Math 301 Midterm Exam #2 Practice Problems Solutions

1. Your answer to the following questions can just be a number. If your number is naturally written as a product, using for instance factorials or binomial coefficients, please leave it that way!

   You should explain your reasoning as best you can: Correct answers will receive full credit, but I can only award partial credit if you show your process.

   Practice note: You should explain every part of your answer as best you can.

   (a) Find an integer x between 0 and 42 that satisfies $4x \equiv 1 \mod 43$.
   11

   (b) Find an integer x between 0 and 30 that satisfies $7x \equiv 10 \mod 31$.
   28

   (c) Find an integer x between 0 and 46 that satisfies $8x \equiv 4 \mod 47$.
   24

   (d) How many graphs have *exactly* 8 (labeled) vertices $\{a, b, c, d, e, f, g, h\}$?
   2^{28}

   (e) How many subgraphs of K_8 (with vertices labeled $\{a, b, c, d, e, f, g, h\}$) have exactly 8 (labeled) vertices?
   2^{28}

   (f) How many walks of 5 steps are there in K_8?
   $8(7^5)$

   (g) How many *closed* walks of 5 steps are there in K_8? Say that two closed walks are definitely different if they have the same start/end vertex.
   $8(7^4)1$

   (h) How many walks of 5 steps are there in C_8?
   $8(2^5)$

   (i) How many *closed* walks of 5 steps are there in C_8? Say that two closed walks are definitely different if they have the same start/end vertex.
   0

   (j) How many *closed* walks of 6 steps are there in C_8? Say that two closed walks are definitely different if they have the same start/end vertex.
   $8\binom{6}{3}$

   (k) How many subgraphs does C_3 have? Say the vertices are labeled $\{a, b, c\}$.
   18

   (l) How many subgraphs does P_3 have? Say the vertices are labeled $\{a, b, c, d\}$.
   34

2. Answer whether each statement is "True" or "False". No justification is needed. No partial credit.

   Practice note: You should try proving each statement and justifying your answer during practice, even though this type of question on an exam would not require it.

(a) The complement of P_6, the path graph on 7 vertices, has an Eulerian walk.
True.

(b) The complement of C_6, the cycle graph on 6 vertices, has an Eulerian walk.
False.

(c) The complement of a disconnected graph of at least 4 vertices is always connected.
True.

(d) The complement of a connected graph of at least 4 vertices is always disconnected.
False.

(e) A graph with n vertices always has at least 2^n subgraphs.
True.

(f) A graph with n edges always has at least 2^n subgraphs.
True.

(g) For every pair of positive integers a, b there exist integers m, n such that $1 = ma+nb$.
False.

(h) For every pair of positive integers a, b there exist integers m, n such that $\gcd(a, b) = ma + nb$.
True.

(i) There exists a graph of 7 vertices of total vertex degree 44.
False.

(j) For every positive integer m and integer $1 \leq a < m$ there exists an integer x so that $ax \equiv 1 \mod m$.
False.

(k) There exists a graph of 7 vertices of degrees $2, 2, 2, 2, 2, 3, 7$.
False.

(l) There exists a graph of 7 vertices of degrees $2, 2, 2, 2, 3, 5, 7$.
False.

(m) There exists a connected graph of 7 vertices of degrees $1, 1, 1, 1, 2, 2, 2$.
True.

(n) K_5 contains a closed Eulerian walk.
True.

(o) K_6 contains a closed Eulerian walk.
False.

(p) Every graph with all even vertex degrees has a Hamiltonian cycle.
False.

(q) If $a \nmid b$ and $b \mid c$ then $a \nmid c$.
False.

(r) If $a \nmid b$ and $b \nmid c$ then $a \nmid c$.
False.

(s) If $a \equiv b \mod c$ and $b \equiv c \mod a$ then $a \equiv c \mod b$.
False.

(t) If $a \equiv 0 \mod b$ and $b \equiv 0 \mod c$ then $a \equiv 0 \mod c$.
True.

(u) There exists a graph of 7 vertices of degrees $2, 2, 2, 2, 3, 3, 6$.
True.

(v) There exists a graph of 7 vertices of degrees $2, 2, 2, 2, 2, 5, 6$.
False.

3. Use the fact that a cycle-free graph on n vertices has at most $n-1$ edges to prove that: If G is a connected graph with at least 5 vertices and no cycles, then its complement $\bar{G}$ has at least one cycle.

   Proof. Let e, $\bar{e}$ be the number of edges in G and $\bar{G}$ respectively. As $\bar{G}$ is the complement of G we know that $\bar{e} = \binom{n}{2} - e$. If G has no cycles, then $e \leq n-1$. If in addition $\bar{G}$ also has no cycles, then $\bar{e} \leq n-1$. But notice then that

$$n - 1 \geq \bar{e} = \binom{n}{2} - e \geq \binom{n}{2} - (n-1) = (n-1)(n/2 - 1)$$

   As $n - 1 > 0$ this can only happen if

$$1 \geq n/2 - 1 \Leftrightarrow 4 \geq n.$$

   But we said that $n > 5$, so this is a contradiction, and so $\bar{G}$ must be cycle-free.

4. (a) Draw a connected graph on 8 vertices for which removing any edge makes the graph disconnected.

   (b) Draw a connected graph on 8 vertices for which removing any edge leaves the graph connected.

   (c) Draw a connected graph on 8 vertices for which removing any *two* edges leaves the graph connected.

   (d) Draw a connected graph on 8 vertices that has no cycles and *is not the path graph.*