The Gaussian Mixture Probability Hypothesis Density Filter

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The Gaussian Mixture Probability Hypothesis Density Filter

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***Abstract*— A new recursive algorithm is proposed for jointly estimating the time-varying number of targets and their states from a sequence of observation sets in the presence of data association uncertainty, detection uncertainty, noise and false alarms. The approach involves modelling the respective collections of targets and measurements as random finite sets and applying the probability hypothesis density (PHD) recursion to propagate the posterior intensity, which is a first order statistic of the random finite set of targets, in time. At present, there is no closed form solution to the PHD recursion. This work shows that under linear, Gaussian assumptions on the target dynamics and birth process, the posterior intensity at any time step is a Gaussian mixture. More importantly, closed form recursions for propagating the means, covariances and weights of the constituent Gaussian components of the posterior intensity are derived. The proposed algorithm combines these recursions with a strategy for managing the number of Gaussian components to increase efficiency. This algorithm is extended to accommodate mildly nonlinear target dynamics using approximation strategies from the extended and unscented Kalman filters.**

***Index Terms*— Multi-target tracking, optimal filtering, point processes, random sets, intensity function.**

I. INTRODUCTION

In a multi-target environment, not only do the states of the targets vary with time, but the number of targets also changes due to targets appearing and disappearing. Often, not all of the existing targets are detected by the sensor. Moreover, the sensor also receives a set of spurious measurements (clutter) not originating from any target. As a result, the observation set at each time step is a collection of indistinguishable partial observations, only some of which are generated by targets. The objective of multi-target tracking is to jointly estimate, at each time step, the number of targets and their states from a sequence of noisy and cluttered observation sets. Multi-target tracking is an established area of study, for details on its techniques and applications, readers are referred to [1], [2]. Up to date overviews are also available in more recent works such as [3]–[5].

An intrinsic problem in multi-target tracking is the unknown association of measurements with appropriate targets [1], [2], [6], [7]. Due to its combinatorial nature, the data association problem makes up the bulk of the computational load in multi-target tracking algorithms. Most traditional multi-target tracking formulations involve explicit associations between measurements and targets. Multiple Hypotheses Tracking (MHT) and its variations concern the propagation of association hypotheses in time [2], [6], [7]. The joint probabilistic data association filter (JPDAF) [1], [8], the probabilistic MHT (PMHT) [9], and the multi-target particle filter [3], [4] use observations weighted by their association probabilities. Alternative formulations that avoid explicit associations between measurements and targets include Symmetric Measurement Equations [10] and Random Finite Sets (RFS) [5], [11]–[14].

The random finite set (RFS) approach to multi-target tracking is an emerging and promising alternative to the traditional association-based methods [5], [11], [15]. A comparison of the RFS approach and traditional multi-target tracking methods has been given in [11]. In the RFS formulation, the collection of individual targets is treated as a *set-valued state*, and the collection of individual observations is treated as a *set-valued observation*. Modelling set-valued states and set-valued observations as RFSs allows the problem of dynamically estimating multiple targets in the presence of clutter and association uncertainty to be cast in a Bayesian filtering framework [5], [11], [15]–[17]. This theoretically optimal approach to multi-target tracking is an elegant generalization of the single-target Bayes filter. Indeed, novel RFS-based filters such as the *multi-target Bayes filter*, the *Probability Hypothesis Density (PHD) filter* [5], [11], [18] and their implementations [16], [17], [19]–[23] have generated substantial interest.

The focus of this paper is the PHD filter, a recursion that propagates the first-order statistical moment, or intensity, of the RFS of states in time [5]. This approximation was developed to alleviate the computational intractability in the multi-target Bayes filter, which stems from the combinatorial nature of the multi-target densities and the multiple integrations on the (infinite dimensional) multi-target state space. The PHD filter operates on the single-target state space and avoids the combinatorial problem that arises from data association. These salient features render the PHD filter extremely attractive. However, the PHD recursion involves multiple integrals that have no closed form solutions in general. A generic sequential Monte Carlo technique [16], [17], accompanied by various performance guarantees [17], [24], [25], have been proposed to propagate the posterior intensity in time. In this approach, state estimates are extracted from the particles representing the posterior intensity using clustering techniques such as K-mean or expectation maximization. Special cases of this so-called particle-PHD filter have also been independently implemented in [21] and [22]. Due to its ability to handle the time-varying number of nonlinear targets with relatively

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low complexity, innovative extensions and applications of the particle-PHD filter soon followed [26]–[31]. The main drawbacks of this approach though, are the large number of particles, and the unreliability of clustering techniques for extracting state estimates. (The latter will be further discussed in Section III-C.)

In this paper, we propose an analytic solution to the PHD recursion for linear Gaussian target dynamics and Gaussian birth model. This solution is analogous to the Kalman filter as a solution to the single-target Bayes filter. It is shown that when the initial prior intensity is a Gaussian mixture, the posterior intensity at any subsequent time step is also a Gaussian mixture. Moreover, closed form recursions for the weights, means, and covariances of the constituent Gaussian components are derived. The resulting filter propagates the Gaussian mixture posterior intensity in time as measurements arrive in the same spirit as the Gaussian sum filter of [32], [33]. The fundamental difference is that the Gaussian sum filter propagates a probability density using the Bayes recursion, whereas the Gaussian mixture PHD filter propagates an intensity using the PHD recursion. An added advantage of the Gaussian mixture representation is that it allows state estimates to be extracted from the posterior intensity in a much more efficient and reliable manner than clustering in the particle-based approach. In general, the number of Gaussian components in the posterior intensity increases with time. However, this problem can be effectively mitigated by keeping only the dominant Gaussian components at each instance. Two extensions to nonlinear target dynamics models are also proposed. The first is based on linearizing the model while the second is based on the unscented transform. Simulation results are presented to demonstrate the capability of the proposed approach.

Preliminary results on the closed form solution to the PHD recursion have been presented as a conference paper [34]. The current paper is a more complete version of this work.

The structure of the paper is as follows. Section II presents the random finite set formulation of multi-target filtering, and the PHD filter. Section III presents the main result of this paper, namely the analytical solution to the PHD recursion under linear Gaussian assumptions. An implementation of the PHD filter and simulation results are also presented. Section IV extends the proposed approach to nonlinear models using ideas from the extended and unscented Kalman filters. Demonstrations with tracking nonlinear targets are also given. Finally, concluding remarks and possible future research directions are given in Section V.

II. PROBLEM FORMULATION

This section presents a formulation of multi-target filtering in the random finite set (or point process) framework. We begin with a review of single-target Bayesian filtering in Section II-A. Using random finite set models, the multi-target tracking problem is then formulated as a Bayesian filtering problem in Section II-B. This provides sufficient background leading to Section II-C, which describes the PHD filter.

A. Single-target filtering

In many dynamic state estimation problems, the state is assumed to follow a Markov process on the state space $\mathcal{X} \subseteq \mathbb{R}^{n_x}$, with *transition density* $f_{k|k-1}(\cdot|\cdot)$, i.e. given a state x_{k-1} at time $k-1$, the probability density of a transition to the state x_k at time k is[1]

$$f_{k|k-1}(x_k|x_{k-1}). \tag{1}$$

This Markov process is partially observed in the observation space $\mathcal{Z} \subseteq \mathbb{R}^{n_z}$, as modelled by the *likelihood function* $g_k(\cdot|\cdot)$, i.e. given a state x_k at time k, the probability density of receiving the observation $z_k \in \mathcal{Z}$ is

$$g_k(z_k|x_k). \tag{2}$$

The probability density of the state x_k at time k given all observations $z_{1:k} = (z_1, \ldots, z_k)$ up to time k, denoted by

$$p_k(x_k|z_{1:k}), \tag{3}$$

is called the *posterior density* (or *filtering density*) at time k. From an initial density $p_0(\cdot)$, the posterior density at time k can be computed using the Bayes recursion

$$p_{k|k-1}(x_k|z_{1:k-1}) = \int f_{k|k-1}(x_k|x)p_{k-1}(x|z_{1:k-1})dx, \tag{4}$$

$$p_k(x_k|z_{1:k}) = \frac{g_k(z_k|x_k)p_{k|k-1}(x_k|z_{1:k-1})}{\int g_k(z_k|x)p_{k|k-1}(x|z_{1:k-1})dx}. \tag{5}$$

All information about the state at time k is encapsulated in the posterior density $p_k(\cdot|z_{1:k})$, and estimates of the state at time k can be obtained using either the MMSE (Minimum Mean Squared Error) criterion or the MAP (Maximum A Posteriori) criterion[2].

B. Random Finite Set Formulation of Multi-target Filtering

Now consider a multiple target scenario. Let $M(k)$ be the number of targets at time k, and suppose that, at time $k-1$, the target states are $x_{k-1,1}, \ldots, x_{k-1,M(k-1)} \in \mathcal{X}$. At the next time step, some of these targets may die, the surviving targets evolve to their new states, and new targets may appear. This results in $M(k)$ new states $x_{k,1}, \ldots, x_{k,M(k)}$. Note that the order in which the states are listed has no significance in the RFS multi-target model formulation. At the sensor, $N(k)$ measurements $z_{k,1}, \ldots, z_{k,N(k)} \in \mathcal{Z}$ are received at time k. The origins of the measurements are not known, and thus the order in which they appear bears no significance. Only some of these measurements are actually generated by targets. Moreover, they are indistinguishable from the false measurements. The objective of multi-target tracking is to jointly estimate the number of targets and their states from measurements with uncertain origins. Even in the ideal case where the sensor observes all targets and receives no clutter, single-target filtering methods are not applicable since there is no information about which target generated which observation.

[1] For notational simplicity, random variables and their realizations are not distinguished.

[2] These criteria are not necessarily applicable to the multi-target case.

Since there is no ordering on the respective collections of target states and measurements at time k, they can be naturally represented as finite sets, i.e.

$$X_k = \{x_{k,1}, \ldots, x_{k,M(k)}\} \in \mathcal{F}(\mathcal{X}), \tag{6}$$

$$Z_k = \{z_{k,1}, \ldots, z_{k,N(k)}\} \in \mathcal{F}(\mathcal{Z}), \tag{7}$$

where $\mathcal{F}(\mathcal{X})$ and $\mathcal{F}(\mathcal{Z})$ are the respective collections of all finite subsets of $\mathcal{X}$ and $\mathcal{Z}$. The key in the random finite set formulation is to treat the target set X_k and measurement set Z_k as the *multi-target state* and *multi-target observation* respectively. The multi-target tracking problem can then be posed as a filtering problem with (multi-target) state space $\mathcal{F}(\mathcal{X})$ and observation space $\mathcal{F}(\mathcal{Z})$.

In a single-target system, uncertainty is characterized by modelling the state x_k and measurement z_k as random vectors. Analogously, uncertainty in a multi-target system is characterized by modelling the multi-target state X_k and multi-target measurement Z_k as *random finite sets* (RFS). An RFS X is simply a finite-set-valued random variable, which can be described by a discrete probability distribution and a family of joint probability densities [11], [35], [36]. The discrete distribution characterizes the cardinality of X, while for a given cardinality, an appropriate density characterizes the joint distribution of the elements of X.

In the following, we describe an RFS model for the time evolution of the multi-target state, which incorporates target motion, birth and death. For a given multi-target state X_{k-1} at time $k-1$, each $x_{k-1} \in X_{k-1}$ either continues to exist at time k with probability[3] $p_{S,k}(x_{k-1})$, or dies with probability $1 - p_{S,k}(x_{k-1})$. Conditional on the existence at time k, the probability density of a transition from state x_{k-1} to x_k is given by (1), i.e. $f_{k|k-1}(x_k|x_{k-1})$. Consequently, for a given state $x_{k-1} \in X_{k-1}$ at time $k-1$, its behavior at the next time step is modelled as the RFS

$$S_{k|k-1}(x_{k-1}) \tag{8}$$

that can take on either $\{x_k\}$ when the target survives, or $\emptyset$ when the target dies. A new target at time k can arise either by spontaneous births (i.e. independent of any existing target) or by spawning from a target at time $k-1$. Given a multi-target state X_{k-1} at time $k-1$, the multi-target state X_k at time k is given by the union of the surviving targets, the spawned targets and the spontaneous births:

$$X_k = \left[\bigcup_{\zeta \in X_{k-1}} S_{k|k-1}(\zeta)\right] \cup \left[\bigcup_{\zeta \in X_{k-1}} B_{k|k-1}(\zeta)\right] \cup \Gamma_k, \tag{9}$$

where

$$\begin{aligned} \Gamma_k &= \text{RFS of spontaneous birth at time } k, \\ B_{k|k-1}(\zeta) &= \text{RFS of targets spawned at time } k \text{ from} \\ &\quad\ \text{a target with previous state } \zeta. \end{aligned}$$

It is assumed that the RFSs constituting the union in (9) are independent of each other. The actual forms of Γ_k and $B_{k|k-1}(\cdot)$ are problem dependent; some examples are given in Section III-D.

[3]Note that $p_{S,k}(x_{k-1})$ is a probability parameterized by x_{k-1}.

The RFS measurement model, which accounts for detection uncertainty and clutter, is described as follows. A given target $x_k \in X_k$ is either detected with probability[4] $p_{D,k}(x_k)$ or missed with probability $1 - p_{D,k}(x_k)$. Conditional on detection, the probability density of obtaining an observation z_k from x_k is given by (2), i.e. $g_k(z_k|x_k)$. Consequently, at time k, each state $x_k \in X_k$ generates an RFS

$$\Theta_k(x_k) \tag{10}$$

that can take on either $\{z_k\}$ when the target is detected, or $\emptyset$ when the target is not detected. In addition to the target originated measurements, the sensor also receives a set K_k of false measurements, or clutter. Thus, given a multi-target state X_k at time k, the multi-target measurement Z_k received at the sensor is formed by the union of target generated measurements and clutter, i.e.

$$Z_k = K_k \cup \left[\bigcup_{x \in X_k} \Theta_k(x)\right] \tag{11}$$

It is assumed that the RFSs constituting the union in (11) are independent of each other. The actual form of K_k is problem dependent; some examples will be illustrated in Section III-D.

In a similar vein to the single-target dynamical model in (1) and (2), the randomness in the multi-target evolution and observation described by (9) and (11) are respectively captured in the *multi-target transition density* $f_{k|k-1}(\cdot|\cdot)$ and *multi-target likelihood*[5] $g_k(\cdot|\cdot)$ [5], [17]. Explicit expressions for $f_{k|k-1}(X_k|X_{k-1})$ and $g_k(Z_k|X_k)$ can be derived from the underlying physical models of targets and sensors using Finite Set Statistics (FISST)[6] [5], [11], [15], although these are not needed for this paper.

Let $p_k(\cdot|Z_{1:k})$ denote the *multi-target posterior density*. Then, the optimal multi-target Bayes filter propagates the multi-target posterior in time via the recursion

$$\begin{aligned} &p_{k|k-1}(X_k|Z_{1:k-1}) \\ &\quad = \int f_{k|k-1}(X_k|X) p_{k-1}(X|Z_{1:k-1}) \mu_s(dX), \end{aligned} \tag{12}$$

$$\begin{aligned} &p_k(X_k|Z_{1:k}) \\ &\quad = \frac{g_k(Z_k|X_k) p_{k|k-1}(X_k|Z_{1:k-1})}{\int g_k(Z_k|X) p_{k|k-1}(X|Z_{1:k-1}) \mu_s(dX)}, \end{aligned} \tag{13}$$

where μ_s is an appropriate reference measure on $\mathcal{F}(\mathcal{X})$ [17], [37]. We remark that although various applications of point process theory to multi-target tracking have been reported in the literature (e.g. [38]–[40]), FISST [5], [11], [15] is the first systematic approach to multi-target filtering that uses RFSs in the Bayesian framework presented above.

The recursion (12)-(13) involves multiple integrals on the space $\mathcal{F}(\mathcal{X})$, which are computationally intractable. Sequential Monte Carlo implementations can be found in [16], [17], [19], [20]. However, these methods are still computationally

[4]Note that $p_{D,k}(x_k)$ is a probability parameterized by x_k.

[5]The same notation is used for multi-target and single-target densities. There is no danger of confusion since in the single-target case the arguments are vectors whereas in the multi-target case the arguments are finite sets.

[6]Strictly speaking, FISST yields the set derivative of the belief mass functional, but this is in essence a probability density [17].

intensive due to the combinatorial nature of the densities, especially when the number of targets is large [16], [17]. Nonetheless, the optimal multi-target Bayes filter has been successfully applied to applications where the number of targets is small [19].

C. The Probability Hypothesis Density (PHD) filter

The PHD filter is an approximation developed to alleviate the computational intractability in the multi-target Bayes filter. Instead of propagating the multi-target posterior density in time, the PHD filter propagates the posterior intensity, a first-order statistical moment of the posterior multi-target state [5]. This strategy is reminiscent of the constant gain Kalman filter, which propagates the first moment (the mean) of the single-target state.

For a RFS X on $\mathcal{X}$ with probability distribution P, its first-order moment is a non-negative function v on $\mathcal{X}$, called the *intensity*, such that for each region $S \subseteq \mathcal{X}$ [35], [36]

$$\int |X \cap S| \, P(dX) = \int_S v(x)dx. \tag{14}$$

In other words, the integral of v over any region S gives the expected number of elements of X that are in S. Hence, the total mass $\hat{N} = \int v(x)dx$ gives the expected number of elements of X. The local maxima of the intensity v are points in $\mathcal{X}$ with the highest local concentration of expected number of elements, and hence can be used to generate estimates for the elements of X. The simplest approach is to round $\hat{N}$ and choose the resulting number of highest peaks from the intensity. The intensity is also known in the tracking literature as the Probability Hypothesis Density (PHD) [18], [41].

An important class of RFSs, namely the Poisson RFSs, are those completely characterized by their intensities. A RFS X is *Poisson* if the cardinality distribution of X, $\Pr(|X| = n)$, is Poisson with mean $\hat{N}$, and for any finite cardinality, the elements x of X are independently and identically distributed according to the probability density $v(\cdot)/\hat{N}$ [35], [36]. For the multi-target problem described in the subsection II-B, it is common to model the clutter RFS [K_k in (11)] and the birth RFSs [Γ_k and $B_{k|k-1}(x_{k-1})$ in (9)] as Poisson RFSs.

To present the PHD filter, recall the multi-target evolution and observation models from Section II-B with

$\gamma_k(\cdot)$	=	intensity of the birth RFS Γ_k at time k,
$\beta_{k\|k-1}(\cdot\|\zeta)$	=	intensity of the RFS $B_{k\|k-1}(\zeta)$ spawned at time k by a target with previous state ζ,
$p_{S,k}(\zeta)$	=	probability that a target still exists at time k given that its previous state is ζ,
$p_{D,k}(x)$	=	probability of detection given a state x at time k,
$\kappa_k(\cdot)$	=	intensity of the clutter RFS K_k at time k

and consider the following assumptions:

A.1. Each target evolves and generates observations independently of one another,

A.2. Clutter is Poisson and independent of target-originated measurements,

A.3. The predicted multi-target RFS governed by $p_{k|k-1}$ is Poisson.

Remark 1. Assumptions A.1 and A.2 are standard in most tracking applications (see for example [1], [2]) and have already been alluded to earlier in Section II-B. The additional assumption A.3 is a reasonable approximation in applications where interactions between targets are negligible [5]. In fact, it can be shown that A.3 is completely satisfied when there is no spawning and the RFSs X_{k-1} and Γ_k are Poisson.

Let v_k and $v_{k|k-1}$ denote the respective intensities associated with the multi-target posterior density p_k and the multi-target predicted density $p_{k|k-1}$ in the recursion (12)-(13). Under assumptions A.1-A.3, it can be shown (using FISST [5] or classical probabilistic tools [37]) that the posterior intensity can be propagated in time via the PHD recursion:

$$\begin{aligned} v_{k|k-1}(x) &= \int p_{S,k}(\zeta) f_{k|k-1}(x|\zeta) v_{k-1}(\zeta) d\zeta \\ &\quad + \int \beta_{k|k-1}(x|\zeta) v_{k-1}(\zeta) d\zeta + \gamma_k(x), \end{aligned} \tag{15}$$

$$\begin{aligned} v_k(x) &= [1 - p_{D,k}(x)] v_{k|k-1}(x) \\ &\quad + \sum_{z \in Z_k} \frac{p_{D,k}(x) g_k(z|x) v_{k|k-1}(x)}{\kappa_k(z) + \int p_{D,k}(\xi) g_k(z|\xi) v_{k|k-1}(\xi) d\xi} \end{aligned} \tag{16}$$

It is clear from (15)-(16) that the PHD filter completely avoids the combinatorial computations arising from the unknown association of measurements with appropriate targets. Furthermore, since the posterior intensity is a function on the single-target state space $\mathcal{X}$, the PHD recursion requires much less computational power than the multi-target recursion (12)-(13), which operates on $\mathcal{F}(\mathcal{X})$. However, as mentioned in the introduction, the PHD recursion does not admit closed form solutions in general, and numerical integration suffers from the 'curse of dimensionality.'

III. The PHD recursion for linear Gaussian models

This section shows that for a certain class of multi-target models, herein referred to as *linear Gaussian multi-target* models, the PHD recursion (15)-(16) admits a closed form solution. This result is then used to develop an efficient multi-target tracking algorithm. The linear Gaussian multi-target models are specified in Section III-A, while the solution to the PHD recursion is presented in Section III-B. Implementation issues are addressed in Section III-C. Numerical results are presented in Section III-D and some generalizations are discussed in Section III-E.

A. Linear Gaussian multi-target model

Our closed form solution to the PHD recursion requires, in addition to assumptions A.1-A.3, a linear Gaussian multi-target model. Along with the standard linear Gaussian model for individual targets, the linear Gaussian multi-target model includes certain assumptions on the birth, death and detection of targets. These are summarized below:

A.4. Each target follows a linear Gaussian dynamical model and the sensor has a linear Gaussian measurement model, i.e.

$$f_{k|k-1}(x|\zeta) = \mathcal{N}(x; F_{k-1}\zeta, Q_{k-1}), \tag{17}$$

$$g_k(z|x) = \mathcal{N}(z; H_k x, R_k), \tag{18}$$

where $\mathcal{N}(\cdot;m,P)$ denotes a Gaussian density with mean m and covariance P, F_{k-1} is the state transition matrix, Q_{k-1} is the process noise covariance, H_k is the observation matrix, and R_k is the observation noise covariance.

A.5. The survival and detection probabilities are state independent, i.e.

$$p_{S,k}(x) = p_{S,k}, \tag{19}$$
$$p_{D,k}(x) = p_{D,k}. \tag{20}$$

A.6. The intensities of the birth and spawn RFSs are Gaussian mixtures of the form

$$\gamma_k(x) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}(x; m_{\gamma,k}^{(i)}, P_{\gamma,k}^{(i)}), \tag{21}$$
$$\beta_{k|k-1}(x|\zeta) = \sum_{j=1}^{J_{\beta,k}} w_{\beta,k}^{(j)} \mathcal{N}(x; F_{\beta,k-1}^{(j)}\zeta + d_{\beta,k-1}^{(j)}, Q_{\beta,k-1}^{(j)}), \tag{22}$$

where $J_{\gamma,k}$, $w_{\gamma,k}^{(i)}$, $m_{\gamma,k}^{(i)}$, $P_{\gamma,k}^{(i)}$, $i = 1,\ldots,J_{\gamma,k}$, are given model parameters that determine the shape of the birth intensity; similarly, $J_{\beta,k}$, $w_{\beta,k}^{(j)}$, $F_{\beta,k-1}^{(j)}$, $d_{\beta,k-1}^{(j)}$, and $Q_{\beta,k-1}^{(j)}$, $j = 1,\ldots,J_{\beta,k}$, determine the shape of the spawning intensity of a target with previous state ζ.

Some remarks regarding the above assumptions are in order:

Remark 2. Assumptions A.4 and A.5 are commonly used in many tracking algorithms [1], [2]. For clarity in the presentation, we only focus on state independent $p_{S,k}$ and $p_{D,k}$, although closed form PHD recursions can be derived for more general cases (see Subsection III-E).

Remark 3. In assumption A.6, $m_{\gamma,k}^{(i)}$, $i = 1,...,J_{\gamma,k}$ are the peaks of the spontaneous birth intensity in (21). These points have the highest local concentrations of expected number of spontaneous births, and represent, for example, airbases or airports where targets are most likely to appear. The covariance matrix $P_{\gamma,k}^{(i)}$ determines the spread of the birth intensity in the vicinity of the peak $m_{\gamma,k}^{(i)}$. The weight $w_{\gamma,k}^{(i)}$ gives the expected number of new targets originating from $m_{\gamma,k}^{(i)}$. A similar interpretation applies to (22), the spawning intensity of a target with previous state ζ, except that the jth peak, $F_{\beta,k-1}^{(j)}\zeta + d_{\beta,k-1}^{(j)}$, is an affine function of ζ. Usually, a spawned target is modelled to be in the proximity of its parent. For example, ζ could correspond to the state of an aircraft carrier at time $k-1$, while $F_{\beta,k-1}^{(j)}\zeta + d_{\beta,k-1}^{(j)}$ is the expected state of fighter planes spawned at time k. Note that other forms of birth and spawning intensities can be approximated, to any desired accuracy, using Gaussian mixtures [42].

B. The Gaussian mixture PHD recursion

For the linear Gaussian multi-target model, the following two propositions present a closed form solution to the PHD recursion (15)-(16). More concisely, these propositions show how the Gaussian components of the posterior intensity are analytically propagated to the next time.

Proposition 1 *Suppose that Assumptions A.4-A.6 hold and that the posterior intensity at time $k-1$ is a Gaussian mixture of the form*

$$v_{k-1}(x) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(x; m_{k-1}^{(i)}, P_{k-1}^{(i)}). \tag{23}$$

Then, the predicted intensity for time k is also a Gaussian mixture, and is given by

$$v_{k|k-1}(x) = v_{S,k|k-1}(x) + v_{\beta,k|k-1}(x) + \gamma_k(x), \tag{24}$$

where $\gamma_k(x)$ is given in (21),

$$v_{S,k|k-1}(x) = p_{S,k} \sum_{j=1}^{J_{k-1}} w_{k-1}^{(j)} \mathcal{N}(x; m_{S,k|k-1}^{(j)}, P_{S,k|k-1}^{(j)}), \tag{25}$$
$$m_{S,k|k-1}^{(j)} = F_{k-1} m_{k-1}^{(j)}, \tag{26}$$
$$P_{S,k|k-1}^{(j)} = Q_{k-1} + F_{k-1} P_{k-1}^{(j)} F_{k-1}^T, \tag{27}$$
$$v_{\beta,k|k-1}(x) = \sum_{j=1}^{J_{k-1}} \sum_{\ell=1}^{J_{\beta,k}} w_{k-1}^{(j)} w_{\beta,k}^{(\ell)} \mathcal{N}(x; m_{\beta,k|k-1}^{(j,\ell)}, P_{\beta,k|k-1}^{(j,\ell)}), \tag{28}$$
$$m_{\beta,k|k-1}^{(j,\ell)} = F_{\beta,k-1}^{(\ell)} m_{k-1}^{(j)} + d_{\beta,k-1}^{(\ell)}, \tag{29}$$
$$P_{\beta,k|k-1}^{(j,\ell)} = Q_{\beta,k-1}^{(\ell)} + F_{\beta,k-1}^{(\ell)} P_{\beta,k-1}^{(j)} (F_{\beta,k-1}^{(\ell)})^T \tag{30}$$

Proposition 2 *Suppose that Assumptions A.4-A.6 hold and that the predicted intensity for time k is a Gaussian mixture of the form*

$$v_{k|k-1}(x) = \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(x; m_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}). \tag{31}$$

Then, the posterior intensity at time k is also a Gaussian mixture, and is given by

$$v_k(x) = (1 - p_{D,k}) v_{k|k-1}(x) + \sum_{z \in Z_k} v_{D,k}(x;z), \tag{32}$$

where

$$v_{D,k}(x;z) = \sum_{j=1}^{J_{k|k-1}} w_k^{(j)}(z) \mathcal{N}(x; m_{k|k}^{(j)}(z), P_{k|k}^{(j)}), \tag{33}$$
$$w_k^{(j)}(z) = \frac{p_{D,k}\, w_{k|k-1}^{(j)} q_k^{(j)}(z)}{\kappa_k(z) + p_{D,k} \sum_{\ell=1}^{J_{k|k-1}} w_{k|k-1}^{(\ell)} q_k^{(\ell)}(z)}, \tag{34}$$
$$q_k^{(j)}(z) = \mathcal{N}(z; H_k m_{k|k-1}^{(j)}, R_k + H_k P_{k|k-1}^{(j)} H_k^T), \tag{35}$$
$$m_{k|k}^{(j)}(z) = m_{k|k-1}^{(j)} + K_k^{(j)}(z - H_k m_{k|k-1}^{(j)}), \tag{36}$$
$$P_{k|k}^{(j)} = [I - K_k^{(j)} H_k] P_{k|k-1}^{(j)}, \tag{37}$$
$$K_k^{(j)} = P_{k|k-1}^{(j)} H_k^T (H_k P_{k|k-1}^{(j)} H_k^T + R_k)^{-1}. \tag{38}$$

Propositions 1 and 2 can be established by applying the following standard results for Gaussian functions:

Lemma 1 *Given F, d, Q, m, and P of appropriate dimensions, and that Q and P are positive definite,*

$$\int \mathcal{N}(x; F\zeta + d, Q) \mathcal{N}(\zeta; m, P) d\zeta = \mathcal{N}(x; Fm + d, Q + FPF^T) \tag{39}$$

Lemma 2 *Given H, R, m, and P of appropriate dimensions, and that R and P are positive definite,*

$$\mathcal{N}(z; Hx, R)\mathcal{N}(x; m, P) = q(z)\mathcal{N}(x; \tilde{m}, \tilde{P}) \tag{40}$$

where

$$q(z) = \mathcal{N}(z; Hm, R + HPH^T) \tag{41}$$
$$\tilde{m} = m + K(z - Hm) \tag{42}$$
$$\tilde{P} = (I - KH)P \tag{43}$$
$$K = PH^T(HPH^T + R)^{-1} \tag{44}$$

Note that Lemma 1 can be derived from Lemma 2, which in turn can be found in [43] or [44] (Section 3.8), though in a slightly different form.

Proposition 1 is established by substituting (17), (19), (21), (22) and (23) into the PHD prediction (15), and replacing integrals of the form (39) by appropriate Gaussians as given by Lemma 1. Similarly, Proposition 2 is established by substituting (18), (20) and (31) into the PHD update (16), and then replacing integrals of the form (39) and product of Gaussians of the form (40) by appropriate Gaussians as given by Lemmas 1 and 2 respectively.

It follows by induction from Propositions 1 and 2 that if the initial prior intensity v_0 is a Gaussian mixture (including the case where $v_0 = 0$), then all subsequent predicted intensities $v_{k|k-1}$ and posterior intensities v_k are also Gaussian mixtures. Proposition 1 provides closed form expressions for computing the means, covariances and weights of $v_{k|k-1}$ from those of v_{k-1}. Proposition 2 then provides closed form expressions for computing the means, covariances and weights of v_k from those of $v_{k|k-1}$ when a new set of measurements arrives. Propositions 1 and 2 are, respectively, the prediction and update steps of the PHD recursion for a linear Gaussian multi-target model, herein referred to as the *Gaussian mixture PHD recursion*. For completeness, we summarize the key steps of the Gaussian mixture PHD filter in Table I.

Remark 4. The predicted intensity $v_{k|k-1}$ in Proposition 1 consists of three terms $v_{S,k|k-1}$, $v_{\beta,k|k-1}$ and γ_k due, respectively, to the existing targets, the spawned targets, and the spontaneous births. Similarly, the updated posterior intensity v_k in Proposition 2 consists of a mis-detection term, $(1 - p_{D,k})v_{k|k-1}$, and $|Z_k|$ detection terms, $v_{D,k}(\cdot; z)$, one for each measurement $z \in Z_k$. As it turns out, the recursions for the means and covariances of $v_{S,k|k-1}$ and $v_{\beta,k|k-1}$ are Kalman predictions, and the recursions for the means and covariances of $v_{D,k}(\cdot; z)$ are Kalman updates.

Given the Gaussian mixture intensities $v_{k|k-1}$ and v_k, the corresponding expected number of targets $\hat{N}_{k|k-1}$ and $\hat{N}_k$ can be obtained by summing up the appropriate weights. Propositions 1 and 2 lead to the following closed form recursions for $\hat{N}_{k|k-1}$ and $\hat{N}_k$:

Corollary 1 *Under the premises of Proposition 1, the mean of the predicted number of targets is*

$$\hat{N}_{k|k-1} = \hat{N}_{k-1}\left(p_{S,k} + \sum_{j=1}^{J_{\beta,k}} w_{\beta,k}^{(j)}\right) + \sum_{j=1}^{J_{\gamma,k}} w_{\gamma,k}^{(j)}, \tag{45}$$

TABLE I

PSEUDO-CODE FOR THE GAUSSIAN MIXTURE PHD FILTER.

given $\{w_{k-1}^{(i)}, m_{k-1}^{(i)}, P_{k-1}^{(i)}\}_{i=1}^{J_{k-1}}$, and the measurement set Z_k.
step 1. (prediction for birth targets)
  $i = 0$.
  for $j = 1, \ldots, J_{\gamma,k}$
    $i := i + 1$.
    $w_{k|k-1}^{(i)} = w_{\gamma,k}^{(j)}, \quad m_{k|k-1}^{(i)} = m_{\gamma,k}^{(j)}, \quad P_{k|k-1}^{(i)} = P_{\gamma,k}^{(j)}$.
  end
  for $j = 1, \ldots, J_{\beta,k}$
    for $\ell = 1, \ldots, J_{k-1}$
      $i := i + 1$.
      $w_{k|k-1}^{(i)} = w_{k-1}^{(\ell)} w_{\beta,k}^{(j)}$,
      $m_{k|k-1}^{(i)} = d_{\beta,k-1}^{(j)} + F_{\beta,k-1}^{(j)} m_{k-1}^{(\ell)}$,
      $P_{k|k-1}^{(i)} = Q_{\beta,k-1}^{(j)} + F_{\beta,k-1}^{(j)} P_{k-1}^{(\ell)} (F_{\beta,k-1}^{(j)})^T$.
    end
  end
step 2. (prediction for existing targets)
  for $j = 1, \ldots, J_{k-1}$
    $i := i + 1$.
    $w_{k|k-1}^{(i)} = p_{S,k}\, w_{k-1}^{(j)}$,
    $m_{k|k-1}^{(i)} = F_{k-1} m_{k-1}^{(j)}, \quad P_{k|k-1}^{(i)} = Q_{k-1} + F_{k-1} P_{k-1}^{(j)} F_{k-1}^T$,
  end
  $J_{k|k-1} = i$.
step 3. (construction of PHD update components)
  for $j = 1, \ldots, J_{k|k-1}$
    $\eta_{k|k-1}^{(j)} = H_k m_{k|k-1}^{(j)}, \quad S_k^{(j)} = R_k + H_k P_{k|k-1}^{(j)} H_k^T$,
    $K_k^{(j)} = P_{k|k-1}^{(j)} H_k^T [S_k^{(j)}]^{-1}, \quad P_{k|k}^{(j)} = [I - K_k^{(j)} H_k] P_{k|k-1}^{(j)}$.
  end
step 4. (update)
  for $j = 1, \ldots, J_{k|k-1}$
    $w_k^{(j)} = (1 - p_{D,k}) w_{k|k-1}^{(j)}$,
    $m_k^{(j)} = m_{k|k-1}^{(j)}, \quad P_k^{(j)} = P_{k|k-1}^{(j)}$.
  end
  $\ell := 0$.
  for each $z \in Z_k$
    $\ell := \ell + 1$.
    for $j = 1, \ldots, J_{k|k-1}$
      $w_k^{(\ell J_{k|k-1}+j)} = p_{D,k}\, w_{k|k-1}^{(j)} \mathcal{N}(z; \eta_{k|k-1}^{(j)}, S_k^{(j)})$.
      $m_k^{(\ell J_{k|k-1}+j)} = m_{k|k-1}^{(j)} + K_k^{(j)}(z - \eta_{k|k-1}^{(j)})$,
      $P_k^{(\ell J_{k|k-1}+j)} = P_{k|k}^{(j)}$,
    end
    $w_k^{(\ell J_{k|k-1}+j)} := \dfrac{w_k^{(\ell J_{k|k-1}+j)}}{\kappa_k(z) + \sum_{i=1}^{J_{k|k-1}} w_k^{(\ell J_{k|k-1}+i)}}$, for $j = 1, \ldots, J_{k|k-1}$.
  end
  $J_k = \ell J_{k|k-1} + J_{k|k-1}$.
output $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}$.

Corollary 2 *Under the premises of Proposition 2, the mean of the updated number of targets is*

$$\hat{N}_k = \hat{N}_{k|k-1}(1 - p_{D,k}) + \sum_{z \in Z_k} \sum_{j=1}^{J_{k|k-1}} w_k^{(j)}(z) \tag{46}$$

In Corollary 1, the mean of the predicted number of targets is obtained by adding the mean number of surviving targets, the mean number of spawnings and the mean number of births. A similar interpretation can be drawn from Corollary 2. When there is no clutter, the mean of the updated number of targets is the number of measurements plus the mean number of targets that are not detected.

C. Implementation issues

The Gaussian mixture PHD filter is similar to the Gaussian sum filter of [32], [33] in the sense that they both propagate Gaussian mixtures in time. Like the Gaussian sum filter, the Gaussian mixture PHD filter also suffers from computation problems associated with the increasing number of Gaussian components as time progresses. Indeed, at time k, the Gaussian mixture PHD filter requires

$$(J_{k-1}(1+J_{\beta,k})+J_{\gamma,k})(1+|Z_k|)=\mathcal{O}(J_{k-1}|Z_k|)$$

Gaussian components to represent v_k, where J_{k-1} is number of components of v_{k-1}. This implies the number of components in the posterior intensities increases without bound.

A simple pruning procedure can be used to reduce the number of Gaussian components propagated to the next time step. A good approximation to the Gaussian mixture posterior intensity

$$v_k(x)=\sum_{i=1}^{J_k} w_k^{(i)}\mathcal{N}(x;m_k^{(i)},P_k^{(i)})$$

can be obtained by truncating components that have weak weights $w_k^{(i)}$. This can be done by discarding those with weights below some preset threshold, or by keeping only a certain number of components with strongest weights. Moreover, some of the Gaussian components are so close together that they could be accurately approximated by a single Gaussian. Hence, in practice these components can be merged into one. These ideas lead to the simple heuristic pruning algorithm shown in Table II.

TABLE II
PRUNING FOR THE GAUSSIAN MIXTURE PHD FILTER.

given $\{w_k^{(i)},m_k^{(i)},P_k^{(i)}\}_{i=1}^{J_k}$, a truncation threshold T, a merging threshold U, and a maximum allowable number of Gaussian terms J_{max}.
Set $\ell=0$, and $I=\{i=1,\dots,J_k|w_k^{(i)}>T\}$.
repeat
  $\ell:=\ell+1$.
  $j:=\arg\max_{i\in I} w_k^{(i)}$.
  $L:=\left\{i\in I \,\middle|\, (m_k^{(i)}-m_k^{(j)})^T(P_k^{(i)})^{-1}(m_k^{(i)}-m_k^{(j)})\le U\right\}$.
  $\tilde{w}_k^{(\ell)}=\sum_{i\in L} w_k^{(i)}$.
  $\tilde{m}_k^{(\ell)}=\frac{1}{\tilde{w}_k^{(\ell)}}\sum_{i\in L} w_k^{(i)}x_k^{(i)}$.
  $\tilde{P}_k^{(\ell)}=\frac{1}{\tilde{w}_k^{(\ell)}}\sum_{i\in L} w_k^{(i)}(P_k^{(i)}+(\tilde{m}_k^{(\ell)}-m_k^{(i)})(\tilde{m}_k^{(\ell)}-m_k^{(i)})^T)$.
  $I:=I\backslash L$.
until $I=\emptyset$.
if $\ell>J_{max}$ then replace $\{\tilde{w}_k^{(i)},\tilde{m}_k^{(i)},\tilde{P}_k^{(i)}\}_{i=1}^{\ell}$ by those of the J_{max} Gaussians with largest weights.
output $\{\tilde{w}_k^{(i)},\tilde{m}_k^{(i)},\tilde{P}_k^{(i)}\}_{i=1}^{\ell}$ as pruned Gaussian components.

TABLE III
MULTI-TARGET STATE EXTRACTION

given $\{w_k^{(i)},m_k^{(i)},P_k^{(i)}\}_{i=1}^{J_k}$.
Set $\hat{X}_k=\emptyset$.
for $i=1,\dots,J_k$
  if $w_k^{(i)}>0.5$,
    for $j=1,\dots,\mathrm{round}(w_k^{(i)})$
      update $\hat{X}_k:=\left[\hat{X}_k,m_k^{(i)}\right]$
    end
  end
end
output $\hat{X}_k$ as the multi-target state estimate.

Having computed the posterior intensity v_k, the next task is to extract multi-target state estimates. In general, such a task may not be simple. For example, in the particle-PHD filter [17], the estimated number of targets $\hat{N}_k$ is given by the total mass of the particles representing v_k. The estimated states are then obtained by partitioning these particles into $\hat{N}_k$ clusters, using standard clustering algorithms. This works well when the posterior intensity v_k naturally has $\hat{N}_k$ clusters. Conversely, when $\hat{N}_k$ differs from the number of clusters, the state estimates become unreliable.

In the Gaussian mixture representation of the posterior intensity v_k, extraction of multi-target state estimates is straightforward since the means of the constituent Gaussian components are indeed the local maxima of v_k, provided that they are reasonably well-separated. Note that after pruning (see Table II) closely spaced Gaussian components would have been merged. Since the height of each peak depends on both the weight and covariance, selecting the $\hat{N}_k$ highest peaks of v_k may result in state estimates that correspond to Gaussians with weak weights. This is not desirable because the expected number of targets due to these peaks is small, even though the magnitudes of the peaks are large. A better alternative is to select the means of the Gaussians that have weights greater than some threshold e.g. 0.5. This state estimation procedure for the Gaussian mixture PHD filter is summarized in Table III.

D. Simulation Results

Two simulation examples are used to test the proposed Gaussian mixture PHD filter. An additional example can be found in [34].

1) Example 1: For illustration purposes, consider a two-dimensional scenario with an unknown and time varying number of targets observed in clutter over the surveillance region $[-1000,1000]\times[-1000,1000]$ (in m). The state $x_k=[\ p_{x,k},p_{y,k},\dot{p}_{x,k},\dot{p}_{y,k}\ ]^T$ of each target consists of position $(p_{x,k},p_{y,k})$ and velocity $(\dot{p}_{x,k},\dot{p}_{y,k})$, while the measurement is a noisy version of the position.

Each target has survival probability $p_{S,k}=0.99$, and follows the linear Gaussian dynamics (17) with

$$F_k = \begin{bmatrix} I_2 & \Delta I_2 \\ 0_2 & I_2 \end{bmatrix}, \qquad Q_k = \sigma_\nu^2 \begin{bmatrix} \frac{\Delta^4}{4} I_2 & \frac{\Delta^3}{2} I_2 \\ \frac{\Delta^3}{2} I_2 & \Delta^2 I_2 \end{bmatrix},$$

where I_n and 0_n denotes, respectively, the $n \times n$ identity and zero matrices, $\Delta = 1s$ is the sampling period, and $\sigma_\nu = 5(m/s^2)$ is the standard deviation of the process noise. Targets can appear from two possible locations as well as spawned from other targets. Specifically, a Poisson RFS Γ_k with intensity

$$\gamma_k(x) = 0.1\mathcal{N}(x; m_\gamma^{(1)}, P_\gamma) + 0.1\mathcal{N}(x; m_\gamma^{(2)}, P_\gamma),$$

where

$$\begin{aligned} m_\gamma^{(1)} &= [\ 250, 250, 0, 0\ ]^T, \\ m_\gamma^{(2)} &= [\ -250, -250, 0, 0\ ]^T, \\ P_\gamma &= \mathrm{diag}([\ 100,\ 100,\ 25,\ 25\ ]^T), \end{aligned}$$

is used to model spontaneous births in the vicinity of $m_\gamma^{(1)}$ and $m_\gamma^{(2)}$. Additionally, the RFS $B_{k|k-1}(\zeta)$ of targets spawned from a target with previous state ζ is Poisson with intensity

$$\begin{aligned} \beta_{k|k-1}(x|\zeta) &= 0.05\mathcal{N}(x; \zeta, Q_\beta), \\ Q_\beta &= \mathrm{diag}([\ 100, 100, 400, 400\ ]^T). \end{aligned}$$

Each target is detected with probability $p_{D,k} = 0.98$, and the measurement follows the observation model (18) with $H_k = [\ I_2\ \ 0_2\ ]$, $R_k = \sigma_\varepsilon^2 I_2$, where $\sigma_\varepsilon = 10m$ is the standard deviation of the measurement noise. The detected measurements are immersed in clutter that can be modelled as a Poisson RFS K_k with intensity

$$\kappa_k(z) = \lambda_c V u(z), \tag{47}$$

where $u(\cdot)$ is the uniform density over the surveillance region, $V = 4 \times 10^6 m^2$ is the 'volume' of the surveillance region, and $\lambda_c = 12.5 \times 10^{-6} m^{-2}$ is the average number of clutter returns per unit volume (i.e. 50 clutter returns over the surveillance region).

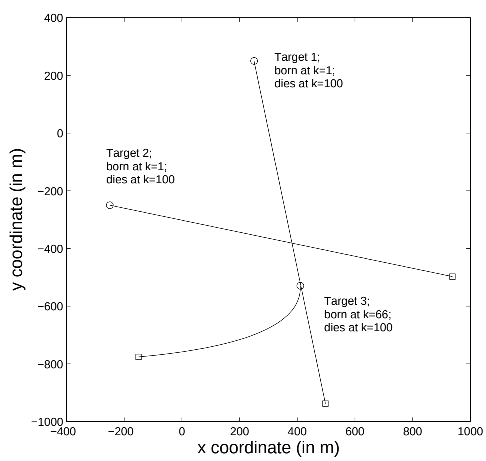

Fig. 1. Target trajectories. '○'– locations at which targets are born; '□'– locations at which targets die.

Figure 1 shows the true target trajectories, while Figure 2 plots these trajectories with cluttered measurements against time. Targets 1 and 2 are born at the same time but at two different locations. They travel along straight lines (their tracks cross at $k = 53$s) and at $k = 66$s target 1 spawns target 3.

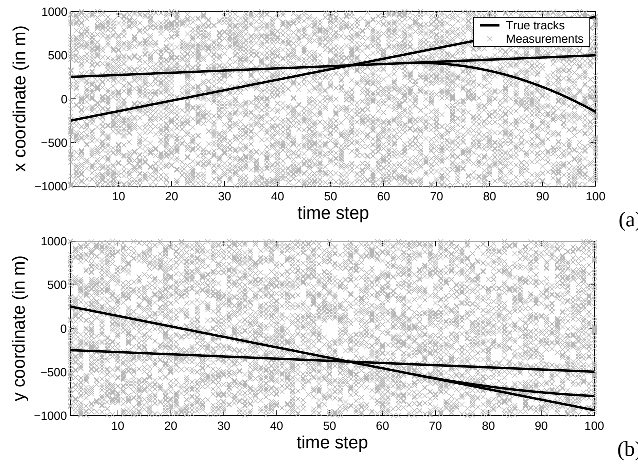

Fig. 2. Measurements and true target positions.

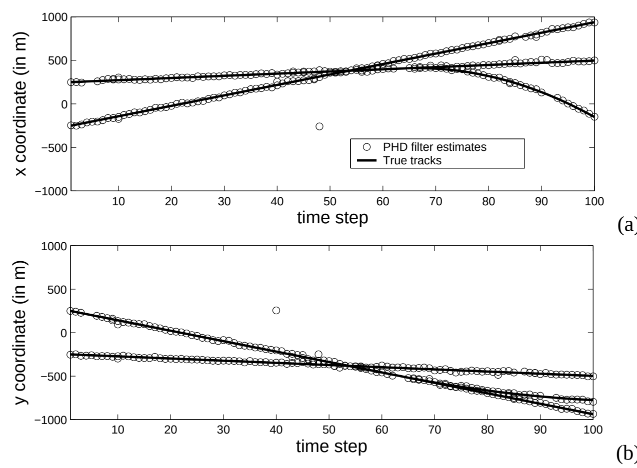

Fig. 3. Position estimates of the Gaussian mixture PHD filter.

The Gaussian mixture PHD filter, with parameters $T = 10^{-5}$, $U = 4$, and $J_{max} = 100$ (see Table II for the meanings of these parameters) is applied. From the position estimates shown in Figure 3, it can be seen that the Gaussian mixture PHD filter provides accurate tracking performance. The filter not only successfully detects and tracks targets 1 and 2, but also manages to detect and track the spawned target 3. The filter does generate anomalous estimates occasionally, but these false estimates die out very quickly.

2) Example 2: In this example we evaluate the performance of the Gaussian mixture PHD filter by benchmarking it against the JPDA filter [1], [8] via Monte Carlo simulations. The JPDA filter is a classical filter for tracking a known and fixed number of targets in clutter. In a scenario where the number of targets is constant, the JPDA filter (given the correct number of targets) is expected to outperform the PHD filter, since the latter has neither knowledge of the number of targets, nor even knowledge that the number of targets is constant. For these reasons, the JPDA filter serves as a good benchmark.

The experiment settings are the same as those of Example 1, but without spawning, as the JPDA filter requires a known and fixed number of targets. The true tracks in this example are those of targets 1 and 2 in Figure 1. Target trajectories are fixed for all simulation trials, while observation noise and clutter are independently generated at each trial.

We study track loss performance by using the following circular position error probability (CPEP) (see [45] for example)

$$\mathrm{CPEP}_k(r) = \frac{1}{|X_k|} \sum_{x \in X_k} \rho_k(x,r),$$

for some position error radius r, where

$$\rho_k(x,r) = \mathrm{Prob}\{\|H\hat{x} - Hx\|_2 > r \text{ for all } \hat{x} \in \hat{X}_k\},$$

$H = [\ I_2\ 0_2\ ]$ and $\|\cdot\|_2$ is the 2-norm. In addition, we measure the expected absolute error on the number of targets for the Gaussian mixture PHD filter:

$$\mathrm{E}\{|\ |\hat{X}_k| - |X_k|\ |\}.$$

Note that standard performance measures such as the mean square distance error are not applicable to multi-target filters that jointly estimate the number of targets and their states (such as the PHD filter).

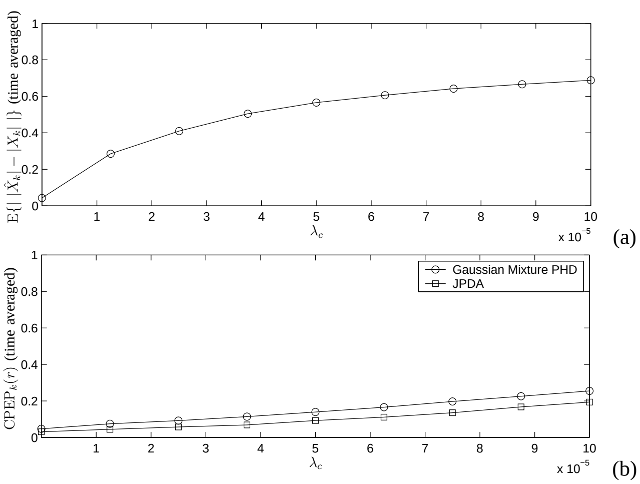

Fig. 4. Tracking performance versus clutter rate. The detection probability is fixed at $p_{D,k} = 0.98$. The CPEP radius is $r = 20$m.

Figure 4 shows the tracking performance of the two filters for various clutter rates λ_c [cf., Eq. (47)] with the CPEP radius fixed at $r = 20$m. Observe that the CPEPs of the two filters are quite close for a wide range of clutter rates. This is rather surprising considering that the JPDA filter has exact knowledge of the number of targets. Figure 4(a) suggests that the occasional overestimation/underestimation of the number of targets is not significant in the Gaussian mixture PHD filter.

Figure 5 shows the tracking performance for various values of detection probability $p_{D,k}$ with the clutter rate fixed at $\lambda_c = 12.5 \times 10^{-6}\mathrm{m}^{-2}$. Observe that the performance gap between the two filters increases as $p_{D,k}$ decreases. This is because the PHD filter has to resolve higher detection uncertainty on top of uncertainty in the number of targets. When detection uncertainty increases ($p_{D,k}$ decreases), uncertainty about the number of targets also increases. In contrast, the JPDA filter's exact knowledge of the number of targets is not affected by the increase in detection uncertainty.

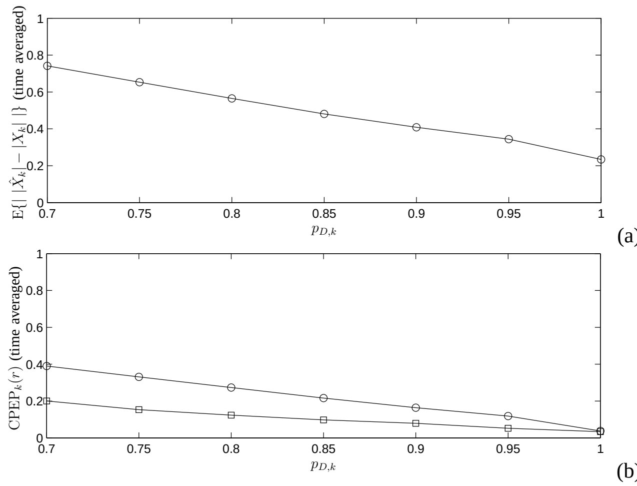

Fig. 5. Tracking performance versus detection probability. The clutter rate is fixed at $\lambda_c = 12.5 \times 10^{-6}\mathrm{m}^{-2}$. The CPEP radius is $r = 20$m.

E. Generalizations to exponential mixture $p_{D,k}$ and $p_{S,k}$

As remarked in Section III-A, closed form solutions to the PHD recursion can still be obtained for a certain class of state-dependent probability of survival and probability of detection. Indeed, Propositions 1 and 2 can be easily generalized to handle $p_{S,k}(x)$ and $p_{D,k}(x)$ of the forms:

$$p_{S,k}(\zeta) = w_{S,k}^{(0)} + \sum_{j=1}^{J_{S,k}} w_{S,k}^{(j)} \mathcal{N}(\zeta; m_{S,k}^{(j)}, P_{S,k}^{(j)}), \quad (48)$$

$$p_{D,k}(x) = w_{D,k}^{(0)} + \sum_{j=1}^{J_{D,k}} w_{D,k}^{(j)} \mathcal{N}(x; m_{D,k}^{(j)}, P_{D,k}^{(j)}), \quad (49)$$

where $J_{S,k}$, $w_{S,k}^{(0)}$, $w_{S,k}^{(i)}$, $m_{S,k}^{(i)}$, $P_{S,k}^{(i)}$, $i = 1, ..., J_{S,k}$ and $J_{D,k}$, $w_{D,k}^{(0)}$, $w_{D,k}^{(i)}$, $m_{D,k}^{(i)}$, $P_{D,k}^{(i)}$, $i = 1, ..., J_{D,k}$ are given model parameters such that $p_{S,k}(x)$ and $p_{D,k}(x)$ lie between 0 and 1 for all x.

The closed form predicted intensity $v_{k|k-1}$ can be obtained by applying Lemma 2 to convert $p_{S,k}(\zeta)v_{k-1}(\zeta)$ into a Gaussian mixture, which is then integrated with the transition density $f_{k|k-1}(x|\zeta)$ using Lemma 1. The closed form updated intensity v_k can be obtained by applying Lemma 2 once to $p_{D,k}(x)v_{k|k-1}(x)$ and twice to $p_{D,k}(x)g_k(x|z)v_{k|k-1}(x)$ to convert these products to Gaussian mixtures. For completeness, the Gaussian mixture expressions for $v_{k|k-1}$ and v_k are given in the following Propositions, although their implementations will not be pursued.

Proposition 3 *Under the premises of Proposition 1, with $p_{S,k}(x)$ given by (48) instead of (19), the predicted intensity $v_{k|k-1}$ is given by (24) but with*

$$v_{S,k|k-1}(x) = \sum_{i=1}^{J_{k-1}} \sum_{j=0}^{J_{S,k}} w_{S,k|k-1}^{(i,j)} \mathcal{N}(x; m_{S,k|k-1}^{(i,j)}, P_{S,k|k-1}^{(i,j)}) \quad (50)$$

$$
\begin{aligned}
w_{S,k|k-1}^{(i,j)} &= w_{k-1}^{(i)} w_{S,k}^{(j)} q_{k-1}^{(i,j)},\\
m_{S,k|k-1}^{(i,j)} &= F_{k-1} m_{k-1}^{(i,j)},\\
P_{S,k|k-1}^{(i,j)} &= Q_{k-1} + F_{k-1} P_{k-1}^{(i,j)} F_{k-1}^T,\\
q_{k-1}^{(i,0)} &= 1,\ m_{k-1}^{(i,0)} = m_{k-1}^{(i)},\ P_{k-1}^{(i,0)} = P_{k-1}^{(i)},\\
q_{k-1}^{(i,j)} &= \mathcal{N}(m_{S,k}^{(j)}; m_{k-1}^{(i)}, P_{S,k}^{(j)} + P_{k-1}^{(i)}),\\
m_{k-1}^{(i,j)} &= m_{k-1}^{(i)} + K_{k-1}^{(i,j)}(m_{S,k}^{(j)} - m_{k-1}^{(i)}),\\
P_{k-1}^{(i,j)} &= (I - K_{k-1}^{(i,j)}) P_{k-1}^{(i)},\\
K_{k-1}^{(i,j)} &= P_{k-1}^{(i)} (P_{k-1}^{(i)} + P_{S,k}^{(j)})^{-1}.
\end{aligned}
$$

Proposition 4 *Under the premises of Proposition 2, with $p_{D,k}(x)$ given by (49) instead of (20),*

$$
v_k(x) = v_{k|k-1}(x) - v_{D,k}(x) + \sum_{z\in Z_k} v_{D,k}(x;z), \qquad (51)
$$

where

$$
\begin{aligned}
v_{D,k}(x) &= \sum_{i=1}^{J_{k|k-1}} \sum_{j=0}^{J_{D,k}} w_{k|k-1}^{(i,j)} \mathcal{N}(x; m_{k|k-1}^{(i,j)}, P_{k|k-1}^{(i,j)}),\\
w_{k|k-1}^{(i,j)} &= w_{D,k}^{(j)} w_{k|k-1}^{(i)} q_{k|k-1}^{(i,j)},\\
q_{k|k-1}^{(i,0)} &= 1,\ m_{k|k-1}^{(i,0)} = m_{k|k-1}^{(i)},\ P_{k|k-1}^{(i,0)} = P_{k|k-1}^{(i)},\\
q_{k|k-1}^{(i,j)} &= \mathcal{N}(m_{D,k}^{(j)}; m_{k|k-1}^{(i)}, P_{D,k}^{(j)} + P_{k|k-1}^{(i)}),\\
m_{k|k-1}^{(i,j)} &= m_{k|k-1}^{(i)} + K_{k|k-1}^{(i,j)}(m_{D,k}^{(j)} - m_{k|k-1}^{(i)}),\\
P_{k|k-1}^{(i,j)} &= (I - K_{k|k-1}^{(i,j)}) P_{k|k-1}^{(i)},\\
K_{k|k-1}^{(i,j)} &= P_{k|k-1}^{(i)} (P_{k|k-1}^{(i)} + P_{D,k}^{(j)})^{-1},\\
v_{D,k}(x;z) &= \sum_{i=1}^{J_{k|k-1}} \sum_{j=0}^{J_{D,k}} w_k^{(i,j)}(z) \mathcal{N}(x; m_{k|k}^{(i,j)}(z), P_{k|k}^{(i,j)}),\\
w_k^{(i,j)}(z) &= \frac{w_{k|k-1}^{(i,j)} q_k^{(i,j)}(z)}{\kappa_k(z) + \sum_{r=1}^{J_{k|k-1}} \sum_{s=0}^{J_{D,k}} w_{k|k-1}^{(r,s)} q_k^{(r,s)}(z)},\\
q_k^{(i,j)}(z) &= \mathcal{N}(z; H_k m_{k|k-1}^{(i,j)}, R_k + H_k P_{k|k-1}^{(i,j)} H_k^T),\\
m_{k|k}^{(i,j)}(z) &= m_{k|k-1}^{(i,j)} + K_k^{(i,j)}(z - H_k m_{k|k-1}^{(i,j)}),\\
P_{k|k}^{(i,j)} &= (I - K_k^{(i,j)} H_k) P_{k|k-1}^{(i,j)},\\
K_k^{(i,j)} &= P_{k|k-1}^{(i,j)} H_k^T (H_k P_{k|k-1}^{(i,j)} H_k^T + R_k)^{-1}.
\end{aligned}
$$

Conceptually, the Gaussian mixture PHD filter implementation can be easily extended to accommodate exponential mixture probability of survival. However, for exponential mixture probability of detection, the updated intensity contains Gaussians with negative and positive weights, even though the updated intensity itself (and hence the sum of the weights) is non-negative. Although these Gaussians can be propagated using Propositions 3 and 4, care must be taken in the implementation to ensure non-negativity of the intensity function after merging and pruning.

IV. EXTENSION TO NONLINEAR GAUSSIAN MODELS

This section considers extensions of the Gaussian mixture PHD filter to nonlinear target models. Specifically, the modelling assumptions A.5 and A.6 are still required, but the state and observation processes can be relaxed to the nonlinear model:

$$
x_k = \varphi_k(x_{k-1}, \nu_{k-1}), \qquad (52)
$$

$$
z_k = h_k(x_k, \varepsilon_k), \qquad (53)
$$

where φ_k and h_k are known nonlinear functions, ν_{k-1} and ε_k are zero-mean Gaussian process noise and measurement noise with covariances Q_{k-1} and R_k, respectively. Due to the nonlinearity of φ_k and h_k, the posterior intensity can no longer be represented as a Gaussian mixture. Nonetheless, the proposed Gaussian mixture PHD filter can be adapted to accommodate nonlinear Gaussian models.

TABLE IV
PSEUDO-CODE FOR THE EK-PHD FILTER.

given $\{w_{k-1}^{(i)}, m_{k-1}^{(i)}, P_{k-1}^{(i)}\}_{i=1}^{J_{k-1}}$, and the measurement set Z_k.

step 1. (construction of birth target components)

follow Step 1. of Table I.

step 2. (prediction for existing targets)

for $j = 1, \ldots, J_{k-1}$

$i := i + 1$.

$w_{k|k-1}^{(i)} = p_{S,k}\, w_{k-1}^{(j)}, \quad m_{k|k-1}^{(i)} = \varphi_k(m_{k-1}^{(j)}, 0),$

$P_{k|k-1}^{(i)} = G_{k-1}^{(j)} Q_{k-1} [G_{k-1}^{(j)}]^T + F_{k-1}^{(j)} P_{k-1}^{(j)} [F_{k-1}^{(j)}]^T,$

where

$$
F_{k-1}^{(j)} = \left.\frac{\partial \varphi_k(x_{k-1}, 0)}{\partial x_{k-1}}\right|_{x_{k-1}=m_{k-1}^{(j)}}, \qquad
G_{k-1}^{(j)} = \left.\frac{\partial \varphi_k(m_{k-1}^{(j)}, \nu_{k-1})}{\partial \nu_{k-1}}\right|_{\nu_{k-1}=0}.
$$

end

$J_{k|k-1} = i$.

step 3. (construction of PHD update components)

for $j = 1, \ldots, J_{k|k-1}$

$\eta_{k|k-1}^{(j)} = h_k(m_{k|k-1}^{(j)}, 0),$

$S_k^{(j)} = U_k^{(j)} R_k [U_k^{(j)}]^T + H_k^{(j)} P_{k|k-1}^{(j)} [H_k^{(j)}]^T,$

$K_k^{(j)} = P_{k|k-1}^{(j)} [H_k^{(j)}]^T [S_k^{(j)}]^{-1},$

$P_{k|k}^{(j)} = [I - K_k^{(j)} H_k^{(j)}] P_{k|k-1}^{(j)},$

where

$$
H_k^{(j)} = \left.\frac{\partial h_k(x_k, 0)}{\partial x_k}\right|_{x_k=m_{k|k-1}^{(j)}}, \qquad
U_k^{(j)} = \left.\frac{\partial h_k(m_{k|k-1}^{(j)}, \varepsilon_k)}{\partial \varepsilon_k}\right|_{\varepsilon_k=0}.
$$

end

step 4. (update)

follow Step 4. of Table I to obtain $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}$.

output $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}$.

In single-target filtering, analytic approximations of the nonlinear Bayes filter include the extended Kalman (EK) filter [46], [47] and the unscented Kalman (UK) filter [48], [49]. The EK filter approximates the posterior density by a Gaussian, which is propagated in time by applying the Kalman recursions to local linearizations of the (nonlinear) mappings φ_k and h_k. The UK filter also approximates the posterior density by a Gaussian, but instead of using the linearized model, it computes the Gaussian approximation of the posterior density at the next time step using the unscented transform. Details for the EK and UK filters are given in [46], [47] and [48], [49], respectively.

TABLE V
PSEUDO-CODE FOR THE UK-PHD FILTER.

---

given $\{w_{k-1}^{(i)}, m_{k-1}^{(i)}, P_{k-1}^{(i)}\}_{i=1}^{J_{k-1}}$ and the measurement set Z_k.

step 1. (construction of birth target components)

follow Step 1. of Table I.

for $j = 1, \ldots, i$

- set
$$\mu := \begin{bmatrix} m_{k|k-1}^{(j)} \\ 0 \end{bmatrix}, \quad C := \begin{bmatrix} P_{k|k-1}^{(j)} & 0 \\ 0 & R_k \end{bmatrix}.$$
- use the unscented transformation (see [46], [47]) with mean μ and covariance C to generate a set of sigma points and weights, denoted by $\{y_k^{(\ell)}, u^{(\ell)}\}_{\ell=0}^{L}$.
- partition $y_k^{(\ell)} = [\ (x_{k|k-1}^{(\ell)})^T, (\epsilon_k^{(\ell)})^T\ ]^T$ for $\ell = 0, 1, \ldots, L$.
- compute
$$z_{k|k-1}^{(\ell)} := h_k(x_{k|k-1}^{(\ell)}, \varepsilon_k^{(\ell)}), \quad \ell = 0, \ldots, L,$$
$$\eta_{k|k-1}^{(j)} = \sum_{\ell=0}^{L} u^{(\ell)} z_{k|k-1}^{(\ell)},$$
$$S_k^{(j)} = \sum_{\ell=0}^{L} u^{(\ell)} (z_{k|k-1}^{(\ell)} - \eta_{k|k-1}^{(j)})(z_{k|k-1}^{(\ell)} - \eta_{k|k-1}^{(j)})^T,$$
$$G_k^{(j)} = \sum_{\ell=0}^{L} u^{(\ell)} (x_{k|k-1}^{(\ell)} - m_{k|k-1}^{(j)})(z_{k|k-1}^{(\ell)} - \eta_{k|k-1}^{(j)})^T,$$
$$K_k^{(j)} = G_k^{(j)} [S_k^{(j)}]^{-1},$$
$$P_{k|k}^{(j)} = P_{k|k-1}^{(j)} - G_k^{(j)} [S_k^{(j)}]^{-1} [G_k^{(j)}]^T.$$

end

step 2. (construction of existing target components)

for $j = 1, \ldots, J_{k-1}$

- $i := i + 1$.
- $w_{k|k-1}^{(i)} = p_{S,k}\, w_{k-1}^{(j)}$.
- set
$$\mu := \begin{bmatrix} m_{k-1}^{(i)} \\ 0 \\ 0 \end{bmatrix}, \quad C := \begin{bmatrix} P_{k-1}^{(i)} & 0 & 0 \\ 0 & Q_{k-1} & 0 \\ 0 & 0 & R_k \end{bmatrix}.$$
- use the unscented transformation with mean μ and covariance C to generate a set of sigma points and weights, denoted by $\{y_k^{(\ell)}, u^{(\ell)}\}_{\ell=0}^{L}$.
- partition $y_k^{(\ell)} = [\ (x_{k-1}^{(\ell)})^T, (\nu_{k-1}^{(\ell)})^T, (\epsilon_k^{(\ell)})^T\ ]^T$ for $\ell = 0, 1, \ldots, L$.
- compute
$$x_{k|k-1}^{(\ell)} := \varphi_k(x_{k-1}^{(\ell)}, \nu_{k-1}^{(\ell)}), \quad \ell = 0, \ldots, L,$$
$$z_{k|k-1}^{(\ell)} := h_k(x_{k|k-1}^{(\ell)}, \epsilon_k^{(\ell)}), \quad \ell = 0, \ldots, L,$$
$$m_{k|k-1}^{(i)} = \sum_{\ell=0}^{L} u^{(\ell)} x_{k|k-1}^{(\ell)},$$
$$P_{k|k-1}^{(i)} = \sum_{\ell=0}^{L} u^{(\ell)} (x_{k|k-1}^{(\ell)} - m_{k|k-1}^{(j)})(x_{k|k-1}^{(\ell)} - m_{k|k-1}^{(j)})^T,$$
$$\eta_{k|k-1}^{(i)} = \sum_{\ell=0}^{L} u^{(\ell)} z_{k|k-1}^{(\ell)},$$
$$S_k^{(i)} = \sum_{\ell=0}^{L} u^{(\ell)} (z_{k|k-1}^{(\ell)} - \eta_{k|k-1}^{(j)})(z_{k|k-1}^{(\ell)} - \eta_{k|k-1}^{(j)})^T,$$
$$G_k^{(i)} = \sum_{\ell=0}^{L} u^{(\ell)} (x_{k|k-1}^{(\ell)} - m_{k|k-1}^{(j)})(z_{k|k-1}^{(\ell)} - \eta_{k|k-1}^{(j)})^T,$$
$$K_k^{(i)} = G_k^{(i)} [S_k^{(i)}]^{-1},$$
$$P_{k|k}^{(i)} = P_{k|k-1}^{(i)} - G_k^{(i)} [S_k^{(i)}]^{-1} [G_k^{(i)}]^T.$$

end

$J_{k|k-1} = i$.

step 3. (update)

follow Step 4. of Table I to obtain $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}$.

output $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}$.

Following the development in Section III-B, it can be shown that the posterior intensity of the multi-target state propagated by the PHD recursions (15)-(16) is a weighted sum of various functions, many of which are non-Gaussian. In the same vein as the EK and UK filters, we can approximate each of these non-Gaussian constituent functions by a Gaussian. Adopting the philosophy of the EK filter, an approximation of the posterior intensity at the next time step can then be obtained by applying the Gaussian mixture PHD recursions to a locally linearized target model. Alternatively, in a similar manner to the UK filter, the unscented transform can be used to compute the components of the Gaussian mixture approximation of the posterior intensity at the next time step. In both cases, the weights of these components are also approximations.

Based on the above observations, we propose two nonlinear Gaussian mixture PHD filter implementations, namely, the *extended Kalman PHD* (EK-PHD) *filter* and the *unscented Kalman PHD* (UK-PHD) *filter*. Given that details for the EK and UK filters have been well-documented in the literature (see e.g. [44], [46]–[49]), the developments for the EK-PHD and UK-PHD filters are conceptually straightforward, though notationally cumbersome, and will be omitted. However, for completeness, the key steps in these two filters are summarized as pseudo codes in Tables IV and V, respectively.

Remark 5. Similar to its single-target counterpart, the EK-PHD filter is only applicable to differentiable nonlinear models. Moreover, calculating the Jacobian matrices may be tedious and error-prone. The UK-PHD filter, on the other hand, does not suffer from these restrictions and can even be applied to models with discontinuities.

Remark 6. Unlike the particle-PHD filter, where the particle approximation converges (in a certain sense) to the posterior intensity as the number of particle tends to infinity [17], [24], this type of guarantee has not been established for the EK-PHD or UK-PHD filter. Nonetheless, for mildly nonlinear problems, the EK-PHD and UK-PHD filters provide good approximations and are computationally cheaper than the particle-PHD filter, which requires a large number of particles and the additional cost of clustering to extract multi-target state estimates.

A. Simulation Results for a Nonlinear Gaussian Example

In this example, each target has a survival probability $p_{S,k} = 0.99$ and follows a nonlinear nearly-constant turn model [50] in which the target state takes the form $x_k = [\ y_k^T, \omega_k\ ]^T$, where $y_k = [\ p_{x,k}, p_{y,k}, \dot{p}_{x,k}, \dot{p}_{y,k}\ ]^T$, and ω_k is the turn rate. The state dynamics are given by

$$y_k = F(\omega_{k-1}) y_{k-1} + G w_{k-1},$$
$$\omega_k = \omega_{k-1} + \Delta u_{k-1},$$

where

$$F(\omega) = \begin{bmatrix} 1 & 0 & \frac{\sin\omega\Delta}{\omega} & -\frac{1-\cos\omega\Delta}{\omega} \\ 0 & 1 & \frac{1-\cos\omega\Delta}{\omega} & \frac{\sin\omega\Delta}{\omega} \\ 0 & 0 & \cos\omega\Delta & -\sin\omega\Delta \\ 0 & 0 & \sin\omega\Delta & \cos\omega\Delta \end{bmatrix}, G = \begin{bmatrix} \frac{\Delta^2}{2} & 0 \\ 0 & \frac{\Delta^2}{2} \\ \Delta & 0 \\ 0 & \Delta \end{bmatrix},$$

$\Delta = 1\text{s}$, $w_k \sim \mathcal{N}(\cdot; 0, \sigma_w^2 I_2)$, $\sigma_w = 15\text{m/s}^2$, and $u_k \sim \mathcal{N}(\cdot; 0, \sigma_u^2)$, $\sigma_u = (\pi/180)\text{rad/s}$. We assume no spawning, and that the spontaneous birth RFS is Poisson with intensity

$$\gamma_k(x) = 0.1\mathcal{N}(x; m_\gamma^{(1)}, P_\gamma) + 0.1\mathcal{N}(x; m_\gamma^{(2)}, P_\gamma),$$

where

$$m_\gamma^{(1)} = [\ -1000, 500, 0, 0, 0\ ]^T,$$
$$m_\gamma^{(2)} = [\ 1050, 1070, 0, 0, 0\ ]^T,$$
$$P_\gamma = \text{diag}([\ 2500, 2500, 2500, 2500, (6 \times \tfrac{\pi}{180})^2\ ]^T).$$

Each target has a probability of detection $p_{D,k} = 0.98$. An observation consists of bearing and range measurements

$$z_k = \begin{bmatrix} \arctan(p_{x,k}/p_{y,k}) \\ \sqrt{p_{x,k}^2 + p_{y,k}^2} \end{bmatrix} + \varepsilon_k,$$

where $\varepsilon_k \sim \mathcal{N}(\cdot; 0, R_k)$ with $R_k = \text{diag}([\sigma_\theta^2, \sigma_r^2]^T)$, $\sigma_\theta = 2 \times (\pi/180)\text{rad/s}$ and $\sigma_r = 20\text{m}$. The clutter RFS follows the uniform Poisson model in (47) over the surveillance region $[-\pi/2, \pi/2]\text{rad} \times [0, 2000]\text{m}$, with $\lambda_c = 3.2 \times 10^{-3}(\text{radm})^{-1}$ (i.e. an average of 20 clutter returns on the surveillance region).

The true target trajectories are plotted in Figure 6. Targets 1 and 2 appear from 2 different locations, 5s apart. They both travel in straight lines before making turns at $k = 16$s. The tracks almost cross at $k = 25$s, and the targets resume their straight trajectories after $k = 34$s. The pruning parameters for the UK-PHD and EK-PHD filters are $T = 1 \times 10^{-5}$, $U = 4$, and $J_{max} = 100$. The results, shown in Figures 7 and 8, indicate that both the UK-PHD and EK-PHD filters exhibit good tracking performance.

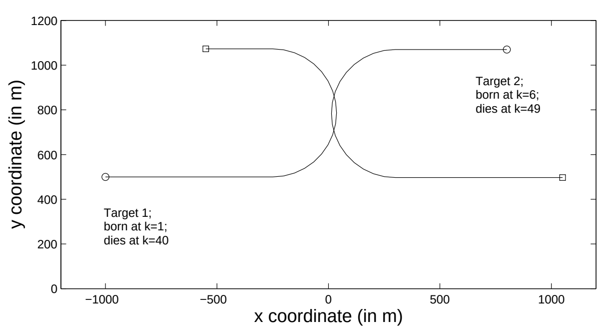

Fig. 6. Target trajectories. '○'– locations at which targets are born; '□'– locations at which targets die.

In many nonlinear Bayes filtering applications, the UK filter has shown better performance than the EK filter [49]. The same is expected in nonlinear PHD filtering. However, this example only has a mild nonlinearity and the performance gap between the EK-PHD and UK-PHD filters may not be noticeable.

V. Conclusions

Closed form solutions to the PHD recursion are important analytical and computational tools in multi-target filtering. Under linear Gaussian assumptions, we have shown that when the initial prior intensity of the random finite set of targets is a Gaussian mixture, the posterior intensity at any time step is

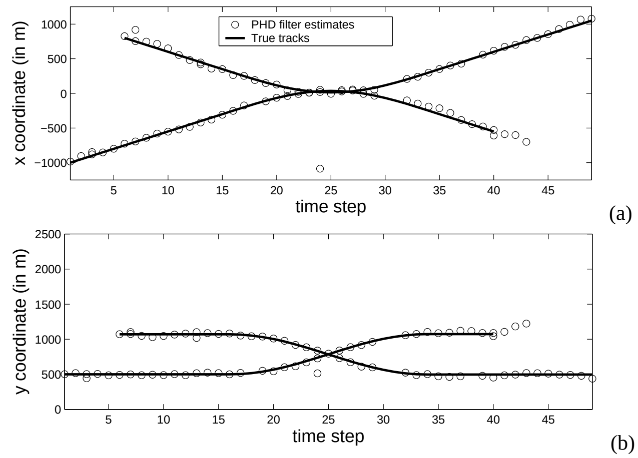

Fig. 7. Position estimates of the EK-PHD filter.

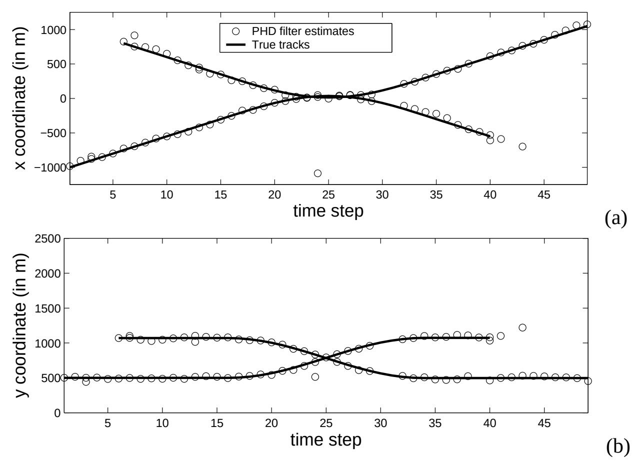

Fig. 8. Position estimates of the UK-PHD filter.

also a Gaussian mixture. More importantly, we have derived closed form recursions for the weights, means, and covariances of the constituent Gaussian components of the posterior intensity. An implementation of the PHD filter has been proposed by combining the closed form recursions with a simple pruning procedure to manage the growing number of components. Two extensions to nonlinear models using approximation strategies from the extended Kalman filter and the unscented Kalman filter have also been proposed. Simulations have demonstrated the capabilities of these filters to track an unknown and time-varying number of targets under detection uncertainty and false alarms.

There are a number of possible future research directions. Closed formed solutions to the PHD recursion for jump Markov linear models are being investigated. In highly non-linear, non-Gaussian models, where particle implementations are required, the EK-PHD and UK-PHD filters are obvious candidates for efficient proposal functions that can improve performance. This also opens up the question of optimal importance functions and their approximations. The efficiency and simplicity in implementation of the Gaussian mixture PHD recursion also suggest possible application to tracking in sensor networks.

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