Linear Algebra I: Homework 5

Due Friday, September 22, 2017

1. a. Find the matrix $\frac{d}{dx}$ acting on the vector space $V = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ of polynomials of degree 2 or less in the ordered basis $B = (x^2, x, 1)$.

   b. Use your matrix from (a) to rewrite the differential equation ($p(x)$ is a vector in V)

   $$\tfrac{d}{dx}p(x) = x$$

   as a matrix equation. Find all solutions of the matrix equation, then write them as elements of V.

   c. Find the matrix for $\frac{d}{dx}$ acting on the same vector space V but now with the ordered basis $C = (x^2 + x, x^2 - x, 1)$.

   d. Use your matrix from (c) to rewrite the differential equation ($p(x)$ is a vector in V)

   $$\tfrac{d}{dx}p(x) = x$$

   as a matrix equation. Find all solutions of the matrix equation, then write them as elements of V.

2. Suppose that A is a square matrix that is **antisymmetric**, meaning that

   $$A = -A^T.$$

   Prove that $\operatorname{tr}(A) = 0$.

3. The **matrix exponential** of a matrix M is given by the Taylor series,

   $$\exp(M) = I + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}M^k.$$

   For $\lambda \in \mathbb{R}$, let A be the matrix:

   $$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

   and let B be the matrix:

   $$B = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$$

a. Find a concise formula for A^k, for any integer k.

b. Find a concise formula for $\exp(A)$.

c. Find B^k, for any integer k.

d. Find a concise formula for $\exp(B)$.

4. Find the determinant of the matrix

$$\begin{pmatrix} 2 & 1 & 3 & 7 \\ 6 & 1 & 4 & 4 \\ 2 & 1 & 8 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

5. Is the determinant of a matrix a linear transformation into the real numbers $\mathbb{R}$? Explain or give a counterexample.