Problem Set 1

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Acknowledgement

I collaborated with Alex, Flora, and martin.

Part 1: Theory

In this problem set, you will use a Mirrlees-style model to characterize the optimal linear income tax with a lump-sump grant. You'll then solve for the optimal tax in a numerical simulation.

1.

We substitute the budget constraint $c = z - T(z)$ into the utilify function:

$$U_i(z - T(z), z) = u\left(z - T(z)\right) - v\left(\frac{z}{w_i}\right).$$

The budget constraint is:

$$c = z - T(z) = z - (tz - b) = (1-t)z + b.$$

Substituting this into the utility function again:

$$U_i((1-t)z + b, z) = u\left((1-t)z + b\right) - v\left(\frac{z}{w_i}\right).$$

Taking the derivative with respect to z and setting it to zero:

$$(1-t)\cdot u'((1-t)z+b)-\frac{1}{w_i}\cdot v'\left(\frac{z}{w_i}\right)=0.$$

This is the FOC for each agent's choice of income.

2.

Uncompensated Elasticity:

$$\epsilon_i=\frac{\partial z_i}{\partial(1-t)}\cdot\frac{1-t}{z_i}$$

Implicit differentiation of the FOC gives us:

$$u'+(1-t)u''\cdot\left(z_i+(1-t)\frac{\partial z_i}{\partial(1-t)}\right)-\frac{v''}{w_i^2}\frac{\partial z_i}{\partial(1-t)}=0$$

Rearranging it by $\frac{\partial z_i}{\partial(1-t)}$ and using the fact $\left(\frac{v'}{u'}\right)^2=(1-t)^2w_i^2$ that gives us:

$$\frac{\partial z_i}{\partial(1-t)}=\frac{w_i^2u'+(1-t)w_i^2u''z_i}{v''-u''\left(\frac{v'}{u'}\right)^2}$$

So multiplying left hand side with $\frac{1-t}{z_i}$ and using the FOC condition $w_i(1-t)u'=v'$ gives us the formula for ε_i.

Income Effect:

Implicitly differentiating FOC by b gives us:

$$(1-t)u''\cdot\left[(1-t)w_i\cdot\frac{\partial z_i}{\partial b}+1\right]-\frac{v''}{w_i}\cdot\frac{\partial z_i}{\partial b}=0.$$

Rearranging it by $\frac{\partial z_i}{\partial b}$ and using the fact that $\left(\frac{v'}{u'}\right)^2=(1-t)^2w_i^2$ gives us:

$$\left[v''-u''\left(\frac{v'}{u'}\right)^2\right]\cdot\frac{\partial z_i}{\partial b}=(1-t)w_iu''.$$

Then multiplying the partial differentiation by $(1-t)$ and rearranging the equation gives us the formula for η_i.

Compensated Elasticity:

We can easily use slutsky equation (elasticity form) to show that compensated elasticity will be same as $\varepsilon_i - \eta_i$. This is because from slutsky equation we have $\varepsilon_i = \varepsilon_i^c + \eta_i$. This can be thought of as adjusting for the income effect from the uncompensated elasticity.

3.

We will use the definition of the uncompensated elasticity:

$$\varepsilon_i = \frac{\partial z}{d(1-t)} \cdot \frac{1-t}{z}.$$

Then we can rearrange it to get:

$$\begin{aligned}
\frac{dz}{d(1-t)} &= \frac{z}{1-t} \cdot \varepsilon_i &(1)\\
&= \frac{z}{1-t} \cdot (\varepsilon_i^c + \eta_i) &(2)\\
&\Rightarrow dz = \frac{z}{1-t}(\varepsilon_i^c + \eta_i) \cdot d(1-t) = -\frac{z}{1-t}(\varepsilon_i^c + \eta_i) \cdot dt &(3)
\end{aligned}$$

4.

Suppose that the uncompensated elasticity is the same for everyone. Also, the total net government revenues are defined as:

$$\int_i (tz_i - b) dv(i).$$

Then we can see that this marginal raise in tax will affect the net revenue in two ways. First, it will have mechanical effect where (assuming z_i is fixed) the government will earn addtional tax by increasing the tax rate: $\int_i (dt \cdot z_i) dv(i)$. But there will be fiscal externality from the behavior component where the tax will affect the behavior of the people and affect their income: $\int_i t \cdot \frac{\partial z_i}{\partial(1-t)} \frac{d(1-t)}{dt} \cdot dt dv(i)$. Rearranging by substituting ε gives us $-\int_i t \cdot \varepsilon \cdot \frac{1}{1-t} z_i dv(i)$.

Then we can easily see that integrating by i will give use the formula in terms of average income $\bar{z}$. We can then see that adding these two components give us the total effect on net government revenues of marginally raising t by a small amount dt.

5.

The impact is:

$$\frac{dR}{db} = \int_i \left(t \cdot \frac{\partial z_i}{\partial b} - 1 \right) dv(i) = \frac{t}{1-t} \int_i \eta_i dv(i) - 1.$$

Intuitively, change in lump-sum benefit is affecting both the income of the individual (behavioral) and also mechanically raising the income of the person (mechanical).

The size of the total fiscal cost would depend on the sign of the income effect. Income effect is negative as having more income makes people to spend more of leisure (not work). Thus the equation tells us that the total fiscal cost of the lump-sump grant will be greater than \$1 per capita.

6.

We will use the $\frac{\partial \mathcal{L}}{\partial b} = 0$ and result we got from question 5:

$$0 = \frac{\partial \mathcal{L}}{\partial b} = \frac{\partial}{\partial b} \int_i G dv(i) + \lambda \underbrace{\left(\frac{t}{1-t} \int_i \eta_i dv(i) - 1 \right)}_{\text{From result in question 5}}.$$

Rearranging this gives us:

$$\frac{t}{1-t} \int_i \eta_i dv(i) - 1 = -\int_i \frac{\overbrace{G' \cdot \frac{\partial U_i}{\partial c}}^{:=g_i}}{\lambda} dv(i) \Rightarrow \frac{t}{1-t} \int_i \eta_i dv(i) = \int_i (1 - g_i) dv(i)$$

Like in the class, we can think of right hand side as the net social value of transferring marginal amount from the individual to the government. On the other hand, the left hand side is the indirect cost of marginal change in the lump-sump we found in question 5 due to change in people's behavior by income effect. In that sense, this equation would mean that the net value government is gaining from taxing individual should be enough to cover the indirect fiscal cost from giving lump-sum benefit to people.

7.

We will use the fact that in optimal policy, $\frac{\partial \mathcal{L}}{\partial t} = 0$.

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial t} && (4)\\
&= -\int_i G' \left(U'_{ic} \left(\frac{\partial z_i}{\partial t} - z_i - t\frac{\partial z_i}{\partial t} \right) + U'_{iz}\frac{\partial z_i}{\partial t} \right) dv(i) + \lambda \int_i \left(z_i + t\frac{\partial z_i}{\partial t} \right) dv(i) && (5)\\
&= -\int_i z_i U'_{ic} G' dv(i) + \lambda \int_i \left(z_i + t\frac{\partial z_i}{\partial t} \right) dv(i) && (6)
\end{aligned}
$$

Rearranging and using g_i and ε_i, we get:

$$\int_i g_i z_i dv(i) = \int_i \left(z_i - \varepsilon_i \frac{t}{1-t} z_i \right) dv(i).$$

After using the fact that $\varepsilon_i = \varepsilon_i^c = \eta_i$ and $g_i + \frac{t}{1-t}\eta_i = \tilde{g}_i$, we can simplify the equation to:

$$\int_i \tilde{g}_i z_i dv(i) = \int_i \left(z_i - \varepsilon_i^c \frac{t}{1-t} z_i \right) dv(i) \Rightarrow -\int_i \varepsilon_i^c \frac{t}{1-t} z_i dv(i) = \int_i \tilde{g}_i z_i - z_i dv(i).$$

Since expectation of $\tilde{g}_i$ is 1, we can denote the right hand side of the equation as $\text{Cov}(\tilde{g}_i, z_i)$. Then assuming elasticity is same for everyone, we can get the equation in the question 7.

8.

Diamond's *many-person Ramsey raule* is basically a statement describing the trade-off between efficiency and distributional concern. On the one hand, the government wants to minimize the distortion from taxation by taxing more on (relative) inelastic goods. On the other hand, the government wants to set lower tax on people with lower marginal utility of consumption.

The equation in question 7 somewhat takes this trade-off into account. The covariance term can be understood as measuring the level of regressiveness of the tax system. If the covariance is negative, it would mean putting more welfare weights on people with less income. The compensated elasticity in the equation takes into account the importance of efficiency in tax system. If elasticity rise, the tax will have to fall to hold the equation. This sort of captures the "taxing more on inelastic good" concept.

Part 2: Numerical application

```r
# Load libraries
library(tidyverse)
library(gt)

# Given parameters
t <- 0.3  # Current tax rate
b <- 5000 # Lump-sum benefit
```

1.

The FOC becomes:

$$1 - t - \left(\frac{z_i}{w_i}\right)^k \cdot \frac{1}{w_i} = 0.$$

Then we can rearrange it by z_i: $z_i = (1-t)^{1/k} w_i^{(k+1)/k}$.

Then we can get: $\frac{\partial z_i}{\partial(1-t)} = \frac{1}{k} w_i^{\frac{k+1}{k}} (1-t)^{\frac{1}{k}-1}$.

Expressions for ε_i can be written as (we can just use the result we got in question 2):

$$\varepsilon_i = \frac{1}{k}.$$

Then you can easily see that the income effect η_i is zero as the functional form is not a function of b. This would mean that $\varepsilon_i = \varepsilon_i^c = \frac{1}{k}$. Since compensated elasticity is 0.3, we also get $k = \frac{10}{3}$.

2.

We apply FOC w.r.t z on individual utility to get:

$$1 - t - \left(\frac{z}{w_i}\right)^k \cdot \frac{1}{w_i} = 0 \Rightarrow z_i = w_i^{1+\frac{1}{k}} \cdot (1-t)^{\frac{1}{k}}.$$

Then by plugging the values mentioned in the question, we can get the implied ability (if we let $k = 10/3$, we will exactly get the number).

```r
implied_ability <- map(c(20000, 50000, 200000), ~ (.x / (1-t)^0.3)^(1/1.3))
↪  |>
  bind_cols() |>
  rename_with(~ c("low", "middle", "high")) |> print()
```

```
# A tibble: 1 x 3
    low middle   high
  <dbl>  <dbl>  <dbl>
1 2209.  4470. 12986.
```

We can also use net government revenue function to get the exogenous expenditures E. We can compute the integration: $\int_i (tz - b) dv(i) = E$.

```r
E <- ((3/10) * 20000 * (3/10)) + ((3/10) * 50000 * (6/10)) + ((3/10) * 200000
↪  * (1/10)) - 5000

print(paste("Exogenous expenditures (E):", E))
```

```
[1] "Exogenous expenditures (E): 11800"
```

3.

Using the result we got in FOC in question 1, we can derive z_i as:

```r
earning_zi <- function(t, w) {
  z = (1-t)^(0.3) * w^(1.3)
  return(z)
}

# We also compute earning for each type under t = 0
map(c(2209, 4470, 12986), ~ earning_zi(0, .x)) |>
  bind_cols() |>
  rename_with(~ c("low", "middle", "high")) |>
  print()
```

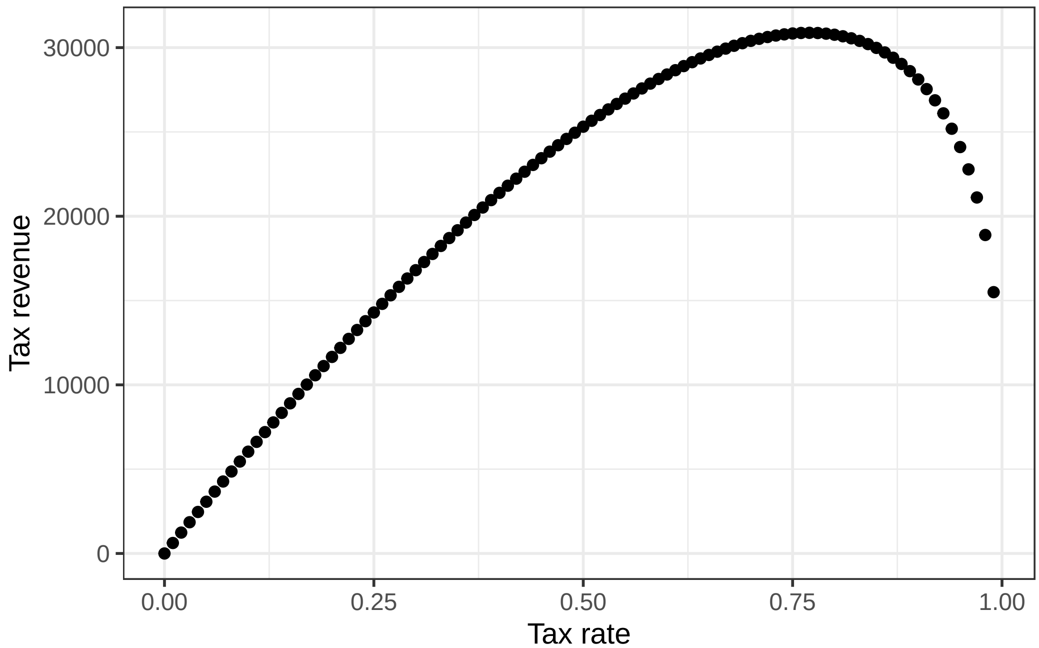
```
# A tibble: 1 x 3
     low middle    high
   <dbl>  <dbl>   <dbl>
1 22256. 55642. 222596.
```

Now we will use the z_i we got above and use it to compute $t \cdot \bar{z}$:

```
# Define population proportions
weights <- c(0.3, 0.6, 0.1)

tax_revenue <- function(t) {
  if (t<0 || t>1) {
    return(-Inf)
  }
  z_i <- (1-t)^(0.3) * c(2209, 4470, 12986)^(1.3)
  z_bar <- sum(weights * z_i)
  return(t * z_bar)
}

# Optimize the tax rate using 'optimize' function
result <- optimize(tax_revenue, interval = c(0,0.99), maximum = TRUE)
optimal_tax_rate <- result$maximum
print(paste("Optimal tax rate: ", optimal_tax_rate))
```

```
[1] "Optimal tax rate:  0.769236810710384"
```

```
# plot the graph
tax_interval <- seq(0, 0.99, by = 0.01)
tax_revenue_interval <- map(tax_interval, tax_revenue) %>%
  unlist()

tax_graph <- tibble(t = tax_interval, revenue = tax_revenue_interval)

tax_graph %>%
  ggplot(aes(t, revenue)) +
  geom_point() +
  scale_x_continuous("Tax rate") +
  scale_y_continuous("Tax revenue") +
  theme_bw()
```



4.

The social welfare is defined as:

$$\int_i G(U_i)dv(i) = \int_i G\left((1-t)z + b - \frac{1}{1+k}\left(\frac{z}{w_i}\right)^{1+k}\right)dv(i) = \int_i \log\left((1-t)z + b - \frac{1}{1+k}\left(\frac{z}{w_i}\right)^{1+k}\right)dv(i),$$

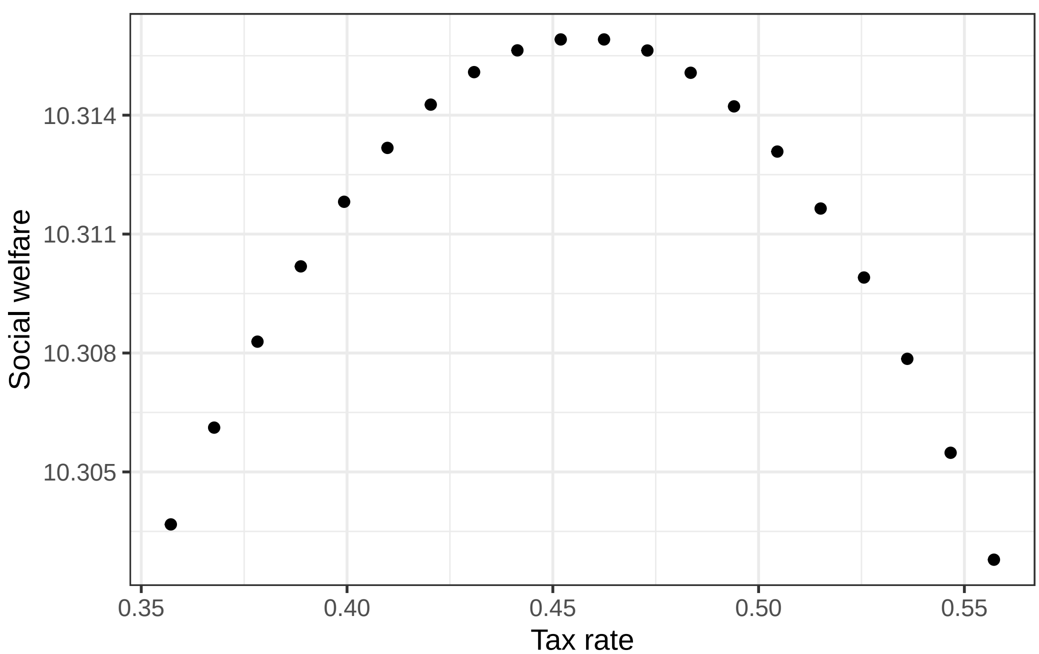
where b is set such that it keeps the exogenous expenditures level $E = 11800$.

```
w_i = c(2209, 4470, 12986)

welfare <- function(t) {
  if (t<0 || t>1) {
    return(-Inf)
  }
  z_i <- (1-t)^(0.3) * c(2209, 4470, 12986)^(1.3)
  b <- sum(t * z_i * weights) - 11800
  welfare_i <- log((1-t)*z_i + b - (3/13)*(z_i/w_i)^(13/3))
  result <- sum(weights * welfare_i)
  return(result)
}
```

```r
# Optimize the tax rate using 'optimize' function
opt_welfare <- optimize(welfare, interval = c(0,0.99), maximum = TRUE)
optimal_tax_rate <- opt_welfare$maximum
print(paste("Optimal tax rate: ", optimal_tax_rate))
```

```
[1] "Optimal tax rate:  0.457185817337985"
```

```r
# plot the graph
tax_interval <- seq(optimal_tax_rate - 0.1, optimal_tax_rate + 0.1,
 ↪  length.out = 20)
welfare_interval <- map(tax_interval, welfare) %>%
  unlist()

tax_graph <- tibble(t = tax_interval, welfare = welfare_interval)

tax_graph %>%
  ggplot(aes(t, welfare)) +
  geom_point() +
  scale_x_continuous("Tax rate") +
  scale_y_continuous("Social welfare") +
  theme_bw()
```



5.

We first calculate the value λ using the formula given (note that the second term in the equation is zero since income effect is zero).

```
k = 10/3

lambda_ftn <- function(t) {
  z_i <- (1-t)^(1/k) * w_i^((k+1)/k)
  b <- sum(t * z_i * weights) - 11800

  g = c()

  for (i in c(1,2,3)) {
    g[i] = 1 / ( (1-t) * z_i[i] + b - (1/(1+k)) * (z_i[i] / w_i[i])^(1+k) )

  }

  result <- sum(weights * g)
  return(result)
}
```

Now use the result we got in the end of Part 1 to get the optimal tax t^*:

$$t^* = \frac{\frac{-\text{Cov}(z_i, \tilde{g}_i)}{\varepsilon^c \cdot \bar{z}}}{1 - \frac{\text{Cov}(z_i, \tilde{g}_i)}{\varepsilon^c \cdot \bar{z}}} \tag{7}$$

Now we create a function to get the naive optimal value of t.

```
k = 10/3
comp_est <- 0.3
w_i = c(2209, 4470, 12986)
weights <- c(0.3, 0.6, 0.1)
g=c()

naive_t <- function(t) {
  lam <- lambda_ftn(t)
  z_i <- (1-t)^(0.3) * w_i^(1.3)
  z_bar <- sum(weights * z_i)
  b <- sum(t * z_i * weights) - 11800
```

```
  for (i in c(1,2,3)) {
    g[i] = 1 / ( (1-t) * z_i[i] + b - (1/(1+k)) * (z_i[i] / w_i[i])^(1+k) )

  }

  g_bar <- g / rep(lam, length(g))


cov_zg <- sum(weights * (z_i - sum(weights * z_i)) * (g_bar - sum(weights *
 ↪  g_bar)))

  result <- ((-cov_zg / (comp_est * z_bar)) / (1- (cov_zg / (comp_est *
 ↪  z_bar))))
}

print(naive_t(0.3))
```

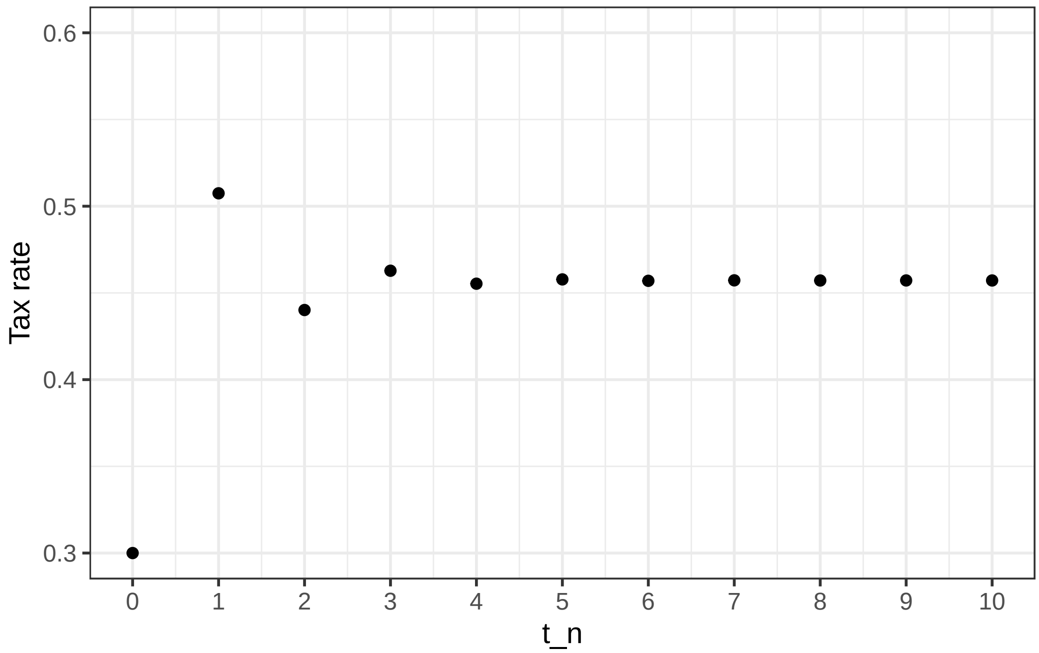
```
[1] 0.5074486
```

Then we do the iteration using tUpdate function:

```
tUpdate <- function(n_iter, initial_value) {
  result <- accumulate(seq_len(n_iter), ~ naive_t(.x), .init = initial_value)
  return(result)
}
print(tUpdate(10, 0.3))
```

```
 [1] 0.3000000 0.5074486 0.4401456 0.4627916 0.4553222 0.4578002 0.4569798
 [8] 0.4572516 0.4571615 0.4571914 0.4571815
```

```
# plot the result
data <- tibble(t = 0:10, tax = tUpdate(10, 0.3))

data %>%
  ggplot(aes(t, tax)) +
  geom_point() +
  scale_y_continuous(name = "Tax rate", limits = c(0.3, 0.6)) +
  scale_x_continuous(name = "t_n", breaks = 0:10) +
  theme_bw()
```



Then you can see that the t_{10} is very similar to the optimum we found in problem 4, implying that it is well converging to the optimum tax rate.