Queueing Theory and Simulation, lecture 13

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June 22, 2021

Contents

1 Status and Plan

1.1 Plan

- Past: recursions
  - Numerical methods to compute $p(n)$ $(=\pi(n))$ for $M^X/M/1$.
  - Recursion to compute $p(n)$ $(=\pi(n))$ for $M/G/1$.
- Now: Simple queueing control.
  - Compute the long-run average cost of an $M/M/1$ (first) and a $M/G/1$ (second) under an N policy.
  - We use many of the concepts we developed earlier.
  - A set of highly elegant and very useful ideas.
- Next:
  - An open network of $M/M/c$ queues
  - Fixed points of the (vector) equation $\lambda = \gamma + \lambda P$ and $v = c + Pv$.

2 N policy for the $M/M/1$ queue

2.1 Model

- Jobs arrive as a Poisson proces with rate λ.
- Service times are iid, $S \sim \mathrm{Exp}(\mu)$.
- $\rho = \lambda/\mu < 1$, system is stable

- As soon as the server becomes idle, it switches off
- The server pays K € to switch on
- The server pays h € per unit time per job in the system.

2.2 Problem for N policy

Three steps:

- System starts empty, pay jobs until N in system.
- Pay K to switch on
- As long as jobs are present, pay each of them

Which N minimes the long-run average cost?

2.3 Steps to Solve:

- Start at zero, find expected cost $W(q)$ to hit q.
- Start at some level q, find expected time $T(q)$ until empty
- Start at some level q, find expected cost $V(q)$ until empty
- Combine all components and apply renewal reward.

We start with finding an expression for $T(q)$ and $V(q)$.

2.4 Analysis, expected time $T(q)$ to hit 0

- If there were no stochasticity, the net outflow is $\mu - \lambda$, hence $T(q) = q/(\mu - \lambda)$.
- But there is stochasticity!
- Use recursion! With this:

$$T(q) = \frac{1}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} T(q+1) + \frac{\mu}{\lambda + \mu} T(q-1). \tag{1}$$

- To solve, *guess* the form $T(q) = aq + b$, and substitute.
- $T(0) = 0 \implies b = 0$. Solving for a gives

$$T(q) = \frac{q}{\mu - \lambda}. \tag{2}$$

2.5 Analysis, expected cost $V(q)$ to hit 0.

- Recursion, $\alpha = \lambda/(\lambda + \mu), \beta = \mu/(\lambda + \mu)$,

$$V(q) = h \frac{q}{\lambda + \mu} + \alpha V(q+1) + \beta V(q-1). \tag{3}$$

- To solve, *guess* the form $V(q) = aq^2 + bq + c$.
- $V(0) = 0 \implies c = 0$. Solving for a, b gives

$$V(q) = \frac{h}{2} \frac{1}{\mu - \lambda} q^2 + \frac{h}{2} \frac{\lambda + \mu}{(\mu - \lambda)^2} q. \tag{4}$$

2.6 Application to $M/M/1$

- Average number of jobs in system is $\mathsf{E}[L]$.
- Hence, average pay is $h\,\mathsf{E}[L]$.
- Set $K = 0$
- $C(1)$? $C(1) = 1/\lambda + T(1)$.
- Renewal reward $h\,\mathsf{E}[L] = V(1)/C(1)$.
- Exercise: show that $V(1)/C(1) = h\rho/(1-\rho)$.

2.7 Analysis, expected cost $W(q)$ to move to q

- Recursion:

$$W(q) = W(q-1) + h\frac{q-1}{\lambda} = h\frac{q(q-1)}{2\lambda}. \tag{5}$$

2.8 Analysis, expected cycle time

- Expected time to start at 0, wait until level N is hit, switch on, process jobs until empty, and restart.

$$C(N) = N/\lambda + T(N).$$

2.9 Long-run average cost

- Final result

$$\frac{W(N) + K + V(N)}{C(N)} = \frac{W(N) + K + V(N)}{N/\lambda + T(N)}.$$

3 N policy for the $M/G/1$ queue.

3.1 Analysis

- Unlike the $M/M/1$ queue: there is no memoryless during service times.
- Focus on job departure times. The inter-arrival times are Exp, so still memoryless.
- Solving is similar to the $M/M/1$ queue
- But, deal with costs due to arrivals during a service time.

3.2 Analysis, expected time $T(q)$ to hit 0

- Y jobs arrive during a service.
- Use recursion:

$$T(q) = \mathsf{E}[S] + \mathsf{E}[T(q+Y-1)]. \tag{6}$$

- Substitute the *guess* $T(q) = aq + b$.
- Using that $T(0) = 0$, solving for a, b gives

$$T(q) = \frac{\mathsf{E}[S]}{1 - \lambda \mathsf{E}[S]} q.$$

- Check that this is $q/(\mu - \lambda)$ for $M/M/1$.

3.3 Analysis, expected cost $V(q)$ to hit 0.

- Use recursion:

$$V(q) = U(q) + \mathsf{E}[V(q+Y-1)]. \tag{7}$$

- $U(q)$ is expected cost of jobs waiting during a service S.

$$U(q) = hq \mathsf{E}[S] + \mathsf{E}[H(S)]$$

- $\mathsf{E}[H(S)]$ is expected cost of new jobs entering during the service.

3.4 Analyis: cost of new arrivals

- Suppose s amount of service remains.
- When a new job arrives, we pay it immediately hs, but nothing while it is in the system.
- What can happen during a period $\delta \ll 1$?
  - $\mathsf{P}[\text{no new arrival}] = 1 - \lambda\delta$, pay $U(s-\delta)$
  - $\mathsf{P}[\text{one new arrival}] = \lambda\delta$, pay $hs + U(s-\delta)$
  - Neglect multiple arrivals as probability is $o(\delta)$.
- Hence:

$$\begin{aligned}
H(s) &= (1-\lambda\delta)\cdot 0 + \lambda\delta \cdot hs + H(s-\delta) + o(\delta), \\
H'(s) &= \lambda hs, \quad H(0) = 0, \\
H(s) &= \lambda hs^2/2 \\
\mathsf{E}[H(S)] &= \lambda h \mathsf{E}\left[S^2\right]/2.
\end{aligned}$$

3.5 Analysis, expected cost $V(q)$ to hit 0.

- Hence

$$\begin{aligned} V(q) &= U(q) + \mathsf{E}[V(q+Y-1)] \\ &= hq\,\mathsf{E}[S] + \mathsf{E}[H(S)] + \mathsf{E}[V(q+Y-1)] \\ &= hq\,\mathsf{E}[S] + \lambda h\,\mathsf{E}\left[S^2\right]/2 + \mathsf{E}[V(q+Y-1)]. \end{aligned}$$

- Substitute $V(q) = aq^2 + bq + c$, and solve for a, b, c, with boundary condition $V(0) = 0$.
- After some work this gives

$$V(q) = \frac{h}{2}\frac{\mathsf{E}[S]}{1-\rho}q^2 + h\frac{1+\rho C_s^2}{2}\frac{\mathsf{E}[S]}{(1-\rho)^2}q.$$

3.6 Long-run average cost

- Final result

$$\frac{W(N)+K+V(N)}{C(N)} \tag{8}$$

$$= h\frac{1+C_s^2}{2}\frac{\rho^2}{1-\rho} + h\rho + h\frac{N-1}{2} + K\frac{\lambda(1-\rho)}{N}. \tag{9}$$

- Setting $N = 1$ gives the PK formula
- Optimal N^*:

$$N^* \approx \sqrt{\frac{2\lambda(1-\rho)K}{h}} \to \sqrt{\frac{2\lambda K}{h}}, \quad \text{as } \mathsf{E}[S] \to 0.$$

- These are the Economic Production Quantity (EPQ) and Economic Order Quantity (EOQ) formulas.