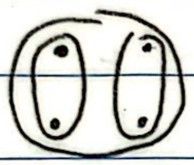
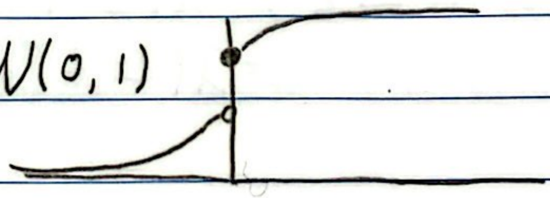
Q1 (a)  $\mathcal{A} = \{\emptyset, \{a,b,c,d\}, \{a,b\}, \{c,d\}\}$

(b) $(a, b]$, or $\{a\} \cup (a+1, a+2)$ etc

(c) $f_n \equiv \frac{1}{n}$, $f \equiv 0$, or $f_n = \begin{cases} 0 & x \in [0, \frac{1}{n}] \\ n & x \in (0, \frac{1}{n}) \end{cases}$

(d) $\frac{1}{2} \circ 0 + \frac{1}{2} \circ \mathcal{N}(0,1)$



(e) Let $\Omega = \{H, T\}$

$X_n = \mathbb{1}\{\omega = H\}$

$X = \mathbb{1}\{\omega = T\}$

Q2 (a) λ the standard Lebesgue measure.

for $\lambda(f^{-1}(E))$ to be well-defined.

$$f^{-1}(E) \in \mathcal{M} \quad \forall E \in \mathcal{M}$$

i.e. f must be measurable

(b) 1. $\varphi(\emptyset) = \lambda(f^{-1}(\emptyset)) = \lambda(\phi) = 0$ ✓

2. $\varphi(E) \geq 0$ as $\lambda(\mathcal{M}) \subseteq [0, \infty]$ ✓

3. $\varphi(\bigsqcup E_i) = \lambda(f^{-1}(\bigsqcup E_i))$

$= \lambda(\bigsqcup f^{-1}(E_i)) = \sum \lambda(f^{-1}(E_i))$

$= \sum \varphi(E_i)$ ✓

so φ is a measure

(c) $\mathcal{L}_\varphi \not\subseteq \mathcal{L}_\lambda$. Consider $f \equiv c$,

then $f^{-1}((-\infty, c-1)) = \emptyset$, so $\varphi((-\infty, c-1)) = 0$

Also $\mathcal{L}_\lambda \not\subseteq \mathcal{L}_\varphi$ with the same f,

$f^{-1}(\{c\}) = \mathbb{R}$, so $\varphi(\{c\}) = \infty$

(d) If $0 \notin E$, then

$$\xi^{-1}(E) = A \cap E.$$

If $0 \in E$, then

$$\xi^{-1}(E) = E \cup A^c \left(= (E \cap A) \cup A^c\right)$$

(e) As is clear from the above, we need A to be measurable.

(f) From the previous parts, we have:

$$\varphi(E) = \begin{cases} \lambda(E \cap A) & \text{if } 0 \notin E \\ \lambda(E \cup A^c) & \text{if } 0 \in E \end{cases}$$

Q3 (a) 1. $\mu_x(\emptyset)=0$

2. $\mu_x(E)\geq 0 \quad \forall E \in 2^{\mathbb{N}}$

3. $\forall E_1 \ldots$ w/ $E_i \cap E_j = \emptyset \; j \neq i$, we have

$\mu_x(\sqcup E_i) = \sum \mu_x(E_i)$

(b) 1. by def'n

2. clear as $0 \leq x$

3. $\mu_x(\sqcup E_i) = \sum_{n \in \sqcup E_i} x^n = \sum_i \sum_{n \in E_i} x^n = \sum_i \mu_x(E_i)$

(c) we need $\mu_x(\mathbb{N}) < \infty$, so

$\sum_{i=0}^{\infty} x^i < \infty$ ie $x<1$

(d) because the σ-alg on which μ_x is defined is $2^{\mathbb{N}}$

(e) Notice that $\mu(\mathbb{N}) \leq \int_{\mathbb{N}} f \, d\mu_x \leq 2\mu(\mathbb{N})$

so a necessary and sufficient condition is that $\mu(\mathbb{N}) < \infty$ ie. $x \in [0,1)$

Scanned with CamScanner

(f) If $x=0$, then $\int f \, d\mu_0 = 2$ clearly.

For other $x<1$ $\int_{\mathbb{N}} f \, d\mu_x = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{2n}$

as the even terms are counted twice.

This yields, as $\sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n$, that

$$\int_{\mathbb{N}} f \, d\mu_x = \frac{1}{1-x} + \frac{1}{1-x^2}$$

Q4 (a) $M_X(t) = \mathbb{E}[e^{tX}]$

$$= \sum_{x=0} e^{-\lambda} e^{tx_i} \left(\frac{\lambda^x}{x!}\right)$$

$$= \sum_{\infty} e^{-\lambda} \frac{1}{x!} (e^t \lambda)^x$$

$$= e^{-\lambda} \sum_{x=0} \frac{1}{x!} (e^t \lambda)^x \quad \text{as } e^y = \sum \frac{1}{x!} y^x,$$

$$= e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t - 1)} \quad \text{as requested.}$$

(b) $\mathbb{E}[X] = \frac{d}{dt} e^{\lambda(e^t-1)} \Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)} \Big|_{t=0}$
$\boxed{= \lambda}$

$$\mathbb{E}[X^2] = \frac{d^2}{dt^2} e^{\lambda(e^t-1)} \Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)} \Big|_{t=0}$$
$$= \lambda + \lambda^2$$

so the variance of X is

$$\mathbb{E}[(X-\lambda)^2] = \mathbb{E}[X^2] - 2\mathbb{E}[X]\lambda + \lambda^2$$
$$= \mathbb{E}[X^2] - \lambda^2$$
$$\boxed{= \lambda}$$

Scanned with CamScanner

(c) $\mathbb{E}[e^{tY}] = \mathbb{E}[e^{t\sum X_i}] = \mathbb{E}[\prod e^{tX_i}]$

as X_i are independent $= \prod \mathbb{E}[e^{tX_i}]$

$= \prod e^{\lambda_i(e^t-1)}$ by (a)

$\Rightarrow \quad M_Y(t) = e^{(\sum \lambda_i)(e^t-1)}$

(d) from (a) we conclude that

Y follows a Poisson distribution w/ param

$\sum \lambda_i$