Queueing Theory and Simulation, lecture 7

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Contents

1 Overview

1.1 Past

- Sakasegawa's formula for $\mathsf{E}[W(G/C/c)]$.
- Analysis of server interruptions (setups, adjustments and failures) and batching on $\mathsf{E}[J]$.

1.2 Now

- End of approximations: $G/G/1$ queue in tandem
- Start with exact work: renewal reward theorem

1.3 Future,

- Some more (very) useful insights (theorems)
- Exact queueing models

1.4 Assignments

- Suppose your partner of that week does not contribute, mail me.
- I'll ask the partner whether your complaint is correct and redistribute the grade.
- But, I do *not* tell you what grade I give to the partner (Privacy matters).
- I organized the questions for assignment 4.

1.5 Exercises

- If you struggle with the exercises, then that is a clear sign that you have to reread (and rethink!)

what you learned during the courses Probability Theory and Probability Distributions. All probability topics we use in this cours on queueing have been taught in these two courses.

2 GG1 queues in tandem

2.1 Arrivals become departures

- Consider stations in tandem: one after the other
- The departures of a station (except the last) move to a downstream station.
- For Sakasegawa's formula we need λ, $\mathsf{E}[S], c, C_a^2, C_s^2$.
- For each station we can characterize λ, $\mathsf{E}[S], c, C_s^2$, but what is C_a^2 for an intermediate station?
- $C_{a,i+1}^2 = C_{d,i}^2$.
- $C_{d,i}^2 \approx (1-\rho_i^2)C_{a,i}^2 + \rho_i^2 C_{s,i}^2$.
- And now we can analyze each station in sequence!

2.2 Extensions to general networks

- There are models to analyze general networks of $G/G/c$ queues, but we need algorithms to get numerical output.
- In one of the last sections of the book we analyze open networks of $M/M/c$ queues
- Closed networks (networks in which the number of jobs remain constant, departures are replaced right away with new arrivals) are useful to analyze production networks.

3 Renewal reward

3.1 Intuition

How much do I earn on average per hour if you pay me:

- Exactly 10 Euro exactly every hour?
- $X \sim U\{0,20\}$ Euro exactly every hour?
- Exactly 10 Euro every $T \sim U\{0,2\}$ hour?
- $X \sim U\{0,20\}$ every $T \sim U\{0,2\}$ hour?

3.2 Renewal reward theorem (RRT)

Assumptions and definitions:

- The reward $\{Y(t)\}$ of a system is non-decreasing.
- We sample the system at times $0 < T_1 < T_2 \ldots$.
- The increment $X_k = Y(T_k) - Y(T_{k-1})$.
- $N(t)$ is the number of samples taken up to time t.
- $\lambda = \lim_{t\to\infty} N(t)/t$ is the sample rate.
- $Y = \lim_{t\to\infty} Y(t)/t$ is the reward rate
- $X = \lim_{n\to\infty} \sum_{k=1}^{n} X_k/n$ is the average reward earned per sample.

Theorem: $Y = \lambda X$.

3.3 A first application of the RRT

For a queuing system $\rho = \lambda\,\mathsf{E}\,[S]$ is equal to the fraction of time the servers are busy.

$$T_k = D_k, \qquad k/D_k \to \delta, \tag{1}$$

$$Y(t) = \int_0^t I_{L(s)>0}\,\mathrm{d}s \qquad Y(t)/t \to \text{fraction busy} \tag{2}$$

$$Y(T_k) = Y(D_k) = \sum_{k=1}^{D(t)} S_k, \qquad \sum_{k=1}^{n} S_k/n \to \mathsf{E}\,[S] \tag{3}$$

$$Y = \lambda X = \delta\,\mathsf{E}\,[S], \tag{4}$$

since λ in the RRT is the departure rate and $X = \mathsf{E}\,[S]$. Finally, in rate stable systems, the arrival rate λ is the departure rate δ.