Linear Algebra I: Homework 9

Due Friday, April 20, 2018

1. Let $\mathbb{R}^2$ have inner product,

$$\langle \vec{u}, \vec{v} \rangle = 3u_1v_1 + 5u_2v_2.$$

   Let $\vec{u} = (1, 1)$, $\vec{v} = (3, 2)$, $\vec{w} = (0, -1)$.

   a. Compute $\langle \vec{u}, \vec{w} \rangle$.

   b. Compute $\langle 3\vec{u}, \vec{v} \rangle$.

   c. Compute $\|\vec{u} - 3\vec{w}\|$.

   d. Find some unit vectors with regards to this inner product $\langle \cdot, \cdot \rangle$ and sketch its unit circle. Hint; it will *not* look like a typical unit circle.

2. Use the Gram-Schmidt process to orthonormalize the basis B with respect to the dot product on $\mathbb{R}^3$:

$$B = \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right)$$

3. Let W be a subspace of a vector space V with inner product $\langle \cdot, \cdot \rangle$. The **orthogonal complement** of W in V is the subspace $W^{\perp}$ of all vectors u which are orthogonal to *every* vector in W.

   a. Let

$$W = \text{span} \left\{ \begin{pmatrix} 1 \\ 4 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

   be a subspace of $\mathbb{R}^4$. Find a matrix equation for which $W^{\perp}$ is the set of all solutions, then solve it to find $W^{\perp}$.

   b. Let R be the subspace defined by the plane $2x + y - z = 0$ in $\mathbb{R}^3$. Find $R^{\perp}$.

4. Let W be a subspace of V, $B = \left\{ \vec{b}_1, \vec{b}_2, \cdots, \vec{b}_k \right\}$ be an orthonormal basis for W and $C = \{\vec{c}_1, \vec{c}_2, \cdots, \vec{c}_\ell\}$ be an orthonormal basis for its orthogonal complement (see #3) $W^{\perp}$.

   Consider the set of vectors

$$U = \left\{ \vec{b}_1 \vec{b}_2, \cdots \vec{b}_k, \vec{c}_1, \vec{c}_2, \cdots, \vec{c}_\ell \right\}.$$

a. Show that the only vector in both W and $W^\perp$ is $\vec{0}$.

b. It turns out that for every vector $\vec{x} \in V$, there is a **unique** way to write it as the sum $\vec{x} = \vec{x}^{||} + \vec{x}^{\perp}$ where $\vec{x}^{||} \in W$ and $\vec{x}^{\perp} \in W^\perp$ (basically, Gram-Schmidt).

   Taking this as a given, explain why U is an *orthonormal basis* for V. Hint; first explain why U spans V. Then, tell me what dot products between different kinds of vectors in U are, and use this to convince me that it is orthonormal (and hence linearly independent, too).

5. Let A be a symmetric matrix.

   a. If $\vec{v}_1, \vec{v}_2$ are eigenvectors of A for eigenvalues λ_1, λ_2 (where $\lambda_1 \neq \lambda_2$), explain why v_1 and v_2 are orthogonal. Hint; remember $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$ and compute $(A\vec{v}_1) \cdot \vec{v}_2$.

   b. Suppose A diagonalizes (in fact, every symmetric matrix A diagonalizes always). Explain why it is possible to find an orthogonal basis of eigenvectors for A. Conclude that it is possible to diagonalize A as

$$A = PDP^T$$

   where P is an **orthogonal** matrix.