{Max}_z \{E_{ht} \mathbf{W}_{t+1} - (a/2)V_{ht}(\mathbf{W}_{t+1})\}, \quad (2.2)$$

i.e.,

$$z_{ht} = E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)/a\sigma^2, \quad (2.3)$$

where ‘ $a$ ’ denotes the risk aversion, which is assumed to be equal for all traders. Let  $z_{st}$  denote the supply of shares per investor and  $n_{ht}$  the fraction of investors of type  $h$  at date  $t$ . Equilibrium of demand and supply implies,

$$\sum n_{ht} \{E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1} - Rp_t)/a\sigma^2\} = z_{st}. \quad (2.4)$$

If there is only one type  $h$ , market equilibrium yields the pricing equation

$$Rp_t = E_{ht}(\mathbf{p}_{t+1} + \mathbf{y}_{t+1}) - a\sigma^2 z_{st}. \quad (2.5a)$$

Now specialize Eq. (2.5a) to the special case of zero supply of outside shares, i.e.,  $z_{st} = 0$ , for all  $t$ . Furthermore, in order to get a bench mark notion of the rational expectations ‘fundamental solution’  $p_t^*$ , put the information set  $\mathbb{F}_t = \{p_t, p_{t-1}, \dots; y_t, y_{t-1}, \dots\}$  and consider the equation

$$Rp_t^* = E_t\{\mathbf{p}_{t+1}^* + \mathbf{y}_{t+1}\}, \quad (2.5b)$$

where  $E_t$  is conditional expectation on the information set  $\mathbb{F}_t$ . In the case where the dividend process  $\{y_t\}$  is IID,  $E_t\{y_{t+1}\} = \bar{y}$  which is a constant. For the special case  $\{y_t\}$  IID we now have a standard notion of ‘fundamental’. Put  $p_t^* = \bar{p}$ , where  $\bar{p}$  solves

$$R\bar{p} = \bar{p} + \bar{y}. \quad (2.5c)$$

Now Eq. (2.5b) typically has infinitely many solutions but only the constant solution  $\bar{p} = \bar{y}/(R - 1)$  of Eq. (2.5c) satisfies the ‘no bubbles’ condition  $\lim_{t \rightarrow \infty} Ep_t/R^t = 0$ . It will be convenient to work with the *deviation*  $x_t$  from the benchmark fundamental  $p_t^*$ , i.e.,

$$x_t = p_t - p_t^*. \quad (2.6)$$

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<sup>1</sup> The assumption of homogeneous, constant beliefs on variance is made primarily for analytical tractability. Notice however that heterogeneity in conditional expectations in fact leads to heterogeneity in conditional variance as well, but we will ignore this second-order effect. Nelson (1992) provides some justification for homogeneity of beliefs on variance in a diffusion context, arguing that the variance can be estimated with infinite precision by repeated sampling within a fixed period of time, whereas this is not the case for the mean. This argument suggests that, in a discrete time approximation to a diffusion model, there should be more disagreement about the mean than about the variance.

We shall now introduce heterogeneous beliefs and study the equilibrium dynamical system. Rewrite Eq. (2.4) for the case of zero supply of outside shares to get

$$Rp_t = \sum n_{ht} E_{ht}(p_{t+1} + y_{t+1}). \tag{2.7}$$

We must be precise about the class of beliefs about deviations from the fundamental solution that we wish to consider. Assume

*Assumption 1.* All beliefs are of the form

$$E_{ht}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}) + f_h(x_{t-1}, \dots, x_{t-L}), \tag{2.8}$$

where  $p_{t+1}^*$  denotes the fundamental,  $E_t(p_{t+1}^* + y_{t+1})$  is the conditional expectation of the fundamental on the information set  $\mathbb{F}_t$ ,  $x_t = p_t - p_t^*$  is the deviation from the fundamental, and  $f_h$  is some *deterministic* function which can differ across trader types  $h$ . Hence, we restrict beliefs to deterministic functions of past deviations from the fundamental. Assumption 1 includes the case of an IID dividend process, with  $E_t y_{t+1} = \bar{y}$  and the corresponding constant fundamental  $p_t^* = \bar{p} = \bar{y}/(R - 1)$ , as a special case. An advantage of our ‘deviation from the fundamental’ formulation is that other, time varying fundamental solutions can easily be incorporated. We may now use Assumption 1, Eq. (2.8), the fact that  $\sum n_{ht} = 1$  for all  $t$  and Eq. (2.5b), to write the equilibrium equation (2.7) in deviations form,

$$Rx_t = \sum n_{ht} f_h(x_{t-1}, \dots, x_{t-L}) \equiv \sum n_{ht} f_{ht}. \tag{2.9a}$$

Recall that all traders are assumed to have common, constant conditional variances  $\sigma^2 = V_{ht}(R_{t+1})$ , on excess returns  $R_{t+1} = p_{t+1} + y_{t+1} - Rp_t$ . Let  $\rho_{ht} = E_{ht}(R_{t+1})$  and consider the goal function

$$\text{Max}_z \{ E_{ht} R_{t+1} z - (a/2) z^2 V_{ht}(R_{t+1}) \} = \text{Max}_z \{ \rho_{ht} z - (a/2) z^2 \sigma^2 \}. \tag{2.10}$$

Note that Eq. (2.10) is equivalent to the objective (2.2) up to a constant, so the optimum choice of shares of the risky asset is the same. Denote the optimum solution of Eq. (2.10) by  $z(\rho_{ht})$ .

Turn now to the adaption of beliefs, i.e., the dynamics of the fractions  $n_{ht}$  of different trader types. First, emphasizing the timing of updating beliefs, we slightly change notation and rewrite Eq. (2.9a) as

$$Rx_t = \sum n_{h,t-1} f_h(x_{t-1}, \dots, x_{t-L}) \equiv \sum n_{h,t-1} f_{ht}. \tag{2.9b}$$

where  $n_{h,t-1}$  denotes the fraction of type  $h$  at the beginning of period  $t$ , *before* the equilibrium price  $x_t$  has been observed. Lagging the fractions by one period as in Eq. (2.9b) will be convenient, when we analyze the asset pricing model with heterogeneous beliefs as an explicit nonlinear difference equation in deviations  $x_t$  from the fundamental and fractions  $n_{ht}$ , in Section 4.

Before we define the ‘fitness function’ we compute the realized excess return over period  $t$  to period  $t + 1$ ,

$$\begin{aligned} R_{t+1} &= p_{t+1} + y_{t+1} - Rp_t = x_{t+1} + p_{t+1}^* + y_{t+1} - Rx_t - Rp_t^* \\ &= x_{t+1} - Rx_t + p_{t+1}^* + y_{t+1} - E_t(p_{t+1}^* + y_{t+1}) + E_t(p_{t+1}^* + y_{t+1}) - Rp_t^* \\ &\equiv x_{t+1} - Rx_t + \delta_{t+1}. \end{aligned} \quad (2.11)$$

Since by Eq. (2.5b),  $E_t(p_{t+1}^* + y_{t+1}) - Rp_t^* = 0$ ,  $\delta_{t+1}$  is a Martingale Difference Sequence (MDS) w.r.t.  $\mathbb{F}_t$ , i.e.,  $E\{\delta_{t+1}|\mathbb{F}_t\} = 0$  for all  $t$ . Indeed we may view the decomposition (2.11) as separating the ‘explanation’ of realized excess returns  $R_{t+1}$  into the *contribution*  $x_{t+1} - Rx_t$  of the theory being exposted here and the conventional Efficient Markets Theory term  $\delta_{t+1}$ .<sup>2</sup>

Let the ‘fitness function’ or the ‘performance measure’  $\pi(R_{t+1}, \rho_{ht})$ , be defined by

$$\pi_{h,t} = \pi(R_{t+1}, \rho_{ht}) = R_{t+1}z(\rho_{ht}) = (x_{t+1} - Rx_t + \delta_{t+1})z(\rho_{ht}), \quad (2.12)$$

i.e., fitness is given by realized profits for trader type  $h$ .<sup>3</sup> Notice that in general, realized returns depend upon stochastic dividends and is given by  $R_{t+1} = x_{t+1} - Rx_t + \delta_{t+1}$ . In Section 4 we will mainly focus on the *deterministic* nonlinear asset pricing dynamics, with  $\delta_{t+1} = 0$  for all  $t$  and constant dividend  $\bar{y}$  per time period. However, Section 4 also contains some numerical simulations with a stochastic dividend process  $y_t = \bar{y} + \varepsilon_t$ , where  $\varepsilon_t$  is IID, with a uniform distribution on a small interval  $[-\varepsilon, +\varepsilon]$ , to investigate the effect of noise upon the asset pricing dynamics. Notice that in the IID dividend case we simply have  $\delta_{t+1} = \varepsilon_{t+1}$ .

More generally, one can introduce additional memory into the performance measure, by considering a weighted average of realized profits, as follows:

$$U_{h,t} = \pi_{h,t} + \eta U_{h,t-1}, \quad (2.13)$$

<sup>2</sup> Studies like Brock et al. (1992) parametrize  $\delta_{t+1}$  by, for example, a GARCH or EGARCH model, and test for the presence of an ‘additional term’  $x_{t+1} - Rx_t$  by bootstrapping the null distribution of objects like trading strategies under the null hypothesis that  $x_{t+1} - Rx_t = 0$  for all  $t$ . Brock et al. (1992) reject the null hypothesis that  $x_{t+1} - Rx_t = 0$ . Hence, it appears that extra structure is needed to ‘explain’ excess returns data. This kind of finding motivates our theoretical work.

<sup>3</sup> In Brock and Hommes (1997c) and recent work in progress, we focus on another, perhaps somewhat more reasonable performance measure, namely  $\pi_{h,t} = \pi(\rho_t, \rho_{ht}) = R_{t+1}z(\rho_{ht}) - (a/2)[z(\rho_{ht})]^2\sigma^2$ , where the second term captures risk adjustment. More generally, one might even consider a weighted average of past risk adjusted realized profits. The non-risk adjusted case considered in the present paper may be regarded as a fitness function that is slightly inconsistent with the traders being myopic mean-variance maximizers of wealth. On the other hand, from a practical viewpoint realized profits may be the most relevant performance measure real traders care about.

where the parameter  $\eta$  represents the ‘memory strength’. In most of the case-studies in Section 4, for analytical tractability we will concentrate on the case  $\eta = 0$ , where fitness is given by most recent past realized profit. However, Section 4 contains one important special case with memory in the performance measure, i.e.,  $\eta > 0$ , where one group of traders has rational expectations (perfect foresight).

Now write type  $h$  beliefs  $\rho_{ht} = E_{ht}R_{t+1} = f_{ht} - Rx_t$  in deviation form. Let the updated fractions  $n_{h,t}$  be given by the discrete choice probability

$$n_{h,t} = \exp[\beta U_{h,t-1}] / Z_t, \quad Z_t \equiv \sum \exp[\beta U_{h,t-1}], \quad (2.14)$$

where the parameter  $\beta$  is the *intensity of choice* measuring how fast agents switch between different prediction strategies; see Brock and Hommes (1997a) for a motivation of using the discrete choice setup for predictor selection. If intensity of choice is infinite, the entire mass of traders uses the strategy that has highest fitness. If intensity of choice is zero, the mass of traders distributes itself evenly across the set of available strategies.

The timing in Eq. (2.14) is important. We can only allow the fitness function that goes into the RHS of Eq. (2.14) to depend upon fitness  $U$  and return  $R$  at date  $t - 1$  and further back in the past in order to ensure that  $n_{h,t}$  only depends upon observable deviations  $x_t$  at date  $t$  and further back in the past. The timing in Eq. (2.14) ensures that past realized profits  $R_{t+1-j}z(\rho_{h,t-j})$ ,  $j \geq 1$ , are indeed *observable* quantities that can be used in predictor selection. In Section 4, we will investigate the asset pricing model with heterogeneous, adaptive beliefs for different simple, linear predictors.

### 3. Numerical analysis in nonlinear dynamics

This section briefly discusses some important numerical tools in the analysis of nonlinear discrete time dynamical systems, such as phase diagrams, bifurcation diagrams and numerical computation of Lyapunov exponents and fractal dimension of strange attractors. Although these tools are standard in nonlinear dynamics, a brief description is included here for nonspecialists. General references on nonlinear dynamics and chaos are, e.g. Guckenheimer and Holmes (1983) and Arrowsmith and Place (1990); economic applications of 2- and 3-D nonlinear discrete dynamical systems include Brock and Hommes (1997a,b), Goeree (1996), Hommes (1991, 1995), Medio (1992) and Vilder (1995, 1996).

A general strategy for analyzing a nonlinear dynamical system starts with a stability analysis of the steady states. In particular, a comparative static analysis of how a steady state becomes unstable as a model parameter is varied can reveal primary bifurcations in a possible route to complicated dynamics. A *bifurcation* is a qualitative change in the dynamics, for example concerning the



existence or stability of a steady state or a cycle. The way in which the eigenvalues of the corresponding jacobian matrix at the steady state cross the unit circle as a parameter changes, characterizes the type of bifurcation that will occur. At the bifurcation value, three (generic) cases arise:

1. An eigenvalue  $\lambda = +1$ : a *saddle-node* bifurcation in which a pair of steady states, one stable and one saddle, is created; or a *pitchfork* bifurcation in which two additional steady states are created.
2. An eigenvalue  $\lambda = -1$ : a *period doubling* or *flip* bifurcation, in which a 2-cycle is created.
3. A pair of complex eigenvalues on the unit circle: a *Hopf* or *Neimark–Sacker* bifurcation, in which an invariant circle with periodic or quasi-periodic dynamics is created.

For our nonlinear asset pricing model, we will present a stability analysis of the steady states, and show that, depending upon the types of traders in the market, a pitchfork, a flip or a Hopf bifurcation can occur as the primary bifurcation, as the intensity of choice to switch predictors increases. The stability of cycles created in these bifurcations depends upon higher-order derivatives at the steady state. In order to investigate this stability, in higher-dimensional systems one has to use a so-called center manifold reduction and compute the corresponding normal form of the bifurcation. However, in concrete examples such as our asset pricing model, this is a highly complicated procedure; see, e.g. Kuznetsov (1995) as a general reference on applied bifurcation theory. Fortunately, in practice stable cycles created in the bifurcations may easily be detected by numerical simulations, especially by plotting phase diagrams.

Secondary and subsequent bifurcations of higher-order cycles are in general difficult to detect analytically. Furthermore, there is a simple, systematic way of finding stable cycles numerically, by means of a so-called *bifurcation diagram*. In a bifurcation diagram, the long-run behavior of the model is plotted as a (multi-valued) function of a parameter. Such a diagram can be obtained by plotting say 500 points of an orbit, after a transient of say 100 periods, for say 1000 equally spaced parameter values in the parameter interval under consideration (see the next section for examples). In addition, one may plot phase diagrams for parameters of interest as suggested in the bifurcation diagram, and see whether in the phase space, e.g., strange attractors, with a complicated fractal structure, can be observed for these parameters.

Phase diagrams and bifurcation diagrams are thus simple, numerical simulation techniques, revealing already a lot of information about the global dynamics. For example, an infinite sequence of period doublings is one of the possible theoretical routes to chaos which may be discovered in such numerical simulations. However, additional numerical analysis should be undertaken in order to investigate whether indeed, as suggested by the phase and bifurcation diagrams,

strange attractors arise. This can be done by numerical computation of the Lyapunov characteristic exponents (LCEs) and the fractal dimension.

LCEs measure the average rate of divergence (or convergence) of nearby initial states, along an attractor in several directions. The largest *Lyapunov characteristic exponent* can be defined as

$$\lambda_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\|(D_x F^n) \delta x\|), \quad (3.1)$$

where  $F$  is an  $n$ -dimensional map,  $D_x F^n$  is the jacobian of the  $n$ th iterate of the map at the point  $x$ ,  $\delta x$  is an initial perturbation vector and  $\| \cdot \|$  denotes the length of a vector in  $\mathbb{R}^n$ . For an  $n$ -dimensional system  $x_{t+1} = F(x_t)$  there exist  $n$  distinct Lyapunov characteristic exponents, ordered as  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ , each measuring the average expansion or contraction along an orbit in the different directions; for a formal definition of LCEs and theoretical background, see, e.g. Eckmann and Ruelle (1985), Guckenheimer and Holmes (1983), (pp. 283–288) or Brock (1986).

Attractors may be characterized by their Lyapunov spectrum. For a stable steady state or a stable cycle, all LCEs are negative. For a quasi-periodic attractor which is topologically equivalent to a circle, the largest LCE  $\lambda_1 = 0$ , whereas all other LCEs are negative. An attractor is called a *strange* or a *chaotic* attractor, if the corresponding largest LCE  $\lambda_1 > 0$ , implying *sensitive dependence upon initial conditions*. Nearby initial states converging to the strange attractor typically diverge at an exponential rate  $\lambda_1$ . For our nonlinear asset pricing model, we compute the LCEs using algorithms described in Benettin et al. (1980a,b).

The notion *strange* attractor refers to the often complicated, fractal geometric structure of attractors exhibiting sensitive dependence. This complicated geometric structure can be quantified by the notion of *fractal dimension*. In the literature, there exist several definitions of fractal dimension; for an overview see, e.g., Falconer (1990). Here we concentrate on the frequently used *box counting* or *capacity* dimension. Let  $\mathcal{A} \subseteq \mathbb{R}^n$  and  $N(\varepsilon)$  the minimum number of  $n$ -dimensional cubes with side  $\varepsilon$  needed to cover  $\mathcal{A}$ . The box counting dimension is defined as

$$D_b = \lim_{\varepsilon \rightarrow 0} \frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}. \quad (3.2)$$

If the set  $\mathcal{A}$  is a line segment, then  $D_b = 1$ ; if  $\mathcal{A}$  is a (filled) square, then  $D_b = 2$ , etc. However, in general  $D_b$  can take noninteger values. In particular, nonlinear dynamical systems often can have attractors with noninteger fractal (e.g., box counting) dimension. A numerical procedure for computing the box counting dimension of an attractor is now straightforward. Take an initial state converging to the attractor, and compute say 10,000 points of that trajectory after some

transient. Next compute  $N(\varepsilon)$  over a range of small  $\varepsilon$ -values and estimate the slope of a straight line by OLS-regression in a  $\log(N(\varepsilon)) - \log(1/\varepsilon)$  plot. In our nonlinear asset pricing model, using the chaotic data analyzer (CDA) by Sprott and Rowlands (1992), we find strange attractors with box counting dimensions close to 2.

#### 4. Some simple belief types

In this section we investigate some simple, but typical examples of our asset pricing model with a few, i.e., two, three or four, different types of beliefs; Brock and Hommes (1997c) develop a notion of ‘large type limit’ (LTL) to study the asset pricing model with many different belief types. Our aim here is to investigate the role of simple belief types in deviations from fundamentals and the primary and secondary bifurcations in possible routes to complex asset price dynamics. All beliefs will be of the simple, linear form

$$f_{ht} = g_h x_{t-1} + b_h, \quad (4.1)$$

where  $g_h$  is the *trend* and  $b_h$  is the *bias* of trader type  $h$ . If  $b_h = 0$ , we call agent  $h$  a pure *trend chaser* if  $g > 0$  (strong trend chaser if  $g > R$ ) and a *contrarian* if  $g < 0$  (strong contrarian if  $g < -R$ ). If  $g_h = 0$ , type  $h$  is said to be purely biased (upward resp. downward biased if  $b_h > 0$  resp.  $b_h < 0$ ). The simple predictors (4.1) could be considered as the simplest idealization of overreacting securities analysts or overreacting investors, as in De Bondt and Thaler (1985), (cf. Thaler, 1994, Part Five). In the special case  $g_h = b_h = 0$ , Eq. (4.1) reduces to *fundamentalists*, believing that prices return to their fundamental value. Fundamentalists do have all past prices and dividends in their information set, but they do *not* know the fractions  $n_{h,t}$  of the other belief types. We will consider one example with a group of *rational* agents, i.e., with predictor

$$f_{Rt} = x_{t+1}. \quad (4.2)$$

Rational agents have perfect foresight. At each date they know not only all past prices and dividends, but also the market equilibrium equation, with all fractions  $n_{h,t}$  of other belief types. Rational agents are thus able to compute  $x_{t+1}$  perfectly. Let us first derive the fitness measure for the simple belief types (4.1) and (4.2). Rewriting Eq. (2.3) in deviations form yields the demand for shares by type  $h$ ,  $z_{h,t-1} = E_{h,t-1}(p_t + y_t - Rx_{t-1})/(a\sigma^2) = (f_{h,t-1} - Rx_{t-1})/(a\sigma^2)$ , by Assumption 1. Realized profits  $\pi_{j,t-1} = R_t z_{h,t-1}$  in Eq. (2.12) for type  $h$  in Eq. (4.1) are then given by, recalling that we have set  $\delta_{t+1} \equiv 0$ , for all  $t$ ,

$$\pi_{j,t-1} = \frac{1}{a\sigma^2} (x_t - Rx_{t-1})(g_h x_{t-2} + b_h - Rx_{t-1}), \quad (4.3)$$

and for the rational agents in Eq. (4.2) by

$$\pi_{R,t-1} = \frac{1}{a\sigma^2}(x_t - Rx_{t-1})^2 - C, \quad (4.4)$$

where  $C \geq 0$  are costs for rational expectations. Accumulated past profits are

$$U_{j,t-1} = \pi_{j,t-1} + \eta U_{j,t-2}, \quad 0 \leq \eta \leq 1, \quad (4.5)$$

where  $\eta$  denotes memory strength. In most cases studied below (except for the important case of rational agents versus trend chasers) we will concentrate on the polar case of  $\eta = 0$ , where the fitness measure is last periods profit.

Brock and Hommes (1997c) show that, for the risk adjusted performance measure, the EMH holds in the asset pricing model with heterogeneous beliefs, i.e., if memory is infinite, i.e.,  $\eta = 1$ , and costs for rational expectations are zero, then all bounded deterministic deviations  $\{x_t\}$  converge to zero as  $t \rightarrow \infty$ . To put it another way, overreacting investors and/or securities analysts would be driven out of the market in an infinite memory world where rational expectations are costlessly available. But since it is argued by Thaler and others that such investors are present in real markets (cf. Thaler (1994) and his references), we should study what kinds of relaxations of perfect rationality can lead to survival of ‘boundedly rational’ traders in equilibrium. In this section, we start tackling this problem for the non risk adjustment fitness measure (2.12), i.e., with realized profits as the performance measure. We investigate several cases to see which predictors may survive the evolutionary equilibrium dynamics.

#### 4.1. Two belief types

We investigate three simple, but typical cases with two belief types.

##### 4.1.1. Perfect foresight versus trend chaser

As a first example consider the case where type 1 are rational agents and type 2 pure trend chasers. We get from Eq. (2.9b)

$$Rx_t = n_{1,t-1}x_{t+1} + n_{2,t-1}gx_{t-1}, \quad (4.6a)$$

and using Eqs. (4.3), (4.4) and (4.5) the updated fractions  $n_{ht}$  are given by

$$n_{1t} = \exp \left[ \beta \left( \frac{1}{a\sigma^2}(x_t - Rx_{t-1})^2 + \eta U_{1,t-2} - C \right) \right] / Z_t \quad (4.6b)$$

$$n_{2t} = \exp \left[ \beta \left( \frac{1}{a\sigma^2}(x_t - Rx_{t-1})(gx_{t-2} - Rx_{t-1}) + \eta U_{2,t-2} \right) \right] / Z_t \quad (4.6c)$$

where  $U_{1t}$  and  $U_{2t}$  are accumulated profits of the two types,  $C \geq 0$  is the cost one must pay each period to obtain rational expectations and  $Z_t$  is a normalization factor as before. Recall the dating convention on which lagged profits are

inserted into the fitness function. In Eq. (4.6a) the old fractions  $n_{j,t-1}$  are used in the equilibrium equation. After the new resulting equilibrium price  $x_t$  has been revealed by Eq. (4.6a), and observed by all agents, it is used in the fitness measure to update the fractions in Eqs. (4.6b) and (4.6c). It will be convenient to introduce the difference in fractions  $m_t$  and a simple computation shows that

$$m_t = n_{1t} - n_{2t} = \tanh\left(\frac{\beta}{2}\left[\frac{1}{a\sigma^2}(x_t - Rx_{t-1})(x_t - gx_{t-2}) + \eta(U_{1,t-2} - U_{2,t-2}) - C\right]\right). \quad (4.6d)$$

The presence of  $x_{t+1}$  on the RHS of Eq. (4.6a) prevents direct application of dynamical systems theory because our system has a ‘forward looking’ element in it. However, it is straightforward to analyze steady states and linearizations. This will be enough to show how the presence of a positive cost  $C > 0$  can generate incentives for traders to ‘free ride’ on those who pay the expenses of attaining rational expectations. Brock and Hommes (1997a) show that in an adaptive beliefs cobweb demand–supply system, a high intensity of choice to switch between a costly rational expectations predictor and a ‘free ride’ naive predictor, leads to market instability and chaotic price fluctuation. In the asset pricing model, depending upon the configuration of types of trading strategies this economic force can generate instability of the fundamental steady state  $E_1$ , with  $x = 0$ , and existence of additional non-fundamental steady states  $E_2$  and  $E_3$ :

*Lemma 1* (Existence of steady states of Eqs. (4.6a), (4.6b), (4.6c) and (4.6d)). Assume  $\eta < 1$ . Let  $D = 1/(a\sigma^2)$ ,  $m^{\text{eq}} = \text{Tanh}(-\beta C/2(1 - \eta))$ ,  $m^* = 1 - 2(R - 1)/(g - 1)$  and  $x^*$  be the positive solution (if it exists) of

$$\text{Tanh}\left[\frac{\beta}{2(1 - \eta)}[D(g - 1)(R - 1)(x^*)^2 - C]\right] = m^*.$$

1. For  $g < R$ ,  $E_1 = (0, m^{\text{eq}})$  is the unique steady state.
2. For  $g > 2R - 1$ , there exist three steady states  $E_1 = (0, m^{\text{eq}})$ ,  $E_2 = (x^*, m^*)$  and  $E_3 = (-x^*, m^*)$ .
3. For  $R < g < 2R - 1$  there are two possibilities:
  - 3.1. if  $m^* < m^{\text{eq}}$  then  $E_1$  is the unique steady state.
  - 3.2. if  $m^* > m^{\text{eq}}$  then there are three steady states  $E_1$ ,  $E_2$  and  $E_3$ .

According to the lemma, when the trendchasers extrapolate only weakly ( $0 < g < R$ ), the fundamental steady state  $E_1 = (0, \text{Tanh}(-\beta C/2))$  is the unique steady state. If there are no costs ( $C = 0$ ) for rational expectations, half of the

traders are of type 1 and half of the traders are of type 2 for any  $\beta$ . When the trendchasers extrapolate very strongly ( $g > 2R - 1$ ) there are two additional nonfundamental steady states  $E_2$  and  $E_3$ , even when there are no costs for rational expectations. Hence, even when there are no information costs and memory is strong ( $\eta$  is close to 1) rational agents do not drive out very strong trendchasers. The case of strongly extrapolating trendchasers ( $R < g < 2R - 1$ ) and positive information costs ( $C > 0$ ) is the most interesting. For  $\beta = 0$ ,  $m^{\text{eq}} = 0 > m^*$ , whereas for large  $\beta$ ,  $m^{\text{eq}} \approx -1 < m^*$ . Hence, as the intensity of choice  $\beta$  increases, a (pitchfork) bifurcation occurs in which two additional steady states  $E_2 = (p^*, m^*)$  and  $E_3 = (-p^*, m^*)$  are created. Notice that, the nonfundamental steady states  $\pm x^*$  converge to the fundamental steady state 0, as the intensity of choice  $\beta \rightarrow \infty$  and/or the memory strength  $\eta \rightarrow 1$  from below.

The assumption of perfect foresight, i.e., the term  $x_{t+1}$  on the RHS of Eq. (4.6a), prevents us to analyze the global dynamics, using numerical tools from nonlinear dynamical systems directly. In the next sections we therefore investigate deviations from full rationality by replacing the perfect foresight agents by fundamentalist.

#### 4.1.2. Fundamentalists versus trend chasers

In our second two belief types example, type 1 are fundamentalists, believing that prices return to their fundamental solution  $x_t = 0$ , whereas type 2 believes in a pure trend  $f_{2t} = gx_{t-1}$  (without bias, i.e.,  $b = 0$ ). In the case where memory is only one lag, i.e.,  $\eta = 0$ , the adaptive belief system becomes

$$Rx_t = n_{2,t-1}gx_{t-1}, \quad (4.7a)$$

$$n_{1t} = \exp \left[ \beta \left( \frac{1}{a\sigma^2} Rx_{t-1}(Rx_{t-1} - x_t) - C \right) \right] / Z_t, \quad (4.7b)$$

$$n_{2t} = \exp \left[ \beta \left( \frac{1}{a\sigma^2} (x_t - Rx_{t-1})(gx_{t-2} - Rx_{t-1}) \right) \right] / Z_t. \quad (4.7c)$$

The difference in fractions is now given by

$$m_t \equiv n_{1t} - n_{2t} = \text{Tanh} \left( \frac{\beta}{2} [ -Dgx_{t-2}(x_t - Rx_{t-1}) - C ] \right), \quad (4.7d)$$

where  $D = 1/(a\sigma^2)$ . Here  $C \geq 0$  is the cost of obtaining access to the belief system type 1. This cost  $C$  may be positive because ‘training’ costs must be borne to obtain enough ‘understanding’ of how markets work in order to believe that they should price according to the EMH fundamental. The adaptive belief system Eqs. (4.7a), (4.7b), (4.7c) and (4.7d) is a third-order difference equation or equivalently a three-dimensional system. We are now ready to explore existence and stability of steady states of Eqs. (4.7a), (4.7b), (4.7c) and (4.7d), and primary

and secondary bifurcations as the intensity of choice  $\beta$  varies. The first result on steady states is:

*Lemma 2* (Existence and stability of steady states for Eqs. (4.7a), (4.7b), (4.7c) and (4.7d)). *Let  $m^{\text{eq}} = \text{Tanh}(-\beta C/2)$ ,  $m^* = 1 - 2R/g$  and  $x^*$  be the positive solution (if it exists) of  $\text{Tanh}[(\beta/2)[Dg(R-1)(x^*)^2 - C]] = m^*$ .*

1. *For  $0 < g < R$ ,  $E_1 = (0, m^{\text{eq}})$  is the unique, globally stable steady state.*
2. *For  $g > 2R$ , there exist three steady states  $E_1 = (0, m^{\text{eq}})$ ,  $E_2 = (x^*, m^*)$  and  $E_3 = (-x^*, m^*)$ ; the steady state  $E_1 = (0, m^{\text{eq}})$  is unstable.*
3. *For  $R < g < 2R$  there are two possibilities:*
  - 3.1. *if  $m^* < m^{\text{eq}}$  then  $E_1$  is the unique, globally stable steady state.*
  - 3.2. *if  $m^* > m^{\text{eq}}$  then there are three steady states  $E_1, E_2$  and  $E_3$ ; the steady state  $E_1 = (0, m^{\text{eq}})$  is unstable.*

Hence, when the trendchasers extrapolate only weakly ( $0 < g < R$ ), then the fundamental steady state  $E_1$  is globally stable. If costs  $C = 0$  half of the traders are of type 1 and half of the traders are of type 2 for any  $\beta$ . This makes sense because the difference in profits is zero at  $x = 0$ . Now if  $C > 0$ , we see that the mass on type 1 decreases to zero as  $\beta$  (or  $C$ ) increases to  $+\infty$ . This makes economic sense. There is no point in paying any cost in a steady state for a trading strategy that yields no extra profit in that steady state. As intensity of choice  $\beta$  increases the mass on the most profitable strategy in net terms, increases. When the trendchasers extrapolate very strongly ( $g > 2R$ ) there are two additional nonzero steady states  $E_2$  and  $E_3$ , even when there are no information costs. The case of strongly extrapolating trendchasers ( $R < g < 2R$ ) and positive information costs for the fundamentalists is the most interesting. For  $\beta = 0$ ,  $m^{\text{eq}} = 0 > m^*$ , whereas for large  $\beta$ ,  $m^{\text{eq}} \approx -1 < m^*$ . Hence, as the intensity of choice increases, a *pitchfork* bifurcation occurs for some  $\beta = \beta^*$ , in which the fundamental steady state  $E_1$  becomes unstable and two additional non-fundamental (stable) steady states  $E_2 = (p^*, m^*)$  and  $E_3 = (-p^*, m^*)$  are created, one above and one below the fundamental. The next result describes what happens to these nonfundamental steady states, as  $\beta$  further increases.

*Lemma 3* (Secondary bifurcation). *Let  $E_2 = (x^*, m^*)$  and  $E_3 = (-x^*, m^*)$  be the nonfundamental steady states as in Lemma 2. Assume  $R < g < 2R$  and  $C > 0$  and let  $\beta^*$  be the pitchfork bifurcation value. There exists  $\beta^{**}$  such that  $E_2$  and  $E_3$  are stable for  $\beta^* < \beta < \beta^{**}$  and unstable for  $\beta > \beta^{**}$ . For  $\beta = \beta^{**}$ ,  $E_2$  and  $E_3$  exhibit a Hopf bifurcation.*

*4.1.2.1. Numerical analysis: case 1.*  $g = 1.2$ ,  $D = a\sigma^2 = 1.0$ ,  $C = 1.0$ ,  $R = 1.1$ ,  $\beta$  varies. Immediately after the secondary Hopf bifurcation, the model has two

attracting invariant circles around the two (unstable) nonfundamental steady states  $E_2$  and  $E_3$ , as illustrated in Fig. 1a. Immediately after this Hopf bifurcation, the model thus has two co-existing (quasi-)periodic attractors, one above and the other one below the fundamental. An important question is whether, as  $\beta$  further increases, the invariant circles break into strange attractors.

Fig. 1 shows plots of the attractors in the  $(x_t, m_t)$  plane without noise (Fig. 1a and b) and with noise (Fig. 1c and d). The noisy attractors have been obtained by using a stochastic dividend process  $y_t = \bar{y} + \varepsilon_t$ , with IID noise  $\varepsilon_t$ , uniformly

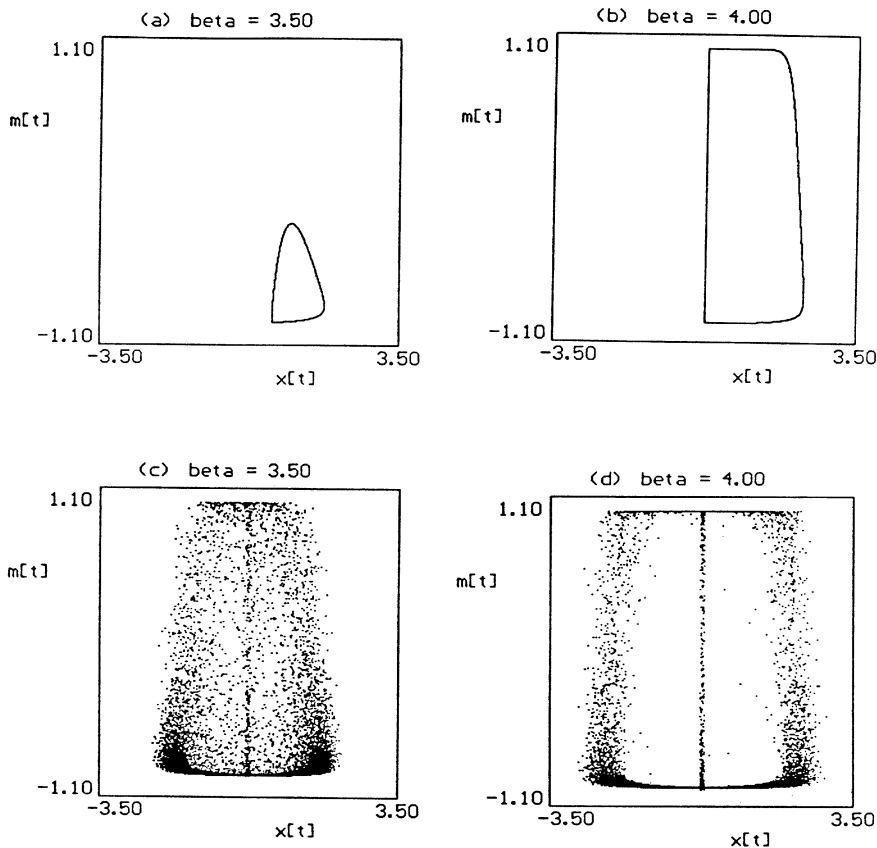


Fig. 1. Trend versus fundamentalists: attractors without noise (a–b) and with noise (c–d). The noisy attractors are generated by the heterogeneous agent model with an IID stochastic dividend process  $y_t = \bar{y} + \varepsilon_t$ , with  $\varepsilon_t$  uniformly distributed on the interval  $[-0.05, 0.05]$ . (a) the attracting invariant circle around the positive unstable nonfundamental steady state; there is a second attracting invariant circle (not shown) around the negative unstable nonfundamental steady state; (b) both attractors (only one is shown) have moved close to the stable manifold  $x \equiv 0$  of the fundamental steady state  $E = (0, \text{Tanh}(-\beta C/2))$ .



distributed on the interval  $[-0.05, 0.05]$ , added to the constant dividend process  $\bar{y}$ . In Fig. 1a, the orbit converges to the attracting invariant ‘circle’ above the fundamental, created after the secondary Hopf bifurcation of the nonfundamental steady states. In Fig. 1b, the invariant circle has grown and roughly has the shape of a ‘square’. In order to understand the dynamics, it will be useful to consider the limiting case  $\beta = +\infty$ , where in each period all agents choose the optimal predictor. We have the following:

*Lemma 4.* Assume  $C > 0$  and  $\beta = +\infty$ . For  $g > R$ , the steady state  $E_1 = (0, -1)$  is locally unstable, with eigenvalues 0 and  $g/R$ . There are two possibilities for the unstable manifold  $W^u(E_1)$ :

1. if  $g > R^2$ , then  $W^u(E_1)$  equals the unstable eigenvector  $[(g/R)^2, g/R, 1]^T$ ;
2. if  $R < g < R^2$  then  $W^u(E_1)$  is bounded; all orbits converge to the (locally unstable) steady state  $E_1$ .

This lemma reveals important insight in the case when the intensity of choice  $\beta$  to switch predictors is large but finite. For  $g > R^2$ , a typical initial state will diverge to infinity. However, for  $R < g < R^2$ , we expect all orbits to remain bounded, moving away from the unstable steady state and later on returning very close to the unstable EMH fundamental. In fact, for  $\beta$  large the system must be close to having a *homoclinic point*, a notion already introduced by Poincaré at the end of last century. Homoclinic points are nowadays recognized as a key feature of chaotic dynamics. Let us briefly recall this important notion. Let  $p$  be a saddle point steady state (or periodic saddle). A point  $q$  is called a homoclinic point if  $q \neq p$  is an intersection point between the stable and the unstable manifolds of  $p$ . For our asset pricing model, the stable manifold  $W^s(E_1)$  contains the vertical segment  $x = 0$ , whereas the unstable manifold  $W^u(E_1)$  moves to the right and then ‘folds back’ close to the stable manifold, as suggested in Fig. 1b. From Lemma 4(2) it follows in fact that for  $R < g < 2R$  and  $\beta = +\infty$  there exist homoclinic points. Therefore, for  $\beta$  large but finite the model must be close to having (transversal) homoclinic points. It is well known that homoclinic orbits imply very complicated dynamical behavior and possibly existence of strange attractors for a large set of parameter values. However, because our system is 3-D, applying recent homoclinic bifurcation theory (Palis and Taken (1993)), as was done in Brock and Hommes (1997a) for the 2-D cobweb adaptive belief system with rational versus naive expectations, is much more delicate. Therefore, we investigate the occurrence of strange attractors by numerical analysis.

Fig. 2 shows some corresponding time series with and without noise. Prices are characterized by switching between an unstable phase of an upward (or downward) trend and a stable phase with prices close to their fundamental value. In the noise free case, this switching seems to be fairly regular. In the

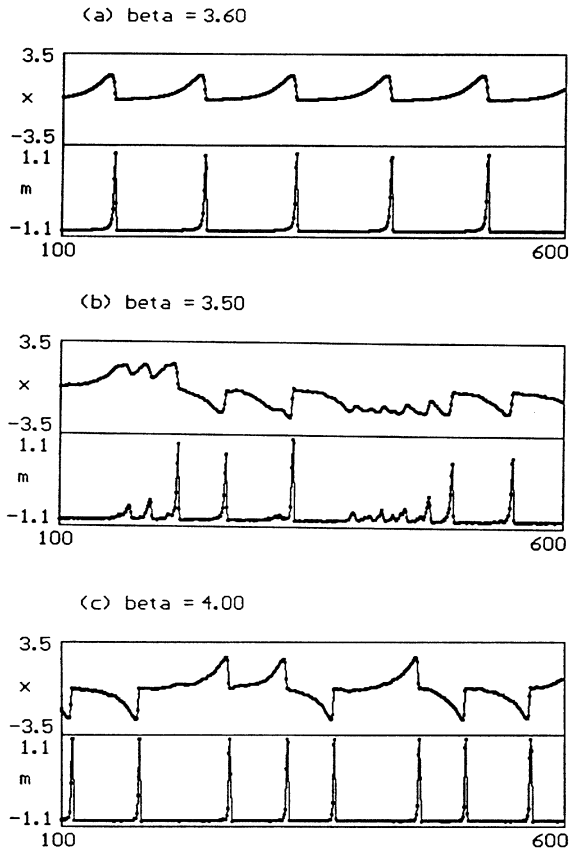


Fig. 2. Trend versus fundamentalists: time series of (deviations from steady state) prices and difference in fractions, without noise (a) and with noise added to the dividend process (b–c).

presence of small noise added to the dividend process however, the switching becomes highly irregular and unpredictable. Periods of ‘optimism’ with above fundamental price fluctuations and periods of ‘pessimism’ with below fundamental price fluctuations interchange irregularly; the higher the intensity of choice, the more frequently switching between ‘optimistic’ and ‘pessimistic’ phases occurs in the case with noise (cf. Fig. 2b and c).

Fig. 3 shows a bifurcation diagram w.r.t. the intensity of choice, suggesting periodic and quasi-periodic dynamics after the primary Hopf bifurcation. Fig. 3b shows the corresponding largest LCE plot, which becomes (slightly) positive for  $\beta > 3.55$ . In a market consisting of fundamentalists and trend chasers, a high intensity of choice thus leads to weakly chaotic asset prices fluctuations, which are highly sensitive to noise, with an irregular switching

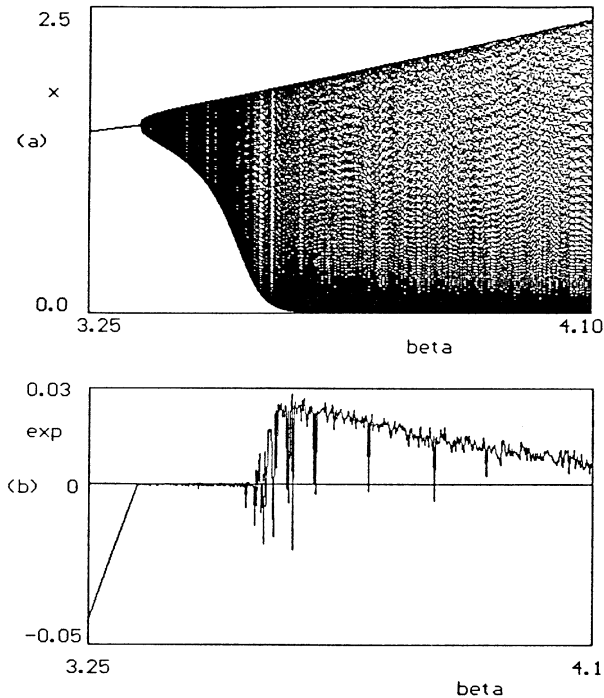


Fig. 3. Trend versus fundamentalists. (a) bifurcation diagram, with periodic and quasi-periodic dynamics after the primary Hopf bifurcation; (b) largest LCE-plot, with (quasi-)periodic dynamics immediately after the Hopf bifurcation and weakly chaotic dynamics for  $\beta > 3.55$ .

between close to the EMH fundamental prices and upward and downward trends.

4.1.3. Fundamentalists versus contrarians

In our third and final two belief types example, type 1 are fundamentalists and type 2 are contrarians, with predictor  $f_{2t} = gx_{t-1}$ , with  $g < 0$ . The adaptive belief system is thus identical to Eqs. (4.7a), (4.7b), (4.7c) and (4.7d), but now with  $g < 0$ .

*Lemma 5* (Existence of steady state and 2-cycle for Eqs. (4.7a), (4.7b), (4.7c) and (4.7d) with  $g < 0$ ). Assume  $g < 0$ . Let  $m^{eq} = \text{Tanh}(-\beta C/2)$ ,  $m^* = 1 + 2R/g$  and let  $x^*$  be the positive solution (if it exists) of  $\text{Tanh}[(\beta/2) [-Dg(R+1)(x^*)^2 - C]] = m^*$ .

1. The fundamental steady state  $E_1 = (0, m^{eq})$  is the unique steady state; it is globally stable for  $-R < g (< 0)$ .

2. For  $g < -2R$  the steady state  $E_1 = (0, m^{\text{eq}})$  is unstable and there exists a period two cycle  $\{(x^*, m^*), (-x^*, m^*)\}$ .
3. For  $-2R < g < -R$  there are two possibilities:
  - 3.1. if  $m^* < m^{\text{eq}}$  then  $E_1$  is the unique, globally stable steady state.
  - 3.2. if  $m^* > m^{\text{eq}}$  then the steady state  $E_1$  is unstable and there exists a period two cycle  $\{(x^*, m^*), (-x^*, m^*)\}$ .

Very strong contrarians ( $g < -2R$ ) may thus lead to the existence of a period two cycle, even when there are no costs for fundamentalists. When costs for fundamentalists are positive, strong contrarians ( $-2R < g < -R$ ) may lead to a two cycle. In particular, as the intensity of choice  $\beta$  increases, a *period doubling bifurcation* occurs in which the fundamental steady state becomes unstable and a (stable) two cycle is created, with one point above and the other one below the fundamental. The next result describes what happens to this 2-cycle as  $\beta$  increases further:

*Lemma 6 (Secondary bifurcation).* Let  $\{(x^*, m^*), (-x^*, m^*)\}$  be the two cycle as in Lemma 5. Assume  $-2R < g < -R$ ,  $C > 0$  and let  $\beta^*$  be the period doubling bifurcation value. There exists  $\beta^{**}$  such that the two cycle is stable for  $\beta^* < \beta < \beta^{**}$  and unstable for  $\beta > \beta^{**}$ . For  $\beta = \beta^{**}$  a Hopf bifurcation of the two-cycle occurs.

After the secondary bifurcation, the model has an attractor consisting of two invariant circles around each of the two (unstable) period two points (Fig. 4a), one lying above and the other one lying below the fundamental. Immediately after this Hopf bifurcation, the price dynamics is either periodic or quasi-periodic, jumping back and forth between the two circles. The next result applies to the case  $\beta = +\infty$ .

*Lemma 7.* Assume  $C > 0$  and  $\beta = +\infty$ . For  $g < -R$ , the fundamental steady state  $E_1 = (0, -1)$  is locally unstable, with eigenvalues 0 and  $g/R$ . The unstable manifold  $W^u(E_1)$  is bounded and all orbits converge to this saddle point  $E_1$ .

In particular, Lemma 7 implies that all points of the unstable manifold  $W^u(E_1)$  converge to  $E_1$  and are thus also on the stable manifold  $W^s(E_1)$ . Hence, for  $\beta = +\infty$  the system has homoclinic orbits. We therefore expect that, with strong contrarians ( $g < -R$ ) a high intensity of choice leads to a system which is close to having a homoclinic intersection between the stable and unstable manifolds of the fundamental steady state. This suggests the occurrence of chaos for high intensity of choice.

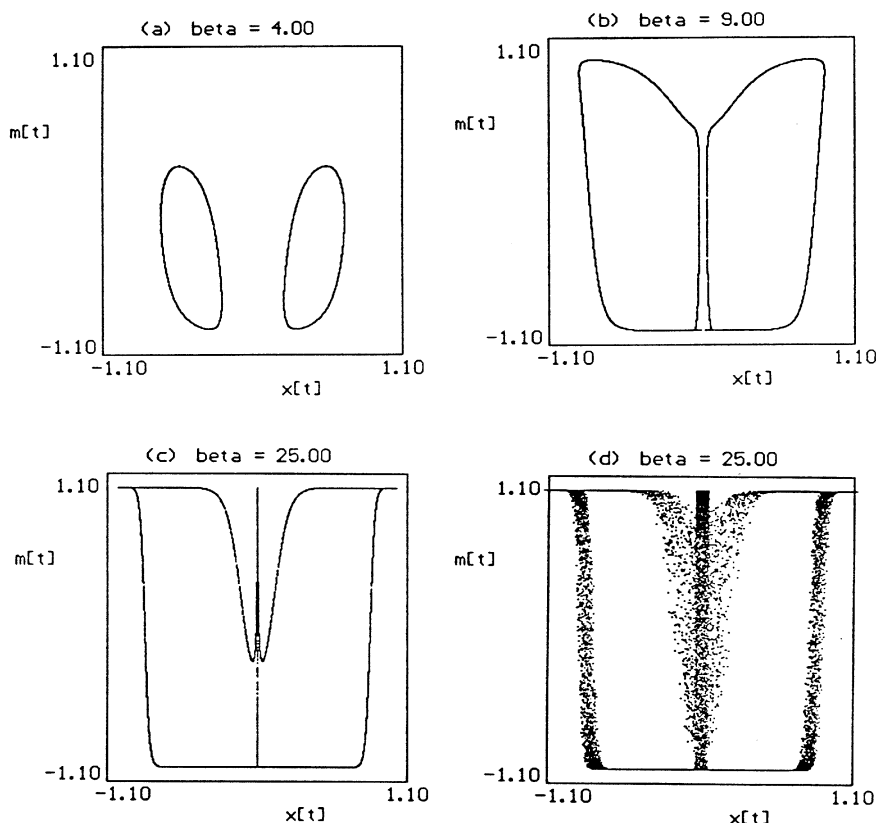


Fig. 4. Contrarians versus fundamentalists: attractors without noise (a–c) and with noise added to dividends (d). (a) The attractor consists of a pair of invariant circles around the unstable 2-cycle; (b–c) the attractors move closer to the stable manifold  $x \equiv 0$  of the fundamental steady state  $E = (0, \tanh(-\beta C/2))$ ; (d) attractor with small noise.

*4.1.3.1. Numerical analysis: Case 2.*  $g = -1.5$ ,  $D = a\sigma^2 = 1.0$ ,  $C = 1.0$ ,  $R = 1.1$ ,  $\beta$  varies. Fig. 4 shows plots of the attractors in the  $(x, m)$  plane without noise (Fig. 4a–c) and with IID noise added to the dividend process (Fig. 4d). In Fig. 4a, the orbit converges to an attractor consisting of the two invariant ‘circles’ created after the secondary Hopf bifurcation of the two cycle. As  $\beta$  increases, the two circles ‘move’ closer to each other. In Fig. 4b–d, the system seems to be already close to having a homoclinic orbit (cf. Lemma 7); the stable manifold  $W^s(E_1)$  contains the vertical segment  $x = 0$ , whereas the unstable manifold  $W^u(E_1)$  has two branches, one moving to the right and one to the left, and then ‘folding back’ close to the stable manifold.

Fig. 5 shows some time series corresponding to the attractors in Fig. 4, without noise (Fig. 5a–d) and with noise added to the dividend process (Fig. 5e).

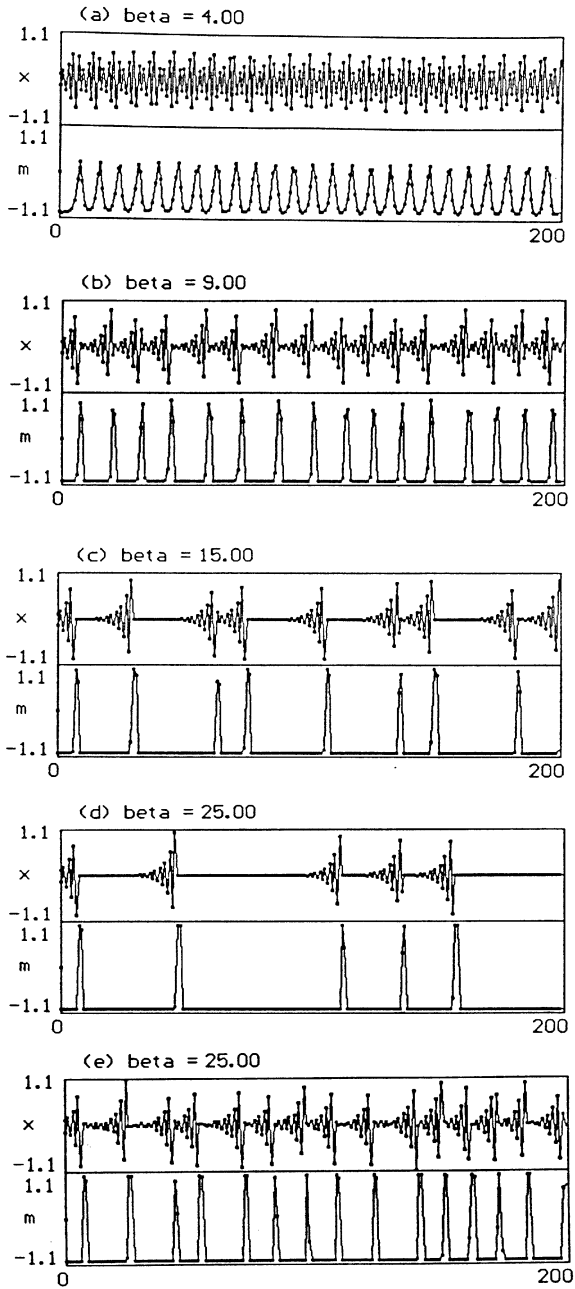


Fig. 5. Contrarians versus fundamentalists: time series of (deviations from steady state) prices and difference in fractions, without noise (a–d) and with noise added to dividends (e).

Prices are characterized by an irregular switching between a stable phase with prices close to their fundamental value and an unstable phase of up and down price oscillations with increasing amplitude. In fact, the dynamical behavior is very similar to the chaotic price fluctuations in the cobweb model with costly rational versus free naive expectations in Brock and Hommes (1997a). In particular, the geometric shape of the (strange) attractors of the 3-D asset pricing model with costly fundamentalism versus contrarians is very similar to the geometric shape of the strange attractors in the 2-D cobweb demand–supply model with costly rational versus naive expectations.

Fig. 6a shows a bifurcation diagram w.r.t. the intensity of choice  $\beta$ . The diagram shows the primary period doubling bifurcation of the steady state, the secondary Hopf bifurcation of the 2-cycle and also suggests breaking of the invariant circle into strange attractors and period doubling routes to chaos. The largest LCE-plot in Fig. 6b shows that  $\lambda_1 > 0$  for  $\beta > 5.8$ , implying that chaos arises. The chaotic region is interspersed with many stable cycles where  $\lambda_1 < 0$ .

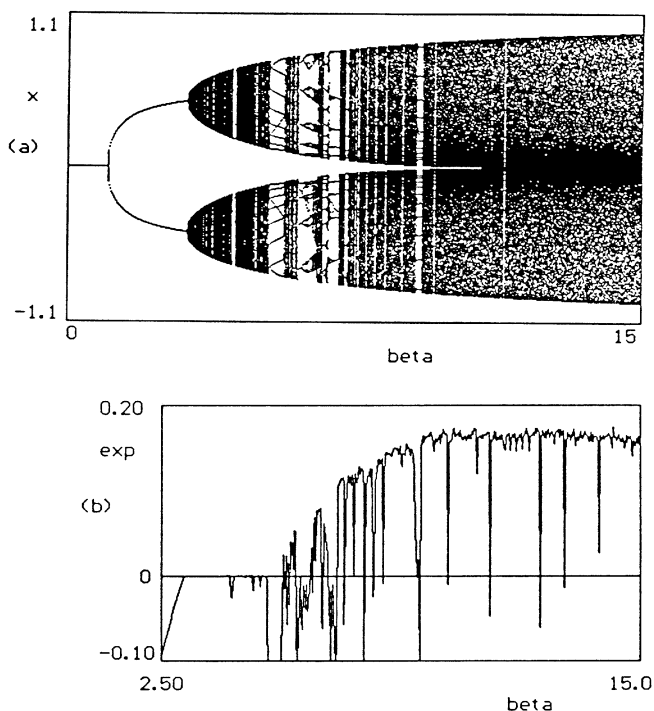


Fig. 6. Contrarians versus fundamentalists. (a) Bifurcation diagram, with primary period doubling and secondary Hopf bifurcation leading to periodic and quasi-periodic and subsequently also chaotic dynamics; (b) largest LCE-plot, with (quasi-)periodic dynamics immediately after the secondary Hopf bifurcation and chaos, interspersed with stable cycles for higher  $\beta$ -values.

In fact, the LCE-plot has a fractal structure. We also computed the box counting dimension of the attractors, which seem to be close to 1. This may be due to the fact that the fundamental steady state has an eigenvalue 0, so there is a strong contraction in the stable direction.

In a market with fundamentalists versus contrarians a high intensity of choice produces chaotic asset price dynamics, with irregular fluctuations around the EMH fundamental. In summary, for our two predictor cases, one can say that the presence of trend chasers or contrarians may lead to market instability and chaos. A rational choice between fundamentalists beliefs and trends or contrarians triggers ‘castles in the air’ and up and down oscillations around the unstable EMH fundamental.

#### 4.2. Three belief types: Fundamentalists versus opposite biases

In this subsection we consider an example with three different belief types. As before, type 1 are fundamentalists, i.e.,  $g_1 = b_1 = 0$ . Types 2 and 3 are purely biased traders, so  $g_2 = g_3 = 0$ ; type 2 is upward biased and type 3 downward biased, i.e.,  $b_2 > 0 > b_3$ . With memory length of one lag, i.e.,  $\eta = 0$ , and no costs for fundamentalism, the adaptive belief system (2.9b)–(2.14) then reduces to

$$Rx_t = n_{2,t-1}b_2 + n_{3,t-1}b_3, \tag{4.8a}$$

$$n_{j,t} = \exp\left(\frac{\beta}{a\sigma^2}(b_j - Rx_{t-1})(x_t - Rx_{t-1})\right)/Z_t, \quad j = 1, 2, 3. \tag{4.8b}$$

System (4.8a) and (4.8b) is thus equivalent to a second-order difference equation in  $x_t$ . The following result describes the stability of the unique steady state:

*Lemma 8.* Assume  $b_2 > 0 > b_3$ . System (4.8a) and (4.8b) has a unique steady state  $E$  which equals the fundamental steady state when  $b_2 = -b_3$ . There exists a  $\beta$ -value  $\beta^*$ , such that  $E$  is stable for  $0 \leq \beta < \beta^*$  and  $E$  is unstable for  $\beta > \beta^*$ . For  $\beta = \beta^*$  the system exhibits a Hopf bifurcation.

Assume for a moment that the biases of types 2 and 3 are exactly opposite, i.e.,  $b_2 = -b_3$ . According to Lemma 4.8, as the intensity of choice to switch predictors increases, the fundamental steady state becomes unstable due to a Hopf bifurcation. In the presence of biased agents, the first step towards complicated price fluctuations is thus different from that in the presence of pure trend chasers or contrarians as in the previous subsection. Next turn to a numerical analysis of what happens as  $\beta$  further increases.

##### 4.2.1. Numerical analysis: Case 3. $b_2 = 0.2, b_3 = -0.2, a\sigma^2 = 1.0, C = 0, R = 1.1, \beta$ varies

Figs. 7 and 8 show some attractors in the  $(x_{t-1}, x_t)$  plane and some corresponding time series. In Fig. 7a the orbit converges to an attracting invariant



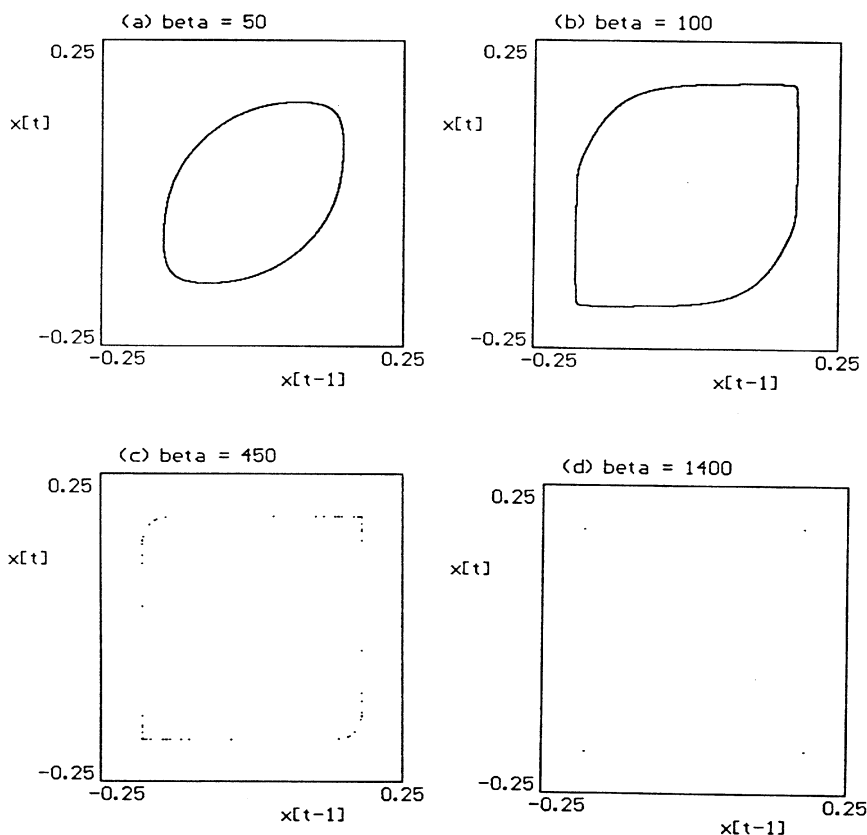


Fig. 7. Purely opposite bias versus fundamentalists: quasi-periodic attractors (a–b) and periodic attractors (c–d).

‘circle’ created in the Hopf bifurcation. As  $\beta$  increases the dynamics remains periodic or quasi-periodic, and the invariant circle slowly changes its shape into a ‘square’. For high values of  $\beta$  orbits converge to a stable 4-cycle. For the limiting case  $\beta = +\infty$ , we indeed have:

*Lemma 9. Assume  $\beta = +\infty$  and biased beliefs are exactly opposite, i.e.,  $b_2 = -b_3 = b > 0$ . System (4.8a) and (4.8) has a stable 4-cycle attracting all orbits, except for hairline cases converging to the unstable fundamental steady state. For all three trader types average profits along the 4-cycle equal  $b^2$ .*

Lemma 9 implies that in a three type world, even when there are no costs and memory is infinite, fundamentalist beliefs cannot drive out opposite purely biased beliefs, when the intensity of choice to switch strategies is high. Hence, the

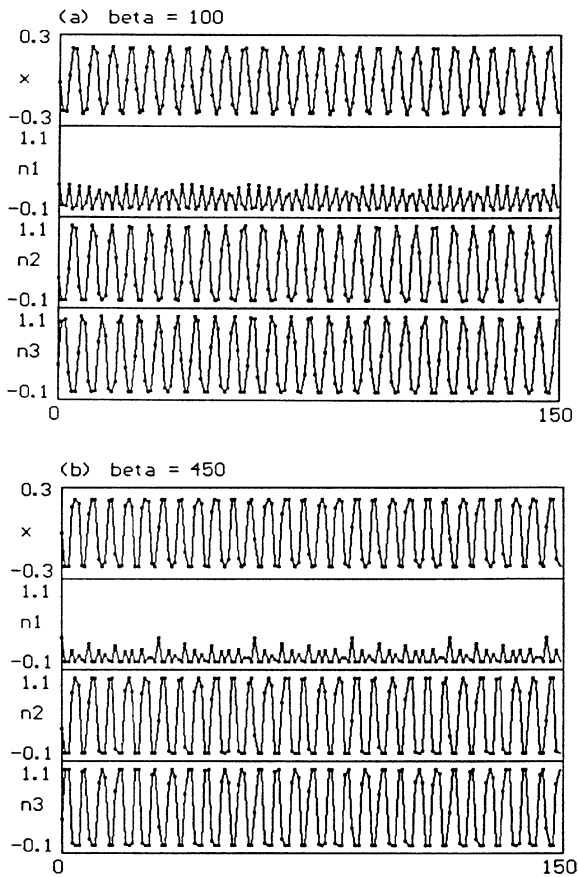


Fig. 8. Purely opposite bias versus fundamentalists: time series of (deviations from steady state) prices and fractions.

market can protect a biased trader from his own folly if he is part of a group of traders whose biases are ‘balanced’ in the sense that they average out to zero over the set of types. Centralized market institutions can make it difficult for unbiased traders to prey on a set of biased traders provided they remain ‘balanced’ at zero. Of course, in a pit trading situation, unbiased traders could learn which types are biased and simply take the opposite side of the trade. This is an example where a centralized trading institution like the New York Stock Exchange could ‘protect’ biased traders, whereas in a pit trading institution, they could be eliminated.

Fig. 9 shows a bifurcation diagram w.r.t.  $\beta$  and the corresponding plot of the largest LCE  $\lambda_1$ , which is either close to 0 (within three digits accuracy) or

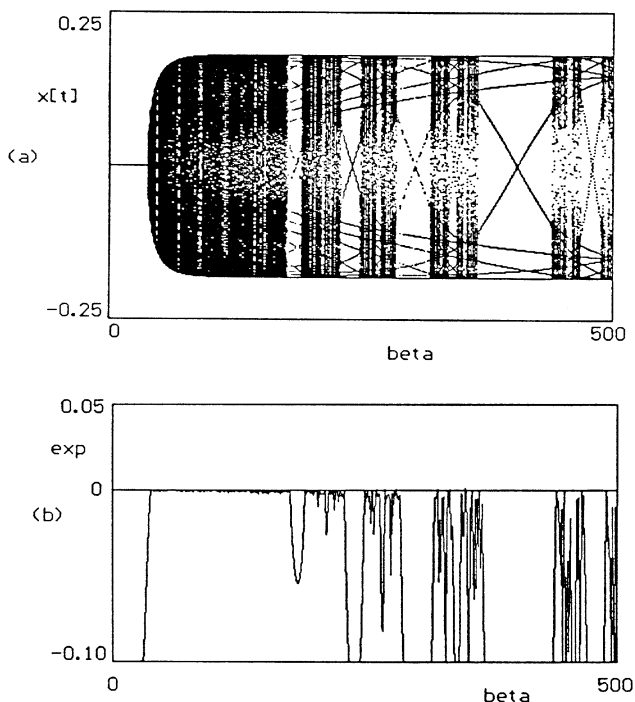


Fig. 9. Purely opposite bias versus fundamentalists. (a) Bifurcation diagram with primary Hopf bifurcation leading to periodic and quasi-periodic dynamics; (b) largest LCE-plot; chaos does not arise.

negative. Hence, in a three type world with fundamentalists and purely biased traders, for high values of the intensity of choice, only regular (quasi-)periodic fluctuations around the unstable fundamental steady state occur. Opposite biases may cause perpetual oscillations around the fundamental, even when there are no costs for fundamentalists, but cannot lead to chaotic movements. Apparently, some (strong) trend chasers or contrarian beliefs are needed to trigger chaotic asset price fluctuations.

#### 4.3. Four belief types: Fundamentalists versus trend versus bias

Finally, we consider an example with four different belief types. As before, type 1 are fundamentalists. Belief parameters for the other three types are:  $g_2 = 1.1$ ,  $b_2 = 0.2$ ;  $g_3 = 0.9$ ,  $b_3 = -0.2$ ;  $g_4 = 1.21$  and  $b_4 = 0$ . Hence, type two is a trend with upward bias, type three a trend with downward bias and type four a strong, pure trend chaser. This example exhibits some typical features observed in many other examples as well. With  $\eta = 0$ , and no costs for the four

predictors, the equilibrium dynamics (2.9b)–(2.14) becomes

$$Rx_t = \sum_{j=1}^4 n_{j,t-1}(g_j x_{t-1} + b_j) \tag{4.9a}$$

$$n_{j,t} = \exp\left(\frac{\beta}{a\sigma^2}(g_j x_{t-2} + b_j - Rx_{t-1})(x_t - Rx_{t-1})\right)/Z_t, \quad j = 1, 2, 3, 4. \tag{4.9b}$$

System (4.9a) and (4.9b) is equivalent to a third-order difference equation in  $x_t$ . We have:

*Lemma 10. With the parameters as above, the fundamental  $x^* = 0$ ,  $n_j^* = 1/4$ , is a steady state of Eqs. (4.9a) and (4.9b). This fundamental steady state is stable for  $0 < \beta < 50$  and unstable for  $\beta > 50$ . At  $\beta = 50$  a Hopf bifurcation occurs.*

Fig. 10 shows some attractors projected into the  $(x_t, x_{t-1})$  plane for different values of the intensity of choice  $\beta$  and Fig. 11 some corresponding time series without noise (Fig. 11a) and with IID noise added to the dividend process (Fig. 11b). For low values of the intensity of choice a stable steady occurs. As  $\beta$  increases, the steady state becomes unstable due to a Hopf bifurcation and an ‘invariant circle’ with quasi-periodic dynamics arises (Fig. 10a). As  $\beta$  further increases, the invariant circle breaks up into a strange attractor (Fig. 10b–d). Fig. 11a shows chaotic time series for both  $x_t$  and the four fractions  $n_{jt}$ . Chaos is characterized by an irregular switching between a stable phase with prices close to the fundamental and an unstable phase of an upward trend where most agents are of type 2. For a very high intensity of choice ( $\beta > 94$ ), at some point almost all traders become fundamentalists, driving prices back to their EMH fundamental steady state. In our numerical simulations, the price then gets ‘stuck’ into the locally unstable fundamental steady state. Fig. 11b shows however, that a small amount of noise added to the dividend process leads to erratic fluctuations around the fundamental again. Also notice that small noise added to the dividend process triggers temporary speculative bubbles more frequently than in the noise free case. The simulations suggest that, as in the two predictor cases in Section 4.1, for high values of the intensity of choice, the system is close to having a homoclinic intersection between the stable and unstable manifolds of the fundamental steady state.

Fig. 12 shows a bifurcation diagram w.r.t.  $\beta$  and the corresponding largest LCE-plot. Positive largest LCEs occur for  $\beta > 85$ , implying the occurrence of chaotic price fluctuations in the four type case. The chaotic price fluctuations exhibit a mixture of the types of behavior that have been observed in the two and three predictor cases before: a switching between close to EMH fundamental prices, oscillations around the fundamental and

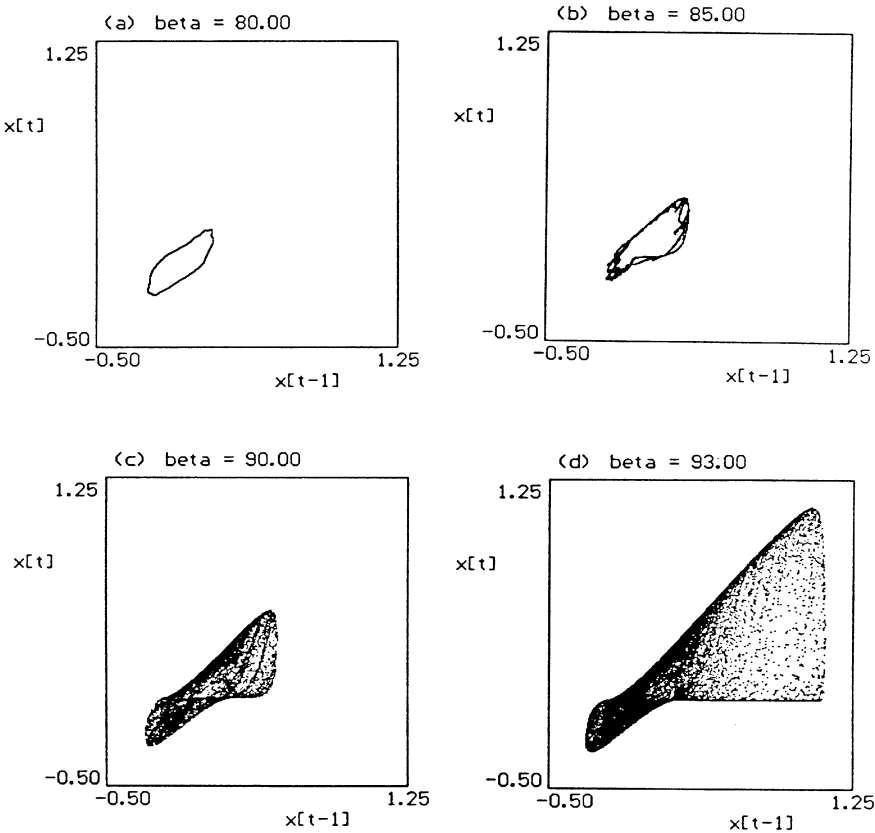


Fig. 10. Four belief types. Quasi-periodic attractor (a) and breaking of the invariant circle into strange attractors, with box counting dimension and largest LCE: (b)  $D_b = 1.74$ ,  $\lambda_1 = 0.04$ ; (c)  $D_b = 2.07$ ,  $\lambda_1 = 0.10$ ; (d)  $D_b = 1.99$ ,  $\lambda_1 = 0.10$ .

trends at irregular intervals. It is important to note that in this four belief type case this irregular switching occurs for not too strong trends and zero costs for all predictors.

5. Concluding remarks.

We have investigated an asset pricing model with heterogeneous beliefs, where agents select a predictor from a finite set, based upon their past performance as measured by realized profits. If all traders would be identical and rational, the model essentially reduces to Lucas (1978) asset pricing model and under the additional assumption that dividends are IID, the asset price

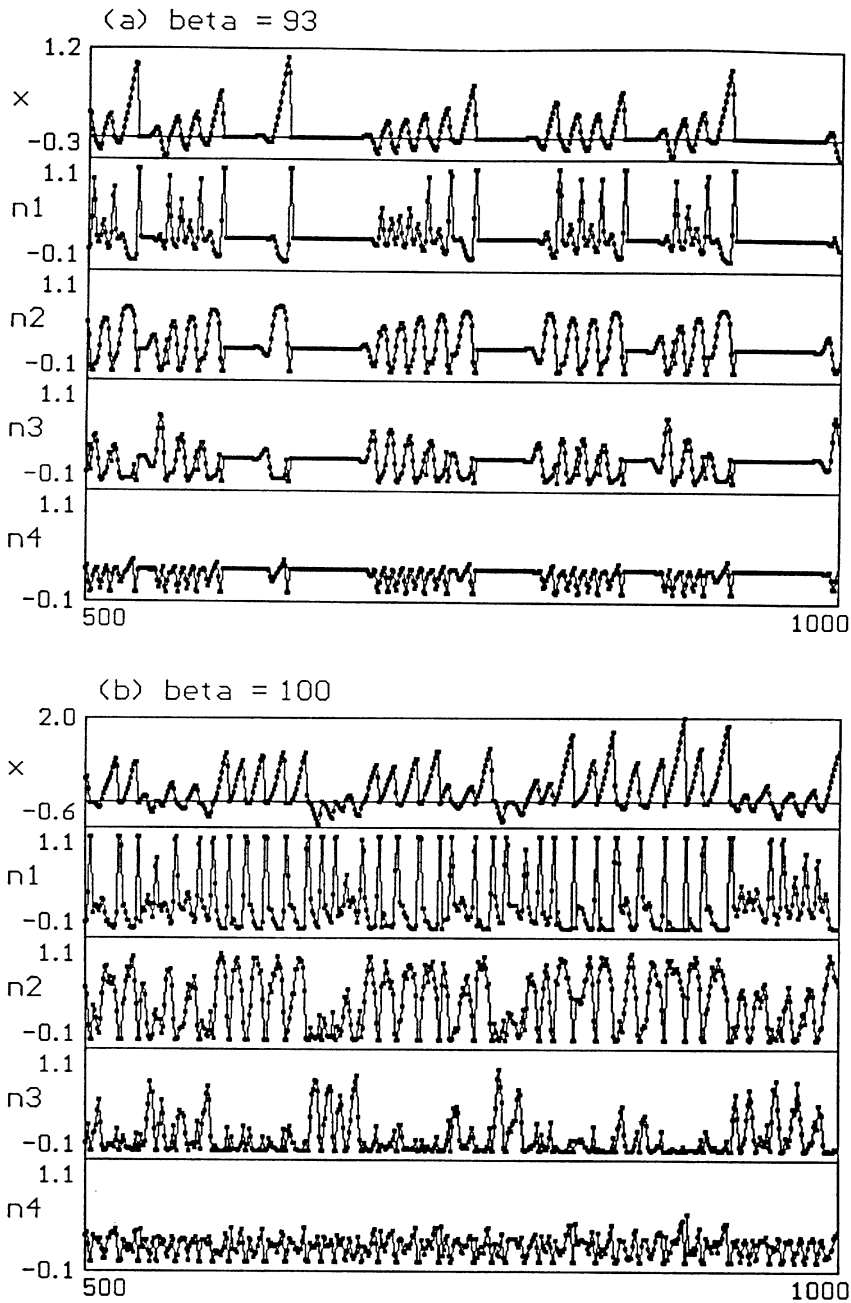


Fig. 11. Four belief types: time series of (deviations from steady state) prices and the four fractions  $n_{j,t}$ , without noise (a–b) and with noise added to dividends (c).

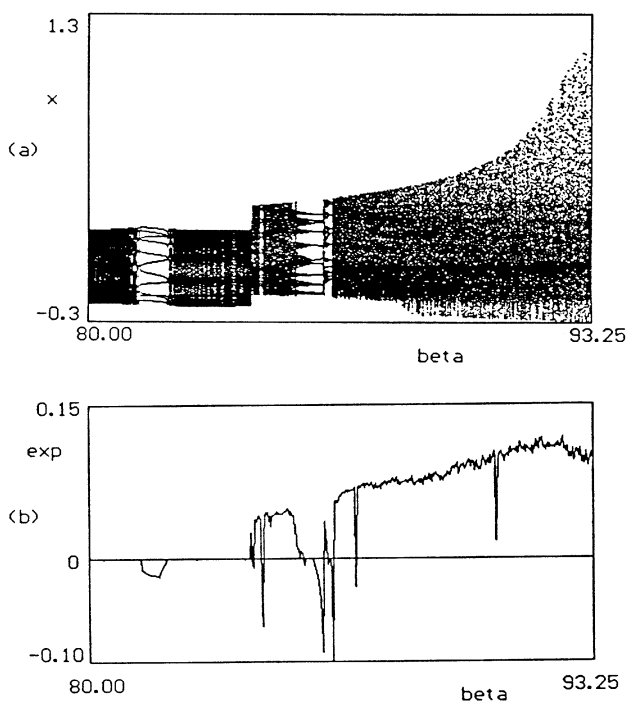


Fig. 12. Four belief types. Bifurcation diagram (a) and corresponding largest LCE-plot (b); chaos arises for  $\beta > 85$ .

dynamics would be extremely simple: one constant price equal to the fundamental at all dates. In contrast, in a heterogeneous agent financial market, the evolutionary dynamics may lead to persistent deviations from the fundamental price, and highly irregular, chaotic asset price fluctuations, when the intensity of choice to switch prediction strategies becomes high.

Several bifurcation routes to complicated asset price fluctuations have been observed, as the intensity of choice to switch predictors increases. The primary bifurcations of the fundamental steady state are closely related to the type of traders that dominate the market, in the following way:

1. Trend chasers give rise to a *pitchfork bifurcation* of the fundamental steady state, in which two additional stable nonfundamental steady states arise, one above and one below the fundamental.
2. Contrarians give rise to a *period doubling bifurcation* to a 2-cycle, with one point above and the other one below the fundamental, with up and down oscillation of prices around the fundamental.
3. Opposite biased beliefs lead to a *Hopf bifurcation* and (quasi-)periodic fluctuations around an unstable fundamental steady state.

The characteristic features of subsequent bifurcation routes to strange attractors, as the intensity of choice increases, are also closely related to the type of traders present in the market. Trend chasers trigger irregular switching between phases of ‘optimism’, with prices following temporary speculative bubbles growing beyond the EMH fundamental, or ‘pessimism’, with prices falling below the fundamental. Contrarians cause a lot of irregular, up and down fluctuations around the fundamental, whereas opposite biases cause oscillatory fluctuations around the fundamental, even when there are no costs for fundamentalists. Our numerical work also suggests that biases alone do not trigger chaotic asset price fluctuations. Sensitivity to initial states and irregular switching between different phases seems to be triggered by trend extrapolators and contrarians. A mixture of fundamentalists, trend chasers, contrarians and biased beliefs leads to strange attractors exhibiting some stylized, qualitative features observed in financial markets, with an irregular switching between phases of close to the EMH fundamental price fluctuations, phases of optimism with upward trends and phases of pessimism with declining asset prices. A key feature of our heterogeneous beliefs model is that the irregular fluctuations in asset prices are triggered by a rational choice, based upon realized profits, in prediction strategies (*Rational Animal Spirits*). In Brock and Hommes (1997b) it has been shown that these rational animal spirits exhibit some quantitative features of asset price fluctuations as well. In particular, the autocorrelation structure of prices and returns in the model is similar to the autocorrelation structure in monthly IBM stock prices and returns.

Our asset pricing model with heterogeneous beliefs is very simple and stylized. In particular, for the case of one risky and one risk-free asset, the equilibrium pricing equation is one dimensional. One may thus question the validity and generality of our results. Do similar results also hold for asset pricing models with more than two assets or even in a general equilibrium framework, with a higher-dimensional equilibrium pricing equation? We cannot answer this question in full generality, but would like to make an important note. In a one-parameter family of higher-dimensional dynamic asset pricing models, there are only a few generic (co-dimension one) bifurcations which can lead to instability of the fundamental steady state. The generic co-dimension one bifurcations are: a saddle node ( $\lambda = +1$ ), a pitchfork ( $\lambda = +1$ ), a period doubling ( $\lambda = -1$ ) or a Hopf bifurcation (complex eigenvalues on the unit circle); see, e.g., Kuznetsov (1995) for a detailed mathematical treatment. We conjecture that also for higher-dimensional asset pricing models, a large fraction of trend chasers typically leads to a pitchfork bifurcation, a large fraction of contrarians to a period doubling bifurcation and a large fraction of opposite biases to a Hopf bifurcation. In addition, possible strange attractors arising in these higher-dimensional versions might bear resemblance to the strange attractors in our simple stylized model. For example, in the presence of trend chasers, one would expect the chaotic asset price fluctuations to be characterized by



temporary speculative bubbles at irregular time intervals. We leave it for future work to see whether this conjecture is true.

There has been a long debate concerning the question whether irrational, speculative traders would be driven out of the market by rational smart money traders. The results in this paper suggest that, in a heterogeneous, many agent nonlinear world, the answer to this question seems to be not as obvious as one might have guessed.

## Acknowledgements

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## Appendix A

*Proof of Lemma 1.* From Eq. (4.6a) we get that a steady state  $(x^*, n_1^*, n_2^*)$  must satisfy

$$Rx^* = n_1^*x^* + n_2^*gx^* = \frac{1+m^*}{2}x^* + \frac{1-m^*}{2}gx^*, \quad m^* = \frac{n_1^* - n_2^*}{2} \quad (\text{A.1})$$

implying  $x^* = 0$  or  $R = (1+m^*)/2 + g(1-m^*)/2$ . Solving the latter for  $m^*$  yields

$$m^* = 1 - 2\frac{R-1}{g-1}. \quad (\text{A.2})$$

The reader may easily check that  $m^* > 1$  for  $g < 1$  and  $m^* < -1$  for  $1 < g < R$ , while  $-1 < m^* < 0$  for  $R < g < 2R-1$  and  $0 < m^* < 1$  for  $g > 2R-1$ . Hence, for  $g < R$ , the only steady state is the fundamental  $x^* = 0$ . From Eqs. (4.3), (4.4) and (4.5) we get the steady state accumulated profits

$$U_1^* = \frac{D(1-R)^2(x^*)^2 - C}{1-\eta} \quad \text{and} \quad U_2^* = \frac{D(1-R)(g-R)(x^*)^2}{1-\eta} \quad (\text{A.3})$$

so that the steady-state difference in fractions

$$m^* = n_1^* - n_2^* = (\exp[\beta U_1^*] - \exp[\beta U_2^*])/Z^* = \text{Tanh}\left(\frac{\beta}{2}(U_1^* - U_2^*)\right),$$

so

$$m^* = \text{Tanh}\left(\frac{\beta}{2(1-\eta)}[D(g-1)(R-1)(x^*)^2 - C]\right). \tag{A.4}$$

The fundamental steady state is then given by  $(0, m^{\text{eq}}) = (0, \text{Tanh}(-\beta C/2(1-\eta)))$  and the nonfundamental steady states are  $(x^*, m^*)$  and  $(-x^*, m^*)$ , where  $m^*$  is given by Eq. (A.2) and  $x^*$  is the positive solution of Eq. (A.4). For  $g > 2R - 1, 0 < m^* < 1$  and Eq. (A.4) always has two solutions  $\pm x^*$ . For  $R < g < 2R - 1, -1 < m^* < 0$  and Eq. (A.4) has solutions  $\pm x^*$  if and only if  $\text{Tanh}(\beta C/2(1-\eta)) = m^{\text{eq}} < m^*$ .  $\square$

*Proof of Lemma 2.* Existence of the steady states is proven in essentially the same way as for Lemma 1, so we concentrate on the stability of the fundamental steady state  $E_1 = (0, m^{\text{eq}})$ . Using  $n_{2,t-1} = (1 - m_{t-1})/2$  and Eq. (4.7d), and substituting into Eq. (4.7a) we get a third order, or equivalently a three dimensional, system. It is easy to derive the characteristic equations. For the fundamental steady state we get a double eigenvalue 0 and a (positive) eigenvalue  $\lambda = (g/R)(1 - m^{\text{eq}})/2$ . From Eq. (4.7a) it is clear that the fundamental steady state is globally stable for  $(0 \leq) g < R$  and locally unstable for  $g > 2R$ . Moreover, the eigenvalue  $\lambda > 1$  if and only if  $m^{\text{eq}} < 1 - 2R/g = m^*$ .  $\square$

*Proof of Lemma 3.* Straightforward computations yield the characteristic equation for the stability of the nonfundamental steady states  $(\pm x^*, m^*)$ :

$$g(\lambda) = \lambda^3 - \lambda^2\left(1 + \frac{b}{R}\right) + b\lambda + b\frac{R-1}{R} = 0, \tag{A.5}$$

where

$$b = \frac{\beta D g^2}{4} (x^*)^2 \phi' \left( \frac{\beta}{2} [Dg(R-1)(x^*)^2 - C] \right), \text{ with } \phi(z) = \tanh(z).$$

At the pitchfork bifurcation value  $\beta = \beta^*, x^* = 0$ , so that  $b = 0$ . For  $\beta = \beta^*$ , Eq. (A.5) has a double eigenvalue 0 and an eigenvalue  $+1$ . For  $\beta$  slightly larger than  $\beta^*$ ,  $b$  is slightly larger than 0 and Eq. (A.5) has three real eigenvalues inside the unit circle. Hence, for  $\beta$  slightly larger than  $\beta^*$ , the nonfundamental steady states  $(\pm x^*, m^*)$  are stable. As  $\beta$  increases,  $b$  also increases and  $b \rightarrow +\infty$  as  $\beta \rightarrow +\infty$ , since  $\phi'((\beta/2)[Dg(R-1)(x^*)^2 - C]) \rightarrow +\infty$ . Hence, one of the

eigenvalues must cross the unit circle at some critical  $\beta = \beta^{**}$  and the nonfundamental steady states  $(\pm x^*, m^*)$  become unstable. Since for all  $\beta > \beta^*$ ,  $g(1) = 2b(1 - 1/R) > 0$  and  $g(-1) = -2 - 2D/R < 0$ , we conclude that for  $\beta = \beta^{**}$  two eigenvalues must be complex, so a Hopf bifurcation occurs for  $\beta = \beta^{**}$ .  $\square$

*Proof of Lemma 4.* From Eq. (4.7d), for  $\beta = +\infty$  we get

$$m_t = \begin{cases} +1 & \text{if } Dg x_{t-2}(R x_{t-1} - x_t) > C, \\ -1 & \text{if } Dg x_{t-2}(R x_{t-1} - x_t) \leq C. \end{cases} \quad (\text{A.6})$$

The fundamental steady state is  $(0, -1)$ , with a double eigenvalue 0 and an unstable eigenvalue  $g/R$  with corresponding eigenvector  $[(g/R)^2, g/R, 1]^T$ . In order to compute the unstable manifold, we iterate this unstable eigenvector. As long as  $m_t = -1$  we get  $x_{t+1} = (g/R)x_t$ , but as soon as  $m_t = +1$  we get  $x_{t+1} = 0$ . Take an initial state  $x_0 = \varepsilon(g/R)^2$ ,  $x_{-1} = \varepsilon(g/R)$ ,  $x_{-2} = \varepsilon$ , with  $\varepsilon > 0$  small,  $m_0 = m_{-1} = m_{-2} = -1$ , on the unstable eigenvector. We get  $x_1 = \varepsilon(g/R)^3$  and the expression determining whether  $m_1 = -1$  or  $m_1 = +1$  is  $C_1 = Dg\varepsilon^2(g/R)^3(R^2 - g)/R$ . As long as  $m_{t-1} = -1$  we get  $x_t = \varepsilon(g/R)^{t+2}$  and the expression determining whether  $m_t = -1$  or  $m_t = +1$  is

$$C_t = Dg\varepsilon^2(g/R)^{2t+1} \left( \frac{R^2 - g}{R} \right). \quad (\text{A.7})$$

From Eq. (A.7) it is clear that for  $g > R^2$ ,  $C_t < 0 < C$  for all  $t$ , and therefore  $m_t = -1$  for all  $t$ . Hence, for  $g > R^2$  the unstable manifold is simply the unstable eigenvector and thus unbounded. For  $R < g < R^2$ , it is clear from Eq. (A.7) that for some smallest  $T > 0$ , we get  $C_T > C$ , so that  $m_T = +1$ . It is then easily verified that  $x_t = 0$ , for all  $t \geq T + 1$ ,  $m_{T+1} = +1$  and  $m_t = -1$ , for all  $t \geq T + 2$ . Hence, for  $R < g < R^2$  the unstable manifold is bounded and all orbits converge to the (saddle-point) fundamental steady state  $(0, -1)$ .  $\square$

*Proof of Lemma 5.* From Eq. (4.7a), with  $g < 0$ , it is immediately clear that  $x = 0$  is the only steady state and that it is globally stable for  $-R < g < 0$ . For  $g < -2R$ , from Eq. (4.7a) and the fact that the steady state fraction  $n_2^{\text{eq}}$ , it is also clear that the steady state is unstable. Moreover, for  $g < -2R$ ,  $0 < m^* < 1$ , so that  $\tanh[(\beta/2)[-Dg(R+1)x^2 - C]] = m^*$  has two solutions  $x^*$  and  $-x^*$ . Since  $(g/R)(1 - m^*)/2 = -1$ , it then follows from Eqs. (4.7a) and (4.7d) that  $\{(x^*, m^*), (-x^*, m^*)\}$  is a period two cycle. Finally, for  $-2R < g < -R$ , the fundamental steady state is stable and  $\tanh[(\beta/2)[-Dg(R+1)x^2 - C]] = m^*$  has solutions  $\pm x^*$  if and only if  $m^* > m^{\text{eq}} = \tanh(-\beta C/2)$ .  $\square$

*Proof of Lemma 6.* The third-order system (4.7a)–(4.7d) is equivalent to the 3-D system  $(x_{t+1}, y_{t+1}, z_{t+1}) = F(x_t, y_t, z_t) = (y_t, z_t, (g/R)((1 - m_t)/2)z_t)$ , with

$m_t = \tanh[(\beta/2)[-Dgx_t(z_t - Ry_t) - C]]$ . Let  $T = -I$ , with  $I$  the identity matrix. The system is symmetric w.r.t. the origin, i.e.  $FT = TF$ . Define  $G = FT$ , then the two points  $\{(x^*, -x^*, x^*), (-x^*, x^*, -x^*)\}$  of the period two orbit of  $F$  are fixed points of  $G$ . Moreover, the stability of the two cycle of  $F$  is determined by the Jacobian matrix  $JG(x^*, -x^*, x^*) = -JF(x^*, -x^*, x^*)$ . A simple computation shows that the characteristic equation for the eigenvalues of  $JF(x^*, -x^*, x^*)$  is

$$g(\lambda) = \lambda^3 - \lambda^2 \left(1 + \frac{b}{R}\right) + b\lambda + b \frac{R-1}{R} = 0, \quad (\text{A.8})$$

where

$$b = \frac{\beta Dg^2}{4} (x^*)^2 \phi' \left( \frac{\beta}{2} [-Dg(R+1)(x^*)^2 - C] \right) \quad \text{with } \phi(z) = \tanh(z).$$

In the same way as in the proof of Lemma 3 it can be shown that (i) for  $\beta > \beta^*$ ,  $\beta$  close to  $\beta^*$  all eigenvalues are real and inside the unit circle and (ii) as  $\beta$  increases to  $+\infty$ , at some  $\beta = \beta^{**}$  two complex eigenvalues cross the unit circle. Hence, the period two cycle loses stability by a Hopf bifurcation at  $\beta = \beta^{**}$ .  $\square$

*Proof of Lemma 7.* The proof is essentially the same as the proof of Lemma 4. For  $g < -R$ ,  $C_t$  in Eq. (A.7) is always positive and for some finite, minimum value  $T$ ,  $C_T > C$ . Consequently, for all  $t \geq T+2$  we get  $(x_t, m_t) = (0, -1) = E_1$ .  $\square$

*Proof of Lemma 8.* We will prove a more general result for the case with  $K$  purely biased types  $b_j$  (including fundamentalists with  $b_1 = 0$ ). The system is

$$Rx_t = \sum_{j=1}^K n_{j,t-1} b_j, \quad (\text{A.9a})$$

$$n_{j,t} = \exp(\alpha b_j(x_t - Rx_{t-1}))/Z_t, \quad \alpha = \beta/(a\sigma^2), \quad 1 \leq j \leq K \quad (\text{A.9b})$$

where Eq. (A.9b) has been obtained from Eq. (4.8b) by subtracting off identical terms from the exponents in both numerator and denominator. The dynamic system (A.9a) and (A.9b) is thus of the form

$$Rx_t = V_\alpha(x_{t-1} - Rx_{t-2}), \quad (\text{A.10})$$

where

$$V_\alpha(y) = \frac{\sum b_j \exp(\alpha b_j y)}{\sum \exp(\alpha b_j y)} = \sum_{j=1}^K b_j n_j \equiv \langle b_j \rangle. \quad (\text{A.11})$$

Steady states of Eqs. (A.9a) and (A.9b) or Eq. (A.10) are determined by

$$Rx^* = V_\alpha(x^* - Rx^*) = V_\alpha(-rx^*), \quad r = R - 1. \quad (\text{A.12})$$

A straightforward computation shows, with  $Z = \sum_{j=1}^K \exp(\alpha b_j y)$ , that

$$\begin{aligned} \frac{d}{dy} V_\alpha(y) &= \sum_{j=1}^K b_j \left( \frac{\alpha b_j \exp(\alpha b_j y)}{Z} - \frac{\exp(\alpha b_j y)}{Z^2} Z'(y) \right) \\ &= \sum_{j=1}^K \left( \alpha n_j b_j^2 - \alpha n_j b_j \sum_{j=1}^K n_j b_j \right) = \sum_{j=1}^K (\alpha n_j b_j^2 - \alpha n_j b_j \langle b_j \rangle) \\ &= \alpha [\langle b_j^2 \rangle - \langle b_j \rangle^2] > 0, \end{aligned} \quad (\text{A.13})$$

where the inequality follows from the fact that the term between square brackets can be interpreted as the variance of the stochastic process where each  $b_j$  is drawn with probability  $n_j$ . Therefore,  $V_\alpha(y)$  is increasing and  $V_\alpha(-rx^*)$  decreasing in  $x^*$ . From Eq. (A.12) it then follows immediately that the steady state  $x^*$  must be unique. From Eq. (A.11) we get  $V_\alpha(0) = \sum_{j=1}^K b_j / K = \bar{b}$ , so that  $x^*$  equals the fundamental steady state if and only if  $\bar{b} = 0$ , i.e., when all biases are exactly balanced.

From Eq. (A.10), the characteristic equation determining the stability of the steady state is easily derived as

$$\lambda^2 - \frac{S}{R} \lambda + S = 0, \quad S = V'_\alpha(-rx^*). \quad (\text{A.14})$$

The eigenvalues are complex for  $0 < S < 4R^2$ . Since  $\alpha = \beta / (a\sigma^2)$  we have  $S = 0$  for  $\beta = 0$ , and if the smallest  $b_j < 0$  and the largest  $b_i > 0$ ,  $S \gg 1$  is large when  $\beta$  is large. We therefore conclude that a Hopf bifurcation occurs for some  $\beta = \beta^*$  for which  $S = 1$ .  $\square$

*Proof of Lemma 9.* From Eq. (4.3) we get that realized profits for each type are given by

$$\pi_{1,t-1} = -Rx_{t-1}(x_t - Rx_{t-1})/(a\sigma^2), \quad (\text{A.15a})$$

$$\pi_{2,t-1} = (b - Rx_{t-1})(x_t - Rx_{t-1})/(a\sigma^2), \quad (\text{A.15b})$$

$$\pi_{3,t-1} = (-b - Rx_{t-1})(x_t - Rx_{t-1})/(a\sigma^2). \quad (\text{A.15c})$$

At each date  $t$ , either  $\pi_2 < \pi_1 < \pi_3$  or  $\pi_3 < \pi_1 < \pi_2$ , except in hairline cases where  $x_t - Rx_{t-1} = 0$ . In these hairline cases, all fractions become equal and it is easily seen from Eq. (4.8a) that the orbit then converges to the unstable, fundamental steady state  $x^* = 0$ . However, for  $\beta = +\infty$  for each initial state with  $x_1 \neq Rx_0$ , after one period all traders are either of type 2 or of type 3, so that either  $x_{t+1} = b/R$  or  $x_{t+1} = -b/R$ . Using the profits (A.15a)–(A.15c) the

reader may then easily compute the next four points in the orbit and conclude that it converges to the 4-cycle  $\{b/R, b/R, -b/R, -b/R\}$ . Realized profits along this 4-cycle for each type are:

$$\text{fundamentalists: } \left\{ b^2 \left( 1 - \frac{1}{R} \right), b^2 \left( 1 + \frac{1}{R} \right), b^2 \left( 1 - \frac{1}{R} \right), b^2 \left( 1 + \frac{1}{R} \right) \right\}, \quad (\text{A.16a})$$

$$\text{upward bias } b_2 = b \left\{ 0, 0, 2b^2 \left( 1 - \frac{1}{R} \right), 2b^2 \left( 1 + \frac{1}{R} \right) \right\}, \quad (\text{A.16b})$$

$$\text{downward bias } b_3 = -b \left\{ 2b^2 \left( 1 - \frac{1}{R} \right), 2b^2 \left( 1 + \frac{1}{R} \right), 0, 0 \right\}. \quad (\text{A.16c})$$

For each of the three types, the accumulated profits along the 4-cycle are thus  $4b^2$ , so average profits are  $b^2$ , for all three types.  $\square$

*Proof of Lemma 10.* It is easily verified that  $x^* = 0$ ,  $n_j^* = 1/4$ ,  $1 \leq j \leq 4$ , is a steady state of Eqs. (4.9a) and (4.9b), since  $\langle b \rangle = \sum_{j=1}^4 b_j = 0$  for our choice of the parameters. The characteristic equation for the stability of this steady state is

$$\lambda \left[ \lambda^2 - \left( \frac{\langle g \rangle}{R} + \frac{\beta}{R} \langle b^2 \rangle \right) \lambda + \beta \langle b^2 \rangle \right] = 0, \quad (\text{A.17})$$

where  $\langle g \rangle = \frac{1}{4} \sum_{j=1}^4 g_j = 0.8025$  and  $\langle b^2 \rangle = \frac{1}{4} \sum_{j=1}^4 b_j^2 = 0.02$ . From Eq. (A.17) it then follows that the steady state is stable for  $0 < \beta < 1/\langle b^2 \rangle$ , unstable for  $\beta > 1/\langle b^2 \rangle$  and that a Hopf bifurcation occurs for  $\beta = 1/\langle b^2 \rangle = 50$ .  $\square$

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