Queueing Theory and Simulation, lecture 9

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June 22, 2021

Contents

1 Overview

1.1 Past

- Construction/simulation and approximation of queueing systems
- Use of sample paths in level crossing arguments to derive $\lambda(n)p(n) = \mu(n+1)p(n+1)$, recursion.
- Use of sample paths to obtain insight in the PASTA property $\lambda(n) = \lambda \implies p(n) = \pi(n)$.

1.2 Now

- Little's law, 5.4
- Simple exact queueing systems, 6.1
- BTW: list of non-trivial typos in the queueing book, see nestor.

1.3 Next

- Many applications: 6.2
- Queues with batch arrivals, 6.3
- MG1 queue, motivation behind Sakasegawa's formula, 6.4

2 Little's law

2.1 Crude population dynamics

For ease, assume

- The Netherlands has (roughly stable) 16 million inhabitants (= average number of people in the system)
- people live on average 80 years (= average sojourn time on this planet)

How many new borns do we have per year? How many people die per week? $16000000/80 = 200000$ per year ≈ 4000 per week.

This is Little's law: $\mathsf{E}[L] = \lambda\,\mathsf{E}[J] \implies \lambda = \mathsf{E}[L] / \mathsf{E}[J]$.

2.2 Mortality rate USA

- Death rate: 869.7 deaths per 100,000 population, see `https://www.cdc.gov/nchs/fastats/deaths.htm`.
- Should we believe this?
- $100000/870 > 115$ year.
- Can this really be true?
- Two plausible hypotheses :-) : They export nearly dead people to other countries, or they hire the RIVM to do the statistics.

2.3 Concepts and main observation

$$
\begin{aligned}
L(s) &= A(s) - D(s) && (1)\\
T &= \text{Time at which system is empty} && (2)\\
J_k &= \int_0^T I_{A_k \leq s < D_k}\, \mathrm{d}s, \quad k \leq A(T) && (3)\\
L(s) &= \sum_{k=1}^{A(T)} I_{A_k \leq s < D_k}, \quad s \leq T && (4)\\
&\implies && (5)\\
\int_0^T L(s)\, \mathrm{d}s &= \int_0^T \sum_{k=1}^{A(T)} I_{A_k \leq s < D_k}\, \mathrm{d}s = \sum_{k=1}^{A(T)} J_k. && (6)
\end{aligned}
$$

2.4 Taking limits

$$\{T_k\} = \text{Sequence of system empty times} \tag{7}$$

$$\frac{1}{T_k}\int_0^{T_k} L(s)\ \mathrm{d}s = \frac{A(T_k)}{T_k}\frac{1}{A(T_k)}\sum_{n=1}^{A(T_k)} J_k \tag{8}$$

$$k \to \infty \implies \tag{9}$$

$$\mathsf{E}[L] = \lambda\,\mathsf{E}[J] \tag{10}$$

3 PASTA, Little's law, memoryless Combined

3.1 M/M/1, expected waiting time

$$\mathsf{E}[L] = \lambda\,\mathsf{E}[J] \tag{11}$$

$$\mathsf{E}[W] = \mathsf{E}[L]\ \mathsf{E}[S], \quad \text{memoryless service} \tag{12}$$

$$\mathsf{E}[J] = \mathsf{E}[W] + \mathsf{E}[S] \tag{13}$$

$$\implies \tag{14}$$

$$\mathsf{E}[L] = \lambda(\mathsf{E}[W] + \mathsf{E}[S]) = \lambda\,\mathsf{E}[W] + \rho \tag{15}$$

$$= \lambda\,\mathsf{E}[L]\ \mathsf{E}[S] + \rho \tag{16}$$

$$\implies \tag{17}$$

$$\mathsf{E}[L] = \rho/(1-\rho) \tag{18}$$

3.2 M/M/1 stationary distribution

$$\lambda(n)p(n) = \mu(n+1)p(n+1) \tag{19}$$

$$\tag{20}$$

For the $M/M/1$ queue: $\lambda(n) = \lambda, \mu(n) = \mu \implies$

$$p(n) = \frac{\lambda}{\mu}p(n-1) = \rho p(n-1) = \rho^n p(0) \tag{21}$$

$$1 = \sum_{n=0}^{\infty} p(n) = p(0)\sum_{n=0}^{\infty}\rho^n = \frac{p(0)}{1-\rho} \tag{22}$$

$$\implies \tag{23}$$

$$p(0) = 1-\rho \implies p(n) = (1-\rho)\rho^n \tag{24}$$

3.3 Specific queueing systems, 6.1

Recall:

- $M/M/1$: $\lambda(n) = \lambda, \mu(n) = \mu$.
- $M/M/\infty$: $\lambda(n) = \lambda, \mu(n) = \mu n$.

- $M/M/c$: $\lambda(n) = \lambda, \mu(n) = \mu \min\{n, c\}$.
- $M/M/c/c$: \$λ(n) = λ \1{n< c}, μ(n) = μ n.

Do the exercises of 6.1