Math 317: Homework 5

Due Friday, March 8, 2019

1. (15.1, 15.2, 15.4) Determine which of the following series converge. Justify your answers.

   a.
$$\sum \left[\sin\left(\frac{n\pi}{6}\right)\right]^n$$

   b.
$$\sum \frac{(-1)^n}{n}$$

   c.
$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$$

   d.
$$\sum_{n=2}^{\infty} \frac{\log n}{n^2}$$

2. (17.4) Prove that the function $\sqrt{x}$ is continuous on its natural domain $[0, \infty)$. *Hint:* Think about Example 5 in §8.

3. (17.5)

   a. Prove that x^m is a continuous function for any $m \in \mathbb{N}$.

   b. Prove that every polynomial function $\sum_{i=0}^{n} a_i x^i$ is continuous.

4. (17.13)

   a. Let $f(x) = 1$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show that f is discontinous at every x in $\mathbb{R}$.

   b. Let $h(x) = x$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show that h is continuous at 0 and at no other point.

5. Let $E \subseteq \mathbb{R}$ be a set of real numbers. An element $s_0 \in E$ is *interior* to E if for some $r > 0$ we have

$$\{s \in S : |s - s_0| < r\} = (s_0 - r, s_0 + r) \subset E.$$

   The set E is *open in* $\mathbb{R}$ if every point in E is interior to E.

   Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and for any subset $V \subset R$, let $f^{-1}(V) \subset R$ be the set;

$$f^{-1}(V) = \{x \in \mathbb{R} : f(x) \in V\}.$$

   Show that f is continuous if and only if, for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is also an open set.