Penn Math Camp Part II

Quiz 3

July 28, 2024

The exam is closed books, 2 hours long, **2pages, 3 questions, 100 points in total.** If you skip one subquestion, you can continue to the next one assuming that the previous result holds.

1. (40 points) This question studies projection matrix. Let A be an $m \times n$ matrix.

   1. (5 points) Write down the definition of a projection matrix.

   Solution: Let A be an $m \times n$ matrix. Then the matrix P is a projection matrix onto $Im(A)$ if for all $\boldsymbol{z} \in \mathbb{R}^m$,

   $$P\boldsymbol{z} \in Im(A) \subseteq \mathbb{R}^m, \quad (I - P)\boldsymbol{z} \in Orthog(A) \subseteq \mathbb{R}^m.$$

   2. (10 points) Prove A^tA is invertible if and only if A is full rank.

   Solution:

   We will show that $Ker(\boldsymbol{X}^t\boldsymbol{X}) = Ker(\boldsymbol{X})$. Consequently, either both matrices are full rank or neither of them is.

   ($\Longleftarrow$) Suppose that $\beta \in Ker(\boldsymbol{X})$, then $\boldsymbol{X}\beta = \mathbf{0}_{n\times 1}$. That means that $(\boldsymbol{X}^t\boldsymbol{X})\beta = \mathbf{0}_{k\times 1}$ and that $\beta \in Ker(\boldsymbol{X}^t\boldsymbol{X})$.

   ($\Longrightarrow$) Suppose that $\beta \in Ker(\boldsymbol{X}^t\boldsymbol{X})$ then $\boldsymbol{X}^t\boldsymbol{X}\beta = \mathbf{0}_{m\times 1}$. This also means that $\beta^t\boldsymbol{X}^t\boldsymbol{X}\beta = 0_{1\times 1} = (\boldsymbol{X}\beta)^t(\boldsymbol{X}\beta) = ||\boldsymbol{X}\beta||^2$. We know that a norm is equal to zero if and only if the vector is zero. Therefore $\boldsymbol{X}\beta = \mathbf{0}_{n\times 1}$ and $\beta \in Ker(\boldsymbol{X})$.

   3. (15 points) If A is an $m \times n$ full rank matrix, prove $P = A(A^tA)^{-1}A^t$ is a projection matrix onto $Im(A)$.

Solution:

(a) First, we show that the matrix is symmetric and idempotent.

i. Symmetric: $P^t = (A(A^tA)^{-1}A^t)^t = A((A^tA)^{-1})^tA^t = A(A^tA)^{-1}A^t = P$.

ii. Idempotent: $PP = A(A^tA)^{-1}A^tA(A^tA)^{-1}A^t = A(A^tA)^{-1}A^t = P$.

(b) Second, we show that $Im(P) = Im(A)$.

i. $Im(P) \subseteq Im(A)$: Let $\boldsymbol{x} \in \mathbb{R}^m$. Therefore $P\boldsymbol{x} = A\boldsymbol{z}$, where $\boldsymbol{z} = (A^tA)^{-1}A^t\boldsymbol{x} \in \mathbb{R}^n$, is contained in $Im(A)$.

ii. $Im(A) \subseteq Im(P)$: Suppose that $\boldsymbol{z} \in Im(A) \subseteq \mathbb{R}^m$, then there exists a $\boldsymbol{x} \in \mathbb{R}^n$ such that $A\boldsymbol{x} = \boldsymbol{z}$. Then

$$\begin{aligned}
P\boldsymbol{z} &= A(A^tA)^{-1}A^t\boldsymbol{z} && \text{(Substituting definition of } P\text{)}\\
&= A(A^tA)^{-1}A^tA\boldsymbol{x} && \text{(Since } \boldsymbol{z} \in Im(A)\text{)}\\
&= A\boldsymbol{x} && \text{(Cancelling out terms)}\\
&= \boldsymbol{z} && \text{(Plugging-in definition of } \boldsymbol{z}\text{)}
\end{aligned}$$

That means that $\boldsymbol{z} \in Im(P)$. Therefore, $Im(A) \subseteq Im(P)$.

4. (10 points) Prove $P = A(A^tA)^{-1}A^t$ is the unique projection matrix onto $Im(A)$.

Solution: Suppose that there exist two matrices P, P' such that for all $\boldsymbol{x} \in \mathbb{R}^m$, then $P\boldsymbol{x}, P'\boldsymbol{x} \in Im(A)$ and $(I-P)\boldsymbol{x}, (I-P')\boldsymbol{x} \in Orthog(A)$. Notice that $P\boldsymbol{x} - P'\boldsymbol{x} = (P-P')\boldsymbol{x} \in Im(A)$, and $(I-P')\boldsymbol{x} - (I-P)\boldsymbol{x} = (P-P')\boldsymbol{x} \in Orthog(A)$. That is, $(P-P')\boldsymbol{x} \in Im(A) \cap Orthog(A)$. Since $Im(A) \cap Orthog(A) = \{0\}$, we know it must be that $(P-P')\boldsymbol{x} = 0$, i.e. $P\boldsymbol{x} = P'\boldsymbol{x}$. Since $\boldsymbol{x}$ is arbitrary, then $P = P'$. To prove this set $\boldsymbol{x} = \boldsymbol{e}_j$ (an elementary basis vector) and use the fact that $P\boldsymbol{e}_j = p_j = p'_j = P'\boldsymbol{e}_j$ for $j \in \{1, \ldots, m\}$, where p_j, p'_j are the j^{th} columns of P, P', respectively.

2. (40 points) This question focuses on the eigenvalues and eigenvectors of a matrix.

1. (15 points) Prove that a symmetric matrix is positive definite if and only if all the eigenvalues are greater than 0, i.e., $x^tAx > 0, \quad \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}_{n\times 1}\} \iff \lambda_i > 0, \quad \forall i \in \{1, 2, ..., n\}$

Solution: ($\Longleftarrow$) Assume the corresponding vectors are $\{v_1, v_2, ..., v_n\}$, since $x^tAx > 0, \quad \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}_{n\times 1}\}$, $v_i^tAv_i = v_i^t\lambda_iv_i = \lambda_iv_i^tv_i = \lambda_i||v_i||^2 > 0$. So $\lambda_i > 0$ holds for all i.

($\Longrightarrow$) Since A is symmetric, there exists an orthogonal matrix Q such that $A = Q^t \Lambda Q$. Since Q is full rank(Q is invertible), $y = Qx \neq 0$ if $x \neq 0$. So $\forall x \in \mathbb{R}^n \setminus \{\mathbf{0}_{n\times 1}\}$, $x^t A x = x^t Q^t \Lambda Q x = y^t \Lambda y = \sum \lambda_i y_i^2 > 0$. So A is positive definite.

Consider the matrix

$$A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

2. (15 points) Compute the eigenvalues of A, and corresponding eigenvectors.

Solution:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -2 \\ 1 & \lambda - 3 \end{pmatrix} = \lambda(\lambda - 3) + 2$$
$$= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

Therefore, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

To compute the eigenvectors, consider $(\lambda I - A)v = 0$. Plug in the eigenvalues, we can find that the eigenvectors corresponding to $\lambda_1 = 1$ is of the form

$$v_1 = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

for any α, and the eigenvectors corresponding to $\lambda_2 = 2$ is of the form

$$v_2 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

for any α.

3. (10 points) Let $x = (3, 2)^T$. Compute $A^{10}x$.

Solution: We could write x as a linear combination of the eigenvectors:

$$x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = v_1 + v_2.$$

Therefore,

$$A^{10}x = A^{10}(v_1 + v_2) = \lambda_1^{10} v_1 + \lambda_2^{10} v_2 = 1^{10} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^{10} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1026 \\ 1025 \end{pmatrix}.$$

3. (20 points) Separating hyperplane.

   1. (5 points) Consider a nonempty set $D \subseteq \mathbb{R}^m$. Let $b \notin D, b \in \mathbb{R}^m$. State the conditions we need for D so that we have the following strict separating hyperplane theorem:

   $$\exists p \in \mathbb{R}^m, p \neq 0, w \in \mathbb{R}, p^T y < w, \forall y \in D, p^T b > w$$

   Solution: D is a closed, convex set.

   2. (5 points) Suppose that A is an $m \times n$ matrix. Define

   $$D = \{Ax | x \in \mathbb{R}^n, x \geq 0\}.$$

   Pick one of the conditions you list above and show that the set D indeed satisfies it.

   Solution:

   We show D is convex. Let $d_1, d_2 \in D$. Then $\exists x_1, x_2 \geq 0$ s.t. $Ax_1 = d_1, Ax_2 = d_2$. Then $\lambda d_1 + (1-\lambda)d_2 = \lambda A x_1 + (1-\lambda)Ax_2 = A(\lambda x_1 + (1-\lambda)x_2)$. Since $\forall \lambda \in [0,1], \lambda x_1 + (1-\lambda)x_2 \geq 0$, then the vector is in the set, and we showed that D is convex.

   3. (10 points) Show that in this case, we also have $p^T y \leq 0, \forall y \in D$.

   Solution:

   First of all, we know that $0 \in D$. Therefore, $p^T(0) = 0$. This means that $w > 0$.

   Then suppose that $p^T y > 0$. Then $\forall w \geq 0, \exists \alpha \in (0, \infty)$ such that $\alpha p^T y > w$.

   Rearranging, $p^T(\alpha y)$. Since $\tilde{y} = \alpha y \in D$ by definition, then $\exists \tilde{y} \in D$ such that $p^T \tilde{y} > w$, which is a contradiction. Therefore $p^T y \leq 0, \forall y \in D$.