Econ 897 (math camp). Part I

Friday Quiz №1

July 12

You should work on the quiz independently. You are not allowed to use any meaterial. You have 2 hours. If you are not able to solve a problem, skip it and move to the next one. You don't have to solve all problems. If you are doing it not in class, please submit your solution on Canvas by Saturday 6PM or send it to me by e-mail (korolkov@sas.upenn.edu). Good luck!

Problem 1 (6 points)**.** Which of the following binary relations are reflexive? Symmetric? Transitive?

(i) $R = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \text{ and } n \text{ has common prime factor}\} \subset \mathbb{N} \times \mathbb{N}$;

(ii) $R = \varnothing \subset \mathbb{N} \times \mathbb{N}$;

(iii) $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m + n = 0\} \subset \mathbb{Z} \times \mathbb{Z}$.

Solution.

(i) Every natural number except 1 has a common prime factor with itself[1], so R is not **reflexive**. Also R is **symmetric** by construction. To see that R is **not transitive** ($3R6$ and $6R2$ but not $3R2$).

(ii) R is **not reflexive** (for every $n \in \mathbb{N}$, $(n, n) \notin \mathbb{R}$. But R is **symmetric** and **transitive** (because it is empty, the first part of implication will never be satisfied so the implication will always be true)[2].

(iii) R is **symmetric**. Indeed, if $m + n = 0$, then $n + m = 0$. R is **not reflexive** ($5 + 5 \neq 0$) and **not transitive** (the same counterexample).

Problem 2 (6 points)**.** Suppose we have two finite sets: A with m elements and B with n elements.

(i) How many different mappings from A to B are there?

(ii) How many of them are bijections?

Solution.

(i) Let's numerate all elements of A. There are n options where the first element can be mapped, n options where the second element can be mapped and so on. Multiplying all options give us n^m different mappings[3].

[1] I didn't penalized if you haven't mentioned 1 case.

[2] That can be a little bit tricky. To understand that $A \to B$ is true if A is false, think of it that way: when is $A \to B$ not true? Only when A is true and B is false. So to prove that R is not symmetric you need to find a pair (m, n) such that $(m, n) \in R$ and $(n, m) \notin R$.

[3] Quite often set of all mapping from A to B is denoted as B^A. That is because it's cardinality (At least for finite sets) is equal to $(\#B)^{\#A}$.

(ii) There are 0 bijections if $m \neq 0$ (because bijections exist only between sets of same cardinality). If $m = n$ there are n options where the first element can be mapped, $n-1$ options where the second element can be mapped and so on. Multiplying all options give us $n!$ different bijections.

Problem 3 (5 points)**.** Does $d(x, y) = (x - y)^2$ define a metric over $\mathbb{R}$?

Solution. No. Let's take $x = 0$, $y = 2$ and $z = 4$. Then $d(x, y) = d(y, z) = 4$ and $d(x, z) = 16$, so triangle inequality is violated.

Problem 4 (11 points, French railway metric)**.** Prove that

$$d(x, y) = \begin{cases} \|x - y\|, \text{ if } x \text{ and } y \text{ are collinear;} \\ \|x\| + \|y\|, \text{ otherwise;} \end{cases}$$

defines a metric over $\mathbb{R}^2$. Here $\|\cdot\|$ is a standard Euclidean norm. Vectors x and y are collinear if for some scalar $\lambda \neq 0$, $x = \lambda y$ (if they lie on the same line).

Solution. Note, that for collinear vectors the metric is just usual Euclidean distance.

Again, it's quite easy to see that this metric is symmetric and is equal to 0 iff the vectors coincide. So we have to check only triangle inequality. Let's check several cases. First cases is when all three vectors are collinear[4]. Then it immediately follows from triangle inequality for Euclidean distance. Second case is that two out of three vectors are collinear. If it is x and y then inequality is

$$\|x - y\| + \|y\| + \|z\| \geqslant \|x\| + \|z\|$$

$$\|x - y\| + \|y\| \geqslant \|x\|$$

which follows directly from norm properties. If x and z are collinear then

$$\|x\| + \|y\| + \|y\| + \|z\| \geqslant \|x - y\|$$

which follows from $\|x - y\| \leqslant \|x\| + \|y\|$. If all three vectors are collinear then

$$\|x\| + \|y\| + \|y\| + \|z\| \geqslant \|x|| + \|z\|$$

which is quite obviously true.

Problem 5 (8 points)**.** Assume we have a sequence (a_n) (in $\mathbb{R}$) such that $a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n \geqslant \cdots \geqslant 0$ and $a_n \to 0$. Sequence (b_n) is defined in the following way: $b_n = \sum_{i=0}^{n} (-1)^i a_i$ Prove that it converges.

Solution. Consider sequences b_{2n} and b_{2n+1}. The first one is non-increasing sequence:

$$b_{2n+2} - b_{2n} = a_{2n+2} - a_{2n+1} \leqslant 0$$

The second one is non-decreasing:

$$b_{2n+3} - b_{2n+1} = -a_{2n+3} + a_{2n+2} \geqslant 0$$

[4]Being collinear is equivalence relation on all non-zero vectors. I omit the case where one of the vectors is zero, because they all are quite trivial.

Also every element of the first sequence is greater than any element of the second sequenceю Indeed, if $m > n$ then

$$b2n - b_{2m+1} \geqslant b_{2m} - b_{2m+1} = a_{2m+1} > 0$$

and if $m \leqslant n$

$$b2n - b_{2m+1} \geqslant b_{2n} - b_{2n+1} = a_{2n+1} > 0$$

That means we have two monotone bounded sequence. They converge, and distance between them converge to 0:

$$b_{2n} - b_{2n+1} = a_{2n+1} \to 0$$

That means they converge to the same number.

Problem 6 (4 points)**.** Assume f and g are continuous functions from X to $\mathbb{R}$. Is $\max(f, g)$ continuous?

Solution. As we discussed in the class, it's enough to check preimage of arbitrary open interval (a, b)[5]:

$$\begin{aligned} \max(f(x), g(x)) \in (a, b) &\Leftrightarrow \big((f(x) \in (a, b) \wedge g(x) \in (-\infty, b)) \vee (g(x) \in (a, b) \wedge f(x) \in (-\infty, b))\big) \\ &\Leftrightarrow \big((x \in f^{-1}(a, b) \wedge x \in g^{-1}(-\infty, b)) \vee \big((x \in g^{-1}(a, b) \wedge x \in f^{-1}(-\infty, b)). \end{aligned}$$

As f and g are continuous functions, $f^{-1}(a, b)$, $g^{-1}(-\infty, b))$, $g^{-1}(a, b)$ and $f^{-1}(-\infty, b)$ are open sets. Then preimage of (a, b) then is a union of two intersections of two open sets. Hence it is open itself.

Problem 7 (6 points)**.** Consider set $X = \{a, b, c, d\}$. Which of the following are topologies on X?

(i) $\{\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$;

(ii) $\{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}\}$;

(iii) $\{\varnothing, X, \{a, c, d\}, \{b, c, d\}\}$.

Solution. Only the first one is topology. In the second one $\{a, b, d\}$ do not belong to topology as it should, in the third one $\{c, d\}$.

Problem 8 (4 points)**.** Show that constant functions ($f : X \to Y$ such that for all $x \in X$ $f(x) = c$ where c is some element of Y) are continuous for any topology on X and any topology on Y.

Solution. Preimage of every open set in Y is either empty (open), either X (open).

[5] $\wedge$ and $\vee$ are logical **and** and logical **or** respcetively.