Math Camp

Hyoungchul Kim

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1 Day 1

Important notes

- Due to housing issue, I might have to miss the first day.
- Due to this problem, I will be manually studying for about 5 to 6 subsections using the notes given by the instructor.

Short notes in class

- Part 1: real calculus.
- Part 2: linear algebra.
- Part 3: probability and optimization.
- Each part lasts two weeks.
- HW assignment every week but not graded.
- Friday: Quiz (2 hours).
- Only thing that matters is math camp exam: Mostly everyone passes if one takes classes.
- Communication done through canvas.
- Part 1 deals with mathematical logic, spaces (metrics, topology), correspondence.

Some notation

$\mathbb{N}$: Natural numbers.
$\mathbb{Z}$: Integer numbers.
$\mathbb{Q}$: Rational numbers.
$\mathbb{R}$: Real numbers.

Binary relation

Binary relation R on set X is a subset of $X \times X$.

1. R is reflexive if: $\forall x \in X$, xRx

2. R is symmetric if: $\forall x, y \in X$, $xRy \implies yRx$
3. R is transitive if: $\forall x, y \in X$, $xRy, yRz \implies xRz$

Def: A binary relation is an equivalence relation if it is reflexive, symmetric and transitive. We denote this as $x \sim y$.

Def: R is antisymmetric if $\forall x, y \in X, xRy$ and $yRx \implies x = y$.

Def: R is partial order on X if it is reflexive, antisymmetric and transitive.

Def: $x, y \in X$ are comparable if xRy or yRx.

Def: A partial order R is a total order if every two elements are comparable.

Def: $f : X \to Y$ is an injection if $\forall a, b \in X, a \neq b \implies f(a) \neq f(b)$.

Def: $f : X \to Y$ is an surjection if $Y = f(X) = \{y \in Y | \exists x \in X : f(x) = y\}$.

Def: $f : X \to Y$ is called bijection if it is both an injection and surjection.

Fact: $f : X \to Y$ is an injection if $\forall y \in Y$, $f^{-1}(\{y\})$ has no more than one element.

Fact: $f : X \to Y$ is an surjection if $\forall y \in Y$, $f^{-1}(\{y\})$ has at least one element.

Fact: $f : X \to Y$ is an bijection if $\forall y \in Y$, $\# f^{-1}(\{y\}) = 1$.

Fact: f have inverse $\iff$ f is a bijection.

Proof. i. f^{-1} exists $\implies$ f is a bijection: $f^{-1}(\{y\}) = \{f^{-1}(y)\}$
. Assume not $\implies$ $\exists a \in f^{-1}(\{y\})\ s.t. a \neq f^{-1}(y)$. But $f(a) = y \implies f^{-1}(f(a)) = f^{-1}(y)$.

ii. f is a bijection $\implies$ f^{-1} exists. Construct g s.t. $g(y)$ is the only element of $f^{-1}(\{y\})$. $f \cdot g : f(g(y)) = y$; $g \cdot f : g(f(x)) = y$. □

Fact: Bijection between two finite sets exists only if $\#A = \#B$.

Def: A and B have equal cardinality if there is a bijection $f : A \to B$.

Fact: Having the same cardinality is an equivalence relation.

Proof. i. reflexivity: id_A is a bijection.

ii. symmetry: $f : A \to B$ is a bijection $\implies$ f^{-1} is also a bijection.

iii. transitivity: $f : A \to B$ and $g : B \to C$ is bijection, then $g \cdot f$ is also bijection. □

Def: A is denumerable if $\#A = \#\mathbb{N}$. (countably infinite) A is countable if it is finite or denumerable. A is uncountable if it is not countable.

Ex: Show $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof. (0,0), (0,1) ...
(1,0), (1,1) ...
Just go diagonal. Then it will get you a mapping that is bijection.

$$f(m,n) = \frac{(m+n)(m+n+1)}{2} + m. \quad \square$$

Corr: $\forall k > 0$, $\mathbb{N}^k$ is denumerable.

Proof. later. $\square$

Prob: Let $A \subset B$. A is infinite and B is denumerable. Then A is denumerable.

Proof. $f : \mathbb{N} \to B$. $f^{-1}(A) \subset \mathbb{N}$ Later solve it. $\square$

Corr: Let $f : A \to B$ be a surjection. If A is denumerable and B is infinite, then B is denumerable.

Proof. Later. $\square$

Def: We say $\#A \leq \#B$ if there is an injection $f : A \to B$. If there is no bijection from A to B, we can say that the cardinality of A is strictly smaller than that of B.

Prob: Prove that $\{0,1\}^{\mathbb{N}}$ is uncountable.

Proof. Later. But just use the fact that if denumerable, its element can be written as a sequence. $\square$

Def: The set of all subsets of A is called power set of A and is noted as 2^A.

Prob: If A is finite $\implies \#2^A = 2^{\#A}$.

Proof. Later. $\square$

Theorem For every $A, \#2^A > \#A$.

Proof. Injection: $x \to \{x\}$ from A to 2^A.
Suppose there is a bijection $f : A \to 2^A$. $B = \{x \in A | x not \in f(x)\} = 2^A$. $\implies \exists y \in A s.t. f(y) = B$.
1. $y \in B \implies y not \in f(y) = B$ by def of B.
2. $y not \in B \implies y \in f(y) = B$ by def of B.
This is contradiction. Thus done. $\square$

Prob: $\#\mathbb{R} = \#2^{\mathbb{N}}$.

Proof. We show that $[0,1] = 2^{\mathbb{N}}$ in cardinality. $\square$

Prob: $\forall n \in \mathbb{N}, \mathbb{R}^n \sim \mathbb{R}$.

Proof. Later. $\square$

Theorem If $\#A \leq \#B$ and vice versa, then the cardinality for both set is same.

Some Exercises

Exercise 1.3.11: Why such partition exists only for equivalence relation? Are all three properties required for existence of such partition?

Proof. car $\square$

Exercise 1.3.20: Let's define a binary relation R on $\mathbb{N}$ in a following way: xRy if y is divisible by x. Is this relation a partial order, a total order or it is not order at all?

Proof. car $\square$

Exercise 1.4.10: Define a bijection in a similar way (using preimages).

Proof. car $\square$

Exercise 1.4.11: Show that $f : X \to Y$ and $g : Y \to Z$ both are bijections, then $g \cdot f : X \to Z$ is a bijection too.

Proof. car $\square$

Exercise 1.4.15: $f : X \to Y$ has an inverse if and only if it is a bijection.

Proof. car $\square$

2 Day 2

PS1 review

PS 1.

Proof. This is false. Let $p(x, y) \implies x + y = 5$. $\square$

PS 2.

Proof. Obvious. $\square$

PS 3.

Proof. (i) $xRy \iff x \neq y$.
(ii) $xRy \iff x < y + 1$.
(iii) $xRy \iff x > y$. $\square$

PS 4.

Proof. Obvious. $\square$

PS extra. Set of all ftns on $\mathbb{R}$ fRg if $\forall x \in \mathbb{R}\, f(x) \leq g(x)$. Is it a total order, partial order or not an order?

Proof. (1) reflexive: fRf
(2) transitive: $fRg\, gRh$.
(3) antisymmetric. $\square$

PS 5.

Proof. No. $\square$

PS 6.

Proof. Check my notes in github. $\square$

PS 7.

Proof. We know that $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$. We just need to show there is bijection between $\mathbb{Q} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. We use the fact there there is bijection from rational number to natural number. $\square$

PS 8.

Proof. (iii) Try to use that $(0,1) \times \mathbb{N} \to \mathbb{R}$ is injection and there is bijection from $(0,1)$ to real number.
(iv) set of all finite seq of $\mathbb{N} = \cup_{n \in \mathbb{N}}\{$all seq of length n$\}$. $\square$

PS 9.

Proof. Later. $\square$

PS 10.

Proof. X is set of a possible bijections from one countable set to another.
(i) Sps A and B have different cardinality $\implies$ zero.
(ii) If they have same cardinality m : $m!$.
(iii) If they have same cardinality to $\mathbb{N}$: continuum.
Let X be set of all bijections from natural number to natural number.
Need to show that $f : 2^{\mathbb{N}} \to X$ and vice versa is injection. $\square$

PS 11.

Proof. No.
i. A is finite. Then 2^A is finite.
ii. If A is infinite. $A \sim \mathbb{N} \implies \#2^A > \mathbb{N}$ $\square$

PS 12.

Proof. i. X is finite. Then $X = \{0, 1, \ldots, n\}$. Later. Just remember to move them or shift them. $\square$

Short notes in class

Def: Vector space over R is a nonempty set V on which two binary operations (scalar multiplication and vector addition is defined). It has following properties:

1. $u + (v + w) = (u + v) + w$.

2. $u + w = w + u$.

3. $\exists 0 \in V s.t. \, u + 0 = u$.

4. $\exists - u \in V s.t. \, u + (-u) = 0$.

5. $\lambda(\mu v) = (\lambda\mu)v$.

6. $1 \cdot u = u$.

7. $(\lambda + \mu) \, u = \lambda u + \mu u$.

8. $\lambda(u + v) = \lambda u + \lambda v$.

Def: V is a v.s. Then norm can be defined as $|| \cdot ||: V \to \mathbb{R}_+$.

1. $|| \, u \, ||= 0 \iff v = 0$.

2. $| \, \alpha v \, ||=| \, \alpha \, | \cdot || \, v \, ||$.

3. $|| \, u + v \, ||\leq|| \, v \, || + || \, u \, ||$.

Ex. Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We can define norm as $|| \, x \, ||= \sqrt{\sum_{i=1}^n X_i^2}$.

Note. Remember that there are many ways to construct norms.

Def: A metric on a set X is a function

$$d : X \times X \to \mathbb{R}_+, \forall x, y \in X.$$

1. $d(x, y) = d(y, x)$.

2. $d(x, y) = 0 \iff x = y$.

3. $d(x, z) \leq d(x, y) + d(y, z)$.

Def: A metric on a set X is a function

$$d : X \times X \to \mathbb{R}_+.$$

s.t. $\forall x, y \in X$,

1. $d(x, y) = d(y, x)$.

2. $d(x, y) = 0 \iff x = y$.

3. $d(x, y) \leq d(x, y) + d(y, z)$.

Def: A pair (X, d) is called a metric space.

Ex: Trivial metric is $d(x, y)$ where it is 0 if $x = y$ and 1 otherwise.

Ex: Normed vector space $(V, || \cdot ||)$. In this case, $d(x, y) =|| \, x - y \, ||$ induces a metric on it.

Def: A sequence is basically a function $f : \mathbb{N} \to X$ and we write it as $(x_n) \in X$.

Def: A cauchy sequence is a sequence such that: $\forall \varepsilon > 0, \exists N, s.t. \forall n, m > N, d(x_n, x_m) < \varepsilon$.

Def: A sequence (x_n) converges to x if

$$\forall \varepsilon > 0, \exists N : \forall i > N, d(x_n, x) < \varepsilon.$$

We denote this as $X_n \to x$; $\lim_{n\to\infty} x_n = x$.

Note: The convergence is unique. Assume it is not unique and the sequence converge to x and y. Then take $\varepsilon = \frac{d(x,y)}{3}$. There will be some N large enough for some i s.t. $d(x_i, x) < \varepsilon, d(x_i, y) < \varepsilon)$. This implies that $d(x, y) < 2\varepsilon$. But this means $d(x, y) < 2\varepsilon = \frac{2}{3} d(x, y) \implies$ contradiction.

Prop: Every converging sequence is a cauchy sequence. But note that vice versa does not hold generally. Think about $x_n = \frac{1}{n}$ in (0,1).

Theorem In $\mathbb{R}$, every cauchy seq. converges.

Def: (X, d) is complete if any cauchy seq. converge (e.g. $\mathbb{R}, \mathbb{R}^n$ with euclidean metric).

Ex: $\mathbb{Q}$ is not complete space. Take any seq. converging to square 2.

Prop: $C\left([a, b]\right)$ with sup metric is a complete space.

Proof. Let f_n be a cauchy seq. For $\forall x \in [a, b], \mid f_n(x) - f_m(x) \mid \leq d(f_n, f_m) = \max \mid f_n(x) - f_m(x) \mid . \implies \forall x \in [a, b], f_n$ is a cauchy seq.

Define f s.t. $\forall x \in [a, b], f(x) = \lim_{n\to\infty} f_n(X) \implies f_n \to f \ldots.$ □

Def: Mapping $A : X \to X$ is a contraction if $\exists \lambda \in [0, 1)$ s.t. $\forall x, y \in X$ $d\left(A(x), A(y)\right) \leq \lambda \cdot d(x, y)$.

Def: $x \in X$ is a fixed point of $A : X \to X$ if $A(x) = x$.

Theorem (Banach fixed-point thm) A is a contraction on complete (X, d). Then A has unqiue fixed point $x^* \in X$.

Bellman eq. $V(x) = max_a\left(u(a, x) + \beta V(x')\right),\ X' = g(a, x)$. $A : f \to max_a\left(u(a, x) + \beta f(x')\right)$.