Details behind theoretical formulation

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This document provide a detailed derivation on the dynamical equations and equilibrium condition. Attempt has not been made to make the material self contained. It is suggested reader refer to LRS4 and LRS1 for more information.

1 Largrangian and Dynamical equations

$$m \equiv \text{mass of DM core}$$
$$M \equiv \text{mass of normal matter}$$

In the following, primed variable denote corresponding quantities for the NS companion. Note

1.1 Modified Lagrangian

$$L = T + T' + T_{orb} - U - U' - W - W' - W_i \tag{1}$$

The presence of DM core modify the self-gravitational potential W by

$$W = W_{NS} + W_{NS-DM}$$

Also the gravitational interaction between the NS, W_i has the following expression:

$$\begin{aligned} W_i = &-\frac{(M+m)(M'+m')}{r} \\ &- \frac{M(m'+M')}{20r^3}\kappa_n \left[a_1^2(3\cos 2\alpha + 1) - a_2^2(3\cos 2\alpha - 1) - 2a_3^2\right] \\ &- \frac{M'(m+M)}{20r^3}\kappa_n' \left[a_1'^2(3\cos 2\alpha' + 1) - a_2'^2(3\cos 2\alpha' - 1) - 2a_3'^2\right] \end{aligned} \tag{2}$$

1.2 Evaluation of W_{NS-DM}

It is purposed that

$$W_{NS-DM} = W_s f,$$

where s stand for spherical and f is given by

$$f = \frac{\mathcal{I}}{2R^2}$$

where

$$\mathcal{I} = A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2 \tag{3}$$

Note $R^3 = a_1 a_2 a_3$

1.2.1 General formula of W_s

Let $\mathbf{R}$ be the position vector of the DM and $\mathbf{r}$ be the position vector of the NS fluid element. Then W_s is given by the following integral:

$$W_s = -m \int \frac{\rho(\mathbf{r})}{\mid \mathbf{R} - \mathbf{r} \mid} d^3x \tag{4}$$

Or using spherical coordinate:

$$W_s = -4\pi m \int r\rho(r)dr \tag{5}$$

1.2.2 Polytropic Variable

To proceed, note the NS is a polytrope and the radius, density and mass can be written as dimensionless Lane-Emden varaible:

$$\xi = \frac{r}{r_0} \tag{6}$$

$$\mu = \frac{m}{m_0} \tag{7}$$

$$\rho = \rho_c \theta^n \tag{8}$$

Here the m refer to the mass profile of the NS, instead of the DM. Also Note that

$$m_0 \equiv 4\pi\rho_c r_0^3 \tag{9}$$

The variable are related in the Lane-Emden equation (first order ODE form) :

$$\frac{d\theta}{d\xi} = -\frac{\mu}{\xi^2} \tag{10}$$

$$\frac{d\mu}{d\xi} = \xi^2\theta^n \tag{11}$$

At the NS surface, we have

$$R = r_0\xi_1 \tag{12}$$

$$M = m_0\mu_1, \tag{13}$$

where $\mu_1 = \mu(\xi_1)$. Make use of the Lane-Emden equation, we have

$$\mu_1 = \xi^2 \mid \theta_1' \mid . \tag{14}$$

The prime here represent the first order derivative with respect to ξ.

1.2.3 Integrating W_s

In term of Lane-Emden variable,

$$\begin{aligned} W_s &= -4\pi m r_0^2 \rho_c \int_0^{\xi_1} \xi\theta^n d\xi \\ &= -m\frac{m_0}{r_0}\int_0^{\xi_1} \xi\theta^n d\xi \\ &= -m\frac{M}{\mu_1}\frac{\xi_1}{R}\int_0^{\xi_1} \xi\theta^n d\xi \\ &= -\frac{mM}{R}\left(\frac{1}{\xi_1 \mid \theta_1' \mid}\int_0^{\xi_1} \xi\theta^n d\xi\right) \end{aligned} \tag{15}$$

Define $b_n \equiv \frac{1}{\xi_1|\theta_1'|}\int_0^{\xi_1} \xi\theta^n d\xi$, we arrive at

$$W_{NS-DM} = -b_n \frac{mM}{R} f \tag{16}$$

The b_n can be obtained numerically.

1.2.4 Alternate expression of W

It could be useful to express W in term of ρ_c instead of R using

$$\rho_c = \frac{\xi_1}{4\pi \mid \theta_1' \mid}\frac{M}{R^3}, \tag{17}$$

Then

$$\begin{aligned} W_{NS-DM} &= -b_n \frac{mM}{R} f \\ &= -b_n m M f \left(\frac{4\pi \mid \theta_1' \mid}{\xi_1}\right)^{1/3} \frac{\rho_c^{1/3}}{M^{1/3}} \\ &= -k_4 m M^{2/3} \rho_c^{1/3} f, \end{aligned} \tag{18}$$

where we have defined $k_4 \equiv b_n \left(\frac{4\pi|\theta_1'|}{\xi_1}\right)^{1/3}$. Given that

$$W_{NS} = -\frac{3}{5-n}\frac{M^2}{R} f = -k_2 M^{5/3} \rho_c^{1/3} f, \tag{19}$$

we arrive at

$$\boxed{W = W_{NS} + W_{NS-DM} = -M^{2/3}\rho_c^{1/3} f\,(k_2 M + k_4 m)} \tag{20}$$

1.3 Modification of the dynamical equations

The only real modification is a new term present in $\ddot{a}_i$ equation due to W_{NS-DM}. We shall obtain this term by Euler-Lagrange Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{21}$$

1.3.1 Differentiation w.r.t a_1

The new term in the Lagrangian is

$$\begin{aligned} W_{NS-DM} &= -b_n \frac{mM}{R} \frac{\mathcal{I}}{2R^2} \\ &= -\frac{b_n}{2} mM \frac{\mathcal{I}}{R^3}. \end{aligned} \tag{22}$$

Then we have to differentiate the $\mathcal{I}/a_1a_2a_3$ w.r.t a_1. We need the relation

$$\frac{\partial \mathcal{I}}{\partial a_i} = \frac{1}{a_i}\left(\mathcal{I} - a_i^2 A_i\right). \tag{23}$$

Then

$$\begin{aligned} \frac{\partial}{\partial a_1}\left(\frac{\mathcal{I}}{a_1a_2a_3}\right) &= \frac{\frac{\partial \mathcal{I}}{\partial a_1} a_1a_2a_3 - \mathcal{I}a_2a_3}{R^6} \\ &= \frac{1}{R^6}\left[\frac{1}{a_1}\left(a_2^2A_2 + a_3^2A_3\right)R^3 - a_2a_3\left(a_1^2A_1 + a_2^2A_2 + a_3^2A_3\right)\right] \\ &= \frac{1}{a_1R^6}\left[\left(a_2^2A_2 + a_3^2A_3\right)R^3 - R^3\left(a_1^2A_1 + a_2^2A_2 + a_3^2A_3\right)\right] \\ &= \frac{1}{a_1R^3}\left(-a_1^2A_1\right) \\ &= -\frac{a_1A_1}{R^3} \\ &= -\frac{A_1}{a_2a_3} \end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned} \frac{\partial W_{NS-DM}}{\partial a_1} &= -\frac{b_n mM}{2}\frac{\partial}{\partial a_1}\left(\frac{\mathcal{I}}{a_1a_2a_3}\right) \\ &= \frac{b_n mMA_1}{2a_2a_3} \end{aligned} \tag{25}$$

This term appear in the right side of Euler-Lagrange Equation as

$$\frac{\partial L}{\partial a_1} = [...] - \frac{b_n mMA_1}{2a_2a_3} \tag{26}$$

1.3.2 Differentiation w.r.t $\dot{a}_1$

The only term involve $\dot{a}_1$ is in the kinetic energy term T

$$T = [...] + \frac{1}{10}\kappa_n M(\dot{a}_1^2 + \dot{a}_2^2 + \dot{a}_3^2) \tag{27}$$

Then

$$\begin{aligned} \frac{\partial L}{\partial \dot{a}_1} &= \frac{\partial T}{\partial \dot{a}_1} \\ &= \frac{1}{5}\kappa_n M\dot{a}_1 \end{aligned} \tag{28}$$

1.3.3 Dynamical Equation of $\ddot{a}_i$

Combing the previous stuff, we have

$$\begin{aligned} \frac{1}{5}\kappa_n M\ddot{a}_1 &= [...] - \frac{b_n mMA_1}{2a_2a_3} \\ \ddot{a}_1 &= [...] - \frac{5m}{2}\frac{b_n}{\kappa_n}\frac{A_1}{a_2a_3} \end{aligned} \tag{29}$$

It is straight forward to obtain similar expression for other two axes. To sum up:

$$\boxed{\begin{aligned} \ddot{a}_1 &= [...] - \frac{5m}{2}\frac{b_n}{\kappa_n}\frac{A_1}{a_2a_3} && \text{(30a)} \\ \ddot{a}_2 &= [...] - \frac{5m}{2}\frac{b_n}{\kappa_n}\frac{A_2}{a_1a_3} && \text{(30b)} \\ \ddot{a}_3 &= [...] - \frac{5m}{2}\frac{b_n}{\kappa_n}\frac{A_3}{a_1a_2} && \text{(30c)} \end{aligned}}$$

2 Evaluation of equilibrium conditions (I)

2.1 Equilibrium conditions

For a NS with DM core, the equilibrium configuration need to be work out. This require quite extensive algebra. Such configuration can be obtained by minimizing the energy function E of the BNS system (Assume circular orbit):

$$E(r, \rho_c, \lambda_1, \lambda_2, \rho_c', \lambda_1, \lambda_2') = U + U' + W + W' + T + T' + T_{orb} + W_i. \tag{31}$$

The equilibrium condition is:

$$\frac{\partial E}{\partial r} = \frac{\partial E}{\partial \rho_c} = \frac{\partial E}{\partial \lambda_1} = \frac{\partial E}{\partial \lambda_2} = 0 \tag{32}$$

2.2 a_1, a_2, a_3 in term of $\rho_c, \lambda_1, \lambda_2$

Following the practice in LRS4, we shall express a_1, a_2, a_3 in term of $\rho_c, \lambda_1, \lambda_2$. The new variable are :

$$\lambda_1 \equiv \left(\frac{a_3}{a_1}\right)^{2/3}, \qquad \lambda_2 \equiv \left(\frac{a_3}{a_2}\right)^{2/3}, \qquad \rho_c = \frac{\xi_1}{4\pi \mid \theta_1' \mid}\frac{M}{a_1a_2a_3}. \tag{33}$$

For convenience, define $k_5 = \frac{\xi_1}{4\pi|\theta_1'|}$, then

$$\rho_c = k_5\frac{M}{a_1a_2a_3}. \tag{34}$$

Eliminate a_3 using λ_1, λ_2:

$$\begin{aligned} \lambda_1^{3/2}a_1 &= \lambda_2^{3/2}a_2 \\ a_2 &= \left(\frac{\lambda_1}{\lambda_2}\right)^{3/2} a_1 \end{aligned} \tag{35}$$

By noting $a_3 = \lambda_1^{3/2} a_1$,

$$\begin{aligned}
\rho_c a_1 a_2 a_3 &= k_5 M \\
\rho_c a_1 \left(\frac{\lambda_1}{\lambda_2}\right)^{3/2} a_1 \lambda_1^{3/2} a_1 &= k_5 M \\
\rho_c \left(\frac{\lambda_1^2}{\lambda_2}\right)^{3/2} a_1^3 &= k_5 M \\
a_1^3 &= \frac{k_5 M}{\rho_c} \left(\frac{\lambda_2}{\lambda_1^2}\right)^{3/2} \\
a_1 &= \left(\frac{k_5 M}{\rho_c}\right)^{1/3} \frac{\lambda_2^{1/2}}{\lambda_1}
\end{aligned} \tag{36}$$

Using Eq. (35),

$$\begin{aligned}
a_2 &= \left(\frac{\lambda_1}{\lambda_2}\right)^{3/2} \left(\frac{k_5 M}{\rho_c}\right)^{1/3} \frac{\lambda_2^{1/2}}{\lambda_1} \\
&= \left(\frac{k_5 M}{\rho_c}\right)^{1/3} \frac{\lambda_1^{1/2}}{\lambda_2}
\end{aligned} \tag{37}$$

Finally for $a_3 = \lambda_1^{3/2} a_1$,

$$\begin{aligned}
a_3 &= \lambda_1^{3/2} \left(\frac{k_5 M}{\rho_c}\right)^{1/3} \frac{\lambda_2^{1/2}}{\lambda_1} \\
&= \left(\frac{k_5 M}{\rho_c}\right)^{1/3} (\lambda_1 \lambda_2)^{1/2}
\end{aligned} \tag{38}$$

To sum up,

$$a_1 = \left(\frac{k_5 M}{\rho_c}\right)^{1/3} \frac{\lambda_2^{1/2}}{\lambda_1} \tag{39a}$$

$$a_2 = \left(\frac{k_5 M}{\rho_c}\right)^{1/3} \frac{\lambda_1^{1/2}}{\lambda_2} \tag{39b}$$

$$a_3 = \left(\frac{k_5 M}{\rho_c}\right)^{1/3} (\lambda_1 \lambda_2)^{1/2} \tag{39c}$$

2.3 Energy Functions in term of $\rho_c, \lambda_1, \lambda_2$

Internal energy U:

$$U = k_1 K \rho_c^{1/n} M. \tag{40}$$

Self-gravitational potential W:

$$W = -M^{2/3} \rho_c^{1/3} f \left(k_2 M + k_4 m\right) \tag{41}$$

Gravitational Interaction Potential W_i:

$$\begin{aligned}
W_i &= [...] - \frac{M(m' + M')}{20 r^3} \kappa_n \left(4a_1^2 - 2a_2^2 - 2a_3^2\right) \\
&= [...] - \frac{M(m' + M')}{10 r^3} \kappa_n \left(\frac{k_5 M}{\rho_c}\right)^{2/3} \left(2\frac{\lambda_2}{\lambda_1^2} - \frac{\lambda_1}{\lambda_2^2} - \lambda_1 \lambda_2\right)
\end{aligned} \tag{42}$$

2.4 The Kinetic Energy Term

The KE term is:

$$T = \frac{\kappa_n M}{10}(a_1^2 + a_2^2)(\Lambda^2 + \Omega^2) - \frac{2\kappa_n M}{5} a_1 a_2 \Lambda\Omega \tag{43}$$

The dependence of Λ, Ω on $\rho_c, \lambda_1, \lambda_2$ is quite complicated and hard to do differentiation. We shall copy the result obtained in LRS4.

2.4.1 The Conserved Quantities

In minimizing the energy function E, we have to hold total angular momentum J_{tot} and fluid circulation $\mathcal{C}$ constant:

$$\begin{aligned} J_{tot} &= J + J' + J_{orb} \\ &= \frac{1}{5}\kappa_n M(a_1^2 + a_2^2)\Omega - \frac{2}{5}\kappa_n M a_1 a_2 \Lambda + [...] + \frac{(M+m)(M'+m')}{(M+m+M'+m')} r^2 \Omega \\ \mathcal{C} &= \frac{1}{5}\kappa_n M(a_1^2 + a_2^2)\Lambda - \frac{2}{5}\kappa_n M a_1 a_2 \Omega \end{aligned} \tag{44}$$

In circular orbit, r is constant and therefore J_{orb} is constant. Thus by requiring $J_{tot} =$ constant, J is automatically conserved, since J and J' are unrelated.

2.4.2 Expression without Λ, Ω given in LRS4

To differentiate T, we use the following expression:

$$T = \frac{5(J+\mathcal{C})^2}{4\kappa_n M(a_1 - a_2)^2} + \frac{5(J-\mathcal{C})^2}{4\kappa_n M(a_1 + a_2)^2} \tag{45}$$

To express $(a_1 - a_2)^2$ in term of $\rho_c, \lambda_1, \lambda_2$

$$\begin{aligned} (a_1 - a_2)^2 &= \left(\frac{k_5 M}{\rho_c}\right)^{2/3} \left(\frac{\lambda_2^{1/2}}{\lambda_1} - \frac{\lambda_1^{1/2}}{\lambda_2}\right)^2 \\ &= \left(\frac{k_5 M}{\rho_c}\right)^{2/3} \left(\frac{\lambda_2^{3/2} - \lambda_1^{3/2}}{\lambda_1 \lambda_2}\right)^2 \end{aligned} \tag{46}$$

Then

$$T = \frac{5}{4\kappa_n M}\left(\frac{\rho_c}{k_5 M}\right)^{2/3}\left[\left(\frac{(J+\mathcal{C})\lambda_1\lambda_2}{\lambda_2^{3/2} - \lambda_1^{3/2}}\right)^2 + \left(\frac{(J-\mathcal{C})\lambda_1\lambda_2}{\lambda_2^{3/2} + \lambda_1^{3/2}}\right)^2\right] \tag{47}$$

Recall J and $\mathcal{C}$ are constant.

2.5 Evaluation of equilibrium condition $\partial E/\partial \rho_c = 0$

Need to differentiate E w.r.t ρ_c:

$$\frac{\partial E}{\partial \rho_c} = \frac{\partial U}{\partial \rho_c} + \frac{\partial W}{\partial \rho_c} + \frac{\partial T}{\partial \rho_c} + \frac{\partial W_i}{\partial \rho_c} \tag{48}$$

The first term is straight forward:

$$\frac{\partial U}{\partial \rho_c} = \frac{U}{n\rho_c} \tag{49}$$

For the second term, we need to ask whether f depend on ρ_c:

$$\begin{aligned} f &= \frac{A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2}{2a_1 a_2 a_3} \\ &= \frac{A_1 (a_1/a_3)^2 + A_2 (a_2/a_3)^2 + A_3}{2(a_1 a_2 a_3/a_3^3)^{2/3}} \\ &= \frac{A_1 \lambda_1^{-3} + A_2 \lambda_2^{-3} + A_3}{2\lambda_1^{-1}\lambda_2^{-1}} \end{aligned} \tag{50}$$

f does not depend on ρ_c. Then

$$\frac{\partial W}{\partial \rho_c} = \frac{W}{3\rho_c} \tag{51}$$

The Third term:

$$\frac{\partial T}{\partial \rho_c} = \frac{2T}{3\rho_c} \tag{52}$$

The Fourth term:

$$\frac{\partial W_i}{\partial \rho_c} = -\frac{2W_i^{M-M'_{tot}}}{3\rho_c} \tag{53}$$

Therefore,

$$\begin{aligned} \partial E/\partial \rho_c &= 0 \\ 0 &= \frac{U}{n\rho_c} + \frac{W}{3\rho_c} + \frac{2T}{3\rho_c} - \frac{2W_i^{M-M'_{tot}}}{3\rho_c} \\ \frac{3}{n}U + W + 2T &= 2W_i^{M-M'_{tot}}. \end{aligned} \tag{54}$$

Which is the viral relation for the NS.

3 Evaluation of equilibrium conditions (II)

The following equilibrium condition:

$$\frac{\partial E}{\partial \lambda_1} = 0 \tag{55}$$

will be evaluated.

3.1 Evaluation of $\partial W/\partial \lambda_1$

From $W = -M^{2/3}\rho_c^{1/3}(k_2 M + k_4 m)f$, need to evaluate $\partial f/\partial \lambda_1$, recall f is given by:

$$f = \frac{\mathcal{I}}{2R^2}, \qquad \mathcal{I} = A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2 \tag{56}$$

Trying to evaluate $\partial f/\partial\lambda_1$:

$$\frac{\partial f}{\partial\lambda_1} = \frac{1}{2}\left(R^{-2}\frac{\partial\mathcal{I}}{\partial\lambda_1} - 2\mathcal{I}R^{-3}\frac{\partial R}{\partial\lambda_1}\right) \tag{57}$$

Since $R^3 = a_1a_2a_3 = k_5M/\rho_c$,

$$\frac{\partial R}{\partial\lambda_1} = 0 \tag{58}$$

Then,

$$\frac{\partial f}{\partial\lambda_1} = \frac{1}{2R^2}\frac{\partial\mathcal{I}}{\partial\lambda_1} \tag{59}$$

Using following relation:

$$\frac{\partial\mathcal{I}}{\partial a_i} = \frac{1}{a_i}\left(\mathcal{I} - A_ia_i^2\right) \tag{60}$$

Then $\partial\mathcal{I}/\partial\lambda_1$ can be obtained as

$$\frac{\partial\mathcal{I}}{\partial\lambda_1} = \frac{\partial\mathcal{I}}{\partial a_1}\frac{\partial a_1}{\partial\lambda_1} + \frac{\partial\mathcal{I}}{\partial a_2}\frac{\partial a_2}{\partial\lambda_1} + \frac{\partial\mathcal{I}}{\partial a_3}\frac{\partial a_3}{\partial\lambda_1} \tag{61}$$

where $\partial a_i/\partial\lambda_1$ is given by

$$\frac{\partial a_1}{\partial\lambda_1} = -R\lambda_1^{-2}\lambda_2^{1/2}, \qquad \frac{\partial a_1}{\partial\lambda_1} = \frac{R}{2}\lambda_1^{-1/2}\lambda_2^{-1}, \qquad \frac{\partial a_3}{\partial\lambda_1} = \frac{R}{2}\lambda_1^{-1/2}\lambda_2^{1/2} \tag{62}$$

Then

$$\begin{aligned}
\frac{\partial\mathcal{I}}{\partial\lambda_1} &= -\frac{R}{a_1}\left(A_2a_2^2 + A_3a_3^2\right)\lambda_1^{-2}\lambda_2^{1/2} + \frac{R}{2a_2}\left(A_1a_1^2 + A_3a_3^2\right)\lambda_1^{-1/2}\lambda_2^{-1} \\
&\quad + \frac{R}{2a_3}\left(A_1a_1^2 + A_2a_2^2\right)\lambda_1^{-1/2}\lambda_2^{1/2} \\
&= -\frac{R}{a_1}\left(A_2a_2^2 + A_3a_3^2\right)a_3^{-4/3}a_1^{4/3}a_3^{1/3}a_2^{-1/3} \\
&\quad + \frac{R}{2a_2}\left(A_1a_1^2 + A_3a_3^2\right)a_3^{-1/3}a_1^{1/3}a_3^{-2/3}a_2^{2/3} \\
&\quad + \frac{R}{2a_3}\left(A_1a_1^2 + A_2a_2^2\right)a_3^{-1/3}a_1^{1/3}a_3^{1/3}a_2^{-1/3} \\
&= -R\left(A_2a_2^2 + A_3a_3^2\right)a_1^{1/3}a_2^{-1/3}a_3^{-1} + \frac{R}{2}\left(A_1a_1^2 + A_3a_3^2\right)a_1^{1/3}a_2^{-1/3}a_3^{-1} \\
&\quad + \frac{R}{2}\left(A_1a_1^2 + A_2a_2^2\right)a_1^{1/3}a_2^{-1/3}a_3^{-1} \\
a_2^{2/3}a_3^{4/3}\frac{\partial\mathcal{I}}{\partial\lambda_1} &= -R\left(A_2a_2^2 + A_3a_3^2\right)a_1^{1/3}a_2^{1/3}a_3^{1/3} + \frac{R}{2}\left(A_1a_1^2 + A_3a_3^2\right)a_1^{1/3}a_2^{1/3}a_3^{1/3} \\
&\quad + \frac{R}{2}\left(A_1a_1^2 + A_2a_2^2\right)a_1^{1/3}a_2^{1/3}a_3^{1/3} \\
&= R^2\left(-A_2a_2^2 - A_3a_3^2 + \frac{1}{2}A_1a_1^2 + \frac{1}{2}A_3a_3^2 + \frac{1}{2}A_1a_1^2 + \frac{1}{2}A_2a_2^2\right) \\
&= R^2\left(A_1a_1^2 - \frac{1}{2}A_2a_2^2 - \frac{1}{2}A_3a_3^2\right)
\end{aligned} \tag{63}$$

It is now more convenience to use the following expression for W,

$$W = -\frac{M}{R}(c_n M + b_n m) f, \qquad\qquad c_n = \frac{3}{5-n} \tag{64}$$

Put all stuff together,

$$\begin{aligned} \frac{\partial W}{\partial \lambda_1} &= -\frac{M}{R}(c_n M + b_n m) \frac{\partial f}{\partial \lambda_1} \\ &= -\frac{M}{R}(c_n M + b_n m) \frac{1}{2R^2} \frac{\partial \mathcal{I}}{\partial \lambda_1} \\ a_2^{2/3} a_3^{4/3} \frac{\partial W}{\partial \lambda_1} &= -\frac{M}{2R^3}(c_n M + b_n m) R^2 \left(A_1 a_1^2 - \frac{1}{2} A_2 a_2^2 - \frac{1}{2} A_3 a_3^2 \right) \end{aligned} \tag{65}$$

$$\boxed{a_2^{2/3} a_3^{4/3} \frac{\partial W}{\partial \lambda_1} = -\frac{M}{2R}(c_n M + b_n m) \left(A_1 a_1^2 - \frac{1}{2} A_2 a_2^2 - \frac{1}{2} A_3 a_3^2 \right)} \tag{66}$$

3.2 Evaluation of $\partial T/\partial \lambda_1$

Recall

$$T = \frac{B_n}{R^2} \left[(J + \mathcal{C})^2 f_-(\lambda)^2 + (J - \mathcal{C})^2 f_+(\lambda)^2 \right] \tag{67}$$

where

$$\begin{aligned} B_n &\equiv \frac{5}{4\kappa_n M} \\ f_\pm(\lambda) &\equiv \frac{\lambda_1 \lambda_2}{\lambda_2^{3/2} \pm \lambda_1^{3/2}} \end{aligned} \tag{68}$$

Trying to differentiate T,

$$\frac{\partial T}{\partial \lambda_1} = \frac{2B_n}{R^2} \left[(J + \mathcal{C})^2 f_-(\lambda) \frac{\partial f_-}{\partial \lambda_1} + (J - \mathcal{C})^2 f_+(\lambda) \frac{\partial f_+}{\partial \lambda_1} \right] \tag{69}$$

Now,

$$
\begin{aligned}
\frac{\partial f_-}{\partial \lambda_1} &= \frac{\lambda_2\left(\lambda_2^{3/2}-\lambda_1^{3/2}\right)-\lambda_1\lambda_2\left(-\frac{3}{2}\lambda_1^{1/2}\right)}{\left(\lambda_2^{3/2}-\lambda_1^{3/2}\right)^2} \\
&= \frac{\lambda_1\lambda_2}{\lambda_2^{3/2}-\lambda_1^{3/2}}\frac{\lambda_1^{-1}\lambda_2^{3/2}-\lambda_1^{1/2}+\frac{3}{2}\lambda_1^{1/2}}{\lambda_2^{3/2}-\lambda_1^{3/2}} \\
&= f_-\frac{\lambda_1^{-1}\lambda_2^{3/2}+\frac{1}{2}\lambda_1^{1/2}}{\lambda_2^{3/2}-\lambda_1^{3/2}} \\
&= f_-\frac{a_1^{2/3}a_3^{-2/3}a_2^{-1}a_3+\frac{1}{2}a_1^{-1/3}a_3^{1/3}}{a_3a_2^{-1}-a_3a_1^{-1}} \\
&= f_-\frac{a_1^{2/3}a_2^{-1}a_3^{-2/3}+\frac{1}{2}a_1^{-1/3}a_3^{-2/3}}{a_2^{-1}-a_1^{-1}} \\
&= f_-\frac{a_1^{5/3}a_3^{-2/3}+\frac{1}{2}a_1^{2/3}a_2a_3^{-2/3}}{a_1-a_2} \\
a_2^{2/3}a_3^{4/3}\frac{\partial f_-}{\partial \lambda_1} &= f_-\frac{a_1^{5/3}a_2^{2/3}a_3^{2/3}+\frac{1}{2}a_1^{2/3}a_2^{5/3}a_3^{2/3}}{a_1-a_2} \\
&= f_-R^2\frac{a_1+\frac{1}{2}a_2}{a_1-a_2}
\end{aligned}
\tag{70}
$$

Similarly,

$$
a_2^{2/3}a_3^{4/3}\frac{\partial f_+}{\partial \lambda_1} = f_+R^2\frac{a_1-\frac{1}{2}a_2}{a_1+a_2} \tag{71}
$$

Note that

$$
\left(\frac{f_\pm(\lambda)}{R}\right)^2 = \frac{1}{(a_1\pm a_2)^2} \tag{72}
$$

Then,

$$
\begin{aligned}
a_2^{2/3}a_3^{4/3}\frac{\partial T}{\partial \lambda_1} &= 2B_n\left[(J+\mathcal{C})^2 f_-^2\frac{a_1+\frac{1}{2}a_2}{a_1-a_2}+(J-\mathcal{C})^2 f_+^2\frac{a_1-\frac{1}{2}a_2}{a_1+a_2}\right] \\
&= 2B_nR^2\left[(J+\mathcal{C})^2\frac{a_1+\frac{1}{2}a_2}{(a_1-a_2)^3}+(J-\mathcal{C})^2\frac{a_1-\frac{1}{2}a_2}{(a_1+-a_2)^3}\right]
\end{aligned}
\tag{73}
$$

Recall J and $\mathcal{C}$ are,

$$
\begin{aligned}
J &= \frac{\kappa_n M}{5}\left(a_1^2+a_2^2\right)\Omega-\frac{2\kappa_n M}{5}a_1a_2\Lambda \\
\mathcal{C} &= \frac{\kappa_n M}{5}\left(a_1^2+a_2^2\right)\Lambda-\frac{2\kappa_n M}{5}a_1a_2\Omega
\end{aligned}
\tag{74}
$$

Then $(J\pm\mathcal{C})^2$ is

$$
\begin{aligned}
(J\pm\mathcal{C})^2 &= \left[\frac{\kappa_n M}{5}(\Omega\pm\Lambda)\left(a_1^2+a_2^2\mp 2a_1a_2\right)\right]^2 \\
&= \left(\frac{\kappa_n M}{5}\right)^2(\Omega\pm\Lambda)^2(a_1\mp a_2)^4
\end{aligned}
\tag{75}
$$

Put all stuff together,

$$
\begin{aligned}
a_2^{2/3}a_3^{4/3}\frac{\partial T}{\partial \lambda_1} &= \frac{10R^2}{4\kappa_n M}\left(\frac{\kappa_n M}{5}\right)^2\left[(\Omega+\Lambda)^2(a_1-a_2)\left(a_1+\frac{1}{2}a_2\right)\right. \\
&\qquad\left.+(\Omega-\Lambda)^2(a_1+a_2)\left(a_1-\frac{1}{2}a_2\right)\right] \\
&= \frac{\kappa_n MR^2}{10}\left[\left(\Omega^2+2\Omega\Lambda+\Lambda^2\right)\left(a_1^2-\frac{1}{2}a_1a_2-\frac{1}{2}a_2^2\right)\right. \\
&\qquad\left.+\left(\Omega^2-2\Omega\Lambda+\Lambda^2\right)\left(a_1^2+\frac{1}{2}a_1a_2-\frac{1}{2}a_2^2\right)\right] \\
&= \frac{\kappa_n MR^2}{10}\left(2a_1^2\Omega^2-a_2^2\Omega^2-2a_1a_2\Omega\Lambda+2a_1^2\Lambda^2-a_2^2\Lambda^2\right)
\end{aligned}
\tag{76}
$$

Define new variable Q_i to replace Λ,

$$
Q_1 \equiv \frac{a_1}{a_2}\Lambda, \qquad\qquad Q_2 \equiv -\frac{a_2}{a_1}\Lambda \tag{77}
$$

Finally,

$$
\boxed{a_2^{2/3}a_3^{4/3}\frac{\partial T}{\partial \lambda_1} = \frac{\kappa_n MR^2}{10}\left(2a_1^2\Omega^2-a_2^2\Omega^2+2a_1^2\Omega Q_2+2a_2^2Q_1^2-a_1^2Q_2^2\right)} \tag{78}
$$

3.3 Evaluation of $\partial W_i/\partial\lambda_1$

Recall

$$
W_i = [...] - \frac{M(m'+M')}{10r^3}\kappa_n R^2\left(2\frac{\lambda_2}{\lambda_1^2}-\frac{\lambda_1}{\lambda_2^2}-\lambda_1\lambda_2\right) \tag{79}
$$

The differentiation is straight forward:

$$
\begin{aligned}
\frac{\partial W_i}{\partial \lambda_1} &= -\frac{\kappa_n M(m'+M')R^2}{10r^3}\left(-4\lambda_1^{-3}\lambda_2-\lambda_2^{-2}-\lambda_2\right) \\
&= \frac{\kappa_n M(m'+M')R^2}{10r^3}\left(4a_1^2a_3^{-2}a_2^{-2/3}a_3^{2/3}+a_2^{4/3}a_3^{-4/3}+a_2^{-2/3}a_3^{2/3}\right)
\end{aligned}
\tag{80}
$$

We have,

$$
\boxed{a_2^{2/3}a_3^{4/3}\frac{\partial W_i}{\partial \lambda_1} = \frac{\kappa_n M(m'+M')R^2}{10r^3}\left(4a_1^2+a_2^2+a_3^2\right)} \tag{81}
$$

3.4 Expression of $\partial E/\partial\lambda_1$

$$
\begin{aligned}
\frac{\partial E}{\partial \lambda_1} &= \frac{\partial U}{\partial \lambda_1}+\frac{\partial W}{\partial \lambda_1}+\frac{\partial T}{\partial \lambda_1}+\frac{\partial W_i}{\partial \lambda_1} \\
0 &= a_2^{2/3}a_3^{4/3}\left(\frac{\partial W}{\partial \lambda_1}+\frac{\partial T}{\partial \lambda_1}+\frac{\partial W_i}{\partial \lambda_1}\right) \\
0 &= -\frac{M}{2R}(c_nM+b_nm)\left(A_1a_1^2-\frac{1}{2}A_2a_2^2-\frac{1}{2}A_3a_3^2\right) \\
&\quad+\frac{\kappa_n MR^2}{10}\left(2a_1^2\Omega^2-a_2^2\Omega^2+2a_1^2\Omega Q_2+2a_2^2Q_1^2-a_1^2Q_2^2\right) \\
&\quad+\frac{\kappa_n M(m'+M')R^2}{10r^3}\left(4a_1^2+a_2^2+a_3^2\right)
\end{aligned}
\tag{82}
$$

4 Evaluation of equilibrium conditions (III)

The following equilibrium condition:

$$\frac{\partial E}{\partial \lambda_2} = 0 \tag{83}$$

will be evaluated.

4.1 Evaluation of $\partial W/\partial \lambda_2$

From $W = -M^{2/3}\rho_c^{1/3}(k_2 M + k_4 m)f$, need to evaluate $\partial f/\partial \lambda_2$, recall f is given by:

$$f = \frac{\mathcal{I}}{2R^2}, \qquad \mathcal{I} = A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2 \tag{84}$$

Trying to evaluate $\partial f/\partial \lambda_1$:

$$\frac{\partial f}{\partial \lambda_2} = \frac{1}{2}\left(R^{-2}\frac{\partial \mathcal{I}}{\partial \lambda_2} - 2\mathcal{I}R^{-3}\frac{\partial R}{\partial \lambda_2}\right) \tag{85}$$

Since $R^3 = a_1 a_2 a_3 = k_5 M/\rho_c$,

$$\frac{\partial R}{\partial \lambda_2} = 0 \tag{86}$$

Then,

$$\frac{\partial f}{\partial \lambda_2} = \frac{1}{2R^2}\frac{\partial \mathcal{I}}{\partial \lambda_2} \tag{87}$$

Using following relation:

$$\frac{\partial \mathcal{I}}{\partial a_i} = \frac{1}{a_i}\left(\mathcal{I} - A_i a_i^2\right) \tag{88}$$

Then $\partial \mathcal{I}/\partial \lambda_2$ can be obtained as

$$\frac{\partial \mathcal{I}}{\partial \lambda_2} = \frac{\partial \mathcal{I}}{\partial a_1}\frac{\partial a_1}{\partial \lambda_2} + \frac{\partial \mathcal{I}}{\partial a_2}\frac{\partial a_2}{\partial \lambda_2} + \frac{\partial \mathcal{I}}{\partial a_3}\frac{\partial a_3}{\partial \lambda_2} \tag{89}$$

where $\partial a_i/\partial \lambda_2$ is given by

$$\frac{\partial a_1}{\partial \lambda_2} = \frac{R}{2}\lambda_1^{-1}\lambda_2^{-1/2}, \qquad \frac{\partial a_2}{\partial \lambda_2} = -R\lambda_1^{1/2}\lambda_2^{-2}, \qquad \frac{\partial a_3}{\partial \lambda_2} = \frac{R}{2}\lambda_1^{1/2}\lambda_2^{-1/2} \tag{90}$$

Then

$$
\begin{aligned}
\frac{\partial \mathcal{I}}{\partial \lambda_2} &= \frac{R}{2a_1}\left(A_2 a_2^2 + A_3 a_3^2\right)\lambda_1^{-1}\lambda_2^{-1/2} - \frac{R}{a_2}\left(A_1 a_1^2 + A_3 a_3^2\right)\lambda_1^{1/2}\lambda_2^{-2} \\
&\quad + \frac{R}{2a_3}\left(A_1 a_1^2 + A_2 a_2^2\right)\lambda_1^{1/2}\lambda_2^{-1/2} \\
&= \frac{R}{2a_1}\left(A_2 a_2^2 + A_3 a_3^2\right) a_3^{-2/3} a_1^{2/3} a_3^{-1/3} a_2^{1/3} \\
&\quad - \frac{R}{a_2}\left(A_1 a_1^2 + A_3 a_3^2\right) a_3^{1/3} a_1^{-1/3} a_3^{-4/3} a_2^{4/3} \\
&\quad + \frac{R}{2a_3}\left(A_1 a_1^2 + A_2 a_2^2\right) a_3^{1/3} a_1^{-1/3} a_3^{-1/3} a_2^{1/3} \\
&= \frac{R}{2}\left(A_2 a_2^2 + A_3 a_3^2\right) a_1^{-1/3} a_2^{1/3} a_3^{-1} - R\left(A_1 a_1^2 + A_3 a_3^2\right) a_1^{-1/3} a_2^{1/3} a_3^{-1} \\
&\quad + \frac{R}{2}\left(A_1 a_1^2 + A_2 a_2^2\right) a_1^{-1/3} a_2^{1/3} a_3^{-1} \\
a_1^{2/3} a_3^{4/3}\frac{\partial \mathcal{I}}{\partial \lambda_2} &= \frac{R}{2}\left(A_2 a_2^2 + A_3 a_3^2\right) a_1^{1/3} a_2^{1/3} a_3^{1/3} - R\left(A_1 a_1^2 + A_3 a_3^2\right) a_1^{1/3} a_2^{1/3} a_3^{1/3} \\
&\quad + \frac{R}{2}\left(A_1 a_1^2 + A_2 a_2^2\right) a_1^{1/3} a_2^{1/3} a_3^{1/3} \\
&= R^2\left(\frac{1}{2}A_2 a_2^2 + \frac{1}{2}A_3 a_3^2 - A_1 a_1^2 - A_3 a_3^2 + \frac{1}{2}A_1 a_1^2 + \frac{1}{2}A_2 a_2^2\right) \\
&= R^2\left(-\frac{1}{2}A_1 a_1^2 + A_2 a_2^2 - \frac{1}{2}A_3 a_3^2\right)
\end{aligned} \tag{91}
$$

It is now more convenience to use the following expression for W,

$$
W = -\frac{M}{R}(c_n M + b_n m) f, \qquad c_n = \frac{3}{5-n} \tag{92}
$$

Put all stuff together,

$$
\begin{aligned}
\frac{\partial W}{\partial \lambda_2} &= -\frac{M}{R}(c_n M + b_n m)\frac{\partial f}{\partial \lambda_2} \\
&= -\frac{M}{R}(c_n M + b_n m)\frac{1}{2R^2}\frac{\partial \mathcal{I}}{\partial \lambda_2} \\
a_1^{2/3} a_3^{4/3}\frac{\partial W}{\partial \lambda_2} &= -\frac{M}{2R^3}(c_n M + b_n m) R^2\left(-\frac{1}{2}A_1 a_1^2 + A_2 a_2^2 - \frac{1}{2}A_3 a_3^2\right)
\end{aligned} \tag{93}
$$

$$
\boxed{a_1^{2/3} a_3^{4/3}\frac{\partial W}{\partial \lambda_2} = -\frac{M}{2R}(c_n M + b_n m)\left(-\frac{1}{2}A_1 a_1^2 + A_2 a_2^2 - \frac{1}{2}A_3 a_3^2\right)} \tag{94}
$$

4.2 Evaluation of $\partial T/\partial \lambda_2$

Recall

$$
T = \frac{B_n}{R^2}\left[(J + \mathcal{C})^2 f_-(\lambda)^2 + (J - \mathcal{C})^2 f_+(\lambda)^2\right] \tag{95}
$$

where

$$
\begin{aligned}
B_n &\equiv \frac{5}{4\kappa_n M} \\
f_\pm(\lambda) &\equiv \frac{\lambda_1\lambda_2}{\lambda_2^{3/2} \pm \lambda_1^{3/2}}
\end{aligned}
\tag{96}
$$

Trying to differentiate T,

$$
\frac{\partial T}{\partial \lambda_2} = \frac{2B_n}{R^2}\left[(J+\mathcal{C})^2 f_-(\lambda)\frac{\partial f_-}{\partial \lambda_2} + (J-\mathcal{C})^2 f_+(\lambda)\frac{\partial f_+}{\partial \lambda_2}\right] \tag{97}
$$

Now,

$$
\begin{aligned}
\frac{\partial f_-}{\partial \lambda_2} &= \frac{\lambda_1\left(\lambda_2^{3/2} - \lambda_1^{3/2}\right) - \lambda_1\lambda_2\left(\frac{3}{2}\lambda_2^{1/2}\right)}{\left(\lambda_2^{3/2} - \lambda_1^{3/2}\right)^2} \\
&= \frac{\lambda_1\lambda_2}{\lambda_2^{3/2} - \lambda_1^{3/2}} \frac{\lambda_2^{1/2} - \lambda_1^{3/2}\lambda_2^{-1} - \frac{3}{2}\lambda_2^{1/2}}{\lambda_2^{3/2} - \lambda_1^{3/2}} \\
&= -f_- \frac{\lambda_1^{3/2}\lambda_2^{-1} + \frac{1}{2}\lambda_2^{1/2}}{\lambda_2^{3/2} - \lambda_1^{3/2}} \\
&= -f_- \frac{a_3 a_1^{-1} a_3^{-2/3} a_2^{2/3} + \frac{1}{2} a_3^{1/3} a_2^{-1/3}}{a_3 a_2^{-1} - a_3 a_1^{-1}} \\
&= -f_- \frac{a_1^{-1} a_2^{2/3} a_3^{-2/3} + \frac{1}{2} a_2^{-1/3} a_3^{-2/3}}{a_2^{-1} - a_1^{-1}} \\
&= -f_- \frac{a_2^{5/3} a_3^{-2/3} + \frac{1}{2} a_1 a_2^{2/3} a_3^{-2/3}}{a_1 - a_2} \\
a_1^{2/3} a_3^{4/3} \frac{\partial f_-}{\partial \lambda_1} &= -f_- \frac{a_1^{2/3} a_2^{5/3} a_3^{2/3} + \frac{1}{2} a_1^{5/3} a_2^{2/3} a_3^{2/3}}{a_1 - a_2} \\
&= -f_- R^2 \frac{a_2 + \frac{1}{2}a_1}{a_1 - a_2}
\end{aligned}
\tag{98}
$$

Similarly,

$$
a_1^{2/3} a_3^{4/3} \frac{\partial f_+}{\partial \lambda_2} = f_+ R^2 \frac{a_2 - \frac{1}{2}a_1}{a_1 + a_2} \tag{99}
$$

Note that

$$
\left(\frac{f_\pm(\lambda)}{R}\right)^2 = \frac{1}{(a_1 \pm a_2)^2} \tag{100}
$$

Then,

$$
\begin{aligned}
a_1^{2/3} a_3^{4/3} \frac{\partial T}{\partial \lambda_1} &= 2B_n\left[(J+\mathcal{C})^2 f_-^2 \frac{a_2 + \frac{1}{2}a_1}{a_2 - a_1} + (J-\mathcal{C})^2 f_+^2 \frac{a_2 - \frac{1}{2}a_1}{a_1 + a_2}\right] \\
&= 2B_n R^2\left[(J+\mathcal{C})^2 \frac{a_2 + \frac{1}{2}a_1}{(a_2 - a_1)^3} + (J-\mathcal{C})^2 \frac{a_2 - \frac{1}{2}a_1}{(a_1 + a_2)^3}\right]
\end{aligned}
\tag{101}
$$

Recall J and $\mathcal{C}$ are,

$$\begin{aligned} J &= \frac{\kappa_n M}{5}\left(a_1^2+a_2^2\right)\Omega - \frac{2\kappa_n M}{5}a_1 a_2 \Lambda \\ \mathcal{C} &= \frac{\kappa_n M}{5}\left(a_1^2+a_2^2\right)\Lambda - \frac{2\kappa_n M}{5}a_1 a_2 \Omega \end{aligned} \tag{102}$$

Then $(J \pm \mathcal{C})^2$ is

$$\begin{aligned} (J \pm \mathcal{C})^2 &= \left[\frac{\kappa_n M}{5}(\Omega \pm \Lambda)\left(a_1^2 + a_2^2 \mp 2a_1 a_2\right)\right]^2 \\ &= \left(\frac{\kappa_n M}{5}\right)^2 (\Omega \pm \Lambda)^2 (a_1 \mp a_2)^4 \end{aligned} \tag{103}$$

Put all stuff together,

$$\begin{aligned} a_1^{2/3} a_3^{4/3} \frac{\partial T}{\partial \lambda_2} &= \frac{10R^2}{4\kappa_n M}\left(\frac{\kappa_n M}{5}\right)^2 \left[(\Omega+\Lambda)^2 (a_2 - a_1)\left(a_2 + \frac{1}{2}a_1\right)\right. \\ &\qquad \left. + (\Omega-\Lambda)^2 (a_1 + a_2)\left(a_2 - \frac{1}{2}a_1\right)\right] \\ &= \frac{\kappa_n M R^2}{10}\left[\left(\Omega^2 + 2\Omega\Lambda + \Lambda^2\right)\left(a_2^2 - \frac{1}{2}a_1 a_2 - \frac{1}{2}a_1^2\right)\right. \\ &\qquad \left. + \left(\Omega^2 - 2\Omega\Lambda + \Lambda^2\right)\left(a_2^2 + \frac{1}{2}a_1 a_2 - \frac{1}{2}a_1^2\right)\right] \\ &= \frac{\kappa_n M R^2}{10}\left(2a_2^2\Omega^2 - a_1^2\Omega^2 - 2a_1 a_2 \Omega\Lambda + 2a_2^2\Lambda^2 - a_1^2\Lambda^2\right) \end{aligned} \tag{104}$$

Define new variable Q_i to replace Λ,

$$Q_1 \equiv \frac{a_1}{a_2}\Lambda, \qquad Q_2 \equiv -\frac{a_2}{a_1}\Lambda \tag{105}$$

Finally,

$$\boxed{a_1^{2/3} a_3^{4/3} \frac{\partial T}{\partial \lambda_2} = \frac{\kappa_n M R^2}{10}\left(2a_2^2\Omega^2 - a_1^2\Omega^2 + 2a_1^2\Omega Q_2 + 2a_1^2 Q_2^2 - a_2^2 Q_1^2\right)} \tag{106}$$

4.3 Evaluation of $\partial W_i/\partial\lambda_2$

Recall

$$W_i = [...] - \frac{M(m'+M')}{10r^3}\kappa_n R^2\left(2\frac{\lambda_2}{\lambda_1^2} - \frac{\lambda_1}{\lambda_2^2} - \lambda_1\lambda_2\right) \tag{107}$$

The differentiation is straight forward:

$$\begin{aligned} \frac{\partial W_i}{\partial \lambda_2} &= -\frac{\kappa_n M(m'+M')R^2}{10r^3}\left(2\lambda_1^{-2} + 2\lambda_1\lambda_2^{-3} - \lambda_1\right) \\ &= -\frac{\kappa_n M(m'+M')R^2}{10r^3}\left(2a_3^{-4/3}a_1^{4/3} + 2a_3^{2/3}a_1^{-2/3}a_3^{-2}a_2^2 - a_3^{2/3}a_1^{-2/3}\right) \end{aligned} \tag{108}$$

We have,

$$\boxed{a_1^{2/3} a_3^{4/3} \frac{\partial W_i}{\partial \lambda_1} = -\frac{\kappa_n M(m'+M')R^2}{10r^3}\left(2a_1^2 + 2a_2^2 - a_3^2\right)} \tag{109}$$

4.4 Expression of $\partial E/\partial \lambda_2$

$$\begin{aligned}
\frac{\partial E}{\partial \lambda_2} &= \frac{\partial U}{\partial \lambda_2} + \frac{\partial W}{\partial \lambda_2} + \frac{\partial T}{\partial \lambda_2} + \frac{\partial W_i}{\partial \lambda_2} \\
0 &= a_1^{2/3} a_3^{4/3} \left(\frac{\partial W}{\partial \lambda_2} + \frac{\partial T}{\partial \lambda_2} + \frac{\partial W_i}{\partial \lambda_2} \right) \\
0 &= -\frac{M}{2R}(c_n M + b_n m) \left(-\frac{1}{2} A_1 a_1^2 + A_2 a_2^2 - \frac{1}{2} A_3 a_3^2 \right) \\
&\quad + \frac{\kappa_n M R^2}{10} \left(2a_2^2 \Omega^2 - a_1^2 \Omega^2 + 2a_1^2 \Omega Q_2 + 2a_1^2 Q_2^2 - a_2^2 Q_1^2 \right) \\
&\quad - \frac{\kappa_n M (m' + M') R^2}{10 r^3} \left(2a_1^2 + 2a_2^2 - a_3^2 \right)
\end{aligned} \tag{110}$$

5 Rearranging the above 3 equilibrium equations

From the result of above equilibrium conditions (I), (II), (III) section, we have

$$-\frac{3}{n} U(\rho_c) = W(\rho_c) + 2T(\rho_c) - 2W_i^{M-M'_{tot}}(\rho_c) \tag{111}$$

$$0 = W(\lambda_1) + T(\lambda_1) + W_i(\lambda_1) \tag{112}$$

$$0 = W(\lambda_2) + T(\lambda_2) + W_i(\lambda_2) \tag{113}$$

where the notation $W, T, W_i(\rho_c, \lambda_1, \lambda_2)$ used here is used to indicate the terms in equilibrium condition come from differentiating the self-gravitational potential W, Kinetic Energy of NS T and gravitational interaction potential W_i with respect to $\rho_c, \lambda_1, \lambda_2$.

5.1 Elimination of A_1, A_2

The A_1, A_2 can be eliminated by (111) $- 2 \times$ (112) $- 2 \times$ (113)

5.1.1 The W term

The W's term are:

$$W(\rho_c) = -\frac{M d_n}{2R^3} \left(A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2 \right) \tag{114a}$$

$$W(\lambda_1) = -\frac{M d_n}{2R^3} \left(A_1 a_1^2 - \frac{1}{2} A_2 a_2^2 - \frac{1}{2} A_3 a_3^2 \right) \tag{114b}$$

$$W(\lambda_2) = -\frac{M d_n}{2R^3} \left(-\frac{1}{2} A_1 a_1^2 + A_2 a_2^2 - \frac{1}{2} A_3 a_3^2 \right) \tag{114c}$$

where $d_n = (c_n M + b_n m)$. Then

$$\begin{aligned}
W(\rho_c) - 2W(\lambda_1) - 2W(\lambda_2) &= -\frac{M d_n}{2R^3} \big(A_1 a_1^2 - 2A_1 a_1^2 + A_1 a_1^2 \\
&\qquad A_2 a_2^2 + A_2 a_2^2 - 2A_2 a_2^2 \\
&\qquad A_3 a_3^2 + A_3 a_3^2 + A_3 a_3^2 \big) \\
&= -\frac{3M d_n}{2R^3} A_3 a_3^2
\end{aligned} \tag{115}$$

5.1.2 The T term

The T's term are: (Note $a_1\Lambda = a_2 Q_1$ and $-a_2\Lambda = a_1 Q_2$)

$$T(\rho_c) = \frac{\kappa_n M}{10}\left(a_1^2\Omega^2 + a_2^2\Omega^2 + a_2^2 Q_1^2 + a_1^2 Q_2^2 + 4a_1^2\Omega Q_2\right) \tag{116a}$$

$$T(\lambda_1) = \frac{\kappa_n M}{10}\left(2a_1^2\Omega^2 - a_2^2\Omega^2 + 2a_1^2\Omega Q_2 + 2a_2^2 Q_1^2 - a_1^2 Q_2^2\right) \tag{116b}$$

$$T(\lambda_2) = \frac{\kappa_n M}{10}\left(2a_2^2\Omega^2 - a_1^2\Omega^2 + 2a_1^2\Omega Q_2 + 2a_1^2 Q_2^2 - a_2^2 Q_1^2\right) \tag{116c}$$

Obviously $2T(\rho_c) - 2T(\lambda_1) - 2T(\lambda_2) = 0$

5.1.3 The W_i term

The W_i's term are:

$$W_i(\rho_c) = -\frac{\kappa_n M(m' + M')}{10r^3}\left(2a_1^2 - a_2^2 - a_3^2\right) \tag{117a}$$

$$W_i(\lambda_1) = \frac{\kappa_n M(m' + M')}{10r^3}\left(4a_1^2 + a_2^2 + a_3^2\right) \tag{117b}$$

$$W_i(\lambda_2) = -\frac{\kappa_n M(m' + M')}{10r^3}\left(2a_1^2 + 2a_2^2 - a_3^2\right) \tag{117c}$$

Then,

$$\begin{aligned} -2\left(W_i(\rho_c) + W_i(\lambda_1) + W_i(\lambda_2)\right) &= -\frac{3\kappa_n M(m' + M')a_3^2}{5r^3} \\ &= -3\frac{(m' + M')}{r^3} I_{33} \\ &= -3\mu_R I_{33}, \end{aligned} \tag{118}$$

where $I_{ij} = (1/5)\kappa_n M a_i^2 \delta_{ij}$ and $\mu_R = (m' + M')/r^3$

5.1.4 The combined equation

Finally we have,

$$\begin{aligned} -\frac{3}{n}U(\rho_c) &= -\frac{3Md_n}{2R^3} A_3 a_3^2 - 3\mu_R I_{33} \\ -\frac{U}{n} &= -\frac{Md_n}{2R^3} A_3 a_3^2 - \mu_R I_{33} \end{aligned} \tag{119}$$

Next define:

$$\mathcal{M}_{ij} = -\frac{Md_n}{2R^3} A_i a_i^2 \delta_{ij} \tag{120}$$

Finally,

$$\boxed{-\frac{U}{n} = \mathcal{M}_{33} - \mu_R I_{33}} \tag{121}$$

5.2 Elimination of A_2, A_3

The A_2, A_3 can be eliminated by $(111) + 2 \times (112)$

5.2.1 The W term

$$\begin{aligned} W(\rho_c) + 2W(\lambda_1) &= -\frac{3Md_n}{2R^3} A_1 a_1^2 \\ &= 3\mathcal{M}_{11} \end{aligned} \tag{122}$$

5.2.2 The T term

$$\begin{aligned} 2T(\rho_c) + 2T(\lambda_1) &= \frac{\kappa_n M}{10}\left(6a_1^2\Omega^2 + 6a_2^2 Q_1^2 + 12a_1^2\Omega Q_2\right) \\ &= \frac{3\kappa_n M}{5}\left(a_1^2\Omega^2 + a_2^2 Q_1^2 + 2a_1^2\Omega Q_2\right) \end{aligned} \tag{123}$$

5.2.3 The W_i term

$$\begin{aligned} -2W_i(\rho_c) + 2W_i(\lambda_1) &= \frac{\kappa_n M(m' + M')}{10r^3} 12a_1^2 \\ &= \frac{6\kappa_n M(m' + M')a_1^2}{5r^3} \\ &= 6\mu_R I_{11} \end{aligned} \tag{124}$$

5.2.4 The combined equation

$$-\frac{3}{n}U = 3\mathcal{M}_{11} + 3\left(I_{11}\Omega^2 + I_{22}Q_1^2 + 2I_{11}\Omega Q_2\right) + 6\mu_R I_{11} \tag{125}$$

$$\boxed{-\frac{U}{n} = \mathcal{M}_{11} + I_{11}\left(\Omega^2 + 2\mu_R + 2\Omega Q_2\right) + I_{22}Q_1^2} \tag{126}$$

5.3 Elimination of A_1, A_3

The A_1, A_3 can be eliminated by $(111) + 2 \times (113)$

5.3.1 The W term

$$\begin{aligned} W(\rho_c) + 2W(\lambda_2) &= -\frac{3Md_n}{2R^3} A_2 a_2^2 \\ &= 3\mathcal{M}_{22} \end{aligned} \tag{127}$$

5.3.2 The T term

$$\begin{aligned} 2T(\rho_c) + 2T(\lambda_2) &= \frac{\kappa_n M}{10}\left(6a_2^2\Omega^2 + 6a_1^2 Q_2^2 - 12a_1 a_2\Omega\Lambda\right) \\ &= \frac{3\kappa_n M}{5}\left(a_2^2\Omega^2 + a_1^2 Q_2^2 - 2a_2^2\Omega Q_1\right) \end{aligned} \tag{128}$$

5.3.3 The W_i term

$$\begin{aligned}-2W_i(\rho_c)+2W_i(\lambda_2) &= -\frac{\kappa_n M(m'+M')}{10r^3}6a_2^2 \\ &= -\frac{3\kappa_n M(m'+M')a_2^2}{5r^3} \\ &= -3\mu_R I_{22} \end{aligned} \tag{129}$$

5.3.4 The combined equation

$$-\frac{3}{n}U = 3\mathcal{M}_{22} + 3\left(I_{22}\Omega^2 + I_{11}Q_2^2 - 2I_{22}\Omega Q_1\right) - 3\mu_R I_{22} \tag{130}$$

$$\boxed{-\frac{U}{n} = \mathcal{M}_{22} + I_{22}\left(\Omega^2 - \mu_R - 2\Omega Q_1\right) + I_{11}Q_2^2} \tag{131}$$

6 The modified Kepler's Law

The equilibrium condition:

$$\frac{\partial E}{\partial r} = 0 \tag{132}$$

yield the modified Kepler's Law.

6.1 Derivation of the modified Kepler's Law

The relevant energy terms are:

$$T = \frac{1}{2}\mu r^2\Omega^2 \tag{133}$$

$$W_i = \frac{M_{tot}M'_{tot}}{r} + \frac{\kappa_n M M'_{tot}}{10r^3}\left(2a_1^2 - a_2^2 - a_3^2\right) + \frac{\kappa'_n M_{tot} M'}{10r^3}\left(2a_1'^2 - a_2'^2 - a_3'^2\right) \tag{134}$$

$$= \frac{M_{tot}M'_{tot}}{r} + \frac{M'_{tot}}{2r^3}\left(2I_{11} - I_{22} - I_{33}\right) + \frac{M_{tot}}{2r^3}\left(2I'_{11} - I'_{22} - I'_{33}\right) \tag{135}$$

where $\mu = M_{tot}M'_{tot}/\left(M_{tot}+M'_{tot}\right)$. Now,

$$\begin{aligned} \frac{\partial T}{\partial r} + \frac{\partial W_i}{\partial r} &= 0 \\ \frac{M_{tot}M'_{tot}}{M_{tot}+M'_{tot}} r\Omega^2 &= \frac{M_{tot}M'_{tot}}{r^2} + \frac{3}{2}\frac{M'_{tot}}{r^4}\left(2I_{11} - I_{22} - I_{33}\right) + \frac{3}{2}\frac{M_{tot}}{r^4}\left(2I'_{11} - I'_{22} - I'_{33}\right) \\ \Omega^2 &= \frac{M_{tot}+M'_{tot}}{r^3} + \frac{3}{2}\frac{M_{tot}+M'_{tot}}{M_{tot}}\frac{\left(2I_{11} - I_{22} - I_{33}\right)}{r^5} \\ &\quad + \frac{3}{2}\frac{M_{tot}+M'_{tot}}{M'_{tot}}\frac{\left(2I'_{11} - I'_{22} - I'_{33}\right)}{r^5} \\ \Omega^2 &= \frac{M_{tot}+M'_{tot}}{r^3}\left(1+\delta+\delta'\right) \end{aligned} \tag{136}$$

where

$$\delta = \frac{3}{2}\frac{(2I_{11} - I_{22} - I_{33})}{M_{tot}r^2} \tag{137}$$

$$\delta' = \frac{3}{2}\frac{(2I'_{11} - I'_{22} - I'_{33})}{M'_{tot}r^2} \tag{138}$$

7 Combining 3 rearranged equilibrium equations into 2 axes equations

7.1 The rearranged 3 equilibrium equations

Recall we have derived:

$$-\frac{U}{n} = \mathcal{M}_{11} + I_{11}\left(\Omega^2 + 2\mu_R + 2\Omega Q_2\right) + I_{22}Q_1^2 \tag{139}$$

$$-\frac{U}{n} = \mathcal{M}_{22} + I_{22}\left(\Omega^2 - \mu_R - 2\Omega Q_1\right) + I_{11}Q_2^2 \tag{140}$$

$$-\frac{U}{n} = \mathcal{M}_{33} - \mu_R I_{33} \tag{141}$$

where $\mathcal{M}_{ij} = -\frac{Md_n}{2R^3}A_i a_i^2 \delta_{ij}$ and $\mu_R = (m' + M')/r^3$

7.2 Axes equations

The equations relating the ellipsoid axes are found by (141) – (139) and (141) – (140)

7.2.1 (141) - (139)

$$\begin{aligned}
&\mathcal{M}_{33} - \mathcal{M}_{11} - I_{11}\left(\Omega^2 + 2\mu_R + 2\Omega Q_2\right) - I_{22}Q_1^2 - \mu_R I_{33} = 0 \\
&\frac{1}{5}\kappa_n M\left[a_1^2\left(\Omega^2 + 2\mu_R + 2\Omega Q_2\right) + a_2^2 Q_1^2 + \mu_R a_3^2\right] = \frac{Md_n}{2R^3}\left(A_1 a_1^2 - A_3 a_3^2\right) \\
&\frac{4\kappa_n R^3}{5d_n}\left[a_1^2\left(\Omega^2 + 2\mu_R + 2\Omega Q_2\right) + a_2^2 Q_1^2 + \mu_R a_3^2\right] = 2\left(A_1 a_1^2 - A_3 a_3^2\right) \\
&\frac{4\kappa_n}{5d_n}\mu_R R^3\left[\frac{Q_1^2}{\mu_R}a_2^2 + \left(\frac{\Omega^2}{\mu_R} + 2 + 2\frac{Q_2\Omega}{\mu_R}\right)a_1^2 + a_3^2\right] = 2\left(A_1 a_1^2 - A_3 a_3^2\right)
\end{aligned} \tag{142}$$

Using modified Kepler's law:

$$\begin{aligned}
\frac{\Omega^2}{\mu_R} &= \frac{M_{tot} + M'_{tot}}{r^3}\left(1 + \delta + \delta'\right)\frac{r^3}{M'_{tot}} \\
&= \left(1 + \frac{M_{tot}}{M'_{tot}}\right)\left(1 + \delta + \delta'\right)
\end{aligned}$$

By defining $h_n \equiv 4\kappa_n/5d_n$,

$$\boxed{h_n \mu_R R^3\left\{\frac{Q_1^2}{\mu_R}a_2^2 + \left[2 + \left(1 + \frac{M_{tot}}{M'_{tot}}\right)\left(1 + \delta + \delta'\right) + 2\frac{Q_2\Omega}{\mu_R}\right]a_1^2 + a_3^2\right\} = 2\left(A_1 a_1^2 - A_3 a_3^2\right)} \tag{143}$$

7.2.2 (141) - (140)

$$\begin{aligned}
\mathcal{M}_{33} - \mathcal{M}_{22} - I_{22}\left(\Omega^2 - \mu_R - 2\Omega Q_1\right) - I_{11}Q_2^2 - \mu_R I_{33} = 0 \\
\frac{1}{5}\kappa_n M\left[a_2^2\left(\Omega^2 - \mu_R - 2\Omega Q_1\right) + a_1^2 Q_2^2 + \mu_R a_3^2\right] = \frac{M d_n}{2R^3}\left(A_2 a_2^2 - A_3 a_3^2\right) \\
\frac{4\kappa_n R^3}{5 d_n}\left[a_2^2\left(\Omega^2 - \mu_R - 2\Omega Q_1\right) + a_1^2 Q_2^2 + \mu_R a_3^2\right] = 2\left(A_2 a_2^2 - A_3 a_3^2\right) \\
h_n \mu_R R^3\left[\frac{Q_2^2}{\mu_R}a_1^2 + \left(\frac{\Omega^2}{\mu_R} - 1 - 2\frac{Q_1\Omega}{\mu_R}\right)a_2^2 + a_3^2\right] = 2\left(A_2 a_2^2 - A_3 a_3^2\right)
\end{aligned} \tag{144}$$

$$\boxed{h_n \mu_R R^3\left\{\frac{Q_2^2}{\mu_R}a_1^2 + \left[\left(1 + \frac{M_{tot}}{M'_{tot}}\right)(1 + \delta + \delta') - 1 - 2\frac{Q_1\Omega}{\mu_R}\right]a_2^2 + a_3^2\right\} = 2\left(A_2 a_2^2 - A_3 a_3^2\right)} \tag{145}$$

8 The equilibrium mean radius

We reference the ellipsoid with DM core to the spherical polytrope without DM core but same mass M with the ellipsoid without DM core

8.1 Spherical polytrope

The equilibrium radius of the spherical polytrope is R_0. To find it's expression, we start from the Viral relation:

$$\frac{3}{n}U + W = 0 \tag{146}$$

The Internal energy and self-gravitational potential are given by,

$$U = k_1 K \rho_c^{1/n} M \tag{147}$$

$$W = -c_n \frac{M^2}{R_0} \tag{148}$$

where,

$$k_1 = \frac{n(n+1)}{5-n}\xi_1|\theta'_1| \tag{149}$$

$$c_n = \frac{3}{5-n} \tag{150}$$

$$\rho_c = \frac{\xi_1}{4\pi|\theta'_1|}\frac{M}{R_0^3} \tag{151}$$

Divide the Viral relation by -W and we have.

$$\begin{aligned}
\frac{3}{n}\frac{U}{-W} &= 1 \\
\frac{3}{n}k_1 K\rho_c^{1/n} M\frac{R_0}{c_n M^2} &= 1 \\
\frac{3}{n}\frac{n(n+1)}{5-n}\xi_1|\theta_1'|K\left(\frac{\xi_1}{4\pi|\theta_1'|}\frac{M}{R_0^3}\right)^{1/n} M\frac{(5-n)R_0}{3M^2} &= 1 \\
(n+1)K\xi_1\xi_1^{1/n}|\theta_1'||\theta_1'|^{-1/n}(4\pi)^{-1/n}M^{\frac{1}{n}+1-2}R_0^{-\frac{3}{n}+1} &= 1 \\
(n+1)K\xi_1^{\frac{n+1}{n}}|\theta_1'|^{\frac{n-1}{n}}(4\pi)^{-\frac{1}{n}}M^{\frac{1-n}{n}}R_0^{\frac{n-3}{n}} &= 1 \\
(n+1)K\xi_1^{\frac{n+1}{n}}|\theta_1'|^{\frac{n-1}{n}}(4\pi)^{-\frac{1}{n}}M^{\frac{1-n}{n}} &= R_0^{\frac{3-n}{n}} \\
\left[\frac{(n+1)K}{4\pi}\right]^{n/(3-n)}(4\pi)^{\frac{n-1}{3-n}}\xi_1^{\frac{n+1}{3-n}}|\theta_1'|^{\frac{n-1}{3-n}}M^{\frac{1-n}{3-n}} &= R_0 \qquad (152)
\end{aligned}$$

Finally,

$$\boxed{R_0 = \xi_1\left(\xi_1^2|\theta_1'|\right)^{(n-1)/(3-n)}\left[\frac{(n+1)K}{4\pi}\right]^{n/(3-n)}\left(\frac{M}{4\pi}\right)^{(1-n)/(3-n)}} \qquad (153)$$

8.2 The Ellipsoid

The Viral relation is

$$\frac{3}{n}U + W + 2T = 2W_i^{M-M'_{tot}} \qquad (154)$$

The relevant energy functions are

$$W = -\frac{Md_n}{R}f \qquad (155)$$

$$W_i^{M-M'_{tot}} = -\frac{MM'_{tot}}{10r^3}\kappa_n\left(2a_1^2 - a_2^2 - a_3^2\right) \qquad (156)$$

By defining

$$g_t \equiv \frac{R}{r^3}\left(2I_{11} - I_{22} - I_{33}\right) \qquad (157)$$

, the interaction energy can be written as

$$W_i^{M-M'_{tot}} = \frac{-M'_{tot}}{2R}g_t \qquad (158)$$

Again divide the Viral relation with $|W|$ and we have,

$$\begin{aligned}
\frac{3}{n}\frac{U}{|W|} - 1 + 2\frac{T}{|W|} &= -\frac{M'_{tot}}{|W|R}g_t \\
\frac{3}{n}k_1 K\rho_c^{1/n} M\frac{R}{Md_n f} &= 1 - 2\frac{T}{|W|} - \frac{M'_{tot}R}{Md_n fR}g_t \\
\frac{3}{n}\frac{n(n+1)}{5-n}\xi_1|\theta'_1|K\left(\frac{\xi_1}{4\pi|\theta'_1|}\frac{M}{R^3}\right)^{1/n}\frac{R}{d_n} &= \left(1-2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t \\
\frac{3}{5-n}(n+1)K\xi_1^{1+\frac{1}{n}}|\theta'_1|^{1-\frac{1}{n}}(4\pi)^{-\frac{1}{n}}M^{\frac{1}{n}}R^{1-\frac{3}{n}}\frac{1}{d_n} &= \left(1-2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t \\
\frac{c_n M}{d_n}(n+1)K\xi_1^{\frac{n+1}{n}}|\theta'_1|^{\frac{n-1}{n}}(4\pi)^{-\frac{1}{n}}M^{\frac{1-n}{n}}R^{\frac{n-3}{n}} &= \left(1-2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t \\
\left[\frac{(n+1)K}{4\pi}\right]^{n/(3-n)}\xi_1\left(\xi_1^2|\theta'_1|\right)^{(n-1)/(3-n)}(4\pi)^{\frac{n-1}{3-n}}M^{\frac{1-n}{3-n}}R^{-1} &= \left\{\frac{d_n}{c_n M}\left[\left(1-2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t\right]\right\}^{n/(3-n)} \\
\xi_1\left(\xi_1^2|\theta'_1|\right)^{(n-1)/(3-n)}\left[\frac{(n+1)K}{4\pi}\right]^{n/(3-n)}\left(\frac{M}{4\pi}\right)^{(1-n)/(3-n)} &= R\left\{\frac{d_n}{c_n M}\left[\left(1-2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t\right]\right\}^{n/(3-n)}
\end{aligned} \tag{159}$$

The result is

$$\boxed{R = R_0\left\{\frac{d_n}{c_n M}\left[\left(1-2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t\right]\right\}^{-n/(3-n)}} \tag{160}$$

9 The R/R' equation

A remainder of the equation is

$$\frac{R}{R'} - \frac{(a_1a_2a_3)^{1/3}}{(a'_1a'_2a'_3)^{1/3}} = 0 \tag{161}$$

where,

$$R = R_0\left\{\frac{d_n}{c_n M}\left[\left(1+2\frac{T}{|W|}\right)f - \frac{M'_{tot}}{Md_n}g_t\right]\right\}^{-n/(3-n)} \tag{162}$$

$$R' = R'_0\left\{\frac{d'_n}{c'_n M'}\left[\left(1+2\frac{T'}{|W'|}\right)f' - \frac{M_{tot}}{M'd'_n}g'_t\right]\right\}^{-n'/(3-n')} \tag{163}$$

Note that for $n = n'$ and $K = K'$,

$$R'_0 = R_0\left(\frac{M}{M'}\right)^{-(1-n)/(3-n)} \tag{164}$$

9.1 The T/W term

Recall

$$T = \frac{\kappa_n M}{10}(a_1^2 + a_2^2)(\Lambda^2 + \Omega^2) - \frac{2\kappa_n M}{5} a_1 a_2 \Lambda\Omega \tag{165}$$

$$|W| = \frac{M d_n}{R} f \tag{166}$$

$$\Lambda = -f_R \frac{a_1 a_2}{a_1^2 + a_2^2}\Omega \tag{167}$$

Rearrange the T term:

$$\begin{aligned} T &= \frac{\kappa_n M}{10}(a_1^2 + a_2^2)\left[1 + f_R^2\left(\frac{a_1 a_2}{a_1^2 + a_2^2}\right)^2\right]\Omega^2 + \frac{2\kappa_n M}{5} f_R \frac{a_1^2 a_2^2}{a_1^2 + a_2^2}\Omega^2 \\ &= \frac{\kappa_n M}{10}\left\{(a_1^2 + a_2^2)\left[1 + f_R^2 \frac{a_1^2 a_2^2}{(a_1^2 + a_2^2)^2}\right] + 4 f_R \frac{a_1^2 a_2^2}{a_1^2 + a_2^2}\right\}\Omega^2 \end{aligned} \tag{168}$$

Recall the modified Kepler's law,

$$\Omega^2 = \frac{M_{tot} + M'_{tot}}{r^3}(1 + \delta + \delta') \tag{169}$$

Then,

$$\boxed{\frac{T}{|W|} = \frac{\kappa_n (M_{tot} + M'_{tot})}{10 d_n f}\frac{R}{r^3}\left\{(a_1^2 + a_2^2)\left[1 + f_R^2 \frac{a_1^2 a_2^2}{(a_1^2 + a_2^2)^2}\right] + 4 f_R \frac{a_1^2 a_2^2}{a_1^2 + a_2^2}\right\}(1 + \delta + \delta')} \tag{170}$$

10 The Q term

Recall $a_1\Lambda = a_2 Q_1$, $-a_2\Lambda = a_1 Q_2$ and $\mu_R = M'_{tot}/r^3$. Then

$$\begin{aligned} \frac{Q_1^2}{\mu_R}a_2^2 &= \left(\frac{a_1}{a_2}\right)^2 \Lambda^2 \frac{r^3}{M'_{tot}} a_2^2 \\ &= \frac{r^3}{M'_{tot}} a_1^2 \left(f_R \frac{a_1 a_2}{a_1^2 + a_2^2}\Omega\right)^2 \\ &= \frac{f_R^2 r^3}{M'_{tot}} a_1^2 \left(\frac{a_1 a_2}{a_1^2 + a_2^2}\right)^2 \Omega^2 \\ &= \frac{M_{tot} + M'_{tot}}{M'_{tot}} f_R^2 a_1^2 \left(\frac{a_1 a_2}{a_1^2 + a_2^2}\right)^2 (1 + \delta + \delta') \end{aligned} \tag{171}$$

The second term,

$$\begin{aligned} \frac{Q_2\Omega}{\mu_R} &= -\frac{a_2}{a_1}\Lambda \frac{r^3}{M'_{tot}}\Omega \\ &= f_R \frac{a_2}{a_1}\frac{r^3}{M'_{tot}}\frac{a_1 a_2}{a_1^2 + a_2^2}\Omega^2 \\ &= \frac{M_{tot} + M'_{tot}}{M'_{tot}} f_R \frac{a_2^2}{a_1^2 + a_2^2}(1 + \delta + \delta') \end{aligned} \tag{172}$$

Similarly,

$$\begin{aligned}
\frac{Q_2^2}{\mu_R}a_1^2 &= a_2^2\Lambda^2\frac{r^3}{M'_{tot}} \\
&= \frac{r^3}{M'_{tot}}a_2^2\left(f_R\frac{a_1a_2}{a_1^2+a_2^2}\Omega\right)^2 \\
&= \frac{f_R^2r^3}{M'_{tot}}a_2^2\left(\frac{a_1a_2}{a_1^2+a_2^2}\right)^2\Omega^2 \\
&= \frac{M_{tot}+M'_{tot}}{M'_{tot}}f_R^2a_2^2\left(\frac{a_1a_2}{a_1^2+a_2^2}\right)^2(1+\delta+\delta') \qquad (173)
\end{aligned}$$

and,

$$\begin{aligned}
\frac{Q_1\Omega}{\mu_R} &= \frac{a_1}{a_2}\Lambda\frac{r^3}{M'_{tot}}\Omega \\
&= -f_R\frac{a_1}{a_2}\frac{r^3}{M'_{tot}}\frac{a_1a_2}{a_1^2+a_2^2}\Omega^2 \\
&= -\frac{M_{tot}+M'_{tot}}{M'_{tot}}f_R\frac{a_1^2}{a_1^2+a_2^2}(1+\delta+\delta') \qquad (174)
\end{aligned}$$