Problem Set 1

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Part 1: Theory

In this problem set, you will use a Mirrlees-style model to characterize the optimal linear income tax with a lump-sump grant. You'll then solve for the optimal tax in a numerical simulation.

1.

We substitute the budget constraint $c = z - T(z)$ into the utilify function:

$$U_i(z - T(z), z) = u\left(z - T(z)\right) - v\left(\frac{z}{w_i}\right).$$

The budget constraint is:

$$c = z - T(z) = z - (tz - b) = (1-t)z + b.$$

Substituting this into the utility function again:

$$U_i((1-t)z + b, z) = u\left((1-t)z + b\right) - v\left(\frac{z}{w_i}\right).$$

Taking the derivative with respect to z and setting it to zero:

$$(1-t) \cdot u'((1-t)z + b) - \frac{1}{w_i} \cdot v'\left(\frac{z}{w_i}\right) = 0.$$

This is the FOC for each agent's choice of income.

2. ?

Uncompensated Elasticity:

$$\epsilon_i = \frac{\partial z_i}{\partial(1-t)} \cdot \frac{1-t}{z_i}$$

Implicit differentiation of the FOC gives us:

$$u' + (1-t)u'' \cdot \left(z_i + (1-t)\frac{\partial z_i}{\partial(1-t)}\right) - \frac{v''}{w_i^2}\frac{\partial z_i}{\partial(1-t)} = 0$$

Rearranging it by $\frac{\partial z_i}{\partial(1-t)}$ and using the fact $\left(\frac{v'}{u'}\right)^2 = (1-t)^2 w_i^2$ that gives us:

$$\frac{\partial z_i}{\partial(1-t)} = \frac{w_i^2 u' + (1-t)w_i^2 u'' z_i}{v'' - u''\left(\frac{v'}{u'}\right)^2}$$

So multiplying left hand side with $\frac{1-t}{z_i}$ and using the FOC condition $w_i(1-t)u' = v'$ gives us the formula for ε_i.

Income Effect:

Implicitly differentiating FOC by b gives us:

$$(1-t)u'' \cdot \left[(1-t)w_i \cdot \frac{\partial z_i}{\partial b} + 1\right] - \frac{v''}{w_i} \cdot \frac{\partial z_i}{\partial b} = 0.$$

Rearranging it by $\frac{\partial z_i}{\partial b}$ and using the fact that $\left(\frac{v'}{u'}\right)^2 = (1-t)^2 w_i^2$ gives us:

$$\left[v'' - u''\left(\frac{v'}{u'}\right)^2\right] \cdot \frac{\partial z_i}{\partial b} = (1-t)w_i u''.$$

Then multiplying the partial differentiation by $(1-t)$ and rearranging the equation gives us the formula for η_i.

Compensated Elasticity:

We can easily use slutsky equation (elasticity form) to show that compensated elasticity will be same as $\varepsilon_i - \eta_i$. This can be thought of as adjusting for the income effect from the uncompensated elasticity. This is because to compensate for the loss from affecting unit tax t, we would have to change lumpsum benefit b to give certain utility level for the consumer.

3.

We will use the definition of the uncompensated elasticity:

$$\varepsilon_i = \frac{\partial z}{d(1-t)} \cdot \frac{1-t}{z}.$$

Then we can rearrange it to get:

$$\begin{aligned}
\frac{dz}{d(1-t)} &= \frac{z}{1-t} \cdot \varepsilon_i && (1)\\
&= \frac{z}{1-t} \cdot (\varepsilon_i^c + \eta_i) && (2)\\
&\Rightarrow dz = \frac{z}{1-t}(\varepsilon_i^c + \eta_i) \cdot d(1-t) = -\frac{z}{1-t}(\varepsilon_i^c + \eta_i) \cdot dt && (3)
\end{aligned}$$

4.

Suppose that the uncompensated elasticity is the same for everyone. Also, the total net government revenues are defined as:

$$\int_i (tz_i - b) dv(i).$$

Then we can see that this marginal raise in tax will affect the net revenue in two ways. First, it will have mechanical effect where (assuming z_i is fixed) the government will earn addtional tax by increasing the tax rate: $\int_i (dt \cdot z_i) dv(i)$. But there will be fiscal externality from the behavior component where the tax will affect the behavior of the people and affect their income: $\int_i t \cdot \frac{\partial z_i}{\partial(1-t)} \frac{d(1-t)}{dt} \cdot dt dv(i)$. Rearranging by substituting ε gives us $-\int_i t \cdot \varepsilon \cdot \frac{1}{1-t} z_i dv(i)$.

Then we can easily see that integrating by i will give use the formula in terms of average income $\bar{z}$. We can then see that adding these two components give us the total effect on net government revenues of marginally raising t by a small amount dt.

5.

The impact is:

$$\frac{dR}{db} = \int_i \left(t \cdot \frac{\partial z_i}{\partial b} - 1 \right) dv(i) = \frac{t}{1-t} \int_i \eta_i dv(i) - 1.$$

Intuitively, change in lump-sum benefit is affecting both the income of the individual (behavioral) and also mechanically raising the income of the person (mechanical).

The size of the total fiscal cost would depend on the sign of the income effect. Income effect is negative as having more income makes people to spend more of leisure (not work). Thus the equation tells us that the total fiscal cost of the lump-sump grant will be greater than \$1 per capita.

6. ?

We will use the $\frac{\partial \mathcal{L}}{\partial b} = 0$ and result we got from question 5:

$$0 = \frac{\partial \mathcal{L}}{\partial b} = \frac{\partial}{\partial b} \int_i G dv(i) + \lambda \underbrace{\left(\frac{t}{1-t} \int_i \eta_i dv(i) - 1 \right)}_{\text{From result in question 5}}.$$

Rearranging this gives us:

$$\frac{t}{1-t} \int_i \eta_i dv(i) - 1 = -\int_i \frac{\overbrace{G' \cdot \frac{\partial U_i}{\partial c}}^{:= g_i}}{\lambda} dv(i) \Rightarrow \frac{t}{1-t} \int_i \eta_i dv(i) = \int_i (1 - g_i) dv(i)$$

Like in the class, we can think of left hand side as the net social cost of transferring marginal amount from the individual to the government. On the other hand, the right hand side is

7.

We will use the fact that in optimal policy, $\frac{\partial \mathcal{L}}{\partial t} = 0$.

$$0 = \frac{\partial \mathcal{L}}{\partial t} \tag{4}$$

8. ?

This aligns with the **many-person Ramsey rule**, stating that tax rates should be set such that the marginal social welfare loss per dollar of revenue is equal across individuals. If g_i^* is decreasing in z_i, higher-income individuals should be taxed more.

Part 2: Numerical application

```
# Load libraries
library(tidyverse)
library(gt)

# Given parameters
t <- 0.3  # Current tax rate
b <- 5000 # Lump-sum benefit
```

1.

The FOC becomes:

$$1 - t - \left(\frac{z_i}{w_i}\right)^k \cdot \frac{1}{w_i} = 0.$$

Then we can rearrange it by z_i: $z_i = (1-t)^{1/k} w_i^{(k+1)/k}$.

Then we can get: $\frac{\partial z_i}{\partial(1-t)} = \frac{1}{k} w_i^{\frac{k+1}{k}} (1-t)^{\frac{1}{k}-1}$.

Expressions for ε_i can be written as (we can just use the result we got in question 2):

$$\varepsilon_i = \frac{1}{k}.$$

Then you can easily see that the income effect η_i is zero as the functional form is not a function of b. This would mean that $\varepsilon_i = \varepsilon_i^c = \frac{1}{k}$. Since compensated elasticity is 0.3, we also get $k = \frac{10}{3}$.

2.

We apply FOC w.r.t z on individual utility to get:

$$1 - t - \left(\frac{z}{w_i}\right)^k \cdot \frac{1}{w_i} = 0 \Rightarrow z_i = w_i^{1+\frac{1}{k}} \cdot (1-t)^{\frac{1}{k}}.$$

Then by plugging the values mentioned in the question, we can get the implied ability (if we also assume some value for k, we will exactly get the number).

```
implied_ability <- map(c(20000, 50000, 200000), ~ (.x / (1-t)^0.3)^(1/1.3))
↪   |>
  bind_cols() |>
  rename_with(~ c("low", "middle", "high")) |> print()
```

```
# A tibble: 1 x 3
    low middle   high
  <dbl>  <dbl>  <dbl>
1 2209.  4470. 12986.
```

We can also use net government revenue function to get the exogenous expenditures E. We can compute the integration: $\int_i (tz - b) dv(i) = E$.

```
E <- ((3/10) * 20000 * (3/10)) + ((3/10) * 50000 * (6/10)) + ((3/10) * 200000
↪   * (1/10)) - 5000

print(paste("Exogenous expenditures (E):", E))
```

```
[1] "Exogenous expenditures (E): 11800"
```

3.

Using the result we got in FOC in question 1, we can derive z_i as:

```
earning_zi <- function(t, w) {
  z = (1-t)^(0.3) * w^(1.3)
  return(z)
}
```

```r
# We also compute earning for each type under t = 0
map(c(2209, 4470, 12986), ~ earning_zi(0, .x)) |>
  bind_cols() |>
  rename_with(~ c("low", "middle", "high")) |>
  print()
```

```
# A tibble: 1 x 3
     low middle    high
   <dbl>  <dbl>   <dbl>
1 22256. 55642. 222596.
```

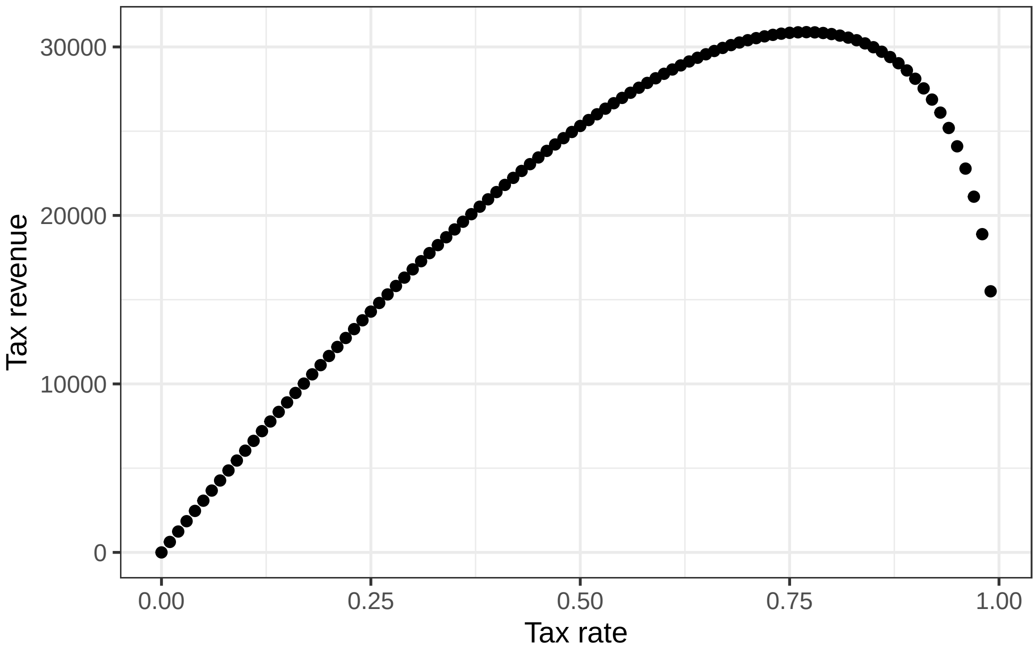
Now we will use the z_i we got above and use it to compute $t \cdot \bar{z}$:

```r
# Define population proportions
weights <- c(0.3, 0.6, 0.1)

tax_revenue <- function(t) {
  if (t<0 || t>1) {
    return(-Inf)
  }
  z_i <- (1-t)^(0.3) * c(2209, 4470, 12986)^(1.3)
  z_bar <- sum(weights * z_i)
  return(t * z_bar)
}

# Optimize the tax rate using 'optimize' function
result <- optimize(tax_revenue, interval = c(0,0.99), maximum = TRUE)
optimal_tax_rate <- result$maximum
print(paste("Optimal tax rate: ", optimal_tax_rate))
```

```
[1] "Optimal tax rate:  0.769236810710384"
```

```r
# plot the graph
tax_interval <- seq(0, 0.99, by = 0.01)
tax_revenue_interval <- map(tax_interval, tax_revenue) %>%
  unlist()

tax_graph <- tibble(t = tax_interval, revenue = tax_revenue_interval)

tax_graph %>%
```

```r
  ggplot(aes(t, revenue)) +
  geom_point() +
  scale_x_continuous("Tax rate") +
  scale_y_continuous("Tax revenue") +
  theme_bw()
```



4.

The social welfare is defined as:

$$\int_i G(U_i)dv(i) = \int_i G\left((1-t)z + b - \frac{1}{1+k}\left(\frac{z}{w_i}\right)^{1+k}\right)dv(i) = \int_i \log\left((1-t)z + b - \frac{1}{1+k}\left(\frac{z}{w_i}\right)^{1+k}\right)dv(i),$$

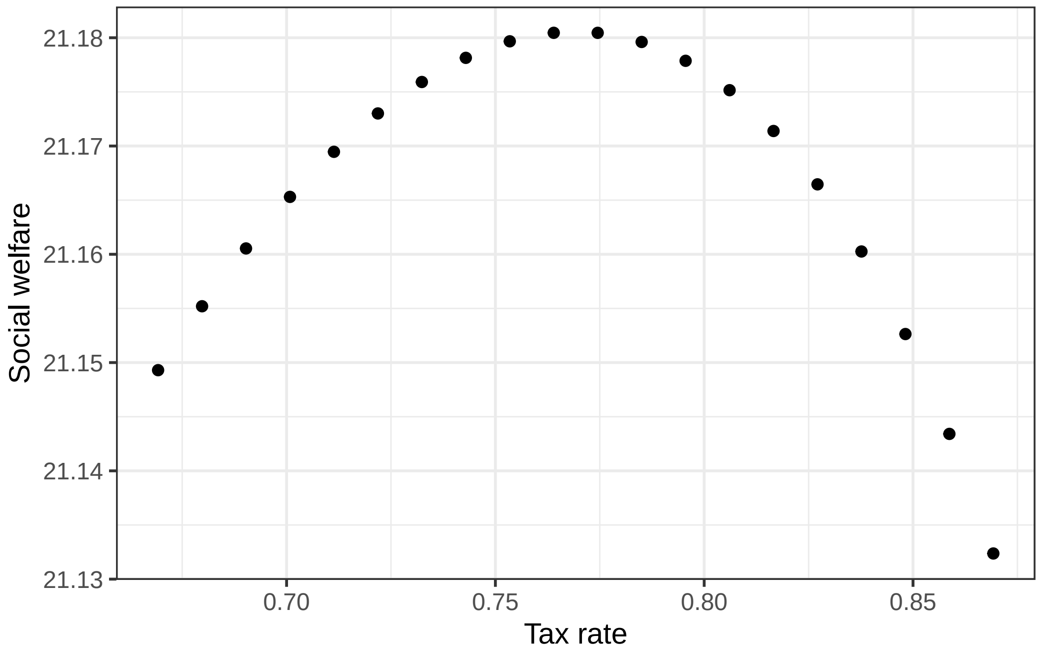
where b is set such that it keeps the exogenous expenditures level $E = 11800$.

```r
w_i = c(2209, 4470, 12986)

welfare <- function(t) {
  if (t<0 || t>1) {
    return(-Inf)
```

```r
  }
  z_i <- (1-t)^(0.3) * c(2209, 4470, 12986)^(1.3)
  b <- sum(t * z_i * w_i) - 11800
  welfare_i <- log((1-t)*z_i + b - (3/13)*(z_i/w_i)^(13/3))
  result <- sum(weights * welfare_i)
  return(result)
}

# Optimize the tax rate using 'optimize' function
opt_welfare <- optimize(welfare, interval = c(0,0.99), maximum = TRUE)
optimal_tax_rate <- opt_welfare$maximum
print(paste("Optimal tax rate: ", optimal_tax_rate))
```

```
[1] "Optimal tax rate:  0.769234566865495"
```

```r
# plot the graph
tax_interval <- seq(optimal_tax_rate - 0.1, optimal_tax_rate + 0.1,
 ↪  length.out = 20)
welfare_interval <- map(tax_interval, welfare) %>%
  unlist()

tax_graph <- tibble(t = tax_interval, welfare = welfare_interval)

tax_graph %>%
  ggplot(aes(t, welfare)) +
  geom_point() +
  scale_x_continuous("Tax rate") +
  scale_y_continuous("Social welfare") +
  theme_bw()
```


21.18
21.17
21.16
21.15
21.14
21.13
Social welfare
0.70
0.75
0.80
0.85
Tax rate

5. ?