Queueing Theory and Simulation, lecture 10

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Contents

1 Overview

1.1 Past

- renewal reward: $Y = \lambda X$
- Little's law $\mathsf{E}[L] = \lambda \mathsf{E}[W]$
- PASTA: $\pi(n) = p(n)$.

1.2 Now

- An exercise of 6.2.
- Rest of 6.2 is self study. The ideas behind server planning are simple but important to remember.
- Batch queues $M^X/M/1$.

1.3 Next

- $\mathsf{E}[W(M/G/1)]$, i.e., Pollaczek-Khinchine equation
- $M^X/M/1$ queue length distribution

1.4 General things

- Do the exercises at the end of the sections! I put the derivations in the exercises to keep the main text clear and clean.

2 Applications, Exercise 6.2.4

2.1 Model

- Fast-food stand with two servers, $S \sim \text{Exp}(5)$.
- When there is no one in the queue, people are reluctant to use the stand, fearing that the food is unsavory.
- People are also reluctant to use the stand when the queue is long.
- $\lambda(0) = 10$, $\lambda(1) = 15$, $\lambda(2) = 15$, $\lambda(3) = 10$, $\lambda(4) = 5$, $\lambda(n) = 0, n \geq 5$.
- Poisson arrivals, but conditional on queue length.
- This is a queue with balking.

2.2 Questions

- Calculate the steady state probabilities.
- What is the average arrival rate?
- $\mathsf{E}[L]$, $\mathsf{E}[Q]$, $\mathsf{E}[J]$ and $\mathsf{E}[J]$

The code looks great!

2.3 Code

Python Code

```python
labda = np.array([10.0, 15.0, 15.0, 10.0, 5.0])
mu = np.array([0., 5., 10., 10., 10., 10.])
c = 2
p = np.ones_like(mu)
for i in range(len(labda)):
    p[i+1] = labda[i] *p[i]/ mu[i+1]
p /= p.sum()
labdaBar = sum(labda[n] * p[n] for n in range(len(labda)))
L = sum(n * p[n] for n in range(len(p)))
Q = sum(max(n - c,0) * p[n] for n in range(len(p)))
J = L / labdaBar
W = Q / labdaBar
```

3 $M^X/M/1$ queue, expected waiting time

3.1 Model

- Jobs arrive as a Poisson process with rate λ
- Jobs arrive at batches of items
- Batch sizes $\{B_i\}$ iid, $B_i \sim B$.
- Items in batch have idd service times $\sim S$.

Goal: find an expression for the expected waiting time $\mathsf{E}[W]$ in the queue. ($\mathsf{E}[J] = \mathsf{E}[W] + \mathsf{E}[S]$)

3.2 Step 1

$$
\begin{aligned}
\rho &= \lambda \mathsf{E}[B]\, \mathsf{E}[S] \\
\mathsf{E}[L] &= \mathsf{E}\left[Q^B\right] \mathsf{E}[B] + \mathsf{E}\left[L_s^B\right] \\
\mathsf{E}\left[W^B\right] &= \mathsf{E}\left[Q^B\right] \mathsf{E}[B]\, \mathsf{E}[S] + \mathsf{E}\left[L_s^B\right] \mathsf{E}[S] \\
&= \mathsf{E}[L]\, \mathsf{E}[S] = \mathsf{E}[W], \\
\mathsf{E}\left[Q^B\right] &= \lambda \mathsf{E}\left[W^B\right], \\
&\implies \\
\mathsf{E}[W] &= \mathsf{E}\left[W^B\right] = \frac{\mathsf{E}\left[L_s^B\right]}{1-\rho} \mathsf{E}[S]
\end{aligned}
$$

Next step: What is $\mathsf{E}\left[L_s^B\right]$?

3.3 Find expected number at the server.

$$
\begin{aligned}
Y(t) &= \int_0^t L_s^B(s)\, \mathrm{d}s, && (1) \\
\mathsf{E}\left[L_s^B\right] &= \lim_t \frac{Y(t)}{t} = Y = \lambda X. && (2) \\
X_k &= Y(D_k) - Y(D_{k-1}), && (3) \\
\mathsf{E}[X_k \mid B_k = b] &= \frac{b(b+1)}{2} \mathsf{E}[S] && (4) \\
\mathsf{E}[X_k \mid B_k] &= \frac{B_k(B_k+1)}{2} \mathsf{E}[S] && (5) \\
X = \mathsf{E}[\mathsf{E}[X_k \mid B_k]] &= \frac{\mathsf{E}\left[B^2\right] + \mathsf{E}[B]}{2} \mathsf{E}[S] && (6) \\
& && (7)
\end{aligned}
$$

3.4 Rest of Section 6.3

- Polishing (see the book) gives:

$$
\mathsf{E}[W] = \frac{1 + C_s^2}{2} \frac{\rho}{1-\rho} \mathsf{E}[B]\, \mathsf{E}[S] + \frac{1}{2} \frac{\rho}{1-\rho} \mathsf{E}[S]. \tag{8}
$$

- This is remarkably similar to Sakasegawa's formula!
- Self study: a derivation of $\mathsf{P}[L_s = i]$ by means of $Y = \lambda X$.