Queueing Theory and Simulation, lecture 8

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Contents

1 Overview

1.1 Past

- Finish approximations, $G/G/1$ queue in tandem
- Start with exact work: renewal reward theorem
- application to queueing: fraction of time the servers are busy is $\lambda \mathsf{E}[S]$.

1.2 Now

Very elegant use of sample paths (ideas based on simuation) to obtain crucial work horses of queueing theory:

- Level crossing
- PASTA

1.3 Future

- Little's law
- All is set in place to analyze exact queueing models

2 Level crossing argument

2.1 Notation and concepts

$$
\begin{aligned}
A(n,t) &= \sum_{k=1}^{\infty} I_{A_k \le t} I_{L(A_k-)=n} = \sum_{k=1}^{A(t)} I_{L(A_k-)=n} && (1)\\
Y(n,t) &= \int_0^t I_{L(s)=n}\,\mathrm{d}s, && (2)\\
p(n,t) &= \frac{Y(n,t)}{t}, && (3)\\
D(n,t) &= \sum_{k=1}^{\infty} I_{D_k \le t} I_{L(D_k)=n} = \sum_{k=1}^{D(t)} I_{L(D_k)=n} && (4)
\end{aligned}
$$

2.2 Useful limits

$$
\begin{aligned}
\lambda(n) &= \lim_{t\to\infty} \frac{A(n,t)}{Y(n,t)} && (5)\\
p(n) &= \lim_{t\to\infty} p(n,t) && (6)\\
\mu(n+1) &= \lim_{t\to\infty} \frac{D(n,t)}{Y(n+1,t)}. && (7)
\end{aligned}
$$

2.3 Level crossing

$$
\begin{aligned}
1 &\ge |A(n,t) - D(n,t)|, && (8)\\
\lim_{t\to\infty} \frac{A(n,t)}{t} &= \lim_{t\to\infty} \frac{D(n,t)}{t}, && (9)\\
\lim_{t\to\infty} \frac{A(n,t)}{t} &= \lim_{t\to\infty} \frac{A(n,t)}{Y(n,t)} \frac{Y(n,t)}{t} = \lambda(n)p(n), && (10)\\
\lim_{t\to\infty} \frac{D(n,t)}{t} &= \lim_{t\to\infty} \frac{D(n,t)}{Y(n+1,t)} \frac{Y(n+1,t)}{t} && (11)\\
&= \mu(n+1)p(n+1), && (12)\\
&\implies && (13)\\
\lambda(n)p(n) &= \mu(n+1)p(n+1). && (14)
\end{aligned}
$$

We have a recursion!

2.4 Specific queueing systems

By making proper choices for $\lambda(n)$ and $\mu(n)$ we can model many different queueing systems

- $M/M/1$: $\lambda(n) = \lambda, \mu(n) = \mu$.
- $M/M/\infty$: $\lambda(n) = \lambda, \mu(n) = \mu n$.
- $M/M/c$: $\lambda(n) = \lambda, \mu(n) = \mu \min\{n, c\}$.
- $M/M/c/c$: $λ(n) = λ \1{n< c}, μ(n) = μ n.

2.5 Implications

$$p(n+1) = \frac{\lambda(n)}{\mu(n+1)} p(n), \tag{15}$$

$$p(n+1) = \frac{\lambda(n)\lambda(n-1)\cdots\lambda(0)}{\mu(n+1)\mu(n)\cdots\mu(1)} p(0) \tag{16}$$

$$1 = p(0)\left(1 + \sum_{n=0}^{\infty} \frac{\lambda(n)\lambda(n-1)\cdots\lambda(0)}{\mu(n+1)\mu(n)\cdots\mu(1)}\right) \tag{17}$$

$$\tag{18}$$

2.6 KPIs

$$\mathsf{E}\left[L\right] = \sum_{n=0}^{\infty} n p(n), \tag{19}$$

$$\mathsf{P}\left[L \geq n\right] = \sum_{i=n}^{\infty} p(i). \tag{20}$$

3 PASTA

3.1 Notation and concepts

$$\frac{A(n,t)}{t} = \frac{A(n,t)}{Y(n,t)} \frac{Y(n,t)}{t} \to \lambda(n)p(n), \quad \text{as } t \to \infty, \tag{21}$$

$$\frac{A(n,t)}{t} = \frac{A(t)}{t} \frac{A(n,t)}{A(t)}, \tag{22}$$

$$\frac{A(n,t)}{A(t)} = \frac{1}{A(t)} \sum_{k=1}^{A(t)} I_{L(A_k-)=n}, \tag{23}$$

$$\pi(n) = \lim_{t\to\infty} \frac{1}{A(t)} \sum_{k=1}^{A(t)} I_{L(A_k-)=n}, \tag{24}$$

$$\implies \tag{25}$$

$$\lambda(n)p(n) = \lambda\pi(n) \tag{26}$$

3.2 PASTA (Poisson arrivals see time averages)

- $\lambda(n) = \lambda \iff p(n) = \pi(n)$.
- Jobs arrive as a Poisson process $\implies \lambda(n) = \lambda$. (Hard to prove)

$\implies$

- Jobs arrive as a Poisson process $\implies p(n) = \pi(n)$

In other words, when you sample a system at times $\{T_k\}$ such that $T_k - T_{k-1}$ are exp. distributed, then sample averages converge to the time averages!