Econ 897 (math camp). Part I

Problem set №4. Solutions

July 15

Problem 1. Cofinite topology on X is defined in following way:

$$\tau = \{A \in 2^X \mid X \setminus A \text{ is finite}\} \cup \{\varnothing, X\}.$$

Show that it is indeed topology.

Solution. The first property is satisfied by construction. To check the second property let's take any set of $A_i \in \tau$ other than X. Then

$$X \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in \mathcal{I}} X \setminus A_i$$

and intersection of any number of finite sets is finite. Now let's take finite number of non-empty[1] sets $A_i \in \tau$. Then

$$X \setminus \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n} X \setminus A_i$$

and union of finite number of finite sets is finite.

Problem 2. Construct homeomorphism between $(0,1)$ and $\mathbb{R}$ (with standard topologies).

Solution. The same function that we used for bijections

$$x \mapsto \arctan x$$

is also continuous and has continuous inverse tan. It establishes homeomorphism between $(-\pi/2, \pi/2)$ and $\mathbb{R}$, and of course $(-\pi/2, \pi/2)$ is homeomorphic to $(0,1)$.

Problem 3. Use the previous problem to show that completeness is not a topological property. There can be d_1 and d_2 on set X such that (X, d_1) is complete, (X, d_2) is not complete, but they induce the same topology.

Hint. We know, that $\mathbb{R}$ is complete and $(0,1)$ is not complete. Use this to construct metric on $\mathbb{R}$ such that there is Cauchy sequence there that do not converge (or metric on $(0,1)$ such that every Cauchy sequence converges to something in $(0,1)$).

Solution. If we take the composition of the homeomorphism above with the metric on $(-\pi/2, \pi/2)$, we get a metric on $\mathbb{R}$:

$$d(x,y) = |\arctan x - \arctan y|$$

With this metric, $\mathbb{R}$ is not complete, though the topology is the same. For example, now $x_n = n$ is a Cauchy sequence.

[1] If one of them is empty, their intersection is empty and hence open.

Problem 4. Show that $\mathrm{Int}(E)$ is the set of points that belong to E with some open neighborhood.

Solution. If $x \in \mathrm{Int}(E)$, then $\mathrm{Int}(E)$ is the neighborhood in E. If x belongs to E with some open neighborhood V, then

$$x \in V \subset \bigcup_{U \subset E, U \in \tau} U = \mathrm{Int}(E)$$

Problem 5. Show that a set without limit points in any topological space is closed.

Solution. The closure of this set is just this set. The closure is closed, so it is closed.

Problem 6. Suppose $x_n \to x$ in a metrical space. Show that

$$\{x_n\}_{n=1}^{\infty} \bigcup \{x\}$$

is compact.

Solution. Take the element of the cover that contains x. After some N, it contains all the points of the sequence. The rest, of course, allows a finite subcover, and that with the first open set are a finite subcover.

Problem 7 (Cantor space)**.** Consider the set $\{0,1\}^{\mathbb{N}}$ (set of all infinite sequence of 0 and 1). Assume each copy $\{0,1\}$ has a discrete topology.

(i) Is $\{0,1\}$ compact?

(ii) What is the set of all open sets in $\{0,1\}^{\mathbb{N}}$ with product topology? What it's cardinality?

(iii) Is $\{0,1\}^{\mathbb{N}}$ compact with product topology?

(iv) Consider a **box topology** on $\{0,1\}^{\mathbb{N}}$: topology with the base of product of **any number** of open sets. What is the set of all open sets in $\{0,1\}^{\mathbb{N}}$ with box topology? What it's cardinality?

(v) Is $\{0,1\}^{\mathbb{N}}$ compact with box topology?

Solution.

(i) Yes (it's finite).

(ii) It is all sets where finite number of elements are fixed, e.g. for some $i_1, \cdots i_n$ and $b_1, \cdots b_n$

$$A = \{a_0 \ldots : a_{i_1} = b_1, \cdots a_{i_n} = b_n\}$$

This is the base of the topology and it has cardinality $\aleph_0$. The cardinality of the whole topology is $2^{\aleph_0}$.

(iii) It is by Tychonoff's theorem.

(iv) This is just discrete topology on $\{0,1\}^{\mathbb{N}}$ and cardinality is $2^{2^{\aleph_0}}$.

(v) No, because it's infinite and the topology is discrete.