Econ 897 (math camp). Part I

Problem set №2. Solutions

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Artemii Korolkov

Problem 1. Let define the following subset of $C([a,b])$

$$X = \{f \in C([a,b]) | f(a) = f(b)\}$$

Is it vector space (over $\mathbb{R}$)?

Solution. Yes.

Problem 2. For which $p > 0$ the following formula

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

define a norm over $\mathbb{R}^n$?

Hint. x^p is convex for $p \geqslant 1$.

Solution. It's easy to see that the first two properties are satisfied. Indeed, $\|x\|_p = 0$ if and only if all components $x_1, \cdots x_n$ are 0. The homogeneity is quite easy to check.

The last property is a little bit harder: it holds for $p \geqslant 1$ and do not hold for $p \in (0,1)$. There are several ways to show it, one is directly applying Minkowski inequality, another is using convexity of $f(x) = x^p$ for $p \geqslant 1$.

Let's do the second way. We know that for convex function f and $\alpha_1 + \alpha_2 = 1$

$$f(\alpha_1 x + \alpha_2 y) \leqslant \alpha_1 f(x) + \alpha_2 f(y)$$

That implies that

$$(x_k + y_k)^p = \left(\alpha_1 \frac{x_k}{\alpha_1} + \alpha_2 \frac{y_k}{\alpha_2}\right)^p \leqslant \alpha_1 \frac{x_k^p}{\alpha_1^p} + \alpha_2 \frac{y_k^p}{\alpha_2^p}$$

Combining we get:

$$\|x+y\|_p{}^p \leqslant \alpha_1 \frac{\|x\|_p^p}{\alpha_1^p} + \alpha_2 \frac{\|y\|_p^p}{\alpha_2^p}$$

Taking $\alpha_1 = \frac{\|x\|_p}{\|x\|_p + \|y|_p}$ and $\alpha_2 = 1 - \alpha_1$ we will get

$$\|x+y\|_p{}^p \leqslant (\alpha_1 + \alpha_2)(\|x\|_p + \|y|_p)^p = (\|x\|_p + \|y|_p)^p$$

which is equivalent to

$$\|x+y\|_p \leqslant \|x\|_p + \|y|_p$$

When $p < 1$ the counterexample with $x = (1,0)$ and $y = (0,1)$ will work.

Bonus question. Does this norm have a limit when $p \to +\infty$? Is this limit a norm?

Solution. When $p \to +\infty$, it terns to sup-metric. Indeed, let for some k

$$|x_k| = \max\{|x_1|, \cdots, |x_n|\}$$

Then

$$\|x\|_p = |x_k| \left(\left(\frac{|x_1|}{|x_k|} \right)^p + \cdots + 1 + \cdots + \left(\frac{|x_n|}{|x_k|} \right)^p \right)^{1/p}$$

As $|x_n| \geqslant |x_k|$ each fraction goes to 0, so overall sum goes to 1. Then $\|x\|_p \to |x_k|$.

Problem 3. Show that if d is metric over some space X, then $\frac{d}{1+d}$ is also a metric over X.

Solution. Given metric is symmetric if and only if d is symmetric. Same with being equal to 0. We have to prove only triangle inequality

$$\frac{d(x,z)}{1+d(x,z)} \leqslant \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$$\begin{aligned} d(x,z) + d(x,z)d(x,y) + d(x,z)d(y,z) + d(x,z)d(x,y)d(y,z) &\leqslant \\ \leqslant d(x,y) + d(x,z)d(x,y) + d(x,y)d(y,z) + d(x,z)d(x,y)d(y,z) &+ \\ + d(y,z) + d(y,z)d(x,y) + d(x,z)d(y,z) + d(x,z)d(x,y)d(y,z) & \end{aligned}$$

$$d(x,z) \leqslant d(x,y) + d(y,z) + d(x,y)d(y,z) + d(y,z)d(x,y) + d(x,z)d(x,y)d(y,z)$$

Applying triangle inequality to this will result into only non-negative parts on RHS remaining.

Problem 4. Prove the following theorem:

Theorem (Squeeze theorem)**.** *Let (x_n), (y_n) and (z_n) be three sequences in $\mathbb{R}$ such that for almost all n (for all except the finite number) the following is true:*

$$x_n \leqslant y_n \leqslant z_n$$

If

$$\lim x_n = \lim z_n = a$$

then y_n also converges a

Solution. Quite naturally, if almost all elements of (x_n) and (z_n) lies in any ϵ-neighborhood of a, then it means that almost all elements of y_n lies there.

Problem 5. Prove that every monotone bounded sequence in $\mathbb{R}$ converges.

Hint. Any bounded set in $\mathbb{R}$ has sup and inf.

Solution. WLOG we can prove for increasing sequence. Then use the property that any bounded set in $\mathbb{R}$ has a sup and show that this sup will be a limit of a sequence.

Problem 6. Is space X from problem 1 complete (with metric induced by sup norm)?

Solution. Yes. Limit of each Cauchy sequence will also lie in this set.

Problem 7. Assume (X, d) is a complete metric space. Can there exist a mapping $A : X \to X$ such that for all $x, y \in X$

$$d(A(x), A(y)) < d(x, y)$$

and it doesn't have a fixed point?

Solution. It can. The key fact that doesn't allow us to use Banach fixed point theorem is that here the «ratio of contraction» can be infinitely close to 1. Let's consider the following example:

$$A(x) = x + \frac{1}{x}$$

and $X = [1, +\infty)$. Then it's easy to see that $A(x) > x$ for all x so it doesn't have a fixed point. Now we have to check that is contraction. For $y > x$ the distance between images is

$$A(y) - A(x) = y + \frac{1}{y} - x - \frac{1}{x} = (y - x) + \frac{x - y}{xy} = (y - x)\left(1 - \frac{1}{xy}\right)$$

$\left(1 - \frac{1}{xy}\right)$ is always less than 1, so the distance between two points actually decreases after applying function.