

Problem Set 1

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Part 1: Theory

In this problem set, you will use a Mirrlees-style model to characterize the optimal linear income tax with a lump-sum grant. You'll then solve for the optimal tax in a numerical simulation.

1.

We substitute the budget constraint $c = z - T(z)$ into the utility function:

$$U_i(z - T(z), z) = u(z - T(z)) - v\left(\frac{z}{w_i}\right).$$

The budget constraint is:

$$c = z - T(z) = z - (tz - b) = (1 - t)z + b.$$

Substituting this into the utility function again:

$$U_i((1 - t)z + b, z) = u((1 - t)z + b) - v\left(\frac{z}{w_i}\right).$$

Taking the derivative with respect to z and setting it to zero:

$$(1 - t) \cdot u'((1 - t)z + b) - \frac{1}{w_i} \cdot v'\left(\frac{z}{w_i}\right) = 0.$$

This is the FOC for each agent's choice of income.

2.

Uncompensated Elasticity:

$$\epsilon_i = \frac{\partial z_i}{\partial(1-t)} \cdot \frac{1-t}{z_i}$$

Implicit differentiation of the FOC gives us:

$$u' + (1-t)u'' \cdot \left(z_i + (1-t) \frac{\partial z_i}{\partial(1-t)} \right) - \frac{v''}{w_i^2} \frac{\partial z_i}{\partial(1-t)} = 0$$

Rearranging it by $\frac{\partial z_i}{\partial(1-t)}$ and using the fact $\left(\frac{v'}{u'}\right)^2 = (1-t)^2 w_i^2$ that gives us:

$$\frac{\partial z_i}{\partial(1-t)} = \frac{w_i^2 u' + (1-t) w_i^2 u'' z_i}{v'' - u'' \left(\frac{v'}{u'}\right)^2}$$

So multiplying left hand side with $\frac{1-t}{z_i}$ and using the FOC condition $w_i(1-t)u' = v'$ gives us the formula for ϵ_i .

Income Effect:

Implicitly differentiating FOC by b gives us:

$$(1-t)u'' \cdot \left[(1-t)w_i \cdot \frac{\partial z_i}{\partial b} + 1 \right] - \frac{v''}{w_i} \cdot \frac{\partial z_i}{\partial b} = 0.$$

Rearranging it by $\frac{\partial z_i}{\partial b}$ and using the fact that $\left(\frac{v'}{u'}\right)^2 = (1-t)^2 w_i^2$ gives us:

$$\left[v'' - u'' \left(\frac{v'}{u'}\right)^2 \right] \cdot \frac{\partial z_i}{\partial b} = (1-t)w_i u''.$$

Then multiplying the partial differentiation by $(1-t)$ and rearranging the equation gives us the formula for η_i .

Compensated Elasticity:

We can easily use slusky equation (elasticity form) to show that compensated elasticity will be same as $\varepsilon_i - \eta_i$. This can be thought of as adjusting for the income effect from the uncompensated elasticity. This is because to compensate for the loss from affecting unit tax t , we would have to change lumpsum benefit b to give certain utility level for the consumer.

3. ?

Suppose the government marginally raises the tax t by dt . Then its effect will be sum of substitution and income effect. So it can be written as:

$$dz_i = -\varepsilon_i^c \cdot \frac{z_i}{1-t} dt - \eta_i \cdot \frac{1}{1-t} dt$$

4.

Suppose that the uncompensated elasticity is the same for everyone. Also, the total net government revenues are defined as:

$$\int_i (tz_i - b) dv(i).$$

Then we can see that this marginal raise in tax will affect the net revenue in two ways. First, it will have mechanical effect where (assuming z_i is fixed) the government will earn additional tax by increasing the tax rate: $\int_i (dt \cdot z_i) dv(i)$. But there will be fiscal externality from the behavior component where the tax will affect the behavior of the people and affect their income: $\int_i t \cdot \frac{\partial z_i}{\partial (1-t)} \frac{d(1-t)}{dt} \cdot dt dv(i)$. Rearranging by substituting ε gives us $-\int_i t \cdot \varepsilon \cdot \frac{1}{1-t} z_i dv(i)$.

Then we can easily see that integrating by i will give use the formula in terms of average income \bar{z} . We can then see that adding these two components give us the total effect on net government revenues of marginally raising t by a small amount dt .

5.

The impact is:

$$\frac{dR}{db} = \int_i \left(t \cdot \frac{\partial z_i}{\partial b} - 1 \right) dv(i) = \frac{t}{1-t} \int_i \eta_i dv(i) - 1.$$

Intuitively, change in lump-sum benefit is affecting both the income of the individual (behavioral) and also mechanically raising the income of the person (mechanical).

The size of the total fiscal cost would depend on the aggregate of the income effect. If income effect is high (people respond sensitively to the change in lump-sum benefit), the government can actually gain more from the unit tax as the income increases. But if it is not that elastic, there will be more cost coming due to the direct effect (spending more on grant).

6. ?

We will use the $\frac{\partial \mathcal{L}}{\partial b} = 0$ and result we got from question 5:

$$0 = \frac{\partial \mathcal{L}}{\partial b} = \frac{\partial}{\partial b} \int_i G dv(i) + \lambda \underbrace{\left(\frac{t}{1-t} \int_i \eta_i dv(i) - 1 \right)}_{\text{From result in question 5}}.$$

Rearranging this gives us:

$$\frac{t}{1-t} \int_i \eta_i dv(i) - 1 = - \int_i \overbrace{G' \cdot \frac{\partial U_i}{\partial c}}^{:=g_i} dv(i) \Rightarrow \frac{t}{1-t} \int_i \eta_i dv(i) = \int_i (1 - g_i) dv(i)$$

Like in the class, we can think of left hand side as the net social cost of transferring marginal amount from the individual to the government. On the other hand, the right hand side is

7. ?

The social value of income:

$$g_i^* = g_i + \frac{t}{1-t} \eta_i$$

Using this:

$$\frac{t}{1-t} = - \frac{\text{Cov}(z_i, g_i^*)}{\epsilon^c \bar{z}}$$

8. ?

This aligns with the **many-person Ramsey rule**, stating that tax rates should be set such that the marginal social welfare loss per dollar of revenue is equal across individuals. If g_i^* is decreasing in z_i , higher-income individuals should be taxed more.

Part 2: Numerical application

```
# Load libraries
library(tidyverse)

# Given parameters
t <- 0.3 # Current tax rate
b <- 5000 # Lump-sum benefit

# for simplicity, we will assume that k = 1
k = 1
```

1. ?

The FOC becomes:

lorem

Expressions can be written as (we can just use the result we got in question 2):

$$\varepsilon_i =$$

2.

We apply FOC w.r.t z on individual utility to get:

$$1 - t - \left(\frac{z}{w_i} \right)^k = 0 \Rightarrow w_i = \frac{z}{(1 - t)^{1/k}}.$$

Then by plugging the values mentioned in the question, we can get the implied ability (if we also assume some value for k , we will exactly get the number).

```
implied_ability <- map(c(20000, 50000, 200000), ~ .x / (0.7)) |>
  bind_cols() |>
  rename_with(~ c("low", "middle", "high")) |> print()
```

```
# A tibble: 1 x 3
      low middle   high
  <dbl> <dbl>   <dbl>
1 28571. 71429. 285714.
```

We can also use net government revenue function to get the exogenous expenditures E . We can compute the integration: $\int_i (tz - b) dv(i) = E$.

```
E <- ((3/10) * 20000 * (3/10)) + ((3/10) * 50000 * (6/10)) + ((3/10) * 200000
  ↪ * (1/10)) - 5000
print(paste("Exogenous expenditures (E):", E))
```

```
[1] "Exogenous expenditures (E): 11800"
```

3. ?

4. ?

```
compute_log_social_welfare <- function(zi, p, t, k, E) {
  # Ensure tax rate is within valid range
  if (t < 0 || t >= 1) {
    stop("Tax rate t must be in the range [0, 1).")
  }
  if (k <= 0) {
    stop("Curvature parameter k must be positive.")
  }

  # Total taxable income
  total_income <- sum(zi*p)

  # Compute budget-balancing lump-sum transfer
  b <- total_income * t - E
```

```

# Compute consumption for each individual
c <- (1 - t) * zi + b

# Compute individual utilities
U <- c - (1 / (1 + k)) * (zi)^(1 + k)

# Ensure utilities are positive (to take log safely)
if (any(U <= 0)) {
  warning("Some utility values are non-positive; log is undefined.")
  U[U <= 0] <- NA # Assign NA for undefined values
}

# Compute log-social-welfare
W_log <- sum(log(U), na.rm = TRUE)

# Return results as a list
return(list(Log_Welfare = W_log, Benefit = b))
}

# Example usage

result <- compute_log_social_welfare(c(20000, 50000, 200000), c(0.3, 0.6,
↪ 0.1), 0.7, 1, 11800)
print(result)

```

```

$Log_Welfare
[1] 0

```

```

$Benefit
[1] 27400

```

5. ?