Multivariate gamma function

In mathematics, the **multivariate gamma function** Γ_p is a generalization of the gamma function. It is useful in multivariate statistics, appearing in the probability density function of the Wishart and inverse Wishart distributions, and the matrix variate beta distribution.[1]

It has two equivalent definitions. One is given as the following integral over the $p \times p$ positive-definite real matrices:

$$\Gamma_p(a) = \int_{S>0} \exp(-\mathrm{tr}(S))\, |S|^{a - \frac{p+1}{2}}\, dS,$$

where $|S|$ denotes the determinant of S. The other one, more useful to obtain a numerical result is:

$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma(a + (1-j)/2).$$

In both definitions, a is a complex number whose real part satisfies $\Re(a) > (p-1)/2$. Note that $\Gamma_1(a)$ reduces to the ordinary gamma function. The second of the above definitions allows to directly obtain the recursive relationships for $p \geq 2$:

$$\Gamma_p(a) = \pi^{(p-1)/2} \Gamma(a) \Gamma_{p-1}(a - \tfrac{1}{2}) = \pi^{(p-1)/2} \Gamma_{p-1}(a) \Gamma(a + (1-p)/2).$$

Thus

- $\Gamma_2(a) = \pi^{1/2} \Gamma(a) \Gamma(a - 1/2)$
- $\Gamma_3(a) = \pi^{3/2} \Gamma(a) \Gamma(a - 1/2) \Gamma(a - 1)$

and so on.

This can also be extended to non-integer values of p with the expression:

$${\displaystyle \Gamma _{p}(a)=\pi ^{p(p-1)/4}{\frac {G(a+{\frac {1}{2}})G(a+1)}{G(a+{\frac {1-p}{2}})G(a+1-{\frac {p}{2}})}}}$$

Where G is the Barnes G-function, the indefinite product of the Gamma function.

The function is derived by Anderson[2] from first principles who also cites earlier work by Wishart, Mahalanobis and others.

There also exists a version of the multivariate gamma function which instead of a single complex number takes a p-dimensional vector of complex numbers as its argument. It generalizes the above defined multivariate gamma function insofar as the latter is obtained by a particular choice of multivariate argument of the former.[3]

Derivatives

We may define the multivariate digamma function as

$$\psi_{p}(a)={\frac{\partial \log \Gamma_{p}(a)}{\partial a}}=\sum_{i=1}^{p}\psi(a+(1-i)/2),$$

and the general polygamma function as

$$\psi_p^{(n)}(a)=\frac{\partial^n \log\Gamma_p(a)}{\partial a^n}=\sum_{i=1}^{p}\psi^{(n)}(a+(1-i)/2).$$

Calculation steps

- Since

$$\Gamma_p(a)=\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma\left(a+\frac{1-j}{2}\right),$$

it follows that

$$\frac{\partial\Gamma_p(a)}{\partial a}=\pi^{p(p-1)/4}\sum_{i=1}^{p}\frac{\partial\Gamma\left(a+\frac{1-i}{2}\right)}{\partial a}\prod_{j=1,j\neq i}^{p}\Gamma\left(a+\frac{1-j}{2}\right).$$

- By definition of the digamma function, ψ,

$$\frac{\partial\Gamma(a+(1-i)/2)}{\partial a}=\psi(a+(i-1)/2)\Gamma(a+(i-1)/2)$$

it follows that

$$\begin{aligned}\frac{\partial\Gamma_p(a)}{\partial a}&=\pi^{p(p-1)/4}\prod_{j=1}^{p}\Gamma(a+(1-j)/2)\sum_{i=1}^{p}\psi(a+(1-i)/2)\\&=\Gamma_p(a)\sum_{i=1}^{p}\psi(a+(1-i)/2).\end{aligned}$$

References

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