Macro hw 2. (Hyoungchul Kim)

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1. (1) (Resource constraint will bind so we substitute C_t)

Functional eq. is as follows

$$V(k) = \max_{\substack{0 \le k' \le Bk^\alpha + (1-\delta)k \\ k' \ge 0}} \left\{ \frac{(Bk^\alpha + (1-\delta)k - k')^{1-\sigma} - 1}{1-\sigma} + \beta V(k') \right\}$$

In this setup, k is the only state variable and k' is the only control variable. (c is also control if we don't substitute)

1. (2)

The functional eq. (fe) becomes

$$v(k) = \max_{0 \le k' \le Bk^\alpha} \left\{ \log(Bk^\alpha - k') + \beta v(k') \right\}$$

first get FOC:

$$\frac{1}{Bk^\alpha - k'} = \beta v'(k') \Rightarrow \frac{1}{Bk^\alpha - k'} = \frac{\beta a_1}{k'}$$

$$\Rightarrow k' = \frac{\beta a_1 B k^\alpha}{1 + a_1 \beta}$$

The plug this into 'fe' to get

$$a_0 + a_1 \log k = \log\left(Bk^\alpha - \frac{\beta a_1 Bk^\alpha}{1 + a_1\beta}\right)$$
$$+ \beta a_0 + \beta a_1 \log\left(\frac{\beta a_1 Bk^\alpha}{1 + a_1 \beta}\right)$$

$$\Rightarrow a_1 = \frac{\alpha}{1 - \alpha\beta}$$

$$\Rightarrow a_0 = \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta) + \frac{\log(B)}{1-\alpha\beta} + \log(1-\alpha\beta)\right]$$

Using FOC, policy ftn becomes

$$g(k) = k' = \alpha\beta B k^\alpha$$

1. (3) Compute steady state stock by

$$k^{ss} = \alpha\beta B k_{ss}^\alpha$$

$$\Rightarrow \underline{k^{ss} = (\alpha\beta B)^{\frac{1}{1-\alpha}}}.$$

1. (4)

Let $V_0(k) = 0$, $\forall k$.

Then solve $V_1(k) = \max_{k'} \{ \log(Bk^\alpha - k') + \beta V_0(k') \}$

As $V_0(k') = 0$ for all k', the optimal solution will have s.t. $k' = 0$.

so

$$V_1(k) = \log B + \alpha \log k$$

for V_2, we get

$$V_2 = \max_{k'} \{ \log(Bk^\alpha - k') + \beta \log B + \alpha\beta \log k' \}$$

$$\text{F.O.C:} \quad -\frac{1}{Bk^\alpha - k'} + \frac{\alpha\beta}{k'} = 0$$

$$\Rightarrow k' = \frac{\alpha\beta B k^\alpha}{1 + \alpha\beta}.$$ Then V_2 becomes

$$V_2(k) = \log\left(\frac{B}{1+\alpha\beta}\right) + \alpha\beta \log\left(\frac{\alpha\beta B}{1+\alpha\beta}\right)$$
$$+ \beta\log(B) + \alpha \log k + \alpha^2\beta \log k$$

we do similarly for V_3 to get:

$$V_3(k) = \max_{k'} \{ \log(Bk^\alpha - k') + \beta V_2 k' \}$$

$\Rightarrow$ (after doing FOC)

$$k' = \frac{\alpha\beta Bk^\alpha + \alpha^2\beta^2 Bk^\alpha}{1 + \alpha\beta + \alpha^2\beta^2}$$

So we get

$$V_3(k) = \beta\left(\log\left(\frac{B}{1+\alpha\beta}\right) + \alpha\beta\log\left(\frac{\alpha\beta B}{1+\alpha\beta}\right) + \beta\log B\right)$$
$$+ \log\left(\frac{Bk^\alpha}{1+\alpha\beta+\alpha^2\beta^2}\right)$$
$$+ (\alpha\beta + \alpha^2\beta^2)\log\left(\frac{(\alpha\beta+\alpha^2\beta^2)Bk^\alpha}{1+\alpha\beta+\alpha^2\beta^2}\right)$$

Note that policy ftn from V_1, V_2, V_3 imply that for n time optimization, we get

$$k' = \frac{(\sum_{i=1}^{n} \alpha^i\beta^i) Bk^\alpha}{\sum_{i=0}^{n} \alpha^i\beta^i}$$

If we do this infinitely ($n \to \infty$) we see that it will converge to our solution in part 2.

$$(\because \quad k' = \left(1 - \frac{1}{\sum_{i=0}^{n} (\alpha\beta)^i}\right) Bk^\alpha$$

$$\boxed{n \to \infty} \quad = \left(1 - \frac{1}{\frac{1}{1-\alpha\beta}}\right) = Bk^\alpha = \alpha\beta Bk^\alpha$$

1.(5). $B \approx 5.5$

Fe: $V(k) = \max_{k'} \{\log(5.5k^{0.3} - k') + 0.6 \cdot V(k')\}$

we let $k^3 = (\alpha\beta B)^{\frac{1}{1-\alpha}} = 1$

So $K = \{k^1, k^2, 1, k^3, k^4, k^5\}$

as $k^i = (1 + 0.1 * (i-3))k^3$,

we get $K = (0.8, 0.9, 1, 1.1, 1.2)$

first let $V_0(k) = 0 \ \forall k \in K$.

Solve

$$V_1(k) = \max_{k' \in K} \{\log(5.5k^{0.3} - k') + 0.6 \times 0\}$$

This yields $k' = g_1(k) = 0.8$ for all $k \in K$.

Then $V_1(0.8) \approx 1.469$

$V_1(0.9) \approx 1.510$

$V_1(1) \approx 1.548$

$V_1(1.1) \approx 1.581$

$V_1(1.2) \approx 1.611$

After V_2, I use computer to get the answers. Note that the value might differ slightly due to finite-precision of the computation.

$V_2(0.8) \approx 2.371$

$V_2(0.9) \approx 2.413$

$V_2(1.0) \approx 2.452$

$V_2(1.1) \approx 2.487$

$V_2(1.2) \approx 2.518$

$V_3(0.8) \approx 2.906$

$V_3(0.9) \approx 2.949$

$V_3(1.0) \approx 2.988$

$V_3(1.1) \approx 3.022$

$V_3(1.2) \approx 3.054$

(I will attach my Julia code for this as well).

1. (6) ~ (10) codes and plots and table results will be separately given. I will write the name of the file here.

1. (6)
plot1-6_policy.png
" _value.png

1. (7)
plot1-7_policy.png
" _value.png

results are somewhat similar but more smooth (especially policy function).

1. (8)
plot1-8_C-value.png
plot1-8_k-value.png

values_output_p1-8.xlsx

1. (9)
plot1-9_C-value.png
plot1-9_k-value.png
plot1-9_policy.png
Plot1-9_value.png

values_output_p1-9.xlsx

The pattern is very similar to previous case. But it seems we have overall lower utility level and consumption.

1. (10).
values_output_p1-10-sigma0.xlsx ($\sigma = 0.5$)
values_output_p1-10-sigma2.xlsx ($\sigma = 2$)

plot1-10_C-value_sigma0.png
" " _sigma2.png
" _k-value_sigma0.png
" " _sigma2.png

plot1-10-policy_sigma2.png
" _value_sigma0.png
plot1-10-policy-sigma0.png
" " _value-sigma2.png

as utility becomes identical to question 9 when $\sigma = 1$, I do not post it here. (check plots in Question 9)

remember that $\frac{1}{\sigma}$ is IES.

So higher σ means lower $\frac{1}{\sigma}$. So people become less willing to make intertemporal substitution. So they want to smooth consumption.

So convergence is slightly slower when σ is higher.

2. (1).

we assume nonnegativity constraint do not bind and resource constraint bind. Then we can rewrite the optimization problem s.t:

$$\max_{k_{t+1}, n_t} \sum \beta^t U\left(k_t^{\alpha}(An_t)^{1-\alpha} - k_{t+1}, n_t\right)$$

Then apply FOC:

$$\partial k_{t+1}: \beta^t U_1\left(k_t^{\alpha}(An_t)^{1-\alpha} - k_{t+1}, n_t\right)$$
$$= \beta^{t+1} U_1\left(k_{t+1}^{\alpha}(An_{t+1})^{1-\alpha} - k_{t+2}, n_{t+1}\right) \cdot \alpha (An_{t+1})^{1-\alpha} k_{t+1}^{\alpha-1}$$

$$\Rightarrow \beta^t U_1(c_t, n_t) = \beta^{t+1} U_1(c_{t+1}, n_{t+1}) \alpha (An_{t+1})^{1-\alpha} k_{t+1}^{\alpha-1}$$

similarly,

$$\partial n_t: \beta^t U_1'(c_t, n_t) \cdot (1-\alpha) k_t^{\alpha} A^{1-\alpha} \cdot n_t^{-\alpha}$$
$$= -\beta^t U_2(c_t, n_t)$$

(★ U_1 is derivative for 1st argument, and U_2 is 2nd argument)

If we make it more simpler:

(1) $U_1(c_t, n_t) = \beta U_1(c_{t+1}, n_{t+1}) \alpha (An_{t+1})^{1-\alpha} \cdot k_{t+1}^{\alpha-1}$

(2) $U_1(c_t, n_t)(1-\alpha) k_t^{\alpha} A^{1-\alpha} n_t^{-\alpha}$
$= -U_2(c_t, n_t)$

(1) eq. is Euler equation relating marginal cost of saving one more unit of capital today to marginal benefit of having capital tomorrow.

(2) eq. is intratemporal optimality condition that relates marginal cost of working today to marginal benefit of working more today.

2. (2)

Then 'fe' are as follows

$$V(k) = \max_{0 \le n,\, 0 \le k' \le k^{\alpha}(An)^{1-\alpha}} \{U(k^{\alpha}(An)^{1-\alpha} - k', n) + \beta V(k')\}$$

where k is state variable and k', n are control variables.

2. (3).

Using the guess, we solve for

$$V(k) = \max_{n, k'} \{\log(k^{\alpha}(An)^{1-\alpha} - k' - n) + \beta(a_0 + a_1 \log k')\}$$

FOC:

$$\partial n: \frac{(1-\alpha) k^{\alpha} A^{1-\alpha} \cdot n^{-\alpha} - 1}{k^{\alpha}(An)^{1-\alpha} - k' - n} = 0$$
$$\Rightarrow n = \left[(1-\alpha) A^{1-\alpha}\right]^{\frac{1}{\alpha}} \cdot k$$

$$\partial k': \frac{-1}{k^{\alpha}(An)^{1-\alpha} - k' - n} + \frac{a_1 \beta}{k'} = 0$$
$$\Rightarrow k' = \frac{k^{\alpha}(An)^{1-\alpha} - n}{1 + \frac{1}{\alpha_1 \beta}}$$

Then substituting for n, we get

$$k' = \frac{a_1 \beta A^{\frac{1-\alpha}{\alpha}} (1-\alpha)^{\frac{1}{\alpha}} \cdot \left(\frac{\alpha}{1-\alpha}\right)}{1 + \beta a_1} k$$

Then we plug the expression for n, k' to value ftn to get

$$a_0 + a_1 \log k = \text{Const} + (a_1 \beta + 1) \log k$$

where "Const" is some constant value (very difficult to calculate ...)

Then as this must hold for all k, we get

$$a_1 = \frac{1}{1-\beta}. \quad (a_0 = \text{const}).$$

Now put this a_1 into k':

$$k' = \frac{a_1 \beta A^{\frac{1-\alpha}{\alpha}} (1-\alpha)^{\frac{1}{\alpha}} \cdot \left(\frac{\alpha}{1-\alpha}\right)}{1 + \beta a_1} k$$

$$\Rightarrow k' = g(k) = \frac{\frac{\beta}{1-\beta} A^{\frac{1-\alpha}{\alpha}} (1-\alpha)^{\frac{1}{\alpha}} \left(\frac{\alpha}{1-\alpha}\right)}{1 + \frac{\beta}{1-\beta}} k$$

Using this, we can also plug in values to get $N(k) = \left((1-\alpha) A^{1-\alpha}\right)^{\frac{1}{\alpha}} k$, $C(k) = \left((1-\alpha) A\right)^{\frac{1-\alpha}{\alpha}} \times (1-\alpha\beta) k$

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2. (4)

We had

$$V(k) = \max_{n, k'} \{U(k^{\alpha}(An)^{1-\alpha} - k', n) + \beta V(k')\}$$

as recursive formulation.

Apply FOC $\left[c = k^{\alpha}(An)^{1-\alpha} - k' \text{ for notation}\right]$

$$\partial k': U_1(c, n) = \beta V'(k') \quad \cdots (1)$$

$$\partial n: U_1(c, n) \cdot k^{\alpha} \cdot A^{1-\alpha} (1-\alpha) n^{-\alpha} \quad \cdots (2)$$
$$= -U_2(c, n)$$

we can see that (2) is the intratemporal optimality condition.

For (1), we use envelope condition.

we assume solution (optimality) hold and differentiate 'fe' by k:

$$V'(k) = U_1(c, n)\left[\alpha k^{\alpha-1}(An)^{1-\alpha} - g'(k)\right] + \beta V'(g(k)) \cdot g'(k)$$

$(\because k' = g(k) \text{ in optimal})$

$$= U_1(c, n) \alpha k^{\alpha-1}(An)^{1-\alpha} - g'(k) \underbrace{\left[U_1(c, n) - \beta V'(g(k))\right]}_{= 0 \text{ by FOC}}$$

$$\therefore V'(k) = U_1(c, n) \alpha k^{\alpha-1} (An)^{1-\alpha}$$

Now switch to next period (k')

then $V'(k') = U_1(c', n') \alpha k'^{\alpha-1} (An')^{1-\alpha}$

plug this to (1) and we get

$$U_1(c, n) = \beta U_1(c', n') \alpha (An')^{1-\alpha} k'^{\alpha-1}$$

Then this is the Euler equation.

2. (5). Using optimality condition in part 4 and utility function in part 3, the condition on left hand side is

$$\frac{1}{C(k) - N(k)} = \frac{1}{\left[(1-\alpha)A\right]^{\frac{1-\alpha}{\alpha}}(1-\alpha\beta)k - (1-\alpha)^{\frac{1}{\alpha}} A^{\frac{1-\alpha}{\alpha}} k}$$

right hand side is

$$\frac{\beta}{C(g(k)) - N(g(k))} \cdot \left[(1-\alpha)A\right]^{\frac{1-\alpha}{\alpha}} \cdot \alpha\beta$$

(where $k' = g(k)$)

As $C(g(k)) - N(g(k))$

$$= (1-\alpha)^{\frac{1-\alpha}{\alpha}} A^{\frac{1-\alpha}{\alpha}} (1-\alpha\beta) g(k) - \left[(1-\alpha) A^{1-\alpha}\right]^{\frac{1}{\alpha}} g(k)$$

$$= \left\{\left((1-\alpha)A\right)^{\frac{1-\alpha}{\alpha}} - \left[(1-\alpha) A^{1-\alpha}\right]^{\frac{1}{\alpha}}\right\} g(k)$$

$$= \{ \quad " \quad \} \times \underbrace{\frac{\frac{\beta}{1-\beta} \cdot A^{\frac{1-\alpha}{\alpha}} \cdot (1-\alpha)^{\frac{1}{\alpha}} \cdot \left(\frac{\alpha}{1-\alpha}\right)}{1 + \frac{\beta}{1-\beta}} k}_{g(k)}$$

After laborious computation, we can show that <u>LHS = RHS</u>.

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3. (1)

We still have k as the only state variable as capital move freely across two sectors.

So we set 'fe' be:

$$V(k) = \max_{\{k_1, k_2, n_1, n_2, k'\}} \{U(k_1^{\alpha} n_1^{1-\alpha}, n_1+n_2) + \beta V(k')\}$$

where
$$\left[\begin{array}{l} k_1+k_2=k, \\ k' = k_2^{\gamma} n_2^{1-\gamma} \\ k_1, k_2, n_1, n_2, k' \geq 0 \end{array}\right]$$

(k_1, k_2, n_1, n_2, k' are control variables)

3. (2)

Now, k_1, k_2 is state variable.

then "fe" becomes

$$V(k_1, k_2) = \max_{\{n_1, n_2, k_1', k_2'\}} \{U(k_1^{\alpha} n_1^{1-\alpha}, n_1+n_2) + \beta V(k_1', k_2')\}$$

s.t.

$$k_1', k_2', n_1, n_2 \geq 0$$
$$k_1' + k_2' = k_2^{\gamma} n_2^{1-\gamma}$$

(rest are control variables, n_1, n_2, k_1', k_2')

3. (3) now all k_1, k_2, n_1, n_2 are state variables.

So

$$V(k_1, k_2, n_1, n_2) = \max_{\{k_1', k_2', n_1', n_2'\}} \{U(k_1^{\alpha} n_1^{1-\alpha}, n_1+n_2) + \beta V(k_1', k_2', n_1', n_2')\}$$

s.t.

$$k_1' + k_2' = k_2^{\gamma} n_2^{1-\gamma}$$
$$k_1', k_2', n_1', n_2' \geq 0.$$

3. (4) Arrow-Debreu Equilibrium consists of prices $\underbrace{\{P_t, r_t, W_t\}_{t=0}^{\infty}}_{①}$

and allocations for the firm ②

$\{k_{1t}^d, k_{2t}^d, n_{1t}^d, n_{2t}^d, Y_t\}_{t=0}^{\infty}$ and

the household $\underbrace{\{C_t, i_t, x_{t+1}, k_{1t}^s, k_{2t}^s, n_{1t}^s, n_{2t}^s\}_{t=0}^{\infty}}_{③}$

s.t.

1. given the ① prices, allocation of the representative firm (maybe two as we have two sectors) ② solves Equation (1)

2. given ① prices, the allocation of the representative household ③ solves Equation (2)

3. Markets clear

$$Y_t = C_t + i_t \quad \text{(goods market)}$$

$$n_{1t}^d = n_{1t}^s, \quad n_{2t}^d = n_{2t}^s \quad \text{(labor)}$$

$$k_{1t}^d = k_{1t}^s, \quad k_{2t}^d = k_{2t}^s \quad \text{(capital service)}$$

(☆ as capital fully depreciate in this economy $i_t = x_{t+1}$).

where

Equation (1):

$$\Pi_1 = \max_{\{Y_{1t}, k_{1t}, n_{1t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} P_t (Y_{1t} - r_{1t} k_{1t} - W_{1t} n_{1t})$$

s.t. $Y_{1t} = F_1(k_{1t}, n_{1t}) \ \forall t \geq 0$

$$Y_{1t}, k_{1t}, n_{1t} \geq 0$$

and also same for sector 2 except we change all '1' subscript to '2'.

Equation (2):

$$\max_{\{C_t, i_t, x_{t+1}, k_{1t}, k_{2t}, n_{1t}, n_{2t}\}} \sum_{t=0}^{\infty} \beta^t U(C_t, n_t)$$

s.t. $\sum_{t=0}^{\infty} P_t (C_t + i_t)$

$$\leq \sum_{t=0}^{\infty} \left[P_t (r_t k_{1t} + W_t n_{1t}) + P_t (r_t k_{2t} + W_t n_{2t})\right] + \Pi_1 + \Pi_2$$

$$x_{t+1} = i_t \quad \forall t \geq 0$$

$$0 \leq n_{1t}, n_{2t}, \quad 0 \leq k_{1t}, k_{2t}, \quad k_t = k_{1t} + k_{2t} \leq x_t \quad \forall t \geq 0$$

$$C_t, x_{t+1} \geq 0 \quad \forall t \geq 0$$

x_0 given.

3. (5) As capital, labor freely allocate across sectors, we will assume the wage and capital return rate is same across sectors.

For firm maximization, note that combined production function $k_{1t}^{\alpha} n_{1t}^{1-\alpha} + k_{2t}^{\gamma} n_{2t}^{1-\gamma}$ is still constant returns to scale.
Using FOC, r_t and W_t becomes

$$r_t = \alpha k_{1t}^{\alpha-1} n_{1t}^{1-\alpha} = \gamma k_{2t}^{\gamma-1} n_{2t}^{1-\gamma}$$
and
$$W_t = (1-\alpha) k_{1t}^{\alpha} n_{1t}^{-\alpha} = (1-\gamma) k_{2t}^{\gamma} n_{2t}^{-\gamma}$$

And as firm is constant returns profit (Π) will be 0.

As capital fully depreciates, we have $x_t = k_t$ and $i_t = k_{t+1}$.
Then household problem simplifies to

$$\max_{\{C_t, n_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t, n_t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} P_t (C_t + k_{t+1}) = \sum_{t=0}^{\infty} P_t (r_t k_t + W_t n_t)$$

Doing FOC gives us

1) ∂C_t: $\dfrac{\beta U_1(C_{t+1}, n_{t+1})}{U_1(C_t, n_t)} = \dfrac{P_{t+1}}{P_t}$

2) ∂n_t: $\dfrac{\beta U_2(C_{t+1}, n_{t+1})}{U_2(C_t, n_t)} = \dfrac{P_{t+1}}{P_t} \cdot \dfrac{W_{t+1}}{W_t}$

3) ∂k_{t+1}: $\dfrac{1}{r_{t+1}} = \dfrac{P_{t+1}}{P_t}$

If we normalize $P_0 = 1$,
we can use the ratio $\dfrac{P_{t+1}}{P_t} = \dfrac{\beta U_1(C_{t+1}, n_{t+1})}{U_1(C_t, n_t)}$ to set equilibrium price P_t.

The equation we have above pins down all the eqm prices

(e.g. $\dfrac{1}{r_{t+1}} = \dfrac{P_{t+1}}{P_t}$, $\dfrac{P_{t+1}}{P_t} \cdot \dfrac{W_{t+1}}{W_t} = \dfrac{\beta U_2(-)}{U_2(-)}$)

4.(1).

We need to consider first when $G_0 = \overline{G}$. In this case, we will have constant $G_t = \overline{G}$ for all t. So as G is like a parameter in this case, it will not be a state nor control variable.

But G becomes state variable if $G_0 \neq \overline{G}$. For simplicity, we will assume $G_0 \neq \overline{G}$ from now on (value function equation for $G_0 = \overline{G}$ is just exactly same except $V(\cdot)$ have no G argument).

$$V(K, G) = \max_{C, \ell, K'} \{ U(C, \ell) + \beta V(K', \overbrace{(1-\rho)\overline{G} + \rho G}^{G'}) \}$$

$$\text{s.t. } C + K' + G = (K)^{\alpha}(\ell)^{1-\alpha}(G)^{\gamma} = Y$$
$$C, K' \geq 0, \quad \ell \geq 0.$$

4.(2)

case 1: $G_0 < \overline{G}$.

By $G_{t+1} = (1-\rho)\overline{G} + \rho G_t$, you can see that government function's growth rate is shrinking.

case 2. $G_0 = \overline{G}$.

Then we all have $G_t = \overline{G}$ for all t.

Value function becomes $(C = K^{\alpha}\ell^{1-\alpha}G^{\gamma} - K' - G)$

$$V(K, G) = \max_{\ell, K'} \{ U(K^{\alpha}\ell^{1-\alpha}G^{\gamma} - K' - G, \ell) + \beta V(K', G') \}$$

Now apply FOC and get euler equation:

1) $\partial K'$: $U_1(\overbrace{K^{\alpha}\ell^{1-\alpha}G^{\gamma} - K'}^{\sim} - G, \ell) = \beta V_1(K', G')$

2) $\partial \ell$: $U_1(\sim, \ell) \cdot (1-\alpha)K^{\alpha}G^{\gamma} \cdot \ell^{-\alpha} + U_2(\sim, \ell) = 0$

Using the standard envelope condition gives us the euler equation

$$U_1(\sim, \ell) = \beta \cdot U_1(K'^{\alpha}\ell'^{1-\alpha}G'^{\gamma} - K'' - G', \ell') \cdot \alpha K' \ell'^{1-\alpha} G'^{\gamma}$$

Since we know the exact form of utility function, we plug it into the equations above to get relation between ℓ, K, G

$$\Rightarrow G^{\gamma} = \frac{\psi(\ell)^{\frac{\alpha x + 1}{x}}}{(1-\alpha)K^{\alpha}}$$

Now let's go back to output at period 0.

$$Y_0 = (K_0)^{\alpha}(\ell_0)^{1-\alpha}(G_0)^{\gamma}.$$

We can see that Y_0 increases as ℓ_0 increases. (If K, G is fixed).

For low gov. spending to cause recession, we need to have that the output (Y) given $G_0 < \overline{G}$ has to be smaller than $\overline{G} = G_0$.

This would mean ℓ has to decrease as G decreases.

Since we had eq. $G^{\gamma} = \frac{\psi(\ell)^{\frac{\alpha x + 1}{x}}}{(1-\alpha)K^{\alpha}}$,

This can happen when $\frac{\alpha x + 1}{x} > 0$, $\psi > 0$, $\gamma \geq 0$.

4.(3). $\boxed{\begin{array}{l}\text{individual state: } k \\ \text{aggregate state: } G, K \\ \text{control variable: } c, \ell, k'\end{array}}$

A recursive competitive equilibrium is characterized by following functions: a value function $V: \mathbb{R}^3 \to \mathbb{R}$ and policy functions $g_c: \mathbb{R}^3 \to \mathbb{R}$, $g_g: \mathbb{R}^3 \to \mathbb{R}$, $g_\ell: \mathbb{R}^3 \to \mathbb{R}$ for household, and pricing functions $w, r: \mathbb{R}^3 \to \mathbb{R}$ and aggregate law of motion $H: \mathbb{R}^2 \to \mathbb{R}$ s.t.

1) given w, r, H, household solve

$$V(k, K, G) = \max_{c, \ell, k'} \{ u(c, \ell) + \beta(k', K', G') \}$$

$$\text{s.t. } c + k' + G = w(S)L(S) + (1 + r(S))k$$
$$G = \theta T(S) + (1-\theta)\tau(S)w(S)L(S)$$
$$K' = H(S)$$

for policy fcns,

$$k' = g_k(k, K, G),$$
$$c = g_c(k, K, G),$$
$$\ell = g_\ell(k, K, G).$$

2) pricing satisfy

① $w(S) = (1-\alpha) K^\alpha (L(S))^{-\alpha} G^\gamma$

② $r(S) = \alpha K^{\alpha - 1} (L(S))^{1-\alpha} G^\gamma$

3) Consistency:

$$H(K, G) = g_k(K, K, G)$$

4) Market clears: for all K, G

$$g_c(K, K, G) + g_k(K, K, G) + G = K^\alpha L(S)^{1-\alpha} G^\gamma$$

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4. (4).

We use the intratemporal optimality condition

$$U_1(c, \ell) \cdot [1 - (1-\theta)\tau] w = U_2(c, \ell)$$

$$\Rightarrow [1 - (1-\theta)\tau] = \psi \ell^{\frac{1}{\chi}}$$

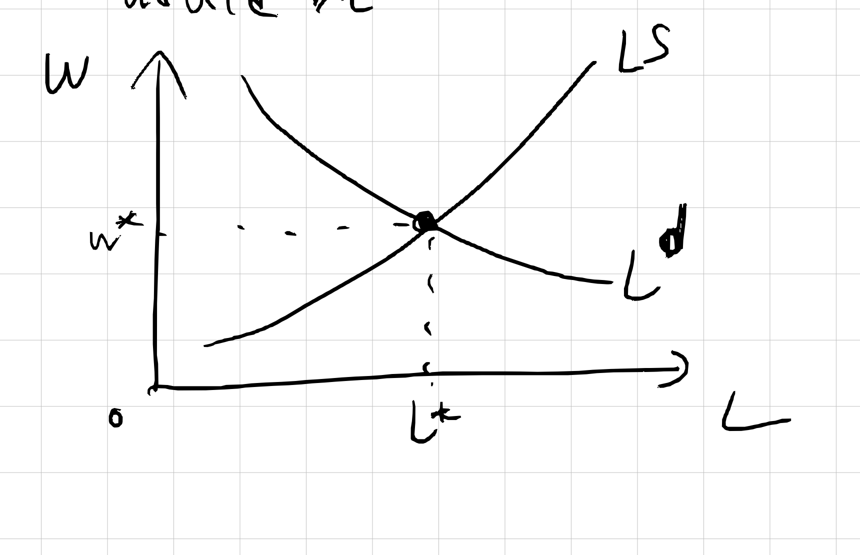
So we set $L^s = \left[\frac{(1-(1-\theta)\tau)}{\psi} w \right]^\chi$

Now we use firm optimality condition wrt ℓ.

$$w(S) = (1-\alpha) K^\alpha L(S)^{-\alpha} G^\gamma$$

$$\Rightarrow L^d = \left[\frac{(1-\alpha) K^\alpha G^\gamma}{w} \right]^{\frac{1}{\alpha}}$$

So in terms of graph it would be



4. (5).

First note that by computing $L^d = L^s$, we can get equilibrium labor as

$$L^* = \left(\frac{[1 - (1-\theta)\tau][(1-\alpha)K^\alpha G^\gamma]}{\psi} \right)^{\frac{\chi}{\alpha\chi + 1}}$$

when $\theta = 1$, we get

$$L^* = \left(\frac{(1-\alpha)K^\alpha G^\gamma}{\psi} \right)^{\frac{\chi}{\alpha\chi+1}}$$

When $\theta = 0$, we get

$$L^* = \left(\frac{(1-\tau)(1-\alpha)K^\alpha G^\gamma}{\psi} \right)^{\frac{\chi}{\alpha\chi+1}}$$

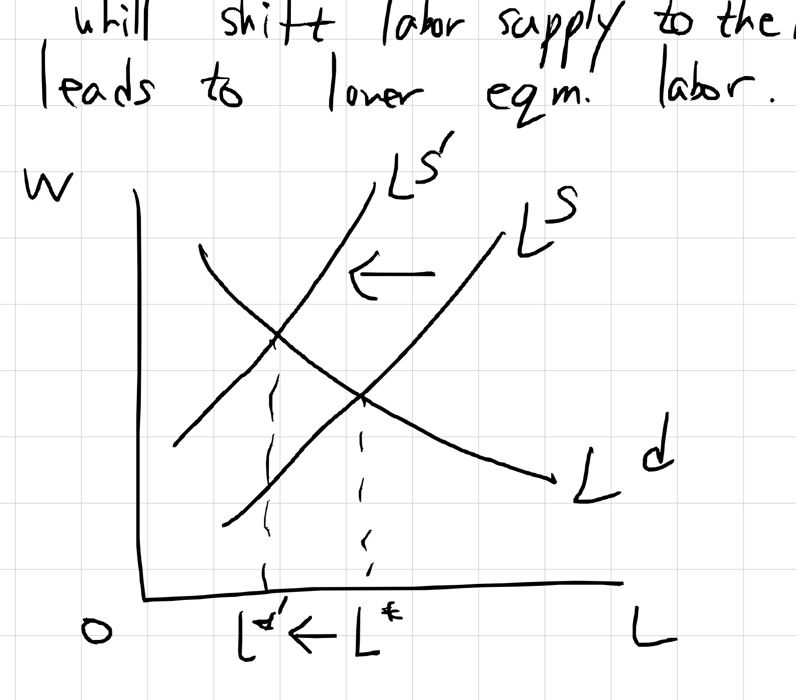
Due to $(1-\tau)$, we can see that equilibrium labor is higher in $\theta = 1$ (lump sum tax).

In social planner, we can use the $G^\gamma = \frac{\psi (L)^{\frac{1}{\chi} + \alpha}}{(1-\alpha)K^\alpha}$, which will give us $L = \left(\frac{(1-\alpha)K^\alpha G^\gamma}{\psi} \right)^{\frac{\chi}{\alpha\chi+1}}$

So it is same as the competitive equilibrium labor when $\theta = 1$.

In graph, for $\theta = 0$, the supply curve is not affected (equilibrium labor same as social planner).

But when $\theta = 1$, the proportional tax will shift labor supply to the left, which leads to lower eqm. labor.



5. (1) We know that $n_t = 1$ in optimum as consumer does not have n_t in the utility function.

Also, in the equilibrium, social planner will be maximizing over aggregate consumption and capital. So the problem becomes:

$$\max_{\{C_{t-1}, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(C_t - \psi C_{t-1})$$

$$\text{s.t. } C_t + K_{t+1} = K_t^{\alpha+\gamma} + (1-\delta)K_t$$

(K_0 is given).
($C_{-1} = 0$)

5. (2) We can recursively write the problem as

$$V(C^0, K) = \max_{C, K' \geq 0} \{U(C - \psi C^0) + \beta V(C, K')\}$$

s.t.

$$C + K' = K^{\alpha+\gamma} + (1-\delta)K$$

(C^0 is past consumption).

Then C^0, K is state variables and C, K' are controls.

5. (3). Use $C = K^{\alpha+\gamma} + (1-\delta)K - K'$,

$$V(C^0, K) = \max_{K' \geq 0} \{U(K^{\alpha+\gamma} + (1-\delta)K - K' - \psi C^0) + \beta V(K^{\alpha+\gamma} + (1-\delta)K - K', K')\}$$

FOC gives us:

$$U'(C - \psi C^0) = -\beta V_1(C, K') + \beta V_2(C, K')$$

Now we need to use envelope condition. (let $K' = g(K)$). We first differentiate V by k:

① $$V_2(C^0, K) = U'(C - \psi C^0)\left[(\alpha+\gamma)K^{\alpha+\gamma-1} + (1-\delta) - g'(K)\right] + \beta V_1(C, K') \cdot \left[(\alpha+\gamma)K^{\alpha+\gamma-1} + (1-\delta) - g'(K)\right] + \beta V_2(C, K') \cdot g'(K)$$

$$= g'(K)\Big[\underbrace{-U'(C-\psi C^0) - \beta V_1(C, K') + \beta V_2(C, K')}_{=0\ (\because \text{FOC})}\Big] + \left[U'(C - \psi C^0) + \beta V_1(C, K')\right] \cdot \left[(\alpha+\gamma)K^{\alpha+\gamma-1} + (1-\delta)\right]$$

② $$V_1(C_{-1}, K) = -\psi U'(C - \psi C^0)$$

Then take ①② to next period to get

①: $$V_2(C, K') = \left[U'(C' - \psi C) + \beta V_1(C', K'')\right] \times \left[(\alpha+\gamma)(K')^{\alpha+\gamma-1} + (1-\delta)\right]$$

②: $$V_1(C, K') = -\psi U'(C' - \psi C)$$

Then plug this to FOC to get Euler equation:

$$U'(C - \psi C_0) = \beta\psi U'(C' - \psi C) + \beta\left[(\alpha+\gamma)(K')^{\alpha+\gamma-1} + (1-\delta)\right]\{U'(C' - \psi C) - \beta\psi U'(C'' - \psi C')\}$$

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5. (4)
recursive competitive equilibrium is set of functions where there is value function $V: \mathbb{R}_+^4 \to \mathbb{R}$ and policy function $g_c: \mathbb{R}_+^4 \to \mathbb{R}$, $g_k: \mathbb{R}_+^4 \to \mathbb{R}$ for representative household and pricing function $W: \mathbb{R}_+^2 \to \mathbb{R}$, $r: \mathbb{R}_+^2 \to \mathbb{R}$ and aggregate law of motion s.t.

① given W and r, household value function equation

$$V(c^0, C^0, k, K) = \max_{c, k' \geq 0} \{U(c - \psi c^0) + \beta V(c, C, k', K')\}$$

s.t.

$$c + k' = W(C^0, K) + (1 + r(C^0, K) - \delta)k$$
$$K' = H_K(C^0, K)$$
$$C = H_C(C^0, K)$$

② Pricing will satisfy:

$$W(C^0, K) = (1-\alpha)K^\alpha K^\gamma$$
$$r(C^0, K) = \alpha K^{\alpha-1} K^\gamma$$

③ Consistency satisfy: for all K, C^0

$$H_K(C^0, K) = g_k(C^0, C^0, K, K)$$
$$H_C(C^0, K) = g_c(C^0, C^0, K, K)$$

④ Market clears: for all K, C^0

$$g_c(C^0, C^0, K, K) + g_k(C^0, C^0, K, K) = K^{\alpha+\gamma} + (1-\delta)K$$

5. (5) The Bellman equation was

$$V(c^0, C^0, k, K) = \max_{k' \geq 0} \{U(w + (1+r-\delta)k - k' - \psi c^0) + \beta V(c, C, k', K')\}$$

Then FOC gives us:

$$U'(w + (1+r-\delta)k - k' - \psi c^0) = -\beta V_c(c, C, k', K') + \beta V_{k'}(c, C, k', K')$$

Now we utilize envelope condition (note that $k' = g_k(c^0, C^0, k, K) = g_k(k)$)

1) $$\frac{\partial V}{\partial k} = U'(w + (1+r-\delta)k - k' - \psi c^0) \cdot \{1 + r - \delta - g_k'(k)\} + \beta V_c(c, C, k', K') \cdot \{1 + r - \delta - g_k'(k)\} + \beta V_k(c, C, k', K') \cdot g_k'(k)$$

$$= \left[U'(w + (1+r-\delta)k - k' - \psi c^0) + \beta V_c(c, C, k', K')\right](1 + r - \delta)$$

So
$$\frac{\partial V(c, C, k', K')}{\partial k} = U'(c' - \psi c)\left[1 + r(C, K') - \delta\right] + \beta V_c(c', C', k'', K'')\left[1 + r(C, K') - \delta\right]$$

2) $$\frac{\partial V}{\partial c} = -\psi U'(c' - \psi c)$$

so $$\frac{\partial V(c', C', k'', K'')}{\partial c} = -\psi U'(c'' - \psi c')$$

Then plug this to FOC to get euler equation:

$$U'(c - \psi c^0) = \beta\left[1 + r(C, K') - \delta\right] \times \{U'(c' - \psi c) - \psi\beta U'(c'' - \psi c')\} + \beta\psi U'(c' - \psi c)$$

where
$$k' = g_k(c^0, C^0, k, K)$$
$$k'' = g_k(c, C, k', K')$$
$$c' = g_c(c^0, C^0, k, K)$$
$$c'' = g_c(c, C, k', K').$$

5. (6) Now we assume ! ∃ competitive equilibrium. We showed in 3, 5 that social planner problem and competitive equilibrium both solve the analogous euler equation. This implies that having consumption habits in the utility function did not change (distort) the pareto efficiency of the equilibrium in competitive case.