# Analysis of Queueing Systems with Sample Paths and Simulation

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CONTENTS

# 1

# INTRODUCTION

In the first section of this chapter we provide some high level motivation why to study queueing systems and an overview of the book. In the second we recall some notation and results of analysis and probability that we use throughout the book.

## 1.1 MOTIVATION AND OVERVIEW

Queueing systems abound, and the analysis and control of queueing systems are major topics in the control, performance evaluation and optimization of production and service systems.

AT MY LOCAL SUPERMARKET, for instance, any customer gets his/her groceries for free when there is a queue of 4 or more customers and there is an unoccupied cashier desk. The manager that controls the occupation of the cashier positions is keen on keeping the queue short. Now this is easy enough: just hire many cashiers. However, the cost of personnel may then outweigh the yearly average cost of paying the customer penalties. Thus, the manager's problem is to control the service capacity such the penalties and the personnel cost are about equally large.

Fast food restaurants also deal with many interesting queueing situations. Consider, for instance, the preparation of hamburgers. Typically, hamburgers are made-to-stock, in other words, they are prepared before the actual demand has arrived. Thus, hamburgers in stock can be interpreted as customers in queue waiting for service, where the service time is the time between the arrival of two customers that buy hamburgers. The hamburgers have a typical lifetime, and they have to be scrapped if they remain on the shelf longer than a specified amount of time. Of course, it is easy to achieve zero scrap cost, simply by keeping no stock at all. However, to prevent lost-sales, it is important to maintain a certain amount of hamburgers in stock. In this case, the manager has to balance the scrap cost against the cost of lost sales.

Service systems, such as hospitals, call centers, courts, and so on, have a certain capacity available to serve customers. The performance of such systems is, in part, measured by the total number of jobs processed per year and the fraction of jobs processed within a certain time frame between receiving and closing the job. Here the problem is to organize the capacity such that the sojourn time, i.e., the typical time a job spends in the system, does not exceed some threshold.

CLEARLY, IN ALL these examples, the performance of the queueing system has to be monitored and controlled. Typically the following performance measures are relevant.

1. The fraction of time $p(n)$ that the system contains $n$ customers. In particular, $1 - p(0)$, i.e., the fraction of time the system contains jobs, is important, as this is a measure of the time-average occupancy of the servers, hence related to personnel cost.

2. The fraction of customers $\pi(n)$ that 'see upon arrival' the system with $n$ customers. This measure relates to customer perception and lost sales, i.e., fractions of arriving customers that do not enter the system.

3. The average, variance, and/or distribution of the waiting time.

4. The average, variance, and/or distribution of the number of customers in the system.

In this book, we will be primarily concerned with making models of queueing systems such that we can compute or estimate these performance measures.

IN CHAPTER 2 WE start with constructing queueing systems in discrete time and continuous time. This serves three goals. First, construction is concrete, so that by specifying the rules to characterize the behavior of the system, you (the reader) develop essential modeling skills. Second, these rules can often be easily implemented in computer code and used to simulate actual queueing system. Simulation is in general the best way to analyze practical queueing systems, as realistic systems seldom yield to mathematics. Third, we will use sample-path arguments to develop the theoretical results of Chapter 6 and onwards, and these sample paths are precisely what simulators produce.

NOTWITHSTANDING THE POWER of simulation, it is often hard to obtain structural understanding of the behavior of queueing systems. Instead, mathematical models, whether exact or approximate, are useful to help reason about and improve queueing systems. In Chapter 4 we use approximations and general results of probability theory to understand how production and service situations are affected by the system parameters such as service speed, batching rules, and outages. As such, the first two chapters illustrate and motivate the study of practical queueing systems,

ONCE IT IS clear what queueing theory is about, the stage is set for a more mathematical treatment of queueing systems. In Chapter 5 we develop some necessary key results. For this, we use sample paths of queueing process as a guiding principle, and assume that sample paths capture the 'normal' stochastic behavior of the system (with probability one.) Since we can also construct sample paths with simulation, sample paths form a direct bridge between the practical aspects of queueing theory of Chapter 2 and the mathematical analysis of the rest of the book.

IN CHAPTER 6 WE develop exact models for single-station queueing systems and in Chapter 7 we extend our study to simple examples of queueing control and queueing networks. Here we combine the material of the earlier chapters with other mathematical tools such as difference and differential

equations, and non-negative matrices. As such, the last chapter in particular provides a stepping stone to Markov chains, Markov decision theory, optimization and dynamic programming.

IN OUR DISCUSSIONS we mostly focus on obtaining an intuitive understanding of the analytical tools. For proofs and/or more extensive results we refer to the bibliography at the end of the book.

WHILE THE MAIN text contains some examples and derivations, most of the examples are delegated to exercises. Also, some of these exercises are consistency checks between results derived for different queueing models, and thereby provide important relations between various parts of the text. We note in passing that, while such checks are trivial in principle, the algebra can be quite difficult at times. The exercises are not meant to be really easy; they should require (some) work. Hints and solutions to all problems. are available at the end of the book.

## 1.2 INDISPENSABLE KNOWLEDGE

THERE ARE SOME fundamental concepts you are supposed to have seen earlier in your career. We recall them here, and use them over and over in the rest of the book, mostly without reference.

WE INTRODUCE THE following notation:

$$\mathbb{1}_A = \begin{cases} 1, & \text{if } A \text{ is true,} \\ 0, & \text{if } A \text{ is false.} \end{cases}, \qquad [x]^+ = x\,\mathbb{1}_{x\geq 0}, \tag{1.2.1}$$

$$f(x-) = \lim_{y\uparrow x} f(y), \qquad f(x+) = \lim_{y\downarrow x} f(y). \tag{1.2.2}$$

The (set) function $\mathbb{1}$ is known as the *indicator function*.

We write $f(h) = o(h)$ for a function $f$ to say that $f$ is such that $f(h)/h \to 0$ as $h \to 0$. If we write $f(h) = o(h)$ it is implicit that $|h| \ll 1$. We call this *small o notation*.

You should know that:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i, \tag{1.2.3a}$$

$$e^x = \lim_{n\to\infty} (1 + x/n)^n, \tag{1.2.3b}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{1.2.3c}$$

$$\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha}. \tag{1.2.3d}$$

FOR A NON-NEGATIVE, integer-valued random variable $X$ with *probability mass function* $f(k) = \mathsf{P}\{X = k\}$, *distribution function* $F(k) = \mathsf{P}\{X \le k\}$ and *survivor function* $G(k) = \mathsf{P}\{X > k\}$ we have

$$X = \sum_{n=0}^{\infty} X \mathbb{1}_{X=n} = \sum_{n=0}^{\infty} n \mathbb{1}_{X=n}, \tag{1.2.4a}$$

$$\mathsf{E}[\mathbb{1}_{X\le x}] = \mathsf{P}\{X \le x\}, \tag{1.2.4b}$$

$$\mathsf{E}[X] = \mathsf{E}\left[\sum_{n=0}^{\infty} n \mathbb{1}_{X=n}\right] = \sum_{n=0}^{\infty} n \mathsf{E}[\mathbb{1}_{X=n}] = \sum_{n=0}^{\infty} n f(n), \tag{1.2.4c}$$

$$\mathsf{E}\big[g(X)\big] = \sum_{n=0}^{\infty} g(n) f(n), \tag{1.2.4d}$$

$$\mathsf{V}[X] = \mathsf{E}\big[X^2\big] - (\mathsf{E}[X])^2. \tag{1.2.4e}$$

(1.2.4f)

Eq. (1.2.4d) is known as the *law of the unconscious statistician*.

Let $X$ be a continuous non-negative random variable with pdf $f$. We write $\mathsf{E}[X] = \int_0^\infty x f(x)\,\mathrm{d}x$ for the expectation of $X$. Recall LOTUS: $\mathsf{E}\big[g(X)\big] = \int_0^\infty g(x) f(x)\,\mathrm{d}x$.

Sometimes we write $\int \mathrm{d}F(x)$ rather than $\int f(x)\,\mathrm{d}x$. The reason behind this notation has to do with jumps in the cdf $F$. As an example, suppose the rv $X$ is such that $\mathsf{P}\big\{Xleqx\big\} = 0$ for $x < 0$, $\mathsf{P}\{X = 0\} = 1/2$, and $\mathsf{P}\{X \le x\} = (2 - e^{-\lambda x})/2$ for $x > 0$. Then $X$ does not have a pdf $f$ everywhere, in particular, $F$ makes a jump of size $1/2$ at 0. For such mixed rvs, we write $\mathrm{d}F(x)$ to include the jumps and the continuous part in an integral. To see how this works, consider some general continuous function $g$. Then, for the $X$ above,

$$\begin{aligned}\mathsf{E}\big[g(X)\big] &= \int_{-\infty}^{0-} g(x) f(x)\,\mathrm{d}x + g(0)/2 + \int_{0+}^{\infty} g(x) f(x)\,\mathrm{d}x \\ &= g(0)/2 + 1/2 \int_{0+}^{\infty} g(x) \lambda e^{-\lambda x}\,\mathrm{d}x,\end{aligned}$$

since $f(x) = 0$ for $x < 0$ and $f(x) = \lambda e^{-\lambda x}/2$ for $x > 0$, but $f(0)$ is not defined.

You should be able to use indicator functions and integration by parts to show that $\mathsf{E}\big[X^2\big] = 2\int_0^\infty y G(y)\,\mathrm{d}y$, where $G(x) = 1 - F(x)$, provided the second moment exists.

For general random variables $X$ and $Y$:

$$\mathsf{E}[X + Y] = \mathsf{E}[X] + \mathsf{E}[Y], \tag{1.2.5}$$

and, if $X$ and $Y$ are independent so that $\mathsf{E}[XY] = \mathsf{E}[X]\,\mathsf{E}[Y]$,

$$\mathsf{V}[X + Y] = \mathsf{V}[X] + \mathsf{V}[Y]. \tag{1.2.6}$$

THE *moment-generating function* $M_X(s)$ of a random variable $X$ is defined for $s \in \mathbb{R}$ sufficiently small as:

$$M_X(s) = \mathsf{E}\big[e^{sX}\big]. \tag{1.2.7a}$$

$M_X(s)$ uniquely characterizes the distribution of $X$. From this definition it follows that:

$$\mathsf{E}[X] = M_X'(0) = \left.\frac{\mathrm{d}M_X(s)}{ds}\right|_{s=0}, \tag{1.2.7b}$$

$$\mathsf{E}\big[X^2\big] = M_X''(0), \tag{1.2.7c}$$

and, if $X$ and $Y$ are independent,

$$M_{X+Y}(s) = M_X(s) \cdot M_Y(s). \tag{1.2.7d}$$

ABOUT *conditional probability* you should know that

$$\mathsf{P}\{A \mid B\} = \frac{\mathsf{P}\{AB\}}{\mathsf{P}\{B\}}, \quad \text{if } \mathsf{P}\{B\} > 0, \tag{1.2.8a}$$

and, if $A = \bigcup_{i=1}^{n} B_i$ with $\mathsf{P}\{B_i > 0\}$ for all $i$,

$$\mathsf{P}\{A\} = \sum_{i=1}^{n} \mathsf{P}\{AB_i\} = \sum_{i=1}^{n} \mathsf{P}\{A \mid B_i\}\, \mathsf{P}\{B_i\}. \tag{1.2.8b}$$

WE SAY THAT a set $\{X_k\}$ of iid rvs are *distributed as the common rv* $X$ when all $X_k$ have the same cdf as $X$.

**Ex 1.2.1.** Show that $|h|^\alpha = o(h)$ for all $\alpha > 1$. Conclude in particular that $ah^2 = o(h)$ for any constant $a$.

**Ex 1.2.2.** Let $c$ be a constant (in $\mathbb{R}$) and the functions $f$ and $g$ both of $o(h)$. Then show that (1) $f(h) \to 0$ when $h \to 0$, (2) $c \cdot f = o(h)$, (3) $f + g = o(h)$, and (4) $f \cdot g = o(h)$.

**Ex 1.2.3.** Why is $e^x = 1 + x + o(x)$?

**Ex 1.2.4.** Why is (1.2.4a) true?

**Ex 1.2.5.** Show that $G(k) = \sum_{m=0}^{\infty} \mathbb{1}_{m>k} f(m)$ for discrete $X$.

**Ex 1.2.6.** For a nonnegative discrete random variable $X$, use indicator functions to prove that $\mathsf{E}[X] = \sum_{k=0}^{\infty} G(k)$.

**Ex 1.2.7.** For discrete $X$, use indicator functions to prove that $\sum_{i=0}^{\infty} iG(i) = \mathsf{E}\left[X^2\right]/2 - \mathsf{E}[X]/2$.

**Ex 1.2.8.** For general non-negative $X$, use indicator functions to prove that $\mathsf{E}[X] = \int_0^\infty x\,\mathrm{d}F(x) = \int_0^\infty G(y)\,\mathrm{d}y$, where $G(x) = 1 - F(x)$.

**Ex 1.2.9.** Use indicator functions to prove that for a general non-negative random variable $X$, $\mathsf{E}\left[X^2\right] = 2\int_0^\infty yG(y)\,\mathrm{d}y$.

**Ex 1.2.10.** Show that $\mathsf{E}\left[X^2\right]/2 = \int_0^\infty yG(y)\,\mathrm{d}y$ for a continuous non-negative random variable $X$ with survivor function $G$.

**Ex 1.2.11.** What is the value of $M_X(0)$?

**Ex 1.2.12.** We have one gift to give to one out of three children. As we cannot divide the gift into parts, we decide to let 'fate decide'. That is, we choose a random number in the set $\{1,2,3\}$. The first child that guesses this number wins the gift. Show that the probability of winning the gift is the same for each child.

# 2

# CONSTRUCTION AND SIMULATION OF QUEUEING SYSTEMS

The first step to analyze a queueing system is to model it. And for this, there is often not a better start than to build a simulation model. For this reason, the aim of this first chapter is to teach you how to construct and simulate queueing processes.

In Section 2.1 we build discrete-time models of queueing systems, which means that we use the number of jobs that arrive and can be served in fixed periods of time to construct the queueing process. Such a period can be an hour, or a day; in fact, any amount of time that makes sense in the context in which the model will be used. Typically, we model the number of arrivals and potential services as random variables, and in many practical settings it is reasonable to take the number of arrivals in a period as Poisson distributed. This being the case, we consider the Poisson distribution in Section 2.2, and once we have an understanding of this process, we can use random number generators to generate (Poisson distributed) random numbers of arrivals and services to drive the simulator.

In Section 2.3 we focus on constructing queueing processes in continuous time. In this setting, the inter-arrival times and service times of individual jobs become of importance, and then exponentially distributed random variables play a fundamental role. We therefore discuss the properties of the exponential distribution in Section 2.4. There we also mention the interesting and close relationship between the exponential distribution and the Poisson distribution.

As will become apparent, both types of constructing queueing processes, the discrete-time and continuous-time models, are easy to implement as computer programs. We include many exercises to show you the astonishing diversity of queueing systems that can be analyzed by simulation. In passing, we develop a number of performance measures to provide insight into the (transient and long-run average) behavior of queueing processes.

## 2.1 QUEUEING PROCESSES IN DISCRETE-TIME

We start with a case to provide motivation to study queueing systems. Then we develop a set of recursions of fundamental importance by which we can simulate the case and evaluate the efficacy of several policies to improve the system.

To illustrate the power of this approach, this section contains many exercises in which you are asked to model different queueing situations by means of such recursions. Once the recursions are obtained, the These recursions are often quite easy so that they can also be useful to discuss

how well the model captures 'reality' with managers, medical doctors, and so on.

AT A MENTAL HEALTH department, five psychiatrists do intakes of future patients to determine the best treatment process once patients 'enter the system'. There are complaints about the time patients have to wait for the intake; the desired waiting time is around two weeks, but the realized waiting time is sometimes more than three months. This is unacceptably long, but ... what to do about it?

THE FIVE PSYCHIATRISTS put forward various suggestions to reduce the waiting times.

1. Give all psychiatrists the same weekly capacity for doing intakes. Motivation: Currently one psychiatrist does 9 intakes per week while two other psychiatrists do only 1. This is not a problem during weeks when all psychiatrists are present; however, psychiatrists take holidays, visit conferences, and so on. So, if the psychiatrist with the most intakes per week goes on leave, this affects the intake capacity considerably.

2. Synchronize holidays.[0] Motivation: to reduce the variation in the service capacity, the psychiatrists plan their holidays consecutively. However, perhaps it is better to work at full capacity or not at all.

3. Control[1] the intake capacity as a function of the waiting time. Motivation: in analogy with supermarkets, scale up (down) capacity when the queue is long (short).

WE NEXT DEVELOP a method to simulate the behavior of the system over time so that we can evaluate the effect of the above suggestions. We use simulation, because the mathematical analysis is too hard.[2]

LET US START with discussing the essentials of the simulation of a queueing system. The easiest way to construct queueing processes is to 'chop up' time in periods[3] and develop recursions for the behavior of the queue from period to period. Using fixed-sized periods has the advantage that we do not have to specify specific inter-arrival times or service times of individual customers; only the number of arrivals in a period and the number of potential services are relevant.

Let us define:

$a_k =$ the number of jobs that arrive *in* period $k$,

$c_k =$ the capacity, i.e., the maximal number of jobs that can be served, during period $k$,

$d_k =$ the number of jobs that depart *in* period $k$,

$L_k =$ the *system length*, i.e., the number of jobs in the system at the *end* of period $k$.[4]

In the sequel we also call $a_k$ the *size of the batch* arriving in period $k$. Note that the definition of $a_k$ is a bit subtle: we may assume that the arriving

[0] *With the insights of Chapter 4 we can immediately see that this is a bad suggestion.*

[1] *People often object to such policies because they believe that they have to do more work. However, this is wrong. Suppose that 1000 patients per year need treatment. This number does not depend on whether they spend time in queue or not.*

[2] *In case of doubt, try to analyze transient multi-server queueing systems with vacations, of which this is an example.*

[3] *The length of these periods depends on context. For the present case, it is appropriate to take weeks. For supermarkets perhaps a length of 5 minutes is best.*

[4] *In this type of queueing system there is not a job in service, we only count the jobs in the system at the end of a period. Thus, the number of jobs in the system and in queue coincide in this model.*

jobs arrive either at the start or at the end of the period. In the first case, the jobs can be served in period $k$, in the latter case, they *cannot* be served in period $k$.

Since $L_{k-1}$ is the system length at the *end* of period $k-1$, it must also be the number of customers at the *start* of period $k$. Assuming that jobs arriving in period $k$ cannot be served in period $k$, the number of customers that depart in period $k$ is therefore

$$d_k = \min\{L_{k-1}, c_k\}, \tag{2.1.1a}$$

Now that we know the number of departures, the number at the end of period $k$ is given by

$$L_k = L_{k-1} - d_k + a_k. \tag{2.1.1b}$$

Like this, if we are given $L_0$, we can obtain $L_1$, and from this $L_2$, and so on.[5]

[5] *In a sense, (2.1.1) is the $F = ma$ of a queueing system: Given initial conditions, we apply the rule at time $k-1$ to get the state at time $k$, and so on.*

IT IS IMPORTANT to realize that the above recursions only construct $\{L_k\}$, i.e., the dynamics of the number of jobs in the system. If we also need information about the *sojourn times*, i.e., the time jobs spend in the system, it is necessary to specify the *service discipline*, i.e., a rule that decides on the order in which jobs in queue are taken into service.[6] In this book we assume henceforth that jobs move to the server in the order in which they arrive. This is known as *First-In-First-Out (FIFO)*; First-Come-First Serve (FCFS) is another much used acronym. There are many other rules, such as Last-In-First-Out, but we do not discuss these here.

[6] *[2.1.10]*

WE NOTE THAT there is a difference between *waiting time* $W$ and sojourn time $J$. The former is the time jobs spend in queue, the latter the time in the system, which is the waiting plus the time at the server $L_s$. In the model (2.1.1) we implicitly include service time, so that we should actually speak about sojourn time, which we henceforth do in this section.

IT IS CLEAR that in (2.1.1) we assume that jobs that arrive in period $k$ cannot be served in period $k$. If the situation is such that jobs *can* be served in the same period as they arrive, then (2.1.1) should be changed to[7]

[7] *[2.1.2]*

$$d_k = \min\{L_{k-1} + a_k, c_k\}. \tag{2.1.2}$$

Which of (2.1.1) or (2.1.2) to choose depends on the context, and what we like to model. If we like to be 'on the safe side', then it is perhaps best to use (2.1.1) because with this rule, we overestimate the queue lengths, while with (2.1.2) we underestimate the queue lengths. In general, no rule is 'best'; what is 'good' depends essentially on the context.

OF COURSE WE are not going to carry out the computations for $\{L_k\}$ by hand. Typically, we use company data to model the arrival process $\{a_k\}$ and the capacity $\{c_k\}$, and feed this data into a computer to carry out the recursions (2.1.1). If we do not have sufficient data, we make a probability model for these data and use the computer to generate random numbers with, hopefully, similar characteristics as the real data. At any rate, from this point on, we assume that it is easy, by means of computers, to obtain numbers $a_1, \ldots, a_n$ for $n \gg 1000$, and so on.

Here we continue with the case of the five psychiatrists and analyze the proposed rules to improve the performance of the system. We mainly want to reduce the long sojourn times.

AS A FIRST STEP in the analysis, we model the arrival process of patients as a Poisson process, see Section 2.2. The duration of a period is taken to be a week. The average number of arrivals per period, based on data of the company, was slightly less than 12 per week; in the simulation we set it to $\lambda = 11.8$ per week.

NEXT, WE MODEL the capacity in the form of a matrix such that row $i$ corresponds to the weekly capacity of psychiatrist $i$:

$$C = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & \dots \\ 3 & 3 & 3 & \dots \\ 9 & 9 & 9 & \dots \end{pmatrix}.$$

Thus, psychiatrists 1, 2, and 3 do just one intake per week, the fourth does 3, and the fifth does 9 intakes per week. The sum over column $k$ is the total service capacity for week $k$ of all psychiatrists together.

With the matrix $C$ it is simple to make other capacity schemes. A more balanced scheme would be like this:

$$C = \begin{pmatrix} 2 & 2 & 2 & \dots \\ 2 & 2 & 2 & \dots \\ 3 & 3 & 3 & \dots \\ 4 & 4 & 4 & \dots \\ 4 & 4 & 4 & \dots \end{pmatrix}.$$

Next, we include the effects of holidays on the capacity. This is easily done by setting the capacity of a certain psychiatrist to 0 in a certain week. Let us assume that just one psychiatrist is on leave in a week, each psychiatrist has one week per five weeks off, and the psychiatrists' holiday schemes rotate. To model this, we set $C_{1,1} = C_{2,2} = \cdots = C_{1,6} = C_{2,7} = \cdots = 0$, i.e.,

$$C = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 0 & \dots \\ 2 & 0 & 2 & 2 & 2 & 2 & \dots \\ 3 & 3 & 0 & 3 & 3 & 3 & \dots \\ 4 & 4 & 4 & 0 & 4 & 4 & \dots \\ 4 & 4 & 4 & 4 & 0 & 4 & \dots \end{pmatrix}.$$

Hence, the total average capacity must be $4/5 \cdot (2+2+3+4+4) = 12$ patients per week. The other holiday scheme—all psychiatrists take holiday in the same week–corresponds to setting entire columns to zero, i.e., $C_{i,5} = C_{i,10} = \cdots = 0$ for week 5, 10, and so on. Note that all these variations in holiday schemes result in the same average capacity.

NOW THAT WE have modeled the arrivals and the capacities, we can use the recursions (2.1.1) to simulate the queue length process for the four different scenarios proposed by the psychiatrists, unbalanced versus balanced capacity, and spread out holidays versus simultaneous holidays.

The results are shown in Fig. 2.1.1. We plot, for each period, the largest and the smallest queue that occurred under all four capacity plans that result from following the first and second suggestions of the psychiatrists. The graphs make clear that these suggestions hardly affect the behavior of the queue length process.

Now we consider Suggestion 3, which comes down to doing more intakes when it is busy, and fewer when it is quiet. A simple rule is to let the capacity for week $k$ depend on the queue length of the week before, for instance,

$$c_k = \begin{cases} 12+e, & \text{if } L_{k-1} \geq 24, \\ 12-e, & \text{if } L_{k-1} \leq 12. \end{cases} \tag{2.1.3}$$

We can take $e = 1$ or 2, or perhaps a larger number; the larger $e$, the larger the control we exercise. We can of course also adapt the thresholds 12 and 24.

Let's consider three different control levels, $e = 1$, $e = 2$, and $e = 5$; when $e = 5$, each psychiatrist does one extra intake. The results, see Fig. 2.1.2, show a striking difference indeed. The queue does not explode anymore; already taking $e = 1$ has a large influence.

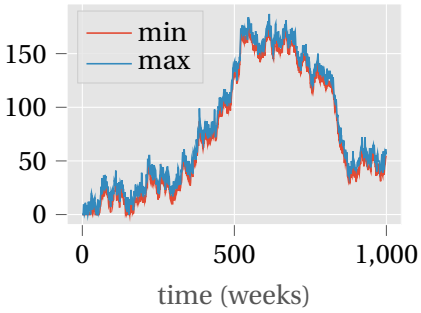



Figure 2.1.1: Effect of capacity and holiday plans. Per time point we plot the maximum and the minimum queue length under the policies.

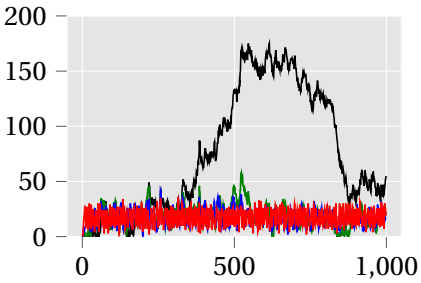



Figure 2.1.2: Controlling the number of intakes.

THIS SIMULATION EXPERIMENT shows that changing holiday plans or spreading the work over multiple servers, i.e., psychiatrists, may not significantly affect the queueing behavior. However, controlling the service rate as a function of the queue length can improve the situation dramatically.

IN GENERAL, with recursions as (2.1.1) we can carry out simple what-if-analyses. For instance, a hospital is considering to buy a second MRI scanner, because the current one is saturated, as it is used from 8 am to 6 pm, and can certainly not serve the forecasted demand. Suppose we can find a percentage of staff willing to work from 8 pm to 11 pm, say, and that patients, 30% perhaps, are also prepared to come to the hospital for an MRI scan between 8 pm and 11 pm.[8] This idea would increase the capacity with about 30% = (3/(18−8)) thereby enabling the hospital to postpone the investment in an MRI scanner with one or two years.

Hence, with simulation we can evaluate the influence on *Key Performance Indicators (KPIs)*, such as average waiting time or longest queue length, of different policies to control the queueing system.

THE READER SHOULD therefore understand that the recursions such as (2.1.1) and control rules such as (2.1.3) as the essential elements of for making a model. Once we have these, we can let the computer do the work of running the recursions many times. Therefore most of the exercises below ask to come up with recursions for specific queueing systems.

As an aside, it may be that the recursions you find are not identical to the recursions of the solutions, because the assumptions you make are slightly different from the ones I make. (In fact, only the recursions completely specify the model, but if the problem statement would contain the recursions, there would be nothing left to practice anymore.)

**Ex 2.1.1.** Suppose that $c_k = 7$ for all $k$, $a_1 = 5$, $a_2 = 4$ $a_3 = 9$, and $L_0 = 8$, compute $L_3$.

[8] *Perhaps some staff and patients even prefer to work/have scans in the evening.*

**Ex 2.1.2.** Prove that the recursion $L_k = [L_{k-1} + a_k - c_k]^+$ generates a system in which jobs can be served in the period they arrive. Then, modify this recursion such that jobs *cannot* be served in the period in which they arrive.

*Serve jobs in the period in which they arrive.*

**Ex 2.1.3.** A queueing system is under periodic review, i.e., at the end of each period the queue length is measured. Jobs arriving at period $k$ cannot be served in period $k$ and the system cannot contain more than $K$ jobs. Develop code to simulate $\{L_k\}$ and compute the amount of jobs lost per period, and the fraction of jobs lost after simulating for $T$ periods.

*Queue with blocking*

**Ex 2.1.4.** All items that are produced by a machine are tested. With probability $p$ an item turns out to be faulty, and is returned to the queue for repair. The production time of new and faulty items is equal. Supposing items cannot be served in the period they arrive, develop a set of recursions to simulate $\{L_k\}$.

*Systems with yield loss*

**Ex 2.1.5.** A fraction $p$ of items do not meet the quality requirements after the first pass at a machine, but require a second pass to repair the problems. Assume that repair jobs need half the service time of new jobs and are served with priority over new jobs. Develop the recursions.

*Rework*

**Ex 2.1.6.** Demand arriving at a single-server queue in a period can be served in that period. At the start of period $k$, the server capacity is set to $c_k$ at cost $\beta c_k$. A job charges $h$ per period in the system. Make a model that computes the cost for the first $T$ periods.

*Cost models*

**Ex 2.1.7.** To ensure that the server capacity is always fully used, a server works at rate $c$ in period $k$ if $L_{k-1} \geq N$ jobs, and not otherwise. Set up the recursions.

*Server controlled by threshold policy*

**Ex 2.1.8.** A machine can switch on and off. Suppose that the jobs that arrive in period $k$ can be served in that period. If the system length hits $N$ (becomes empty) in period $k$, the machine switches on (off) in period $k+1$. It costs $K$ to switch on the machine. There is also a cost $\beta$ per period when the machine is on, and it costs $h$ per period per customer in the system. Make a model to compute the cost for the first $T$ periods. Assume the machine is off at $k = 0$.

*Cost of an N policy.*

**Ex 2.1.9.** Suppose jobs be served in the period in which they arrive. Write $\tau = \min\{k : L_k = 0\}$ for the time to hit 0. When jobs *cannot* be served in the period in which they arrive, modify the definition of $\tau$ such that it still captures the idea of hitting 0.

**Ex 2.1.10.** Take $d_k = \min\{L_{k-1} + a_k, c_k\}$, and assume that jobs are served in FIFO sequence. Find expressions for the shortest (longest) possible sojourn time $J_k^-$ ($J_k^+$) of a job that arrives at time $k$.

*Estimating sojourn time, rather than queue length.*

**Ex 2.1.11.** A machine serves with two queues such that jobs in the first queue get priority over jobs in the other queue. Find the recursions.

*Priority queueing*

**Ex 2.1.12.** One server serves two queues. Each queue receives service capacity in proportion to the queue length. Derive the recursions.

*Proportionally fair queueing*

*Queues with reserved service capacity*

**Ex 2.1.13.** Consider a single-server that serves two parallel queues. Queue $i$ has minimal guaranteed service capacity $r^i$ each period, such that $c_k \geq r^1 + r^2$. Extra capacity beyond the reserved capacity is given to queue 1 with priority. (An example is a psychiatrist who reserves capacity for different patient groups.) Formulate the recursions.

*Queue with protected service capacity and lost capacity*

**Ex 2.1.14.** Consider a single-server that serves two parallel queues. Queue $i$ receives a minimal service capacity $r^i$ every period. Reserved capacity unused for one queue cannot be used to serve the other queue. Any extra capacity beyond the reserved capacity, i.e, $c_k - r^2$, is given to queue 1 with priority. Formulate the recursions.

An example is the operation room of a hospital. There is a weekly capacity, part of which is reserved for emergencies. It might not be possible to assign this reserved capacity to other patient groups, because it should be available at all times for emergency patients. A result of this is that unused capacity is lost. In practice it may not be as extreme as in the model, but still part of the unused capacity is lost. 'Use it, or lose it' often applies to service capacity.

*Tandem networks*

**Ex 2.1.15.** Consider a production network with two production stations in tandem, that is, the jobs processed at station 1 move at the end of period $k$ to station 2. What are the recursions?

*A tandem queue with blocking*

**Ex 2.1.16.** Consider a production network with two production stations in tandem and blocking: the server at station 1 is not allowed to produce more than the amount $M$ station 2 can maximally contain. We assume also that work jobs moves first from station 1 to station 2 before jobs from station 2 leave. Thus, besided the regular conditions, we need to impose $c^1_{k+1} \leq M - L^2_k$. Find recursions.

*Merging streams*

**Ex 2.1.17.** Consider a production situation with two stations A and B that send their products to station C. Derive the recursions.

*Splitting streams*

**Ex 2.1.18.** Consider a paint mixing machine that produces products for two downstream packaging machines A and B, each with its own queue. In the simplest model, the content of the queue at the mixing machine is proportional to the demands $\lambda^A$ and $\lambda^B$ for the packaging machines. Provide the recursions.

*Queues with setup times*

**Ex 2.1.19.** One server serves two parallel queues, one at a time. After serving one queue until it is empty, the server moves to the other queue, which requires one period setup time. Make a model.

An example: a nurse takes blood samples at two departments in a hospital. It takes time to walk from one location to another. An interesting, but hard question: what rule would minimize average waiting time?

## 2.2 POISSON DISTRIBUTION

In this section we first introduce the Poisson process to model the arrival process of jobs[0] that need to receive service at a station.[1]

In the exercises we derive numerous properties of the Poisson process; the rest of the book we will use these results time and again.

[0] *A job can be anything that requires service; it can be a customer, an item to make, a car to repair, and so on.*

[1] *A station is a general name for a shop, a machine, a mechanic, and so on.*

CONSIDER A STREAM OF CUSTOMERS that enter a shop during a time interval of duration $t = 1$ hour.[2] If we chop up the hour into small periods of 1 seconds, it is reasonable to assume that the probability of an arrival in a specific period is small and the probability of two or more arrivals is very small. Moreover, it is reasonable to assume that the probability that a customer enters in a certain second is the same for each second.

More formally, to model such an arrival process, we make a few assumptions. We think about the interval as composed of $n$ many small subintervals, all with equal length $t/n$. We take $n$ so large that we can neglect the probability that two or more customers arrive in one small interval. Moreover, we assume that the arrival of a customer in a certain subinterval is independent of the (non)-arrival of customers in any of the other subintervals, and that the probability of an arrival is given by a (very) small probability $p$.

In precise terms, we model the arrivals of customers as a set of *identically and independently distributed (iid)* random variables $\{B_i\}$ such that $B_i = 1$ denotes an arrival in interval $i$, and $B_i = 0$ no arrival. The arrival probability is $\mathsf{P}\{B_i = 1\} = p$.

*Bernoulli distributed*

It is well-known that the total number of arrivals $N_n(t)$ that occur in the $n$ intervals is *binomially distributed*, i.e.,

$$\mathsf{P}\{N_n(t) = k\} = \binom{n}{k} p^k (1-p)^{n-k}.$$

It is easy to show[3] that the expected number of arrivals during $[0, t]$ is $\mathsf{E}[N_n(t)] = \sum_{i=1}^{n} \mathsf{E}[B_i] = np$.

LET US TAKE the limit $n \to \infty$ and $p \to 0$, but such that the expected number of arrivals $np$ during $[0, t]$ remains constant. Specifically, we take the limit such that $pn = \lambda t$, where the constant $\lambda$ has the interpretation of the *arrival rate* of customers. In this case,[4]

$$\lim_{\substack{n\to\infty, p\to 0 \\ np=\lambda t}} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}. \tag{2.2.1}$$

We say that the random variable $N(t)$ associated with this distribution is *Poisson distributed*, i.e.,

$$\mathsf{P}\{N(t) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!},$$

and we write $N(t) \sim \text{Pois}(()\lambda t)$.[5] With $N(t)$, we define $N(s, t] := N(t) - N(s)$ as the number of customers arriving in the time period $(s, t]$.[6]

THE FAMILY of random variables $N_\lambda = \{N(t), t \geq 0\}$ is a *Poisson process* with rate $\lambda$.[7] We remark that $N_\lambda$ has *stationary and independent increments*. Stationarity means that the distributions of the number of arrivals are the same for all intervals of equal length, that is, $N(s, t]$ has the same distribution as $N(u, v]$ if $t - s = v - u$. Independence means, roughly speaking, that knowing that $N(s, t] = n$, does not help to make any predictions about the value of $N(u, v]$ if the intervals $(s, t]$ and $(u, v]$ do not overlap.[8]

[2] *The unit of time $t$ depends on the context. For example, when we model a shop, it is reasonable to take $t$ in the order of one hour, but when we think about a certain type of operations at a hospital, the $t$ might be weeks.*

[3] *[2.2.1]*

[4] *[2.2.2]*

[5] *[2.2.3]*

[6] *Note that $[0, t]$ is closed at both ends, but $(s, t]$ is open at the left.*

[7] *A random process is a much more complicated mathematical object than a random variable: a process is a (possibly uncountable) set of random variables indexed by time, not just* one *random variable.*

[8] *We refer to the literature on (mathematical) probability theory for further background.*

We next address a number of useful properties of the Poisson process. If $\Delta t \ll 1$ then for all $t$,[9]

[9] [2.2.4]–[2.2.6]

$$\mathsf{P}\{N(t+\Delta t) = n \,|\, N(t) = n\} = 1 - \lambda\Delta t + o(\Delta t), \tag{2.2.2a}$$
$$\mathsf{P}\{N(t+\Delta t) = n+1 \,|\, N(t) = n\} = \lambda\Delta t + o(\Delta t), \tag{2.2.2b}$$
$$\mathsf{P}\{N(t+\Delta t) \geq n+2 \,|\, N(t) = n\} = o(\Delta t). \tag{2.2.2c}$$

The next equation says that if you know that an arrival occurred during $[0, t]$, the arrival is distributed uniformly on the interval $[0, t]$. If $s \in [0, t]$,[10]

[10] [2.2.10]

$$\mathsf{P}\{N(s) = 1 \,|\, N(t) = 1\} = \frac{s}{t}. \tag{2.2.2d}$$

Note that this is independent of $\lambda$. Finally,[11]

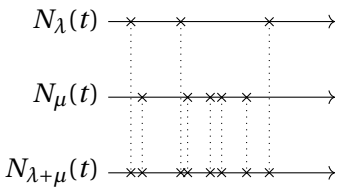
[11] [2.2.7]–[2.2.9]

$$\mathsf{E}[N(t)] = \lambda t, \tag{2.2.2e}$$
$$\mathsf{E}\left[(N(t))^2\right] = \lambda^2 t^2 + \lambda t, \tag{2.2.2f}$$
$$\mathsf{V}[N(t)] = \lambda t. \tag{2.2.2g}$$

THE RELATIVE VARIABILITY of service times is a very important concept in queueing theory. For instance, suppose the standard deviation of customer inter-arrival times is 1 minute. When the mean inter-arrival time is 1 hour, we are inclined to call the process regular, while if the mean is 1 minute, we would call it irregular. To differentiate between such cases, define the *square coefficient of variation (SCV)* of a random variable $X$ as

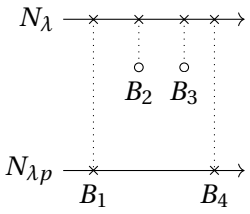
$$C^2 := \frac{\mathsf{V}[X]}{(\mathsf{E}[X])^2}. \tag{2.2.3}$$

MERGING AND SPLITTING of arrival processes often occurs in practice. Consider, for instance, the arrival processes $N_\lambda$ of men and $N_\mu$ of women at a shop, see the figure at the right. Each cross represents an arrival; in the upper line it corresponds to a man, in the middle line to a woman, and in the lower line to an arrival of a general customer at the shop. Thus, the shop 'sees' the merged process of these two arrival processes. In fact, this merged process $N_{\lambda+\mu}$ is also a Poisson process[12] with rate $\lambda + \mu$.




[12] [2.2.12]

We can also *split*, or *thin*, a stream into several sub-streams. Model the stream of people passing by a shop as a Poisson process $N_\lambda$. In the figure at the right, we mark these arrivals as crosses at the upper line. With probability $p$ a person decides, independent of anything else, to enter the shop; the crosses at the lower line are the customers that enter the shop. The Bernoulli random variable $B_1 = 1$ so that the first passerby enters the shop; the second passerby does not enter as $B_2 = 0$, and so on.




The concepts of *merging* and *thinning* are useful to analyze queueing networks, see Section 7.3. Suppose the departure stream of a machine splits into two sub-streams, e.g., a fraction $p$ of the jobs moves on to another machine and the rest $(1-p)$ of the jobs leaves the system. Then we can model the arrival stream at the second machine as a thinned stream (with probability $p$) of the departures of the first machine. Merging occurs where the output streams of various stations arrive at another station.

WITH MOMENT-GENERATING FUNCTIONS we can simplify the derivations above; the last few exercises of this section show how to apply this.[13]

[13] *In general, it is hard to obtain closed-form expressions for the moment-generating function, but when it works, it is an easy and slick technique.*

**Ex 2.2.1.** Show that $\mathsf{E}[N_n(t)] = \sum_{i=1}^{n} \mathsf{E}[B_i] = np$.

**Ex 2.2.2.** Show (2.2.1)

**Ex 2.2.3.** What is the difference between $N_n(t)$ and $N(t)$?

**Ex 2.2.4.** Show (2.2.2a).

**Ex 2.2.5.** Show (2.2.2b).

**Ex 2.2.6.** Show (2.2.2c)

**Ex 2.2.7.** Show (2.2.2e).

**Ex 2.2.8.** Show (2.2.2f).

**Ex 2.2.9.** Show (2.2.2g).

**Ex 2.2.10.** Show (2.2.2d).

**Ex 2.2.11.** Show that the SCV of $N(t) \sim P(\lambda t)$ is equal to $1/(\lambda t)$. What does this mean for $t$ large?

**Ex 2.2.12.** If the Poisson arrival processes $N_\lambda$ and $N_\mu$ are independent, show with a conditioning argument that the merged process $N_\lambda + N_\mu$ is a Poisson process with rate $\lambda + \mu$.

**Ex 2.2.13.** If the Poisson arrival processes $N_\lambda$ and $N_\mu$ are independent, show that

$$\mathsf{P}\{N_\lambda(t) = 1 \,|\, N_\lambda(t) + N_\mu(t) = 1\} = \frac{\lambda}{\lambda + \mu}.$$

Note that the RHS does not depend on $t$.

**Ex 2.2.14.** Show with conditioning that thinning the Poisson process $N_\lambda$ by means of Bernoulli random variables with success probability $p$ results in a Poisson process $N_{\lambda p}$.

**Ex 2.2.15.** Show that $M_{N(t)}(s) = \exp(\lambda t(e^s - 1))$.

**Ex 2.2.16.** Use $M_{N(t)}$ to find $\mathsf{E}[N(t)]$ and $\mathsf{V}[N(t)]$.

**Ex 2.2.17.** Show with moment-generating functions that thinning the Poisson process $N_\lambda$ by means of Bernoulli random variables with success probability $p$ results in a Poisson process $N_{\lambda p}$.

**Ex 2.2.18.** If the Poisson arrival processes $N_\lambda$ and $N_\mu$ are independent, use moment-generating functions to show that $N_\lambda + N_\mu$ is a Poisson process with rate $\lambda + \mu$.

**Ex 2.2.19.** Use moment-generating functions to prove (2.2.1).

## 2.3 QUEUEING PROCESS IN CONTINUOUS TIME

In Section 2.1 we modeled time as progressing in discrete chunks. However, we can also model queueing systems in continuous time, so that jobs can arrive at any moment in time and have arbitrary service times. In this section, we develop a set of recursions to construct the waiting times of jobs served in the sequence in which they arrive. First we concentrate on a situation in which there is one server available; at the end we extend this to multiple servers, and discuss some computational aspects.

Let's imagine that a machine starts working on a one-hour job at 9 am. When the next job arrives before 10 am, this second job has to wait in queue until the first job finishes at 10 am. Suppose next that the one-hour job arrives at 9 am but has to wait 5 hours before its production can start. When the next job arrives before $9+5+1=3$ pm, it has to wait, while if it arrives after 3 pm, it finds the machine free, and its service can start right away.

MORE GENERALLY, suppose we are given a sequence $\{X_k\}$ of *inter-arrival* times between jobs and a sequence $\{S_k\}$ of *service times*. When job $k-1$ has to wait a time $W_{k-1}$ in queue, then adds its service time $S_{k-1}$ to the waiting time, and a time $X_k$ elapses between the arrival time of job $k-1$ and $k$, then job $k$ has to wait in queue

$$W_k = [W_{k-1} + S_{k-1} - X_k]^+. \tag{2.3.1}$$

Observe[0] that with this recursion it is easy to compute the sequence of waiting times $\{W_k\}$ in queue: from the initial condition $W_0 = 0$ and $S_0 = 0$ we can obtain $W_1$, and then $W_2$, and so on, see Fig. 2.3.1.

[0] *Note that we assume that job k 'reveals' its service time at the moment it arrives.*

Henceforth, we always assume (implicitly) that the $\{X_k\}$ are iid, the $\{S_k\}$ are iid, and the $\{X_k\}$ are independent of the $\{S_k\}$.

The *sojourn time* $J_k$, is the time job $k$ spends in the entire system. Thus,

$$J_k = W_k + S_k,$$

and a bit of thought will show that $J_k$ can also be found from

$$J_k = [J_{k-1} - X_k]^+ + S_k. \tag{2.3.2}$$

IT IS CLEAR that we need a sequence $\{X_k\}$ of inter-arrival times, but such times are not always *measured*. However, if we know the number of arrivals $A(t)$ as a function of time $t$, we can reconstruct $\{X_k\}$. For instance, if we know that $A(s) = k-1$ and $A(t) = k$, then the arrival time $A_k$ of the $k$th job must lie somewhere in $(s,t]$. Specifically, we define the *arrival time* of job $k$ as[1] [2]

$$A_k = \min\{t : A(t) \geq k\}, \quad A_0 = 0.$$

[1] *If we want to be mathematically precise, we must take* inf *rather than* min.

[2] *[2.3.2]*

Once we have the sequence of arrival times $\{A_k\}$, the sequence of *inter-arrival times* $\{X_k, k = 1,2,\ldots\}$ between consecutive customers follows as

$$X_k = A_k - A_{k-1}.$$

Conversely, if the basic data consists of the inter-arrival times $\{X_k\}$, we find the arrival times with the recursion[3]

[3] *[2.3.3]*

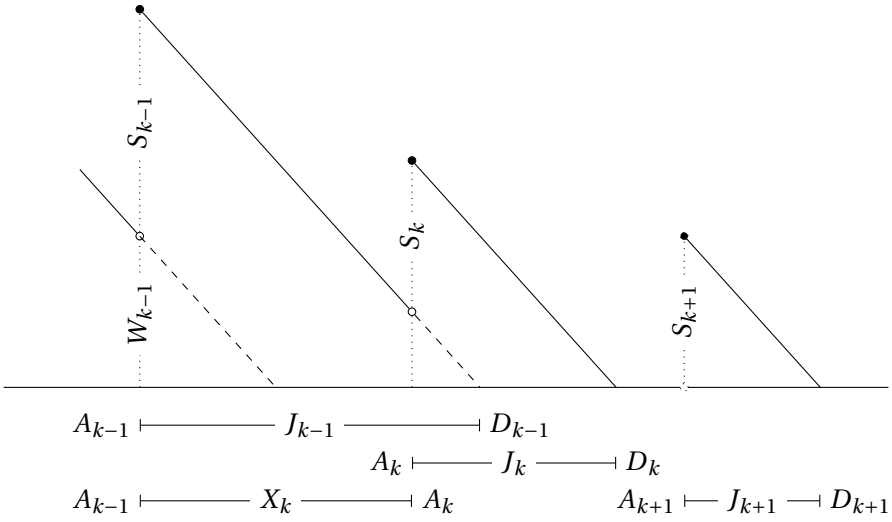



Figure 2.3.1: Construction of the single-server queue in continuous time. The virtual waiting time process is shown by the solid lines with slope $-1$.

$$A_k = A_{k-1} + X_k, \quad A_0 = 0. \tag{2.3.3}$$

And then, from the arrival times $\{A_k\}$, we can define $A(t)$ as [4] [5]

$$A(t) = \max\{k : A_k \le t\}. \tag{2.3.4}$$

Observe that the function $t \to A(t)$ is right-continuous.[6]

The *virtual waiting time process* $\{V(t)\}$ is the time a job would have to wait if it would arrive at time $t$.[7] To construct[8] $\{V(t)\}$, we simply draw lines that start at points $(A_k, W_k)$ and have slope -1, until the lines hit the $x$-axis, in which case the virtual waiting time remains zero until the next arrival occurs. Fig. 2.3.1 shows the graph of the virtual waiting time process.

AFTER MOVING TO the server and completing its service, a job leaves the system. The *departure time of the system* is job $k$ is given by[9]

$$D_k = A_k + J_k.$$

With the sequence $\{D_k\}$, the number of departures $D(t)$ up to time $t$ can be computed as[10]

$$D(t) = \max\{k : D_k \le t\} = \sum_{k=1}^{\infty} \mathbb{1}_{D_k \le t}.$$

ONCE WE HAVE THE ARRIVAL and departure processes it is easy to compute the *number of jobs in the system* at time $t$ as, see Fig. 2.3.2,

$$L(t) = A(t) - D(t) + L(0), \tag{2.3.5}$$

where $L(0)$ is the number of jobs in the system at time $t = 0$; typically we assume that $L(0) = 0$.

In a queueing system, a job can be in queue or in service. We therefore distinguish between the number of jobs in the system $L(t)$, the number of jobs in queue $Q(t)$, and the number in service $L_s(t)$. Clearly, $L(t) = Q(t) + L_s(t)$.

[4] *[2.3.4]*

[5] *For the mathematically inclined, we should consider $\{A(t,\omega), \omega \in \Omega\}$ as a counting process labeled by the sample-set in the sample space $\Omega$, and not just look at one sample $A(t,\omega)$.*

[6] *You might want to prove this.*

[7] *In other words, it is the waiting time observed by job that arrive 'virtually'.*

[8] *[2.3.9]*

[9] *[2.3.5] and [2.3.7]*

[10] *[2.3.5]*

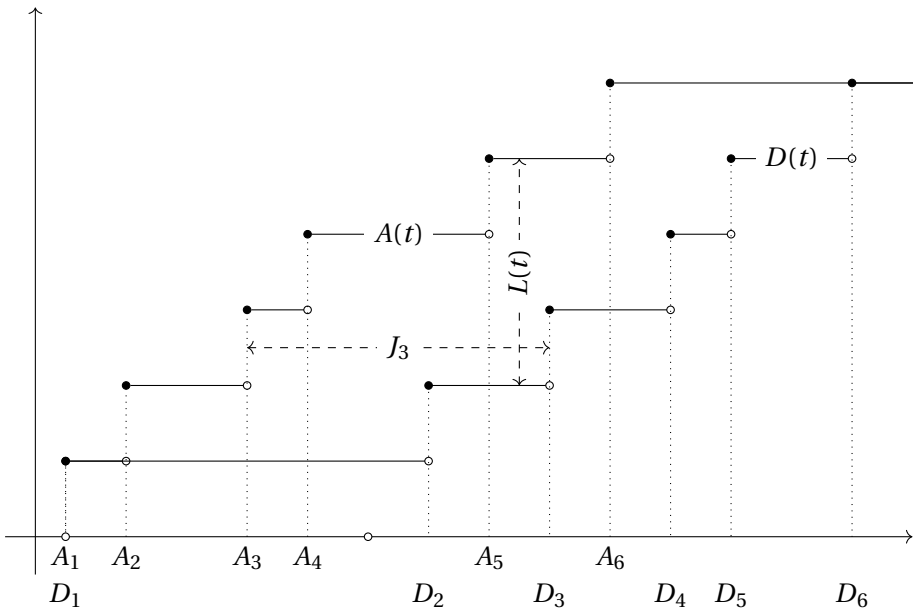



Figure 2.3.2: Relation between the arrival process $\{A(t)\}$, the departure process $\{D(t)\}$, the number in the system $\{L(t)\}$ and the sojourn times $\{J_k\}$.

It is important to realize that the queue length process $\{Q(t)\}$ at general time moments $t$ can be quite different from the queue length process $\{Q(A_k-)\}$ as observed by arriving jobs.[11]

[11] *Observe that we write $A_k-$, and not $A_k$; we need to be careful about left and right limits at jump epochs.*

LET US NEXT CONSTRUCT a multi-server FIFO queue in which the service of the first job in line starts when a server becomes free; if a server is free when a job arrives, the job's service starts right away.

Suppose there are $c$ servers available, each with its own waiting line, like in a supermarket. When job $k$ arrives, it sees a waiting time $w_k(i)$ at line $i$; write $w_k = (w_k(1), \ldots, w_k(c))$ for the vector of waiting times. Of course, the job selects the line with the shortest waiting time.[12] Thus, it joins the end of line $s_k = \operatorname{argmin}\{w_k(i) : i = 1, \ldots, c\}$.

[12] *This is not necessarily the same as the shortest queue.*

To formulate a recursion for $w_k$, let $e(i)$ be the $i$th unit vector[13], and $\mathbf{1} = (1, \ldots, 1)$. The waiting time of job $k$ becomes $W_k = w_k(s_k)$, and, in analogy with (2.3.1), the vector $w_k$ updates as

[13] *A 1 at place $i$ and zeros elsewhere.*

$$\begin{aligned} s_k &= \operatorname{argmin}\{w_k(i) : i = 1, \ldots, c\}, \\ w_{k+1} &= [w_k + S_k e(s_k) - X_{k+1}\mathbf{1}]^+, \end{aligned}$$

where $[\cdot]^+$ applies element-wise.

It is useful to analyze the algorithmic complexity of this algorithm. For job $k$, we need to find the minimum in $w_k$, and compute and subtract $X_{k+1}\mathbf{1}$. The number of computational operations for this is $c$, as $w_k$ and $\mathbf{1}$ contain $c$ elements. For a simulation with $N$ jobs, the total amount of operations is $N \times c$. However, by using a different implementation,[14] the complexity can be reduced to $N \times \log_2 c$, which is considerably faster when $c$ and $N$ are large.

[14] *With event stacks.*

**Ex 2.3.1.** Show that we can also define $A(t)$ as $A(t) = \sum_{k=1}^{\infty} \mathbb{1}_{A_k \le t}$.

**Ex 2.3.2.** Are the following mappings correct:

$$A_k : \mathbb{N} \to \mathbb{R}, \quad \text{job id (integer) to arrival time (real number)},$$
$$A(t) : \mathbb{R} \to \mathbb{N}, \quad \text{time (real number) to number of jobs (integer)?}$$

**Ex 2.3.3.** What is the meaning of $A_{A(t)}$, and of $A(A_n)$?

**Ex 2.3.4.** In view of the above, can $A(t)$ be defined as $\min\{k : A_k \geq t\}$ or as $\min\{k : A_k > t\}$?

*Practice with definitions.*

**Ex 2.3.5.** Assume that $X_1 = 10$, $X_2 = 5$, $X_3 = 6$ and $S_1 = 17$, $S_2 = 20$ and $S_3 = 5$, compute the arrival times, waiting times in queue, the sojourn times and the departure times for these three customers.

**Ex 2.3.6.** Assume that $X_k = 10$ minutes and $S_k = 11$ minutes for all $k$, i.e., $X_k$ and $S_k$ are deterministic and constant. Compute $A_k$, $W_k$, $D_k$ as functions of $k$. Then find expressions for $A(t)$ and $D(t)$.

**Ex 2.3.7.** Suppose that $X_k \in \{1,3\}$ such that $\mathsf{P}\{X_k = 1\} = \mathsf{P}\{X_k = 3\}$ and $S_k \in \{1,2\}$ with $\mathsf{P}\{S_k = 1\} = \mathsf{P}\{S_k = 2\}$. If $W_0 = 3$, what are the distributions of $W_1$ and $W_2$?

**Ex 2.3.8.** Explain the following recursions for a single server queue:

$$A_k = A_{k-1} + X_k, \qquad D_k = \max\{A_k, D_{k-1}\} + S_k, \qquad J_k = D_k - A_k.$$

**Ex 2.3.9.** Provide a specification of the virtual waiting time process $\{V(t)\}$ for all $t$.

**Ex 2.3.10.** In [2.3.6], find an expression for $L(A_k-)$ (The meaning of $-$ sign is defined in (1.2.2).)

**Ex 2.3.11.** Explain that if we know $\tilde{A}(t)$, i.e. the number of jobs that departed from the queue up to time $t$, then

$$Q(t) = A(t) - \tilde{A}(t), \qquad L_s(t) = \tilde{A}(t) - D(t) = L(t) - Q(t).$$

**Ex 2.3.12.** Consider a multi-server queue with $m$ servers. Suppose that at some time $t$ it happens that $\tilde{A}(t) - D(t) < m$, where $\tilde{A}(t)$ is the number of jobs that departed from the queue up to time $t$, but $A(t) - D(t) > m$. How can this occur?

**Ex 2.3.13.** Show that $L(t) = \sum_{k=1}^{\infty} \mathbb{1}_{A_k \leq t < D_k}$ when the system starts empty.

**Ex 2.3.14.** Show that $L(A_k) > 0 \implies A_k \leq D_{k-1}$.

**Ex 2.3.15.** With the recursions (2.3.2) it is apparently easy to compute the waiting time (in queue), but it is less simple to compute the number of jobs in queue or in the system. In this exercise we develop an algorithm to compute the number of jobs in the system as seen by arrivals. Explain why the following (algorithmic efficient) procedure works:

$$L(A_k-) = L(A_{k-1}-) + 1 - \sum_{i=k-1-L(A_{k-1}-)}^{k-1} \mathbb{1}_{D_i < A_k}.$$

**Ex 2.3.16.** In [2.3.15], why do we take $i = k-1-L(A_{k-1}-)$ in the sum, and not $i = k-2-L(A_{k-1}-)$?

**Ex 2.3.17.** Compute $\mathsf{P}\{S - X \le u\}$ for $S$ and $X$ independent and $S \sim \text{Unif}[0,7]$ and $X \sim \text{Unif}[0,10]$.

**Ex 2.3.18.** Implement the recursions for the multi-server queue in code, and run it on an example.

## 2.4 EXPONENTIAL DISTRIBUTION

In Section 2.2 we introduce the Poisson process to model the arrival process of jobs, and we use the Poisson distribution in simulations to generate the random number of jobs arriving in a period. Likewise, to model and simulate the continuous-time single-server queueing process of Section 2.3, we need to specify distributions for the inter-arrival times $\{X_k\}$ and service times $\{S_k\}$. In particular for the inter-arrival times the exponential distribution is useful as it is closely related to the Poisson distribution. In this section we concentrate on the properties of the exponential distribution; one of the most important is the memory-less property.

We say that a random variable $X$ is *exponentially distributed* with mean $1/\lambda$ if

$$\mathsf{P}\{X \le t\} = 1 - e^{-\lambda t},$$

and then we write $X \sim \text{Exp}(()\lambda)$. The following properties hold:[0]

[0] *[2.4.1]– [2.4.4]*

$$\mathsf{E}[X] = \lambda^{-1}, \tag{2.4.1a}$$

$$\mathsf{V}[X] = \lambda^{-2}, \tag{2.4.1b}$$

$$C^2 = 1, \tag{2.4.1c}$$

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda. \tag{2.4.1d}$$

WE NOW INTRODUCE another fundamental concept. A random variable $X$ is called *memoryless* when it satisfies

$$\mathsf{P}\{X > s + t \,|\, X > s\} = \mathsf{P}\{X > t\}.$$

In words, the probability that $X$ is larger than some time $s+t$, conditional on it being larger than a time $s$, is equal to the probability that $X$ is larger than $t$. Stated differently, even given that $X$ did not occur before $s$, the probability that it takes $t$ time units more to occur, is the same as if we did not have to wait for $s$ time units to pass.

The remarkable fact is that an exponentially distributed random variable is memoryless[1]. The reverse—a continuous memoryless random variable is exponential—can also be proven, but that is harder[2].

[1] *[2.4.7]*

[2] *Yushkevich and Dynkin [1969, Appendix 3]*

The life span of human beings is not memoryless: take $X$ as the life span of an arbitrary person born in 1880, and $s = t = 100$ years. Then $\mathsf{P}\{X > 100\}$ was small, but not zero, but $\mathsf{P}\{X > 200 \,|\, X > 100\} = 0$.[3] However, if $X$ is the time to the next patient with a broken arm at the emergency room of a hospital, what can we say about $X$ when we know that an hour earlier a patient came in with broken arm? Not much, as most of us will agree.

[3] *Even if you believe that Elvis Presley is still alive.*

The characteristic timescales we consider in queueing theory range from minutes to a week.[4] On these timescales it often turns out that it is reasonable to model inter-arrival times of jobs as memoryless, hence exponentially distributed.

[4] *And of course there are exceptions.*

THERE IS A close link between the Poisson process $N$ and the exponential distribution. In fact, a counting process $\{N(t)\}$ is a *Poisson process* with rate $\lambda$ if and only if the inter-arrival times $\{X_k\}$ are iid and $X_k \sim \text{Exp}(\lambda)$.[5]

[5] *[2.4.8]–[2.4.12].*

Thus, if you find it reasonable to model inter-arrival times as memoryless, then the number of arrivals in an interval is necessarily Poisson distributed. And, if you find it reasonable that the occurrence of an event in a small time interval is constant over time and independent from one interval to another, then the arrival process is Poisson, and the inter-arrival times are exponential.

THE GEOMETRIC DISTRIBUTION is the discrete-time analog of exponential distribution. As such, geometric rvs are also memoryless, but in discrete time. To demonstrate this, consider a machine that produces items. An item fails with probability $p$, and is correct with probability $q = 1 - p$, independent of the correctness of any other item. Let $X$ be the number of items produced until a failure occurs. Then $\mathsf{P}\{X = m\} = pq^{m-1}$, from which easily follows that

$$\mathsf{P}\{X > n+m \,|\, X > m\} = \frac{\mathsf{P}\{X > n+m\}}{\mathsf{P}\{X > m\}} = \frac{pq^{n+m}}{pq^m} = q^n = \mathsf{P}\{X > n\}. \tag{2.4.2}$$

**Ex 2.4.1.** Show (2.4.1a).

**Ex 2.4.2.** If $X \sim \text{Exp}(\lambda)$, show that $\mathsf{E}\left[X^2\right] = \frac{2}{\lambda^2}$.

*A useful intermediate step to compute the variance.*

**Ex 2.4.3.** Show (2.4.1b).

**Ex 2.4.4.** Show (2.4.1d).

**Ex 2.4.5.** Use $M_X(t)$ to show (2.4.1a) and (2.4.1b).

**Ex 2.4.6.** Show (2.4.1c).

**Ex 2.4.7.** If $X \sim \text{Exp}(\lambda)$, show that $X$ is memoryless.

**Ex 2.4.8.** If $N_\lambda$ is a Poisson process with rate $\lambda$, show that the time $X_1$ to the first arriving job is $\text{Exp}(\lambda)$.

*Poisson $\implies$ Exponential.*

**Ex 2.4.9.** Assume that inter-arrival times $\{X_i\}$ are iid and $\sim \text{Exp}(\lambda)$. Let the arrival time of the $i$th job be $A_i = \sum_{k=1}^{i} X_k$. Show that $\mathsf{E}[A_i] = i/\lambda$.

*In this and the next exercises we show that exponential $\implies$ Poisson.*

**Ex 2.4.10.** Prove that $A_i$ has density $f_{A_i}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{i-1}}{(i-1)!}$.

*Continuation of [2.4.9].*

**Ex 2.4.11.** Use $f_{A_i}$ of [2.4.10] to show that $\mathsf{E}[A_i] = i/\lambda$.

*Continuation of [2.4.10].*

**Ex 2.4.12.** If the inter-arrival times $\{X_i\}$ are iid $\sim \text{Exp}(\lambda)$, prove that the number $N(t)$ of arrivals during the interval $[0, t]$ is Poisson distributed.

*Continuation of [2.4.10].*

**Ex 2.4.13.** If $X \sim \text{Exp}(\lambda), S \sim \text{Exp}(\mu)$ and independent, show that $Z = \min\{X, S\} \sim \text{Exp}(\lambda + \mu)$, hence $\mathsf{E}[Z] = (\lambda + \mu)^{-1}$.

*This result can be anticipated when you think about merging Poisson processes.*

**Ex 2.4.14.** If $X \sim \text{Exp}(\lambda)$, $S \sim \text{Exp}(\mu)$ and independent, show that

$$\mathsf{P}\{X \le S\} = \frac{\lambda}{\lambda + \mu}.$$

*Now think about splitting Poisson processes.*

# 3

# FROM TRANSIENT TO STEADY-STATE ANALYSIS

With the tools developed in Chapter 2 we can *simulate* queueing processes. We now make a start with developing *mathematical models* of queueing systems. However, as we will see in Section 3.2, the mathematical characterization of the *transient behavior* of even simple queueing system is already extremely complicated. Thus, we have to lower our goals, and for this reason we will focus on the long-run average behavior of queueing systems.

In Section 3.3 we formally define the concepts of arrival and service rate and use these to define stability and load. The notions of arrival and service rate are crucial because they capture our intuition that when jobs can be served faster than they arrive, on average, the queue does not systematically drift to infinity. Once stability is ensured, we can properly define a number of measures to characterize the performance of the queueing system, such as the long-run average waiting time, see Section 3.4.

Before introducing these definitions, however, we introduce in Section 3.1 some commonly used notation to characterize the type of queueing process.

Finally, in Section 3.5 we provide an overview of the relations introduced in the previous and this chapter.

## 3.1 KENDALL'S NOTATION

As is apparent from Sections 2.1 and 2.3, the construction of a queueing process for a single station involves three main elements: the distribution of job inter-arrival times, the distribution of the service times, and the number of servers present to process jobs.

To characterize the type of queueing process it is common to use *Kendall's abbreviation* $A/B/c/K$, where $A$ is the distribution of the iid inter-arrival times, $B$ the distribution of the iid service times, $c$ the number of servers, and $K$ the system size, i.e., the total number of customers that can be simultaneously present, whether in queue or in service.[0] In this notation, it is assumed that the service discipline is FIFO (First in first out)[1], i.e., jobs are served in the order in which they arrive.

The most important inter-arrival and service distributions are the exponential distribution, denoted with the shorthand[2] $M$, and the general distribution[3], denoted with $G$. We write $D$ for a deterministic (constant) random variable.

A FEW IMPORTANT EXAMPLES are the following queueing processes: $M/M/1$ in which interarrival times and service times are exponentially distributed; $M/G/1$ with expoential interarrival times and general service times; and $G/G/c$ in which the interarrival and service times are not specified.

[0] *The meaning of K differs among authors. Sometimes it stands for the capacity of the queue, not the entire system. In this book K corresponds to the system's size.*

[1] *FCFS (First come first serve) is an other much used accronym for this service discipline.*

[2] *which stands for Markov or Memoryless*

[3] *with the implicit assumption that the first moment is finite*

A model that is often used to determine the number of beds needed in (a ward of) a hospital is the $M/M/c/c$ queue. The motivation is as follows. Practice tells us that patient inter-arrival times are memoryless, hence exponentially distributed. Data of patients treatment times shows that these times are often also well-described by an exponential distribution. Next, there are $c$ beds available, and each bed can serve one patient. When all beds are occupied, the hospital is 'full'.

WHEN AT AN arrival epoch more than one job arrive simultaneously (like a group of people entering a restaurant), we say that a *batch of jobs* arrives. Likewise, the server can work in batches, for instance, when an oven processes multiple jobs at the same time. We write $A^X/B^Y/c$ to indicate that we consider batch arrivals and batch services, and then $X$ stands for the distribution of the size of the arriving batch, and $Y$ for the size of the service batch. When $X \equiv 1$ or $Y \equiv 1$, i.e., single batch arrivals or single batch services, we suppress the $X$ or $Y$ in the queueing formula.

## 3.2 QUEUEING PROCESSES AS REGULATED RANDOM WALKS

In this section, we provide an elegant construction of a queueing process based on *random walk*. This serves two goals. The first is to show that queueing theory is essentially based on concepts of fundamental interest in probability theory (the random walk), hence is strongly related to many other applications of random walks, such as finance, inventory theory, and insurance theory. The second is to show that it is (very) hard to characterize the transient behavior of a queueing process. Thus, in the rest of the book we will only study queueing systems in steady-state, by which we mean that we consider the long-run average limits of the system.

IN THE CONSTRUCTION of queueing processes as set out in Section 2.1 we are given two sequences of iid rvs: the number of arrivals $\{a_k\}$ and the service capacities $\{c_k\}$ per period which we use in the recursion[0] $L_k = [L_{k-1} + a_k - c_k]^+$ to compute the number of jobs in the system. By removing the $[\cdot]^+$ operator, we obtain a random walk $\{Z_k, k = 0, 1, \ldots\}$ with $Z_k$ given by

[0] *[2.1.2]*

$$Z_k = Z_{k-1} + a_k - c_k. \tag{3.2.1}$$

To see that $\{Z_k\}$ is indeed a random walk, observe that $Z$ makes jumps of size $a_k - c_k, k = 1, \ldots$, and $\{a_k - c_k\}$ is a sequence of iid rvs. Observe that $\{Z_k\}$ is 'free', i.e., it can take positive and negative values, whereas $\{L_k\}$ is restricted to the non-negative integers.

It is interesting to express $\{L_k\}$ in terms of $\{Z_k\}$. From (3.2.1), it is evident that $a_k - c_k = Z_k - Z_{k-1}$, hence,

$$L_k = [L_{k-1} + a_k - c_k]^+ = [L_{k-1} + Z_k - Z_{k-1}]^+.$$

It follows that $L_k - Z_k = \max\{L_{k-1} - Z_{k-1}, -Z_k\}$, and, by completing the recursion,[1] we obtain

[1] *[3.2.1]*

$$L_k = Z_k - \left(\min_{1 \le i \le k} Z_i\right) \wedge 0, \tag{3.2.2}$$

where $a \wedge b = \min\{a, b\}$.

This recursion for $L_k$ leads to nice graphs. In Fig. 3.2.1 we take $a_k \sim$ Bern(0.3) and $c_k \sim$ Bern(0.4). We can clearly see that $L_k$ is equal to $Z_k$ minus the minimum of $Z_i, i \le k$.

In Fig. 3.2.2, $a_k \sim$ Bern(0.49) the random walk behaves according to

$$Z_k = Z_{k-1} + 2a_k - 1. \tag{3.2.3}$$

Thus, if $a_k = 1$, the random walk increases by one step, while if $a_k = 0$, the random walk decreases by one step, so that $Z_k \neq Z_{k-1}$ always. Observe that this is slightly different from a random walk that satisfies (3.2.1); there, $Z_k = Z_{k-1}$, if $a_k = c_k$.

WHAT CAN WE say about the transient (time-dependent) behavior of $\{L_k\}$ with the above? To obtain insight into this question, let us suppose that $a_k \sim \text{Pois}(\lambda)$ and $c_k \sim \text{Pois}(\mu)$.

$$Z_k = Z_0 + N_{\lambda,k} - N_{\mu,k}.$$

Writing $Z_0 = m$ and with a bit of work, we can show[2] that[3]

$$\mathsf{P}_m\{Z_k = n\} = e^{-(\lambda+\mu)k}(\lambda k)^{n-m} \sum_{j=0}^{\infty} \frac{(\lambda\mu k^2)^j}{j!(n-m+j)!}. \tag{3.2.4}$$

It is apparent that only formally the distribution of $L_k$ can be obtained from this and (3.2.2), but we will not obtain a simple expression. Consequently, we cannot find useful expressions for $\mathsf{P}\{L_k = n\}$ as a function of $n$.[4]

Now observe that $Z_k$ is just the free $M/M/1$ queue, which is just about the simplest queueing system to analyze; other queueing systems are (much) more complicated. As we already have to give up our attempts to analyze the transient $M/M/1$ queue, we have to give up our hope, but contend ourselves henceforth with the analysis of queueing systems in the limit as $t \to \infty$.

**Ex 3.2.1.** Show that $L_k$ satisfies (3.2.2).

**Ex 3.2.2.** Show that, when $Z_0 = m$, $\mathsf{P}\{Z_k = n\}$ satisfies (3.2.4).

**Ex 3.2.3.** Suppose for the $G/G/1$ that a job sees $n$ jobs in the system upon arrival. Use the central limit theorem to estimate the distribution of the waiting time in queue for this job.

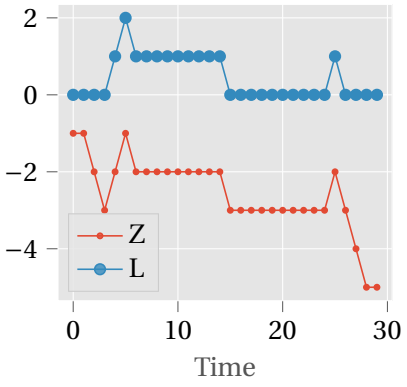



Figure 3.2.1: An instance of (3.2.1).

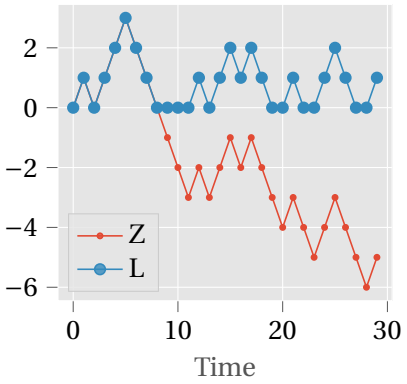



Figure 3.2.2: An instance of (3.2.3).

[2] *[3.2.2]*

[3] *Note that the difference of two Poisson random variables is not Poisson, in contrast to the sum.*

[4] *We refer to $k \to \mathsf{P}\{L_k = n\}$ as the transient behavior of $\{L_k\}$.*

*When we start with a really large queue, we characterize the time to clear the system in Section 7.1.*

## 3.3 RATE, STABILITY AND LOAD

In this section, we develop a number of measures to characterize the performance of queueing systems in steady-state. In particular, we define the load, which is, arguably, the most important performance measure of a queueing system to check.

WE FIRST FORMALIZE the arrival rate and departure rate in terms of the arrival and departure processes $\{A(t)\}$ and $\{D(t)\}$, see Section 2.3. The *arrival rate* is the long-run average number of jobs that arrive per unit time along a sample path, i.e.,[0]

[0] *This limit does not necessarily exist if $A(t)$ is some pathological function.*

$$\lambda = \lim_{t\to\infty} \frac{A(t)}{t}. \tag{3.3.1}$$

Likewise, the *departure rate* is

$$\delta = \lim_{t\to\infty} \frac{D(t)}{t}. \tag{3.3.2}$$

Henceforth we assume that both limits are finite.

OBSERVE THAT, if the system is empty[1] at time 0, i.e., $L(0) = 0$, the number of departures must be smaller than or equal to the number of arrivals, i.e., $D(t) \le A(t)$ for all $t$. Therefore,

[1] *Why is this necessary to require?*

$$\delta = \lim_{t\to\infty} \frac{D(t)}{t} \le \lim_{t\to\infty} \frac{A(t)}{t} = \lambda.$$

It is intuitively obvious that when $\lambda > \delta$, the system length process $L(t) = L(0) + A(t) - D(t) \to \infty$ as $t \to \infty$. Combining the above two inequalties, we say that a system is *rate-stable* if

$$\lambda = \delta. \tag{3.3.3}$$

In words: the system is rate-stable whenever jobs leave the system just as fast as they arrive in the long run.

SUPPOSE THAT WE are given a sequence of iid inter-arrival times $\{X_k\}$ distributed as the common rv $X$. Then we can relate the arrival rate $\lambda$ to the mean inter-arrival time $\mathsf{E}[X]$ as follows.[2] Observe that at time $t = A_n$ precisely $n$ arrivals occurred. Then, with [2.3.3] we see that $A(A_n) = n$, and therefore,

[2] *The existence of the limit (3.3.1) is also guaranteed under this assumption.*

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \frac{A_n}{n} = \frac{A_n}{A(A_n)}.$$

Below we show that $A_n \to \infty$ as $n \to \infty$ implies that $\lim_{n\to\infty} A_n/A(A_n) = \lim_{t\to\infty} t/A(t)$. With this, it follows from (3.3.1) that the average inter-arrival time between two consecutive jobs is

$$\mathsf{E}[X] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = \lim_{n\to\infty} \frac{A_n}{A(A_n)} = \lim_{t\to\infty} \frac{t}{A(t)} = \frac{1}{\lambda}, \tag{3.3.4}$$

where we take $t = A_n$ in the limit for $t \to \infty$. In words, the arrival rate $\lambda$ is the *inverse* of the expected inter-arrival time $\mathsf{E}[X]$.

IN (3.3.4) WE REPLACE the limit with respect to $n$ by a limit with respect to $t$. To show that this is allowed, observe that $A_{A(t)}$ is the arrival time of the last job before time $t$ and that $A_{A(t)+1}$ is the arrival time of the first job after time $t$. Therefore,

$$A_{A(t)} \le t < A_{A(t)+1} \Leftrightarrow \frac{A_{A(t)}}{A(t)} \le \frac{t}{A(t)} < \frac{A_{A(t)+1}}{A(t)} = \frac{A_{A(t)+1}}{A(t)+1}\frac{A(t)+1}{A(t)}.$$

Now $A(t)$ is a counting process such that $A(t) \to \infty$ as $t \to \infty$, therefore,

$$\lim_{n\to\infty} \frac{A_n}{n} = \lim_{t\to\infty} \frac{A_{A(t)}}{A(t)} = \lim_{t\to\infty} \frac{A_{A(t)+1}}{A(t)+1},$$

where the third limit follows trivially from the second. Finally, because $(A(t)+1)/A(t) \to 1$, we arrive at the equality $\lim_{t\to\infty} t/A(t) = \lim_{n\to\infty} A_n/n$.

CONSIDER THE $G/G/1$ QUEUE.[3] Let $S_k$ be the required service time of the $k$th job to be served, so that $U_n = \sum_{k=1}^{n} S_k$ is the total service time of the first $n$ jobs. Letting $U(t) = \max\{n : U_n \leq t\}$, we define the *service*, or *processing*, rate as

$$\mu := \lim_{t\to\infty} \frac{U(t)}{t}.$$

Similar to the relation $\mathsf{E}[X] = 1/\lambda$, we have the relation

$$\mathsf{E}[S] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} S_k = \lim_{n\to\infty} \frac{U_n}{n} = \lim_{n\to\infty} \frac{U_n}{U(U_n)} = \lim_{t\to\infty} \frac{t}{U(t)} = \frac{1}{\mu}.$$

ONCE WE HAVE the arrival and service rate, we define the *load* for a server in a $G/G/c$ queue as the rate at which work arrives per server:

$$\text{Server load } = \frac{\lambda\,\mathsf{E}[S]}{c} = \frac{\lambda}{c\mu} = \frac{\mathsf{E}[S]}{c\,\mathsf{E}[X]}.$$

Next, we define the *utilization* as

$$\text{Server utilization } = \frac{\delta\,\mathsf{E}[S]}{c}, \tag{3.3.5}$$

as this the rate at which work leaves the system after being served by one of the $c$ servers. Finally, for ease we write

$$\rho := \frac{\lambda\,\mathsf{E}[S]}{c}.$$

When the system is stable, so that $\lambda = \delta$, the utilization, load and $\rho$ are all equal, and we will use these interchangably. However, However, when jobs are rejected, or when customers leave the queue before being served, $\lambda > \delta$. In such case, care is required. Moreover, the utilization is necessarily at most 1, while the load per server $\lambda\,\mathsf{E}[S]/c$ can exceed one.

It is easy to check with a simulation of the $G/G/1$ queue that $L(t)$ increases at rate $\lambda - \mu$ when $\lambda > \mu$, while $L(t) \approx L(0) + (\lambda - \mu)t$ when $\lambda < \mu$ and $L(0)$ large, until the system is empty.[4]

For this reason the utilization is, perhaps, the most important performance measure to check: when $\rho \geq 1$, we are 'in trouble', when $\rho < 1$, we are 'safe'. In the sequel we tacitly assumte that $\rho < 1$, unless stated otherwise.

**Ex 3.3.1.** Can you make an arrival process such that $A(t)/t$ does not have a limit?

**Ex 3.3.2.** If the system starts empty, then we know that the number $L(t)$ in the system at time $t$ is equal to $A(t) - D(t)$. Show that the system is rate-stable if $L(t)$ remains finite, or, more generally, $L(t)/t \to 0$ as $t \to \infty$.

**Ex 3.3.3.** Show that $\mathsf{E}[X_k - S_k] > 0$ implies that $\rho < 1$.

**Ex 3.3.4.** Consider a queueing system with $c$ servers with identical production rates $\mu$. What would be a reasonable stability criterion for this system?

[3] *See [3.3.4] for an extension to $G/G/c$ queues.*

[4] *It turns out that, when $\mathsf{E}[X] = \mathsf{E}[S]$ but $\mathsf{V}[X - S] > 0$, the queue length process behaves in a very peculiar way. This is due to the fact that the symmetric random walk has some unexpected behavior, and Section 3.2 shows that queueing systems and random walks are intimately related.*

## 3.4 (LIMITS OF) EMPIRICAL PERFORMANCE MEASURES

In Section 2.3 we use the arrival process $\{A(t)\}$ and the service times $\{S_k\}$ to construct the waiting times $\{W_k\}$, sojourn times $\{J_k\}$, and the number in the system $\{L(t)\}$. If the queueing system is rate-stable, we can sensibly define several long-run average performance measures.

DEFINE THE expected *waiting time in queue* as

$$\mathsf{E}[W] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} W_k.$$

Note that this is the limit of waiting times as *observed by arriving jobs*: the first job has to wait $W_1$ in queue, the second $W_2$, and so on.[0] The *expected sojourn time* is defined likewise. The *distribution* of the waiting time as seen by arrivals can be found by counting:

$$\mathsf{P}\{W \le x\} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{W_k \le x}. \tag{3.4.1}$$

For the distribution of $J$ a similar definition applies. The *average number of jobs* in the system is given by

$$\mathsf{E}[L] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} L(A_k-), \tag{3.4.2}$$

since $L(A_k-)$ is the number of jobs in the system at the arrival epoch of the $k$th job. The distribution of $L$ follows also from counting, compare (3.4.1).

A RELATED SET of performance measures follows by tracking the system's behavior over time and taking the *time-average*.[1]

Assuming the limit exists, we use (2.3.5) to define the *time-average number of jobs* as [2]

$$\mathsf{E}[L] = \lim_{t\to\infty} \frac{1}{t} \int_0^t L(s)\,\mathrm{d}s. \tag{3.4.3}$$

The *time-average fraction of time the system contains at most m jobs* is defined as

$$\mathsf{P}\{L \le m\} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}_{L(s)\le m}\,\mathrm{d}s.$$

PROVING THE EXISTENCE of the above limits requires a considerable amount of mathematics.[3] Here we sidestep all such fundamental issues, and simply assume all is well defined in our examples. The limiting random variables are known as the *steady-state* or *stationary* limits.

BESIDES THAT THE transient analysis of queueing systems is difficult,[4] there is an another reasons why much queueing theory is concerned with the analysis of the system in steady state. Steady-state models provide formulas[5], rather than numbers, with which we can compute performance measures of reasonable quality. Hence, for many practical purposes, a steady-state analysis suffices.

**Ex 3.4.1.** Design a queueing system to show that the average number of jobs in the system as seen by the server can be very different from what customers see upon arrival.

[0] *We colloquially say that a statistic based on the sampling of arriving jobs is 'as seen by arrivals'.*

[1] *Now we say that such performance measures are as 'seen by the server'.*

[2] *Even though the symbols are the same, this expectation is not necessarily the same as* (3.4.2).

[3] *See Asmussen [2003] if you are interested.*

[4] *See Section 3.2*

[5] *i.e., structural insight*

**Ex 3.4.2.** If $L(t)/t \to 0$ as $t \to \infty$, can it still be true that $\mathsf{E}[L] > 0$?

**Ex 3.4.3.** Consider the discrete-time model specified by (2.1.1). We assume that $a_k$ is a *batch* of jobs arriving in period $k$ *after* the departures $d_k$ have left. Provide a simulation to estimate $\mathsf{P}\{L \le m\}$ for a situation in which the first job of the batch sees $L_{k-1} - d_k$ jobs in the system, the second sees $L_{k-1} - d_k + 1$ jobs, and so on.

*Performance measures obtained by sampling in discrete-time queueing models require some extra attention.*

## 3.5 GRAPHICAL SUMMARY

Here is, in graphical form, an overview to show the relation between the concepts developed for the $G/G/1$ queue in this and the previous chapter.

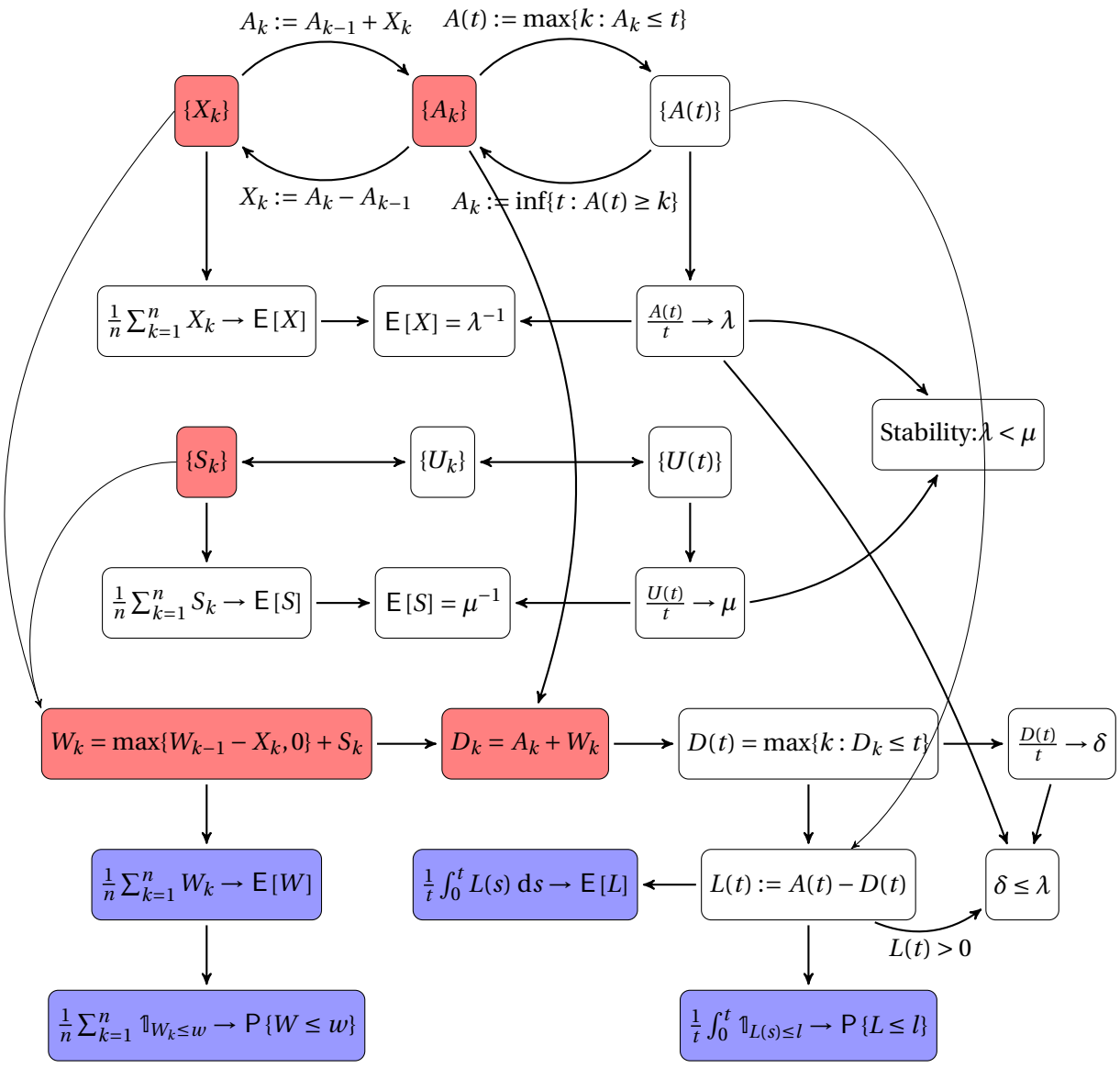



Figure 3.5.1: The relations between, on the one hand, the construction of the $G/G/1$ queue from the primary data such as the inter-arrival times $\{X_k; k \ge 0\}$ and, on the other hand, the different performance measures.

# 4

# APPROXIMATE QUEUEING MODELS

In the previous two chapters we learned how to construct and simulate queueing processes. Simulation is a powerful tool but one of its limitations is that it does not easily provide insight into structural behavior of systems. For this we need theoretical models, and the derivation of such models form the contents of the remainder of the book.

IN THIS CHAPTER we discuss two formulas that might be considered as the most important formulas to understand the behavior of queueing systems. The first is Sakasegawa's formula that approximates the expected queueing time in a $G/G/c$; the second characterizes the propagation of variability through a tandem network of $G/G/c$ queues. With a bit of exaggeration, it is justified to say that the entire philosophy behind lean manufacturing and the world-famous Toyota production system are based on the principles that can be derived from these two formulas.

Here we take these formulas for granted, but focus on the insights they provide into the performance of queueing systems and how to use them to guide improvement procedures for production and service systems. In Section 6.4 we provide the theoretical background of Sakasegawa's formula.

In Section 4.1 we introduce Sakasegawa's formula and discuss the main insights it offers. Then we illustrate how to use this formula to estimate waiting times in three queueing settings in which the service process is interrupted. In the first case, Section 4.2, the server has to produce jobs from different families, and there is a change-over time required to switch from one production family to another. As such setups reduce the time the server is available, the load must increase. In fact, to reduce the load, the server produces in batches of fixed sizes. In the second case, in Section 4.3, the server sometimes requires small adjustments, for instance, to prevent the production quality to degrade below a certain level. Clearly, such adjustments are typically not required during a job's service; however, they can occur between any two jobs. As a consequence, the number of jobs served between two such adjustments (or setups) is not constant, hence different from batch production where batch sizes are constant. In the third example, in Section 4.4, quality problems or break downs can occur during a job's service. These make job service times more variable, which leads to longer expected queueing times. In the final Section 4.5, we concentrate on tandem queues.

In passing, we use some interesting results of probability theory and the Poisson process, which we use again in, for instance, Chapter 7.

## 4.1 $G/G/c$ QUEUE: SAKASEGAWA'S FORMULA

In this section, we discuss Sakasegawa's formula by which we can estimate the expected waiting time in queue for the $G/G/c$ queue. In the exercises

we show how to use this formula, in Section 6.4 we provide the theoretical underpinning.

While there is no expression available to compute the exact expected waiting time for the $G/G/c$ queue, Sakasegawa's formula provides a reasonable approximation. This takes the form

$$\mathsf{E}[W] = \frac{C_a^2 + C_s^2}{2} \frac{\rho^{\sqrt{2(c+1)}-1}}{1-\rho} \frac{\mathsf{E}[S]}{c}, \tag{4.1.1}$$

where $C_a^2 = \mathsf{V}[X]/(\mathsf{E}[X])^2$ and $C_s^2 = \mathsf{V}[S]/(\mathsf{E}[S])^2$ are the SCV of the interarrival time and service time, respectively, and the utilization of the station[0] is given (3.3.5), i.e., $\rho = \lambda \mathsf{E}[S]/c$, where $\lambda$ is the rate at which jobs arrive at the system, $\mathsf{E}[S]$ is the expected service time, and $c$ the number of servers.

[0] *Here is a subtle point. When there are multiple machines, the utilization of each machine need not be equal to $\rho$. For instance, if there is a preference to choose the 'left-most' machine whenever it is free, then the utilization of this machine is larger than $\rho$. Only when the machines have the same speed, and the* routing *is such that each machine receives a fraction $\lambda/c$ of jobs, the machines have the same utilization.*

IT IS CRUCIAL to memorize the insights into the performance of queueing systems that this formula offers. Even though (4.1.1) is an approximation, it proves to be exceedingly useful when designing queueing systems and analyzing the effect of certain changes.

First, we see that $\mathsf{E}[W] \sim (1-\rho)^{-1}$. Consequently, when $\rho$ is large, the waiting time is (very) large. And, not only is $\mathsf{E}[W]$ large, it is also extremely *sensitive* to the actual value of $\rho$. Clearly, such situations must be avoided, and therefore, when trying to improve a queueing system, the first focus should be on reducing $\rho$.

Second, $\rho = \lambda \mathsf{E}[S]/c$. Thus, when $\rho$ must be made smaller, we have only three options.[1] We can reduce the arrival rate $\lambda$ of jobs, for instance by blocking demand, or sending it elsewhere such as to another machine.[2] We can make $\mathsf{E}[S]$ smaller by replacing a server with a faster one or by shifting some of the processing steps of a job to other servers, thereby making job sizes smaller. Finally, we can add servers, which reduce $\rho$ quite significantly when $c$ is small.[3] If technically possible, adapting $c$ is a very effective mechanism to control waiting times.

[1] *And not more!*

[2] *And if customers have a choice, they will take their own measures, simply by going elsewhere.*

[3] *This is precisely what you see in supermarkets.*

Third, $\mathsf{E}[W] \sim C_a^2$ and $\mathsf{E}[W] \sim C_s^2$, which implies that when job interarrival or service times are very variable, $\mathsf{E}[W]$ is large. Thus, after ensuring that $\rho$ is sufficiently small, it becomes important to concentrate on reducing on $C_a^2$ and $C_s^2$.[4]

[4] *It works the other way too: systems with low variability can deal with higher load.*

Finally, $\mathsf{E}[W] \sim \mathsf{E}[S]/c$. This says that, from the perspective of a job in queue, average job service times are $c$ times as short as 'its own service time'.

CLEARLY, SAKASEGAWA'S EQUATION requires an estimate of $C_a^2$ and $C_s^2$. Now it is not always easy in practice to determine the actual service time distribution, one reason being that service times are often only estimated by a planner, but not actually measured. Similarly, the actual arrival moments of jobs are often not registered, only just the date, or perhaps the hour, that a customer arrived. Hence, it is often not possible to directly estimate $C_a^2$ and $C_s^2$ from the information that is available.

However, when the number of arrivals per period $\{a_n\}$ has been logged for some time, we can use $\{a_n\}$ to estimate $C_a^2$ as[5]

$$C_a^2 \approx \frac{\tilde{\sigma}^2}{\tilde{\lambda}},$$

[5] *This can of course also be used to estimate $C_s^2$.*

where

$$\tilde{\lambda} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i, \quad \tilde{\sigma}^2 = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i^2 - \tilde{\lambda}^2.$$

We derive this results in steps.[6]

[6] *It is based an argument in Cox [1962].*

First recall some results of earlier sections. Let the inter-arrival time $X$ have mean $1/\lambda$ and variance $\sigma^2$, so that

$$C_a^2 = \frac{\mathsf{V}[X_i]}{(\mathsf{E}[X_i])^2} = \frac{\sigma^2}{1/\lambda^2} = \lambda^2\sigma^2.$$

Let $A_k$ be the arrival time of the $k$th arrival, see (2.3.3), and $A(t)$ the number of arrivals up to time $t$, see (2.3.4). Consider the following useful relation between $A(t)$ and $A_k$, see [2.4.10],

$$\mathsf{P}\{A(t) < k\} = \mathsf{P}\{A_k > t\}.$$

Since the inter-arrival times have finite mean and second moment by assumption, we use [3.2.3] to see that

$$\lim_{k\to\infty} \frac{A_k - k/\lambda}{\sigma\sqrt{k}} = N(0,1),$$

where $N(0,1)$ is a standard normal random variable with distribution $\Phi(\cdot)$. Similarly, for an $\alpha$ yet to be determined,[7]

[7] *This is a common trick: suppose that there exists a constant to take of the scaling, and then try an extra relation to 'ferret out' the scaling constant.*

$$\frac{A(t) - \lambda t}{\alpha\sqrt{t}} \to N(0,1).$$

Thus, $\mathsf{E}[A(t)] = \lambda t$ and $\mathsf{V}[A(t)] = \alpha^2 t$.

Using that $\mathsf{P}\{N(0,1) \le y\} = \mathsf{P}\{N(0,1) > -y\}$, we have that

$$\begin{aligned}\Phi(y) &\approx \mathsf{P}\left\{\frac{A_k - k/\lambda}{\sigma\sqrt{k}} \le y\right\} \\ &= \mathsf{P}\left\{\frac{A_k - k/\lambda}{\sigma\sqrt{k}} > -y\right\} \\ &= \mathsf{P}\left\{A_k > \frac{k}{\lambda} - y\sigma\sqrt{k}\right\}.\end{aligned}$$

We can use this relation between the distributions of $A(t)$ and $A_k$ to see that $\mathsf{P}\{A_k > t_k\} = \mathsf{P}\{A(t_k) < k\}$ where we define for ease

$$t_k = \frac{k}{\lambda} - y\sigma\sqrt{k}.$$

With this we get,

$$\begin{aligned}\Phi(y) &\approx \mathsf{P}\{A_k > t_k\} \\ &= \mathsf{P}\{A(t_k) < k\} \\ &= \mathsf{P}\left\{\frac{A(t_k) - \lambda t_k}{\alpha\sqrt{t_k}} < \frac{k - \lambda t_k}{\alpha\sqrt{t_k}}\right\}.\end{aligned}$$

Since $(A(t_k) - \lambda t_k)/\alpha\sqrt{t_k} \to N(0,1)$ as $t_k \to \infty$, the above implies that

$$\frac{k - \lambda t_k}{\alpha\sqrt{t_k}} \to y,$$

as $t_k \to \infty$. Using the above definition of $t_k$, the LHS of this equation can be written as

$$\frac{k-\lambda t_k}{\alpha\sqrt{t_k}} = \frac{\lambda\sigma\sqrt{k}}{\alpha\sqrt{k/\lambda+\sigma\sqrt{k}}}y.$$

Since $t_k \to \infty$ is implied by (and implies) $k \to \infty$, we therefore want that $\alpha$ is such that

$$\frac{\lambda\sigma\sqrt{k}}{\alpha\sqrt{k/\lambda+\sigma\sqrt{k}}}y \to y,$$

as $k \to \infty$. This is precisely the case when

$$\alpha = \lambda^{3/2}\sigma.$$

Finally, for $t$ large (or, by the same token $k$ large),

$$\frac{\sigma_k^2}{\lambda_k} = \frac{\mathsf{V}[A(t)]}{\mathsf{E}[A(t)]} \approx \frac{\alpha^2 t}{\lambda t} = \frac{\alpha^2}{\lambda} = \frac{\lambda^3\sigma^2}{\lambda} = \lambda^2\sigma^2 = C_a^2,$$

where the last equation follows from the above definition of $C_a^2$.

**Ex 4.1.1.** In a manufacturing setting, the Poisson process is not always a suitable model for the arrival process of jobs at a production station. Can you provide an example to see why this is the case?

**Ex 4.1.2.** Consider a single-server queue at which every minute a customer arrives, precisely at the first second and $S \equiv 50$ s. What are $\rho$, $\mathsf{E}[L]$, $C_a^2$, and $C_s^2$?

*This is an important example to memorize.*

**Ex 4.1.3.** Consider the same single-server system as in [4.1.2], but now the customer service time is stochastic: with probability 1/2 a customer requires 1 minute and 20 seconds of service, and with probability 1/2 the customer requires only 20 seconds of service. What are $\rho$, $C_a^2$, and $C_s^2$?

**Ex 4.1.4.** (Hall 5.19) When a bus reaches the end of its line, it undergoes a series of inspections. The entire inspection takes 5 minutes on average, with a standard deviation of 2 minutes. Buses arrive with inter-arrival times uniformly distributed on [3,9] minutes, hence, 10 buses arrive per hour on average. There is one mechanic available for the inspection. Use (4.1.1) to estimate $\mathsf{E}[W]$.

Next, assuming that buses arrive as a Poisson process, estimate $\mathsf{E}[W]$. Why is the queueing time smaller in the first setting?

**Ex 4.1.5.** Show for the $G/G/c/K$ queue that $\beta = 1 - \mathsf{E}[L_s]\,\mu/\lambda$, where $\mu = 1/\mathsf{E}[S]$ is the service rate, $\beta$ the long-run fraction of customers lost, and $\mathsf{E}[L_s]$ the average number of busy/occupied servers.

**Ex 4.1.6.** A machine serves two types of jobs. The processing time of jobs of type $i$, $i = 1,2$, is exponentially distributed with parameter $\mu_i$. The type $T$ of a job is random and independent of anything else, and such that $\mathsf{P}\{T=1\} = p = 1-q = 1-\mathsf{P}\{T=2\}$. (An example is a desk serving men and women, both requiring different average service times, and $p$ is the

probability that the customer in service is a man.) Show that the expected processing time and variance are given by

$$\mathsf{E}[S] = p\,\mathsf{E}[S_1] + q\,\mathsf{E}[S_2]$$
$$\mathsf{V}[S] = p\,\mathsf{V}[S_1] + q\,\mathsf{V}[S_2] + pq(\mathsf{E}[S_1] - \mathsf{E}[S_2])^2.$$

Thus, even if $\mathsf{V}[S_1] = \mathsf{V}[S_2] = 0$, still $\mathsf{V}[S] > 0$ if $\mathsf{E}[S_1] \neq \mathsf{E}[S_2]$. Mixing different jobs types increases variability, hence queueing times.

## 4.2 SETUPS AND BATCH PROCESSING

In some cases, machines have to be setup before they can start producing items. Consider, for instance, a machine that paints red and blue items.[0] When the machine requires a color change, it may be necessary to clean up the machine, which takes time. Another example is an oven that needs a temperature change when different item types require different production temperatures. Service operations form another setting with setup times: when servers (personnel) have to move from one part of a building to another, the time spent moving cannot be spent on serving customers.[1]

In all such cases, the setups consume a significant amount of time; in fact, setup times of an hour or longer are not uncommon. Clearly, in such situations, it is necessary to produce in batches: a server processes a batch of jobs of one type or at one location, then changes from type or location, starts serving a batch of another type or at another location, and so on.

Here we focus on the effect of change-over, or setup, times on the average sojourn time of jobs. First we make a model and provide a list of elements required to compute the expected sojourn time of an item, then we illustrate how to use these elements in a concrete case.

ASSUME THERE ARE two job families, e.g., red and blue, each served by the same single server. Jobs arrive at rate $\lambda_r$ and $\lambda_b$. The SCV of job inter-arrival times is given by $C_a^2$. Jobs of both type require the same average *net processing time* of $\mathsf{E}[S_0]$, provided the server is already setup for the correct job color, and have variance $\mathsf{V}[S_0]$. Setup times are iid and have mean $\mathsf{E}[R]$ and variance $\mathsf{V}[R]$, and are assumed to be independent of job service times.

A job's sojourn time is built up as follows. First, an arrival is assembled with other jobs of the same color to form a batch of constant size $B$. For simplicity, $B$ is the same for both colors.[2] Once a batch is complete, the batch enters a queue (of batches). After some time the batch reaches the head of the queue, and is ready to move to the machine as soon as it becomes available. To serve a batch, the machine first performs a setup, and then processes each job individually until the batch is finished. Once a job's service is completed, it may, or may not[3], have to wait for the other jobs in the same batch before it can leave. Let us make a model of this.

WHEN A JOB of type $i$ arrives, the expected time for its batch to be formed is given by[4]

$$\mathsf{E}[W_i] = \frac{B-1}{2\lambda_i}, \quad i \in \{r, b\}. \tag{4.2.1}$$

[0] *In realistic problems there can be hundreds of families.*

[1] *Often, the setup time depends on the sequence in which the families are produced, for example, a switch in color from white to black takes less cleaning time and cost than from black to white. The problem then becomes to determine first a production sequence that minimizes the sum of the setup times to produce the families, and then find suitable batch sizes to minimize the average waiting times. This problem becomes yet more realistic (and very hard) when jobs have due-dates too.*

[2] *In general, B should depend on the arrival rate of the job type. For instance, when red jobs arrive much faster than blue jobs, it takes much longer to fill a blue batch than a red batch when batch sizes are equal.*

[3] *Depending on the type of production.*

[4] *[4.2.2]*

When the batch is complete, it joins the queue, so we next compute the average time in queue.[5]

[5] *Note that we shift interpretation: batches (not individual items) form a queue and move as a whole to the server.*

NOW THAT WE have a batch of jobs, we need to estimate the average time a batch spends in queue. For this we can use Sakasegawa's formula; we only have to find expressions for each of its components.

The total arrival rate of jobs is $\lambda = \lambda_b + \lambda_r$. Thus, $\lambda_B = \lambda / B$ is the arrival rate of *batches*. It is easy to see that the expected service time of a batch is

$$\mathsf{E}[S_B] = \mathsf{E}[R] + B\,\mathsf{E}[S_0]\,. \tag{4.2.2}$$

Sometimes it useful to convert the effects of the setup time into an *effective processing time* of an individual job:

$$\mathsf{E}[S] = \frac{\mathsf{E}[S_B]}{B} = \frac{\mathsf{E}[R]}{B} + \mathsf{E}[S_0]\,. \tag{4.2.3}$$

With the batch arrival rate and expected batch service time, the load becomes $\rho = \lambda_B \mathsf{E}[S_B]$

It is essential that $B$ is sufficiently large to ensure that $\rho < 1$. With (4.2.2) this leads to the condition that $B$ must be larger than some minimal batch size, i.e.,

$$B > B_m = \frac{\lambda\,\mathsf{E}[R]}{1 - \lambda\,\mathsf{E}[S_0]}.$$

Now that we have identified the service time of a batch and the load, we only need the squared coefficients of variation for the batch inter-arrival times $C^2_{a,B}$ and the batch service times[6] $C^2_{s,B}$ for Sakasegawa's formula. We find that[7]

[6] *Once again, for the* batches, *not the jobs!*

[7] *[4.2.3] [4.2.4]*

$$C^2_{a,B} = \frac{C^2_a}{B}, \qquad C^2_{s,B} = \frac{\mathsf{V}[S_B]}{(\mathsf{E}[S_B])^2}, \tag{4.2.4}$$

where

$$\mathsf{V}[S_B] = \mathsf{V}[R] + B\,\mathsf{V}[S_0]\,.$$

By dividing by $B$ we see that the variance of the effective processing time is

$$\mathsf{V}[S] = \mathsf{V}[R]/B + \mathsf{V}[S_0]\,. \tag{4.2.5}$$

Indeed, the variance of the effective service times is larger than the variance of the net processing times.

IT IS LEFT to find a rule to determine what happens to an item after it has been processed. If the job has to wait until all jobs in the batch are served, the expected time it spends at the server is $\mathsf{E}[R] + B\,\mathsf{E}[S_0]$. However, if the item can leave right after being served, the expected time at the server is[8]

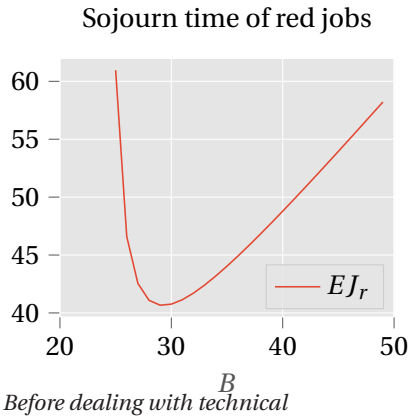
[8] *[4.2.5]*

$$\mathsf{E}[R] + \frac{B-1}{2}\,\mathsf{E}[S_0] + \mathsf{E}[S_0] = \mathsf{E}[R] + \frac{B+1}{2}\,\mathsf{E}[S_0]\,; \tag{4.2.6}$$

the first component is the average time a job has to wait before its service starts, the second is its service time. Our model is complete![9]

[9] *The model in [2.1.10] is more general as it allows the server to work at varying rates.*

WE CAN OBTAIN A NUMBER of important insights from the above model. Using [4.2.1] we plot the sojourn time $\mathsf{E}[J_r]$ of a red job for various values of $B$ in the figure at the right.




*Before dealing with technical derivations, let us first see how to apply the model.*

First we see a sharp decline of $\mathsf{E}[J_r]$. The reason for this is that the load $\rho$ decreases as a function of $B$. Since we know that $\mathsf{E}[W] \sim (1-\rho)^{-1}$, it is indeed essential to stay away from critically loading the server.

When $B$ becomes quite large, we see that the sojourn time increases linearly. This follows right away from (4.2.1) and (4.2.6), because the time to (dis)assemble batches is linear in $B$.

Finally, observe that the graph is not symmetric around the minimum. It is much worse to take $B$ too small than too large.

**Ex 4.2.1.** Jobs arrive at $\lambda = 3$ per hour at a machine with $C_a^2 = 1$; service times are exponential with an average of 15 minutes. Assume $\lambda_r = 0.5$ per hour, hence $\lambda_b = 2.5$ per hour. Between any two batches, the machine requires a cleanup of 2 hours, with a standard deviation of 1 hour. First find the smallest batch size that can be allowed, then compute the average time a red job spends in the system in case $B = 30$ jobs.

**Ex 4.2.2.** Show that the average time a job has to wait to fill the batch (to which this job belongs) is given by (4.2.1).

**Ex 4.2.3.** Explain that the SCV of the batch inter-arrival times is given by (4.2.4).

**Ex 4.2.4.** Show that $C_{s,B}^2$ takes the form as in (4.2.4).

**Ex 4.2.5.** Show that, when items can leave right after being served, the time at the server is given by (4.2.6)

## 4.3 SERVER ADJUSTMENTS

In Section 4.2 we studied the effect of setup times between batches with constant size $B$. However, the server can be interrupted for other reasons than setups. For instance, parts in a machine, such as knives, may need to be replaced or adjusted. Such small tasks can be carried out in between jobs, but it is often hard to predict when it is required. Another example is a GP who has to make a number of unexpected phone calls between seeing two patients. In all such cases, the number of jobs[0] served between any two adjustments[1] is no longer constant. But we know from Sakasegawa's formula that randomness in service times affects queueing times, and we also know from (4.2.3) that the batch size affects the service time. Hence, unplanned adjustments must have an effect on sojourn times.

In this section we develop a simple model to understand the impact of such outages on job sojourn times; we use the same notation as in Section 4.2 and follow the same line of reasoning. With this model, we can analyze a number of trade-offs, such doing fewer, but longer adjustments, or planning adjustments instead just waiting until it becomes necessary at an unexpected moment.[2] With the model we can make graphs of the sojourn time as a function of adjustment rate, so that we can optimize for the adjustment rate.

[0] *You can compare this to a batch of jobs.*

[1] *And this to a setup.*

[2] *[4.3.2]*

It is important to realize that setups and adjustments are both types of *non-preemptive outages*, i.e., they occur *between* jobs, not *during* job service times.

WE ASSUME THAT an adjustment (repair) can occur between any two jobs with some constant probability $p > 0$. Let us write $F$ for the Bernouilli rv associated with such an adjustment: $F = 1$ when an adjustment is necessary, and $F = 0$ otherwise. Clearly, the number of jobs served between two adjustments[3] forms a sequence of geometric iid rvs $\{B_i\}$ such that $\mathsf{P}\{B = k\} = (1-p)^{k-1}p$.[4] As a consequence, $\mathsf{E}[B] = 1/p$ and $\mathsf{V}[B] = (1-p)/p^2$.[5] Suppose also that the adjustment times $\{R_i\}$ are iid rvs with mean $\mathsf{E}[R]$ and variance $\mathsf{V}[R]$.

[3] *Also called a run*

[4] *Recall from (2.4.2) that the run-length $B$ is memoryless under these assumptions.*

[5] *In practice, we measure the average number of jobs $\mathsf{E}[B]$ served between a number of adjustments, and take $p = 1/\mathsf{E}[B]$.*

TO COMPUTE $\mathsf{E}[W]$ with Sakasegawa's formula, we only have to find out how the adjustments affect $\mathsf{E}[S]$ and $C_s^2$. (As the adjustments do not affect the job arrival process, $\lambda$ and $C_a^2$ remain the same.)

It is simple[6] to see that the average effective processing time is

$$\mathsf{E}[S] = \mathsf{E}[S_0] + p\,\mathsf{E}[R] = \mathsf{E}[S_0] + \mathsf{E}[R]\,/\,\mathsf{E}[B]\,. \tag{4.3.1}$$

[6] *[4.3.3]*

As there is one server, $\rho = \lambda\,\mathsf{E}[S]$.

With this expression for $\mathsf{E}[S]$ and an expression[7] for $\mathsf{E}\left[S^2\right]$, we find[8]

$$\begin{aligned}\mathsf{V}[S] &= \mathsf{V}[S_0] + p\,\mathsf{V}[R] + p(1-p)(\mathsf{E}[R])^2 \\ &= \mathsf{V}[S_0] + \mathsf{V}[R]\,/\,\mathsf{E}[B] + (\mathsf{E}[R])^2 C_B^2\,/\,\mathsf{E}[B]\,,\end{aligned} \tag{4.3.2}$$

[7] *[4.3.4]*

[8] *[4.3.5]*

where $C_B^2$ is the SCV of the runlength $B$. We can now fill in Sakasegewa's formula!

BEFORE CONSIDERING SOME examples, let us compare the results of this model to those of Section 4.3. The expected effective service time is the same: an amount $\mathsf{E}[R]\,/\,\mathsf{E}[B]$ gets added to the net processing time $\mathsf{E}[S_0]$. However, the impact on the variance is different. By comparing (4.2.5) to (4.3.2) we see that the latter has an extra (positive) term. This supports our intuition: Unexpected interruptions have a larger effect on variability than expected (planned) interruptions.

**Ex 4.3.1.** Jobs arrive as a Poisson process with rate $\lambda = 9$ per working day. The machine works two 8 hour shifts a day. Work not processed on a day is carried over to the next day. Job service times are 1.5 hours, on average, with standard deviation of 0.5 hours. Outages occur on average between 30 jobs. The average duration of an outage is 5 hours and has a standard deviation of 2 hours. Compute $\mathsf{E}[J]$.

*We first show how to apply the model before we deal with the derivations.*

**Ex 4.3.2.** In the setting of [4.3.1] we can perhaps choose to do an adjustment after every 20 jobs. For simplicity, assume that then adjustments never occur at random. Also assume that an adjustment takes less time, for instance 4.5 hour and are constant. What is $\mathsf{E}[J]$ now?

*In maintenance this is a common problem. Should we wait until a failure occurs, or should we plan the repairs?*

**Ex 4.3.3.** The effective processing time $S = S_0 + R\,\mathbb{1}_{F=1}$. Use this to derive (4.3.1).

*Because planned adjustment can be prepared.*

**Ex 4.3.4.** Recall that $\mathsf{V}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2$ for any random variable. Hence, for $\mathsf{V}[S]$ we need $\mathsf{E}\left[S^2\right]$. Show that

$$\mathsf{E}\left[S^2\right] = \mathsf{E}\left[S_0^2\right] + 2p\,\mathsf{E}[S_0]\,\mathsf{E}[R] + p\,\mathsf{E}\left[R^2\right].$$

**Ex 4.3.5.** Derive (4.3.2).

## 4.4 SERVER FAILURES

In Sections 4.2 and 4.3 we assumed that servers are never interrupted while serving a job. However, this assumption is not always satisfied, for instance, a machine may fail in the midst of processing of a job. In this section, we develop a model[0] to compute the influence on the mean waiting time of such *preemptive outages*, again based on Sakasegawa's formula for the $G/G/1$ queue.

As in the previous sections, we derive expressions for the expectation and variance of the effective processing time $S$.

Supposing that $N$ interruptions occur during the net service time $S_0$ of a job, and the repair times $\{R_i\}$ are a sequence of iid rvs with mean $\mathsf{E}[R]$, the effective service time becomes[1]

$$S = S_0 + S_N = S_0 + \sum_{i=1}^{N} R_i. \tag{4.4.1}$$

In general, $N$ is a random number, so we need to adjust for this in the computation of the mean and variance of $S$.

IT IS COMMON to assume that the time between two interruptions is memoryless with mean $1/\lambda_f$, where $\lambda_f$ is the *failure rate*. Then, if the net service time $S_0$ is a constant, the expected number of failures $\mathsf{E}[N]$ is $\lambda_f S_0$. More generally, if $S_0$ is also a random variable, we find[2] that $\mathsf{E}[N] = \lambda_f \mathsf{E}[S_0]$.

Define the *availability* as

$$A := \frac{m_f}{m_f + \mathsf{E}[R]} \tag{4.4.2}$$

where $m_f$ is the mean time to fail. Since, $m_f = 1/\lambda_f$, it is easy to show[3] that $A = (1 + \lambda_f \mathsf{E}[R])^{-1}$, from which we obtain [4] $\mathsf{E}[S] = \mathsf{E}[S_0]/A$. With this, the utilization becomes

$$\rho = \lambda \mathsf{E}[S] = \lambda \frac{\mathsf{E}[S_0]}{A}.$$

Since $A \in (0,1)$, the utilization increases due to failures.

WITH SOME WORK, we obtain[5] from (4.4.1),

$$\mathsf{E}[S] = (1 + \lambda_f \mathsf{E}[R])\,\mathsf{E}[S_0], \qquad \mathsf{V}[S] = \frac{\mathsf{V}[S_0]}{A^2} + \lambda_f \mathsf{E}\left[R^2\right]\mathsf{E}[S_0]. \tag{4.4.3}$$

By assuming that repair times are exponentially distributed, it is possible to find neat expression for the SCV of the service times: we find[6]

$$C_s^2 = C_0^2 + 2A(1-A)\frac{\mathsf{E}[R]}{\mathsf{E}[S_0]}, \tag{4.4.4}$$

where $C_0^2$ is the SCV of $S_0$.

[0] *There is not much point in combining the models of Section 4.2–Section 4.4 into one large model. For instance, when a machine can be setup, while a part is being repaired, there may not be time lost on the setup. All depends on the specific strategies to combine outages.*

[1] *In an insurance context, if we interpret $\{R_i\}$ as a set of claims, then $\sum_{i=1}^{N} R_i$ is the total claim size of $N$ claims. Likewise, in an inventory system, this sum is the total demand of $N$ customers.*

[2] *[4.4.5]*

[3] *[4.4.2]*

[4] *[4.4.6]*

[5] *[4.4.7]–[4.4.13]*

[6] *[4.4.16]*

**Ex 4.4.1.** Suppose we have a machine with memoryless failure behavior, with a mean-time-to-fail of 3 hours. Regular service times are deterministic with an average of 10 minutes, jobs arrive as a Poisson process with rate of 4 per hour. Repair times are exponential with a mean duration of 30 minutes. What is the average sojourn time?

**Ex 4.4.2.** Derive (4.4.2).

We now derive (4.4.3) in two ways. One is mainly based on using indicator variables, the other on Adam's and Eve's law. Choose whichever you like best, or both.

WITH INDICATORS.

**Ex 4.4.3.** Suppose that the number of failures is equal to the number $n$, show that $\mathsf{E}\left[\sum_{i=1}^{n} R_i\right] = n\,\mathsf{E}[R]$.

**Ex 4.4.4.** Show that $\mathsf{E}\left[\sum_{i=1}^{N} R_i\right] = \mathsf{E}[R]\,\mathsf{E}[N]$ when $N$ is a rv with finite expectation.

*This result is known as Wald's equation.*

**Ex 4.4.5.** Show that $\mathsf{E}[N] = \lambda_f \mathsf{E}[S_0]$ and $\mathsf{E}\left[N^2\right] = \lambda_f^2 \mathsf{E}\left[S_0^2\right] + \lambda_f \mathsf{E}[S_0]$ if we assume for the sake of simplicity that $S_0$ has the density $g$.

*[4.4.5]– [4.4.10] are of fundamental importance in insurance, inventory and queueing theory. They also provide relations between the topics discussed in Section 2.2, Section 6.6, and Section 7.2.*

**Ex 4.4.6.** Show that $\mathsf{E}[S] = \mathsf{E}[S_0] + \lambda_r \mathsf{E}[S_0]\,\mathsf{E}[R]$, and conclude that $\mathsf{E}[S] = \mathsf{E}[S_0]\,/\,A$.

**Ex 4.4.7.** The derivation of $C_s^2$ is a bit more involved. To understand why, explain first that $\mathsf{V}[S] \neq \mathsf{V}[S_0] + \mathsf{V}\left[\sum_{i=0}^{N} R_i\right]$.

**Ex 4.4.8.** Show that

$$\mathsf{E}\left[S^2\right] = \mathsf{E}\left[S_0^2\right] + 2\mathsf{E}\left[S_0 \sum_{i=1}^{N} R_i\right] + \mathsf{E}\left[\sum_{i=1}^{N} R_i^2\right] + \mathsf{E}\left[\sum_{i=1}^{N} \sum_{j \neq i} R_i R_j\right].$$

**Ex 4.4.9.** Show that $\mathsf{E}\left[S_0 \sum_{i=1}^{N} R_i\right] = \lambda_f \mathsf{E}[R]\,\mathsf{E}\left[S_0^2\right]$.

**Ex 4.4.10.** Show that $\mathsf{E}\left[\sum_{i=1}^{N} R_i^2\right] = \lambda_f \mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right]$.

**Ex 4.4.11.** Show that $\mathsf{E}\left[\sum_{i=1}^{N} \sum_{j \neq i} R_i R_j\right] = \lambda_f^2 \mathsf{E}\left[S_0^2\right] (\mathsf{E}[R])^2$.

**Ex 4.4.12.** Combine the above to see that

$$\mathsf{E}\left[S^2\right] = \frac{\mathsf{E}\left[S_0^2\right]}{A^2} + \lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0].$$

**Ex 4.4.13.** Show that

$$\mathsf{V}[S] = \frac{\mathsf{V}[S_0]}{A^2} + \lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0].$$

WITH ADAM'S AND Eve's law.

**Ex 4.4.14.** Derive (4.4.3).

IT REMAINS TO polish the above to find (4.4.4).

**Ex 4.4.15.** Show that

$$C_s^2 = \frac{\mathsf{V}[S]}{(\mathsf{E}[S])^2} = C_0^2 + \frac{\lambda_f \mathsf{E}\left[R^2\right] A^2}{\mathsf{E}[S_0]},$$

**Ex 4.4.16.** With the above assumption that $R$ is exponentially distributed, show that

$$C_s^2 = C_0^2 + 2A(1-A)\frac{\mathsf{E}[R]}{\mathsf{E}[S_0]}.$$

## 4.5 $G/G/1$ QUEUES IN TANDEM

Consider two $G/G/1$ stations in tandem.[0] Suppose we have the financial means to reduce the variability of the processing times at one of the stations, but not at both. Then we like to improve the one that has the most impact on the total sojourn time in the line.

For the waiting time of the first machine, we can use Sakasegawa's formula, but to apply this to the second machine, we need $C_{a,2}^2$, i.e., the SCV of the inter-arrival times at the second station. Now, noting that the output of the first machine forms the input of the second machine, it is clear that $C_{a,2}^2 = C_{d,1}^2$, where $C_{d,1}^2$ is the SCV of the *inter-departure* times of *first* station.

In this section, we present a formulto approximate $C_{d,1}^2$, and with this, we can model the propagation of variability through a tandem networka[1] of $G/G/c$ stations.

Let us consider the inter-departure times of a $G/G/1$ queue. Suppose that the utilization $\rho$ is very high. Then the server will seldom be idle, so that most of the inter-departure times are equal to the service times. However, if the utilization is low, the server will be idle most of the time, and the inter-departure times must be approximately equal to the inter-arrival times.

We obtain an approximation for the SCV $C_d^2$ of the inter-departure times by interpolating between these two extremes[2]:

$$C_d^2 \approx (1-\rho^2)C_a^2 + \rho^2 C_s^2. \tag{4.5.1}$$

Combining Sakasegawa's formula with this expression provides us with a very useful insight for a line of queues. If we reduce $C_{s,1}^2$, i.e., the SCV of service times at the first station, $\mathsf{E}[W_1]$ and $C_{d,1}^2$ become smaller. Since $C_{a,2}^2 = C_{d,1}^2$, $\mathsf{E}[W_2]$ becomes lower too, but also $C_{d,2}^2$, and so on. In other words, the entire chain benefits from an improvement in service variability at the first station.[3]

**Ex 4.5.1.** Consider two $G/G/1$ stations in tandem. Suppose $\lambda = 2$ per hour, $C_{a,1}^2 = 2$, $C_{s,1}^2 = C_{s,2}^2 = 0.5$, and $\mathsf{E}[S_1] = 20$ minutes and $\mathsf{E}[S_2] = 25$ minutes. Compute $\mathsf{E}[J] = \mathsf{E}[J_1] + \mathsf{E}[J_2]$.

[0] *'In tandem' means 'in line', one station after the other.*

[1] *The literature provides algorithms to deal with networks of $G/G/1$ queues in which the output of several stations form the input of another station and rework is allowed. In Section 7.3 we analyze such networks, but only for $M/M/c$ stations.*

[2] *For the $G/G/c$ there is the generalization $C_d^2 \approx 1+(1-\rho^2)(C_a^2-1)+\frac{\rho^2}{\sqrt{c}}(C_s^2-1)$. It is simple to see that this reduces to (4.5.1) for the $G/G/1$ queue.*

[3] *Try to improve at the start.*

# 5

# FUNDAMENTAL TOOLS

To develop mathematical models of queueing systems we need a few concepts that are fundamentally important and have a general interest beyond queueing. All these concepts rely on *sample-path constructions* of queueing, or more general stochastic, systems. We will see that sample paths, which are in fact realizations of simulations of queueing systems, form an elegant and unifying principle.

Here we keep the discussion in these notes mostly at an intuitive level; we refer to El-Taha and Stidham Jr. [1998] for proofs and further background.

## 5.1 RENEWAL REWARD THEOREM

We introduce the renewal reward theorem and provide graphical motivation for its validity.[0] This result proves to be extremely useful: we will apply it regularly in the sequel of the book and here we use it to provide another definition of the utilization $\rho$.

In Fig. 5.1.1, we consider a strictly increasing set of epochs $\{T_k, k = 0,1,\ldots\}$. Let $N = \{N(t), t \geq 0\}$ be the associated counting process such that $N(t) = \max\{k : T_k \leq t\}$. Let $\{Y(t), t \geq 0\}$ be a non-decreasing right-continuous (deterministic) process, and define $X_k := Y(T_k) - Y(T_{k-1})$, i.e,. the increment[1] in $Y$ between the epochs $T_{k-1}$ and $T_k$.

The *renewal reward theorem* states that when $\lim_{t\to\infty} N(t)/t = \lambda$ with $0 < \lambda < \infty$, $Y(t)/t$ has a limit iff $n^{-1}\sum_{k=1}^{n} X_k$ has a limit, and then $Y = \lambda X$.[2]

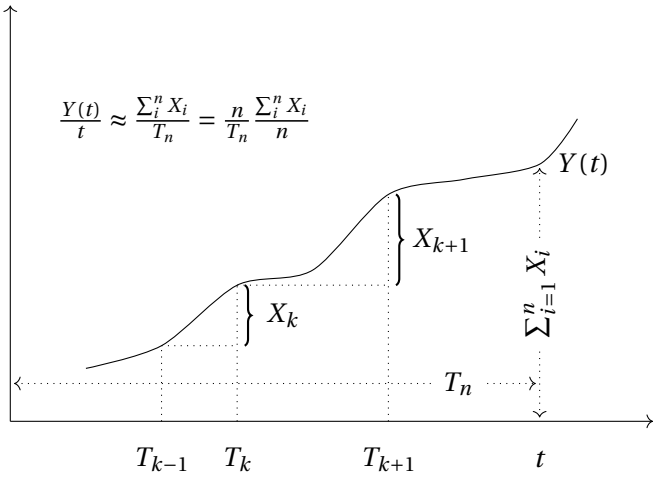



Figure 5.1.1: A graphical 'proof' of $Y = \lambda X$.

WITH THE RENEWAL REWARD THEOREM we can relate the utilization $\rho = \lambda \mathsf{E}[S]$ to the limiting fraction of time the servers of a $G/G/1$ queue are busy. In fact, with this theorem[3]

[0] *El-Taha and Stidham Jr. [1998] give a (simple) proof based on similar arguments as used in Section 3.3.*

[1] *Here $X_k$ is not an inter-arrival time between jobs.*

[2] *In essence the renewal reward theorem is very simple: it states that when* customers *arrive at rate $\lambda$ and each* customer *pays an average amount $X$, the* system *earns money at rate $Y = \lambda X$.*

[3] *[5.1.1]*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{1}_{L(s)>0}\,\mathrm{d}s = \delta\,\mathsf{E}[S] = \lambda\,\mathsf{E}[S]\,. \tag{5.1.1}$$

Clearly, the LHS is the long-run fraction of time the servers are busy and the RHS follows from rate-stability, i.e., $\delta = \lambda$.

We can derive[4] this relation also by considering the fact that

$$\sum_{k=1}^{A(t)} S_k \geq \int_0^t \mathbb{1}_{L(s)>0}\,\mathrm{d}s \geq \sum_{k=1}^{D(t)} S_k. \tag{5.1.2}$$

Then, by taking appropriate limits $\lambda\,\mathsf{E}[S] \geq \rho \geq \delta\,\mathsf{E}[S]$, and rate-stability completes the argument.

[4] *[5.1.2]*

**Ex 5.1.1.** Use the renewal reward theorem to prove (5.1.1).

**Ex 5.1.2.** Explain (5.1.2) and derive $\lambda\,\mathsf{E}[S] \geq \rho \geq \delta\,\mathsf{E}[S]$ from this.

## 5.2 LEVEL CROSSING AND BALANCE EQUATIONS

Consider a sample path[0] of $L := \{L\}$. We say that the sample path *up-crosses level n* at time $t$ when the number of jobs $L$ in the system changes from $n$ to $n+1$ due to an arrival, in other words, when $L(t-) = n$ and $L(t) = n$. The sample path *down-crosses level n* at time $t$ when $L$ changes from $n+1$ to $n$ due to a departure, that is, $L(t-) = n+1$ and $L(t) = n$. Clearly, the number of up-crossings and down-crossings cannot differ by more than 1 at any time, because it is only possible to down-cross level $n$ after an up-crossing (or the other way around). This simple idea will prove to be a key stepping stone in the analysis of queueing systems.

[0] *The graph of the number of jobs in the system obtained by simulation.*

To establish this section's main result (5.2.6), we need a few definitions that are quite subtle, but we will provide intuitive and natural interpretations.[1] After this, we will generalize the principle of level-crossing to *balance equations* which allow us to deal with more general types of transitions.[2]

[1] *They have practical significance too.*

[2] *Fig. 5.5.1 at the end of this chapter provides a graphical summary.*

LET US SAY that the system is in *state n* at time $t$ when $L(t) = n$. Then define[3]

$$A(n,t) := \sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=n} = \sum_{k=1}^{\infty} \mathbb{1}_{A_k\leq t}\,\mathbb{1}_{L(A_k-)=n} \tag{5.2.1a}$$

as the number of arrivals up to time $t$ that saw the system in state $n$.[4] Next, let

$$Y(n,t) := \int_0^t \mathbb{1}_{L(s)=n}\,\mathrm{d}s \tag{5.2.1b}$$

be the total time the system spends in state $n$ during $[0,t]$, and

$$p(n,t) := \frac{1}{t}\int_0^t \mathbb{1}_{L(s)=n}\,\mathrm{d}s = \frac{Y(n,t)}{t}, \tag{5.2.1c}$$

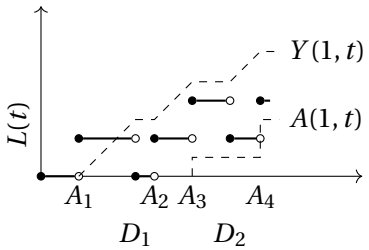
be the fraction of time in state $n$ during $[0,t]$.

With the above definitions, we consider the limits,[5]

$$\lambda(n) := \lim_{t\to\infty}\frac{A(n,t)}{Y(n,t)}, \qquad p(n) = \lim_{t\to\infty} p(n,t), \tag{5.2.2}$$

[3] *Note the $A_k-$; since $L(t)$ is right-continuous, we concentrate on what an* arrival *sees just before it's arrival*

[4] *[5.2.1],[5.2.2]*




*We subtracted 1/2 from the graph of $A(1,t)$, for otherwise this overlaps with the graph of $Y$.*

[5] *Again assuming they exist.*

as the arrival rate *while* the system is in state $n$, and the long-run fraction of time spent in state $n$. To see the rationale behind this definition of $\lambda(n)$, recall that $A(t)/t$ is the number of arrivals during $[0,t]$ divided by $t$, while $A(n,t)/Y(n,t)$ is the number of arrivals during $[0,t]$ that see $n$ jobs divided by the time the system contains $n$ jobs.

Similar to the definition for $A(n,t)$, let[6]

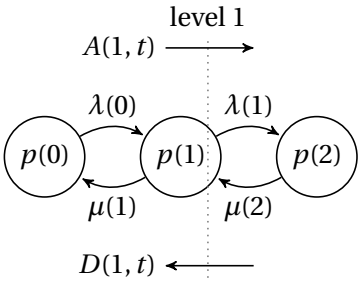
$$D(n,t) := \sum_{k=1}^{D(t)} \mathbb{1}_{L(D_k)=n} = \sum_{k=1}^{\infty} \mathbb{1}_{D_k \le t} \mathbb{1}_{L(D_k)=n} \tag{5.2.3}$$

denote the number of departures up to time $t$ that *leave n customers behind.* Then, define[7]

$$\mu(n+1) := \lim_{t\to\infty} \frac{D(n,t)}{Y(n+1,t)},$$

as *the departure rate from state* $n+1$.

[6] *But now we take* $D_k$, *not* $D_k-$.




[7] *It is easy to get confused here: to leave n jobs behind, the system must contain* $n+1$ *jobs just prior to the departure.*

THE ABOVE DEFINITIONS may seem a bit abstract, but they have simple and practical interpretations.

Consider the sorting process of post parcels at a distribution center of a post-delivery company. Each day tens of thousands of incoming parcels have to be sorted to their final destination. Incoming parcels are deposited on a conveyor belt, and from there, they are carried to outlets, so-called chutes, from where the parcels are sent to a specific region of the Netherlands. Employees pick up the parcels from the chutes and put the parcels in containers. Sometimes parcels arrive a bit faster than the capacity of the employees and then a queue of parcels builds up in the chute. When the chute overflows, parcels are directed to an overflow container and are sorted the next day. The target of the sorting center is to deliver at least 99% of the parcels within one day.

Suppose a chute can contain at most 20 parcels, say. Then, each parcel on the belt that 'sees' 20 parcels in its chute will be sent to the overflow container. Clearly then, $A(20,t)/A(t)$ is the fraction of rejected parcels up to time $t$. For this reason, sorting centers continuously track $A(20,\cdot)$ to adapt server capacity and control the fraction of rejected parcels.

For a second example, suppose it costs $w$ to have a job in queue for a unit of time. The total cost up to time $t$ is then $w\sum_{n=0}^{\infty} nY(n,t)$ and the average cost is $w\sum_{n=0}^{\infty} np(n,t)$.

CONTINUING WITH THE theoretical discussion, observe that customers arrive and depart as single units. Thus, if $\{T_k\}$ is the ordered set of arrival and departure epochs of the customers, then $L(T_k) = L(T_k-) \pm 1$. But this implies that[8]

$$|A(n,t) - D(n,t)| \le 1. \tag{5.2.4}$$

From this observation it follows immediately that

$$\lim_{t\to\infty} \frac{A(n,t)}{t} = \lim_{t\to\infty} \frac{D(n,t)}{t}. \tag{5.2.5}$$

With this equation we obtain two nice identities. The first we develop here, the other in Section 5.3.

[8] *Think about this. This is simple, but has profound consequences.*

Clearly,[9] $\lim_{t\to\infty} A(n,t)/t$ is the rate of arrivals that see the system in

[9] *Supposing the limits exist.*

state $n$. With (5.2.1) we see that

$$\lim_{t\to\infty}\frac{A(n,t)}{t}=\lim_{t\to\infty}\frac{A(n,t)}{Y(n,t)}\frac{Y(n,t)}{t}=\lambda(n)p(n).$$

Similarly, the rate of jobs that leave $n$ jobs behind is

$$\lim_{t\to\infty}\frac{D(n,t)}{t}=\lim_{t\to\infty}\frac{D(n,t)}{Y(n+1,t)}\frac{Y(n+1,t)}{t}=\mu(n+1)p(n+1).$$

Combining this with (5.2.5) we arrive at the *level-crossing equations*[10]

$$\lambda(n)p(n)=\mu(n+1)p(n+1). \tag{5.2.6}$$

[10] *This result is of fundamental importance in queueing (and inventory) theory.*

SUPPOSE WE CAN specify[11] the arrival and service rates $\lambda(n)$ and $\mu(n)$, then we can easily compute the long-run fraction of time $p(n)$ that the system contains $n$ jobs. To see this, rewrite (5.2.6) as

$$p(n+1)=\frac{\lambda(n)}{\mu(n+1)}p(n). \tag{5.2.7}$$

[11] *In Section 6.1 and onward we will model many queueing situations by making suitable choices for $\lambda(n)$ and $\mu(n)$.*

A straightaway recursion then leads to

$$p(n+1)=\frac{\lambda(n)\lambda(n-1)\cdots\lambda(0)}{\mu(n+1)\mu(n)\cdots\mu(1)}p(0).$$

Thus, $p(n)$, $n\geq 1$, is just $p(0)$ times a constant, which is based on arrival and service rates.

To determine $p(0)$ we can use the fact that the numbers $p(n)$ represent probabilities. Hence, from the normalizing condition $\sum_{n=0}^{\infty}p(n)=1$, we get $p(0)=G^{-1}$ with $G$ being the *normalization constant*

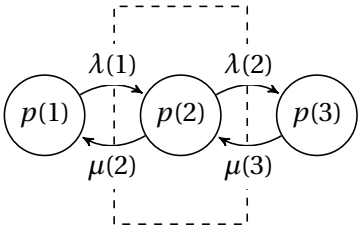
$$G=1+\sum_{n=0}^{\infty}\frac{\lambda(n)\lambda(n-1)\cdots\lambda(0)}{\mu(n+1)\mu(n)\cdots\mu(1)}. \tag{5.2.8}$$

Finally, once we have $p(n)$, it is easy to compute the time-average[12] number of jobs in the system and the long-run fraction of time the system contains at least $n$:

$$\mathsf{E}[L]=\sum_{n=0}^{\infty}np(n),\qquad\qquad \mathsf{P}\{L\geq n\}=\sum_{i=n}^{\infty}p(i).$$

[12] *It is important to realize that this is not necessarily the same as what jobs see upon arrival.*

IT IS IMPORTANT to realize that the level-crossing argument cannot always be used, as it is not always possible to split the state space into two disjoint parts by 'drawing a line' between two states. For a more general approach, we focus on a single state and count how often this state is entered and left. Specifically, define $I(n,t)=A(n-1,t)+D(n,t)$ as the number of times the queueing process enters state $n$ either due to an arrival from state $n-1$ or due to a departure leaving $n$ jobs behind. Similarly, $O(n,t)=A(n,t)+D(n-1,t)$ counts how often state $n$ is left either by an arrival (to state $n+1$) or a departure (to state $n-1$).




Just like (5.2.4), it is evident that $|I(n,t)-O(n,t)|\leq 1$, and this implies[13]

$$\lambda(n-1)p(n-1)+\mu(n+1)p(n+1)=(\lambda(n)+\mu(n))p(n). \tag{5.2.9}$$

[13] *[5.2.10]*

These equations hold for any $n\geq 1$ and are known as the *balance equations*. We will use these equations to analyze queueing networks.

**Ex 5.2.1.** Show that $A(t) = \sum_{n=0}^{\infty} A(n,t)$.

**Ex 5.2.2.** If $\lambda > \delta$ can it happen that $\lim_{t\to\infty} A(n,t)/t > 0$ for some (finite) $n$?

**Ex 5.2.3.** Consider the following (silly) queueing process. At times $0, 2, 4, \ldots$ customers arrive, each customer requires 1 unit of service, and there is one server. Find an expression for $A(n,t)$, when $L(0) = 0$.

*What acronym would describe this queueing situation?*

**Ex 5.2.4.** Find an expression for $Y(n,t)$.

*Continuation of [5.2.3]*

**Ex 5.2.5.** Compute $p(n)$ and $\lambda(n)$.

*Continuation of [5.2.4]*

**Ex 5.2.6.** Compute $D(n,t)$ and $\mu(n+1)$ for $n \geq 0$.

*Continuation of [5.2.5]*

**Ex 5.2.7.** Compute $\lambda(n)p(n)$ for $n \geq 0$, and check $\lambda(n)p(n) = \mu(n+1)p(n+1)$.

*Continuation of [5.2.6]*

**Ex 5.2.8.** Derive $\mathsf{E}[L] = \sum_{n=0}^{\infty} np(n)$ from (3.4.3).

**Ex 5.2.9.** Consider a single server that serves one queue and serves only in batches of 2 jobs at a time (so never 1 job or more than 2 jobs). At most 3 jobs fit in the system. Single jobs arrive as a Poisson process with $\lambda$. Due to blocking, we take $\lambda(n) = \lambda$ for $n < 3$ and $\lambda(n) = 0$ for $n \geq 3$. The batch service times are exponentially distributed with mean $1/\mu$, so that by the memoryless property, $\mu(n) = \mu$.

Make a graph of the state-space and show, with arrows, the transitions that can occur and use level-crossing arguments to express the steady-state probabilities $p(n), n = 0, \ldots, 3$ in terms of $\lambda$ and $\mu$.

*When level-crossing arguments can be applied, the analysis often becomes quite easy.*

*What is the acronym for this system?*

**Ex 5.2.10.** Show (5.2.9) from $|I(n,t) - O(n,t)| \leq 1$.

## 5.3 POISSON ARRIVALS SEE TIME AVERAGES

Suppose jobs arrive exactly at the start of an hour and require 59 minutes of service. If we sample the server occupation at job arrival times, the server is always free, while if we track the server over time, it is nearly always busy. Thus, what jobs see upon arrival is *in general not equal* to what the server perceives. However, when jobs arrive as a Poisson process, both sampling methods produce the same number, a result known as the *Poisson arrivals see time averages* (*PASTA*) property. Here we will discuss this property in more detail.

RECALL FROM SECTION 5.2 that

$$\frac{A(n,t)}{t} = \frac{A(n,t)}{Y(n,t)} \frac{Y(n,t)}{t} \to \lambda(n)p(n), \quad \text{as } t \to \infty. \tag{5.3.1a}$$

Rather than multiplying and dividing the LHS by $Y(n,t)$, we can also multiply and divide by $A(t)$ to get another interesting result:

$$\frac{A(n,t)}{t} = \frac{A(t)}{t} \frac{A(n,t)}{A(t)}. \tag{5.3.1b}$$

It is clear that $A(t)/t \to \lambda$, so let us interpret $A(n,t)/A(t)$. For this, observe first

$$\frac{A(n,t)}{A(t)} = \frac{1}{A(t)} \sum_{k=1}^{\infty} \mathbb{1}_{A_k \le t, L(A_k-)=n} = \frac{1}{A(t)} \sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=n}.$$

Then, since $A(t) \to \infty$ as $t \to \infty$, it follows from the renewal reward theorem[0] that the following limit can be defined:

[0] [5.3.4]

$$\pi_n := \lim_{t\to\infty} \frac{1}{A(t)} \sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=n}.$$

That is, $\pi(n)$ is the long-run fraction of arrivals that observe $n$ customers in the system.

But, in (5.3.1) the LHSs are equal. Hence,

$$\lambda\pi(n) = \lambda(n)p(n), \tag{5.3.2}$$

which implies the main result:

$$\lambda(n) = \lambda \iff \pi(n) = p(n).$$

In words, if the arrival rate does not depend on the state of the system, i.e., $\lambda(n) = \lambda$, the sample probabilities $\{\pi(n)\}$ are equal to the time-average probabilities $\{p(n)\}$.[1]

[1] *Thus, what arrivals see agrees with what the server sees.*

As the above example showed, this property is not satisfied in general. However, when the arrival process is Poisson we have[2] that $\lambda(n) = \lambda$, and its implication $\pi(n) = p(n)$ is known as the *PASTA* property.

[2] *A rigorous proof of this is hard, see e.g., El-Taha and Stidham Jr. [1998]*

WITH SIMILAR REASONING, we can also establish a relation between $\pi(n)$ and $\delta(n)$, i.e., statistics as obtained by the departures. Define, analogous to $\pi(n)$,

$$\delta(n) := \lim_{t\to\infty} \frac{D(n,t)}{D(t)}$$

as the long-run fraction of jobs that leave $n$ jobs *behind*. From (5.2.5),

$$\frac{A(t)}{t}\frac{A(n,t)}{A(t)} = \frac{A(n,t)}{t} \approx \frac{D(n,t)}{t} = \frac{D(t)}{t}\frac{D(n,t)}{D(t)}.$$

Taking limits at the left and right, and using (3.3.2), we obtain for the $G/G/c$ queue[3]

$$\lambda\pi(n) = \delta\delta(n). \tag{5.3.3}$$

[3] *Because customers arrive and leave as single units in a $G/G/c$ queue.*

Consequently, for the rate-stable $G/G/c$ queue the statistics obtained by arrivals is the same as statistics obtained by departures, i.e.,

$$\lambda = \delta \iff \pi(n) = \delta(n). \tag{5.3.4}$$

**Ex 5.3.1.** Show that $\pi(0) = 1$ and $\pi(n) = 0$, for $n > 0$. *Continuation of [5.2.3]*

**Ex 5.3.2.** Check that (5.3.2) holds. *Continuation of [5.3.1]*

**Ex 5.3.3.** When $\lambda \neq \delta$, is $\pi(n) \ge \delta(n)$?

**Ex 5.3.4.** Use the renewal-reward theorem to prove that $A(n,t)/t \to \lambda\pi(n)$.

**Ex 5.3.5.** In a small Midwestern town there lived a retired railroad engineer named William Johnson.[4] The main line on which he had worked for so many years passed through the town. Mr. Johnson suffered from insomnia and would often wake up at any odd hour of the night and be unable to fall asleep again. He found it helpful, in such cases, to take a walk along the deserted streets of the town, and his way always led him to the railroad crossing. He would stand there thoughtfully watching the track until a train thundered by through the dead of the night. The sight always cheered the old railroad man, and he would walk back home with a good chance of falling asleep.

After a while he made a curious observation; it seemed to him that most of the trains he saw at the crossing were traveling eastward, and only a few were going west. Knowing very well that this line was carrying equal numbers of eastbound and westbound trains, and that they alternated regularly, he decided at first that he must have been mistaken in this reckoning. To make sure, he got his little note-book, and began putting down "E" or "W," depending on which way the first train to pass was traveling. At the end of a week, there were five "E's" and only two "W's" and the observations of the next week gave essentially the same proportion. Could it be that he always woke up at the same hour of night, mostly before the passage of eastbound trains?

Being puzzled by this situation, he decided to undertake a rigorous statistical study of the problem, extending it also to the daytime. He asked a friend to make a long list of arbitrary times such as 9:35 a.m., 12:00 noon, 3:07 p.m., and so on, and he went to the railroad crossing punctually at these times to see which train would come first. However, the result was the same as before. Out of one hundred trains he met, about seventy-five were going east and only twenty-five west. In despair, he called the depot in the nearest big city to find whether some of the westbound trains had been rerouted through another line, but this was not the case. He was, in fact, assured that the trains were running exactly on schedule, and that equal numbers of trains daily were going each way. This mystery brought him to such despair that he became completely unable to sleep and was a very sick man.

Can you reveal the mystery thereby solving the sleeping problems of Mr. Johnson?

[4] *This is an interesting riddle from 'Puzzle Math' by G. Gamov and M. Stern.*

## 5.4 LITTLE'S LAW

There is an important relation between the average sojourn time of a job and the long-run time-average number of jobs contained in the system. This relation is called *Little's law*, and is one of the most useful results in queueing theory.[0] The aim of this section is to prove this law under some simple conditions.

WE START BY defining a few intuitively useful concepts.[1] Since $\mathbb{1}_{A_k \le s < D_k}$ is 1 if job $k$ is present in the system and 0 otherwise, $\int_0^t \mathbb{1}_{A_k \le s < D_k} \, \mathrm{d}s$ is the total time job $k$ spent in the sytem up to time $t$.[2] With this, the sojourn time of

[0] *In fact, to analyze throughput in any input-output system.*

[1] *Consult Fig. 2.3.2.*

[2] *An intuitive way to see this is by assuming that a job pays 1 Euro per unit time in the system. Then $\int_0^t \mathbb{1}_{A_k \le s < D_k} \, ds$ is what job k paid up to time $t$.*

the $k$th job then becomes

$$J_k = \int_0^\infty \mathbb{1}_{A_k \le s < D_k} \, \mathrm{d}s.$$

It is also clear that $L(s) = \sum_{k=1}^\infty \mathbb{1}_{A_k \le s < D_k}$ is the number of jobs in the system at time $s$. By combining these two facts, the LHS of the equation

$$\int_0^t L(s) \, \mathrm{d}s = \int_0^t \sum_{k=1}^\infty \mathbb{1}_{A_k \le s < D_k} \, \mathrm{d}s = \sum_{k=1}^\infty \int_0^t \mathbb{1}_{A_k \le s < D_k} \, \mathrm{d}s \tag{5.4.1}$$

acquires the interpretation of the total time jobs have spent in the system up to time $t$.[3]

[3] *Or, the total amount paid by the jobs up to time $t$.*

CONSIDER A DEPARTURE time $T$ at which the system is empty so that $A(T) = D(T)$. Then, $J_k = \int_0^T \mathbb{1}_{A_k \le s < D_k} \, \mathrm{d}s$ for any $k \le A(T)$, and $L(s) = \sum_{k=1}^{A(T)} \mathbb{1}_{A_k \le s < D_k}$ for $s \le T$. Now realize[4] that the areas $\int_0^T L(s) \, \mathrm{d}s$ and $\sum_{k=1}^{A(T)} J_k$ are the same, that is,

$$\int_0^T L(s) \, \mathrm{d}s = \sum_{k=1}^{A(T)} J_k. \tag{5.4.2}$$

[4] *[5.4.5]*

Assuming that there exists an infinite number of times $\{T_i\}$, $T_i \to \infty$, such that the system is empty, i.e., $A(T_i) = D(T_i)$, it is then evident that

$$\frac{1}{T_i} \int_0^{T_i} L(s) \, \mathrm{d}s = \frac{A(T_i)}{T_i} \frac{1}{A(T_i)} \sum_{k=1}^{A(T_i)} J_k.$$

Taking the limit $i \to \infty$ of both sides, provided they exist, gives *Little's law*:

$$\mathsf{E}[L] = \lambda \, \mathsf{E}[J].$$

Note that Little's law need not hold at an arbitrary moment in time,[5] it is a statement about *averages*.

[5] *[5.4.3] and [5.4.4]*

WITH PASTA AND Little's law it is easy to derive expressions for $\mathsf{E}[J]$ and $\mathsf{E}[W]$ for the $M/M/1$ queue.[6] With PASTA, it follows that the waiting time in queue is

$$\mathsf{E}[W] = \mathsf{E}[L] \, \mathsf{E}[S]. \tag{5.4.3}$$

[6] *For the $M/G/1$ queue, things are bit subtler, see [5.4.6] and Section 6.4.*

Combining this with $\mathsf{E}[J] = \mathsf{E}[W] + \mathsf{E}[S]$ leads right away to[7]

$$\mathsf{E}[J] = \frac{\mathsf{E}[S]}{1-\rho}, \qquad \mathsf{E}[L] = \frac{\rho}{1-\rho}, \qquad \mathsf{E}[Q] = \frac{\rho^2}{1-\rho}. \tag{5.4.4}$$

[7] *[5.4.7]*

IN THE ABOVE proof of Little's law we assumed that there is an increasing sequence of epochs $\{T_k, k = 0, 1, \ldots\}$ at which the system is empty. However, in many practical queueing situations the system is never empty. In [5.4.9] and [5.4.10] we show that Little's law also holds under the weaker condition of rate-stability.

**Ex 5.4.1.** Use the (physical) dimensions of the components of Little's law to check that $\mathsf{E}[J] \neq \lambda \, \mathsf{E}[L]$.

*With this check, you can prevent making an often-made mistake.*

**Ex 5.4.2.** Consider the server of the (stable) $G/G/1$ queue as a system by itself. Jobs arrive at rate $\lambda$ and stay $\mathsf{E}[S]$ in this system. Use Little's law to conclude that $\rho = \lambda \, \mathsf{E}[S]$.

**Ex 5.4.3.** Compute $\mathsf{E}[W(M/M/1)]$ when $\lambda = 5/h$ and $\mu = 6/h$.

*$\mathsf{E}[W(M/M/1)]$ is the waiting time in queue for the $M/M/1$ queue*

**Ex 5.4.4.** Suppose that, at the moment you join the system of [5.4.3], the number of customers in the system is 10. What is your expected waiting time?

**Ex 5.4.5.** Derive (5.4.2).

**Ex 5.4.6.** While (5.4.3) is true for the $M/M/1$, it is *not true for the $M/G/1$ queue*. Why is that?

**Ex 5.4.7.** Derive (5.4.4).

**Ex 5.4.8.** Explain that $\sum_{k=1}^{A(t)} J_k \geq \int_0^t L(s)\,\mathrm{d}s \geq \sum_{k=1}^{D(t)} J_k$.

**Ex 5.4.9.** Take suitable limits to show that $\lambda\,\mathsf{E}[J] \geq \mathsf{E}[L] \geq \delta\,\mathsf{E}[J]$. Where do you need the strong law of large numbers?

*Continuation of [5.4.8]*

**Ex 5.4.10.** Suppose that $A(t) = \lambda t$ and $D(t) = [A(t) - 10]^+$ so that the system is never empty for $t > 0$. Conclude that Little's law is still true.

## 5.5 GRAPHICAL SUMMARY

We finish this chapter with providing two summaries in graphical form to clarify how all concepts developed in this chapter relate.

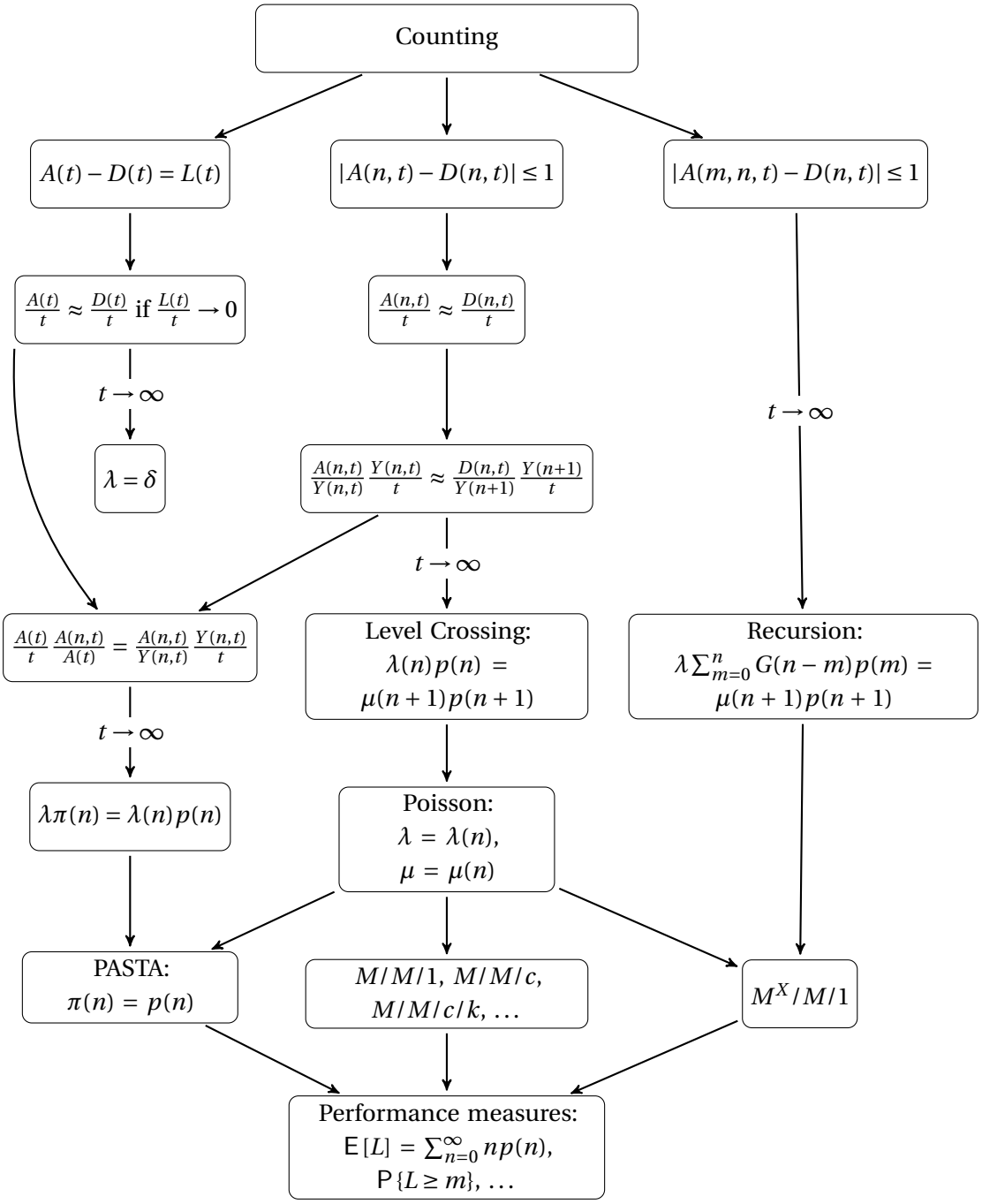



Figure 5.5.1: With level-crossing arguments we can derive a number of useful relations. This figure presents an overview of these relations that we derive in this and the next sections.

# 6

# EXACT QUEUEING MODELS

In this chapter we use the concepts of Chapter 5 to model and analyze many queueing systems in steady state. With sample-path analysis and level-crossing, it will be see in Section 6.1. that it is nearly trivial to derive useful expressions for the $M(n)/M(n)/1$ queue of which the $M/M/1$, the $M/M/c$, and the $M/M/1/K$ queue are just special cases.

In Section 6.2 we combine these results with Little's law and PASTA to compute the most important performance measures for numerous examples.

We then focus on finding the expected waiting time for batch queues, in Section 6.3, and the $M/G/1$ queue, in Section 6.4. In the last two sections of this chapter, Section 6.5 and Section 6.6, we derive expressions for the queue length distributions of the batch queue and the $M/G/1$ queue.

Quite a number of exercises in this chapter are targeted on *checking* that the results for some general queueing system reduce to those for special cases. The reader should understand the importance of such checks. These exercises are simple in a sense—it is perfectly clear what to do, there is no model to make for instance—, but the algebra can be a bit tough at times.

## 6.1 $M/M/1$ QUEUE AND ITS VARIATIONS

In this section we devevelop many different queueing models by using the level-crossing equations (5.2.7) and a judicious choice of $\lambda(n)$ and $\mu(n)$. However, we need to require that the time to the next arrival or departure is memoryless[0]. Here we present the results; the exercises ask you to derive the formulas.[1] In Section 6.2 we show how to apply the models we derive here.

As said, the inter-arrival times and service times need to be memoryless. Specifically, this means that[2]

$$\begin{aligned} \mathsf{P}\{L(t+\Delta t) = n+1 \,|\, L(t) = n\} &= \lambda(n)\Delta t + o(\Delta t), \quad n \geq 0, \\ \mathsf{P}\{L(t+\Delta t) = n-1 \,|\, L(t) = n\} &= \mu(n)\Delta t + o(\Delta t), \quad n \geq 1. \end{aligned}$$

In other words, the arrival rate $\lambda(n)$ and service rate $\mu(n)$ may depend on the number of jobs in the system, but not more.

FOR THE $M/M/1$ QUEUE we take $\lambda(n) = \lambda$ for the arrival rate and $\mu(n) = \mu$ for the service rate. Therefore, the level-crossing equations become

$$p(n+1) = \frac{\lambda(n)}{\mu(n+1)} p(n) = \frac{\lambda}{\mu} p(n) = \rho\, p(n), \quad \rho = \frac{\lambda}{\mu}.$$

As this holds for any $n \geq 0$, $p(n+1) = \rho^{n+1} p(0)$. Normalization with (5.2.8) and (1.2.3d) gives that $p(0) = 1-\rho$, hence

$$p(n) = (1-\rho)\rho^n, \quad n \geq 0. \tag{6.1.1}$$

[0] *hence, exponential*

[1] *The main challenge is not to make computational errors*

[2] *Compare the construction of the Poisson process in [2.2.5].*

It is now easy to compute the most important performance measures. The utilization is $\rho = \lambda/\mu$, and with a bit of algebra,[3]

[3] *[6.1.1]*

$$\mathsf{E}[L_s] = \rho, \qquad \mathsf{E}[L] = \frac{\rho}{1-\rho}, \qquad \mathsf{P}\{L > n\} = \rho^{n+1}, \tag{6.1.2}$$

recall that $\mathsf{E}[L_s]$ is the time-average number of jobs in service, $\mathsf{E}[L]$ is the number in the system. For the number in queue,

$$\mathsf{E}[Q] = \mathsf{E}[L] - \mathsf{E}[L]\, s = \frac{\rho^2}{1-\rho}.$$

IN THE $M/M/1/K$ QUEUE jobs are blocked[4] when $L \geq K$. We can implement this by taking $\lambda(n) = \lambda\, \mathbb{1}_{n<K}$, which gives $\lambda p(n) = \mu p(n+1)$ for $n < K$, and $p(n) = 0$ for $n > K$. Then, using (1.2.3d), it follows right away that

[4] *Or they are not willing to join the system*

$$p(n) = \frac{1-\rho}{1-\rho^{K+1}}\rho^n, \quad \rho = \frac{\lambda}{\mu}.$$

Obseve that, since jobs are blocked, the utilization is no longer equal to the load. The utilization is now $\lambda p(K)/\mu \neq \lambda/\mu = \rho$. Note further that, contrary to the $M/M/1$ case,

$$\mathsf{E}[L_s] = \sum_{n=0}^{K} \min\{n, 1\} p(n) = 1 - p(0) \neq \rho.$$

The rest of the performance measures follow also easily.

CONSIDER NEXT THE $M/M/c$ QUEUE. For this, we set $\lambda(n) = \lambda$, and $\mu(n) = \mu\min\{n, c\}$. As in (3.3.5), we take $\rho = \lambda/(c\mu)$. After some calculations we obtain[5]

[5] *[6.1.4]*

$$p(n) = \frac{1}{G}\frac{1}{\Pi_{k=1}^{n}\min\{c,k\}}(c\rho)^n, \quad n \geq 1, \tag{6.1.3a}$$

$$G = \frac{1}{p(0)} = \sum_{n=0}^{c-1}\frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{(1-\rho)c!}, \tag{6.1.3b}$$

$$\mathsf{E}[Q] = \sum_{n=c}^{\infty}(n-c)p(n) = \frac{(c\rho)^c}{c!G}\frac{\rho}{(1-\rho)^2} \tag{6.1.3c}$$

$$\mathsf{E}[L_s] = \sum_{n=0}^{\infty}\min\{n,c\}p(n) = \frac{\lambda}{\mu}. \tag{6.1.3d}$$

Observe that the expected number of busy servers $\mathsf{E}[L_s]$ is equal to the load $\lambda\,\mathsf{E}[S] = \lambda/\mu$ (and not to $\rho$).

THE $M/M/c/K$ QUEUE is a multi-server queue with blocking and is a simple generalization of the $M/M/c$ and $M/M/1/K$ queue. Now, $\lambda(n) = \lambda\,\mathbb{1}_{n<K}$ and $\mu(n) = \mu\min\{n, c\}$. The probabilities $p(n)$ are given by (6.1.3a). Mind again that now the utilization is not the same as the load. The computation of the normalization is numerically trivial.

Here is an example in code.[6] We use just the recursion $\lambda(n)p(n) = \mu(n+1)p(n+1)$ to compute $p(n)$, as this is easier to code, more generic, and, in this case, also numerically more efficient.

[6] *The other models of this section are just as easily implemented.*

Python Code

```
>>> import numpy as np

>>> mu, c, K = 2, 4, 8
>>> labda = 3 * np.ones(K)
>>> mu = np.arange(K + 1)
>>> mu = np.minimum(mu, c)

>>> p = np.ones(K + 1, dtype=float)

>>> for n in range(K):
...     p[n + 1] = p[n] * labda[n] / mu[n + 1]
...
>>> p /= p.sum()  # normalize
>>> ELs = sum(p[n] * min(n, c) for n in range(len(p)))
```

THE $M/M/c/c$ QUEUE is a special case of the $M/M/c/K$ queue: the number of servers is exactly equal to the number of jobs that are allowed to enter the system.[7] This system is often used to determine the number of beds needed by a hospital: the beds act as servers and the patients as jobs. Given the arrival rate of patients, and the average time they occupy a bed[8], the problem is to find the number of beds $c$ such that the loss probability $p(c)$ is less than some threshold, 1% say.

For hospitals it is reasonable to assume Poisson arrivals since patients arrive independently from a large population.[9] Also, there are typically many patients in the hospital, hence the recovery times are quite accurately described by an exponential distribution.

Take $\lambda(n) = \lambda \mathbb{1}_{n<c}$, and $\mu(n) = n\mu$. As for the $M/M/c$ queue, $\rho = \lambda/(c\mu)$. Then some algebra[10] gives

$$p(n) = \frac{1}{G}\frac{(c\rho)^n}{n!}, 0 \le n \le c, \qquad G = \sum_{n=0}^{c} \frac{(c\rho)^n}{n!},$$

$$\mathsf{E}[Q] = 0, \qquad \mathsf{E}[L_s] = \frac{\lambda}{\mu}\big(1 - p(c)\big).$$

The last two results are easy. As there are as many servers as jobs, jobs cannot be in queue. Next, observing that $\lambda(1-p(c))$ is the rate of accepted jobs[11], the load is $\lambda(1-p(c))/\mu$, and this in turn is equal to the average number of busy servers.

THE $M/M/\infty$ QUEUE is simple to analyze, as is has *ample* servers. Any job that arrives finds a free server, hence, its service can start right away. Therefore, $\lambda(n) = \lambda$ and $\mu(n) = n\mu$ for all $n \ge 0$. By taking the limit $c \to \infty$ in the expressions of the $M/M/c$ queue (or the $M/M/c/c$ queue) we get

$$p(n) = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \qquad \mathsf{E}[L] = \mathsf{E}[L_s] = \frac{\lambda}{\mu}.$$

We see that the number of busy servers in the $M/M/\infty$ queue is Poisson distributed with parameter $\lambda\,\mathsf{E}[S]$. We mention in passing—but do not prove it—that the same results also hold for the $M/G/\infty$ queue.

[7] *This queueing model is also known as the Erlang B model.*

[8] *i.e., the expected service time*

[9] *PASTA implies $\pi(c) = p(c)$, hence the loss fraction is indeed $p(c)$.*

[10] *[6.1.6]*

[11] *Since, by PASTA, a fraction $p(c)$ is lost.*

WITH A FINITE CALLING POPULATION the number of jobs is fixed, $N$ say. This model is useful to analyze repair systems and inventory systems of spare parts. To see this, consider a factory with $N$ machines and one mechanic.[12] A machine can be in one of two states: working or failed. When a machine breaks down, it moves to the repair department and waits until it is repaired. When $n$ machines are in repair, there are $N-n$ machines still working. Thus, if $\lambda$ is the rate at which a machine can fail, $\lambda(N-n)$ is the rate at which any of the working machines can fail. Since there is one mechanic: $\mu(n)=\mu$.

[12] *When there are also N servers available, we obtain the Ehrenfest model of diffusion, which is used to explain the second law of thermodynamics.*

With this model for $\lambda(n)$ and $\mu(n)$ the probabilities become [13]

$$p(n)=\frac{N!}{(N-n)!}\frac{\rho^n}{G}, \qquad G=\sum_{n=0}^{N}\rho^n\frac{N!}{(N-n)!}.$$

[13] *[6.1.8]*

As the expression for $G$ cannot be simplified, there is not much point trying to derive simple expressions for $\mathsf{E}[L]$.

BALKING CUSTOMERS LEAVE when they find the queue too long at the moment they arrive. A simple example model with customer balking is $\mu(n)=\mu$ and

$$\lambda(n)=\begin{cases}\lambda, & \text{if } n=0,\\ \lambda/2, & \text{if } n=1,\\ 0, & \text{if } n>1.\end{cases}$$

Balking is not necessarily the same as blocking. In the latter case, $\lambda(n)=\lambda\,\mathbb{1}_{n<K}$; in the former case, a fraction of the customers may already choose not the join the system at a lower level than $K$.

As the steady-state probabilities depend entirely on the form of $\lambda(n)$ and $\mu(n)$, we have to use the recursion (5.2.7) and the code above to compute $p(n)$. With this, the rate at which customers *enter* the system becomes $\lambda=\sum_{n=0}^{\infty}\lambda(n)p(n)$, and this is the arrival rate we need to use in Little's law.[14]

[14] *[6.2.4]*

**Ex 6.1.1.** Use moment-generating functions to derive $\mathsf{E}[L]$ and $\mathsf{V}[L]$ for the $M/M/1$ queue, and show that $\mathsf{P}\{L>n\}=\rho^{n+1}$.

**Ex 6.1.2.** Numerically check that in [4.1.5], $\beta=1-\mathsf{E}[L_s]\,\mu/\lambda$ for the $M/M/3/8$ queue with $\lambda=3$ and $\mu=2$.

**Ex 6.1.3.** Explain that for the $M/M/1$ queue $\mathsf{E}[Q]=\sum_{n=1}^{\infty}(n-1)\pi(n)$ and use this to find that $\mathsf{E}[Q]=\rho^2/(1-\rho)$.

**Ex 6.1.4.** Derive the expressions for the $M/M/c$ queue.

**Ex 6.1.5.** Check that the performance measures of the $M/M/c$ queue reduce to those of the $M/M/1$ queue if $c=1$.

**Ex 6.1.6.** Derive the expressions for the $M/M/c/c$ queue.

**Ex 6.1.7.** Derive the expressions for the $M/M/\infty$ queue.

**Ex 6.1.8.** Find the steady state probabilities for a single-server queue with a finite calling population with $N$ jobs.

**Ex 6.1.9.** Derive the steady state probabilities $p(n)$ for a queue with a finite calling population with $N$ jobs and $N$ servers. What happens if $N\to\infty$?

**Ex 6.1.10.** Show that as $K \to \infty$, the performance measures of the $M/M/1/K$ converge to those of the $M/M/1$ queue.

**Ex 6.1.11.** Derive the expression for $\mathsf{E}[L]$ in (6.1.2) by means of indicator variables.

**Ex 6.1.12.** Derive $\mathsf{E}[L]$ and $\mathsf{E}\left[L^2\right]$ by differentiating the LHS and RHS of $\sum_{n=0}^{\infty} \rho^n = (1-\rho)^{-1}$.

## 6.2 APPLICATIONS OF LEVEL-CROSSING, LITTLE'S LAW AND PASTA

In this section we apply the tools developed above to a number of interesting queueing systems. Specifically, we discuss in detail how to plan the number of cashiers at a supermarket such that the average queue length remains small. The exercises consider a number of other cases.

LET US DISCUSS the example of cashier planning at a supermarket. We keep the models simple, but we include all necessary steps to solve the a realistic planning problem.

We start with specifying the *arrival process*. It is reasonable to model it as a Poisson process, as there are many potential customers, each choosing with a small probability to go to the supermarket at a certain moment in time. Thus, we only have to characterize the arrival rate. Estimating this for a supermarket is easy because the cash registers track all customers payments. This provides us with the departure time of each customer, hence we can use this to estimate the average number of arrivals per hour.

It is common to use a *demand profile* which shows the average number of customers arriving per hour. Then we model the arrivals as a Poisson process with an arrival rate that is constant during a certain hour as specified by the demand profile.

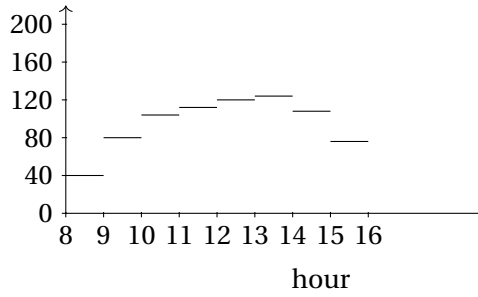



Figure 6.2.1: A demand profile.

It is also easy to find the *service distribution* from the cash registers. The first item scanned after a payment determines the start of a service, and the payment closes the service. For ease, we model the service time distribution as exponential with mean 1.5 minutes.

For the *service objective* we prefer to keep the average number of people in the system such that $\mathsf{E}[L] \leq 5$. (Other KPIs are also possible to use.)

Let us model the situation as an $M/M/1$ queue with a fast server, so that the objective becomes easy to compute. With this we can find a formula to convert the demand profile into a *load profile* to specify the minimal number of servers per hour needed to meet the service objective.

Since $\mathsf{E}[L] = \rho/(1-\rho)$, and $\rho = \lambda \mathsf{E}[S]/c$, solving for $c$ in the inequality $\mathsf{E}[L] \leq 5$ gives

$$c \geq \frac{6}{5} \lambda \mathsf{E}[S] \approx 0.03\lambda,$$

where, with the above estimate, $\mathsf{E}[S] = 1.5/60$ hour. It's easy to apply this formula. For instance, we see in the demand profile Fig. 6.2.1 that $\lambda = 120$ customers per hour between 12 and 13, hence $c \approx 3.8 = 4$.

THE LAST STEP in the planning process is to *cover the load profile with service shifts*. This is typically not easy since shifts have to satisfy all kinds of restrictions with respect to breaks and durations. Moreover, shifts can

also have different costs: evening shifts are typically more expensive per hour.

The usual way to solve such covering problems is by means of an integer problem. For instance, suppose only 4 *shift plans* are available as shown at the right.[0]

For the optimization, let $x_i$ be the number of shifts of type $i$ and $c_i$ the cost of this type. Then the problem is to solve $\min \sum_i c_i x_i$, such that for all hours $t$ the shifts cover the load, i.e., $\sum_i x_i \mathbb{1}_{t \in s_i} \geq 0.03\lambda_t$. (We write $t \in s_i$ if hour $t$ is covered by shift type $i$.)

*0*

1. $++-++$
2. $+++-+$
3. $++-+++$
4. $+++-++$,

*Shift plans. A + indicates a working hour and − a break of an hour.*

FINALLY, WE NEED to address the quality of our approximation: to simplify we used the $M/M/1$ queue with a fast server, while we know that in a supermarket there are often multiple parallel cashiers. Specifically, we are interested in the ratio $\mathsf{E}[L(M/M/c]$ and $\mathsf{E}[L(M/M/1)]$ where in the $M/M/1$ queue the server works at rate $c$.

To understand this better, we consider a numerical example. Suppose that we have an $M/M/3$ queue with $\lambda = 5$ per day and $\mu = 2$ per day per server, and we compare $\mathsf{E}[L]$ of this queue to that of an $M/M/1$ queue with the same arrival rate but with a service rate of $\mu = 3 \cdot 2 = 6$.

Fig. 6.2.2 shows the ratio of $\mathsf{E}[L]$ for both queues as a function of $\rho$. Clearly, when $\rho$ is small, this ratio is about 3. This is reasonable, because the service time in the fast $M/M/1$ is 3 times as small as the service time in the $M/M/3$ queue. Thus, when $\rho$ is small, the time in queue is small, hence $\mathsf{E}[J] \approx \mathsf{E}[S]$. However, when $\rho$ is large, $\mathsf{E}[J] \gg \mathsf{E}[S]$, so that in this case, the ratio between the queue lengths becomes approximately 1.

In our supermarket model, $\rho = 5/6$, hence the approximation is not too bad.

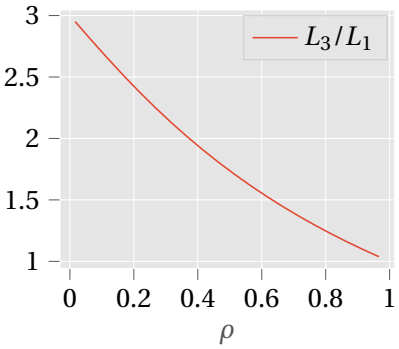



Figure 6.2.2: The ratio of $\mathsf{E}[L]$ for the $M/M/3$ and $M/M/1$ queue.

THE EXERCISES REQUIRE quite some modeling skills. Hence, they may be quite hard even thought the formulas are simple.

**Ex 6.2.1** (Hall 5.6)**.** An $M/M/1$ queue has been found to have an average waiting time in queue of $\mathsf{E}[W] = 1$ minute. The arrival rate is known to be 5 customers per minute. What are the service rate and utilization? Calculate $\mathsf{E}[Q]$, $\mathsf{E}[L]$ and $\mathsf{E}[W]$. Finally, the queue operator would like to provide chairs for waiting customers. He would like to have a sufficient number so that all (waiting and in service) customers can sit down at least 90 percent of the time. How many chairs should he provide?

**Ex 6.2.2** (Hall 5.3)**.** After observing a queue with two servers for several days, the following steady-state probabilities have been determined: $p(0) = 0.4$, $p(1) = 0.3$, $p(2) = 0.2$, $p(3) = 0.05$ and $p(4) = 0.05$. The arrival rate is 10 customers per hour. Determine $\mathsf{E}[L]$, $\mathsf{E}[Q]$, $\mathsf{E}[J]$, $\mathsf{E}[W]$, $\mathsf{V}[L]$ and $\mathsf{V}[Q]$. Finally, compute the service time and the utilization.

**Ex 6.2.3.** (Hall 5.14) An airline phone reservation line has one server and a buffer for two customers. The arrival rate is 6 customers per hour, and a service rate of just 5 customers per hour. Arrivals are Poisson and service times are exponential. Estimate $\mathsf{E}[Q]$ and the average number of customers served per hour. Then, estimate $\mathsf{E}[Q]$ for a buffer of size 5. What is the impact of the increased buffer size on the number of customers served per hour?

**Ex 6.2.4** (Hall 5.8)**.** The queueing system at a fast-food stand behaves in a peculiar fashion. When there is no one in the queue, people are reluctant to use the stand, fearing that the food is unsavory. People are also reluctant to use the stand when the queue is long. This yields the following arrival rates (in numbers per hour): $\lambda(0) = 10$, $\lambda(1) = 15$, $\lambda(2) = 15$, $\lambda(3) = 10$, $\lambda(4) = 5$, $\lambda(n) = 0, n \geq 5$. The stand has two servers, each of which can operate at 5 per hour. Service times are exponential, and the arrival process is Poisson. Calculate the steady state probabilities. Next, what is the average arrival rate? Finally, determine $\mathsf{E}[L]$, $\mathsf{E}[Q]$, $\mathsf{E}[J]$ and $\mathsf{E}[J]$.

**Ex 6.2.5** (Hall 5.10)**.** A repair/maintenance facility would like to determine how many employees should be working in its tool crib. The customers are actually maintenance workers at the facility, and are compensated at the same rate as the tool crib employees. The service time is exponential, with mean 4 minutes, and customers arrive by a Poisson process with rate 28 per hour. What is $\mathsf{E}[W]$ for $c = 1, 2, 3$, or 4 servers? How many employees should work in the tool crib?

*Recall, in queueing systems always somebody has to wait, either a customer in queue or a server being idle. If it is very expensive to let customers wait, the number of servers must be high, whereas if servers are relatively expensive, customers have to do the waiting.*

**Ex 6.2.6** (Hall 5.22)**.** At a large hotel, taxi cabs arrive at a rate of 15 per hour, and parties of riders arrive at the rate of 12 per hour. Whenever taxicabs are waiting, riders are served immediately upon arrival. Whenever riders are waiting, taxicabs are loaded immediately upon arrival. A maximum of three cabs can wait at a time (other cabs must go elsewhere). First find an appropriate way to model this queueing system as an $M/M/1$ queue. Then calculate the expected number of cabs waiting and the expected number of parties waiting, and the average waiting times of cabs and parties. What would be the impact of allowing four cabs to wait at a time? Assume that all members of a party of riders can be served by a single cab, that is, the parties do not exceed the capacity of a cab and all members of a party have the same destination.

*This is a bit hard, but important problem; it is the same as an inventory system with backlogging in which taxis act as products on hand and parties as customers.*

**Ex 6.2.7.** Suppose cabs are not allowed to wait. What is the expected waiting time for a party of riders?

*Continuation of [6.2.6]*

**Ex 6.2.8.** Suppose cabs can contain at most 4 riders, and the size of a party (i.e., a batch) has distribution $B_k$ with $\mathsf{P}\{B_k = i\} = 1/7$ for $i = 1, \ldots, 7$. Parties of riders have the same destination, so riders of different parties cannot be served by one taxi. Provide a set of recursions to simulate this system.

*Continuation of [6.2.6]. This one is quite hard.*

## 6.3 $M^X/M/1$ QUEUE: EXPECTED WAITING TIME

Sometimes jobs arrive in batches, rather than as single units. For instance, when a car or a bus arrives at a fast-food restaurant, a batch consists of the number of people in the vehicle. When the batches arrive as a Poisson process and the individual items within a batch have exponential service times we use the shorthand $M^X/M/1$ for this queueing model. Here we derive expressions for the load, the expected waiting time and number of jobs in the system.

ASSUME THAT JOBS arrive as a Poisson process with rate $\lambda$ and each *job* contains multiple *items*. Denote by $B_k$ the batch size of the $k$th job. We

assume that $\{B_k\}$ is a sequence of iid discrete rvs that distributed as the common rv $B$. The service times of the items in the batch are also iid with common rv $S$. The utilization is therefore $\rho = \lambda \mathsf{E}[B]\,\mathsf{E}[S]$; As always we require that $\rho < 1$.

SUPPOSE THAT AN arriving job joins the end of the queue (if present), and once the queue in front of it has been cleared, it moves in its entirety to the server.[0] Thus, all items in one batch spend the same time in queue. Once the batch moves to the server, the server processes the items one after another until the batch is finished. Write $Q^B$ for the number of batches in queue and $L_s^B$ for the number of items (if any) at the server.

[0] *This is the same batching model as in Section 4.2.*

Observe that the average time $\mathsf{E}\left[W^B\right]$ a batch spends in queue is[1]

$$\mathsf{E}\left[W^B\right] = \mathsf{E}\left[L_s^B\right]\mathsf{E}[S] + \mathsf{E}\left[Q^B\right]\mathsf{E}[B]\,\mathsf{E}[S]\,. \tag{6.3.1}$$

[1] *First the items at the server have to be served, then the ones in queue.*

But since $\mathsf{E}[L] = \mathsf{E}\left[L_s^B\right] + \mathsf{E}\left[Q^B\right]\mathsf{E}[B]$,

$$\mathsf{E}\left[W^B\right] = \mathsf{E}[L]\,\mathsf{E}[S] = \mathsf{E}[W]\,,$$

where $\mathsf{E}[W]$ is the average time an *item* spends in queue. Hence, the average time a job spends in queue is the same as the average time an item spends in queue.

With Little's law, $\mathsf{E}\left[Q^B\right] = \lambda\,\mathsf{E}\left[W^B\right]$. Substituting this in (6.3.1), we obtain

$$\mathsf{E}[W] = \mathsf{E}\left[W^B\right] = \frac{\mathsf{E}\left[L_s^B\right]}{1-\rho}\mathsf{E}[S]\,. \tag{6.3.2}$$

It remains to find an expression for $\mathsf{E}\left[L_s^B\right]$.

FOR THIS WE can use the renewal reward theorem. With (5.4.1) as inspiration, define $Y(t) := \int_0^t L_s^B(s)\,\mathrm{d}s$, so that $Y(t)/t$ is the time-average number of items at the *server*. By taking $D_k$ as the departure time of the $k$th batch, let $X_k = Y(D_k) - Y(D_{k-1})$. A bit of work[2] shows then that

[2] *[6.3.4]*

$$X = \frac{\mathsf{E}\left[B^2\right] + \mathsf{E}[B]}{2}\mathsf{E}[S]\,. \tag{6.3.3}$$

Since $Y = \lim_{t\to\infty} Y(t)/t = \mathsf{E}\left[L_s^B\right]$, and $Y = \lambda X$ (where we use that $\lambda = \delta$), it follows that[3]

[3] *[6.3.5], [6.3.6]*

$$\mathsf{E}\left[L_s^B\right] = \lambda\frac{\mathsf{E}\left[B^2\right]}{2}\mathsf{E}[S] + \frac{\rho}{2} \tag{6.3.4}$$

$$= \frac{1+C_s^2}{2}\rho\,\mathsf{E}[B] + \frac{\rho}{2}, \tag{6.3.5}$$

where $C_s^2 = \mathsf{V}[B]/(\mathsf{E}[B])^2$ is the SCV of the batch size distribution. Finally, substituting this into (6.3.2) gives

$$\mathsf{E}[W] = \frac{1+C_s^2}{2}\frac{\rho}{1-\rho}\mathsf{E}[B]\,\mathsf{E}[S] + \frac{1}{2}\frac{\rho}{1-\rho}\mathsf{E}[S]\,. \tag{6.3.6}$$

RATHER THAN USING the time an entire batch spends at the server, as in the above use of the renewal-reward equation, we can also concentrate on the time spent by single items at the server. This will provide us with a

simple expression for $\mathsf{P}\{L_s = i\}$, which in turn leads to another derivation of (6.3.4).

If $L_s(t)$ is the number of items (of the batch in service) at the server at time $t$, then $Y_i(t) = \int_0^t \mathbb{1}_{L_s(s)=i}\,\mathrm{d}s$ is the total time there are $i$ items at the server. It follows[4] that $\int_{D_{k-1}}^{D_k} \mathbb{1}_{L_s(s)=i}\,\mathrm{d}s = \mathbb{1}_{B_k \geq i} S_{k,i}$, where $S_{k,i}$ is the service time of the $i$th item of this batch.

[4] *Fig. 6.3.1*

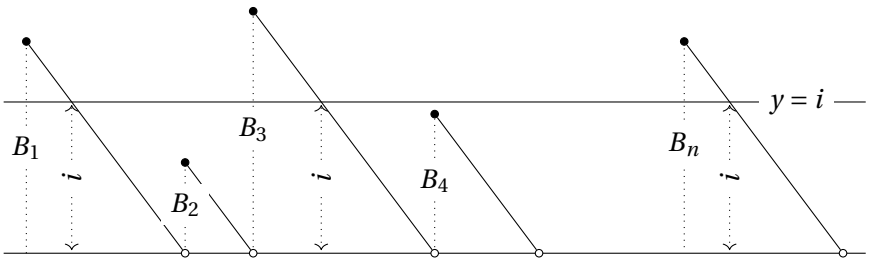



Figure 6.3.1: A batch crosses the line $y = i$ iff it contains at least $i$ items. Thus, during the service of a batch with $i$ or more items, there is precisely one service of an $i$-th item.

By sampling $Y(\cdot)$ at departure times $\{D_k\}$, we see that $X_k = Y_i(D_k) - Y_i(D_{k-1}) = S_{k,i}\,\mathbb{1}_{B_k \geq i}$. Since the $\{S_{k,i}\}$ are iid with $\mathsf{E}\left[S_{k,i}\right] = \mathsf{E}[S]$, and the $\{B_k\}$ are iid, we obtain

$$X = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_{k,i}\,\mathbb{1}_{B_k \geq i} = \mathsf{E}[S\,\mathbb{1}_{B \geq i}].$$

But $B$ and $S$ are independent by assumption, hence $X = \mathsf{E}[S]\,\mathsf{E}[\mathbb{1}_{B\geq i}] = \mathsf{E}[S]\,G(i-1)$, since $G$ is the survivor function of $B$. As for $Y$, by construction, $Y_i(t)/t \to \mathsf{P}\{L_s = i\}$. Using the renewal reward theorem and rate stability ($\delta = \lambda$) we conclude that

$$\mathsf{P}\{L_s = i\} = \lambda\,\mathsf{E}[S]\,G(i-1) = \rho\frac{G(i-1)}{\mathsf{E}[B]}$$
$$\mathsf{E}[L_s] = \sum_{i=0}^{\infty} i\,\mathsf{P}\{L_s = i\} = \frac{\rho}{\mathsf{E}[B]} \sum_{i=1}^{\infty} i\,G(i-1).$$

Simplifying[5] the summation yields again (6.3.6).

[5] *[6.3.7]*

**Ex 6.3.1.** A common operational problem is a machine that receives batches of various sizes. Management likes to know how a reduction of the variability of the batch sizes would affect the average queueing time. Suppose, for the sake of an example, that the batch size $\mathsf{P}\{B = 1\} = \mathsf{P}\{B = 2\} = \mathsf{P}\{B = 3\} = 1/3$. Batches arrive at rate $\lambda = 1/h$. The average processing time for an item is 25 minutes. Compute by how much $\mathsf{E}[L]$ would decrease if $B \equiv 2$.

*Of course, it is up to management to decide whether such reductions outweigh any efforts to reduce the variation in batch sizes.*

**Ex 6.3.2.** Show that $\mathsf{E}\left[W(M^X/M/1)\right] \geq \mathsf{E}[W(M/M/1)]$ even when the loads are the same. What do you conclude?

*Relate this to Sakasegawa's formula.*

**Ex 6.3.3.** Compute $\mathsf{E}[L]$ when $B$ is geometrically distributed with $\mathsf{E}[B] = 1/p$. Check that if $\mathsf{E}[B] = 1$ the $M/M/1$ queue results.

**Ex 6.3.4.** Show (6.3.3).

*This exercise uses similar tools as in Section 4.4.*

**Ex 6.3.5.** Complete the argumentation that leads to (6.3.4).

**Ex 6.3.6.** Derive (6.3.5).

**Ex 6.3.7.** Show that $\sum_{i=1}^{\infty} iG(i-1) = (\mathsf{E}\left[B^2\right] + \mathsf{E}[B])/2$.

## 6.4 $M/G/1$ QUEUE: EXPECTED WAITING TIME

When the service times are not really well approximated by the exponential distribution, the $M/G/1$ queue may be a more suitable model than the $M/M/1$ queue. In this section we derive with sample path arguments the fundamentally important Pollaczek-Khinchine (PK) formula for the average waiting time in an $M/G/1$ queue. At the end of the section we relate Sakasegawa's formula to the PK formula.

WE START WITH the simple observation that before an arriving job gets access to the server, it first has to wait until the job in service (if any) completes and then it has wait for the queue to be cleared. From PASTA it then follows that $\mathsf{E}[W] = \mathsf{E}[S_r] + \mathsf{E}[Q]\,\mathsf{E}[S]$, where $\mathsf{E}[S_r]$ is the (time-)average remaining service time of the job in service. With Little's law $\mathsf{E}[Q] = \lambda\,\mathsf{E}[W]$ and writing $\rho = \lambda\,\mathsf{E}[S]$, we find that

$$\mathsf{E}[W] = \frac{\mathsf{E}[S_r]}{1-\rho}.$$

TO MAKE FURTHER progress we need to find an expression for $\mathsf{E}[S_r]$ for generally distributed service times.[0] For this, we can use the renewal reward theorem, just as in Section 6.3.

Consider the $k$th job of some sample path of the $M/G/1$ queueing process. Let the job's service start[1] at time $\tilde{A}_k$, so that it departs at time $D_k = \tilde{A}_k + S_k$, see Fig. 6.4.1. At time $s$, the remaining service time of job $k$ is $(D_k - s)\,\mathbb{1}_{\tilde{A}_k \le s < D_k}$. By adding up all these times, we find[2] that

$$Y(t) = \int_0^t (D_{D(s)+1} - s)\,\mathbb{1}_{L(s)>0}\,\mathrm{d}s$$

is the total remaining service time as seen by the server up to time $t$. Hence, as $t \to \infty$, $Y(t)/t$ converges to the (time-average) remaining service time $\mathsf{E}[S_r]$.

We also see in Fig. 6.4.1 that $X_k = Y(D_k) - Y(D_{k-1})$ is the area under the triangle. By choosing $T_k = D_k$ as the epochs to inspect $Y(\cdot)$, $X = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} S_k^2/2 = \mathsf{E}\left[S^2\right]/2$. By the renewal-reward theorem, $Y = \delta X$. But $\delta = \lambda$,[3] and therefore

$$\mathsf{E}[S_r] = \frac{\lambda}{2}\mathsf{E}\left[S^2\right].$$

Replacing the above expression in the above expression for $\mathsf{E}[W]$, we obtain the *Pollaczek-Khinchine formula*

$$\mathsf{E}[W] = \frac{1}{2}\frac{\lambda\,\mathsf{E}\left[S^2\right]}{1-\rho} = \frac{1+C_s^2}{2}\frac{\rho}{1-\rho}\mathsf{E}[S], \tag{6.4.1}$$

where the second equation follows after some rewriting.[4]

[0] *For the $M/M/1$ queue, $\mathsf{E}[S_r] = \mathsf{E}[S]$, but not for the $M/G/1$ queue.*

[1] *Usually this is later than the arrival time $A_k$.*

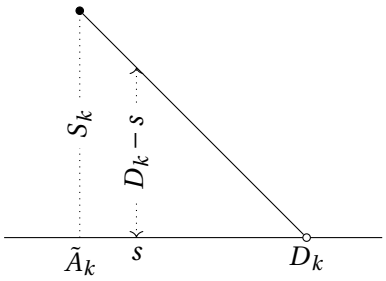



Figure 6.4.1: Remaining service time.

[2] *[6.4.6]*

[3] *by rate-stability*

[4] *[6.4.7]*

THE PK FORMULA is an *exact* result for $\mathsf{E}[W]$, and it reduces to $\mathsf{E}[W]$ for the $M/M/1$ queue by taking[5] $C_s^2 = 1$ in the factor $(1 + C_s^2)/2$. In a similar vein, Sakasegawa's formula for the $G/G/c$ queue reduces to the PK formula for $\mathsf{E}[W]$ for the $M/G/1$ queue when the number of servers $c = 1$ and setting[6] $C_a^2 = 1$ in the factor $(C_a^2 + C_s^2)/2$. Again, recall that Sakasegawa's formula is an approximation, while the PK formula is exact.

[5] *Recall, $C_s^2 = 1$ for the $M/M/1$ queue.*

[6] *Recall, $C_a^2 = 1$ for the $M/G/1$ queue.*

**Ex 6.4.1.** Consider a workstation with just one machine. We model the job arrival process as a Poisson process with rate $\lambda = 3$ per day. The average service time $\mathsf{E}[S] = 2$ hours, $C_s^2 = 1/2$, and the shop is open for 8 hours per day. Show that $\mathsf{E}[W] = 4.5h$. What would you propose to reduce $\mathsf{E}[W]$ to 2h?

**Ex 6.4.2.** Compare $\mathsf{E}[W(M/D/1)]$ to $\mathsf{E}[W(M/M/1)]$.

**Ex 6.4.3.** Compute $\mathsf{E}[J]$ for the $M/G/1$ queue with $S \sim U[0, \alpha]$.

**Ex 6.4.4.** Find an expression in terms of $\lambda$ and $\mathsf{E}[S]$ for the acceptance and loss probabilities of the $M/G/1/1$ queue. Why is this expression not valid for the $G/G/1/1$ queue.

**Ex 6.4.5.** Show that for the $M/G/1$ queue, the expected idle time $\mathsf{E}[I] = 1/\lambda$ and the expected busy time $\mathsf{E}[U] = \mathsf{E}[S]/(1-\rho)$.

**Ex 6.4.6.** Explain the expression for $Y(t)$.

**Ex 6.4.7.** Complete the algebra in (6.4.1).

**Ex 6.4.8.** Show for the $M/G/1$ that $\mathsf{E}[S_r] = \rho\,\mathsf{E}[S_r \mid S_r > 0]$.

**Ex 6.4.9.** It is an easy mistake to think that $\mathsf{E}[S_r] = \mathsf{E}[S]$ when service times are exponential. Why is this wrong?

**Ex 6.4.10.** Show for the $M/G/1$ that with probability $\rho$ a job leaves behind a busy server.

**Ex 6.4.11.** Here is a third way to derive the PK formula, inspired by [6.5.2]; this is a bit more technical than the regular exercises, hence you may skip it. Suppose that the waiting time $W$ has a the density $f$, that is, the cdf $F(x) = \mathsf{P}\{W \le x\}$ has a derivative $f(x)$ for all $x > 0$. ($F$ is not differentiable at 0 because $\mathsf{P}\{W = 0\} > 0$, while $\mathsf{P}\{W \le x\} = 0$ for $x < 0$.) Consider some level $x > 0$. Explain that the density $f(x)$ must satisfy the (integral) equation $\lambda \int_0^x f(y)G(x-y)\,\mathrm{d}y = f(x)$. Use this to derive the PK formula. Why is this not a formal proof?

## 6.5 $M^X/M/1$ QUEUE LENGTH DISTRIBUTION

In Sections 6.3 and 6.4 we established the Pollaczek-Khinchine formula for the waiting times of the $M^X/M/1$ queue and the $M/G/1$ queue, respectively. To compute more difficult performance measures such as the loss probability $\mathsf{P}\{L > n\}$, we need expressions for the stationary distribution $\pi(n)$. Here we present a numerical, recursive, scheme to compute these probabilities.

WE CAN AGAIN use level-crossing arguments to find $\{\pi(n)\}$. However, the reasoning that led to the level-crossing equation (5.2.5) needs to be generalized because, at the arrival of a batch containing many items, multiple levels will be up-crossed, see Fig. 6.5.1.

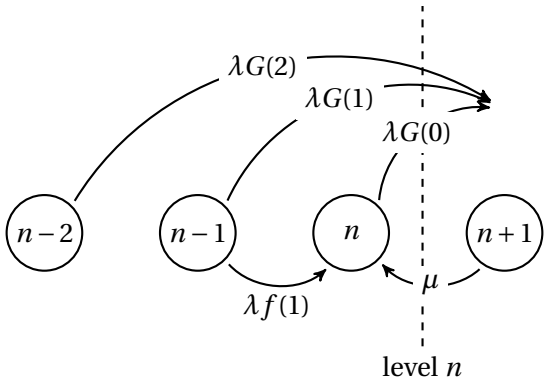



Figure 6.5.1: When $L(t) = n$, the arrival of any batch crosses level $n$ from below, but when $L(t) = n-1$, only a batch larger than 1 ensures a crossing of level $n$, and so on.

LET US FIRST guess what could be the appropriate generalization of $\lambda\pi(n) = \mu\pi(n+1)$.[0] The down-crossings are easy: since items are served one-by-one, we still must have $\mu\pi(n+1)$. For the upcrossings, let us focus for the moment on the arrivals that see $m$, with $m \le n$, in the system. The arrival stream of such customers is thinned with probability $\pi(m)$[1]. Thus, the arrival rate of jobs that see $m$ upon arrival is $\lambda\pi(m)$. By PASTA the arrival process $\{N(u), u \ge t\}$ is independent of $\{L(s), s < t\}$, hence this thinned process is also Poisson.

When an arriving job sees $m$ in the system, and level $n$ has to be up-crossed, the batch size of this job must contain more than $n-m$ items. The rate at which such jobs arrive must be $G(n-m)\pi(m)\lambda$ as $G(n-m)$ is the probability that the job size is larger than $n-m$. Noting that the batch sizes are independent of the arrival times, we obtain yet again a Poisson process, but with rate $\lambda\pi(m)G(n-m)$.

Finally, to compute the up-crossing rate of level $n$, we merge[2] all these thinned Poisson processes to form a Poisson process with rate $\lambda\sum_{m=0}^{n}\pi(m)G(n-m)$.

By level-crossing, the up-crossing and down-crossing rates must match, and therefore,

$$\lambda \sum_{m=0}^{n} \pi(m)G(n-m) = \mu\pi(n+1). \tag{6.5.1}$$

It is easy to see that (6.5.1) holds for the $M/M/1$ queue.[3] Moreover, we can use it to derive $\mathsf{E}[L]$.[4]

WE FORMALIZE the above reasoning as follows. Define

$$A(m,n,t) := \sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=m}\,\mathbb{1}_{B_k>n-m}$$

[0] *i.e., the level-crossing equations of the $M/M/1$ queue.*

[1] *Compare [2.2.14] and [2.2.17]*

[2] *Compare [2.2.12] and [2.2.18].*

[3] *[6.5.1]*

[4] *[6.5.2]*

as the number of jobs up to time $t$ that see $m$ in the system upon arrival and have batch size larger than $n-m$. From Fig. 6.5.1 we see that $A(n,m,t)$ counts the number of up-crossings of level $n$ up to time $t$.

As earlier, we are interested in the limit $A(n,m,t)/t$ as $t \to \infty$. If we follow the reasoning of Section 5.3, we obtain by multiplying and dividing,

$$\frac{A(m,n,t)}{t} = \frac{A(t)}{t}\frac{A(m,t)}{A(t)}\frac{A(m,n,t)}{A(m,t)}. \tag{6.5.2}$$

We already know that $A(t)/t \to \lambda$ and $A(m,t)/A(t) \to \pi(m)$; so, let us provide an interpretation for the third term.

If we fill in the definitions, we get

$$\frac{A(m,n,t)}{A(m,t)} = \frac{\sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=m, B_k>n-m}}{\sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=m}}.$$

In words, this is the fraction of those jobs that see $m$ in the system upon arrival and have size $> n-m$. But recall that we assumed that $\{B_k\}$ are independent of what jobs see upon arrival. Thus, we can just as well count all jobs of size $> n-m$ along a sample path instead of those that see $m$ upon arrival; in the limit the statistics must be the same. Therefore

$$\lim_{t\to\infty} \frac{\sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=m, B_k>n-m}}{\sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=m}} = \lim_{t\to\infty} \frac{\sum_{k=1}^{A(t)} \mathbb{1}_{B_k>n-m}}{A(t)} = G(n-m).$$

By level-crossing, the number of up- and down-crossing must satisfy

$$\left| \sum_{m=0}^{n} A(m,n,t) - D(n,t) \right| \le 1.$$

Then, by combining the above, dividing by $t$, and taking the limit $t \to \infty$, we obtain (6.5.2).

IT IS LEFT to find the normalization constant. As (6.5.1) does not lead to a closed form expression for $\pi(n)$, such as (6.1.1), we have to specify a criterion to stop this iterative procedure. We discuss three different strategies to cope with this problem.

For actual computations we suppose henceforth that the batch size $B$ is finite and bounded by some $N$. Then, as $G(n) = 0$ for $n \ge N$, we can simplify (6.5.1) to

$$\mu\pi(n+1) = \lambda \sum_{m=0}^{\min\{n,N-1\}} \pi(n-m)G(m).$$

THE SIMPLEST STRATEGY is to 'ignore the problem'. Just set $\pi(0) = 1$ to start the iteration, then compute the first $M$, 10 say, terms, and take $\sum_{i=0}^{M}\pi(i)$ as the normalization constant. With this, $\mathsf{E}[L] \approx \sum_{n=0}^{M} n\pi(n)$.

Python Code

```
import numpy as np


def compute_pi(f, M):
    pi = np.ones(M + 1)
```

```
6       F = np.cumsum(f)
7       G = np.ones_like(F) - F
8       N = len(G)
9
10      for n in range(M):
11          pi[n + 1] = sum(pi[n - m] * G[m] for m in range(min(n + 1, N)))
12          pi[n + 1] *= labda / mu
13      return pi / pi.sum()
14
15  M = 10
16  labda, mu = 1, 3
17  f = np.array([0, 1, 1, 1])
18  f = f / f.sum()
19
20  pi = compute_pi(f, M)
21  EL = sum(n * pi[n] for n in range(len(pi)))
```

When running this code, we get $\mathsf{E}[L] = 2.501$.

But is $M = 10$ large enough for this estimate to be approximately correct? Using (6.3.6) to compute exact value, we obtain $\mathsf{E}[L] = 3.333$. Clearly, simply stopping at some arbitrary value is not a viable procedure.

A better criterion is perhaps to stop iterating at some $M$ when $\pi(M) \ll 1$.[5]

[5] *In line 6 we choose to implement* p *as a dictionary, because we do not yet know how many terms we need before satisfying the stopping criterion.*

Python Code

```
1   def compute_pi_2(f, eps):
2       F = np.cumsum(f)
3       G = np.ones_like(F) - F
4       N = len(G)
5
6       pi, n = {}, 0
7       pi[n] = 1
8
9       while pi[n] > eps:
10          pi[n + 1] = sum(pi[n - m] * G[m] for m in range(min(n + 1, N)))
11          pi[n + 1] *= labda / mu
12          n += 1
13
14      norm = sum(pi[n] for n in pi.keys())
15      return {n: pi[n] / norm for n in pi.keys()}
16
17
18  pi = compute_pi_2(f, eps=0.001)
19  EL = sum(n * pi[n] for n in pi.keys())
```

Now we find $\mathsf{E}[L] = 3.304$. This is indeed better, but $\epsilon = 0.001$ is not yet small enough: the second digit is wrong.[6]

[6] *Observe how important it is to have exact results such as* (6.3.6) *to check the results of numerical algorithms.*

THE SECOND METHOD is exact, but more involved: we block demand above some level $M$. There are three common policies to decide which items in a batch to accept.

1. Complete acceptance: accept all batches that arrive when the system contains $K$ or fewer items, and reject the entire batch otherwise.[7]

[7] *[6.5.4]*

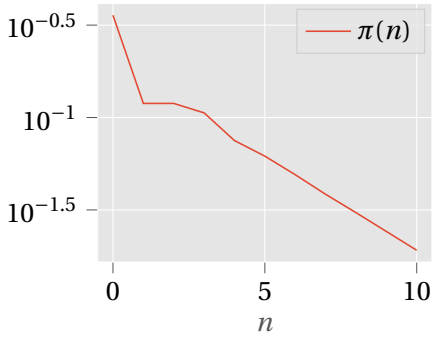
2. Partial acceptance: accept whatever fits of a batch, and reject the rest.[8]

3. Complete rejection: if a batch does not fit entirely into the system, it will be rejected completely.[9]

[8] [6.5.5]

[9] [6.5.6]

Once we have recursions similar to (6.5.1) we can again obtain $\{\pi(n)\}$. Since the recursions stop after level $M$, there is a finite number of recusions, hence we can solve them numerically without much problems.




THE THIRD METHOD relies on the fact that the numbers $\pi(n)$ decrease geometrically fast for $n$ sufficiently large. Let us check this for the numerical example we studied above. In the figure at the right we plot $\log(\pi(n))$ as a function of $n$. Clearly, $\pi(n), n \geq 5$ seems to decrease as a straight line.

To provide some intuition for this fact, observe that for $N < n \leq 2N-1$,

$$\pi(n+1) = \frac{\lambda}{\mu} \sum_{m=0}^{N-1} \pi(n-m)G(m) < \frac{\lambda}{\mu} \sum_{m=0}^{N-1} G(m) = \rho < 1.$$

as $\pi(n) < 1$ for $n \leq N$. But then, for $2N < n \leq 3N-1$,

$$\pi(n+1) = \frac{\lambda}{\mu} \sum_{m=0}^{N-1} \pi(n-m)G(m) < \frac{\lambda}{\mu} \sum_{m=0}^{N-1} G(m)\rho = \rho^2,$$

and so on. However, this is a crude upper bound for the decay rate, because $\pi(n) < 1$ is not very sharp.

We can do better by substituting the form $\pi(n) = C\alpha^n$ into (6.5.1) for $k > N$. Then $\alpha$ should be a root of the equation $\lambda \sum_{m=0}^{N-1} \alpha^m G(N-1-m) = \mu\alpha^N$. Now this is a polynomial (in $\alpha$) of degree $N$, which has $N$ roots in general. We leave it to the interested reader to study the theoretical ramifications of this method, for instance, why should $\alpha$ be the largest root?

**Ex 6.5.1.** Show that (6.5.1) reduces to $\lambda\pi(n) = \mu\pi(n+1)$ for the $M/M/1$ case.

**Ex 6.5.2.** Use (6.5.1) in $\mathsf{E}[L] = \sum_{n=0}^{\infty} n\pi(n)$ to derive (6.3.6).

*This is an important check both on (6.3.6) and (6.5.1).*

**Ex 6.5.3.** Why is $\pi(n) = \delta(n)$ not true for the $M^X/M/1$ queue?

**Ex 6.5.4.** Derive a set of recursions for the $M^X/M/1/K$ queue with complete acceptance.

**Ex 6.5.5.** Derive a set of recursions analogous to (6.5.1) to compute $\pi(n)$ for the $M^X/M/1/K$ queue with partial acceptance.

**Ex 6.5.6.** Derive a set of recursions for the $M^X/M/1/K$ queue with complete rejection.

## 6.6 $M/G/1$ QUEUE LENGTH DISTRIBUTION

In Section 6.5 we used level-crossing arguments to find a recursive for the stationary distribution $\pi(n)$ of the $M^X/M/1$ queue. Here we find a similar recursion to compute $\pi(n)$ of $M/G/1$ queue. However, we cannot simply copy the ideas of Section 6.5 to the present situation, because in the $M^X/M/1$ queue the service times of the items are exponential, hence memoryless, while in the $M/G/1$ this is not the case.

IF WE WANT to characterize the state at all moments in time, we need to keep track of the number of jobs in the system *and* the remaining service time of the job in service (if any), because service times are not memoryless. But, by sampling at job departure times $\{D_k\}$,[0] we can restrict the state description to just the number in the system. We say that the system is in state $n$ at time $D_k$ when $L(D_k) = n$.[1]

Let $Y_k$ denote the number of arrivals during the service time of job $k$. Note that, because the service times are iid, $\{Y_k\}$ is a sequence of iid random variables. Let $Y$ be the common random variable with (pmf) $f(j) = \mathsf{P}\{Y = j\}$ and $G$ as survivor function.

[0] *So that the remaining service time is guaranteed to be* 0

[1] *Not* $L(D_k-)$.

LET US CONCENTRATE on the down-crossing rate of level $n$.[2] Suppose we start the service of job $k$ when the system is in state $n+1$.[3] When $Y_k = 0$, the system contains one job less after the departure of job $k$,[4] that is, $L(D_k) = n$. However, if $Y_k \geq 1$, $L(D_k) \geq n+1$. Consequently, a down-crossing of level $n$ can only occur at time $D_k$ when $L(D_{k-1} = n+1)$ *and* $Y_k = 0$. It follows that the number of down-crossings up to time $t$ is

[2] *Recall that level n lies between states n and* $n+1$.

[3] *Thus,* $L(D_{k-1}) = n+1$.

[4] *Namely, job k left.*

$$D(n+1,0,t) = \sum_{k=1}^{D(t)} \mathbb{1}_{L(D_{k-1})=n+1}\, \mathbb{1}_{Y_k=0}.$$

FOR THE UP-CROSSINGS of level $n$, assume first that the system is in state $m$, $0 < m \leq n$, when the service of job $k$ starts.[5] When $Y_k = 1$, it must be that $L(D_k) = m$ because job $k$ left but one new job arrived in the meantime; thus, level $n$ is *not* crossed. In fact, level $n$ can only be up-crossed when $Y_k > n - m + 1$. Thus,

[5] *i.e.,* $L(D_{k-1}) = m$.

$$D(m,n,t) = \sum_{k=1}^{D(t)} \mathbb{1}_{L(D_{k-1})=m}\, \mathbb{1}_{Y_k>n-m+1}$$

counts the number of up-crossings of level $n$ for $m, 0 < m \leq n$.

When the system is in state $L(D_{k-1}) = 0$, there is a slight subtlety. We must first wait for job $k$ to arrive[6] because job $k-1$ left an empty system behind.[7] Once it arrived, $L(D_k) = 0$ when $Y_k = 0$, $L(D_k) = 1$ when $Y_k = 1$, and so on. Therefore,

[6] *But, as this is an arrival epoch, it is not captured by a change in the system state.*

[7] *[6.6.1]–[6.6.2]*

$$D(0,n,t) = \sum_{k=1}^{D(t)} \mathbb{1}_{L(D_{k-1})=0}\, \mathbb{1}_{Y_k>n}$$

counts the number of up-crossings of that occur when $m = 0$.

BY LEVEL-CROSSING we have that

$$D(n+1,0,t) = D(0,n,t) + \sum_{m=1}^{n} D(m,n,t) \pm 1 \text{ at most.}$$

Let us divide by $t$ and take the limit $t \to \infty$. Using (5.2.3), we get

$$\frac{D(m,n,t)}{t} = \frac{D(t)}{t}\,\frac{D(m,t)}{D(t)}\,\frac{D(m,n,t)}{D(m,t)}, \quad 0 \leq m \leq n.$$

As before, $D(t)/t \to \delta$ and $D(m,t)/D(t) \to \delta(m)$. Then, by a reasoning similar to Section 6.5, $D(m,n,t)/D(m,t) \to G(n-m+1)$ for $0 < m \leq n$, and

$D(0,n,t)/D(0,t) \to G(n)$. Noting that $\pi(m) = \delta(m)$, see (5.3.4), and dividing by $\delta$, we arrive at a recursion for $\{\pi(n)\}$:

$$\pi(n+1)f(0) = \pi(0)G(n) + \sum_{m=1}^{n} \pi(m)G(n+1-m). \tag{6.6.1}$$

Fig. 6.6.1 shows an example for level $n = 3$.

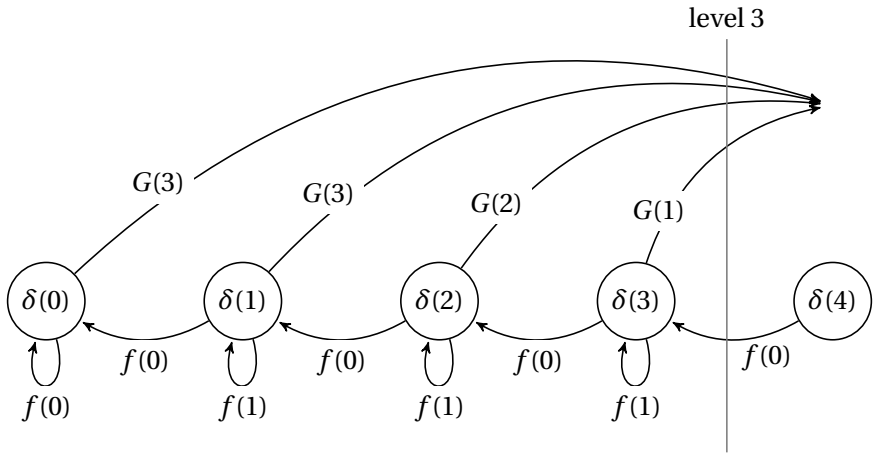



Figure 6.6.1: Level crossing at departure moments.

FOR THE EVALUATION of the above recursion we can just follow the scheme of Section 6.5, but there is an important difference, here the $\{f(k)\}$ needs to be computed.

We use a conditioning argument to find an expression for $\mathsf{P}\{Y = j\}$. Since jobs arrive as a Poisson process, $Y|S \sim P(\lambda S)$; hence,

$$\mathsf{P}\{Y = j \,|\, S = x\} = e^{-\lambda x}\frac{(\lambda x)^j}{j!}.$$

We write $\mathsf{P}\{S \in \mathrm{d}x\} = \mathrm{d}F(x) = F(x+\mathrm{d}x) - F(x)$ for the probability that the service time lies in the (infinitesimal) interval $[x, x+\mathrm{d}x]$.[8] When $F$ has a density $f$, then $\mathrm{d}F(x) = f(x)\,\mathrm{d}x$. With this,

$$\mathsf{P}\{Y = j\} = \int_0^\infty \mathsf{P}\{Y = j \,|\, S = x\}\,\mathrm{d}F(x) = \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^j}{j!}\,\mathrm{d}F(x).$$

In simple cases we can carry out the integration by hand. A simple example is when $S \sim \mathrm{Exp}(()\mu)$.[9] Another is to take $S \equiv s$, i.e., a constant. In this case, all probability mass is concentrated on $s$, so that $\mathrm{d}F(x) = 0$ for $x \neq s$ but $\mathrm{d}F(s) = \infty$.[10] With this, $f(j) = e^{-\lambda s}(\lambda s)^j/j!$.

WHEN WE CANNOT obtain a closed-form expression for the integral we need numerical methods. One simple method is as follows. Make a grid of size $\mathrm{d}x$, for some small number $\mathrm{d}x$, e.g. $\mathrm{d}x = 1/100$. Then take $i$ such that $i\,\mathrm{d}x = x$, and write $\mathrm{d}F(i) = F((i+1)\,\mathrm{d}x) - F(i\,\mathrm{d}x)$. With this,

$$\mathsf{P}\{Y_k = j\} \approx \sum_{i=0}^{\infty} e^{-\lambda i\,\mathrm{d}x}\frac{(\lambda i\,\mathrm{d}x)^j}{j!}\,\mathrm{d}F(i).$$

Let's try a numerical experiment.

[8] *We shamelessly use infinitesimals here. We refer the interested student to any book on measure theory, to add a pile of technical details and hide the intuition.*

[9] *[6.6.3]*

[10] *The density is the, so-called, $\delta$ function concentrated on $s$.*

Python Code

```
import numpy as np

labda = 3
mu = 4
j = 5
dx = 1 / 100


def F(x):
    return 1 - np.exp(-mu * x)


def dF(i):
    return F((i + 1) * dx) - F(i * dx)


def term(i):
    return (
        np.exp(-labda * i * dx)
        * (labda * i * dx) ** j
        / np.math.factorial(j)
        * dF(i)
    )


approx = sum(term(i) for i in range(50))
exact = mu / (labda + mu) * (labda / (labda + mu)) ** j
```

The value of the approximation is 0.0011159 while the exact value[11] is 0.0082619. The difference is significant. We can sum over more terms, for instance to 500. This gives 0.0080988. This is much better, but still the second digit is not correct, even though we evaluated 500 factorials, powers, and exponentials.[12]

To add precision we can make $\mathrm{d}x$ smaller and add yet more terms. However, it seems safer to use a real numerical integrator.

[11] *[6.6.3]*

[12] *This must be numerically quite inefficient.*

Python Code

```
from scipy.integrate import quad

def g(x):
    return (
        np.exp(-labda * x)
        * (labda * x) ** j
        / np.math.factorial(j)
        * mu
        * np.exp(-mu * x)
    )

approx_2 = quad(g, 0, np.inf)
```

Now we obtain 0.0082619, which is indeed correct.

**Ex 6.6.1.** If $L(D_{k-1}) = 0$, what is $\mathsf{E}[D_k - D_{k-1}]$?

**Ex 6.6.2.** Show that if $LD_{k-1} = 0$ and $S_k \sim \text{Exp}(()\mu)$, the density of $D_k - D_{k-1}$ is

$$f_{X+S}(t) = \frac{\lambda\mu}{\mu-\lambda}(e^{-\lambda t} - e^{-\mu t}).$$

**Ex 6.6.3.** If $S \sim \text{Exp}(()\mu)$, show that

$$f(j) = \mathsf{P}\{Y_k = j\} = \frac{\mu}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^j.$$

**Ex 6.6.4.** If $S \sim \text{Exp}(()\mu)$, show that

$$G(j) = \sum_{k=j+1}^{\infty} f(k) = \left(\frac{\lambda}{\lambda+\mu}\right)^{j+1}.$$

**Ex 6.6.5.** Check that the queue length distribution $\{\pi(n)\}$ of the $M/M/1$ queue satisfies (6.6.1).

*This is a nice exercise to test your algebra skills.*

# 7

# QUEUEING CONTROL AND OPEN NETWORKS

In this chapter we study two topics: the control of an $M/M/1$ queue and $M/G/1$ by an $N$-policy in Sections 7.1 and 7.2, and open networks of $M/M/c$ stations in Section 7.3. As we will see, the analysis of the $N$-policy requires to solve an equation of the type $v = c + Pv$, where $v$ and $c$ are vectors and $P$ a (stochastic) matrix, while for the networks we need to solve an equation of the type $\lambda = \gamma + \lambda P$, where $\lambda$ and $\gamma$ are (lying) vectors and $P$ is again a (stochastic) matrix. Since these equations are just the transpose of each other, they allow us to study these two, seemingly unrelated, topics at once. We concentrate in Section 7.4 on the solution of these equations. The analysis in this chapter illustrates many tools and results of the previous chapters; as such, everything comes together here.

We point out that the techniques developed in this chapter extend (way) beyond just queueing theory; they are worth memorizing. The concepts we introduce here can for instance be generalized to (optimal) stopping problems, which find many applications beyond queueing, such as in finance, inventory theory, decision theory, and so on. As another set of extensions, it is possible to make the matrix $P$ and the vector $c$ depend on an action one can take in certain states. This idea underlies Markov decision theory, which in turn provides the theoretical basis of a number of machine learning tools such as $Q$ learning, reinforcement learning, and so on. Thus, while this chapter closes our journey on the study of queueing systems, it is a first step toward a much longer journey on the diverse applications of probability theory.

## 7.1 N-POLICIES FOR THE $M/M/1$ QUEUE

In the queueing systems we analyzed up to now, the server is always present to start serving jobs at the moment they arrive. However, in cases in which there is a cost associated with changing the server from idle to busy, this condition is typically not satisfied. For instance, the cost to heat up an oven after being idle can be quite significant; in other cases, the operator of a machine has to move from one machine to another, which takes time.

TO REDUCE THE average cost, we can use an $N$-policy[0], which works as follows. As soon as the system becomes empty,[1] the server switches off. Then it waits until $N$ jobs have arrived,[2] and then it switches on. The server processes jobs until the system is empty again, then switches off, and remains idle until $N$ new jobs have arrived, and so on.[3]

Note that under an $N$-policy, even though the load remains the same, the server has longer busy and idle times. In fact, some type of servers seem to use such policies on purpose. At hospitals, for instance, doctors prefer to let patients wait until the waiting room is quite full. Like this, the doctor (server) does not have to wait for short times for patients that might be late,

[0] *[2.1.8]*

[1] *and the server idle*

[2] *or N or more items in case of batch arrivals*

[3] *Thus, an N-policy* controls *the server.*

but instead can collect idle times into one long stretch, and do something (they find more) useful instead.

Suppose it costs $K$ Euro to set up the server, independent of the time it has been idle, and each job charges $h$ Euros per unit time while in the system. Then it makes sense to first build up a queue of $N$ jobs right after the server becomes idle,[4] and after some time switch on the server to process jobs until the system is empty again.[5] The problem is to find a switching threshold $N$ that minimizes long-run average cost.

[4] *To reduce time-average setup cost.*

[5] *To reduce time-average queueing costs of jobs.*

In this section we solve this problem for the $M/M/1$ queue, in the next section we generalize to the $M/G/1$ queue. In passing, we obtain a third way to compute the time-average number $\mathsf{E}[L]$ of jobs in the system; the first resulted from Little's law,[6] the second from the analysis of the $M/M/1$ queue in Section 6.1.

[6] *See [5.4.7]*

AS A FIRST STEP, we concentrate on the expected time $T(q)$ it takes to clear the system when there are $q$ jobs in the system and the server is on. As a matter of fact, we present three different ideas to obtain $T(q)$.

The first idea is this. We know that jobs arrive at rate $\lambda$ and they are served at rate $\mu > \lambda$. Clearly, the net 'drain rate' of the queue is $\mu - \lambda$; hence we guess that $T(q) = q/(\mu - \lambda)$.[7] Observe that this reasoning ignores the stochasticity in arrival times and services. As such, such quick reasoning should not be applied in general; things can go very, very wrong.

[7] *By analogy: when you have a 'queue' of $q$ km to cycle, and your speed is $v$ km/h, then it takes $q/v$ h to complete the trip.*

For the second idea, consider a regular $M/M/1$ queue for moment. When a job arrives to an empty $M/M/1$ queue, it takes a busy time $\mathsf{E}[U]$ to get rid of this job and all jobs that arrive during the service. In other words, it takes $\mathsf{E}[U]$ time on average to move from 1 job in the system to 0, i.e., one less, jobs. But then, when there are $q$ jobs in the system, it must take $T(q) = q\mathsf{E}[U]$ units of time to move from state $q$ to state 0. By [6.4.5], $\mathsf{E}[U] = \mathsf{E}[S]/(1-\rho) = 1/(\mu - \lambda)$, since $\mathsf{E}[S] = 1/\mu$ for the $M/M/1$ queue. Again, we see that $T(q) = q/(\mu - \lambda)$.

THE THIRD IDEA is the most powerful. Consider an arbitrary moment in time at which $q > 0$ and the server is busy. Now either of two events happens first: a new job enters the system, or the job in service leaves. The probability of an arrival to occur first is $\alpha = \lambda/(\lambda + \mu)$,[8] the probability of a departure first is $\beta = 1 - \alpha = \mu/(\lambda + \mu)$. Moreover, the expected time to either an arrival or a departure, whichever is first, is $1/(\mu + \lambda)$.[9] Therefore, $T(q)$ must satisfy the following recursion:[10]

[8] *[2.4.14] and [6.6.3]*

[9] *[2.4.13]*

[10] *i.e., a difference equation.*

$$T(q) = \frac{1}{\lambda + \mu} + \alpha T(q+1) + \beta T(q-1). \tag{7.1.1}$$

In words, the system stays in state $q$ for an expected time $1/(\lambda + \mu)$ until an arrival or departure occurs. Then, it moves to state $q+1$ or $q-1$, and from there it takes $T(q+1)$ or $T(q-1)$ until the system is empty. Observe that this reasoning depends crucially on the memoryless property.

To solve this equation, we substitute the guess $T(q) = aq + b$ and solve for $a$ and $b$. It is clear that $T(0) = 0$, hence $b = 0$.[11] It remains to solve for $a$. Filling in $T(q) = aq$ gives

[11] *If the system is empty, it takes no time to clear.*

$$aq = \alpha(aq + a) + \beta(aq - a) + 1/(\lambda + \mu).$$

Noticing that $\alpha+\beta=1$, this reduces to $0=a(\alpha-\beta)+1/(\lambda+\mu)$. Solving this for $a$ gives right away that $a=1/(\mu-\lambda)$.

We note that the solution of (7.1.1) is unique once $T(0)$ is fixed. To see this, observe that in (7.1.1), $T(q+1)$ is a function of $T(q)$ and $T(q-1)$. Thus, $T(1)$ follows from $T(0)$, $T(2)$ from $T(1)$ and $T(0)$, and so on.

WITH THE SAME LINE of reasoning we can compute the expected cost $V(q)$, $q\geq 1$, to clear the system. Noting that the queueing cost is $hq$ per unit time when there are $q$ jobs in the system, it costs $hq/(\lambda+\mu)$ until an arrival or departure occurs. Hence, $V(q)$ satisfies the relation,

$$V(q)=h\frac{q}{\lambda+\mu}+\alpha V(q+1)+\beta V(q-1). \tag{7.1.2}$$

We need a guess for $V(q)$ to solve this equation. Now observe that the last term in (7.1.1) is a constant, and that $T(q)$ is a linear function in $q$. As in (7.1.2) the last term is linear in $q$, let us try a quadratic in $q$ for $V(q)$, i.e., $V(q)=aq^2+bq+c$. As $V(0)=0$, it follows already that $c=0$. Substituting $V(q)=aq^2+bq$ gives[12]

[12] *[7.1.1]*

$$V(q)=\frac{h}{2}\frac{1}{\mu-\lambda}q^2+\frac{h}{2}\frac{\lambda+\mu}{(\mu-\lambda)^2}q.$$

AS AN IMMEDIATE application of the above, let us rederive $\mathsf{E}[L]=\rho/(1-\rho)$, i.e., (6.1.2), for the $M/M/1$ queue. For this, we consider a *busy cycle* that results under the $N=1$ policy. Specifically, a cycle starts when a job arrives at an empty system. The server then switches on, and a busy period $U$ starts. After some time,[13] the system becomes empty again, and the server idles for a period $I$. The cycle stops when a new job arrives. Write $C(1)$ for the expected duration of a busy cycle. Clearly,[14]

[13] *In expectation* $\mathsf{E}[U]=T(1)$.

[14] *[6.4.5]*

$$C(1)=\mathsf{E}[I]+\mathsf{E}[U]=1/\lambda+T(1).$$

With the renewal-reward theorem it is simple to see that[15]

[15] *[7.1.2]*

$$\frac{V(1)}{C(1)}=\frac{V(1)}{1/\lambda+T(1)}=h\,\mathsf{E}[L].$$

After some algebra we get that $V(1)/C(1)=h\rho/(1-\rho)$, thereby completing the argument.[16]

[16] *[7.1.3]*

LET US NEXT analyze the cost under a general $N$-policy. As we already have expressions for the time and cost while the server is on, we only have to consider the time and cost while the server is off.

Clearly, right after the server switches off, we need $N$ independent inter-arrival times to reach level $N$, which takes $N/\lambda$ units of time in expectation.[17].

[17] *[2.4.9]*

For the cost during the build up the queue, we use again a recursive procedure. Write $W(q)$ for the accumulated queueing cost[18] from the moment the server becomes idle up to the arrival time of the $q$th job (the job that sees $q-1$ jobs in the system). Then,[19]

[18] *Here $W$ is not the waiting time in queue.*

[19] *[7.1.4]*

$$W(q)=W(q-1)+h\frac{q-1}{\lambda}=h\frac{q(q-1)}{2\lambda}. \tag{7.1.3}$$

It remains to assemble all results. As the switching cost is $K$, then, by the renewal-reward theorem, the time-average cost of the $N$-policy is equal to

$$\frac{W(N)+K+V(N)}{C(N)} = \frac{W(N)+K+V(N)}{N/\lambda + T(N)},$$

since the expected cycle duration is

$$C(N) = N/\lambda + T(N).$$

FINDING THE OPTIMAL $N$ is easy. Observe that $V(N)$ and $W(N)$ are quadratic in $N$, while $C(N)$ is linear in $N$. Hence, the average cost is a convex function of $N$. In Section 7.2 we derive the general expression and identify the optimal level $N^*$.

**Ex 7.1.1.** Derive the expression for $V(q)$.

**Ex 7.1.2.** Explain that $V(1)/C(1) = h\,\mathsf{E}[L]$.

**Ex 7.1.3.** Show that $V(1)/C(1) = h\rho/(1-\rho)$.

**Ex 7.1.4.** Explain the recursion for $W(q)$ and solve it.

## 7.2 N-POLICIES FOR THE $M/G/1$ QUEUE

Interestingly, we can extend the analysis of Section 6.6 and Section 7.1 to compute the average cost of the $M/G/1$ queue under an $N$-policy. At the end of this section we will find the threshold $N^*$ that minimizes the long-run average cost under the $N$-policy.

Throughout we consider the $M/G/1$ queueing process at moments at which services start. This is similar to Section 6.6: When the system is not empty, these are departure epochs.

IT IS EASY to obtain an expression for the clearing time $T(q)$ when a job starts its service with $q$ jobs in the system. As in Section 6.6, write $Y$ for the number of jobs that arrive during a service time. Then, analogous to (7.1.1), $T(q)$ satisfies the relation

$$T(q) = \mathsf{E}[S] + \mathsf{E}\big[T(q+Y-1)\big], \tag{7.2.1}$$

because first the job in service must leave, and then, when $Y = k$, it takes $T(q+k-1)$ to clear the system.

Again, we guess that $T(q) = aq + b$. But $b = 0$ since $T(0) = 0$. Substituting $aq$ into (7.2.1) gives $qa = \mathsf{E}[S] + aq + a\mathsf{E}[Y] - a$. Noting that $\mathsf{E}[Y] = \lambda\,\mathsf{E}[S]$,[0] we obtain

[0] *[4.4.5]*

$$a = \frac{\mathsf{E}[S]}{1-\lambda\,\mathsf{E}[S]} \implies T(q) = \frac{\mathsf{E}[S]}{1-\lambda\,\mathsf{E}[S]}\,q.$$

TO FIND THE cost to clear the queue, we first concentrate on the expected queueing cost $U(q)$ that accrue when $q$ jobs are present and a service starts. Given $S = s$, the cost for the $q$ jobs is $h$ times the area of the rectangle with base $s$ and height $q$. Next, during the service, $\lambda s$ new jobs arrive in

expectation. Since the jobs arrive as a Poisson process, the arrival times are distributed uniformly over the service time. Therefore the cost of the new jobs is $h$ times the area of the *triangle* with base $s$ and height $\lambda s$. The total cost is therefore $hqs + h\lambda s^2/2$. Taking the expectation over $S$ gives

$$U(q) = hq\,\mathsf{E}[S] + \frac{1}{2}\lambda h\,\mathsf{E}\left[S^2\right].$$

LET $V(q)$ BE the expected queueing costs to clear the system just after a service starts and $q$ jobs are in the system. By analogy with (7.2.1), $V(q)$ must be the solution of

$$V(q) = U(q) + \mathsf{E}\left[V(q+Y-1)\right]. \tag{7.2.2}$$

Now note that as in (7.1.2), $U(q)$ has a term linear in $q$ and a constant term. As before, we substitute the form $V(q) = aq^2 + bq + c$, assemble terms with the same power in $q$, and solve for the coefficients. After some work we arrive at[1]

[1] *[7.2.3]–[7.2.4]*

$$V(q) = \frac{h}{2}\frac{\mathsf{E}[S]}{1-\rho}q^2 + h\frac{1+\rho C_s^2}{2}\frac{\mathsf{E}[S]}{(1-\rho)^2}q.$$

NOW THAT WE have the expected clearing time and cost, we can compute the long-run average cost under a general $N$-policy. The cycle time $C(N) = N/\lambda + T(N)$, and the queueing cost $W(N)$ until level $N$ is reached while the server is idle is given by (7.1.3). Combining all this results in the long-run average cost[2]

[2] *[7.2.7]*

$$\frac{W(N)+K+V(N)}{C(N)} = h\frac{1+C_s^2}{2}\frac{\rho^2}{1-\rho} + h\rho + h\frac{N-1}{2} + K\frac{\lambda(1-\rho)}{N}. \tag{7.2.3}$$

From this, the PK-formula follows directly.[3]

[3] *[7.2.6]*

MINIMIZING OVER $N$ gives that

$$N^* \approx \sqrt{\frac{2\lambda(1-\rho)K}{h}}.$$

The expression for $N^*$, called the *Economic Production Quantity (EPQ)*, is a famous result in inventory theory. Specifically, it is the optimal production quantity for a production-inventory system in which a machine switches on at cost $K$ and items pay holding cost $h$ per unit time. By taking $\mathsf{E}[S] \to 0$, hence $\rho \to 0$, we see that $N^*$ reduces to the *Economic Order Quantity (EPQ)* $\sqrt{2\lambda K/h}$.

**Ex 7.2.1.** Explain intuitively that the system is rate-stable for any $N$.

**Ex 7.2.2.** Why doesn't the utilization $\rho$ depend on $N$?

**Ex 7.2.3.** Simplify $aq^2 + bq = a\,\mathsf{E}\left[(q+Y-1)^2\right] + b\,\mathsf{E}\left[q+Y-1\right] + H(q)$, and assemble powers in $q$ to obtain:

$$a = \frac{h}{2}\frac{\mathsf{E}[S]}{1-\mathsf{E}[Y]} = \frac{h}{2}\frac{\mathsf{E}[S]}{1-\rho},$$
$$b(1-\mathsf{E}[Y]) = a(\mathsf{E}\left[Y^2\right] - 2\mathsf{E}[Y] + 1) + \frac{1}{2}h\lambda\,\mathsf{E}\left[S^2\right].$$

**Ex 7.2.4.** Derive the expression for $V(q)$ with the previous exercise.

*In a sense, this is trivial, as it is just algebra, but it is hard to get the details right.*

**Ex 7.2.5.** Check that $V(q)$ reduces to that of the $M/M/1$ queue.

**Ex 7.2.6.** Derive the PK formula from (7.2.3).

**Ex 7.2.7.** Derive (7.2.3).

## 7.3 OPEN SINGLE-CLASS PRODUCT-FORM NETWORKS

Up to now our analysis focused on single-station queueing systems. However, jobs, or patients, may need several 'processing' steps at different stations. In such cases, jobs move from one queueing station to another. In this section we analyze such queueing networks in simple settings. We start with two stations in tandem, and then extend to general networks. Throughout we assume that external jobs[0] arrive as a Poisson processes, and that service times at all stations are exponentially distributed. Under this condition we are able to obtain closed-form expressions for the stationary distribution of jobs at each station.[1]

[0] *i.e., new jobs that have not visited any other station before.*

[1] *Recall, Section 4.5 only considers the expected sojourn times in tandem networks of $G/G/c$ queues, not the distribution of the number of jobs at each station in a network of $M/M/1$ queues.*

WE START WITH stating the remarkable, and crucially important, result that the *inter-departure times* of jobs of an $M/M/1$ queue are exponentially distributed with rate $\lambda$, just as the inter-arrival times.[2] This property makes the analysis of a *tandem network* of stations, i.e., stations placed in sequence, particularly easy.

[2] *[7.3.1]*

When the first station is an $M/M/1$ queue, jobs arrive as a Poisson process, but its output process is also a Poisson process. Therefore the second station, i.e., the station immediately downstream of station 1, receives jobs as a Poisson process. If job service times at the second station are exponentially distributed, then this station is also an $M/M/1$ station. But then its departure process is a Poisson process in turn, and the third station receives jobs as a Poisson process, and so on.

With this insight, it follows from (6.1.2) that the expected total sojourn time in a tandem network of $M$ stations is equal to

$$\mathsf{E}[J] = \sum_{i=1}^{M} \mathsf{E}[J_i] = \sum_{i=1}^{M} \frac{\mathsf{E}[S_i]}{1-\rho_i},$$

where $\mathsf{E}[J_i]$ is the sojourn time at station $i$ with $\rho_i = \lambda \mathsf{E}[S_i]$.

IT IS NOT DIFFICULT to extend the above result to general networks of $M/M/1$ queues. For this, we first need to model such networks more formally. In particular, we assume that the probability that a job moves to station $j$ after completing its service at station $i$ is independent of anything else, and is given by the number $P_{ij} \in [0,1]$.[3] We assemble all these probabilities in a *routing matrix* $P$ such that $P_{ij}$ is the element of $P$ on the $i$th row and $j$th column. Define $P_{i0} = 1 - \sum_j P_{ij}$ as the probability that a job leaves the network from station $i$. For the network to be stable, we need to require, see Section 7.4,

[3] *This is called Markov routing.*

$$P_{i0} \in [0,1] \text{ for all } i, \text{ and } P_{i0} > 0 \text{ for at least one } i. \tag{7.3.1}$$

Consider station $i$, say, and assume that jobs arrive as a Poisson process with rate $\lambda_i$. Since service times are exponentially distributed, the departure process is also Poisson with rate $\lambda_i$. Then, after departure, jobs are sent with probability $P_{ij}$ to station $j$. It follows from [2.2.17] that jobs sent to station $j$ from station $i$ form a thinned Poisson process with rate $\lambda_i P_{ij}$.

Now take the perspective of station $j$. This station receives a merged 'stream' of thinned Poisson process from all other stations. But, by [2.2.12], this merged process is also a Poisson process with rate $\sum_{i=1}^{M} \lambda_i P_{ij}$. Assuming that external jobs arrive at station $j$ as a Poisson process with rate $\gamma_j$, then by merging this process with the other Poisson stream, we obtain a Poisson process with rate $\gamma_j + \sum_{i=1}^{M} \lambda_i P_{ij}$.

By assumption, service times at station $j$ are exponential, hence, the departure process of this station is Poisson. But then the thinned process resulting from routing its departing jobs to yet other stations is again Poisson, and so on.

We arrive at the intuitively clear fact that this network behaves as a set of $M/M/1$ queues. Below we will give a formal proof of this fact for two stations.

Note that we allow for external jobs arriving at any station. Therefore, this is an *open network*. This differs from so-called *closed networks*; in such networks jobs do not enter or leave the network.

IT IS EVIDENT that, when the network is stable, all jobs that enter a station eventually must leave that station. This insight leads us to the *traffic equations*, which state that for each station $i$ the departure rate must equal the arrival rate, i.e.,

$$\lambda_i = \gamma_i + \sum_{j=1}^{M} \lambda_j P_{ji}, \quad i = 1, \ldots, M. \tag{7.3.2}$$

For the *network as whole*, the total external arrival rate is given by $|\gamma| = \sum_{i=1}^{M} \gamma_i$. Hence, this is also the departure rate of the network.[4]

[4] *Not for an individual station $i$, because $\delta_i = \lambda_i$.*

LET US FOR the moment assume that we can solve the traffic equations, in other words, for given $\gamma = (\gamma_1, \ldots, \gamma_M)$ and routing matrix $P$ we can find a set of numbers $\lambda = (\lambda_1, \ldots, \lambda_M)$ that satisfies (7.3.2). Then, we can define $\rho_i = \lambda_i \mathsf{E}[S_i]$ as the utilization of (the server of) station $i$. Clearly, we assume that $\rho_i < 1$ for all stations $i$.

Evidently, the average total number of jobs in this network of $M/M/1$ stations is equal to the sum of the average number of jobs at each station, hence, $\mathsf{E}[L] = \sum_{i=1}^{M} \mathsf{E}[L_i]$. With this we can also find an expression for the expected total sojourn time in the network $\mathsf{E}[J]$. Applying Little's law first to the network as a whole and then to each station individually, we see that

$$|\gamma| \mathsf{E}[J] = \mathsf{E}[L] = \sum_{i=1}^{M} \mathsf{E}[L_i] = \sum_{i=1}^{M} \lambda_i \mathsf{E}[J_i] = \sum_{i=1}^{M} \lambda_i \frac{\mathsf{E}[S_i]}{1-\rho_i}.$$

Dividing by $|\gamma|$ gives $\mathsf{E}[J]$ in terms of the *visit ratios* $\lambda_i/|\gamma|$,

$$\mathsf{E}[J] = \sum_{i=1}^{M} \frac{\lambda_i}{|\gamma|} \frac{\mathsf{E}[S_i]}{1-\rho_i}.$$

This is intuitive: the visit ratio $\lambda_i/|\gamma|$ tells how often station $i$ is visited relative to the total number of arrivals. Thus, $\mathsf{E}[J_i]\,\lambda_i/|\gamma|$ is the amount of time an 'average' job spends at station $i$ before leaving the network.

ABOVE WE DERIVED expressions for the average waiting time in a network of $M/M/1$ queues, but it is possible to obtain a much stronger result. In fact, the stationary probability $p(n)$ that the system contains $n=(n_1,n_2,\ldots,n_M)$ at stations $1,\ldots,M$ can be written as the *product* of the probabilities of the individual stations, specifically,

$$p(n) = \mathsf{P}\{L_1 = n_1,\ldots,L_M = n_M\} = \Pi_{i=1}^{M} p(n_i).$$

where $p(n_i) = (1-\rho_i)\rho_i^{n_i}$ is the stationary probability that station $i$ contains $n_i$ jobs, compare (6.1.1). In other words, the stationary probability $p(n_i)$ of station $i$ is *independent* of the state of the other stations. Hence, in stationarity, we can concentrate on each station individually. As a consequence, we can easily compute the excess probability $\mathsf{P}\{L_1 > n_1\}$ for each station $i$.

As a matter of fact, any stable open network of $M/M/c$ stations[5] admits a *product-form solution*, a result known as *Jackson's theorem*.

[5] *Note, $M/M/c$ queues, not just $M/M/1$ queues.*

LET US PROVE this result for the case of two $M/M/1$ stations in tandem. Let $p(i,j) = \mathsf{P}\{L_1 = i, L_2 = j\}$ be the probability that station 1 contains $i$ jobs and station 2 contains $j$ jobs. Specifically, we have to show that when $p(i,j)$ has the form $(1-\rho_1)(1-\rho_2)\rho_1^i\rho_2^j$, the balance equations are satisfied for all $i,j \geq 0$, see Section 5.2.

Here we focus on the case $i,j \geq 1$, see Fig. 7.3.1. As the normalization factor appears at both sides of the balance equations, we write $p(i,j) = \rho_1^i\rho_2^j$ for ease. Then it follows that, Clearly, the balance equations hold for $i,j \geq 1$.

$$\begin{aligned}
\text{rate out} &= (\lambda+\mu_1+\mu_2)p(i,j)\\
&= \lambda\rho_1^i\rho_2^j + \mu_1\rho_1^i\rho_2^j + \mu_2\rho_1^i\rho_2^j\\
&= \mu_2\rho_1^i\rho_2^{j+1} + \lambda\rho_1^{i-1}\rho_2^j + \mu_1\rho_1^{i+1}\rho_2^{j-1}\\
&= \mu_2 p(i,j+1) + \lambda p(i-1,j) + \mu_1 p(i+1,j-1)\\
&= \text{rate in.}
\end{aligned}$$

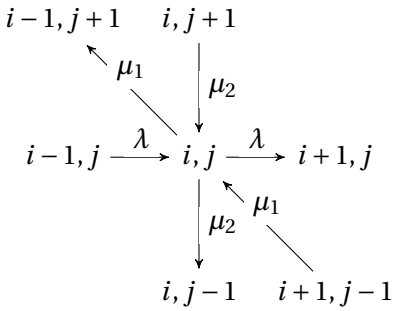



Figure 7.3.1: Transitions in two stations in tandem.

It remains to check the boundary[6] to complete the claim.

[6] *[7.3.4]–[7.3.6]*

**Ex 7.3.1.** Show that the inter-departure times $D_k - D_{k-1} \sim \text{Exp}(()\lambda)$.

**Ex 7.3.2.** Why do we require (7.3.1)?

**Ex 7.3.3.** We have a two-station single-server open network. Jobs enter the network at the first station with rate $\gamma$. A fraction $\alpha$ returns from station 1 to itself; the rest moves to station 2. At station 2 a fraction $\beta_2$ returns to station 2 again, a fraction $\beta_1$ goes to station 1.

First, compute $\lambda$, then analyze what happens if $\alpha \to 1$ or $\beta_1 \to 0$.

**Ex 7.3.4.** Check that $p(0,0)$ satisfy the balance equation for state $(0,0)$.

**Ex 7.3.5.** Check that $p(i,0)$ satisfy the balance equation for state $(i,0)$.

**Ex 7.3.6.** Check that $p(0,j)$ satisfy the balance equation for state $(0,j)$.

## 7.4 ON $\lambda = \gamma + \lambda P$

There is a remarkable similarity between (7.1.2) and (7.3.2). To see this, write the former in matrix form as $v = Pv + h$, with

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ \beta & 0 & \alpha & 0 & \dots \\ 0 & \beta & 0 & \alpha & 0 \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad h = \begin{pmatrix} 0 \\ 1/(\lambda+\mu) \\ 2/(\lambda+\mu) \\ \vdots \end{pmatrix},$$

and the latter as $\lambda = \gamma + \lambda P$.[0] Clearly, the former is just the transpose of the latter. In this final section we concentrate on finding a general condition to ensure that these equations have a solution.

[0] *A similar form can be found for* (7.2.2).

We remark that, formally speaking, the matrix $P$ in $v = Pv + h$ is an infinite matrix. Rather than dealing with the ensuing technical complications, we bypass them by restricting the analysis to an $M/M/1/K$ queue under an $N$-policy, with $K \gg N$ for any $N$ that can be 'reasonable'.[1]

[1] *What is reasonable depends on the context of course, but $N = 1000$ seems ridiculously large for any practical queueing system.*

For ease of writing, we only regard $\lambda = \gamma + \lambda P$.

TO START, define iteratively, $\lambda^0 = 0$, $\lambda^n = \gamma + \lambda^{n-1} P$, for $n \geq 1$. Then, by substitution, we find

$$\lambda^n = \gamma + \lambda^{n-1} P = \gamma + (\gamma + \lambda^{n-2} P)P = \cdots = \gamma \sum_{m=0}^{n-1} P^m,$$

where $P^0$ as the identity matrix. Since $P$ is a non-negative matrix, $P^n$ is also non-negative for any $n$, hence, $\sum_{m=0}^{n} P^m$ is a monotone increasing sequence (of matrices).[2] Thus, for $\lambda = \gamma + \lambda P$ to have a (unique) solution, this sum must remain finite as $n \to \infty$.

[2] *Formally, non-decreasing sequence.*

Now, observe the similarity with $\sum_{i=0}^{n} x^i$ with $|x| < 1$. From (1.2.3d) we see that this converges to $1/(1-x)$ for $n \to \infty$. By the same token, if there is an $\epsilon > 0$ such that for all $i, j$, $0 \leq P_{ij}^m \leq (1-\epsilon)^m$, then $0 \leq \sum_{m=0}^{n} P_{ij}^m \leq \sum_{m=0}^{n} (1-\epsilon)^m \leq 1/\epsilon$.

It remains to find a condition to ensure that such $\epsilon$ exists. For this, we discuss two ways.

FROM [7.3.2] WE KNOW that $P_{i0} = 1 - \sum_{j=1}^{M} P_{ij}$ is the probability that a job leaves the network from station $i$. Next, consider a station $k$ with $P_{ki} > 0$. Then the probability that a job starts at $k$ and leaves the network after a visit to $i$ is $P_{ki} P_{i0} > 0$. Consequently, $P_{k0}^2 = \sum_{j=0}^{M} P_{kj} P_{j0} \geq P_{ki} P_{i0} > 0$.

More generally, we say that $M \times M$ matrix $P$ is *transient* when it is possible to leave the network from any station in at most $M$ steps. In other words, for any station $j$ there is a sequence of intermediate stations $j_1, j_2, \dots, j_{M-1}$ such that $P_{j0}^M \geq P_{j j_1} P_{j_1 j_2} \cdots P_{j_{M-1} 0} > 0$. Using this, it is not hard to prove that $P^n \leq (1-\epsilon)^n$ when $P$ is transient.[3]

[3] *[7.4.2]*

FOR THE SECOND method, let us make the simplifying assumption that $P$ is a diagonalizable matrix with $M$ different eigenvalues.[4] In this case, there exists an invertible matrix $V$ with the (left) eigenvectors of $P$ as its rows and a diagonal matrix $\Lambda$ with the eigenvalues of $P$ on its diagonal such that $VP = \Lambda V$. Hence, premultiplying with $V^{-1}$, $P = V^{-1} \Lambda V$. But then $P^2 =$

[4] *The argument below applies just as well to matrices reduced to Jordan normal form, but adds only notational clutter.*

$V^{-1}\Lambda V \cdot V^{-1}\Lambda V = V^{-1}\Lambda^2 V$, and in general $P^n = V^{-1}\Lambda^n V$. Clearly, if each eigenvalue $\lambda_i$ is such that its modulus $|\lambda_i| < 1$, then $\Lambda^n \to 0$ exponentially fast, hence $P^n \to 0$ exponentially fast.

So, let us prove that all eigenvalues of a finite, transient routing matrix $P$ have modulus less than 1. For this we use *Gerschgorin's disk theorem*. Define the Gerschgorin disk of the $i$th row of the matrix $P$ as the disk in the complex plan:

$$B_i = \left\{ z \in \mathbb{C}; |z - p_{ii}| \leq \sum_{j \neq i} |p_{ij}| \right\}.$$

In words, this is the set of complex numbers that lies within a distance $\sum_{j\neq i} |p_{ij}|$ of the point $p_{ii}$. Next, assume for notational simplicity that for each row $i$ of $P$ we have that $\sum_j p_{ij} < 1$.[5] Then this implies for all $i$ that

[5] *Otherwise apply the argument to* $P^M$.

$$1 > \sum_{j=1}^{M} p_{ij} = p_{ii} + \sum_{j \neq i} p_{ij}.$$

Since all elements of $P$ are non-negative, so that $|p_{ij}| = p_{ij}$, it follows that

$$-1 < p_{ii} - \sum_{j \neq i} p_{ij} \leq p_{ii} + \sum_{j \neq i} p_{ij} < 1.$$

With this and using that $p_{ii}$ is a real number (so that it lies on the real number axis) it follows that the disk $B_i$ lies strictly within the complex unit circle $\{z \in \mathbb{C}; |z| \leq 1\}$. As this applies to any row $i$, the union of the disks $\cup_i B_i$ also lies strictly within the complex unit circle. Now we invoke Gerschgorin's theorem, which states that all eigenvalues of the matrix $P$ must lie in $\cup_i B_i$, to conclude that all eigenvalues of $P$ lie strictly in the unit circle, hence all eigenvalues have modulus smaller than 1.

HERE WE FINISH our discussion of queueing systems, but there are many other interesting extensions to learn, for which we refer to the following books.

- You can find really nice discussions of networks of $M/M/\infty$, chemical reactions, population dynamics and Petri nets in Baez and Biamonte [2019], which is freely available on arXiv.
- Simple queueing networks (networks that satisfy so-called local balance) can be modeled as electrical networks. For this, see Doyle and Laurie Snell [1984], which you can download for free from the homepage of Doyle.
- In more general terms, queueing systems or networks are examples of Markov processes. A particularly nice book on these topics is Norris [1997]. The material of this chapter can be couched in the theory of martingales and optimal stopping. Besides that this is nice theory, it is widely used in quantitative finance.

**Ex 7.4.1.** What is the routing matrix $P$ for a tandem network with $M$ stations? Show that $P^M = 0$.

**Ex 7.4.2.** Show that when $P$ is transient, $P^n \leq (1-\epsilon)^n$ for some $\epsilon > 0$.

**Ex 7.4.3.** Why does the assumption of [7.4.2] does not apply to an infinite transient matrix $P$?

**Ex 7.4.4.** Show that the matrix $P$ used in Section 7.1 is transient for a finite system.

# 8

## HINTS

**h.1.2.6.** Write $\sum_{k=0}^{\infty} G(k) = \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \mathsf{P}\{X = m\}$, reverse the summations. Then realize that $\sum_{k=0}^{\infty} \mathbb{1}_{k<m} = m$.

**h.1.2.7.** $\sum_{i=0}^{\infty} iG(i) = \sum_{n=0}^{\infty} \mathsf{P}\{X = n\} \sum_{i=0}^{\infty} i\,\mathbb{1}_{n \geq i+1}$, and reverse the summations.

**h.1.2.8.** $\mathsf{E}[X] = \int_0^{\infty} x\,\mathrm{d}F(x) = \int_0^{\infty} \int_0^{\infty} \mathbb{1}_{y \leq x}\,\mathrm{d}y\,\mathrm{d}F(x)$.

**h.1.2.9.** $\int_0^{\infty} yG(y)\,\mathrm{d}y = \int_0^{\infty} y \int_0^{\infty} \mathbb{1}_{y \leq x} f(x)\,\mathrm{d}x\,\mathrm{d}y$.

**h.1.2.12.** For the second child, condition on the event that the first does not choose the right number. Use the definition of conditional probability: $\mathsf{P}\{A \mid B\} = \mathsf{P}\{AB\} / \mathsf{P}\{B\}$ provided $\mathsf{P}\{B\} > 0$.

**h.2.1.2.** Modify $d_k$ in (2.1.1) to incorporate the changed service behavior. Then, substitute $d_k$ in the expression for $L_k$.

**h.2.1.4.** When the machine makes $n$ items, and each of these is faulty with probability $p$, then the number of faulty items is $\mathrm{Bin}(n, p)$.

**h.2.1.5.** Make a queue for the new and repaired items and use [2.1.11].

**h.2.1.6.** Use [2.1.2] for the dynamics of $\{L_k\}$.

**h.2.1.8.** Introduce a variable $I_k \in \{0, 1\}$ to keep track of the state of the server. Then, $I_{k+1} = \mathbb{1}_{L_k \geq N} + I_k \mathbb{1}_{0<L_k<N}$ implements the N-policy. Now use [2.1.7] to find $\{c_k\}$ and the switching cost.

**h.2.1.9.** When $a_k > 0$ for all $k$, then $L_k \geq a_k > 0$ for all $k$.

**h.2.1.10.** Consider a numerical example. Suppose $L_{k-1} = 20$. Suppose that the capacity is $c_k = 3$ for all $k$. Then a job that arrives in the $k$th period, must wait at least 20/3 (plus rounding) periods before it can leave the system. Now generalize this numerical example.

**h.2.1.11.** Compute the number of jobs that depart from queue 1. Subtract the used capacity for these jobs from the total capacity to get the capacity remaining for queue 2.

**h.2.1.19.** Introduce an extra variable $p_k$ that specifies which queue is being served. If $p_k = 0$, the server is moving from one queue to the other, if $p_k = 1$, the server is at queue 1, and if $p_k = 2$ it is at queue 2. Develop a table with conditions on $L_k^i$ and $p_k$ to specify $p_{k+1}$.

**h.2.2.1.** Use that $\mathsf{E}[X + Y] = \mathsf{E}[X] + \mathsf{E}[Y]$.

**h.2.2.2.** First find $p$, $n$, $\lambda$ and $t$ such that the rate at which an event occurs in both processes are the same. Then consider the binomial distribution and use the standard limit $(1 - x/n)^n \to e^{-x}$ as $n \to \infty$.

**h.2.2.4.** Use the definition of the conditional probability and small $o$ notation.

**h.2.2.5.** Use [2.2.4] and [1.2.1].

**h.2.2.6.** See the hint for [2.2.5], and use $\sum_{i=2}^{\infty} x^i/i! = \sum_{i=0}^{\infty} x^i/i! - x - 1 = e^x - x - 1$.

**h.2.2.7.** Use (1.2.4d). Note that the term with $n = 0$ does not contribute in the following summation

$$\sum_{n=0}^{\infty} n\frac{\lambda^n}{n!} = \sum_{n=1}^{\infty} n\frac{\lambda^n}{n!} = \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = \lambda e^{\lambda},$$

where we apply a change of notation in the second to last step.

**h.2.2.9.** Use (1.2.4e), [2.2.8], and [2.2.7].

**h.2.2.10.** Observe that

$$\mathbb{1}_{N(0,s]+N(s,t]=1}\,\mathbb{1}_{N(0,s]=1} = \mathbb{1}_{1+N(s,t]=1}\,\mathbb{1}_{N(0,s]=1} = \mathbb{1}_{N(s,t]=0}\,\mathbb{1}_{N(0,s]=1}.$$

Use independence and (1.2.4b).

**h.2.2.12.** Use sets $\{N_\lambda(t) = i\}$ to decompose $\{N_\lambda(t) + N_\mu(t) = n\}$. With this observe that

$$\mathbb{1}_{N_\lambda(t)+N_\mu(t)=n} = \sum_{i=0}^{n} \mathbb{1}_{N_\lambda(t)=i, N_\mu(t)=n-i}.$$

Take expectations left and right, use (1.2.4b), and independence of $N_\lambda$ and $N_\mu$. Near the end of the computation, use (1.2.3a).

**h.2.2.13.** Use the standard formula for conditional probability and that $N_\lambda(t) + N_\mu(t) \sim \mathrm{P}((\lambda + \mu)t)$. Interpret the result.

**h.2.2.14.** Suppose that $N_1$ is the thinned stream, and $N$ the original stream. Condition on the total number of arrivals $N(t) = n$ up to time $t$. Then, realize that the probability that a person is of type 1 is $p$. Hence, when you consider $n$ people in total, the number $N_1(t)$ of type 1 people is binomially distributed. Thus, given that $n$ people arrived, the probability of $k$ 'successes' (i.e., arrivals of type 1), is

$$\mathrm{P}\{N_1(t) = k \,|\, N(t) = n\} = \binom{n}{k} p^k (1-p)^{n-k}.$$

Use (1.2.8) to decompose the $\{N_1 = k\}$, and (1.2.3c) at the end.

**h.2.2.15.** Use (1.2.4d) with $f(x) = e^{sx}$.

**h.2.2.16.** Use (1.2.7b) and (1.2.7c).

**h.2.2.17.** Dropping the dependence of $N$ on $t$ for the moment for notational convenience, consider the random variable

$$Y = \sum_{i=1}^{N} Z_i,$$

with $N \sim P(\lambda)$ and $Z_i \sim B(p)$. Show that the moment-generating function of $Y$ is equal to the moment-generating function of a Poisson random variable with parameter $\lambda p$.

**h.2.2.18.** Use (1.2.7d) and (1.2.7a).

**h.2.2.19.** Solve [2.2.15] and [2.2.17] first. Perhaps [2.2.2] is also useful.

**h.2.3.1.** What is $\mathbb{1}_{A_k \le t}$ if $A_k \le t$?

**h.2.3.4.** Compare this to Definition (2.3.4), or use a numerical example.

**h.2.3.5.** BTW, such simple test cases are also very useful to test computer code. The numbers in the exercise are one such simple case. You can check the results by hand; if the results of the simulator are different, there is a problem.

**h.2.3.6.** Observe that jobs arrive faster than they are served.

**h.2.3.7.** Use (2.3.2).

**h.2.3.9.** Make a plot of the function $A_{A(t)} - t$. What is the meaning of $V(A_{A(t)})$? What is $V(A_{A(t)}) + A_{A(t)} - t$?

**h.2.3.13.** Use Boolean algebra. Write, for notational ease, $A = \mathbb{1}_{A_k \le t}$ and $\bar{A} = 1 - A = \mathbb{1}_{A_k > t}$, and define something similar for $D$. Then show that $A - D = A\bar{D} - \bar{A}D$, and show that $\bar{A}D = 0$. Finally sum over $k$.

**h.2.3.14.** Use that $L(A_k) > 0$ means that the system contains at least one job at the time of the $k$th arrival, and that $A_k \le D_{k-1}$ means that job $k$ arrives before job $k-1$ departs.

**h.2.3.17.** It's evident that $f_X(x) = \mathbb{1}_{0 \le x \le 10}/10$ and $f_S(s) = \mathbb{1}_{0 \le s \le 7}/7$. Let $T = S - X$. Then $X = S - T$, and

$$f_T(t) = \int f_S(s) f_X(s-t)\,\mathrm{d}s = \frac{1}{70}\int \mathbb{1}_{0 \le s \le 7}\,\mathbb{1}_{0 \le s-t \le 10}\,\mathrm{d}s.$$

Now work out the integral.

**h.2.4.1.**

$$\mathsf{E}[X] = \int_0^\infty t f(t)\,\mathrm{d}t = \int_0^\infty t\lambda e^{-\lambda t}\,\mathrm{d}t,$$

where $f(t) = \lambda e^{-\lambda t}$ is the density function of $X$.

**h.2.4.2.** Use [2.4.1].

**h.2.4.3.** Use [2.4.1] and [2.4.2].

**h.2.4.5.** Use (1.2.7b) and (1.2.7c).

**h.2.4.6.** Use the definition (2.2.3)

**h.2.4.7.** Simplify $\mathsf{P}\{X > t+h \mid X > t\}$ with $\mathsf{P}\{A \mid B\} = \mathsf{P}\{AB\}/\mathsf{P}\{B\}$.

**h.2.4.8.** $N(t) = 0 \iff X_1 > t$.

**h.2.4.9.** $\mathsf{E}[A_i] = \mathsf{E}\left[\sum_{k=1}^{i} X_k\right]$

**h.2.4.10.** Why is $M_{A_i}(t) = \mathsf{E}\left[e^{tA_i}\right] = \prod_{k=1}^{i} \mathsf{E}\left[e^{tX_k}\right]$?

**h.2.4.11.** Use [2.4.1], or $\mathsf{E}[X] = \frac{\mathrm{d}}{\mathrm{d}t} M_X(t)|_{t=0}$.

**h.2.4.12.** $\mathsf{P}\{N(t) = k\} = \mathsf{P}\{A_k \le t\} - \mathsf{P}\{A_{k+1} \le t\}$.

**h.2.4.13.** $\min\{X, S\} > x \iff X > x \,\&\, S > x$.

**h.2.4.14.** Use the joint distribution of $X$ and $S$, or use [2.2.13].

**h.3.2.1.** Use recursion and use subsequently, $\max\{\max\{a,b\},c\} = \max\{a,b,c\}$, $L_0 = Z_0$, $\max\{-a,-b\} = -\min\{a,b\}$.

**h.3.2.2.** Use the ideas of [2.2.12].

**h.3.2.3.** Let $W_n = \sum_{k=1}^{n} S_k$. Since $\{S_k\}$ are assumed to be iid for the $G/G/1$ queue, $W_n$ has mean $\mu_n = n\,\mathsf{E}[S]$ and $\sigma_n^2 = n\,\mathsf{V}[S]$.

**h.3.3.1.** As a start, the function $\sin(t)$ does not have a limit as $t \to \infty$. However, the time-average $\sin(t)/t \to 0$. Now you need to make some function whose time-average does not converge, hence it should grow fast, or fluctuate wilder and wilder.

**h.3.3.3.** Remember that $\{X_k\}$ and $\{S_k\}$ are sequences of iid random variables. What are the implications for the expectations?

**h.3.3.4.** What is the rate in, and what is the service capacity?

**h.3.4.1.** Consider a queueing system with constant service and constant inter-arrival times.

**h.4.1.1.** Think about the inter-departure time of a machine that produces items every 10 minutes. What inter-arrival times will a down-stream machine see?

**h.4.1.4.** What is $\lambda$? What is $C_a^2$ in the $G/G/1$ setting; what is it in the $M/G/1$ setting?

**h.4.1.5.** Why is $\lambda\beta$ the rate at which jobs are lost? What, then, is the rate at which jobs are accepted to the system? Observe that $\mu\,\mathsf{E}[L_s]$ is the rate at which jobs leave the system.

**h.4.1.6.** Let $S$ be the processing (or service) time at the server, and $S_i$ the service time of a type $i$ job. Then,

$$S = \mathbb{1}_{T=1} S_1 + \mathbb{1}_{T=2} S_2.$$

**h.4.2.2.** An arbitrary job has to wait half the time it takes to form a batch.

**h.4.2.3.** Use (1.2.5) and (1.2.6).

**h.4.3.1.** Get the units right. Compute the load, and then the rest.

**h.4.3.2.** Realize that we now deal with setups and batch processing.

**h.4.3.4.** Recall that $\mathbb{1}_{F=1}^2 = \mathbb{1}_{F=1}$. Expand the square, use that $S_0$, $R$ and $F$ are independent. Finally, $\mathsf{E}[\mathbb{1}_{F=1}] = p$.

**h.4.3.5.** First compute $\mathsf{E}\left[S^2\right]$. See [4.3.4]. Don't make the error to assume that, since $R$ and $F$ are independent, $VR\,\mathbb{1}_{F=1} = \mathsf{V}[R]\,\mathsf{V}[\mathbb{1}_{F=1}] = \mathsf{V}[R]\,p(1-p)$.

**h.4.4.1.** Mind to work in a consistent set of units, e.g., hours. It is easy to make mistakes.

**h.4.4.2.** Observe that $m_f = 1/\lambda_f$ and $m_r = \mathsf{E}[R]$.

**h.4.4.3.** Is it relevant that $R_1, \ldots, R_n$ are mutually independent?

**h.4.4.4.** Use (1.2.4) and [4.4.3]. Otherwise, use conditional expectation and Adam's law.

**h.4.4.5.** The joint density of $S_0 = s, N = k$ is $g(s)\,e^{-\lambda s}(\lambda s)^k/k!$, since by assumption $S_0$ and $N|S_0 \sim \text{Pois}(\lambda_f s)$. Use [2.2.8].

**h.4.4.6.** Realize that $\mathsf{E}[N] = \lambda_f\,\mathsf{E}[S_0]$. Then use [4.4.2].

**h.4.4.9.** Use (1.2.4) and [4.4.5].

**h.4.4.10.** Use Wald's equation, which we derived in [4.4.4].

**h.4.4.14.** Condition on $S_0$ and $N$.

**h.4.4.15.** Just realize that $\mathsf{E}[S] = \mathsf{E}[S_0]\,/\,A$, and use the above.

**h.5.1.1.** Observe that $Y(t) = \int_0^t \mathbb{1}_{L(s)>0}\,\mathrm{d}s$ is the total amount of time the server has been busy up to the time $t$. Then take $T_k = D_k$ as the epochs at which to inspect $Y(t)$, and realize that $Y(D_k) = \sum_{i=1}^k S_i$. Use (3.3.3), i.e., rate stability.

**h.5.1.2.** For the direction of the inequalities (5.1.2), observe that $t$ can lie half way a service interval, and $A(t) \geq D(t)$. Divide by $t$ in (5.1.2). Then for the LHS, multiply by $A(t)/A(t)$ and take appropriate limits. Similar for the RHS but multiply by $D(t)/D(t)$.

**h.5.3.3.** Use that $\lambda \geq \delta$ always holds. Thus, when $\lambda \neq \delta$, it must be that $\lambda > \delta$. What are the consequences of this inequality; how does the queue length behave as a function of time?

**h.5.3.4.** Check that the conditions of the renewal reward theorem are satisfied in the above proof of (5.3.1b). Then define

$$Y(t) := A(n,t) = \sum_{k=1}^{A(t)} \mathbb{1}_{L(A_k-)=n}$$
$$X_k := Y(A_k) - Y(A_{k-1}) = A(n,A_k) - A(n,A_{k-1}) = \mathbb{1}_{L(A_k-)=n}.$$

**h.5.** See https://archive.org/stream/PuzzleMath-English-GeorgeGamov/puzzlemath_djvu.txt.

**h.5.4.2.** Recall that $\rho := \lim_{t\to\infty} t^{-1}\int_0^t L_s(s)\,\mathrm{d}s$ for the $G/G/1$ queue.

**h.5.4.5.** Substitute the definition of $L(s)$ in the LHS, then reverse the integral and summation.

**h.6.1.1.** First show that $M_L(s) = (1-\rho)\sum_n e^{sn}\rho^n$, then use (1.2.3d). Similarly, $\mathsf{P}\{L \geq n\} = \sum_{k\geq n} p(k)$.

**h.6.1.5.** Fill in $c = 1$. Realize that this is a check on the formulas.

**h.6.1.6.** Realize that the $M/M/c/c$ queue is similar to the $M/M/c$ queue. However, there cannot be more than $c$ jobs in the system.

**h.6.1.7.** Use that for any $x$, $x^n/n! \to 0$ as $n \to \infty$.

**h.6.1.10.** Use that $\sum_{i=0}^{n} x^i = (1-x^{n+1})/(1-x)$. BTW, is it necessary for this expression to be true that $|x| < 1$? What should you require for $|x|$ when you want to take the limit $n \to \infty$?

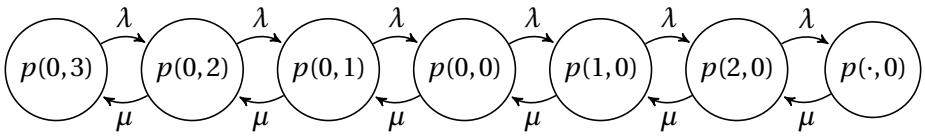
**h.6.2.1.** $\mathsf{E}[Q]$ follows right away from an application of Little's law. For the other quantities we need to find $\mathsf{E}[S]$. Use the expression for $\mathsf{E}[W(M/M/1)]$ to solve for $\rho$. Then, since $\lambda$ is known, $\mathsf{E}[S]$ follows.

**h.6.2.3.** This is an $M/M/1/3$ queue; there is room for 1 customer in service and two in queue.

**h.6.2.4.** This is a queue with balking.

**h.6.2.5.** We are dealing with an $M/M/c$ queue. And, what is the implication of the remark about the compensation rate?

**h.6.2.6.** Let $p_{ij}$ be the fraction of time that the system contains $i$ riders and $j$ taxi cabs. When a group arrives, they take a taxi, so the number of taxis decreases by one. If there are no taxis, the group has to wait. When a new taxi arrives, the number of groups is reduced by one, and so on, until there are 3 taxis waiting and no groups of people. Thus group arrivals acts as job arrivals, and taxi arrivals as services. Here is an overview of the transitions where $\mu$ is the rate at which cabs arrive, and $\lambda$ is the arrival rate of parties of riders.




Now map this system to an $M/M/1$ queue.

**h.6.3.2.** Use (6.3.2) and that $\mathsf{V}[B] \geq 0$.

**h.6.3.3.** $\mathsf{P}\{B = k\} = q^{k-1}p$ with $q = 1-p$. Use generating functions to compute $\mathsf{E}[B]$ and $\mathsf{E}\left[B^2\right]$.

**h.6.3.4.** Suppose the size of the $k$th batch is $B_k = 2$. Writing $S_1$ for the service time of the first item of this batch and $S_2$ for the service of the second item, $X_k = 2S_1 + 1S_2$. Now consider general batch sizes $B$, but realize that $B$ is a random number, hence, the sum is over a random number of random variables.

**h.6.3.7.** Use [1.2.6] and [1.2.7].

**h.6.4.2.** Use that $\mathsf{V}[S] = 0$ for the $M/D/1$ queue.

**h.6.4.4.** The rate of accepted jobs is $\lambda\pi(0)$, hence $\rho = \lambda\pi(0)\,\mathsf{E}[S]$. But also $\rho = 1-\pi(0)$. Now solve for $\pi(0)$.

**h.6.4.5.** Inter-arrivals times are memoryless for the $M/G/1$; for $\mathsf{E}[U]$ use the renewal-reward theorem to see that $\rho = \mathsf{E}[U]/(\mathsf{E}[I] + \mathsf{E}[U])$ for the $G/G/1$.

**h.6.4.7.** Use [6.3.6].

**h.6.4.8.** Use the PASTA property.

**h.6.4.9.** Realize that when estimating $\mathsf{E}[S_r]$ along a sample path, $S_r = 0$ for jobs that arrive at an empty system.

**h.6.4.10.** Apply PASTA and (5.3.4).

**h.6.5.2.** Show first that

$$\mu\,\mathsf{E}[L] = \mu \sum_{n=0}^{\infty} n\pi(n) = \lambda\frac{\mathsf{E}\left[B^2\right]}{2} + \lambda\,\mathsf{E}[B]\,\mathsf{E}[L] + \lambda\frac{\mathsf{E}[B]}{2}.$$

**h.6.6.2.** Do [6.6.1] first.

**h.6.6.3.** Use [2.4.1] to simplify the integral, or use [2.2.13] and [2.4.14].

**h.6.6.4.** Use [6.6.3].

**h.6.6.5.** Solve [6.6.3] and [6.6.4] first. Use shorthands: $\alpha = \lambda/(\lambda+\mu) \implies \mu/(\lambda+\mu) = 1-\alpha \implies \alpha/(1-\alpha) = \lambda/\mu = \rho$.

**h.7.1.1.** Fill $V(q) = aq^2 + bq$ into (7.1.2). Match the coefficients of $q^2$, $q$.

**h.7.1.2.** What is the cost of one cycle? What is the duration of one cycle?

**h.7.1.4.** Use (1.2.3d).

**h.7.2.2.** Use the argumentation that leads to (5.1.2).

**h.7.2.3.**

$$\begin{aligned} aq^2 &= aq^2, \\ bq &= 2aq\,\mathsf{E}[Y] - 2aq + bq + hq\,\mathsf{E}[S], \\ 0 &= a\,\mathsf{E}\left[Y^2\right] - 2a\,\mathsf{E}[Y] + a + b\,\mathsf{E}[Y] - b + \frac{1}{2}\lambda h\,\mathsf{E}\left[S^2\right]. \end{aligned}$$

**h.7.2.4.** Use [4.4.5] to see that $\mathsf{E}\left[Y^2\right] = \lambda^2\,\mathsf{E}\left[S^2\right] + \lambda\,\mathsf{E}[S]$.

**h.7.2.6.** Take $K = 0$ and $N = 1$, and realize that the LHS is $h\,\mathsf{E}[L]$.

**h.7.3.1.** Conditioning on the server being idle or busy at a departure leads to $f_D(t) = f_{X+S}(t)\,\mathsf{P}\{\text{server is idle}\} + f_S(t)\,\mathsf{P}\{\text{server is busy}\}$. Next, use [6.6.2].

**h.7.3.2.** What does it mean when $P_{i0} > 0$?

**h.7.3.4.** Realize that an arrival is required to leave state $(0,0)$, and a departure at the second queue is necessary to enter state $(0,0)$.

**h.7.4.1.** After visiting $M$ stations, any job must have left the tandem network.

**h.7.4.2.** Define $Q = P^M$. As $P$ is transient, $Q_{ij} < 1$ for all $i, j$. Then show that $Q^n \to 0$.

# 9

## SOLUTIONS

**s.1.2.1.** Take $f(x) = h^\alpha$. Then,

$$\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h^\alpha}{h} = \lim_{h\to 0} h^{\alpha-1}.$$

**s.1.2.2.** In fact (1) is trivial: $|f(h)| \le |f(h)/h|$ when $|h| < 1$. But it is given that the RHS goes to zero. For (2) and (3):

$$\lim_{h\to 0} \frac{cf(h)}{h} = c \lim_{h\to 0} \frac{f(h)}{h} = 0, \text{ as } f = o(h),$$
$$\lim_{h\to 0} \frac{f(h)+g(h)}{h} = \lim_{h\to 0} \frac{f(h)}{h} + \lim_{h\to 0} \frac{g(h)}{h} = 0.$$

For (4), use the Algebraic Limit Theorem and multiply with $h/h$,

$$\lim_{h\to 0} \frac{f(h)g(h)}{h} = \lim_{h\to 0} h \frac{f(h)}{h} \frac{g(h)}{h} = \lim_{h\to 0} h \lim_{h\to 0} \frac{f(h)}{h} \lim_{h\to 0} \frac{g(h)}{h} = 0$$

**s.1.2.3.** When $|x| \ll 1$, the terms with $n \ge 2$ in (1.2.3c) are $x^n = o(x)$. Then applying $x^n + x^m = o(x)$ to the Taylor series gives the result.

**s.1.2.4.** To see (1.2.4a), note first that $X\,\mathbb{1}_{X=n} = n\,\mathbb{1}_{X=n}$ because $X = n$ when $\mathbb{1}_{X=n} = 1$, and second that $\sum_{n=0}^{\infty} \mathbb{1}_{X=n} = 1$, since $X$ takes one of the values in $\mathbb{N}$, and events $\{X = n\}$ and $\{X = m\}$ are non-overlapping when $n \ne m$.

**s.1.2.5.** This is just rewriting the definition:

$$G(k) = \mathsf{P}\{X > k\} = \sum_{m=k+1}^{\infty} \mathsf{P}\{X = m\} = \sum_{m=k+1}^{\infty} f(m) = \sum_{m=0}^{\infty} \mathbb{1}_{m>k} f(m).$$

**s.1.2.6.** Observe first that $\sum_{k=0}^{\infty} \mathbb{1}_{m>k} = m$, since $\mathbb{1}_{m>k} = 1$ if $k < m$ and $\mathbb{1}_{m>k} = 0$ if $k \ge m$. With this,

$$\begin{aligned}\sum_{k=0}^{\infty} G(k) &= \sum_{k=0}^{\infty} \mathsf{P}\{X > k\} = \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \mathsf{P}\{X = m\} \\ &= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathbb{1}_{m>k}\, \mathsf{P}\{X = m\} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{1}_{m>k}\, \mathsf{P}\{X = m\} \\ &= \sum_{m=0}^{\infty} m\, \mathsf{P}\{X = m\} = \mathsf{E}[X].\end{aligned}$$

There are some technical details with respect to the interchange of the summations. Since the summands are positive, this is allowed. For further detail, see Capiński and Zastawniak [2003].

**s.1.2.7.**

$$\begin{aligned}\sum_{i=0}^{\infty} iG(i) &= \sum_{i=0}^{\infty} i \sum_{n=i+1}^{\infty} \mathsf{P}\{X=n\} = \sum_{n=0}^{\infty} \mathsf{P}\{X=n\} \sum_{i=0}^{\infty} i\, \mathbb{1}_{n\geq i+1} \\ &= \sum_{n=0}^{\infty} \mathsf{P}\{X=n\} \sum_{i=0}^{n-1} i = \sum_{n=0}^{\infty} \mathsf{P}\{X=n\} \frac{(n-1)n}{2} \\ &= \sum_{n=0}^{\infty} \frac{n^2}{2} \mathsf{P}\{X=n\} - \frac{\mathsf{E}[X]}{2} = \frac{\mathsf{E}\left[X^2\right]}{2} - \frac{\mathsf{E}[X]}{2}.\end{aligned}$$

**s.1.2.8.**

$$\begin{aligned}\mathsf{E}[X] &= \int_0^{\infty} x\,\mathrm{d}F(x) = \int_0^{\infty} \int_0^{x} \mathrm{d}y\,\mathrm{d}F(x) \\ &= \int_0^{\infty} \int_0^{\infty} \mathbb{1}_{y\leq x}\,\mathrm{d}y\,\mathrm{d}F(x) = \int_0^{\infty} \int_0^{\infty} \mathbb{1}_{y\leq x}\,\mathrm{d}F(x)\,\mathrm{d}y \\ &= \int_0^{\infty} \int_y^{\infty} \mathrm{d}F(x)\,\mathrm{d}y = \int_0^{\infty} G(y)\,\mathrm{d}y.\end{aligned}$$

**s.1.2.9.**

$$\begin{aligned}\int_0^{\infty} yG(y)\,\mathrm{d}y &= \int_0^{\infty} y \int_y^{\infty} f(x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\infty} y \int_0^{\infty} \mathbb{1}_{y\leq x} f(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^{\infty} f(x) \int_0^{\infty} y\,\mathbb{1}_{y\leq x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\infty} f(x) \int_0^{x} y\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^{\infty} f(x) \frac{x^2}{2}\,\mathrm{d}x = \frac{\mathsf{E}\left[X^2\right]}{2}.\end{aligned}$$

**s.1.2.10.** Use integration by parts.

$$\int_0^{\infty} yG(y)\,\mathrm{d}y = \frac{y^2}{2}G(y)\Big|_0^{\infty} - \int_0^{\infty} \frac{y^2}{2} g(y)\,\mathrm{d}y = \int_0^{\infty} \frac{y^2}{2} f(y)\,\mathrm{d}y = \frac{\mathsf{E}\left[X^2\right]}{2},$$

since $g(y) = G'(y) = -F'(y) = -f(y)$. Note that we used $\frac{y^2}{2}G(y)\Big|_0^{\infty} = 0 - 0 = 0$, which follows from our assumption that $\mathsf{E}\left[X^2\right]$ exists, implying that $\lim_{y\to\infty} y^2 G(y) = 0$.

**s.1.2.11.** $M_X(0) = \mathsf{E}\left[e^{0X}\right] = \mathsf{E}\left[e^0\right] = \mathsf{E}[1] = 1$.

**s.1.2.12.** The probability that the first child to guess wins is 1/3. What is the probability for child number two? Well, for him/her to win, it is necessary that child one does not win and that child two guesses the right number of the remaining numbers. Assume, without loss of generality that child 1 chooses 3 and that this is not the right number. Then

$$\begin{aligned}&\mathsf{P}\{\text{Child 2 wins}\} \\ &= \mathsf{P}\left\{\text{Child 2 guesses the right number and child 1 does not win}\right\} \\ &= \mathsf{P}\left\{\text{Child 2 guesses the right number} \,\middle|\, \text{child 1 does not win}\right\} \cdot \mathsf{P}\{\text{Child 1 does not win}\} \\ &= \mathsf{P}\left\{\text{Child 2 makes the right guess in the set } \{1,2\}\right\} \cdot \frac{2}{3} \\ &= \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.\end{aligned}$$

Similar conditional reasoning gives that child 3 wins with probability 1/3.

**s.2.1.1.** $d_1 = 7$, $L_1 = 8-7+5 = 6$, $d_2 = 6$, $L_2 = 6-6+4 = 4$, $d_3 = 4$, $L_3 = 4-4+9 = 9$.

**s.2.1.2.**

$$\begin{aligned} d_k &= \min\{L_{k-1} + a_k, c_k\}, \\ L_k &= L_{k-1} + a_k - d_k = L_{k-1} + a_k - \min\{L_{k-1} + a_k, c_k\} \\ &= \max\{L_{k-1} + a_k - c_k, 0\}. \end{aligned}$$

When jobs cannot be served in the period in which they arrive, then $L_k = [L_{k-1} - c_k]^+ + a_k$.

**s.2.1.3.** Here is the python code.

Python Code

```
>>> a = [0, 10, 3, 6]
>>> c = [0, 5, 5, 5]
>>> L = [0] * len(a)
>>> d = [0] * len(a)
>>> l = [0] * len(a)  # loss

>>> K = 8

>>> for k in range(1, len(a)):
...     d[k] = min(L[k - 1], c[k])
...     Lp = L[k - 1] + a[k] - d[k]  #  without loss
...     L[k] = min(Lp, K)  #  chop off at K
...     l[k] = Lp - L[k]  #  lost
...

>>> print(L)
[0, 8, 6, 7]
>>> print(l)
[0, 2, 0, 0]
>>> print(sum(l) / sum(a))  # fraction lost.
0.10526315789473684
```

**s.2.1.4.** Here is the code.

Python Code

```
>>> import numpy as np

>>> np.random.seed(3)

>>> a = [0, 4, 8, 2, 1]
>>> c = [3] * len(a)
>>> L = [0] * len(a)
>>> d = [0] * len(a)
>>> p = 0.2

>>> L[0] = 2

>>> for k in range(1, len(a)):
```

```
...     produced = min(L[k - 1], c[k])
...     faulty = np.random.binomial(produced, p)
...     d[k] = produced - faulty
...     L[k] = L[k - 1] + a[k] - d[k]
...
>>> print(d)
[0, 2, 2, 3, 3]
>>> print(L)
[2, 4, 10, 9, 7]
```

Observe that faulty items do not leave the system.

An intesting challenge: can you use these recursions to *prove* that the long-run average service capacity $n^{-1}\sum_{i=1}^{n} c_i$ must be larger than $\gamma/(1-p)$, where $\gamma = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} a_k$ is the arrival rate of new jobs?

**s.2.1.5.** In code:

Python

```
a = [0, 4, 8, 2, 1]
c = [3] * len(a)
L_R = [0] * len(a)  # repair jobs
d_R = [0] * len(a)  # departing repair jobs
L_N = [0] * len(a)  # new jobs
d_N = [0] * len(a)  # departing new jobs
p = 0.2

L_R[0] = 2
L_N[0] = 8

for k in range(1, len(a)):
    l = L_R[k - 1]
    if l % 2 == 1:
        l -= 1
    d_R[k] = min(l, 2 * c[k])
    c_N = c[k] - d_R[k] / 2  # capacity left for new jobs
    d_N[k] = min(L_N[k - 1], c_N)
    a_R = int(p * d_N[k] + 0.5)  # rounding
    a_N = a[k]
    L_R[k] = L_R[k - 1] + a_R - d_R[k]
    L_N[k] = L_N[k - 1] + a_N - d_N[k]

print(L_R)
print(L_N)
```

I need a trick to get around the division by 2 in line 17 (because of $L_R[k]$ is odd I might end up with 0.5). For this, I check whether $L_R[k]$ is odd or not. If so, it must be at least be 1, and then I can subtract 1 to make $l$ even. This $l$ I can use to compute the number of departures.

Here is a general observation: the number of ways to organize the repairs is countless. Do we serve them with priority or not, do we bin them and make new items instead, do we do the repairs on another, separate, station? Such differences need to be reflected in the simulation model.

**s.2.1.6.** $\sum_{k=1}^{T}\left(\beta c_k + hL_k\right)$.

**s.2.1.7.** Take $c_k = c\,\mathbb{1}_{L_{k-1}>=N}$.

**s.2.1.8.** $I_0 = 0$, $d_k = \min\{L_{k-1} + a_k, I_k c_k\}$, from which $L_k$ and $I_{k+1}$ follow, and so on. Next, observing that the machine switches on in period $k$ iff $I_{k-1} = 0$ and $I_k = 1$, the cost is $\sum_{k=1}^{T}\left(\beta I_k + hL_k + K[I_k - I_{k-1}]^+\right)$.

Interestingly, if arrivals cannot be served in the period in which they arrive, it might be that the system is never empty. For instance, if $a_k \sim$ Unif4, 10, then $L_k \geq a_k \geq 4$ for all $k$.

**s.2.1.9.** Define $\tau$ now as $\min\{k : L_k = a_k\}$.

**s.2.1.10.**

$$J_k^- = \min\left\{m : \sum_{i=k}^{k+m} c_i > L_{k-1}\right\}, \qquad J_k^+ = \min\left\{m : \sum_{i=k}^{k+m} c_i \geq L_{k-1} + a_k\right\}.$$

With a while loop serves well to compute these bounds Supposing that we already computed $\{L_k\}$, then for given $k$,

Python Code

```
J_min, tot_service = 0, 0
while tot_service <= L[k - 1]:
    tot_service += c[k + J_min]
    J_min += 1

J_min -= 1 #  We ran one too far
```

**s.2.1.11.**

$$d_k^1 = \min\{L_{k-1}^1, c_k\}, \qquad c_k^2 = c_k - d_k^1, \qquad d_k^2 = \min\{L_{k-1}^2, c_k^2\}, \qquad L_k^i = L_{k-1}^i + a_k^i - d_k^i.$$

**s.2.1.12.** Let $c_k^i$ be the capacity allocated to queue $i$ in period $k$. The fair rule gives that

$$c_k^1 = \text{round}\frac{L_{k-1}^1}{L_{k-1}^1 + L_{k-1}^2}, \qquad c_k^2 = c_k - c_k^1,$$
$$d_k^i = \min\{L_{k-1}^i, c_k^i\}, \qquad L_k^i = L_{k-1}^i + a_k^i - d_k^i.$$

In general, rules to distribute capacity $c_k$ over the queues can be based on game-theoretic ideas, such as the principle of *equal division of the contested sum*, see the work of Aumann and Maschler.

**s.2.1.13.** Queue 2 minimally needs $c_k^2 = \min\{L_{k-1}^2, r^2\}$, and with this,

$$d_k^1 = \min\{L_{k-1}^1, c_k - c_k^2\}, \qquad d_k^2 = \min\{L_{k-1}^2, c_k - d_k^1\}.$$

**s.2.1.14.** Queue 1 uses all capacity except $r^2$, queue 2 gets the left-over:

$$\begin{aligned} d_k^1 &= \min\{L_{k-1}^1, c_k - r^2\}, \\ c_k^2 &= \min\{c_k - r^1, c_k - d^1 k\} = c_k - \max\{r^1, d_k^1\}, \\ d_k^2 &= \min\{L_{k-1}^2, c_k^2\}. \end{aligned}$$

**s.2.1.15.** Let $a_k^1$ be the external arrivals at station $A$, and the departures of station 1 are the arrivals at station 2: $a_k^2 = d_k^1$. Thus,

$$d_k^i = \min\{L_{k-1}^i, c_k^i\}, \qquad L_k^i = L_{k-1}^i - d_k^i + a_k^i.$$

Python

```
a1 = [0, 2, 3, 8, 0, 9]
c1 = [3] * len(a1)
c2 = [2] * len(a1)
L1 = [0] * len(a1)
L2 = [0] * len(a1)

for k in range(1, len(a1)):
    d1 = min(L1[k-1], c1[k])
    L1[k] = L1[k - 1] + a1[k] - d1
    d2 = min(L2[k-1], c2[k])
    L2[k] = L2[k - 1] + d1 - d2
```

**s.2.1.16.** Using [2.1.14] with $d_k^1 = \min\{L_{k-1}^1, c_k^1, M - L_{k-1}^2\}$ is nearly correct. But, what if $L_{k-1}^2 > M$ for the first few periods? Therefore, take instead $d_k^1 = \min\{L_{k-1}^1, c_k^1, [M - L_{k-1}^2]^+\}$.

Python

```
a1 = [0, 2, 3, 8, 0, 9]
c1 = [3] * len(a1)
c2 = [2] * len(a1)
L1 = [0] * len(a1)
L2 = [0] * len(a1)

M2 = 10

if L2[0] > M2:
    print("The starting value of L2 is too large")
    exit(0)

for k in range(1, len(a1)):
    d1 = min(L1[k], c1[k], M2 - L2[k - 1])
    L1[k] = L1[k - 1] + a1[k] - d1
    d2 = min(L2[k], c2[k])
    L2[k] = L2[k - 1] + d1 - d2
```

**s.2.1.17.** Take $a_k^C = d_k^A + d_k^B$ and use (2.1.1) for each station.

**s.2.1.18.** At the mixing machine $d_k = \min\{L_k, c_k\}$. Therefore, in this very simple model, $a_k^A = d_k\lambda^A/(\lambda^A+\lambda^B)$. Now use [2.1.15].

**s.2.1.19.** With $d_k^i = \min\{L_{k-1}^i, c_k^i \mathbb{1}_{p_k=i}\}$ we can specify the evolution of queue $i$. So it remains to deal with $p_{k+1}$. For this, we use the following 'truth table'. This can be implemented in code to specify what $p_{k+1}$ has to become.

| $\mathbb{1}_{L_k^1>0}$ | $\mathbb{1}_{L_k^2>0}$ | $p_k$ | $p_{k+1}$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 0 | 2 | 2 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 0 | 2 | 0 (switch over time) |
| 0 | 1 | 0 | 2 |
| 0 | 1 | 1 | 0 (switch over time) |
| 0 | 1 | 2 | 2 |
| 1 | 1 | 0 | 1 (to break ties) |
| 1 | 1 | 1 | 1 |
| 1 | 1 | 2 | 2 |

Here is an important warning: in such tables it is very, very easy to miss one (or more) cases. Here, each of the indicators has 2 possible values, and $p_k$ has 3, so there are $2\times 2\times 3 = 12$ cases, all of which are covered in the table.

**s.2.2.1.**

$$\mathsf{E}[N_n(t)] = \mathsf{E}\left[\sum_{i=1}^n B_i\right] = \sum_{i=1}^n \mathsf{E}[B_i] = n\mathsf{E}[B_i] = np.$$

**s.2.2.2.**

$$\begin{aligned}\binom{n}{k}\left(\frac{\lambda t}{n}\right)^k\left(1-\frac{\lambda t}{n}\right)^{n-k} &= \frac{n!}{k!(n-k)!}\left(\frac{\lambda t}{n}\frac{n}{n-\lambda t}\right)^k\left(1-\frac{\lambda t}{n}\right)^n\\ &= \frac{(\lambda t)^k}{k!}\left(\frac{n}{n-\lambda t}\right)^k\frac{n!}{n^k(n-k)!}\left(1-\frac{\lambda t}{n}\right)^n\\ &= \frac{(\lambda t)^k}{k!}\left(\frac{n}{n-\lambda t}\right)^k\frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}\left(1-\frac{\lambda t}{n}\right)^n.\end{aligned}$$

Observe now that, as $\lambda t$ is finite, $n/(n-\lambda t)\to 1$ as $n\to\infty$. Also for any finite $k$, $(n-k)/n\to 1$. Finally, use (1.2.3b) to see that $\left(1-\frac{\lambda t}{n}\right)^n \to e^{-\lambda t}$.

**s.2.2.3.** $N_n(t)$ is a binomially distributed random variable with parameters $n$ and $p$. The maximum value of $N_n(t)$ is $n$. The random variable $N(t)$ models the number of arrivals that can occur during $[0,t]$. As such it is not necessarily bounded by $n$. Thus, $N_n(t)$ and $N(t)$ cannot represent the same random variable.

**s.2.2.4.** Write $N(s,t]$ for the number of arrivals in the interval $(s,t]$. First we make a few simple observations: $N(t+\Delta t] = N(t) + N(t,t+\Delta t]$, hence

$$\mathbb{1}_{N(t+\Delta t)=n,N(t)=n} = \mathbb{1}_{N(t)+N(t,t+\Delta t]=n,N(t)=n} = \mathbb{1}_{N(t,t+\Delta t]=0,N(t)=n}.$$

Thus,

$$\begin{aligned}
\mathsf{P}\{N(t+\Delta t)=n \mid N(t)=n\} &= \frac{\mathsf{P}\{N(t+\Delta t)=n, N(t)=n\}}{\mathsf{P}\{N(t)=n\}} \\
&= \frac{\mathsf{P}\{N(t,t+\Delta t]=0, N(t)=n\}}{\mathsf{P}\{N(t)=n\}} \\
&= \frac{\mathsf{P}\{N(t,t+\Delta t]=0\}\,\mathsf{P}\{N(t)=n\}}{\mathsf{P}\{N(t)=n\}} \quad \text{(independence)} \\
&= \mathsf{P}\{N(t,t+\Delta t]=0\} = \mathsf{P}\{N(0,\Delta t]=0\} \quad \text{(stationarity)} \\
&= e^{-\lambda\Delta t}(\lambda\Delta t)^0/0! = e^{-\lambda\Delta t} = 1-\lambda\Delta t + o(\Delta t).
\end{aligned}$$

**s.2.2.5.**

$$\begin{aligned}
\mathsf{P}\{N(t+\Delta t)=n+1 \mid N(t)=n\} &= \frac{\mathsf{P}\{N(t+\Delta t)=n+1, N(t)=n\}}{PN(t)=n} \\
&= \mathsf{P}\{N(t,t+\Delta t]=1\} = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^1}{1!} \\
&= (1-\lambda\Delta t+o(\Delta t))\lambda\Delta t = \lambda\Delta t - \lambda^2\Delta t^2 + o(\Delta t) \\
&= \lambda\Delta t + o(\Delta t).
\end{aligned}$$

**s.2.2.6.**

$$\begin{aligned}
\mathsf{P}\{N(t+\Delta t)\geq n+2 \mid N(t)=n\} &= \mathsf{P}\{N(t,t+\Delta t]\geq 2\} \\
&= e^{-\lambda\Delta t}\sum_{i=2}^{\infty}\frac{(\lambda\Delta t)^i}{i!} = e^{-\lambda\Delta t}\left(\sum_{i=0}^{\infty}\frac{(\lambda\Delta t)^i}{i!} - \lambda\Delta t - 1\right) \\
&= e^{-\lambda\Delta t}(e^{\lambda\Delta t}-1-\lambda\Delta t) = 1-e^{-\lambda\Delta t}(1+\lambda\Delta t) \\
&= 1-(1-\lambda\Delta t+o(\Delta t))(1+\lambda\Delta t) = 1-(1-\lambda^2\Delta t^2+o(\Delta t)) \\
&= \lambda^2\Delta t^2 + o(\Delta t) = o(\Delta t),
\end{aligned}$$

where we expand

$$(1-\lambda\Delta t+o(\Delta t))(1+\lambda\Delta t) = 1+\lambda\Delta t-\lambda\Delta t-\lambda^2\Delta^2 t^2+o(\Delta t).$$

We can also use the results of the previous parts to see that

$$\begin{aligned}
\mathsf{P}\{N(t+\Delta t)\geq n+2 \mid N(t)=n\} &= \mathsf{P}\{N(t,t+\Delta t]\geq 2\} = 1-\mathsf{P}\{N(t,t+\Delta t]<2\} \\
&= 1-\mathsf{P}\{N(t,t+\Delta t]=0\}-\mathsf{P}\{N(t,t+\Delta t]=1\} \\
&= 1-(1-\lambda\Delta t+o(\Delta t))-(\lambda\Delta t+o(\Delta t)) \\
&= o(\Delta t).
\end{aligned}$$

**s.2.2.7.** When a random variable $N$ is Poisson distributed with parameter $\lambda t$,

$$\begin{aligned}
\mathsf{E}[N] &= \sum_{n=0}^{\infty} n e^{-\lambda t}\frac{(\lambda t)^n}{n!} = \sum_{n=1}^{\infty} n e^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t}\lambda t\sum_{n=1}^{\infty}\frac{(\lambda t)^{n-1}}{(n-1)!} \\
&= e^{-\lambda t}\lambda t\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!} = e^{-\lambda t}\lambda t e^{\lambda t} = \lambda t.
\end{aligned}$$

**s.2.2.8.**

$$\begin{aligned}\mathsf{E}\left[N^2\right] &= \sum_{n=0}^{\infty} n^2 e^{-\lambda t}\frac{(\lambda t)^n}{n!} = e^{-\lambda t}\sum_{n=1}^{\infty} n\frac{(\lambda t)^n}{(n-1)!} = e^{-\lambda t}\sum_{n=0}^{\infty}(n+1)\frac{(\lambda t)^{n+1}}{n!}\\ &= e^{-\lambda t}\lambda t\sum_{n=0}^{\infty} n\frac{(\lambda t)^n}{n!} + e^{-\lambda t}\lambda t\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!} = (\lambda t)^2 + \lambda t.\end{aligned}$$

**s.2.2.9.** $\mathsf{V}[N] = \mathsf{E}\left[N^2\right] - (\mathsf{E}[N])^2 = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t.$

**s.2.2.10.** From the hint,

$$\begin{aligned}\mathsf{P}\{N(0,s] = 1 \,|\, N(0,t] = 1\} &= \frac{\mathsf{P}\{N(0,s] = 1, N(0,t] = 1\}}{\mathsf{P}\{N(0,t] = 1\}} = \frac{\mathsf{P}\{N(0,s] = 1, N(s,t] = 0\}}{\mathsf{P}\{N(0,t] = 1\}}\\ &= \frac{\mathsf{P}\{N(0,s] = 1\}\,\mathsf{P}\{N(s,t] = 0\}}{\mathsf{P}\{N(0,t] = 1\}} = \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} = \frac{s}{t}.\end{aligned}$$

**s.2.2.11.**

$$SCV = \frac{\mathsf{V}[N(t)]}{(\mathsf{E}[N(t)])^2} = \frac{\lambda t}{(\lambda t)^2} = \frac{1}{\lambda t}.$$

The relative variability of the Poisson process goes down as $t \to \infty$.

**s.2.2.12.**

$$\begin{aligned}\mathsf{P}\left\{N_\lambda(t) + N_\mu(t) = n\right\} &= \sum_{i=0}^{n}\mathsf{P}\left\{N_\mu(t) = n-i\right\}\mathsf{P}\{N_\lambda(t) = i\}\\ &= \sum_{i=0}^{n}\frac{(\mu t)^{n-i}}{(n-i)!}\frac{(\lambda t)^i}{i!}e^{-(\mu+\lambda)t} = e^{-(\mu+\lambda)t}\sum_{i=0}^{n}\frac{(\mu t)^{n-i}}{(n-i)!}\frac{(\lambda t)^i}{i!}\\ &= e^{-(\mu+\lambda)t}\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}(\mu t)^{n-i}(\lambda t)^i \quad \text{(binomial formula)}\\ &= \frac{((\mu+\lambda)t)^n}{n!}e^{-(\mu+\lambda)t}.\end{aligned}$$

**s.2.2.13.** With the above:

$$\begin{aligned}\mathsf{P}\left\{N_\lambda(t) = 1 \,\middle|\, N_\lambda(t) + N_\mu(t) = 1\right\} &= \frac{\mathsf{P}\left\{N_\lambda(t) = 1, N_\lambda(t) + N_\mu(t) = 1\right\}}{\mathsf{P}\left\{N_\lambda(t) + N_\mu(t) = 1\right\}}\\ = \frac{\mathsf{P}\left\{N_\lambda(t) = 1, N_\mu(t) = 0\right\}}{\mathsf{P}\left\{N_{\lambda+\mu}(t) = 1\right\}} &= \frac{\mathsf{P}\{N_\lambda(t) = 1\}\,\mathsf{P}\left\{N_\mu(t) = 0\right\}}{\mathsf{P}\left\{N_{\lambda+\mu}(t) = 1\right\}}\\ = \frac{\lambda t\exp(-\lambda t)\exp(-\mu t)}{((\lambda+\mu)t)\exp(-(\lambda+\mu)t)} &= \frac{\lambda t\exp(-(\lambda+\mu)t)}{((\lambda+\mu)t)\exp(-(\lambda+\mu)t)} = \frac{\lambda}{\lambda+\mu}.\end{aligned}$$

Given that a customer arrived in $[0,t]$, the probability that it is of the first type is $\lambda/(\lambda+\mu)$.

**s.2.2.14.**

$$\begin{aligned}
\mathsf{P}\{N_1(t)=k\} &= \sum_{n=k}^{\infty} \mathsf{P}\{N_1(t)=k, N(t)=n\} = \sum_{n=k}^{\infty} \mathsf{P}\{N_1(t)=k \,|\, N(t)=n\}\, \mathsf{P}\{N(t)=n\} \\
&= \sum_{n=k}^{\infty} \mathsf{P}\{N_1(t)=k \,|\, N(t)=n\}\, e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad \text{by the hint} \\
&= e^{-\lambda t}\sum_{n=k}^{\infty} \frac{p^k(1-p)^{n-k}}{k!(n-k)!}(\lambda t)^n = e^{-\lambda t}\frac{(\lambda t p)^k}{k!}\sum_{n=k}^{\infty}\frac{(\lambda t(1-p))^{n-k}}{(n-k)!} \\
&= e^{-\lambda t}\frac{(\lambda t p)^k}{k!}\sum_{n=0}^{\infty}\frac{(\lambda t(1-p))^{n}}{n!} = e^{-\lambda t}\frac{(\lambda t p)^k}{k!}e^{\lambda t(1-p)} = e^{-\lambda t p}\frac{(\lambda t p)^k}{k!}.
\end{aligned}$$

**s.2.2.15.** Since $N(t)$ is Poisson distributed with parameter $\lambda t$,

$$\begin{aligned}
M_{N(t)}(s) &= \mathsf{E}\left[e^{sN(t)}\right] = \sum_{k=0}^{\infty} e^{sk}\,\mathsf{P}\{N(t)=k\} = \sum_{k=0}^{\infty} e^{sk}\frac{(\lambda t)^k}{k!}e^{-\lambda t} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(e^s\lambda t)^k}{k!} = \exp(-\lambda t + e^s\lambda t) = \exp(\lambda t(e^s-1)).
\end{aligned}$$

**s.2.2.16.** Using the expression for the moment-generating function of [2.2.15],

$$M'_{N(t)}(s) = \lambda t e^s \exp(\lambda t(e^s-1)).$$

Hence $\mathsf{E}[N(t)] = M'_{N(t)}(0) = \lambda t$. Next, $M''_{N(t)}(s) = (\lambda t e^s + (\lambda t e^s)^2)\exp(\lambda t(e^s-1))$, hence $\mathsf{E}\left[(N(t))^2\right] = M''(0) = \lambda t + (\lambda t)^2$, and thus, $\mathsf{V}[N(t)] = \mathsf{E}\left[(N(t))^2\right] - (\mathsf{E}[N(t)])^2 = \lambda t + (\lambda t)^2 - (\lambda t)^2 = \lambda t$.

**s.2.2.17.** Consider $Y = \sum_{i=1}^{N} Z_i$. Suppose that $N = n$, so that $n$ arrivals occurred. Then we throw $n$ independent coins with success probability $p$. It is clear that $Y$ is indeed a thinned Poisson random variable.

Model the coins as a generic Bernoulli distributed random variable $Z$. We first need

$$\mathsf{E}\left[e^{sZ}\right] = e^0\,\mathsf{P}\{Z=0\} + e^s\,\mathsf{P}\{Z=1\} = (1-p) + e^s p.$$

Suppose that $N = n$, then since the outcomes $Z_i$ of the coins are iid,

$$\mathsf{E}\left[e^{s\sum_{i=1}^{n} Z_i}\right] = \left(\mathsf{E}\left[e^{sZ}\right]\right)^n = \left(1 + p(e^s-1)\right)^n,$$

where we use (1.2.7d).

With (1.2.4a),

$$\begin{aligned}
\mathsf{E}\left[e^{sY}\right] &= \mathsf{E}\left[\sum_{n=0}^{\infty} e^{s\sum_{i=1}^{N} Z_i}\,\mathbb{1}_{N=n}\right] = \mathsf{E}\left[\sum_{n=0}^{\infty} e^{s\sum_{i=1}^{n} Z_i}\,\mathbb{1}_{N=n}\right] = \sum_{n=0}^{\infty}\mathsf{E}\left[e^{s\sum_{i=1}^{n} Z_i}\,\mathbb{1}_{N=n}\right] \\
&= \sum_{n=0}^{\infty}\mathsf{E}\left[e^{s\sum_{i=1}^{n} Z_i}\right]\mathsf{E}[\mathbb{1}_{N=n}], \quad \text{by independence of } Z_i \text{ and } N, \\
&= \sum_{n=0}^{\infty}\left(1+p(e^s-1)\right)^n \mathsf{P}\{N=n\} \\
&= \sum_{n=0}^{\infty}\left(1+p(e^s-1)\right)^n e^{-\lambda}\frac{\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\left(1+p(e^s-1)\right)^n\lambda^n}{n!} \\
&= e^{-\lambda}\exp(\lambda(1+p(e^s-1))) = \exp(\lambda p(e^s-1)).
\end{aligned}$$

**s.2.2.18.**

$$\begin{aligned} M_{N_\lambda(t)+N_\mu(t)}(s) &= M_{N_\lambda(t)}(s) \cdot M_{N_\mu(t)}(s) = \exp(\lambda t(e^s-1)) \cdot \exp(\mu t(e^s-1)) \\ &= \exp((\lambda+\mu)t(e^s-1)). \end{aligned}$$

**s.2.2.19.** Take $Y = \sum_{i=1}^n Z_i$ with $Z_i \sim B(p)$. Then,

$$M_Y(s) = \mathsf{E}\left[e^{s\sum_{i=1}^n Z_i}\right] = \left(\mathsf{E}\left[e^{sZ}\right]\right)^n = \left(1+p(e^s-1)\right)^n.$$

Recall that $p = \lambda t/n$. Then, with (1.2.3b),

$$\lim_{n\to\infty}\left(1+\frac{\lambda t}{n}(e^s-1)\right)^n = \exp(\lambda t(e^s-1)).$$

**s.2.3.1.** For every $A_k \le t$, we have that $\mathbb{1}_{A_k \le t} = 1$, and else the indicator is 0. Hence, in the summation we count the number of times $A_k \le t$.

**s.2.3.2.** Yes, it is true.

**s.2.3.3.** $A(t)$ is the number of arrivals during $[0,t]$. Suppose that $A(t) = n$. This $n$th job arrived at time $A_n$. Thus, $A_{A(t)}$ is the arrival time of the last job that arrived before or at time $t$. In a similar vein, $A_n$ is the arrival time of the $n$th job. Thus, the number of arrivals up to time $A_n$, i.e., $A(A_n)$, must be $n$.

**s.2.3.4.** Suppose $A_3 = 10$ and $A_4 = 20$. Take $t = 15$. Then $\min\{k : A_k \ge 15\} = 4$ since $A_3 < t = 15 < A_4$. On the other hand $\max\{k : A_k \le t\} = 3$. And, indeed, at time $t = 15$, 3 jobs arrived, not 4. So defining $A(t)$ as $\min\{k : A_k \ge t\}$ is not OK. This example also shows that in general $A(t) \ne \min\{k : A_k > t\}$. So, neither definition is correct.

**s.2.3.5.** Let's feed it to the computer. Mind that in Python (just like in C, and so on), arrays start at index 0, not at index 1.

Python Code

```
>>> X = [0, 10, 5, 6]
>>> S = [0, 17, 20, 5]
>>> A = [0, 0, 0, 0]
>>> for i in range(1, len(X)):
...     A[i] = A[i - 1] + X[i]
...
>>> A
[0, 10, 15, 21]

>>> W = [0, 0, 0, 0]
>>> for i in range(1, len(X)):
...     W[i] = max(W[i - 1] + S[i - 1] - X[i], 0)
...
>>> W
[0, 0, 12, 26]

>>> J = [0, 0, 0, 0]
>>> for i in range(1, len(X)):
```

```
...     J[i] = W[i] + S[i]
...
>>> J
[0, 17, 32, 31]

>>> D = [0, 0, 0, 0]
>>> for i in range(1, len(X)):
...     D[i] = A[i] + W[i] + S[i]
...
>>> D
[0, 27, 47, 52]
```

---

**s.2.3.6.** $A_0 = 0$, $A_1 = 10$, $A_2 = 20$, and so on. Hence, $A_k = 10k$. $W_0 = 0$, $W_1 = \max\{0+0-10, 0\} = 0$. $W_2 = \max\{0+11-10, 0\} = 1$. $W_3 = \max\{1+11-10, 0\} = 2$. Hence, $W_k = k-1$ for $k \geq 1$. Thus, $J_k = k-1+11 = k+10$ for $k \geq 1$, and $D_k = 10k+k+10 = 11k+10$. Note that $W_k$ increases linearly as a function of $k$. All in all, $A(t) = \lfloor t/10 \rfloor$, and $D(t) = \lfloor (t-10)/11 \rfloor$.

**s.2.3.7.** First find the distribution of $Y_k := S_{k-1} - X_k$ so that we can write $W_k = [W_{k-1} + Y_k]^+$. Use independence of $\{S_k\}$ and $\{X_k\}$:

$$\mathsf{P}\{Y_k = -2\} = \mathsf{P}\{S_{k-1} - X_k = -2\} = \mathsf{P}\{S_{k-1} = 1, X_k = 3\} = \mathsf{P}\{S_{k-1} = 1\}\,\mathsf{P}\{X_k = 3\} = \frac{1}{4}.$$

Dropping the dependence on $k$ for ease, we get

$$\begin{aligned}
\mathsf{P}\{Y = -2\} &= \mathsf{P}\{S - X = -2\} = \mathsf{P}\{S = 1, X = 3\} = \mathsf{P}\{S = 1\}\,\mathsf{P}\{X = 3\} = \frac{1}{4},\\
\mathsf{P}\{Y = -1\} &= \mathsf{P}\{S = 2\}\,\mathsf{P}\{X = 3\} = \frac{1}{4},\\
\mathsf{P}\{Y = 0\} &= \mathsf{P}\{S = 1\}\,\mathsf{P}\{X = 1\} = \frac{1}{4},\\
\mathsf{P}\{Y = 1\} &= \mathsf{P}\{S = 2\}\,\mathsf{P}\{X = 1\} = \frac{1}{4}.
\end{aligned}$$

With this

$$\begin{aligned}
\mathsf{P}\{W_1 = 1\} &= \mathsf{P}\{W_0 + Y = 1\} = \mathsf{P}\{3 + Y = 1\} = \mathsf{P}\{Y = -2\} = \frac{1}{4},\\
\mathsf{P}\{W_1 = 2\} &= \mathsf{P}\{3 + Y = 2\} = \mathsf{P}\{Y = -1\} = \frac{1}{4},\\
\mathsf{P}\{W_1 = 3\} &= \mathsf{P}\{3 + Y = 3\} = \mathsf{P}\{Y = 0\} = \frac{1}{4},\\
\mathsf{P}\{W_1 = 4\} &= \mathsf{P}\{3 + Y = 4\} = \mathsf{P}\{Y = 1\} = \frac{1}{4}.
\end{aligned}$$

And, then

$$\begin{aligned}
\mathsf{P}\{W_2 = 1\} = \mathsf{P}\{W_1 + Y = 1\} &= \sum_{i=1}^{4} \mathsf{P}\{W_1 + Y = 1 \mid W_1 = i\}\,\mathsf{P}\{W_1 = i\}\\
&= \sum_{i=1}^{4} \mathsf{P}\{i + Y = 1 \mid W_1 = i\}\,\frac{1}{4} = \sum_{i=1}^{4} \mathsf{P}\{Y = 1 - i \mid W_1 = i\}\,\frac{1}{4}\\
&= \frac{1}{4}\sum_{i=1}^{4} \mathsf{P}\{Y = 1 - i\} = \frac{1}{4}(\mathsf{P}\{Y = 0\} + \mathsf{P}\{Y = -1\} + \mathsf{P}\{Y = -2\}) = \frac{3}{16}.
\end{aligned}$$

**s.2.3.8.** Of course, the service of job $k$ cannot start before it arrives. Hence, it cannot leave before $A_k + S_k$. Therefore it must be that $D_k \geq A_k + S_k$. But the service of job $k$ can also not start before the previous job, i.e. job $k-1$, left the server. Thus job $k$ cannot start before $D_{k-1}$. To clarify it somewhat further, define $S'_k$ as the earliest start of job $k$. Then it must be that $S'_k = \max\{A_k, D_{k-1}\}$—don't confuse the earliest start $S'_k$ and the service time $S_k$—and $D_k = S'_k + S_k$.

**s.2.3.9.** There is a funny way to do this. Recall from a previous exercise that if $A(t) = n$, then $A_n$ is the arrival time of the $n$th job. Thus, the function $A_{A(t)}$ provides us with arrival times as a function of $t$. When $t = A_{A(t)}$, i.e., when $t$ is the arrival time of the $A(t)$th job, we set $V(t) = V(A_{A(t)}) = W_{A(t)}$, i.e., the virtual waiting time at the arrival time $t = A_{A(t)}$ is equal to the waiting time of the $A(t)$th job. Between arrival moments, the virtual waiting time decreases with slope 1, until it hits 0. Thus,

$$V(t) = [V(A_{A(t)}) - (t - A_{A(t)})]^+ = [W_{A(t)} + (A_{A(t)} - t)]^+.$$

The notation may be a bit confusing, but it is in fact very simple. Take some $t$, look back at the last arrival time before time $t$, which is written as $A_{A(t)}$. (In computer code these times are easy to find.) Then draw a line with slope $-1$ from the waiting time that the last arrival saw.

**s.2.3.10.** Recall that $A(t) = \lfloor t/10 \rfloor$, and $D(t) = \lfloor (t-10)/11 \rfloor$. Hence, since $A_k = 10k$ so that $A_k- = 10k-$,

$$L(A_k-) = k - 1 - D(A_k-) = k - 1 - D(10k-) = k - 1 - \left\lfloor \frac{(10k-) - 10}{11} \right\rfloor.$$

The computation is a bit tricky since sometimes arrivals and departures coincide. (Consider for instance $t = 120$.)

**s.2.3.11.** We defined $\tilde{A}(t)$ as the amount of people up to time $t$ that left the queue and moved to the server. It is then clear to see that the number of jobs in queue $Q(t)$ is equal to the number amount of jobs that have arrived, i.e., $A(t)$, minus the number of jobs that left the queue, i.e., $\tilde{A}(t)$. Using the same reasoning for $L_s(t)$, the second line also follows.

**s.2.3.12.** In this case, there are servers idling while there are still customers in queue. If such events occur, we say that the server is not work-conservative.

**s.2.3.13.**

$$L(t) = A(t) - D(t) = \sum_{k=1}^{\infty} \mathbb{1}_{A_k \leq t} - \sum_{k=1}^{\infty} \mathbb{1}_{D_k \leq t} = \sum_{k=1}^{\infty} [\mathbb{1}_{A_k \leq t} - \mathbb{1}_{D_k \leq t}].$$

Write for the moment $A = \mathbb{1}_{A_k \leq t}$ and $\bar{A} = 1 - A = \mathbb{1}_{A_k > t}$, and likewise for $D$. Now we can use Boolean algebra to see that $\mathbb{1}_{A_k \leq t} - \mathbb{1}_{D_k \leq t} = A - D = A(D + \bar{D}) - D = AD + A\bar{D} - D = A\bar{D} - D(1 - A) = A\bar{D} - D\bar{A}$. But $D\bar{A} = 0$ since $D\bar{A} = \mathbb{1}_{D_k \leq t}\mathbb{1}_{A_k > t} = \mathbb{1}_{D_k \leq t < A_k}$ which would mean that the arrival time $A_k$ of the $k$th job would be larger than its departure time $D_k$. As $A\bar{D} = \mathbb{1}_{A_k \leq t < D_k}$

$$L(t) = \sum_{k=1}^{\infty} [\mathbb{1}_{A_k \leq t} - \mathbb{1}_{D_k \leq t}] = \sum_{k=1}^{\infty} \mathbb{1}_{A_k \leq t < D_k}.$$

Boolean algebra is actually a really nice way to solve logical puzzles. If you are interested, you can find some examples on my homepage.

**s.2.3.14.** In a sense, the claim is evident, for, if the system contains a job when job $k$ arrives, it cannot be empty. But if it is not empty, then at least the last job that arrived before job $k$, i.e., job $k-1$, must still be in the system. That is, $D_{k-1} \geq A_k$. A more formal proof proceeds along the following lines. Using that $A(A_k) = k$ and $D(D_{k-1}) = k-1$,

$$L(A_k) > 0 \Leftrightarrow A(A_k) - D(A_k) > 0 \Leftrightarrow k - D(A_k) > 0 \Leftrightarrow k > D(A_k)$$
$$\Leftrightarrow k-1 \geq D(A_k) \Leftrightarrow D(D_{k-1}) \geq D(A_k) \Leftrightarrow D_{k-1} \geq A_k,$$

where the last relation follows from the fact that $D(t)$ is a counting process, hence monotone non-decreasing.

**s.2.3.15.** Let $L(A_k-)$, i.e., the number of jobs in the system as 'seen by' job $k$. It must be that $L(A_k-) = k-1-D(A_k)$. To see this, assume first that no job has departed when job $k$ arrives. Then job $k$ must see $k-1$ jobs in the system. In general, if at time $A_k$ the number of departures is $D(A_k)$, then the above relation for $L(A_k-)$ must hold. Applying this to job $k-1$ we get that $L(A_{k-1}-) = k-2-D(A_{k-1})$.

For the computation of $L(A_k-)$ we do not have to take the departures before $A_{k-1}$ into account as these have already been 'incorporated in' $L(A_{k-1}-)$. Therefore,

$$L(A_k-) = L(A_{k-1}-) + 1 - \sum_{i=k-1-L(A_{k-1}-)}^{k-1} \mathbb{1}_{D_i < A_k}.$$

Suppose $L(A_{k-1}) = 0$, i.e., job $k-1$ finds an empty system at its arrival and $D_{k-1} > A_k$, i.e., job $k-1$ is still in the system when job $k$ arrives. In this case, $L(A_k-) = 1$, which checks with the formula. Also, if $L(A_{k-1}-) = 0$ and $D_{k-1} < A_k$ then $L(A_k-) = 0$. This also checks with the formula.

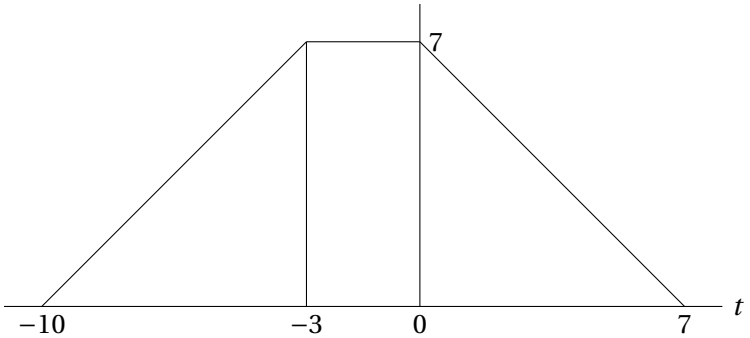
If you are interested in the number of jobs in queue, then take $[L_k - 1]^+$, for if $L_k \geq 1$ one job is in service (and the rest is in queue), while if $L_k = 0$ the system is empty, hence the queue is 0.

**s.2.3.16.** The reason to start at $k-1-L(A_{k-1}-)$ is that the number in the system as seen by job $k$ is $k-1-D(A_k)$ (not $k-2-D(A_k)$). Hence, the jobs with index from $k-1-L(A_{k-1}-), k-L(A_{k-1}-), \ldots, k-1$, could have left the system between the arrival of job $k-1$ and job $k$.

**s.2.3.17.** Note that for the integrand in the hint,

$$\mathbb{1}_{0\leq s\leq 7}\,\mathbb{1}_{0\leq s-t\leq 10} = \mathbb{1}_{0\leq s\leq 7}\,\mathbb{1}_{t\leq s\leq t+10} = \mathbb{1}_{\max\{0,t\}\leq s\leq \min\{7,10+t\}} \implies$$
$$f_T(t) = \frac{1}{70}\int \mathbb{1}_{\max\{0,t\}\leq s\leq \min\{7,t+10\}}\,\mathrm{d}s = \frac{1}{70}[\min\{7,t+10\} - \max\{0,t\}]^+.$$

Draw the graphs of $\min\{7, t+10\}$ and $\max\{0, t\}$ to see that this is the graph of $f_T$:



From this graph,

$$\mathsf{P}\{S - X \le t\} = F_T(t) = \int_{-\infty}^{t} f_T(u)\,\mathrm{d}u$$
$$= \begin{cases} 0, & t \le -10, \\ (t+10)^2/140, & -10 \le t \le -3 \\ 7^2/140 + 7(t+3)/70, & -3 \le t \le 0 \\ 7^2/140 + 7\cdot 3/70 + 7t/70 - t^2/140 & 0 \le t \le 7 \\ 1 & 7 \le t. \end{cases}$$

Solving the integral is, in principle, not hard, but it's easy to make a mistake. Let's use Wolfram Alpha to check this. Type this at the prompt:

```
\int_{0}^{10} \int_0^7 Boole[u<= s-x] ds dx.
```

**s.2.3.18.** Here is my solution in python.

Python Code

```
>>> import numpy as np

>>> m = 3
>>> N = 10

>>> one = np.ones(m, dtype=int)  #  vector with ones

>>> X = np.ones(N + 1, dtype=int)
>>> S = 5 * np.ones(N, dtype=int)
>>> w = np.zeros(m, dtype=int)
>>> W = J = A = D = 0

>>> for k in range(1, N):
...     s = w.argmin()  # server chosen
...     W = w[s]  # waiting time
...     J = W + S[k]  # sojourn time
...     A += X[k]  # arrival time
...     D = A + J  # departure time
...     print(k, S[k], W, w)
...     # now update w
...     w[s] += S[k]
...     w = np.maximum(0, w - X[k + 1] * one)
...
1 5 0 [0 0 0]
2 5 0 [4 0 0]
3 5 0 [3 4 0]
4 5 2 [2 3 4]
5 5 2 [6 2 3]
6 5 2 [5 6 2]
7 5 4 [4 5 6]
8 5 4 [8 4 5]
9 5 4 [7 8 4]
>>> #
```

**s.2.4.1.** Let $I_j = \int_0^\infty e^{-x} x^j \, \mathrm{d}x$. Then, $I_j = j!$, since by recursion,

$$I_j = \int_0^\infty e^{-x} x^j \, \mathrm{d}x = -e^{-x} x^j \Big|_0^\infty + j \int_0^\infty e^{-x} x^{j-1} \, \mathrm{d}x = j I_{j-1},$$
$$I_0 = \int_0^\infty e^{-x} \, \mathrm{d}x = 1.$$

By the change of variable $\lambda x \to y$, $\int_0^\infty e^{-\lambda x} x^j \, \mathrm{d}x = I_j / \lambda^{j+1} = j! / \lambda^{j+1}$. Hence, $\mathsf{E}[X] = \lambda 1! / \lambda^2$.

**s.2.4.2.** $\mathsf{E}\left[X^2\right] = \int_0^\infty t^2 \lambda e^{-\lambda t} \, \mathrm{d}t = \lambda 2! / \lambda^3$.

**s.2.4.3.** $\mathsf{V}[X] = \mathsf{E}\left[X^2\right] - (\mathsf{E}[X])^2 = \frac{2}{\lambda^2} - (\frac{1}{\lambda})^2 = \lambda^{-2}$.

**s.2.4.4.**

$$M_X(t) = \mathsf{E}\left[\exp(tX)\right] = \int_0^\infty e^{tx} f(x) \, \mathrm{d}x = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, \mathrm{d}x = \frac{\lambda}{\lambda - t}.$$

This last integral only converges when $\lambda - t > 0$.

**s.2.4.5.** $M_X'(t) = \lambda / (\lambda - t)^2 \implies M_X'(0) = 1/\lambda$, $M_X''(t) = 2\lambda / (\lambda - t)^3$

**s.2.4.6.** $C^2 = \mathsf{V}[X] / (\mathsf{E}[X])^2 = 1/\lambda^2 / (1/\lambda^2)$.

**s.2.4.7.** By the definition of conditional probability

$$\mathsf{P}\{X > t + h \,|\, X > t\} = \frac{\mathsf{P}\{X > t + h, X > t\}}{\mathsf{P}\{X > t\}} = \frac{\mathsf{P}\{X > t + h\}}{\mathsf{P}\{X > t\}}$$
$$= \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = \mathsf{P}\{X > h\}.$$

**s.2.4.8.** $\mathsf{P}\{X_1 > t\} = \mathsf{P}\{N(t) = 0\} = e^{-\lambda t}$.

**s.2.4.9.** $\mathsf{E}[A_i] = \mathsf{E}\left[\sum_{k=1}^i X_k\right] = i \, \mathsf{E}[X] = i/\lambda$, as $X_i$ iid.

**s.2.4.10.** Using the iid property of the $\{X_i\}$,

$$M_{A_i}(t) = \mathsf{E}\left[e^{tA_i}\right] = \mathsf{E}\left[\exp\left(t \sum_{k=1}^i X_k\right)\right] = \prod_{k=1}^i \mathsf{E}\left[e^{tX_k}\right] = \left(\frac{\lambda}{\lambda - t}\right)^i.$$

From a table of moment-generating functions it follows immediately that $A_i \sim \Gamma(i, \lambda)$, i.e., $A_i$ is Gamma distributed.

**s.2.4.11.**

$$\mathsf{E}[A_i] = \int_0^\infty t f_{A_i}(t) \, \mathrm{d}t = \int_0^\infty t \lambda e^{-\lambda t} \frac{(\lambda t)^{i-1}}{(i-1)!} \, \mathrm{d}t = \frac{1}{(i-1)!} \int_0^\infty e^{-\lambda t} (\lambda t)^i \, \mathrm{d}t = \frac{i!}{(i-1)! \lambda}.$$

In the last step we use the recursion of [2.4.1]. As a check,

$$\mathsf{E}[A_i] = \frac{\mathrm{d}}{\mathrm{d}t} M_{A_i}(t) \Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\lambda}{\lambda - t}\right)^i \Big|_{t=0} = i \left(\frac{\lambda}{\lambda - t}\right)^{i-1} \frac{\lambda}{(\lambda - t)^2} \Big|_{t=0}$$

**s.2.4.12.** With the density of $A_{k+1}$ and applying partial integration,

$$\begin{aligned}\mathsf{P}\{A_{k+1} \le t\} &= \lambda \int_0^t \frac{(\lambda s)^k}{k!} e^{-\lambda s}\,\mathrm{d}s = \lambda \frac{(\lambda s)^k}{k!}\frac{e^{-\lambda s}}{-\lambda}\Big|_0^t + \lambda \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!} e^{-\lambda s}\,\mathrm{d}s \\ &= -\frac{(\lambda t)^k}{k!} e^{-\lambda t} + \mathsf{P}\{A_k \le t\}.\end{aligned}$$

**s.2.4.13.** $\mathsf{P}\{Z > x\} = \mathsf{P}\{\min\{X, S\} > x\} = \mathsf{P}\{X > x, S > x\} = \mathsf{P}\{X > x\}\,\mathsf{P}\{S > x\} = e^{-\lambda x} e^{-\mu x}$, as $X$ and $S$ independent.

**s.2.4.14.**

$$\begin{aligned}\mathsf{P}\{X \le S\} &= \mathsf{E}[\mathbb{1}_{X\le S}] = \int_0^\infty \int_0^\infty \mathbb{1}_{x\le y} f_{X,S}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \lambda\mu \int_0^\infty \int_0^\infty \mathbb{1}_{x\le y} e^{-\lambda x} e^{-\mu y}\,\mathrm{d}y\,\mathrm{d}x = \lambda\mu \int_0^\infty e^{-\mu y} \int_0^y e^{-\lambda x}\,\mathrm{d}x\,\mathrm{d}y \\ &= \mu \int_0^\infty e^{-\mu y}(1 - e^{-\lambda y})\,\mathrm{d}y = \mu \int_0^\infty (e^{-\mu y} - e^{-(\lambda+\mu)y})\,\mathrm{d}y \\ &= 1 - \frac{\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}.\end{aligned}$$

With [2.2.13] it's immediate.

**s.3.2.1.** From the recursion and the hints,

$$\begin{aligned}L_k - Z_k &= \max\{\max\{L_{k-2} - Z_{k-2}, -Z_{k-1}\}, -Z_k\} \\ &= \max\{L_{k-2} - Z_{k-2}, -Z_{k-1}, -Z_k\} \\ &= \max\{L_0 - Z_0, -Z_1, \ldots, -Z_k\} \\ &= \max\{0, -Z_1, \ldots, -Z_k\} \\ &= -\min\{0, Z_1, \ldots, Z_k\}.\end{aligned}$$

**s.3.2.2.** With the hint,

$$\begin{aligned}\mathsf{P}_m\{Z_k = n\} &= \mathsf{P}\{m + N_{\lambda,k} - N_{\mu,k} = n\} = \mathsf{P}\{N_{\lambda,k} = n - m + N_{\mu,k}\} \\ &= \sum_{j=0}^\infty \mathsf{P}\{N_{\lambda,k} = n - m + j, N_{\mu,k} = j\} = \sum_{j=0}^\infty e^{-\lambda k} \frac{(\lambda k)^{n-m+j}}{(n-m+j)!} e^{-\mu k} \frac{(\mu k)^j}{j!} \\ &= e^{-(\lambda+\mu)k} (\lambda k)^{n-m} \sum_{j=0}^\infty \frac{(\lambda\mu k^2)^j}{j!(n-m+j)!}.\end{aligned}$$

**s.3.2.3.** Under conditions you can find on the internet, $(W_n - \mu_n)/\sigma_n \to \mathrm{Norm}(0,1)$ as $n \to \infty$, where $\mathrm{Norm}(0,1)$ is a normally distributed random variable with $\mu = 0$ and $\sigma^2 = 1$. But then

$$\frac{W_n - \mu_n}{\sigma_n} \approx \mathrm{Norm}(0,1) \iff W_n - \mu_n \approx \sigma_n \mathrm{Norm}(0,1) \iff W_n - \mu_n \approx \mathrm{Norm}(0, \sigma_n^2) \iff$$
$$W_n \approx \mu_n + \mathrm{Norm}(0, \sigma_n^2) \iff W_n \approx \mathrm{Norm}(\mu_n, \sigma_n^2) = \mathrm{Norm}(n\,\mathsf{E}[S], n\,\mathsf{V}[S]),$$

**s.3.3.1.** If $A(t) = 3t^2$, then clearly $A(t)/t = 3t$. This does not converge to a limit.

Another example, let the arrival rate $\lambda(t)$ be given as follows:

$$\lambda(t) = \begin{cases} 1 & \text{if } 2^{2k} \le t < 2^{2k+1} \\ 0 & \text{if } 2^{2k+1} \le t < 2^{2(k+1)}, \end{cases}$$

for $k = 0, 1, 2, \ldots$. Let $A(t) = \lambda(t)\,t$. Then $A(t)/t$ does not have a limit. Of course, these examples are quite pathological, and are not representable for 'real life cases'. (Although this is also quite vague. What, then, is a real-life case?)

For the mathematically interested, we seek a function whose Cesàro limit does not exist.

**s.3.3.2.** Since $L(t) = L(0) + A(t) - D(t)$,

$$\lambda = \lim_{t\to\infty} \frac{A(t)}{t} = \lim_{t\to\infty} \frac{D(t)+L(t)}{t} = \lim_{t\to\infty} \frac{D(t)}{t} + \lim_{t\to\infty} \frac{L(t)}{t} = \delta.$$

Hence, $\lambda = \delta$ when $L(t)/t \to 0$.

**s.3.3.3.** $0 > \mathsf{E}[S_k - X_k] = \mathsf{E}[S_k] - \mathsf{E}[X_k] = \mathsf{E}[S] - \mathsf{E}[X]$, where we use the fact that the $\{S_k\}$ and $\{X_k\}$ are iid sequences. Hence,

$$\mathsf{E}[X] > \mathsf{E}[S] \iff \frac{1}{\mathsf{E}[S]} > \frac{1}{\mathsf{E}[X]} \iff \mu > \lambda.$$

**s.3.3.4.** The criterion is that $c$ must be such that $\lambda < c\mu$. (Thus, we interpret the number of servers as a *control*, i.e., a 'thing' we can change, while we assume that $\lambda$ and $\mu$ cannot be easily changed.) To see this, we can take two different points of view. Imagine that the $c$ servers are replaced by one server that works $c$ times as fast. The service capacity of these two systems (i.e., the system with $c$ servers and the system with one fast server) is the same, i.e., $c\mu$, where $\mu$ is the rate of one server. For the system with the fast server, the utilization is defined as $\rho = \lambda/c\mu$, and for stability we require $\rho < 1$. Another way to see it is to assume that the stream of jobs is split into $c$ smaller streams, each with arrival rate $\lambda/c$. In this case, applying the condition that $(\lambda/c)/\mu < 1$ per server leads to the same condition that $\lambda/(c\mu) < 1$.

**s.3.4.1.** Take $L(0) = 0$, $X_k = 10$ and $S_k = 10 - \epsilon$ for some tiny $\epsilon > 0$. Then $L(t) = 1$ nearly all of the time. In fact, $\lim_{t\to\infty} t^{-1} \int_0^t L(t)\,\mathrm{d}t = 1 - \epsilon/10$. However, $L(A_k-) = 0$ for all $k$.

**s.3.4.2.**

$$\mathsf{E}[L] = \lim_{t\to\infty} \frac{1}{t} \int_0^t L(s)\,\mathrm{d}s \neq \lim_{t\to\infty} \frac{L(t)}{t}.$$

If $L(t) = 1$ for all $t$, $\mathsf{E}[L] = 1$, but $L(t)/t \to 0$.

**s.3.4.3.** The idea is like this. The dictionary `L_count` counts the number of jobs that see 0, 1, and so on, in the system.

Here is the code. As an aside, it was hard to make it this simple.

Python Code

```
>>> from collections import defaultdict

>>> L_count = defaultdict(int)

>>> a = [0, 2, 5, 1, 2]
>>> c = [0, 1, 1, 1, 1]

>>> d = [0] * len(a)
>>> L = [0] * len(a)

```

```
>>> for k in range(1, len(a)):
...     d[k] = min(L[k - 1], c[k])
...     L[k] = L[k - 1] + a[k] - d[k]
...     for i in range(a[k]):
...         L_count[L[k - 1] - d[k] + i] += 1
...

>>> # normalize
>>> tot = sum(L_count.values())
>>> L_dist = {k: v / tot for k, v in L_count.items()}

>>> print(L_count)
defaultdict(<class 'int'>, {0: 1, 1: 2, 2: 1, 3: 1, 4: 1, 5: 3, 6: 1})
>>> print(L_dist)
{0: 0.1, 1: 0.2, 2: 0.1, 3: 0.1, 4: 0.1, 5: 0.3, 6: 0.1}
```

**s.4.1.1.** When processing times at a station are nearly constant, and the jobs of this station are sent to a second station for further processing, the inter-arrival times at the second station must be roughly equal. But then the inter-arrival times are not well approximated by the exponential distribution, consequently, the arrival process is not well described by a Poisson process.

**s.4.1.2.** $\rho = \lambda \mathsf{E}[S] = 1/60 \cdot 50 = 5/6$. Since job arrivals do not overlap any job service, the number of jobs in the system is 1 for 50 seconds, then the server is idle for 10 seconds, and so on. Thus $\mathsf{E}[L] = 1 \cdot 5/6 = 5/6$. There is no variance in the inter-arrival times, and also not in the service times, thus $C_a^2 = C_s^2 = 0$. Also $\mathsf{E}[W] = 0$ since $\mathsf{E}[Q] = 0$.

**s.4.1.3.** Again $\mathsf{E}[S]$ is 50 seconds, so that $\rho = 5/6$. Also $C_a^2 = 0$. For the $C_s^2$ we have to do some work.

$$\begin{aligned}
\mathsf{E}[S] &= \frac{20}{2} + \frac{80}{2} = 50 \\
\mathsf{E}\left[S^2\right] &= \frac{400}{2} + \frac{6400}{2} = 3400 \\
\mathsf{V}[S] &= \mathsf{E}\left[S^2\right] - (\mathsf{E}[S])^2 = 3400 - 2500 = 900 \\
C_s^2 &= \frac{\mathsf{V}[S]}{(\mathsf{E}[S])^2} = \frac{900}{2500} = \frac{9}{25}.
\end{aligned}$$

**s.4.1.4.** First the $G/G/1$ case. Observe that in this case, the inter-arrival time $X \sim U[3,9]$, that is, never smaller than 3 minutes, and never longer than 9 minutes.

Python Code

```
>>> a = 3.0
>>> b = 9.0
>>> EX = (b + a) / 2.0  # expected inter-arrival time
>>> EX
6.0
>>> labda = 1.0 / EX  # per minute
>>> labda
0.16666666666666666
>>> VA = (b - a) * (b - a) / 12.0
>>> CA2 = VA / (EX * EX)
```

```
>>> CA2
0.08333333333333333

>>> ES = 5.0
>>> sigma = 2
>>> VS = sigma * sigma
>>> CS2 = VS / (ES * ES)
>>> CS2
0.16

>>> rho = labda * ES
>>> rho
0.8333333333333333

>>> W = (CA2 + CS2) / 2.0 * rho / (1.0 - rho) * ES
>>> W
3.041666666666665
```

Now the $M/G/1$ case. In that case $C_a^2 = 1$.

Python Code

```
>>> W = (1.0 + CS2) / 2.0 * rho / (1.0 - rho) * ES
>>> W
14.499999999999993
```

The arrival process with uniform inter-arrival times is much more regular than a Poisson process. In the first case, bus arrivals are spaced in time at least with 3 minutes.

**s.4.1.5.** Observe that $\lambda(1-\beta)$ is the net arrival rate, as jobs are lost at a rate $\lambda\beta$. The rate at which the station carries out work is $\mu\,\mathsf{E}[L_s]$. Since all jobs that enter the system, must also leave the system (recall, the queue is finite), it follows that $\mu\,\mathsf{E}[L_s] = \lambda(1-\beta)$, from which the expression for $\beta$ readily follows.

**s.4.1.6.** With the hint,

$$\begin{aligned}\mathsf{E}[S] &= \mathsf{E}[\mathbb{1}_{T=1}S_1] + \mathsf{E}[\mathbb{1}_{T=2}S_2] \\ &= \mathsf{E}[\mathbb{1}_{T=1}]\,\mathsf{E}[S_1] + \mathsf{E}[\mathbb{1}_{T=2}]\,\mathsf{E}[S_2], \quad \text{by the independence of } T, \\ &= \mathsf{P}\{T=1\}\,\mathsf{E}[S_1] + \mathsf{P}\{T=2\}\,\mathsf{E}[S_2] \\ &= p\,\mathsf{E}[S_1] + q\,\mathsf{E}[S_2].\end{aligned}$$

For the variance, we need some algebra. Since,

$$\mathbb{1}_{T=1}\,\mathbb{1}_{T=2} = 0 \text{ and } \mathbb{1}_{T=1}^2 = \mathbb{1}_{T=1},$$

we get

$$\begin{aligned}
\mathsf{V}[S] &= \mathsf{E}\left[S^2\right] - (\mathsf{E}[S])^2 \\
&= \mathsf{E}\left[(\mathbb{1}_{T=1}S_1 + \mathbb{1}_{T=2}S_2)^2\right] - (\mathsf{E}[S])^2 \\
&= \mathsf{E}\left[\mathbb{1}_{T=1}S_1^2 + \mathbb{1}_{T=2}S_2^2\right] - (\mathsf{E}[S])^2 \\
&= p\,\mathsf{E}\left[S_1^2\right] + q\,\mathsf{E}\left[S_2^2\right] - (\mathsf{E}[S])^2 \\
&= p\,\mathsf{V}[S_1] + p(\mathsf{E}[S_1])^2 + q\,\mathsf{V}[S_2] + q(\mathsf{E}[S_2])^2 - (\mathsf{E}[S])^2 \\
&= p\,\mathsf{V}[S_1] + p(\mathsf{E}[S_1])^2 + q\,\mathsf{V}[S_2] + q(\mathsf{E}[S_2])^2 - p^2(\mathsf{E}[S_1])^2 - q^2(\mathsf{E}[S_2])^2 - 2pq\,\mathsf{E}[S_1]\,\mathsf{E}[S_2] \\
&= p\,\mathsf{V}[S_1] + q\,\mathsf{V}[S_2] + pq(\mathsf{E}[S_1])^2 + pq(\mathsf{E}[S_2])^2 - 2pq\,\mathsf{E}[S_1]\,\mathsf{E}[S_2], \quad \text{as } p = 1-q \\
&= p\,\mathsf{V}[S_1] + q\,\mathsf{V}[S_2] + pq(\mathsf{E}[S_1] - \mathsf{E}[S_2])^2.
\end{aligned}$$

**s.4.2.1.** First check the load.

Python Code

```
>>> labda = 3  # per hour
>>> ES0 = 15.0 / 60  # hour
>>> ES0
0.25
>>> ER = 2.0
>>> Bmin = labda * ER / (1 - labda * ES0)
>>> Bmin
24.0
```

Python Code

```
>>> B = 30
>>> ES = ES0 + ER / B
>>> rho = labda * ES
>>> rho
0.95
```

The time to form a red batch is

Python Code

```
>>> labda_r = 0.5
>>> EW_r = (B - 1) / (2 * labda_r)
>>> EW_r  # in hours
29.0
```

And the time to form a blue batch is

Python Code

```
>>> labda_b = labda - labda_r
>>> EW_b = (B - 1) / (2 * labda_b)
>>> EW_b  # in hours
5.8
```

The time a batch spends in queue.

Python Code

```
>>> Cae = 1.0
>>> CaB = Cae / B
>>> CaB
0.03333333333333333
>>> Ce = 1.0  # SCV of service times
>>> VS0 = Ce * ES0 * ES0
>>> VS0
0.0625
>>> VR = 1.0 * 1.0  # Var setups is sigma squared
>>> VS = B * VS0 + VR
>>> VS
2.875
>>> ESb = B * ES0 + ER
>>> ESb
9.5
>>> CeB = VS / (ESb * ESb)
>>> CeB
0.03185595567867036
>>> EW = (CaB + CeB) / 2 * rho / (1 - rho) * ESb
>>> EW
5.8833333333333275
```

The time to unpack the batch, i.e., the time at the server.

Python Code

```
>>> Eunpack = ER + (B - 1) / 2 * ES0 + ES0
>>> Eunpack
5.875
```

The overall time red jobs spend in the system.

Python Code

```
>>> total = EW_r + EW + Eunpack
>>> total
40.758333333333326
```

**s.4.2.2.** Suppose a batch is just finished. The first job of a new batch needs to wait, on average, $B-1$ inter-arrival times until the batch is complete, the second $B-2$ inter-arrival times, and so on. The last job does not have to wait at all. Thus, the total time to form a batch is $(B-1)/\lambda_r$. An arbitrary job can be anywhere in the batch, hence its expected time is half the total time.

**s.4.2.3.** The variance of the inter-arrival time of batches is $B$ times the variance of job inter-arrival times. The inter-arrival times of batches is also $B$ times the inter-arrival times of jobs. Thus,

$$C_{a,B}^2 = \frac{B\mathsf{V}[X]}{(B\mathsf{E}[X])^2} = \frac{\mathsf{V}[X]}{(\mathsf{E}[X])^2}\frac{1}{B} = \frac{C_a^2}{B}.$$

**s.4.2.4.** The variance of a batch is $\mathsf{V}\left[R+\sum_{i=1}^{B} S_{0,i}\right] = \mathsf{V}[R] + B\mathsf{V}[S_0]$, since the normal service times $S_{0,i}, i = 1,\ldots,B$, of the jobs are independent, and also independent of the setup time $R$ of the batch.

**s.4.2.5.** First, wait until the setup is finished, then wait (on average) for half of the batch (minus the job itself) to be served, and then the job has to be served itself, that is, $\mathsf{E}[R] + \frac{B-1}{2}\mathsf{E}[S_0] + \mathsf{E}[S_0]$.

**s.4.3.1.** First we determine the load.

Python Code

```
>>> EB = 30
>>> p = 1 / EB
>>> ES0 = 1.5
>>> labda = 9.0 / (2 * 8)  # arrival rate per hour
>>> ER = 5.0
>>> ES = ES0 + p * ER
>>> ES
1.6666666666666667
>>> rho = labda * ES
>>> rho
0.9375
```

So, at least the system is stable.

Python Code

```
>>> VS0 = 0.5 * 0.5
>>> VR = 2.0 * 2.0
>>> VS = VS0 + p * VR + p * (1 - p) * ER * ER
>>> VS
1.1888888888888887
>>> Ce2 = VS / (ES * ES)
>>> Ce2
0.4279999999999999
```

And now we can fill in the waiting time formula.

Python Code

```
>>> Ca2 = 1  # Poisson arrivals
>>> EW = (Ca2 + Ce2) / 2 * rho / (1 - rho) * ES
>>> EW
17.849999999999998
>>> EJ = EW + ES
>>> EJ
19.516666666666666
```

**s.4.3.2.** We can use the model of Section 4.2.

Python Code

```
>>> B = 20
>>> ER = 4.5
>>> VR = 0
>>> ES = ES0 + ER / B
>>> ES
1.725
>>> rho = labda * ES
```

```
>>> rho
0.9703125
>>> VS = VS0 + VR / B
>>> Ce2 = VS / (ES * ES)
>>> Ce2
0.08401596303297626
>>> EW = (Ca2 + Ce2) / 2 * rho / (1 - rho) * ES
>>> EW
30.55855263157897
>>> EJ = EW + ES
>>> EJ
32.28355263157897
```

Comparing this to the results of [4.3.1], we see that the load becomes somewhat higher. Since $\rho$ becomes close to one, doing adjustments regularly is not a good idea.

**s.4.3.3.** Taking expectations and using the independence of $R$ and $F$ gives the result.

**s.4.3.4.**

$$\begin{aligned}\mathsf{E}\left[S^2\right] &= \mathsf{E}\left[(S_0 + R\,\mathbb{1}_{F=1})^2\right] \\ &= \mathsf{E}\left[S_0^2 + 2S_0R\,\mathbb{1}_{F=1} + R^2\,\mathbb{1}_{F=1}\right] \\ &= \mathsf{E}\left[S_0^2\right] + 2p\,\mathsf{E}[S_0]\,\mathsf{E}[R] + p\,\mathsf{E}\left[R^2\right].\end{aligned}$$

**s.4.3.5.**

$$\begin{aligned}\mathsf{V}[S] &= \mathsf{E}\left[S^2\right] - (\mathsf{E}[S])^2 \\ &= \mathsf{E}\left[S_0^2\right] + 2p\,\mathsf{E}[S_0]\,\mathsf{E}[R] + p\,\mathsf{E}\left[R^2\right] \\ &\quad - (\mathsf{E}[S_0])^2 - 2p\,\mathsf{E}[S_0]\,\mathsf{E}[R] - (p\,\mathsf{E}[R])^2 \\ &= \mathsf{V}[S_0] + p(\mathsf{E}\left[R^2\right] - (\mathsf{E}[R])^2) + (\mathsf{E}[R])^2 p(1-p) \\ &= \mathsf{V}[S_0] + p\,\mathsf{V}[R] + (\mathsf{E}[R])^2 p(1-p) \\ &= \mathsf{V}[S_0] + p\,\mathsf{V}[R] + p^3(\mathsf{E}[R])^2\frac{1-p}{p^2} \\ &= \mathsf{V}[S_0] + p\,\mathsf{V}[R] + p^3(\mathsf{E}[R])^2\,\mathsf{V}[B] \\ &= \mathsf{V}[S_0] + p\,\mathsf{V}[R] + p(\mathsf{E}[R])^2 C_B^2.\end{aligned}$$

Now replace $p$ by $1/\mathsf{E}[B]$.

**s.4.4.1.** Let's first check that $\rho < 1$.

Python Code

```
>>> labda = 4.0
>>> ES0 = 10.0 / 60  # in hours
>>> labda_f = 1.0 / 3
>>> ER = 30.0 / 60  # in hours
>>> A = 1.0 / (1 + labda_f * ER)
>>> A
0.8571428571428571
>>> ES = ES0 / A
```

```
>>> ES
0.19444444444444445
>>> rho = labda * ES
>>> rho
0.7777777777777778
```

Python Code

```
>>> Ca2 = 1.0
>>> C02 = 0.0  # deterministic service times
>>> Ce2 = C02 + 2 * A * (1 - A) * ER / ES0
>>> Ce2
0.7346938775510207
>>> EW = (Ca2 + Ce2) / 2 * rho / (1 - rho) * ES
>>> EW
0.5902777777777779
>>> EW + ES  # = EJ
0.7847222222222223
```

**s.4.4.2.** The time to fail is the time in between two interruptions. By assumption, the failure times are Exp(()$\lambda_f$), hence $m_f = 1/\lambda_f$. The expected duration of an interruption is $\mathsf{E}[R]$. With this

$$A = \frac{m_f}{m_f + \mathsf{E}[R]} = \frac{1/\lambda_f}{1/\lambda_f + \mathsf{E}[R]}.$$

**s.4.4.3.** The expectation of the *fixed* sum of random variables is the sum of the expectations. Since the $\{R_i\}$ are iid, $\mathsf{E}\left[\sum_{i=1}^{n} R_i\right] = n\,\mathsf{E}[R]$.

**s.4.4.4.**

$$\begin{aligned}\mathsf{E}\left[\sum_{i=1}^{N} R_i\right] &= \mathsf{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{N=n}\left(\sum_{i=1}^{n} R_i\right)\right] = \sum_{n=0}^{\infty} \mathsf{E}\left[\mathbb{1}_{N=n} n\,\mathsf{E}[R]\right] \\ &= \mathsf{E}[R]\sum_{n=0}^{\infty} n\,\mathsf{E}[\mathbb{1}_{N=n}] = \mathsf{E}[R]\sum_{n=0}^{\infty} n p_n = \mathsf{E}[R]\,\mathsf{E}[N].\end{aligned}$$

**s.4.4.5.** If $S_0 = s$, then the expected number of failures that arrive is $\sim P(\lambda_f s)$. Therefore, $\mathsf{E}[N] = \mathsf{E}[\lambda_f S_0] = \lambda_f\,\mathsf{E}[S_0]$.

In more detail, and with the hint,

$$\mathsf{E}[N] = \int_0^\infty \sum_{k=0}^{\infty} k e^{-\lambda_f s}\frac{(\lambda_f s)^k}{k!} g(s)\,\mathrm{d}s = \int_0^\infty \lambda_f s g(s)\,\mathrm{d}s = \lambda_f\,\mathsf{E}[S_0].$$

Next, [2.2.8],

$$\mathsf{E}[N^2] = \int_0^\infty \sum_{k=0}^{\infty} k^2 e^{-\lambda_f s}\frac{(\lambda_f s)^k}{k!} g(s)\,\mathrm{d}s = \int_0^\infty (\lambda_f^2 s^2 + \lambda_f s) g(s)\,\mathrm{d}s.$$

**s.4.4.6.**

$$\mathsf{E}[S] = \mathsf{E}[S_0] + \mathsf{E}[N]\,\mathsf{E}[R] = \mathsf{E}[S_0] + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}[R] = \mathsf{E}[S_0]\,(1 + \lambda_f\,\mathsf{E}[R]).$$

**s.4.4.7.** Observe that $S_0$ and $N$ are not independent. In fact, when $S_0 = s$, the number of failures $N$ is Poisson distributed with mean $\lambda_f s$.

**s.4.4.8.** Just work out the square of $S_0 + \sum_{i=1}^{N} R_i$ and take expectations. Realize that $(\sum_i R_i)^2 = \sum_i R_i^2 + \sum_i \sum_{j \neq i} R_i R_j$.

**s.4.4.9.**

$$\begin{aligned}\mathsf{E}\left[S_0 \sum_{i=1}^{N} R_i\right] &= \mathsf{E}\left[S_0 \sum_{n=0}^{\infty} \mathbb{1}_{N=n} \sum_{i=1}^{N} R_i\right] = \mathsf{E}\left[S_0 \sum_{n=0}^{\infty} \mathbb{1}_{N=n} \sum_{i=1}^{n} R_i\right] \\ &= \mathsf{E}[R]\,\mathsf{E}\left[S_0 \sum_{n=0}^{\infty} \mathbb{1}_{N=n} n\right] = \mathsf{E}[R] \sum_{n=0}^{\infty} n\,\mathsf{E}[S_0\,\mathbb{1}_{N=n}].\end{aligned}$$

Now observe that $S_0$ and $\mathbb{1}_{N=n}$ are not independent. Thus, we need to use the joint distribution of [4.4.5], and then,

$$\mathsf{E}[S_0\,\mathbb{1}_{N=n}] = \int_0^{\infty} s g(s) e^{-\lambda_f s} \frac{(\lambda_f s)^n}{n!}\,\mathrm{d}s.$$

Continuing with the previous relation:

$$\begin{aligned}\mathsf{E}[R] \sum_{n=0}^{\infty} n\,\mathsf{E}[S_0\,\mathbb{1}_{N=n}] &= \mathsf{E}[R] \sum_{n=0}^{\infty} n \int_0^{\infty} s g(s) e^{-\lambda_f s} \frac{(\lambda_f s)^n}{n!}\,\mathrm{d}s = \mathsf{E}[R] \int_0^{\infty} s g(s) \sum_{n=0}^{\infty} n e^{-\lambda_f s} \frac{(\lambda_f s)^n}{n!}\,\mathrm{d}s \\ &= \mathsf{E}[R] \int_0^{\infty} s g(s) \lambda_f s\,\mathrm{d}s = \lambda_f\,\mathsf{E}[R]\,\mathsf{E}\left[S_0^2\right].\end{aligned}$$

**s.4.4.10.** With the hint,

$$\mathsf{E}\left[\sum_{i=1}^{N} R_i^2\right] = \mathsf{E}\left[R^2\right]\mathsf{E}[N] = \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right].$$

**s.4.4.11.** Since the $\{R_i\}$ are iid,

$$\begin{aligned}\mathsf{E}\left[\sum_{i=1}^{N} \sum_{j \neq i} R_i R_j\right] &= \mathsf{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{N=n} \sum_{i=1}^{N} \sum_{j \neq i} R_i R_j\right] = \mathsf{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{N=n} \sum_{i=1}^{n} \sum_{j \neq i} R_i R_j\right] \\ &= \mathsf{E}\left[\sum_{n=0}^{\infty} \mathbb{1}_{N=n} n(n-1)\right](\mathsf{E}[R])^2 = (\mathsf{E}\left[N^2\right] - \mathsf{E}[N])(\mathsf{E}[R])^2 \\ &= (\lambda_f^2\,\mathsf{E}\left[S_0^2\right] + \lambda_f\,\mathsf{E}[S_0] - \lambda\,\mathsf{E}[S_0])(\mathsf{E}[R])^2.\end{aligned}$$

where we use [4.4.5] in the last line.

**s.4.4.12.** It is just algebra based on the results above. Observe that $(1/A) = 1 + \lambda_f\,\mathsf{E}[R]$,

$$\begin{aligned}\mathsf{E}\left[S^2\right] &= \mathsf{E}\left[S_0^2\right] + 2\,\mathsf{E}\left[S_0 \sum_{i=1}^{N} R_i\right] + \mathsf{E}\left[\sum_{i=1}^{N} R_i^2\right] + \mathsf{E}\left[\sum_{i=1}^{N} \sum_{j \neq i} R_i R_j\right] \\ &= \mathsf{E}\left[S_0^2\right] + 2\lambda_f\,\mathsf{E}[R]\,\mathsf{E}\left[S_0^2\right] + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right] + \lambda_f^2\,\mathsf{E}\left[S_0^2\right](\mathsf{E}[R])^2 \\ &= \mathsf{E}\left[S_0^2\right]/A + \lambda_f\,\mathsf{E}[R]\,\mathsf{E}\left[S_0^2\right] + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right] + \lambda_f^2\,\mathsf{E}\left[S_0^2\right](\mathsf{E}[R])^2 \\ &= \mathsf{E}\left[S_0^2\right]/A + \lambda_f\,\mathsf{E}[R]\,\mathsf{E}\left[S_0^2\right](1 + \lambda_f\,\mathsf{E}[R]) + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right] \\ &= \mathsf{E}\left[S_0^2\right]/A + \lambda_f\,\mathsf{E}[R]\,\mathsf{E}\left[S_0^2\right]/A + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right] \\ &= (1 + \lambda_f\,\mathsf{E}[R])\,\mathsf{E}\left[S_0^2\right]/A + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right] \\ &= \mathsf{E}\left[S_0^2\right]/A^2 + \lambda_f\,\mathsf{E}[S_0]\,\mathsf{E}\left[R^2\right].\end{aligned}$$

**s.4.4.13.**

$$\mathsf{V}[S] = \mathsf{E}\left[S^2\right] - (\mathsf{E}[S])^2 = \frac{\mathsf{E}\left[S_0^2\right]}{A^2} + \lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0] - \frac{(\mathsf{E}[S_0])^2}{A^2}.$$

**s.4.4.14.** When doing the computations by hand, I worked from bottom to top. However, for the solutions, the current sequence is perhaps easier understand.

$$\begin{aligned}
\mathsf{E}[S|S_0, N] &= \mathsf{E}\left[S_0 + \sum_{i=1}^{N} R \middle| S_0, N\right] = S_0 + N\mathsf{E}[R], \text{ since } \{R_i\} \text{ are iid,} \\
\mathsf{E}[S|S_0] &= \mathsf{E}[\mathsf{E}[S|S_0, N]|S_0] = \mathsf{E}[S_0 + N\mathsf{E}[R]|S_0] = S_0 + \lambda_f S_0 \mathsf{E}[R] \\
\mathsf{E}[S] &= \mathsf{E}[\mathsf{E}[S|S_0]] = \mathsf{E}\left[S_0(1 + \lambda_f \mathsf{E}[R])\right] = \mathsf{E}[S_0](1 + \lambda_f \mathsf{E}[R]) = \mathsf{E}[S_0]/A. \\
\mathsf{V}[S|S_0, N] &= \mathsf{V}\left[S_0 + \sum_{i}^{N} R_i\right] = N\mathsf{V}[R], \\
\mathsf{V}[S|S_0] &= \mathsf{E}[\mathsf{V}[S|S_0, N]|S_0] + \mathsf{V}[\mathsf{E}[S|S_0 N]|S_0] = \lambda_f S_0 \mathsf{V}[R] + \mathsf{V}[S_0 + N\mathsf{E}[R]|S_0] \\
&= \lambda_f S_0 \mathsf{V}[R] + (\mathsf{E}[R])^2 \mathsf{V}[N|S_0] = \lambda_f S_0 \mathsf{V}[R] + (\mathsf{E}[R])^2 \lambda_f S_0, \\
\mathsf{V}[S] &= \mathsf{E}[\mathsf{V}[S|S_0]] + \mathsf{V}[\mathsf{E}[S|S_0]] = \lambda_f (\mathsf{V}[R] + (\mathsf{E}[R])^2) \mathsf{E}[S_0] + \mathsf{V}\left[(1 + \lambda_f \mathsf{E}[R]) S_0\right] \\
&= \lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0] + (1 + \lambda_f \mathsf{E}[R])^2 \mathsf{V}[S_0] = \lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0] + \mathsf{V}[S_0]/A^2.
\end{aligned}$$

**s.4.4.15.** Using [4.4.8]–[4.4.13]

$$\begin{aligned}
C_s^2 &= \frac{\mathsf{V}[S]}{(\mathsf{E}[S])^2} = \frac{V(S)A^2}{(\mathsf{E}[S_0])^2} = \frac{\mathsf{E}\left[S_0^2\right] + \lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0] A^2 - (\mathsf{E}[S_0])^2}{(\mathsf{E}[S_0])^2} \\
&= \frac{\mathsf{E}\left[S_0^2\right] - (\mathsf{E}[S_0])^2}{(\mathsf{E}[S_0])^2} + \frac{\lambda_f \mathsf{E}\left[R^2\right] \mathsf{E}[S_0] A^2}{(\mathsf{E}[S_0])^2} = C_0^2 + \frac{\lambda_f \mathsf{E}\left[R^2\right] A^2}{\mathsf{E}[S_0]}.
\end{aligned}$$

**s.4.4.16.** When repair times are exponentially distributed with mean $\mathsf{E}[R]$ we have, by [2.4.2], that $\mathsf{E}\left[R^2\right] = 2(\mathsf{E}[R])^2$. Since $A = 1/(1 + \lambda_f \mathsf{E}[R])$,

$$\begin{aligned}
\lambda_f \mathsf{E}\left[R^2\right] A^2 &= 2\lambda_f (\mathsf{E}[R])^2 A^2 = 2\frac{\lambda_f \mathsf{E}[R]}{1 + \lambda_f \mathsf{E}[R]} A\mathsf{E}[R] \\
&= 2\left(1 - \frac{1}{1 + \lambda_f \mathsf{E}[R]}\right) A\mathsf{E}[R] = 2(1 - A)A\mathsf{E}[R].
\end{aligned}$$

**s.4.5.1.** First station 1.

Python Code

```
>>> labda = 2.0
>>> S1 = 20.0 / 60
>>> rho1 = labda * S1
>>> ca1 = 2.0
>>> cs1 = 0.5
>>> EW1 = (ca1 + cs1) / 2 * rho1 / (1 - rho1) * S1
>>> EJ1 = EW1 + S1
>>> EJ1
1.1666666666666665
```

Now station 2. We first need to compute $C_{d1}^2$.

Python Code

```
>>> cd1 = (1 - rho1 ** 2) * ca1 + rho1 ** 2 * cs1
>>> cd1
1.3333333333333335
>>> labda = 2
>>> S2 = 25.0 / 60
>>> rho2 = labda * S2
>>> ca2 = cd1  # here we use our formula
>>> cs2 = 0.5
>>> EW2 = (ca2 + cs2) / 2 * rho2 / (1 - rho2) * S2
>>> EJ2 = EW2 + S2
>>> EJ2
2.3263888888888897
>>> EJ1 + EJ2
3.4930555555555562
```

**s.5.1.1.** It is evident that $X_k = Y(D_k) - Y(D_{k-1}) = S_k$, hence $X = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} X_k = \mathsf{E}[S]$. In the relation $Y = \lambda X$, the $\lambda$ is $\delta$ since we consider departure epochs $T_k = D_k$, rather than $A_k$. Thus, with the renewal reward theorem $Y = \lambda X$, we get that $Y = \lim_{t\to\infty} t^{-1}\int_0^t \mathbb{1}_{L(s)>0}\,\mathrm{d}s = \delta\,\mathsf{E}[S]$. Observe that the $Y$ is the long-run fraction of time the server is busy.

**s.5.1.2.** For the LHS, as $A(t)\to\infty$ as $t\to\infty$,

$$\lim_{t\to\infty}\frac{1}{t}\sum_{k=1}^{A(t)} S_k = \lim_{t\to\infty}\frac{A(t)}{t}\frac{1}{A(t)}\sum_{k=1}^{A(t)} S_k = \lim_{t\to\infty}\frac{A(t)}{t}\cdot\lim_{t\to\infty}\frac{1}{A(t)}\sum_{k=1}^{A(t)} S_k = \lambda\,\mathsf{E}[S]\,.$$

Apply similar limits to the RHS. The limit of the middle term gives the fraction of time the server is busy.

**s.5.2.1.**

$$A(t) = \sum_{k=1}^{\infty}\mathbb{1}_{A_k\le t} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\mathbb{1}_{L(A_k-)=n}\,\mathbb{1}_{A_k\le t} = \sum_{n=0}^{\infty} A(n,t).$$

**s.5.2.2.** If $\lambda > \delta$, then $L(t)\to\infty$. But then there must be a last time, $s$ say, that $L(s) = n+1$, and $L(t) > n+1$ for all $t > s$. Hence, after time $s$ no job will see the system with $n$ jobs. Thus $A(n,t) = A(n,s)$ for all $t > s$. As this is finite, $\lim_{t\to\infty} A(n,t)/t = 0$.

**s.5.2.3.** It is the $D/D/1$ queue. Next, $A_k = 2(k-1)$ as jobs arrive at $t = 0,2,4,\ldots$ We also know that $L(s) = 1$ if $s\in[2k,2k+1)$ and $L(s) = 0$ for $s\in[2k-1,2k)$ for $k = 0,1,2,\ldots$. Thus, $L(A_k-) = L(2k-) = 0$. Hence, $A(0,t)\approx t/2$ for $t\gg 0$, and $A(n,t) = 0$ for $n\ge 1$.

**s.5.2.4.** Observe that the system never contains more than 1 job. Hence, $Y(n,t) = 0$ for all $n\ge 2$. Then we see that $Y(1,t) = \int_0^t \mathbb{1}_{L(s)=1}\,\mathrm{d}s$. Now observe that for our queueing system $L(s) = 1$ for $s\in[0,1)$, $L(s) = 0$ for $s\in[1,2)$, $L(s) = 1$ for $s\in[2,3)$, and so on. Thus, when $t < 1$, $Y(1,t) = \int_0^t \mathbb{1}_{L(s)=1}\,\mathrm{d}s = \int_0^t 1\,\mathrm{d}s = t$. When $t\in[1,2)$,

$$L(t) = 0 \implies \mathbb{1}_{L(t)=0} \implies Y(1,t) \text{ does not change.}$$

Continuing to $[2,3)$ and so on gives

$$Y(1,t) = \begin{cases} t & t \in [0,1), \\ 1 & t \in [1,2), \\ 1+(t-2) & t \in [2,3), \\ 2 & t \in [3,4), \\ 2+(t-4) & t \in [4,5), \end{cases}$$

and so on. Since $Y(n,t) = 0$ for all $n \geq 2$, $L(s) = 1$ or $L(s) = 0$ for all $s$, therefore,

$$Y(0,t) = t - Y(1,t).$$

**s.5.2.5.** From the other exercises:

$$\begin{aligned} \lambda(0) &\approx \frac{A(0,t)}{Y(0,t)} \approx \frac{t/2}{t/2} = 1, \\ \lambda(1) &\approx \frac{A(1,t)}{Y(1,t)} \approx \frac{0}{t/2} = 0, \\ p(0) &\approx \frac{Y(0,t)}{t} \approx \frac{t/2}{t} = \frac{1}{2}, \\ p(1) &\approx \frac{Y(1,t)}{t} \approx \frac{t/2}{t} = \frac{1}{2}. \end{aligned}$$

For the rest $\lambda(n) = 0$, and $p(n) = 0$, for $n \geq 2$.

**s.5.2.6.** $D(0,t) = \sum_{k=1}^{\infty} \mathbb{1}_{D_k \leq t, L(D_k)=0}$. From the graph of $\{L(s)\}$ we see that all jobs leave an empty system behind. Thus, $D(0,t) \approx t/2$, and $D(n,t) = 0$ for $n \geq 1$. With this, $D(0,t)/Y(1,t) \sim (t/2)/(t/2) = 1$, and so,

$$\mu(1) = \lim_{t\to\infty} \frac{D(0,t)}{Y(1,t)} = 1,$$

and $\mu(n) = 0$ for $n \geq 2$.

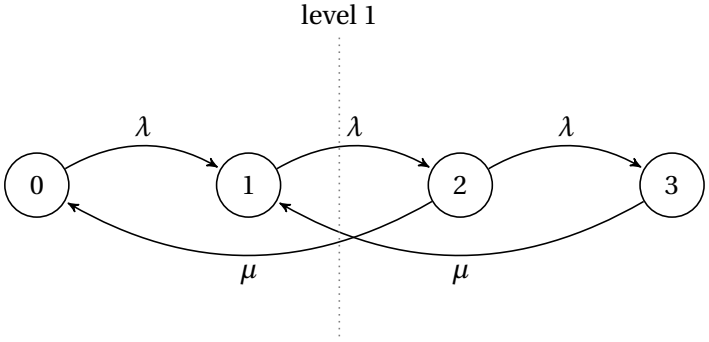
**s.5.2.7.** $\lambda(0)p(0) = 1 \cdot 1/2 = 1/2$, $\lambda(n)p(n) = 0$ for $n > 1$, as $\lambda(n) = 0$ for $n > 0$.
From [5.2.6], $\mu(1) = 1$, hence $\mu(1)p(1) = 1 \cdot 1/2 = 1/2$. Moreover, $\mu(n) = 0$ for $n \geq 2$.
Clearly, for all $n$ we have $\lambda(n)p(n) = \mu(n+1)p(n+1)$.

**s.5.2.8.** Noting that $L(s) = \sum_{n=0}^{\infty} n\, \mathbb{1}_{L(s)=n}$,

$$\frac{1}{t}\int_0^t L(s)\,\mathrm{d}s = \frac{1}{t}\int_0^t \left(\sum_{n=0}^{\infty} n\, \mathbb{1}_{L(s)=n}\right) \mathrm{d}s = \sum_{n=0}^{\infty} \frac{n}{t}\int_0^t \mathbb{1}_{L(s)=n}\,\mathrm{d}s,$$

Now apply (5.2.1c).

**s.5.2.9.** It is the $M/M^2/1/3$ queue.



With level-crossing:

$$\begin{aligned}
&\lambda p(0) = \mu p(2), \quad \text{the level between 0 and 1,}\\
&\lambda p(1) = \mu p(2) + \mu p(3), \quad \text{see level 1,}\\
&\lambda p(2) = \mu p(3), \quad \text{the level between 2 and 3.}
\end{aligned}$$

Solving this in terms of $p(0)$ gives $p(2) = \rho p(0)$, $p(3) = \rho p(2) = \rho^2 p(0)$, and

$$\lambda p(1) = \mu(p(2) + p(3)) = \mu(\rho + \rho^2)p(0) = (\lambda + \lambda^2/\mu)p(0),$$

hence $p(1) = p(0)(\mu + \lambda)/\mu$.

**s.14.**

$$\lim_{t\to\infty} \frac{I(n,t)}{t} = \lim_{t\to\infty} \frac{A(n-1,t)}{t} + \lim_{t\to\infty} \frac{D(n,t)}{t} = \lambda(n-1)p(n-1) + \mu(n+1)p(n+1)$$

and

$$\lim_{t\to\infty} \frac{O(n,t)}{t} = \lim_{t\to\infty} \frac{A(n,t)}{t} + \lim_{t\to\infty} \frac{D(n-1,t)}{t} = \lambda(n)p(n) + \mu(n)p(n)$$

**s.5.3.1.** All arrivals see an empty system. Hence, $A(0,t)/A(t) \approx (t/2)/(t/2) = 1$, and $A(n,t) = 0$ for $n > 0$. Thus, $\pi(0) = \lim_{t\to\infty} A(0,t)/A(t) = 1$ and $\pi(n) = 0$ for $n > 0$. Recall from the other exercises that $p(0) = 1/2$. Hence, statistics as obtained via time averages are not necessarily the same as statistics obtained at arrival moments (or any other point process).

**s.5.3.2.** From the relevant previous exercises, $\lambda = \lim_{t\to\infty} A(t)/t = 1/2$. $\lambda(0) = 1$, $p(0) = 1/2$, and $\pi(0) = 1$. Hence,

$$\lambda\pi(0) = \lambda(0)p(0) \implies \frac{1}{2} \times 1 = 1 \times \frac{1}{2}.$$

For $n > 0$ it's easy, everything is 0.

**s.5.3.3.** The assumptions lead us to conclude that $\lambda > \delta$. As a consequence, the queue length must increase in the long run (jobs come in faster than they leave). Therefore, $A(n,t)/t \to 0$ for all $n$, and also $D(n,t)/t \to 0$. Consequently, $\pi(n) = \delta(n) = 0$, which is the only sensible reconciliation with (5.3.3).

**s.5.3.4.** First we check the conditions. The counting process here is $\{A(t)\}$ and the epochs at which $A(t)$ increases are $\{A_k\}$. By assumption, $A_k \to \infty$, hence $A(t) \to \infty$ as $t \to \infty$. Moreover, by assumption $A(t)/t \to \lambda$. Also $A(n,t)$ is evidently non-decreasing and $A(n,t) \to \infty$ as $t \to \infty$.

From the definitions in the hint,

$$X = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} X_k = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \mathbb{1}_{L(A_k-)=n} = \pi(n).$$

Since $Y = \lim_{t\to\infty} Y(t)/t = \lim_{t\to\infty} A(n,t)/t$ it follows from the renewal reward theorem that

$$Y = \lambda X \implies \lim_{t\to\infty} \frac{A(n,t)}{t} = \lambda X = \lambda\pi(n).$$

**s.5.** The problem is known as the 'inspection paradox'. You should be aware of such biases in data science.

**s.5.4.1.** The dimension of $\lambda$ is a *rate* and $\mathsf{E}[J]$ is time, hence the dimension of $\lambda\,\mathsf{E}[J]$ is a number, just like $\mathsf{E}[L]$.

**s.5.4.2.** Since the queue is stable by assumption, jobs depart at rate $\lambda$, hence any job enters and leaves the server. The average time the server is busy is $\lambda\,\mathsf{E}[S]$, while the time-average number of jobs at the server is $\mathsf{E}[L_s]$. By Little's law, $\mathsf{E}[L_s] = \lambda\,\mathsf{E}[S]$.

**s.5.4.3.** $\mathsf{E}[W] = \mathsf{E}[J] - \mathsf{E}[S] = 1/6/(1-5/6) - 1/6 = 5/6h$.

**s.5.4.4.** When you arrive, you first have to wait for the job in service to finish and then for the 9 jobs in queue. By the memoryless property, at the moment you arrive, the remaining service time of the job in service is still $\mathsf{E}[S]$ Thus, you have to wait $10/\mu = 5/3h \neq 5/6h$.

**s.5.4.5.**

$$\begin{aligned}\int_0^T L(s)\,\mathrm{d}s &= \int_0^T \sum_{k=1}^{A(T)} \mathbb{1}_{A_k \le s < D_k}\,\mathrm{d}s \\ &= \sum_{k=1}^{A(T)} \int_0^T \mathbb{1}_{A_k \le s < D_k}\,\mathrm{d}s = \sum_{k=1}^{A(T)} J_k.\end{aligned}$$

**s.5.4.6.** By the memoryless property of the (exponential) distributed service times of the $M/M/1$ queue, the duration of a job in service, if any, is $\mathrm{Exp}()\mu)$ also at an arrival moment. Therefore, at an arrival moment, all jobs in the system (whether in service or not) have the same expected duration. Hence, the expected time to spend in queue is the expected number of jobs in the system times the expected service time of each job, i.e., $\mathsf{E}[J] = \mathsf{E}[L]\,\mathsf{E}[S]$. Note that we use PASTA to see that the expected number of jobs in the system at an arrival is $\mathsf{E}[L]$. For the $M/G/1$ queue, the job in service (if any) does not have the same distribution as a job in queue. Hence, for the $M/G/1$ queue, the *expected time in queue is not* $\mathsf{E}[L]\,\mathsf{E}[S]$.

**s.5.4.7.**

$$\begin{aligned}\mathsf{E}[J] &= \mathsf{E}[L]\,\mathsf{E}[S] + \mathsf{E}[S] = \lambda\,\mathsf{E}[J]\,\mathsf{E}[S] + \mathsf{E}[S] = \rho\,\mathsf{E}[J] + \mathsf{E}[S],\\ \mathsf{E}[L] &= \lambda\,\mathsf{E}[J] = \frac{\lambda\,\mathsf{E}[S]}{1-\rho} = \frac{\rho}{1-\rho},\\ \mathsf{E}[W] &= \mathsf{E}[J] - \mathsf{E}[S] = \frac{\mathsf{E}[S]}{1-\rho} - \mathsf{E}[S] = \frac{\rho}{1-\rho}\,\mathsf{E}[S] = \mathsf{E}[L]\,\mathsf{E}[S]\\ \mathsf{E}[Q] &= \lambda\,\mathsf{E}[W] = \frac{\rho^2}{1-\rho},\\ \mathsf{E}[L_s] &= \mathsf{E}[L] - \mathsf{E}[Q] = \frac{\rho}{1-\rho} - \frac{\rho^2}{1-\rho} = \rho,\end{aligned}$$

**s.5.4.8.** Intuitively, the left term is all the work that arrived up to time $t$, the middle term is all the work that has been processed, and the right term all the work that left. Any job that is half-way its service counts for full at the left, for half in the middle expression, and not in the right.

More formally, for any job $k$ and time $t$, we have $J_k\,\mathbb{1}_{A_k \le t} \ge \int_0^t \mathbb{1}_{A_k \le s < D_k}\,\mathrm{d}s \ge J_k\,\mathbb{1}_{D_k \le t}$. (To see this, fix $k$, and check the three cases $t < A_k$, $A_k \le t < D_k$, $D_k < t$.) Then,

$$\sum_{k=1}^{\infty} J_k\,\mathbb{1}_{A_k \le t} \ge \int_0^t \sum_{k=1}^{\infty} \mathbb{1}_{A_k \le t < D_k}\,\mathrm{d}s \ge \sum_{k=1}^{\infty} J_k\,\mathbb{1}_{D_k \le t}.$$

Finally, note that $\sum_{k=1}^{\infty} J_k \mathbb{1}_{A_k \le t} = \sum_{k=1}^{A(t)} J_k$ and $\sum_{k=1}^{\infty} J_k \mathbb{1}_{D_k \le t} = \sum_{k=1}^{D(t)} J_k$, and use the definition of $L(s)$.

**s.5.4.9.**

$$\lim_{t\to\infty} \frac{A(t)}{t} \frac{1}{A(t)} \sum_{k=1}^{A(t)} J_k \ge \lim_{t\to\infty} \frac{1}{t} \int_0^t L(s)\,\mathrm{d}s \ge \lim_{t\to\infty} \frac{D(t)}{t} \frac{1}{D(t)} \sum_{k=1}^{D(t)} J_k.$$

We use the strong law of large numbers to conclude that the limits converges to $n^{-1}\sum_{k=1}^{n} J_k \to \mathsf{E}[J]$, and we assume that $\{J_k, k \ge N\}$ is a sequence of iid random variables for $N$ sufficiently large.

**s.5.4.10.** $A(t) = D(t) + 10$ for $t > 10$. However, since $\delta = \lambda$, Little's law follows by [5.4.9].

**s.6.1.1.**

$$M_L(s) = \mathsf{E}\left[e^{sL}\right] = \sum_{n=0}^{\infty} e^{sn} p(n) = (1-\rho)\sum_{n} e^{sn}\rho^n = \frac{1-\rho}{1-e^s\rho},$$

where we assume that $s$ is such that $e^s\rho < 1$. Then,

$$\begin{aligned}
M_L'(s) &= (1-\rho)\frac{1}{(1-e^s\rho)^2} e^s\rho \implies \mathsf{E}[L] = M_L'(0) = \frac{\rho}{1-\rho},\\
\mathsf{E}\left[L^2\right] &= M''(0) = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho},\\
\mathsf{V}[L] &= \mathsf{E}\left[L^2\right] - (\mathsf{E}[L])^2 = \frac{\rho(1+\rho)}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^2} = \frac{\rho}{(1-\rho)^2},\\
\mathsf{P}\{L \ge n\} &= \sum_{k=n}^{\infty} p(k) = \sum_{k=n}^{\infty} p(0)\rho^k = (1-\rho)\sum_{k=n}^{\infty}\rho^k = (1-\rho)\rho^n \sum_{k=0}^{\infty}\rho^k = (1-\rho)\rho^n \frac{1}{1-\rho} = \rho^n.
\end{aligned}$$

**s.6.1.2.** Use the results of [2.2.18]; use PASTA to conclude that $\beta = p(K)$.

**s.6.1.3.** The fraction of time the system contains $n$ jobs is $\pi(n)$ (by PASTA). When the system contains $n > 0$ jobs, the number in queue is one less, i.e., $n-1$.

$$\mathsf{E}[Q] = \sum_{n=1}^{\infty} (n-1)\pi(n) = \sum_{n=1}^{\infty} n\pi(n) - \sum_{n=1}^{\infty}\pi(n) = \mathsf{E}[L] - (1-\pi(0)) = \mathsf{E}[L] - \rho.$$

**s.6.1.4.** For $n \ge 1$,

$$\begin{aligned}
p(n) &= \frac{\lambda(n-1)}{\mu(n)} p(n-1) = \frac{\lambda}{\min\{c,n\}\mu} p(n-1) = \frac{1}{\min\{c,n\}}(c\rho)p(n-1)\\
&= \frac{1}{\min\{c,n\}\min\{c,n-1\}}(c\rho)^2 p(n-2)\\
&= \frac{1}{\Pi_{k=1}^{n}\min\{c,k\}}(c\rho)^n p(0).
\end{aligned}$$

To obtain the normalization constant $G = 1/p(0)$,

$$\begin{aligned}
1 &= \sum_{n=0}^{\infty} p(n) = p(0) + \sum_{n=1}^{c-1} p(n) + \sum_{n=c}^{\infty} p(n) \\
&= p(0) + p(0) \sum_{n=1}^{c-1} \frac{(c\rho)^n}{n!} + p(0) \sum_{n=c}^{\infty} \frac{c^c}{c!} \rho^n \\
&= p(0) \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + p(0) \sum_{n=c}^{\infty} \frac{(c\rho)^c}{c!} \rho^{n-c} \\
&= p(0) \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + p(0) \frac{(c\rho)^c}{c!} \sum_{n=0}^{\infty} \rho^n \\
&= p(0) \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + p(0) \frac{(c\rho)^c}{c!(1-\rho)}.
\end{aligned}$$

Next,

$$\begin{aligned}
\mathsf{E}[Q] &= \sum_{n=c}^{\infty} (n-c) p(n) = \sum_{n=c}^{\infty} (n-c) \frac{c^c}{c!} \rho^n p(0) \\
&= \frac{c^c \rho^c}{Gc!} \sum_{n=c}^{\infty} (n-c) \rho^{n-c} = \frac{c^c \rho^c}{Gc!} \sum_{n=0}^{\infty} n \rho^n \\
&= \frac{c^c \rho^c}{Gc!} \frac{\rho}{(1-\rho)^2}.
\end{aligned}$$

The derivation of the expected number of jobs in service becomes easier if we pre-multiply the normalization constant $G$:

$$\begin{aligned}
G\mathsf{E}[L_s] &= G\left( \sum_{n=0}^{c} n p(n) + \sum_{n=c+1}^{\infty} c p(n) \right) \\
&= \sum_{n=1}^{c} n \frac{(c\rho)^n}{n!} + \sum_{n=c+1}^{\infty} c \frac{c^c \rho^n}{c!} = \sum_{n=1}^{c} \frac{(c\rho)^n}{(n-1)!} + \frac{c^{c+1}}{c!} \sum_{n=c+1}^{\infty} \rho^n \\
&= \sum_{n=0}^{c-1} \frac{(c\rho)^{n+1}}{n!} + \frac{(c\rho)^{c+1}}{c!} \sum_{n=0}^{\infty} \rho^n = c\rho \left( \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!(1-\rho)} \right).
\end{aligned}$$

Observe that the RHS is precisely equal to $\rho c G$, so that $\mathsf{E}[L_s] = c\rho$.

**s.6.1.5.** Take $c = 1$

$$\begin{aligned}
p(0) &= \frac{1}{G} \frac{(c\rho)^0}{0!} = \frac{1}{G}, \\
p(n) &= \frac{1}{G} \frac{c^c \rho^n}{c!} = \frac{1}{G} \frac{1^1 \rho^n}{1!} = \frac{\rho^n}{G}, \quad n = 1, 2, \ldots \\
G &= \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{(1-\rho)c!} = \sum_{n=0}^{0} \frac{\rho^0}{0!} + \frac{\rho}{(1-\rho)} = 1 + \frac{\rho}{1-\rho} = \frac{1}{1-\rho}, \\
\mathsf{E}[L] &= \frac{(c\rho)^c}{c!G} \frac{\rho}{(1-\rho)^2} = \frac{\rho}{1/(1-\rho)} \frac{\rho}{(1-\rho)^2} = \frac{\rho^2}{1-\rho}, \\
\mathsf{E}[L] &= \sum_{n=0}^{c} n p(n) + \sum_{n=c+1}^{\infty} c p(n) = p(1) + 1 \sum_{n=2}^{\infty} p(n) = 1 - p(0) = \rho.
\end{aligned}$$

Everything is in accordance to the formulas we derived earlier for the $M/M/1$ queue.

**s.6.1.6.** In the expressions for the $M/M/c$ queue, just neglect the parts that deal with states $n > c$. We see that $p(n) = p(0)(c\rho)^n/n!$. For the normalization $1 = \sum_{n=0}^{c} p(n) = p(0)\sum_{n=0}^{c} \frac{(c\rho)^n}{n!}$. Next,

$$\begin{aligned}
\mathsf{E}[L_s] &= \sum_{n=0}^{c} np(n) = \sum_{n=1}^{c} np(n) \\
&= G^{-1} \sum_{n=1}^{c} n\frac{(\lambda/\mu)^n}{n!} = G^{-1} \sum_{n=1}^{c} \frac{(\lambda/\mu)^n}{(n-1)!} \\
&= \frac{\lambda}{\mu G} \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} = \frac{\lambda}{G\mu}\left(G - \frac{(\lambda/\mu)^c}{c!}\right) \\
&= \frac{\lambda}{\mu}\left(1 - \frac{1}{G}\frac{(\lambda/\mu)^c}{c!}\right) \\
&= \frac{\lambda}{\mu}\big(1 - p(c)\big).
\end{aligned}$$

**s.6.1.7.** By taking the limit $c \to \infty$, note first that in (6.1.3b),

$$\frac{(c\rho)^c}{(1-\rho)c!} = \frac{(\lambda/\mu)^c}{(1-\rho)c!} \to 0, \quad \text{as } c \to \infty.$$

Hence

$$G = \sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{(1-\rho)c!} \to \sum_{n=0}^{\infty} \frac{(c\rho)^n}{n!} = e^{\lambda/\mu}.$$

Next, for any fixed $n$, eventually $c > n$, and then, as $\rho = \lambda/(\mu c)$,

$$p(n) = \frac{1}{G}\frac{(c\rho)^n}{n!} = \frac{1}{G}\frac{(\lambda/\mu)^n}{n!} \to e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad \text{as } c \to \infty.$$

**s.6.1.8.**

$$\begin{aligned}
p(n+1) &= \frac{(N-n)\lambda}{\mu} p(n) = \rho(N-n)p(n) \\
&= \rho^2(N-n)(N-(n-1))p(n-1) \\
&= \rho^3(N-n)(N-(n-1))(N-(n-2))p(n-2) \\
&= \rho^{n+1}(N-n)(N-(n-1))\cdots(N-(0))p(0) \\
&= \rho^{n+1}\frac{N!}{(N-(n+1))!}p(0).
\end{aligned}$$

**s.6.1.9.** Take $\lambda(n) = (N-n)\lambda$ and $\mu(n) = n\mu$. Then

$$\begin{aligned}
p(n+1) &= \frac{\lambda(n)}{\mu(n+1)}p(n) = \frac{(N-n)\lambda}{(n+1)\mu}p(n) = \frac{(N-n)(N-(n-1))}{(n+1)n}\frac{\lambda^2}{\mu^2}p(n-1) \\
&= \frac{N!}{(N-(n+1))!}\frac{1}{(n+1)!}\rho^{n+1}p(0) = \binom{N}{n+1}\rho^{n+1}p(0).
\end{aligned}$$

Therefore, $G = \sum_{k=0}^{N} \rho^k \binom{N}{k}$.

**s.6.1.10.** To take the limit $K \to \infty$—mind, not the limit $n \to \infty$—, write

$$G = \frac{1-\rho^{K+1}}{1-\rho} = \frac{1}{1-\rho} - \frac{\rho^{K+1}}{1-\rho}.$$

Since $\rho^{K+1} \to 0$ as $K \to \infty$ (recall, $\rho < 1$), we get

$$G \to \frac{1}{1-\rho},$$

as $K \to \infty$. Therefore, $p(n) = \rho^n/G \to \rho^n(1-\rho)$, and the latter are the steady-state probabilities of the $M/M/1$ queue. Finally, if the steady-state probabilities are the same, the performance measures (which are derived from $p(n)$) must be the same.

**s.6.1.11.**

$$\begin{aligned}
\mathsf{E}[L] &= \sum_{n=0}^{\infty} np(n) = \sum_{n=0}^{\infty}\sum_{i=1}^{n} \mathbb{1}_{i\le n}p(n) && n = \sum_{i=1}^{n}\mathbb{1}_{i\le n}\\
&= \sum_{n=0}^{\infty}\sum_{i=1}^{\infty} \mathbb{1}_{i\le n}p(n) && i > n \implies \mathbb{1}_{i\le n} = 0\\
&= \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} \mathbb{1}_{i\le n}p(n) = \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} p(n) && n < i \implies \mathbb{1}_{i\le n} = 0\\
&= \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} (1-\rho)\rho^n && p(n) = (1-\rho)\rho^n\\
&= \sum_{i=1}^{\infty}\sum_{n=0}^{\infty} (1-\rho)\rho^{n+i} && n \to n+i\\
&= \sum_{i=1}^{\infty}(1-\rho)\rho^i \sum_{n=0}^{\infty}\rho^n && \rho^{n+i} = \rho^i\rho^n\\
&= \sum_{i=1}^{\infty}(1-\rho)\rho^i \frac{1}{1-\rho} = \sum_{i=1}^{\infty}\rho^i = \sum_{i=0}^{\infty}\rho^{i+1} && i \to i+1\\
&= \rho\sum_{i=0}^{\infty}\rho^i = \frac{\rho}{1-\rho}.
\end{aligned}$$

Note that, since the summands are positive, we can use Fubini's theorem to justify the interchange of the summations.

**s.6.1.12.** Differentiate the left- and RHS of $(1-\rho)^{-1} = \sum_{n=0}^{\infty}\rho^n$ with respect to $\rho$ and then multiply with $\rho$ to get

$$\frac{\rho}{(1-\rho)^2} = \sum_{n=0}^{\infty} n\rho^n.$$

Then multiply both sides by $1-\rho$ and use that $p(n) = (1-\rho)\rho^n$ to get $\mathsf{E}[L]$.

Differentiating and multiplying with $\rho$ a second time yields

$$\begin{aligned}
\rho\frac{(1-\rho)^2 + \rho 2(1-\rho)}{(1-\rho)^4} &= \rho\frac{1-2\rho+\rho^2+2\rho-2\rho^2}{(1-\rho)^4} = \rho\frac{1-\rho^2}{(1-\rho)^4}\\
&= \rho\frac{1+\rho}{(1-\rho)^3} = \sum_{n=0}^{\infty} n^2\rho^n,
\end{aligned}$$

and hence for $\mathsf{E}\left[L^2\right]$,

$$\begin{aligned}
(1-\rho)\sum_{n=0}^{\infty} n^2\rho^n = \rho\frac{1+\rho}{(1-\rho)^2} &= \frac{\rho}{(1-\rho)^2} + \frac{\rho^2}{(1-\rho)^2} = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{(1-\rho)^2} - \frac{\rho^2}{(1-\rho)^2}\\
&= \frac{2\rho^2}{(1-\rho)^2} + \rho\frac{(1-\rho)}{(1-\rho)^2} = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho}.
\end{aligned}$$

Recall that $p(n) = (1-\rho)\rho^n$.

**s.6.2.1.** With Little's law:

Python Code

```
>>> labda = 5. # per minute
>>> W = 1.
>>> Q = labda*W
>>> Q
5.0
```

We can use the expression for $\mathsf{E}[W]$ to solve for $\rho$, but we can also do a simple search.

Python Code

```
>>> from scipy.optimize import bisect

>>> EW = 1

>>> def find_W(rho): # return W -1 for given rho
...     ES = rho / labda
...     return rho / (1 - rho) * ES - EW
...

>>> rho = bisect(find_W, 0, 0.999)
>>> rho
0.8541019662513663
>>> ES = rho/labda
>>> ES
0.17082039325027326
>>> J = W + ES
>>> J
1.1708203932502732
>>> L = labda*J
>>> L
5.854101966251366
```

Next, find $n$ such that $\sum_{j=0}^{n} p_j > 0.9$.

Python Code

```
>>> n, total, p = 0, 0.0, 1 - rho
>>> while total <= 0.9:
...     total += p
...     n += 1
...     p *= rho
...
>>> total
0.9061042738302431
>>> n
15
```

Note that we increased $n$ one time too often. As a check, use that $(1-\rho)\sum_{j=0}^{n}\rho^j = 1-\rho^{n+1}$.

Python Code

```
>>> n -= 1 # get the right n
>>> 1-rho**(n) # this must be too small.
```

```
0.8900649689616529
>>> 1-rho**(n+1) # this must be OK.
0.9061042738302427
```

**s.6.2.2.** Start with the data, then do the computations.

Python Code

```
>>> c = 2
>>> p = [0.4, 0.3, 0.2, 0.05, 0.05]
>>> labda = 10.0 / 60 # per hour

>>> EL = sum(n * p[n] for n in range(len(p)))
>>> EL
1.05
>>> EQ = sum(max(n - c, 0) * p[n] for n in range(len(p)))
>>> EQ
0.15000000000000002
>>> W = EQ / labda  # in minutes
>>> W
0.9000000000000001
>>> J = EL / labda  # in minutes
>>> J
6.300000000000001
>>> var_L = sum((n - EL) ** 2 * p[n] for n in range(len(p)))
>>> var_L
1.2475
>>> var_Q = sum((max(n - c, 0) - EQ) ** 2 * p[n] for n in range(len(p)))
>>> var_Q
0.22750000000000004
>>> ES = J - W
>>> rho = labda * ES / c
>>> rho
0.45
```

The expected number of busy servers is the average number in the system minus the average number in queue. It should also be equal to $\sum_{n=0}^{\infty} \min\{n, c\} p(n)$. Let's check.

Python Code

```
>>> EBusy = EL - EQ
>>> EBusy
0.9
>>> u = sum(min(n, c) * p[n] for n in range(len(p)))
>>> u
0.8999999999999999
```

**s.6.2.3.** We import `numpy` and convert the lists to arrays is to fix the output precision to 3, otherwise we get long floats in the output.

First the case with $b = 2$.

Python Code

```
>>> import numpy as np
>>> np.set_printoptions(precision=3)
```

```
>>> labda, mu, c = 6.0,  5.0, 1
>>> rho = labda / mu
>>> K = c + 2


>>> p = np.array([rho ** n for n in range(K + 1)]) # range(n) is up to n
>>> G = sum(p)
>>> p /= G  # normalize
>>> p
array([0.186, 0.224, 0.268, 0.322])
>>> L = sum(n * p[n] for n in range(len(p)))
>>> L
1.725782414307004
>>> Q = sum(max(n - c,0) * p[n] for n in range(len(p)))
>>> Q
0.9120715350223545
>>> lost = labda * p[-1]  # the last element of p
>>> labda - lost # accepted, hence served
4.06855439642325
```

Now increase the buffer $b$ to 5.

Python Code

```
>>> K = c + 5
>>> p = np.array([rho ** n for n in range(K + 1)]) # range(n) is up to n
>>> G = sum(p)
>>> p /= G  # normalize
>>> lost = labda * p[-1]  # the last element of P
>>> labda - lost # accepted, hence served
4.612880368265357
```

We see that since the server is overloaded, the acceptance is not much affected by increasing the buffer space. We need an extra server.

**s.6.2.4.** The computations.

Python Code

```
import numpy as np

labda = np.array([10.0, 15.0, 15.0, 10.0, 5.0])
mu = np.array([0., 5., 10., 10., 10., 10.])
c = 2

p = np.ones_like(mu)
for i in range(len(labda)):
    p[i+1] = labda[i] *p[i]/ mu[i+1]

p /= p.sum()
labdaBar = sum(labda[n] * p[n] for n in range(len(labda)))
L = sum(n * p[n] for n in range(len(p)))
Q = sum(max(n - c,0) * p[n] for n in range(len(p)))
```

```
J = L / labdaBar
W = Q / labdaBar
```

**s.6.2.5.** What is the load?

Python Code

```
>>> labda = 28.0 / 60  # arrivals per minute
>>> ES = 4.0
>>> labda * ES
1.8666666666666667
```

Clearly, we need at least two servers.

Python Code

```
>>> from math import factorial


>>> def Q(labda, ES, c):
...     rho = labda * ES / c
...     G = sum([(c * rho) ** n / factorial(n) for n in range(c)])
...     G += (c * rho) ** c / (1.0 - rho) / factorial(c)
...     return (c * rho) ** c / (factorial(c) * G) * rho / (1.0 - rho) ** 2
...
>>> Q(labda, ES, c=2)/labda  # in minutes, Little's law
27.034482758620694
>>> Q(labda, ES, c=3)/labda  # in minutes, Little's law
1.3542675591474136
>>> Q(labda, ES, c=4)/labda  # in minutes, Little's law
0.26778942672317635
```

Since both types of workers cost the same amount of money per unit time, it is best to divide the amount of waiting/idleness equally over both types of workers. The average cost of workers waiting in queue is proportional to $\mathsf{E}[Q]$. At the crib, the load is $\lambda \mathsf{E}[S]$, hence, the average number of idle server is $c - \lambda \mathsf{E}[S]$.

Python Code

```
>>> c = 2
>>> Q(labda, ES, c) + (c - labda * ES)
12.749425287356324
>>> c = 3
>>> Q(labda, ES, c) + (c - labda * ES)
1.7653248609354597
>>> c = 4
>>> Q(labda, ES, c) + (c - labda * ES)
2.2583017324708154
```

**s.6.2.6.** From the figure in the hint, the situation with the taxis corresponds to an $M/M/1$ queue, only the states have a 'different name'. Let $l$ be the number of jobs in an M/M/1 queue. To make a mapping from $l$ to the number of parties $i$ and number of taxis $j$, observe that $l = 3 - j + i$ and $\mathbb{1}_{j>0}\mathbb{1}_{i>0} = 0$, because riders don't wait when there are taxis available. To see this, consider the following table

| $j$ | $i$ | $l$ |
|---|---|---|
| 3 | 0 | 0 |
| 2 | 0 | 1 |
| 1 | 0 | 2 |
| 0 | 0 | 3 |
| 0 | 1 | 4 |
| 0 | 2 | 5 |

With this mapping, the expected number of taxis $T$ is $\mathsf{E}[T] = \sum_{l=0}^{3}(3-l)p(l)$.

Python Code

```
>>> labda = 12.0  # per hour
>>> mu = 15.0  # per hour
>>> rho = labda / mu
>>> n = 3  # number of taxis

>>> ET, p = 0, 1 - rho
>>> for l in range(n):
...     ET += (n - l) * p
...     p *= rho
...
>>> ET
1.0479999999999998
```

For the expected number of groups waiting note that $l = 3 - j + i$ always. Taking expectations, we see that $\mathsf{E}[L] = 3 - \mathsf{E}[T] + \mathsf{E}[G]$, where $\mathsf{E}[G]$ is the expected number of groups.

Python Code

```
>>> EL = rho/(1-rho)
>>> EG = EL-n + ET
>>> EG
2.048000000000001
```

Computing the waiting times is tricky. For the taxis, the rate at which 'jobs' arrive, is the arrival rate of the riders. For the riders, the rate at which 'jobs' arrive is the arrival rate of taxis.

Python Code

```
>>> ET/labda # Waiting for taxis
0.08733333333333332
>>> EG/mu # waiting time for groups
0.1365333333333334
```

What would be the impact of allowing 4 cabs?

Python Code

```
>>> n = 4  # number of taxis

>>> ET, p = 0, 1 - rho
>>> for l in range(n):
...     ET += (n - l) * p
...     p *= rho
...
```

```
>>> ET
1.6383999999999999
>>> EG = EL-n + ET
>>> EG
1.6384000000000007
```

**s.6.2.7.** This is a case with $n = 0$, hence $\mathsf{E}[G] = \mathsf{E}[L]$.

**s.6.2.8.** We concentrate on departure epochs of the taxis. Thus, the $k$th period is the time between the departure of taxi $k-1$ and taxi $k$. During the $k$th epoch $a_k$ batches can arrive. The system starts with $a_0$ batches in queue.

More generally, consider the arrival of taxi $k$. Let this taxi see $Q_{s,k}$ riders at the head of the line. Let $b$ be the index of the first group in the queue, hence group $b-1$ stands at the head of the line. Then,

$$\begin{aligned}
d_k &= \min\{Q_{s,k}, 4\}, && \text{riders served by taxi } k,\\
Q'_s &= Q_{s,k} - d_k, && \text{riders remaining behind at head of line,}\\
Q'_k &= Q_k + a_{k+1}, && \text{groups just before arrival taxi } k+1,\\
h_k &= \mathbb{1}_{Q'_s>0}\,\mathbb{1}_{Q'_k>0} && \text{If } h_k = 1 \text{ a group can move to the head of the line,}\\
Q_{k+1} &= Q'_k - h_k && \text{queue of groups seen by taxi } k+1,\\
Q_{s,k+1} &= Q'_s(1-h_k) + h_k B_b && \text{riders at head of the line seen by taxi } k+1,\\
b &= b + h_k && \text{increase index of served batches by one,}
\end{aligned}$$

**s.6.3.1.** Start with the simple case. $B \equiv 2 \implies \mathsf{V}[B] = 0 \implies C_s^2 = 0$, $\rho = \lambda\,\mathsf{E}[B]\,\mathsf{E}[S] = 1{\cdot}2{\cdot}25/60 = 5/6$. Hence,

$$\mathsf{E}[L] = \frac{1}{2}\frac{5/6}{1/6}2 + \frac{1}{2}\frac{5/6}{1/6} = 5 + \frac{5}{2}.$$

Now the other case. $\mathsf{E}\left[B^2\right] = (1+4+9)/3 = 14/3$. Hence, $\mathsf{V}[B] = 14/3 - 4 = 2/3$. Hence, $C_s^2 = \frac{1}{6}$. And thus,

$$\mathsf{E}[L] = \frac{1+1/6}{2}\frac{5/6}{1/6}2 + \frac{1}{2}\frac{5/6}{1/6} = \frac{7}{6}5 + \frac{5}{2}.$$

The ratio between $\mathsf{E}[L]$ is 10/9. A reduction of about 10% in waiting time can achieved by working in fixed batch sizes.

**s.6.3.2.**

$$\frac{\mathsf{E}\left[W(M^X/M/1)\right]}{\mathsf{E}[W(M/M/1)]} = \frac{\mathsf{E}\left[L_s(M^X/M/1)\right]}{\mathsf{E}[L_s(M/M/1)]} = \frac{\mathsf{E}\left[L_s(M^X/M/1)\right]}{\rho} = \frac{\mathsf{E}\left[B^2\right]}{2\,\mathsf{E}[B]} + \frac{1}{2}.$$

The RHS is $\geq 1$, because $\mathsf{V}[B] \geq 0 \implies \mathsf{E}[B]^2 \geq (\mathsf{E}[B])^2$. Clearly, if variability increases, the average waiting time increases

**s.6.3.3.** We need $\mathsf{V}[B]$ and $\mathsf{E}[B]$. For this,

$$\begin{aligned}
M_B(s) &= \mathsf{E}\left[e^{sB}\right] = \sum_{k=0}^{\infty} e^{sk}\,\mathsf{P}\{B=k\} \\
&= \sum_{k=0}^{\infty} e^{sk} p q^{k-1} = \frac{p}{q}\sum_{k=0}^{\infty}(qe^s)^k = \frac{p}{q}\frac{1}{1-qe^s}, \\
\mathsf{E}[B] &= M_B'(0) = \frac{p}{q}\frac{q}{(1-qe^s)^2}\bigg|_{s=0} = \frac{p}{(1-q)^2} = \frac{1}{p}, \\
\mathsf{E}\left[B^2\right] &= M_B''(0) = \frac{2}{p^2} - \frac{1}{p}, \\
\mathsf{V}[B] &= \mathsf{E}\left[B^2\right] - (\mathsf{E}[B])^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p}, \\
C_s^2 &= \frac{\mathsf{V}[B]}{(\mathsf{E}[B])^2} = p^2\left(\frac{1}{p^2} - \frac{1}{p}\right) = 1-p, \\
(1+C_s^2)/2 &= 1 - p/2, \\
\mathsf{E}[L] &= \left(1-\frac{p}{2}\right)\frac{\rho}{1-\rho}\frac{1}{p} + \frac{1}{2}\frac{\rho}{1-\rho} = \frac{\rho}{1-\rho}\frac{1}{p}.
\end{aligned}$$

$\mathsf{E}[B] = 1 \implies p = 1 \implies \mathsf{E}[L] = \rho/(1-\rho)$.

**s.6.3.4.** Suppose the $k$th job has batch size $B$, then

$$X_k = \int_{D_{k-1}}^{D_k} L_s^B(s)\,\mathrm{d}s = BS_{k,1} + (B-1)S_{k,2} + \cdots + S_{B,B}.$$

Hence, since $S_{k,i}$ are iid,

$$\begin{aligned}
\mathsf{E}[X_k|B_k = B] &= B\,\mathsf{E}[S] + (B-1)\,\mathsf{E}[S] + \cdots \mathsf{E}[S] = B(B+1)/2 \cdot \mathsf{E}[S], \\
\mathsf{E}[X_k] &= \mathsf{E}[\mathsf{E}[X_k|B_k = B]] = \mathsf{E}[B(B+1)/2]\,\mathsf{E}[S].
\end{aligned}$$

**s.6.3.5.** From [6.3.4], rate stability ($\delta = \lambda$), $\rho = \lambda\,\mathsf{E}[B]\,\mathsf{E}[S]$,

$$\mathsf{E}\left[L_s^B\right] = Y = \delta X = \lambda\frac{\mathsf{E}\left[B^2\right] + \mathsf{E}[B]}{2}\mathsf{E}[S] = \lambda\frac{\mathsf{E}\left[B^2\right]}{2}\mathsf{E}[S] + \frac{\rho}{2}.$$

**s.6.3.6.** We have

$$\frac{\mathsf{E}\left[B^2\right]}{(\mathsf{E}[B])^2} = \frac{\mathsf{E}\left[B^2\right] - (\mathsf{E}[B])^2 + (\mathsf{E}[B])^2}{(\mathsf{E}[B])^2} = \frac{\mathsf{V}[B] + (\mathsf{E}[B])^2}{(\mathsf{E}[B])^2} = C_s^2 + 1.$$

**s.6.3.7.**

$$\begin{aligned}
\sum_{i=1}^{\infty} iG(i-1) &= \sum_{i=0}^{\infty}(i+1)G(i) = \sum_{i=0}^{\infty} iG(i) + \sum_{i=0}^{\infty} G(i) \\
&= (\mathsf{E}\left[B^2\right] - \mathsf{E}[B])/2 + \mathsf{E}[B].
\end{aligned}$$

**s.6.4.1.** $\rho = \lambda\,\mathsf{E}[S] = (3/8)\cdot 2 = 3/4$, $\mathsf{E}[W] = 4.5$ h. If we were able to reduce all service variability, i.e., $C_s^2 = 0$, then still $\mathsf{E}[W] = 3$h. Hence, we have to increase capacity, or reduce $\mathsf{E}[S]$. Another possibility is to plan the arrival of jobs such that $C_a^2 = 0$. However, typically this is not possible. Would you accept that the supermarket plans your visits?

**s.6.4.2.** $\mathsf{V}[S]=0 \implies C_s^2=0 \implies \mathsf{E}[W(M/D/1)]=\mathsf{E}[W(M/M/1)]/2$.

**s.6.4.3.**

$$\mathsf{E}[S]=\alpha/2, \qquad \mathsf{E}\left[S^2\right]=\int_0^\alpha x^2\,\mathrm{d}x/\alpha=\alpha^2/3,$$
$$\mathsf{V}[S]=\alpha^2/3-\alpha^2/4=\alpha^2/12, \qquad C_s^2=(\alpha^2/12)/(\alpha^2/4)=1/3,$$
$$\rho=\lambda\alpha/2,$$
$$\mathsf{E}[W]=\frac{1+C_s^2}{2}\frac{\lambda\alpha/2}{1-\lambda\alpha/2}\frac{\alpha}{2}, \qquad \mathsf{E}[J]=\mathsf{E}[W]+\frac{\alpha}{2}.$$

**s.6.4.4.**

$$\lambda\pi(0)\,\mathsf{E}[S]=1-\pi(0) \iff \pi(0)=\frac{1}{1+\lambda\,\mathsf{E}[S]} \iff 1-\pi(0)=\frac{\lambda\,\mathsf{E}[S]}{1+\lambda\,\mathsf{E}[S]}.$$

There is another way to derive this. The system can contain at most 1 job. Necessarily, if the system contains a job, this job must be in service. All jobs that arrive while the server is busy are rejected. Just after a departure, the average time until the next arrival is $1/\lambda$, and then a new service starts with an average duration of $\mathsf{E}[S]$. After this departure, a new cycle starts. Thus, $\rho=\mathsf{E}[S]/(1/\lambda+\mathsf{E}[S])=\lambda\,\mathsf{E}[S]/(1+\lambda\,\mathsf{E}[S])$.

For the $G/G/1/1$ queue the PASTA property does not hold, hence $p(0)\neq\pi(0)$ in general.

**s.6.4.5.** Right after the server becomes free, the time to a new arrival is $\sim \mathrm{Exp}()\lambda)$, with mean $1/\lambda$. For $\mathsf{E}[U]$, solve the expression of the hint with $\mathsf{E}[I]=1/\lambda$. To see why $\rho=\mathsf{E}[U]/(\mathsf{E}[I]+\mathsf{E}[U])$, apply the renewal reward equation. Take $Y(t)=\int_0^t \mathbb{1}_{L_s(s)=1}\,\mathrm{d}t$. Taking as sampling epochs the departures that leave an empty system behind, $X_k=U_k$ and $T_k=I_k+U_k$, where $U_k$ is the $k$th busy time, and $I_k$ the $k$th idle time. Then $Y(t)/t\to\rho$, $X_k/k\to\mathsf{E}[U]$, and $k/(I_k+U_k)\to 1/(\mathsf{E}[I]+\mathsf{E}[U])$, which is the $\lambda$ we use in the renewal-reward theorem.

We can also obtain $\mathsf{E}[U]$ by means of a recursion. The first customer starts a busy time of average duration $\mathsf{E}[S]$. However, during this service $\lambda\,\mathsf{E}[S]$ new jobs arrive, in expectation. Each of these jobs restarts the busy-period. Hence, $\mathsf{E}[U]=\mathsf{E}[S]+\lambda\,\mathsf{E}[S]\,\mathsf{E}[U]$.

**s.6.4.6.** At time $s$, the number of departures is $D(s)$. Thus, $D(s)+1$ is the first job to depart after time $s$. The departure time of this job is $D_{D(s)+1}$, hence the remaining service time at time $s$ is $D_{D(s)+1}-s$, provided this job is in service.

**s.6.4.7.**

$$\lambda\,\mathsf{E}\left[S^2\right]=\frac{\mathsf{E}\left[S^2\right]}{(\mathsf{E}[S])^2}\lambda(\mathsf{E}[S])^2=\frac{\mathsf{E}\left[S^2\right]}{(\mathsf{E}[S])^2}\rho\,\mathsf{E}[S]=(1+C_s^2)\rho\,\mathsf{E}[S].$$

**s.6.4.8.** The probability to find the server busy upon arrival is $\rho$, and $S_r>0 \iff L>0$. Therefore,

$$\mathsf{E}[S_r]=\rho\,\mathsf{E}[S_r\,|\,S_r>0]+(1-\rho)\,\mathsf{E}[S_r\,|\,S_r=0]=\rho\,\mathsf{E}[S_r\,|\,S_r>0].$$

**s.6.4.9.** For the $M/M/1$, service times are memoryless, hence, $\mathsf{E}[S_r\,|\,S_r>0]=\mathsf{E}[S]$. But, from [6.4.8], $\mathsf{E}[S_r]=\rho\,\mathsf{E}[S_r\,|\,S_r>0]$ for the $M/G/1$ queue. The difference is due to the fact that only jobs that arrive at a busy system see $S_r>0$.

**s.6.4.10.** $\rho=\sum_{i=1}^\infty p(i)=\sum_{i=1}^\infty \pi(i)=\sum_{i=1}^\infty \delta(i)$.

**s.6.4.11.** This is not a proof because we assume the existence of a density $f$.

The rate of arriving jobs that see a waiting time $y < x$ is equal to $\lambda f(y)$. To up-cross level $x$ from state $y$, the service time must be at least $x-y$, which has probability $G(x-y) = \mathsf{P}\{S > x-y\}$. Adding up the rates for all possible waiting times below $x$ gives that $\lambda \int_0^x f(y)G(x-y)\,\mathrm{d}y$ is the up-crossing rate of level $x$. The downcrossing rate is $f(x)$ because a fraction $f(x)$ of waiting time is served per unit time at level $x$. Equaling these rates gives the integral equation $\lambda \int_0^x f(y)G(x-y)\,\mathrm{d}y = f(x)$. For $x = 0$, we need to consider the atom $\mathsf{P}\{X = 0\}$. In that case, $\lambda\,\mathsf{P}\{W = 0\} = f(x+)$. We can combine all this in one equation $\lambda \int_0^x G(x-y)\,\mathrm{d}F(y) = f(x)$.

To derive the PK formula, use the above integral equation. Noting that $x\,dF(x) = 0$ and using [1.2.8] and [1.2.9], we get

$$\begin{aligned}
\mathsf{E}[W] &= \int_0^\infty x f(x)\,\mathrm{d}x = \lambda \int_0^\infty x \int_0^x G(x-y)\,\mathrm{d}F(y)\,\mathrm{d}x = \\
&= \lambda \int_0^\infty \int_0^\infty x\,\mathbb{1}_{y\le x} G(x-y)\,\mathrm{d}F(y)\,\mathrm{d}x \\
&= \lambda \int_0^\infty \int_0^\infty x\,\mathbb{1}_{y\le x} G(x-y)\,\mathrm{d}x\,\mathrm{d}F(y) \\
&= \lambda \int_0^\infty \int_0^\infty (u+y)G(u)\,\mathrm{d}u\,\mathrm{d}F(y) \\
&= \lambda \int_0^\infty (\mathsf{E}\left[S^2\right]/2 + y\,\mathsf{E}[S])\,\mathrm{d}F(y) \\
&= \lambda\,\mathsf{E}\left[S^2\right]/2 + \lambda\,\mathsf{E}[S]\,\mathsf{E}[W].
\end{aligned}$$

This gives (6.4.1) right away.

**s.6.5.1.** In the $M/M/1$ queue, $G(0) = 1$ and $G(1) = G(2) = \cdots = 0$. Thus, $\sum_{m=0}^{n} G(n-m)\pi(m) = G(0)\pi(n) = \pi(n)$.

**s.6.5.2.** We use that $\mu\pi(n) = \lambda \sum_{i=0}^{n-1} \pi(i)G(n-1-i)$ and the results of the exercises of Section 6.3 to see that

$$\begin{aligned}
\mu\,\mathsf{E}[L] &= \sum_{n=0}^{\infty} n\,\mu\pi(n), \quad \text{now substitute for } \mu\pi(n) \text{ the recursion (6.5.1)}, \\
&= \lambda \sum_{n=0}^{\infty} n \sum_{i=0}^{n-1} \pi(i)G(n-1-i) = \lambda \sum_{n=0}^{\infty} n \sum_{i=0}^{\infty} \mathbb{1}_{i<n}\pi(i)G(n-1-i) \\
&= \lambda \sum_{i=0}^{\infty} \pi(i) \sum_{n=0}^{\infty} \mathbb{1}_{i<n} nG(n-1-i) = \lambda \sum_{i=0}^{\infty} \pi(i) \sum_{n=i+1}^{\infty} nG(n-1-i) \\
&= \lambda \sum_{i=0}^{\infty} \pi(i) \sum_{n=0}^{\infty} (n+i+1)G(n) = \lambda \sum_{i=0}^{\infty} \pi(i) \left[ \sum_{n=0}^{\infty} nG(n) + (i+1) \sum_{n=0}^{\infty} G(n) \right] \\
&= \lambda \sum_{i=0}^{\infty} \pi(i) \sum_{n=0}^{\infty} nG(n) + \lambda\,\mathsf{E}[B] \sum_{i=0}^{\infty} \pi(i)(i+1) \\
&= \lambda \sum_{i=0}^{\infty} \pi(i) \frac{\mathsf{E}\left[B^2\right] - \mathsf{E}[B]}{2} + \lambda\,\mathsf{E}[B]\,(\mathsf{E}[L]+1) \\
&= \lambda \frac{\mathsf{E}\left[B^2\right] - \mathsf{E}[B]}{2} + \lambda\,\mathsf{E}[B]\,\mathsf{E}[L] + \lambda\,\mathsf{E}[B] \\
&= \lambda \frac{\mathsf{E}\left[B^2\right]}{2} + \lambda\,\mathsf{E}[B]\,\mathsf{E}[L] + \lambda \frac{\mathsf{E}[B]}{2}.
\end{aligned}$$

Dividing both sides by $\mu$ and using that $\lambda \mathsf{E}[B]/\mu = \rho$, we obtain

$$EL = \lambda \frac{\mathsf{E}[B^2]}{2} \mathsf{E}[S] + \rho \mathsf{E}[L] + \frac{\rho}{2}.$$

By bringing $\rho \mathsf{E}[L]$ to the LHS, the RHS becomes equal to (6.3.4).

**s.6.5.3.** Jobs can arrive in batches larger than 1, but items leave one by one.

**s.6.5.4.** The complete-acceptance policy is actually quite simple. As any batch will be accepted when $n \le K$, the queue length is not bounded. Only when the number of items in the system is larger than $K$, we do not accept jobs.

$$\mu\pi(n+1) = \begin{cases} \lambda \sum_{m=0}^{n} \pi(m) G(n-m), & \text{for } n \le K, \\ \lambda \sum_{m=0}^{K} \pi(m) G(n-m), & \text{for } n > K. \end{cases}$$

**s.6.5.5.** For the partial acceptance case, any job is accepted, but the system only admits whatever fits. As level $n \in 0, 1, \ldots, K-1$ is still up-crossed by any batch of size at least $n-m$ when the system is in state $m$, the formula for the up-crossing rate is identical to the case without this acceptance policy. Hence, $\mu\pi(n+1) = \lambda \sum_{m=0}^{n} \pi(m) G(n-m)$, for $n = 0, 1, \ldots, K-1$.

**s.6.5.6.** Suppose a batch of size $k$ arrives when the system contains $m$ items. When $m+k \le K$, the batch can be accepted since the entire batch will fit into the queue, otherwise it will be rejected. Further, level $n, 0 \le n < K$, can only be crossed when $m+k > n$. Thus, for $n = 0, \ldots, K-1$,

$$\begin{aligned} \mu\pi(n+1) &= \lambda \sum_{m=0}^{n} \pi(m) \mathsf{P}\{n-m < B \le K-m\} \\ &= \lambda \sum_{m=0}^{n} \pi(m) [G(n-m) - G(K-m)], \end{aligned}$$

Let us check this for some simple cases. First, when $K \to \infty$, then $G(K-m) \to 0$, so we get our earlier result. Second, take $n = K$. Then $G(n-m) - G(K-m) = 0$ for all $m$, so that the RHS is 0, as it should. Third, take $n = 0$ and $K = 1$, then $\mu\pi(1) = \lambda\pi(0)$. This also makes sense.

**s.6.6.1.** After job $k-1$ left, job $k$ first has to arrive. Hence, $\mathsf{E}[D_k - D_{k-1}] = \mathsf{E}[X_k + S_k] = 1/\lambda + \mathsf{E}[S]$, where we use that $X_k$ is memoryless.

**s.6.6.2.** Since $X \sim \text{Exp}(()\lambda)$ and $S \sim \text{Exp}(()\mu)$, and $X$ and $S$ are independent, their joint density is $f_{X,S}(x,y) = \lambda\mu e^{-\lambda x - \mu y}$. With this,

$$\begin{aligned}
\mathsf{P}\{X+S \le t\} &= \lambda\mu \int_0^\infty \int_0^\infty e^{-\lambda x-\mu y} \mathbb{1}_{x+y\le t}\,\mathrm{d}x\,\mathrm{d}y \\
&= \lambda\mu \int_0^t \int_0^{t-x} e^{-\lambda x-\mu y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda\mu \int_0^t e^{-\lambda x} \int_0^{t-x} e^{-\mu y}\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda \int_0^t e^{-\lambda x}(1-e^{-\mu(t-x)})\,\mathrm{d}x \\
&= \lambda \int_0^t e^{-\lambda x}\,\mathrm{d}x - \lambda e^{-\mu t} \int_0^t e^{(\mu-\lambda)x}\,\mathrm{d}x \\
&= 1 - e^{-\lambda t} - \frac{\lambda}{\mu-\lambda} e^{-\mu t}(e^{(\mu-\lambda)t}-1) \\
&= 1 - e^{-\lambda t} - \frac{\lambda}{\mu-\lambda} e^{-\lambda t} + \frac{\lambda}{\mu-\lambda} e^{-\mu t} \\
&= 1 - \frac{\mu}{\mu-\lambda} e^{-\lambda t} + \frac{\lambda}{\mu-\lambda} e^{-\mu t}.
\end{aligned}$$

The density $f_{X+S}(t)$ is the derivative of this expression with respect to $t$, hence,

$$\begin{aligned}
f_{X+S}(t) &= \frac{\lambda\mu}{\mu-\lambda} e^{-\lambda t} - \frac{\mu\lambda}{\mu-\lambda} e^{-\mu t} \\
&= \frac{\lambda\mu}{\lambda-\mu}(e^{-\mu t} - e^{-\lambda t}).
\end{aligned}$$

Conditioning is much faster:

$$\begin{aligned}
f_{X+S}(t) = \mathsf{P}\{X+S \in \mathrm{d}t\} &= \int_0^t \mathsf{P}\{S+x \in \mathrm{d}t\}\,\mathsf{P}\{X \in \mathrm{d}x\} \\
&= \int_0^t f_S(t-x) f_X(x)\,\mathrm{d}x = \int_0^t \mu e^{-\mu(t-x)} \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \lambda\mu e^{-\mu t} \int_0^t e^{x(\mu-\lambda)}\,\mathrm{d}x = \frac{\lambda\mu}{\mu-\lambda} e^{-\mu t}\left(e^{(\mu-\lambda)t}-1\right).
\end{aligned}$$

**s.6.6.3.** Use conditional probability to see that

$$\begin{aligned}
\mathsf{P}\{Y_n = j\} &= \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!}\,\mathrm{d}F(x) = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} \mu e^{-\mu x}\,\mathrm{d}x = \frac{\mu}{j!}\lambda^j \int_0^\infty e^{-(\lambda+\mu)x} x^j\,\mathrm{d}x \\
&= \frac{\mu}{j!}\left(\frac{\lambda}{\lambda+\mu}\right)^j \int_0^\infty e^{-(\lambda+\mu)x}((\lambda+\mu)x)^j\,\mathrm{d}x = \frac{\mu}{j!}\left(\frac{\lambda}{\lambda+\mu}\right)^j \frac{j!}{\lambda+\mu}.
\end{aligned}$$

Method 2. Consider the Poisson process with rate $\lambda+\mu$, and thin with probability $\mu/(\lambda+\mu)$. Then the probability that $j$ 'failures' occur before a 'success' is precisely $\mathsf{P}\{Y=j\}$.

**s.6.6.4.** Take $\alpha = \lambda/(\lambda+\mu)$ so that $f(j) = (1-\alpha)\alpha^j$.

$$G(j) = \sum_{k=j+1}^\infty f(k) = (1-\alpha) \sum_{k=j+1}^\infty \alpha^k = (1-\alpha)\alpha^{j+1} \sum_{k=0}^\infty \alpha^k = \alpha^{j+1}.$$

**s.6.6.5.** Observe that $f(j) = (1-\alpha)\alpha^j$, and $G(j) = \alpha^{j+1}$. As the normalization factor cancels at both sides, we drop the normalization and just write $\pi(n) = \rho^n$ to simplify the algebra.

For $n = 0$: $f(0)\pi(1) = \pi(0)G(0) \iff (1-\alpha)\rho = 1\,\alpha$, and this checks with the hint. For $n \geq 1$:

$$\begin{aligned}(1-\alpha)\rho^{n+1} &= \pi(0)G(n) + \sum_{m=1}^{n} \pi(m)G(n+1-m) = \alpha^{n+1} + \sum_{m=1}^{n} \rho^m \alpha^{n-m+2} \\ &= \alpha^{n+1} + \alpha^{n+2} \sum_{m=1}^{n} (\rho/\alpha)^m = \alpha^{n+1} + \alpha^{n+1}\rho \sum_{m=0}^{n-1} (\rho/\alpha)^m = \alpha^{n+1} + \alpha^{n+1}\rho \frac{1-(\rho/\alpha)^n}{1-\rho/\alpha} \\ &= \alpha^{n+1} - \alpha^{n+1}(1-(\rho/\alpha)^n), \quad \text{as } 1-\rho/\alpha = -\rho, \\ &= \alpha^{n+1}(\rho/\alpha)^n = \alpha\rho^n.\end{aligned}$$

Since $\rho = \alpha/(1-\alpha)$ we see that the left- and RHSs are the same.

**s.7.1.1.**

$$aq^2 + bq = \alpha a(q^2+2q+1) + \alpha b(q+1) + \beta a(q^2-2q+1) + \beta b(q-1) + hq/(\lambda+\mu).$$

Matching the coefficients of $q^2$, $q$

$$\begin{aligned} a &= \alpha a + \beta a && \implies a = a, \\ b &= \alpha a 2 + \alpha b - \beta a 2 + \beta b + h/(\lambda+\mu), && \implies a = \frac{h}{2}\frac{1}{\mu-\lambda}, \\ 0 &= \alpha a + \alpha b + \beta a - \beta b, && \implies b = \frac{a}{\beta-\alpha}. \end{aligned}$$

**s.7.1.2.** On the one hand, the cost of the jobs in the system during one cycle must be $V(1)$. The duration of one cycle is $C(1)$. By the renewal-reward theorem, the time-average cost is then $V(1)/C(1)$. On the other hand, if the time-average number of jobs in the system is $\mathsf{E}[L]$, and each job pays $h$ per unit time, the time-average cost must be $h\mathsf{E}[L]$.

**s.7.1.3.**

$$\begin{aligned}\frac{V(1)}{1/\lambda + T(1)} &= \frac{a+b}{1/\lambda + 1/(\mu-\lambda)} = \left(\frac{h}{2}\frac{1}{\mu-\lambda} + \frac{h}{2}\frac{\lambda+\mu}{(\mu-\lambda)^2}\right)\frac{\lambda(\mu-\lambda)}{\mu} \\ &= \frac{h}{2}\rho\left(1 + \frac{\lambda+\mu}{\mu-\lambda}\right) = \frac{h}{2}\rho\frac{2\mu}{\mu-\lambda} = h\frac{\rho}{1-\rho}.\end{aligned}$$

**s.7.1.4.** The cost up to the $q$th job is the cost up to the arrival of job $q-1$ plus the cost while there are $q-1$ jobs in the system. The time between the arrival of job $q-1$ and $q$ is $1/\lambda$. $W(q) = h\sum_{i=1}^{q}(i-1)$.

**s.7.2.1.** When we switch on the server, the queue 'drains' at rate $\mu - \lambda > 0$, with $\mu = 1/\mathsf{E}[S]$. Consequently, no matter how large $N$, $T(N) < \infty$. And, whenever the system is empty, the stochastic process restarts. As such cycles start over and over again, and the queue length can never 'escape to infinity'.

**s.7.2.2.** The total number $A(t)$ of job that arrive during $[0,t]$ does not depend on $N$. Thus, in (5.1.2), $\sum_{k=1}^{A(t)} S_k$ does not depend on $N$. Now use rate-stability.

**s.7.2.3.** In the hint, the first equation is superfluous. In the second, $bq$ cancels at both sides, by which we find $a$. The third now follows.

**s.7.2.4.** For $b$, using the expressions for $\mathsf{E}[Y]$ and $\mathsf{E}\left[Y^2\right]$,

$$\begin{aligned}
b(1-\mathsf{E}[Y]) &= a(\mathsf{E}\left[Y^2\right]-2\mathsf{E}[Y]+1)+\frac{1}{2}h\lambda\,\mathsf{E}\left[S^2\right] \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\mathsf{E}[Y])}(\mathsf{E}\left[Y^2\right]-2\mathsf{E}[Y]+1)+\frac{1}{2}h\lambda\,\mathsf{E}\left[S^2\right] \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(\lambda^2\,\mathsf{E}\left[S^2\right]+\lambda\,\mathsf{E}[S]-2\lambda\,\mathsf{E}[S]+1+\lambda\,\mathsf{E}\left[S^2\right]\frac{1-\lambda\,\mathsf{E}[S]}{\mathsf{E}[S]}\right) \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(\lambda^2\,\mathsf{E}\left[S^2\right]-\lambda\,\mathsf{E}[S]+1+\frac{\lambda\,\mathsf{E}\left[S^2\right]}{\mathsf{E}[S]}-\lambda^2\,\mathsf{E}\left[S^2\right]\right) \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(1+\frac{\lambda\,\mathsf{E}\left[S^2\right]}{\mathsf{E}[S]}-\lambda\,\mathsf{E}[S]\right) \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(1+\frac{\lambda\,\mathsf{E}\left[S^2\right]}{\mathsf{E}[S]}-\lambda\frac{(\mathsf{E}[S])^2}{\mathsf{E}[S]}\right) \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(1+\lambda\frac{\mathsf{V}[S]}{\mathsf{E}[S]}\right) \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(1+\lambda\frac{\mathsf{V}[S]}{(\mathsf{E}[S])^2}\,\mathsf{E}[S]\right) \\
&= \frac{h\,\mathsf{E}[S]}{2(1-\lambda\,\mathsf{E}[S])}\left(1+\rho C_s^2\right).
\end{aligned}$$

Divide now both sides by $1-\mathsf{E}[Y]$.

**s.7.2.5.** For $a$, multiply the numerator and denominator by $\mu = 1/\mathsf{E}[S]$. For $b$, multiply by $\mu^2 = 1/(\mathsf{E}[S])^2$, use that $C_s^1 = 1$ because the service times are exponentially distributed, and note that

$$\frac{1+\rho}{1-\rho} = \frac{\mu+\lambda}{\mu-\lambda}.$$

**s.7.2.6.** $h\,\mathsf{E}[L] = h\,\mathsf{E}[Q] + h\,\mathsf{E}[L_s]$. $\mathsf{E}[Q] = \lambda\,\mathsf{E}[W]$ and $\mathsf{E}[L_s] = \lambda\,\mathsf{E}[S]$.

**s.7.2.7.** Note first that $C(N) = N(1/\lambda + \mathsf{E}[S]/(1-\rho)) = N/(\lambda(1-\rho))$. Then,

$$\begin{aligned}
\frac{V(N)+K+W(N)}{C(N)} &= \left(aN^2+bN+K+hN(N-1)/2\lambda\right)\frac{\lambda(1-\rho)}{N} \\
&= \frac{h}{2}\rho N+\frac{h}{2}\frac{\rho}{1-\rho}(1+\rho C_s^2)+\frac{h}{2}(N-1)(1-\rho)+K\frac{\lambda(1-\rho)}{N} \\
&= \frac{h}{2}\frac{\rho}{1-\rho}(1+\rho C_s^2)+\frac{h}{2}(N-1+\rho)+K\frac{\lambda(1-\rho)}{N} \\
&= \frac{h}{2}\frac{\rho}{1-\rho}(\rho+\rho C_s^2+1-\rho)+\frac{h}{2}(N-1+\rho)+K\frac{\lambda(1-\rho)}{N} \\
&= \frac{h}{2}\frac{\rho^2}{1-\rho}(1+C_s^2)+\frac{h}{2}\rho+\frac{h}{2}(N-1+\rho)+K\frac{\lambda(1-\rho)}{N} \\
&= \frac{h}{2}\frac{\rho^2}{1-\rho}(1+C_s^2)+h\rho+h\frac{N-1}{2}+K\frac{\lambda(1-\rho)}{N}.
\end{aligned}$$

**s.7.3.1.**

$$f_D(t) = (1-\rho) f_{X+S}(t) + \rho\mu e^{-\mu t} = (1-\rho)\frac{\mu\lambda}{\lambda-\mu}\left(e^{-\mu t} - e^{-\lambda t}\right) + \rho\mu e^{-\mu t}$$
$$= \left(1 - \frac{\lambda}{\mu}\right)\frac{\mu\lambda}{\lambda-\mu}\left(e^{-\mu t} - e^{-\lambda t}\right) + \rho\mu e^{-\mu t}$$
$$= \frac{\mu-\lambda}{\mu}\frac{\mu\lambda}{\lambda-\mu}\left(e^{-\mu t} - e^{-\lambda t}\right) + \frac{\lambda}{\mu}\mu e^{-\mu t} = -\lambda\left(e^{-\mu t} - e^{-\lambda t}\right) + \lambda e^{-\mu t}$$

**s.7.3.2.** A job leaving station $i$ has to go somewhere, to another station, perhaps to station $i$ again, or leave the network. As $\sum_{i=1}^{M} P_{ij}$ is the probability to be sent to some station in the network, $P_{i0}$ is the probability a job leaves the network from station $i$.

**s.7.3.3.**

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \beta_1 & \beta_2 \end{pmatrix}, \qquad (\lambda_1, \lambda_2) = (\gamma, 0) + (\lambda_1, \lambda_2)P.$$

Solving first for $\lambda_2$ leads to $\lambda_2 = (1-\alpha)\lambda_1 + \beta_2\lambda_2$, so that

$$\lambda_2 = \frac{1-\alpha}{1-\beta_2}\lambda_1.$$

Next, using this and that $\lambda_1 = \alpha\lambda_1 + \beta_1\lambda_2 + \gamma$ gives

$$\gamma = \lambda_1(1-\alpha) - \beta_1\lambda_2 = \lambda_1\left(1 - \alpha - \beta_1\frac{1-\alpha}{1-\beta_2}\right)$$
$$= \lambda_1(1-\alpha)\left(1 - \frac{\beta_1}{1-\beta_2}\right) = \lambda_1(1-\alpha)\frac{1-\beta_1-\beta_2}{1-\beta_2}.$$

Hence,

$$\lambda_1 = \frac{\gamma}{1-\alpha}\frac{1-\beta_2}{1-\beta_1-\beta_2}, \qquad \lambda_2 = \frac{1-\alpha}{1-\beta_2}\lambda_1 = \frac{\gamma}{1-\beta_1-\beta_2}.$$

We want of course that $\lambda_1 < \mu_1$ and $\lambda_2 < \mu_2$. With the above expressions this leads to conditions on $\alpha$, $\beta_1$ and $\beta_2$. Note that we have three parameters, and two equations; there is not a single condition from which the stability can be guaranteed. If $\alpha \uparrow 1$, the arrival rate at node 1 explodes. If $\beta_1 = 0$ no jobs are sent from node 2 to node 1.

**s.7.3.4.** The rate into state $(0,0)$ is $\mu_2 p(0,1) = \mu_2\rho_2$. The rate out of state $(0,0)$ is $\lambda p(0,0) = \lambda$. Since $\rho_2 = \lambda/\mu_2$, these two rates are the same.

**s.7.3.5.**

$$\text{rate out } = (\lambda+\mu_1)p(i,0) = \mu_1\rho_1^i + \lambda\rho_1^i = \lambda\rho_1^{i-1} + \mu_2\rho_1^i\rho_2$$
$$= \lambda p(i-1,0) + \mu_2 p(i,1) = \text{ rate in.}$$

**s.7.3.6.** We show that the rate out is the rate in.

$$\text{rate out } = (\lambda+\mu_2)p(0,j) = \mu_2\rho_2^j + \lambda\rho_2^j = \mu_1\rho_1\rho_2^{j-1} + \mu_2\rho_2^{j+1}$$
$$= \mu_1 p(1,j-1) + \mu_2 p(0,j+1) = \text{ rate in.}$$

**s.7.4.1.** $P_{i,i+1} = 1$, and $P_{ij} = 0$ for $j \neq i+1$. As $P$ is an upper-triangular matrix of dimension $M \times M$, $P^M = 0$.

**s.7.4.2.** $P$ transient $\implies Q_{i0} = P^M_{i0} > 0 \implies \sum_{j=1}^M Q_{ij} < 1$. Since $M$ is a finite number, there exists an $\epsilon > 0$ such that $\max\{\sum_{j=1}^M Q_{ij}\} < 1-\epsilon$. Writing $\mathbf{1}$ for the vector $(1,1,\ldots,1)$, this means that $Q\mathbf{1} < (1-\epsilon)\mathbf{1}$. But then, $Q^2\mathbf{1} = Q(Q\mathbf{1}) < (1-\epsilon)Q\mathbf{1} < (1-\epsilon)^2\mathbf{1}$. As $Q \geq 0$ element wise, we see that $0 \leq Q^n < (1-\epsilon)^n$. And then $P^n = Q^{n/M} < (1-\epsilon)^{n/M}$.

**s.7.4.3.** The above argumentation is not necessarily valid for matrices $P$ that are infinite, since $\inf\{P^M_{ik}\}$ need not be strictly positive for all $i, j$.

**s.7.4.4.** The probability to move from any state $n$ straightaway to 0 is $\beta^n > 0$. (Why do we require $n$ to be finite?)

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# NOTATION

$a_k$ = Number of arrivals in $k$th period

$A(t)$ = Number of arrivals in $[0, t]$

$A_k$ = Arrival time of $k$th job

$B$ = General batch size

$B_k$ = Batch size of $k$th job

$\tilde{A}_k$ = Start of service of $k$th job

$c_k$ = Service/production capacity in $k$th period

$d_k$ = Number of departures in $k$th period

$c$ = Number of servers

$C_a^2$ = Squared coefficient of variation of inter-arrival times

$C_s^2$ = Squared coefficient of variation of service times

$D(t)$ = Number of departures in $[0, t]$

$D_k$ = Departure time of $k$th job

$\delta$ = Departure rate

$F$ = Distribution of service time of a job

$I$ = Idle time of single server

$J_k$ = Sojourn time of $k$th job

$L(t)$ = Number of customers/jobs in system at time $t$

$L_k$ = Number in system as end of $k$th period

$L_s(t)$ = Number of customers/jobs in service at time $t$

$\lambda$ = Arrival rate

$\mu$ = Service rate

$N(t)$ = Number of arrivals in $[0, t]$

$N(s, t)$ = Number of arrivals in $(s, t]$

$p(n)$ = Long-run time average that system contains $n$ jobs

$\pi(n)$ = Stationary probability that an arrival sees $n$ jobs in system

$Q(t)$ = Number of customers/jobs in queue at time $t$

$Q_k$ = Queue length as seen by $k$th job, or at end $k$th period

$\rho$ = utilization of a single server

$S$ = Generic service time

$S_k$ = Service time of $k$th job

$U$ = Busy time of single server

$W$ = Generic waiting time (in queue)

$W_k$ = Waiting time in queue of $k$th job

$X$ = Generic inter-arrival time between two consecutive jobs

$X_k$ = Inter-arrival time between job $k-1$ and job $k$

# INDEX