Upenn Math Camp Part II

Quiz 4

August 5, 2024

The exam will be 2 hours long and be over 100 points. Please answer all questions. If you are not able to answer an item of a question you can continue to the next assuming that the previous item holds.

1. **(45 points)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function. We say f has increasing differences (also known as supermodularity) if $\forall y_2 > y_1$, the difference

$$f(x, y_2) - f(x, y_1)$$

is weakly increasing in x. Assume f is twice differentiable. In this exercise you will show that if a function is supermodular, then $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$ for all $x, y \in \mathbb{R}$.

Fix $(x, y) \in \mathbb{R}^2$. Define the function

$$g(s, h) = [f(x + s, y + h) - f(x + s, y)] - [f(x, y + h) - f(x, y)]$$

(a) (5 points) Show that if f has increasing differences, then $\forall s, h > 0,\ g(s, h) \geq 0$.

Solution:

If $s > 0, h > 0$, then $x + s > x$ and $y + h > y$. Therefore by the definition of increasing differences:

$$f(x + s, y + h) - f(x + s, y) > f(x, y + h) - f(x, y)$$

Subtracting the RHS, we get $g(s, h) \geq 0$.

(b) (10 points) Show that $g(0, 0) = 0, (Dg)_{(0,0)}(v) = 0, \forall v \in \mathbb{R}^2$

Solution:

Substituting into the definition:

$$g(0,0) = (f(x,y) - f(x,y)) - (f(x,y) - f(x,y)) = 0$$

Applying the chain rule, the jacobian of g is:

$$J_{(s,h)} = \left[\frac{\partial f(x+s,y+h)}{\partial x} - \frac{\partial f(x+s,y)}{\partial x}, \frac{\partial f(x+s,y+h)}{\partial y} - \frac{\partial f(x,y+h)}{\partial y}\right]$$

Substituting in $(s,h) = (0,0)$ we get that $J_{(s,h)} = (0,0)$ and therefore $Dg_{(s,h)}(v) = 0$, for all $v \in \mathbb{R}^2$.

[Hint: Use the chain rule to compute jacobian]

(c) (10 points) Compute the hessian matrix H associated with $D^2 g$, evaluated at $(0,0)$.

Solution:

Applying the chain rule again:

$$H_{(s,h)} = \begin{bmatrix} \frac{\partial^2 f(x+s,y+h)}{\partial x^2} - \frac{\partial^2 f(x+s,y)}{\partial x^2} & \frac{\partial^2 f(x+s,y+h)}{\partial x \partial y} \\ \frac{\partial^2 f(x+s,y+h)}{\partial x \partial y} & \frac{\partial^2 f(x+s,y+h)}{\partial y^2} - \frac{\partial^2 f(x,y+h)}{\partial y^2} \end{bmatrix}$$

Evaluating it at $(s,h) = (0,0)$.

$$H_{(0,0)} = \begin{bmatrix} 0 & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial x \partial y} & 0 \end{bmatrix}$$

(d) (10 points) Write down the Taylor expansion $g(s\mathbf{1})$ at point $g(\mathbf{0}_{2\times 1})$, use results in (b) to Show that $\lim_{s\to 0} \frac{g(s,s)}{s^2} = \frac{1}{2} D^2 g_{(0,0)}(\mathbf{1})(\mathbf{1})$.

Where $\mathbf{1}_{2\times 1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution: Define a vector $\mathbf{1}_{2\times 1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. By a first order Taylor expansion

$$g(s,s) = g(0,0) + Dg_{(0,0)}(s\mathbf{1}) + \frac{1}{2} D^2 g_{(0,0)}(s\mathbf{1})(s\mathbf{1}) + R_{(0,0)}(s,s)$$

The residual is a function of the different arguments of the function. Since the first and second terms are zero by the previous part. Then the only term remaining on the right hand side are the second derivative and the residual.

$$g(s,s) = \frac{1}{2} D^2 g_{(0,0)}(s\mathbf{1})(s\mathbf{1}) + R_{(0,0)}(s,s)$$

We can divide by s^2 and use the bilinear property of the second derivative to cancel out the s terms.

$$\frac{g(s,s)}{s^2} = \frac{1}{2} D^2 g_{(0,0)}\left(\mathbf{1}\right)\left(\mathbf{1}\right) + \frac{R_{(0,0)}(s,s)}{s^2}$$

Therefore, since the limit of the second term is zero as a result of Taylor's theorem,

$$\lim_{s\to 0} \frac{g(s,s)}{s^2} = \frac{1}{2} D^2 g_{(0,0)}\left(\mathbf{1}\right)\left(\mathbf{1}\right) + \lim_{s\to 0} \frac{R_{(0,0)}(s,s)}{s^2} = \frac{1}{2} D^2 g_{(0,0)}\left(\mathbf{1}\right)\left(\mathbf{1}\right)$$

(e) (10 points) Compute $D^2 g_{(0,0)}\left(\mathbf{1}\right)\left(\mathbf{1}\right)$ use equation in (c). Show that $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$.

Solution: To conclude notice that if $s, h > 0$ then the differences are increasing, by the result in part one. That means that $g(s,h) \geq 0$. This implies that

$$\lim_{s\downarrow 0} \frac{g(s,s)}{s^2} \geq 0$$

Since the unconstrained limit exists, it is equal to the limit from above. Therefore,

$$D^2 g_{(0,0)}\left(\mathbf{1}\right)\left(\mathbf{1}\right) \geq 0$$

In terms of the hessian, the above expression is equal to $\begin{bmatrix} 1 & 1 \end{bmatrix} H \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is equal to $2\frac{\partial^2 f(x,y)}{\partial x \partial y}$. Putting the results together this implies that term with cross partial derivatives needs to be greater than or equal to zero.

(f) (Bonus Question) Show if a function is twice differentiable and $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$, then it is supermodular. Here we finish proving that supermodularity is equivalent to $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$ for $f \in \mathbb{C}^2$.

Solution: We define function $g(y) = \frac{\partial f}{\partial x}(x,y)$. $\forall y_2 > y_1$, the mean value theorem shows $g(y_2) - g(y_1) = g'(\tilde{y})(y_2 - y_1) = \frac{\partial^2 f}{\partial x \partial y}(x, \tilde{y})(y_2 - y_1)$. Since $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0$, $g(y_2) - g(y_1) = \frac{\partial f}{\partial x}(x, y_2) - \frac{\partial f}{\partial x}(x, y_1) \geq 0$.

Define function $m(x) = f(x, y_2) - f(x, y_1)$. $m'(x) = \frac{\partial f}{\partial x}(x, y_2) - \frac{\partial f}{\partial x}(x, y_1) \geq 0$. So $m(x)$ is increasing in x for all $y_2 > y_1$. $f(x,y)$ is supermodular.

2. **(30 points)** This part focuses on concavity.

(a) (10 points) Write down the two definitions of concave function. Two definitions of quasi concave function.

Solution: See notes.

(b) (10 points) Prove that a function is concave then it is quasi concave.

Solution: See notes.

(c) (10 points) Suppose $f : (a, b) \to \mathbb{R}$ is concave. Fix any x_0 and define

$$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \forall x \in (a, b)\backslash\{x_0\}.$$

Prove $g(x)$ is decreasing.

Solution:

Pick any $x < x'$. There are three cases $x_0 < x < x'$, $x < x_0 < x'$ and $x < x' < x_0$. We show the case $x_0 < x < x'$, other cases are similar. Because

$$x = \frac{x' - x}{x' - x_0} x_0 + \frac{x - x_0}{x' - x_0} x',$$

we have

$$f(x) \geq \frac{x' - x}{x' - x_0} f(x_0) + \frac{x - x_0}{x' - x_0} f(x').$$

But this is equivalent to

$$\frac{f(x') - f(x_0)}{x' - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0}.$$

Note: We did not assume f to be differentiable. It is incomplete to take the derivative. (But it is a good strategy, especially when you feel difficult to prove it without differentiation).

3. **(25 points)** Consider the functions:

$$F : \mathbb{R}_+ \to [0, 1], F \in C^2, F' > 0$$

$$\sigma(v) : \mathbb{R}_+ \to \mathbb{R}_+, \sigma \in C^2, \sigma' > 0$$

$$U_v(b) = (v - b)F(\sigma^{-1}(b))$$

Define $b^*(v) = \text{argmax}_{b \in \mathbb{R}_+} U_v(b)$

(a) (5 points) Assume that there exists a unique $b^*(v) > 0$. Write the first order condition.

Solution:

Using the product rule:

$$\frac{\partial U_v}{\partial b} = (-1)F(\sigma^{-1}(b)) + (v-b)\frac{\partial}{\partial b}F(\sigma^{-1}(b)) = 0$$

Using the chain rule:

$$\frac{\partial U_v}{\partial b} = -F(\sigma^{-1}(b)) + (v-b)F'(\sigma^{-1}(b))\frac{\partial}{\partial b}\sigma^{-1}(b) = 0$$

Using the inverse function theorem:

$$\frac{\partial U_v}{\partial b} = -F(\sigma^{-1}(b)) + (v-b)F'(\sigma^{-1}(b))\frac{1}{\sigma'\left(\sigma^{-1}(b)\right)} = 0$$

(b) (10 points) The above first order condition pins down implicit function $b*(v)$. Assume we have the second order condition that $\frac{\partial^2}{\partial b^2}U_v(b^*) < 0$. Use the implicit function to show that $b^*(v)$ is increasing in v.

Solution:

Since σ is strictly increasing, then σ^{-1} is also strictly increasing.

Apply the implicit function theorem. Let $H(b,v) := \frac{\partial U_v}{\partial b}(b) = 0$. Then

$$B = \frac{\partial H}{\partial b} = \frac{\partial^2}{\partial b^2}U_v(b^*) < 0. \text{ Then } B^{-1} < 0$$

$$A = \frac{\partial H}{\partial v} = \frac{\partial^2}{\partial b \partial v}U_v(b^*) = F'(\sigma^{-1}(b))\frac{\partial}{\partial b}\sigma^{-1}(b) > 0$$

Using the implicit function theorem:

$$\frac{\partial b^*(v)}{\partial v} = -B^{-1}A > 0$$

Therefore $b^*(v)$ is an increasing function of v.

(c) (5 points) Assume in addition that $\sigma(v)$ is a function such that $\forall v \in [0,1], b^*(v) = \sigma(v)$. Show that this implies:

$$\frac{v - \sigma(v)}{\sigma'(v)} = \frac{F(v)}{F'(v)}, \forall v \in [0,1].$$

Solution:

From the first order condition, we have:

$$(v - b^*(v))F'(\sigma^{-1}(b^*(v)))\frac{1}{\sigma'(\sigma^{-1}(b^*(v)))} - F(\sigma^{-1}(b^*(v))) = 0$$

Substituting $b^*(v) = \sigma(v)$, then $\sigma^{-1}(b^*(v)) = v$. The equation simplifies to:

$$(v - \sigma(v))F'(v)\frac{1}{\sigma'(v)} - F(v) = 0$$

Rearranging we get that:

$$\frac{v - \sigma(v)}{\sigma'(v)} = \frac{F(v)}{F'(v)}$$

Alternatively,

$$\sigma(v) = v - \sigma'(v)\frac{F(v)}{F'(v)}.$$

Since $\sigma' > 0, F' > 0, F \geq 0$, then $\sigma(v) \leq v$.

(d) (5 points) Show that $b^*(v) \leq v$. (We can do this without imposing assumption in a-c).

Solution:

For $b > v$, since $v - b < 0$ and $F \in [0, 1]$, we have $U_v(b) < 0$.

Notice that $U_v(v) = 0$ when evaluated at $b = v$. Therefore $b > v$ cannot be the maximum that maximizes $U_v(b)$. Hence it must be that $b^*(v) \leq v$.