Derivation of sectoral gravity model

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Solving within a sector

- Just to make this simple, I will supress the sector index l for now.

I solve the following maximization problem:

$$\max_{c_{ij}} \left\{ \sum_{i}^{N} (\beta_i)^{\frac{1-\sigma}{\sigma}} (c_{ij})^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

subject to:

$$\sum_{i=1}^{N} p_{ij} c_{ij} \leq Y_j.$$

Solving the lagrangian gives the following first order conditions:

$$\frac{\partial L}{\partial c_{ij}} = (\beta_i)^{\frac{1-\sigma}{\sigma}} (c_{ij})^{\frac{\sigma-1}{\sigma}-1} - \lambda p_{ij} = 0.$$

Now divide this by i''s FOC, we get:

$$\frac{p_{ij}}{p_{i'j}} = \left(\frac{\beta_i}{\beta_{i'}}\right)^{\frac{1-\sigma}{\sigma}} \cdot \left(\frac{c_{ij}}{c_{i'j}}\right)^{-\frac{1}{\sigma}}.$$

I can then rearrange this to get the expresson for $c_{i'j}$:

$$c_{i'j} = \left(\frac{\beta_i}{\beta_{i'}}\right)^{\sigma-1} \left(\frac{p_{ij}}{p_{i'j}}\right)^{\sigma} \cdot c_{ij}.$$

Now I can substitute this back into the budget constraint to get the expression for c_{ij}:

$$\sum_{i=1}^{N} p_{ij} \left(\left(\frac{\beta_i}{\beta_{i'}} \right)^{\sigma-1} \left(\frac{p_{ij}}{p_{i'j}} \right)^{\sigma} \cdot c_{ij} \right) = Y_j.$$

This then becomes:

$$c_{ij} = (p_{ij})^{-\sigma} \cdot \left(\frac{\beta_i}{P_j} \right)^{1-\sigma} \cdot Y_j$$

where P_j is the price index: $P_j = \left[\sum_{i=1}^{N} (\beta_i p_{ij})^{1-\sigma} \right]^{\frac{1}{1-\sigma}}$.

Multiplying it by p_{ij} will give us the nominal demand X_{ij}:

$$X_{ij} = p_{ij} \cdot c_{ij} = (p_{ij})^{1-\sigma} \cdot \left(\frac{\beta_i}{P_j} \right)^{1-\sigma} \cdot Y_j.$$

Now I can substitute $p_{ij} = p_i t_{ij}(1+\tau_{ij})(1-z_i)(1-(1-\phi_i)s_i)$:[1]

$$X_{ij} = \left[p_i t_{ij}(1+\tau_{ij})(1-z_i)(1-(1-\phi_i)s_i) \right]^{1-\sigma} \cdot \left(\frac{\beta_i}{P_j} \right)^{1-\sigma} \cdot Y_j.$$

Now I impose market clearing condition $\sum_{j=1}^{N} X_{ij} = Y_i$ and substitute X_{ij} with the expression above:

$$Y_i = \sum_{j=1}^{N} \left[p_i t_{ij}(1+\tau_{ij})(1-z_i)(1-(1-\phi_i)s_i) \right]^{1-\sigma} \cdot \left(\frac{\beta_i}{P_j} \right)^{1-\sigma} \cdot Y_j.$$

From now on, let's denote Y_j as E_j (expenditure equals to income) to avoid confusion. If I divide both sides by the total sectoral income Y I get the following expression:

$$\frac{Y_i}{Y} = \sum_{j=1}^{N} \left[p_i t_{ij}(1+\tau_{ij})(1-z_i)(1-(1-\phi_i)s_i) \right]^{1-\sigma} \cdot \left(\frac{\beta_i}{P_j} \right)^{1-\sigma} \cdot \frac{E_j}{Y}.$$

Now I can define $\Pi_i^{1-\sigma} = \sum_j \left(\frac{t_{ij}(1+\tau_{ij})}{P_j} \right)^{1-\sigma} \cdot \frac{E_j}{Y}$. Then the expression above becomes:

[1] Remember that substitution would have also happened for the price index.

$$\frac{Y_i}{Y} = \left(\beta_i p_i (1-z_i)(1-(1-\phi_i)s_i)\Pi_i\right)^{1-\sigma}, \forall i.$$

Solve for $(\beta_i p_i (1-z_i)(1-(1-\phi_i)s_i))^{1-\sigma}$:

$$(\beta_i p_i (1-z_i)(1-(1-\phi_i)s_i))^{1-\sigma} = \frac{Y_i/Y}{\Pi_i^{1-\sigma}}.$$

Now put substitute the LHS into the price index expression:

$$P_j = \left[\sum_{i=1}^{N} (\beta_i p_i t_{ij}(1+\tau_{ij})(1-z_i)(1-(1-\phi_i)s_i))^{1-\sigma}\right]^{\frac{1}{1-\sigma}}.$$

I can then gets inward multilateral resistance terms:

$$P_j^{1-\sigma} = \sum_i \left(\frac{t_{ij}(1+\tau_{ij})}{\Pi_i}\right)^{1-\sigma} \cdot \frac{Y_i}{Y}.$$

Lastly, I add the income and expenditure definitions and market clearing conditions. For this case, I add back the sector index l:

$$E_j^l = \alpha_j^l Y_j = \alpha_j^l \sum_l Y_j^l,$$

$$p_j^l = \frac{\left(Y_j^l/Y^l\right)^{\frac{1}{1-\sigma}}}{\beta_j^l (1-z_j^l)(1-(1-\phi_j^l)s_j^l)\Pi_j^l}.$$

Then done! I have derived the sectoral gravity model.

References