Math 317: Homework 7

Due Friday, April 5, 2019

1. (28.8) Let $f(x) = x^2$ when x is rational and $f(x) = 0$ otherwise.

   a. Show that f is continuous at 0.

   b. Show that f is discontinuous at all $x \neq 0$.

   c. Prove that f is differentiable at 0. (It is insufficient to simply claim $f'(x) = 2x$)

2. (28.14) Suppose f is differentiable at a.

   a. Prove that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$.

   b. Prove that $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$.

3. (29.5) Let f be a real-valued function which is differentiable on $\mathbb{R}$. Prove that if $|f(x) - f(y)| < (x - y)^2$ for all $x, y \in \mathbb{R}$, then f is a constant function.

4. (29.9) Show that $ex \leq e^x$ for all $x \in \mathbb{R}$.

5. Let f be real-valued and differentiable on $\mathbb{R}$, and let $g(x) = f(x + 1) - f(x)$. Suppose additionally that $\lim_{x\to\infty} f'(x) = 0$.

   Prove that $\lim_{x\to\infty} g(x) = 0$.