Math 317 Midterm Exam #2 Practice Problems

1. Give an example for each of the following, or explain conclusively and clearly why one cannot exist, stating any facts, definitions, and theorems that apply.
   (a) A series which converges, but does not converge absolutely.
   (b) A function which is uniformly continuous on a set S and unbounded on S.
   (c) A function f and a sequence (x_n) so that (x_n) converges, but $(f(x_n))$ diverges.
   (d) A power series $g(x)$ with radius of convergence 1 that converges for $x = -1$.
   (e) A function which is uniformly continuous but not continuous.
   (f) A uniformly convergent function series that is not a power series.
   (g) A sequence (x_n) for which $\liminf x_n \neq \limsup x_n$.
   (h) A sequence (s_n) for which $\liminf s_n$ is undefined.

2. Answer whether each statement is true or false. If the statement is true, give a brief explanation. If the statement is false, provide a counterexample.
   (a) Any function $f : \{3\} \to \mathbb{R}$ is continuous at 3.
   (b) If (x_n) is a sequence of real numbers and (x_{n_k}) is any subsequence, then
   $$\limsup_{n\to\infty} x_n \leq \limsup_{k\to\infty} x_{n_k}.$$
   (c) If a function $f(x)$ is continuous on a set S and (x_n) is a sequence in S which converges to a real number x, then $\lim f(x_n) = f(x)$.
   (d) Every sequence of continuous functions converges pointwise to a continuous function.
   (e) Any function $h : \mathbb{R} \to \mathbb{R}$ satisfying
   $$\lim_{c\to 0} h(x+c) - h(x-c) = 0$$
   is continuous.
   (f) The equation $2x^{14} - 10x^7 + 2 = 5x^{61} + 4x^{19} - 1$ has a solution in the interval $[0, 1]$.

3. Define the series $\sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$. Explain why the series converges uniformly to a continuous function $W_0(x)$ on the interval $(-1/e, 1/e)$. (The function $W_0(x)$ actually can be defined on a larger domain and is called the main branch of the *Lambert W function.* One nice property of this function is that, $W_0(xe^x) = x$ whenever $x \geq -1$.)

4. Find the interval of convergence of the power series
$$\sum_{n=0}^{\infty} \frac{2^n}{\sqrt{n}} x^n$$