Drive to Survive: An Analytics Approach

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15.095 - Final Report

1 Introduction

Formula 1 is a highly strategic sport. To be competitive, Formula 1 teams have to apply cutting-edge decision support technologies to design and implement increasingly complex strategies. Such strategies encompass many decisions throughout a season, spanning long-term decisions (such as budget allocation and tyre compound selection for the season), tactical decisions (such as tyre and nominal pit strategy for each weekend), and operational decisions (such as deciding exactly when to pit and perform an undercut vs. overcut against an opponent in real-time during a race). Optimizing these decisions is no longer just a benefit, but a necessity for modern F1 teams to stay competitive.

To address these strategy creation challenges, this research project will develop data-driven optimization methods to address the real-time pit stop strategy for a Formula 1 team. We will make use of the vast amounts of data, applied in machine learning and optimization methods, to enhance F1 strategies. From a technical standpoint, such optimization is complicated by decision space scale and the different decision interdependencies. This project will develop tractable algorithms to optimize these complex—and intertwined—decisions in fast computational times, with the ultimate goal of providing a strong competitive edge to an F1 team.

2 Problem Introduction

2.1 Problem

For this specific research project, we consider race-time (operational) decisions that a team has to make. The goal of this project is to prescribe a strategy (when to pit, tyres to use, and how to drive) that minimizes the total race time for a specific driver/team. These decisions are often the most difficult to make, but provide the most tangible impacts to a team's success.

Most race-time strategy revolves around tyre management. In every Formula 1 race, there are three tyre compounds (standardized across all teams) that a team can choose from. The tyre compounds range from soft to medium to hard. The underlying trade-off is that softer tyres are much faster fresh, but degrade extremely quickly relative to harder tyres. As tyres degrade, lap-time increases in a nonlinear fashion. Tyres degrade so quickly, it is always faster to pit to replace tyres at least once per race. The fresher tyres will allow you to save 30 seconds or more of what is lost in the pit stop. In some cases, it actually may be faster to stop twice to have consistently fresh tyres throughout the race and have lost the time of two pit stops (one-stop vs two-stop strategy). Previous studies have addressed the first two questions of when to pit and which tyres to use, as this cardinality of the strategy space is not particularly large.

Nobody has addressed the last decision variable — how a driver should drive during a race. It is well known that, at any given moment, F1 drivers are driving in one of three styles: conservatively - sacrifices lap time and minimizes tyre degradation, aggressively - minimizes lap times at the expense of tyre degradation, and balanced - somewhere between the two. In this analysis, we consider the driving method on a lap per lap basis, making the set of possible strategy extremely large (on the order of 10^{28} possible strategies).

2.2 Dataset

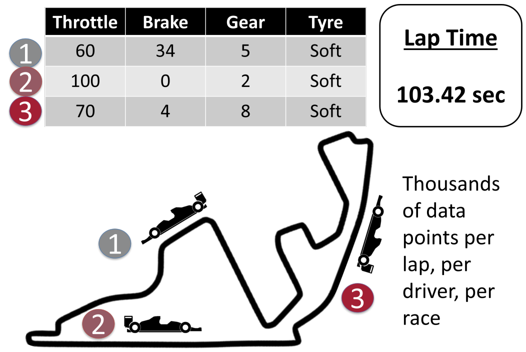


Figure 1: Dataset Visualization

The data we use to do this is taken from previous Grand Prix and consists of thousands of time-series data points per lap, per driver, per race with four important features: throttle, brake, gear, and tyre compounds. Additionally, for every race, driver, and lap we know the lap time that was achieved. The throttle and brake data acquired is presented on a normalized basis as 0 - no throttle and 100 - max throttle. The Gear a categorical feature consisting of 8 forward gears and one reverse. The tyre compound is also a categorical feature consisting of soft, medium, and hard tyres.

3 Method

The goal of this project is to prescribe a strategy (when to pit, tyres to use, and how to drive) that minimizes the total race time for a specific driver/team. To do this effectively, we must completely understand how all of the race-time decisions affect lap time, and then optimize this interaction. There are two underlying problems that we face. The first problem is that we do not have driver style labels for a given lap or otherwise.

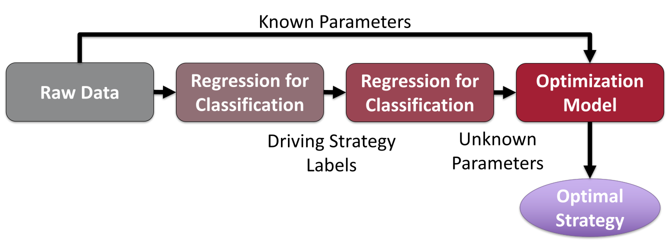


Figure 2: Predict and Prescript Method

We employ a Mixed Integer Linear Regression-Classification Model using parametric regression and sparse hyperplane techniques to identify which driving strategy is being used at any given moment. This approach is a combination of supervised and unsupervised learning. This is a lower-fidelity regression used only to find out driving style labels. We augment our initial dataset with the labeling from this classification model, then feed our augmented dataset into a nonlinear regression to learn the effect racing strategies have on tyre degradation among all tyre types and, in turn, the strategies' respective changes to long-term lap time (tyre dependent). Once the non-linear regression finds all interaction parameters, we have everything we need to solve the Mixed Integer Nonlinear Optimization problem and prescribe a race optimal-time strategy.

3.1 Data Preprocessing

Because the level of analysis considered here is lap decisions, the data has to be on a lap-per-lap basis as well. This was accomplished by aggregating the data points via their first and second moments per lap (mean and standard deviation of throttle, gear, and brake). From there all data points were standardized. The given lap time, driver, and tyre information is obviously already known on a per lap basis. To begin, we first need to generate some hypothesis class to approximate a future lap time so we can perform parametric regression. To this end, we start with this simple regression model represented below.

$$\hat{\Sigma}^i = \overbrace{Px^i + \nu f^i + \mu_0}^{\text{Known}} + \overbrace{\sum_{d \in \mathcal{D}} \mu_d \, p_d^i + \sum_{s \in \mathcal{S}} \lambda_s \chi_s^i + \sum_{t \in \mathcal{T}} \alpha_t z_t^i + \beta(\phi^i)}^{\text{Unknown}} \tag{1}$$

Above the lap time $\hat{\Sigma}^i$ of a data sample i represented by different known and unknown features. The known features are simply the pit stop time P, the additional time due to amount of fuel left ν, and the lap record of a specific track μ_0. The unknown features are the following: the base time for each driver $d \in \mathcal{D}$ (μ_d), the additional time due to driving style $s \in \mathcal{S}$ employed (χ_c), the base time for each tyre compound $t \in \mathcal{T}$ used (α_t), and the additional time added on due to tyre degradation ($\beta(\phi^i)$ - convex function). Note that some covariates are known, while others are not. Specifically, which driver and which tyre is active in the data sample is known (binary features - μ_d & χ_s^i). Other features are not known, such as the driving style for each lap (χ_s^i) and the current tyre degradation value (ϕ^i). However, although the tyre degradation is not explicitly known, we do know that the FiA requires that the tyres degrade in the same manner, but at different rates. Specifically, the soft tyre degrades linearly for 15 laps, the medium tyre degrades linearly for 18 laps, and the hard tyre degrades linearly for 22 laps. After those cutoffs, the tyres then degrade in an exponential fashion. To handle this, the tyre degradation value is found by adding $1/15, 1/18, 1/22$ for the soft, medium, and hard tyre respectively for each lap it is driven. This is a mild assumption to obtain some unit-less value of degradation that can be compared across the three tyres. The last variable, driving strategy (χ_s^i) is unknown, and will be uncovered in the next section. Using everything discussed above, we can understand how to preprocess the known lap times σ^i. Specifically, since some covariates and parameters are known, we preprocess the lap times in the following way.

$$y^i = \Sigma^i - (Px^i + \nu f^i + \mu_0) \tag{2}$$

Now, y^i above is the part of the lap times that we need to explain using the unknown features and parameters.

3.2 Mixed Integer Linear Regression for Classification

All covariates are known except for the driving style (χ_s). As we mentioned before, we assume that there are three driving styles: conservative, balanced, and aggressive. We face a fundamental problem. We cannot perform regression in the normal sense because we do not know both λ and χ_s^i. However, there is a way we can still approach this problem because we have access to driving data (throttle, brake, and gear commands) to aid our analysis. The method's underlying hypothesis is that driving styles are in some way linked to the driving data that we have. We do not have any prior labels on the driving styles, as such, we cannot train a supervised classification model. One idea would be to perform some unsupervised learning technique, such as clustering. Unfortunately, there were no clearly defined clusters and, even if there were, there would be no guarantee that the clusters would actually map correctly to the driving styles.

We created a brand new approach that we call "Regression for Classification". The ultimate objective of which is to simultaneously classify the driving style of each datasample and to perform parametric regression to learn the parameters including λ using the power of optimization. Fundamentally, the idea is that we could let the model try to classify and regress in such a way that the classifications best explain the variance in the data. In principle, this should mean that the λ values will have the most variance. As such, we can expect that the classifications will naturally best explain the three driving styles. This is a mixture of both unsupervised and supervised learning. It is unsupervised because we have no driving style labels to start out; it is also somewhat supervised because we know that the labels will help explain the known lap times.

To begin, we first need to classify the laps given the data that we have. It was determined that a simple hyperplane classification would suffice for this analysis. However, since this analysis has three different classes, two hyperplanes are necessary. We assume there exists a gradient in the data such that, if you follow that gradient, the driving becomes more aggressive. Similarly, if you went the other way, the driving becomes more conservative. As such, two parallel, offset hyperplanes will be used in this approach. This will be handled with the following optimization problem.

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{N} |y_i - (\mu^T p_d^i + \lambda^T \chi_s^i + \alpha^T z^i + \tau \phi^i)| && \text{(3a)}\\
\text{s.t.} \quad & \kappa^T x^i + \kappa_a \leq M \chi_a^i \quad \forall i && \text{(3b)}\\
& -(\kappa^T x^i + \kappa_a) \leq M(1 - \chi_a^i) \quad \forall i && \text{(3c)}\\
& \chi_b^i = (1 - \chi_a^i)(1 - \chi_c^i) \quad \forall i && \text{(3d)}\\
& \kappa^T x^i + \kappa_c \leq M(1 - \chi_c^i) \quad \forall i && \text{(3e)}\\
& -(\kappa^T x^i + \kappa_c) \leq M \chi_c^i \quad \forall i && \text{(3f)}\\
& \chi_a^i + \chi_b^i + \chi_c^i = 1 \quad \forall i && \text{(3g)}\\
& \sum_{p=1}^{P} \kappa_p = 1 && \text{(3h)}\\
& |\kappa|_0 = k && \text{(3i)}\\
& \sum_{i} \chi_s^i \geq b \quad \forall s \in \mathcal{S} && \text{(3j)}
\end{aligned}
$$

In the above, χ_s^i is a binary decision variable, κ is a continuous decision vector, and κ_a & κ_c are continuous hyperplane offsets. We can see that, using the big M notation, $\chi_a^i = 1$ iff $\kappa^T x^i + \kappa_a \geq 0$ and $\chi_c^i = 1$ iff $\kappa^T x^i + \kappa_c \leq 0$. of course they cannot both be true because $\chi_a^i + \chi_b^i + \chi_c^i = 1$, so the offsets will naturally be located such that $\kappa_c \leq \kappa_a$. Finally we have that $\chi_b^i = (1 - \chi_a^i)(1 - \chi_c^i)$, which means that $\chi_b = 1$ iff $\kappa^T x^i + \kappa_a \leq 0$ and $\kappa^T x^i + \kappa_c \geq 0$ (between the two hyperplanes). To ensure that the hyperplanes are interpretable, we also require the following $|\kappa|_0 = k$ (hyperplane sparsity) and $\sum_i \chi_s^i \geq b$ (mini bucket requirement). We put this together into a full regression model in the following section. Additionally, above we use the L_1 norm to measure the empirical error between true and approximated values. One comment to make is that we assume that $\beta(\phi^i) = \tau \phi^i$ to aid in the speed of the model. The result of this model will give us the driving style classifications for each data sample.

3.3 Nonlinear Regression

After completing the Classification for Regression model, we now have classifications for all driving strategies. Before we can move forward to a nonlinear regression model, we first need to make an assumption about how driving strategy affects tyre degradation. We made the assumption that conservative driving would produce 20% less degradation, balanced is nominal tyre degradation, and aggressive driving would produce 20% more tyre degradation. With more time, more sophisticated analysis could be conducted to try and analyze the true impact. After this was completed, all covariates are now completely known. The next step is to perform a nonlinear regression model. It is know, that tyre degradation occcurs linearly for the distances mentioned before, and then exponetially after. As such, the function

$\beta(\phi)$ was assumed to be this form

$$\beta(\omega) = \begin{cases} \tau\,\omega & \text{if } \omega < 1 \\ \tau\, e^{\gamma(\omega-1)} & \text{if } \omega \geq 1 \end{cases} \tag{4}$$

Assuming this nonlinear (convex) form of β, a nonlinear regression model is solved using Python's Curve-Fit API to learn all parameters. These parameters can then be used in a prescriptive optimization model.

3.4 Nonlinear Mixed-Integer Optimization Model

$$\min \quad \sum_{l=1}^{L} \Sigma^l \tag{5a}$$

$$\text{s.t.} \quad y_t^l \leq x^l \quad \forall l \in |L|,\ \forall t \in \mathcal{T} \tag{5b}$$

$$\sum_{t\in\mathcal{T}} y_t^l = x^l \quad \forall l \in |L| \tag{5c}$$

$$\sum_{t\in\mathcal{T}} z_t^l = 1 \quad \forall l \in |L| \tag{5d}$$

$$z_t^{l+1} = z_t^l + g_t^l + y_t^l - \sum_{j\in\mathcal{T}:\, j\neq t} y_t^l \quad \forall l \in |L|,\ \forall t \in \mathcal{T} \tag{5e}$$

$$\chi_c^l + \chi_b^l + \chi_a^l = 1 \quad \forall l \in |L| \tag{5f}$$

$$w_t^0 = \omega_t \quad \forall t \in \mathcal{T} \tag{5g}$$

$$w_t^{l+1} = w_t^l + z_t^l\,(\gamma_t^c \chi_c^l + \gamma_t^n \chi_b^l + \gamma_t^a \chi_a^l) \quad \forall l \in |L-1|,\ \forall t \in \mathcal{T} \tag{5h}$$

$$w_t^l \leq 1 \quad \forall l \in |L-1|,\ \forall t \in \mathcal{T} \tag{5i}$$

$$\phi^l = \sum_{t\in\mathcal{T}} \epsilon_t^l \quad \forall l \in |L|,\ \forall t \in \mathcal{T} \tag{5j}$$

$$\Sigma_d^l = \nu f^l + P x^l + \mu_d + \sum_{s\in\mathcal{S}} \lambda_s \chi_s^l + \sum_{t\in\mathcal{T}} \alpha_t z_t^l + \beta(\phi^l) \quad \forall l \in |L| \tag{5k}$$

$$h^l \geq \beta(\phi^l) \quad \forall l \in |L| \tag{5l}$$

For brevity, the specifics of the model are adressed in the apendix. However, the model answers the three decisions we set to perscribe: when to pit (x^L), which tyres to use (z_t^l), and how to drive on each lap (χ_s^l).

4 Results

Note that all data is taken from last year's 2020 Abu Dahbi Grand Prix. As such, we expect our data and strategies to match the optimal strategy for the 2021 Abu Dahbi Grand Prix occurring December 12, 2021.

4.1 Mixed Integer Linear Regression for Classification

The MILRC model was solved with the following parameters. We required the hyperplane only have support of 3 and that the divisions had to each have at least 100 datapoints. After solving the model, it found that the three most "telling" features for driving aggressiveness were: "Mean of Throttle", "Std

Dev of Gear", and "Std Dev of Brake". Intuitively, this makes sense. We would expect a higher average throttle correlates with more aggressive driving. In addition, a higher Std Dev of Brake & Gear means the driver is acting more aggressively. The model even goes a step further by allowing us to examine the κ coefficients to understand feature importance. It was found that the average throttle command was the most important, followed by Std Dev of Gear, and then Std Dev of Brake. The division of the datapoints is in the figure shown below.

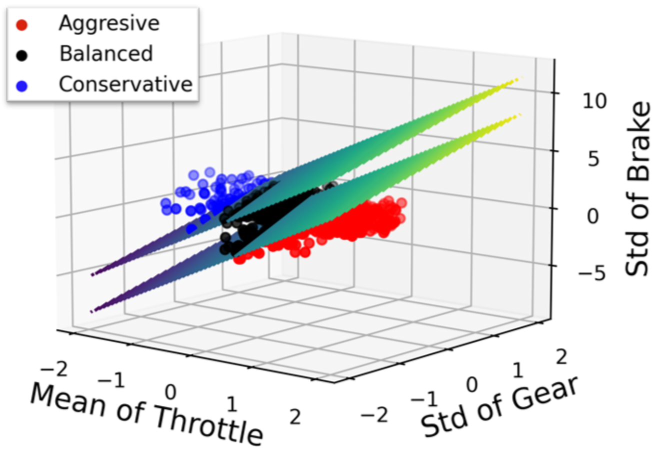


Figure 3: Driving Strategy Classifications

In the model, the λ parameters were clearly organized such that $\lambda_a \leq \lambda_b \leq \lambda_c$, indicating that the suggested hypothesis was correct.

4.2 Nonlinear Regression

After finding the driving strategy labels, we then performed nonlinear regression to learn the parameters. We can first begin by analyzing the lap time contribution vs lap number for each tyre & driver strategy combination. This can be seen in the figure below, which is a combination of the α_t and $\beta(\phi)$ function.

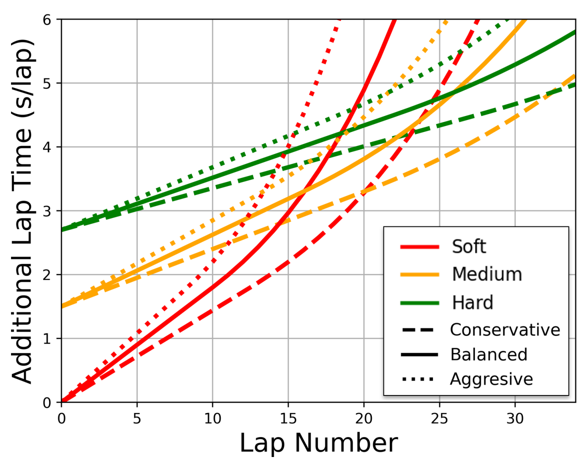


Figure 4: Driving Strategy Classifications

We can see that the the tyre compounds do follow the expected trend where softer tyres start out initially faster, but also degrade more rapidly than the other compounds. Additionally, we can see how the specific driving style affects the overall lap time due to degradation. The specifics of how much time is saved for each lap depending on driving strategy is shown in the table below.

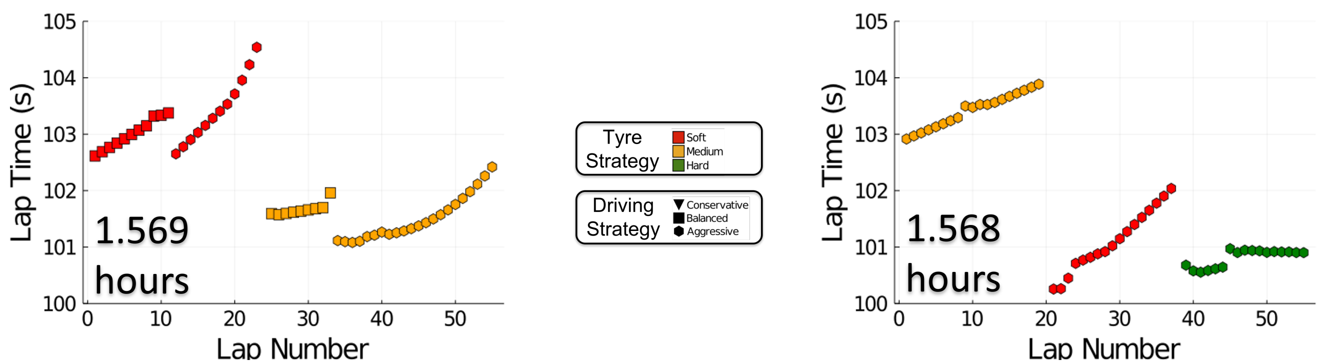
Table 1: Driving Strategy Parameters

Conservative ($\lambda_\mathbf{c}$)	**Balanced** ($\lambda_\mathbf{b}$)	**Aggressive** ($\lambda_\mathbf{a}$)
+1.55 (s/lap)	+0.74 (s/lap)	0.00 (s/lap)

After examining the graph and table, we can understand why this strategy problem can be very hard. Obviously driving aggressively does mean that you are quicker initially than driving conservatively, but as we can see above time can quickly be lost do to tyre degradation. This leads to a very delicate balancing act to figure out what are the best decisions. For brevity, the driver offset values are left out, but they all determine the additional time due to each drivers base car and skill-set.

4.3 Nonlinear Mixed-Integer Optimization Model

Using all the parameters that are learned in the previous sections, we were then able to inform the optimization model and solve for the best strategies. The optimization mode was ran twice. One was ran to find the optimal one stop strategy and the other was ran to find the optimal two stop strategy. Frequently, teams would like to understand both options, even if one is, on paper, slightly better. This is normally because it may play into their game-theoretic strategy. Below we can see a visual representation of the two optimal strategies.



(a) One-Stop Strategy

(c) Two-Stop Strategy

Figure 5: Optimal Strategies for 2021 Abu Dahbi Grand Prix

Initially we can see that the best strategy is actually a two stop strategy, as it has a slightly less total time than the optimal one-stop strategy. However, it is only by a matter of 3 seconds. One interesting observation is that the optimal driving style is always aggressive for the two-stop strategy, while the optimal one-stop strategy initially starts out driving conservatively and then moves to aggressive driving towards the end of the stint. This makes sense, as two stops means tyres are more fresh, so they can take more degradation. This is in contrast to a one-stop strategy where tyres have to be properly preserved throughout the race.

5 Conclusions

In this research project, our goal was to prescribe an operational race-time strategy for a Formula 1 team (when to pit, tyres to use, and how to drive) that minimizes the total race time for a specific driver/team. To this end, we performed a predictive, regression framework to understand how decisions affect lap time and then created a prescriptive, optimization framework to generate the optimal decisions. Along the way, we had to generate new approaches to meet our objectives. First, we were presented with a problem where we did not know covariates and parameters. Instead of performing a clustering then regression framework, we used the power of optimization to generate a novel Regression-for-Classification framework and learn the driving strategy labels. We then employed nonlinear regression to learn the true parameters and nonlinear tyre degradation function. From there, we formulated a Nonlinear Mixed-Integer Optimization Model to find the optimal strategies from all the found parameters. We were able to successfully perform all of this and meet our objectives of optimally prescribing the best strategies for a Formula 1 team. Along the way, we exemplified the power of optimization and how it can be used in novel ways to augment normal ML methods, such as our Regression-for-Classification framework. For future work, there are many possible directions of extension: considering stochastic events such as safety cars & crashes, improving modeling of tyre degradation, and/or including additional features such as real-time engine setting updates.

6 Contributions

Spencer: He formulated the Regression-for-Classification model and the Optimization model. He also made the Predictive and Prescriptive parts of the poster. He wrote the Methods, Results, and Conclusions of the final report.

Harry: He conducted the Nonlinear Regression part of the project. He made the "Why this Problem", "Why this method", and "Dataset" parts of the poster. He wrote the Introduction, Problem Introduction, and the intro to the methods section of the final report.

Appendix A Nonlinear Mixed-Integer Optimization Model

A.1 Notations

1. $\mathcal{T}$ = set of tyres that can be used
2. L = the number of laps in a given race

A.2 Strategy Model

Pit Stop Variables

1. $x^l \in \{0, 1\}$: equals 1 if the vehicle goes to pit on lap l and 0 otherwise
2. $y_t^l \in \{0, 1\}$: equals 1 if the vehicle changes to tyre t during pit stop on lap l and 0 otherwise
3. $z_t^l \in \{0, 1\}$: equals 1 if vehicle uses tyre t on lap l and 0 otherwise
4. $g_t^l \geq 0$: necessary to enforce active tyre, described later

$$y_t^l \leq x^l \quad \forall l \in |L|, \; \forall t \in \mathcal{T} \tag{6a}$$

$$\sum_{t \in \mathcal{T}} y_t^l = x^l \quad \forall l \in |L| \tag{6b}$$

The above constraint ensures that we can only change a tyre when we decide to pit. The second constraint requires specifically that we change a tyre whenever we pit. In principle, this is not the only reason you would come to a pit, but it is an OK assumption for strategic planning purposes.

$$\sum_{t \in \mathcal{T}} z_t^l = 1 \quad \forall l \in |L| \tag{7a}$$

The above constraint ensures that exactly one tyre is used for every lap of the race.

$$z_t^{l+1} = z_t^l + g_t^l + y_t^l - \sum_{j \in \mathcal{T}:\, j \neq t} y_t^l \quad \forall l \in |L|, \; \forall t \in \mathcal{T} \tag{8a}$$

The above constraint enforces the active tyre used for every lap. Suppose that $z_t^l = 0$. If $y_t^l = 1$, then clearly $z_t^{l+1} = 1$ and $g_l^t = 0$. Suppose that $z_t^l = 0$ and $y_j^l = 1$ $(j \neq t)$ then we can see that $z_t^{l+1} = 0$ and $g_t^l = 1$. Last case is that $z_t^l = 1$ and $y_j^l = 1$, then clearly $z_t^{l+1} = 0$ and $g_t^l = 0$. If you have any questions, let me know. But basically g_t^l provides a feasible way to make the math work out. $g_t^l = 1$ if and only if the $z_t^l = 0$ and $y_j^l = 1$ for some $j \neq t$.

Driver Strategy Variables

1. $\chi_c^l \in \{0, 1\}$: equals 1 if the vehicle used conservative driving on lap l and 0 otherwise
2. $\chi_b^l \in \{0, 1\}$: equals 1 if the vehicle used balanced driving on lap l and 0 otherwise
3. $\chi_a^l \in \{0, 1\}$: equals 1 if the vehicle used aggressive driving on lap l and 0 otherwise

$$\chi_c^l + \chi_b^l + \chi_a^l = 1 \quad \forall l \in |L| \tag{9a}$$

We require that one driving strategy be selected for each lap.

Tyre Wear Variables

1. $w_t^l \geq 0$: the normalized tyre degradation for tyre t during lap l

2. $\phi^l \geq 0$: the tyre degradation of the tyre actively used during lap l

$$w_t^0 = \omega_t \quad \forall t \in \mathcal{T} \tag{10a}$$
$$w_t^{l+1} = w_t^l + z_t^l \left(\gamma_t^c \chi_c^l + \gamma_t^n \chi_b^l + \gamma_t^a \chi_a^l\right) \quad \forall l \in |L-1|,\ \forall t \in \mathcal{T} \tag{10b}$$

The first of the above constraints sets the initial degradation for each tyre. The second constraint adds to the degradation for each tyre depending on if the tyre is used (z_t^l) and on what driving condition is selected (χ^l). The degradation values for w_t are unitless variables that hold the tyre degradation across each different tyre compound. The value γ_t^c is the amount of tyre degradation that accumulates using tyre t during driving mode c.

$$\phi^l = \sum_{t \in \mathcal{T}} z_t^l w_t^l \quad \forall l \in |L|,\ \forall t \in \mathcal{T} \tag{11a}$$

The above constraint sets ϕ^l to the tyre degradation value of the active tyre that is used on lap l.

Lap Time

1. $\Sigma_d^l \geq 0$: the lap time on lap l for the vehicle for driver d

$$\Sigma_d^l = \nu f^l + P x^l + \mu_d + \sum_{s \in \mathcal{S}} \lambda_s \chi_s^l + \sum_{t \in \mathcal{T}} \alpha_t z_t^l + \beta(\phi^l) \quad \forall l \in |L| \tag{12a}$$

The above constraint sets the lap time for each lap. The value μ_0 is a constant that sets the theoretical minimum lap time that could be achieved for a given driver and car with no fuel, no degradation, aggressive driving mode, and on the softest tyre. The constant P is the time it takes to take a pit stop. The value α_t is the amount of additional time that is added on depending on which tyre it is used. We can expect that $\alpha_t = 0$ for any tyre t that is the softest compound. The constant ν is the amount of extra time per lap added on per amount of fuel. The constants λ_c, λ_n, and λ_a is the additional time added on depending on which driving mode is selected. We could take $\lambda_a = 0$. The $\beta(\phi^l)$ term is the additional time that is added on due to the tyre degradation level (ϕ^l) during lap l. A convex assumption on $\beta(\phi^l)$ is a good assumption, as it tends to follow some exponential distribution.

Objective Function

$$\min \quad \sum_{l=1}^{L} \Sigma_d^l \tag{13a}$$

The above objective function seeks to minimize the gap time at the end of the race (Ω^{L+1}). For a deterministic analysis, this is equivalent to minimizing the sum of all the lap times.

Linearization

Some of the constraints that are initially listed in the model are nonlinear, how they can all be linearized through standard methods as they are all binary times continuous or binary times binary. Let these be the linearized variables.

1. $\epsilon_t^l = z_l^t\, w_l^t$: tyre degradation of tyre t on lap l if tyre t is used on lap l and 0 otherwise

2. $\eta_{tc}^l = z_l^t\, \chi_c^t$: equals 1 if tyre t is used on lap l and conservative strategy is used and 0 otherwise

3. $\eta^l_{tb} = z^t_l \, \chi^t_b$: equals 1 if tyre t is used on lap l and balanced strategy is used and 0 otherwise

4. $\eta^l_{ta} = z^t_l \, \chi^t_a$: equals 1 if tyre t is used on lap l and aggressive strategy is used and 0 otherwise

5. $h^l \geq \beta(\phi^l)$: h^l will equal $\beta(\phi^l)$ at optimality

Full Model

$$\min \quad \sum_{l=1}^{L} \Sigma^l \tag{14a}$$

$$\text{s.t.} \quad y^l_t \leq x^l \quad \forall l \in |L|, \, \forall t \in \mathcal{T} \tag{14b}$$

$$\sum_{t \in \mathcal{T}} y^l_t = x^l \quad \forall l \in |L| \tag{14c}$$

$$\sum_{t \in \mathcal{T}} z^l_t = 1 \quad \forall l \in |L| \tag{14d}$$

$$z^{l+1}_t = z^l_t + g^l_t + y^l_t - \sum_{j \in \mathcal{T}: j \neq t} y^l_t \quad \forall l \in |L|, \, \forall t \in \mathcal{T} \tag{14e}$$

$$\chi^l_c + \chi^l_b + \chi^l_a = 1 \quad \forall l \in |L| \tag{14f}$$

$$w^0_t = \omega_t \quad \forall t \in \mathcal{T} \tag{14g}$$

$$w^{l+1}_t = w^l_t + z^l_t \, (\gamma^c_t \chi^l_c + \gamma^n_t \chi^l_b + \gamma^a_t \chi^l_a) \quad \forall l \in |L-1|, \, \forall t \in \mathcal{T} \tag{14h}$$

$$\phi^l = \sum_{t \in \mathcal{T}} \epsilon^l_t \quad \forall l \in |L|, \, \forall t \in \mathcal{T} \tag{14i}$$

$$\Sigma^l_d = \nu f^l + P x^l + \mu_d + \sum_{s \in \mathcal{S}} \lambda_s \chi^l_s + \sum_{t \in \mathcal{T}} \alpha_t z^l_t + \beta(\phi^l) \quad \forall l \in |L| \tag{14j}$$

$$h^l \geq \beta(\phi^l) \quad \forall l \in |L| \tag{14k}$$

The full model is summarized above.