Linear Algebra I: Homework 5

Due Friday, March 2, 2018

1. Let $\vec{v}$ be the column vector in $\mathbb{R}^4$ which points from the point $P = (2, -2, 1, 3)$ to $Q = (0, -4, 3, 1)$.

   a. Calculate the magnitude $\|\vec{v}\|$

   b. Calculate the angle between $\vec{v}$ and the vector,

$$\vec{w} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

2. This question pertains to an example of something called the *cross product* of two vectors in $\mathbb{R}^3$. The cross product $\vec{u} \times \vec{v}$ is defined as,

$$\begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \times \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

   where a, b, c are the coefficients of the variables i, j, k in the determinant,

$$\det \begin{pmatrix} i & j & k \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{pmatrix}.$$

   This problem has you work out a specific example.

   a. If i, j, k are arbitrary variables, calculate:

$$\det \begin{pmatrix} i & j & k \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

   b. Let $\vec{v}$ be the vector

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

   where a is the coefficient of i in your answer to (a), b is the coefficient of j in your answer to (a), and z is the coefficient of c in your answer to (a).

   Calculate the dot product,

$$\vec{v} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

c. Calculate the dot product,

$$\vec{v} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

3. Consider the vectors,

$$\vec{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{u} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

Find the set of **all** vectors $\vec{w}$ which are orthogonal to both of $\vec{v}$ and $\vec{u}$.

4. Consider the matrix,

$$S = \begin{pmatrix} 4 & 3 \\ 0 & -2 \end{pmatrix}$$

a. Calculate $\det(S)$.

b. For λ a variable, solve the equation $\det(S - \lambda I) = 0$, where I is the 2×2 identity matrix.

5. For any angle φ, multiplication by the matrix,

$$M_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

defines a linear transformation (a.k.a. a linear function) that rotates vectors $\vec{v}$ in $\mathbb{R}^3$ around the x-axis by φ radians counterclockwise.

a. If you have any vector $\vec{v}$ and you know that $\|\vec{v}\| = 3$, calculate $\|M_\varphi \vec{v}\|$. **Hint:** Think about how rotating something changes its length. The same principles apply here!

b. Calculate the angle between $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

c. Calculate the angle between $M_{\pi/3}\begin{pmatrix}0\\2\\0\end{pmatrix}$ and $M_{\pi/3}\begin{pmatrix}0\\1\\1\end{pmatrix}$. **Hint:** You're welcome to calculate this using matrix multiplication, but that might get a little exhausting. I encourage you to think about how this rotation will affect the angle between the vectors.