Econ 897 (math camp). Part I

Problem set №6. Solutions

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Problem 1. Which of the following sets are connected?

(i) $\{(x,y) \in \mathbb{R}^2 \mid x^2 = y^2 + 1\}$;

(ii) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 > 1\}$;

(iii) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + 4y^2 > 1, x^2 + y^2 \leq 1000\}$;

(iv) $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$;

Solution. Set (i) is not connected; all others are connected.

Problem 2. Show that a topological space X is connected if and only if all continuous functions $f\colon X \to \{0,1\}$ are constant (the topology on $\{0,1\}$ is the discrete topology).

Solution. For any f

$$X = f^{-1}(\{0\}) \bigsqcup f^{-1}(\{1\})$$

If f is not constant and continuous, then we have a disjoint union with two sets that both are nonempty, closed, and open. Thus, X is not connected.

If X is not connected, then $X = A \sqcup B$ for some A, B that are nonempty and open. If we set $f(A) = 0$ and $f(B) = 1$, we obtain a continuous non-constant function.

Problem 3. Show that if for some collection of path-connected sets $\{X_\alpha\}_{\alpha \in A}$

$$\bigcap_{\alpha \in A} X_\alpha \neq \varnothing$$

then

$$\bigcup_{\alpha \in A} X_\alpha$$

is path-connected.

Solution. Take any z in the $\bigcup_{\alpha \in A} X_\alpha$. There is continuous path from it to every point. Combining two paths we will obtain path between any two points. (See the next problem for details on how to combine two paths).

Problem 4. Let's define a following binary relation on X (where X is a topological space): xRy is there exist connected set $E \subset X$ such that $x, y \in E$. Is it equivalence relation? Will it be an equivalence relation if we will replace word «connected» with word «path-connected»? If they both are equivalence relations, how do their equivalence classes are related to each other?

Solution. For connectedness: xRx because set $\{x\}$ is connected, symmetry by construction, transitivity by previous problem.

For path-connectedness: xRx because constant $f : [0,1] \to \{x\}$ is continuous, symmetry by taking composition $f \circ g$ where $g\colon [0,1] \to [0,1]$ ($g\colon \alpha \mapsto 1-\alpha$), transitivity by constructing from f between x and y and g between y and z

$$u(\alpha) = \begin{cases} f(2\alpha) \text{ if } \alpha \le \frac{1}{2}; \\ g(2\alpha - 1) \text{ if } \alpha > \frac{1}{2}; \end{cases}$$

We know that if two points are in one path-connected set, then they are in one connected set. The opposite is necessarily true. So

$$X = \bigsqcup_{\alpha \in A} X_\alpha$$

where each X_α is connected and for each α

$$X = \bigsqcup_{\beta \in B_\alpha} X_{\alpha\beta}$$

where each $X_{\alpha\beta}$ is path-connected.

Problem 5. Show that the path from x to y we constructed for convex spaces is continuous in any normed space.

Solution. Recall that path is $f\colon [0,1] \to V$

$$f\colon \alpha \mapsto (1-\alpha)x + \alpha y = x - \alpha(x-y)$$

It is continuous at every point α_0 as

$$\|x - \alpha(x-y) - x + \alpha_0(x-y)\| = |\alpha_0 - \alpha|\|x-y\|$$

which is less than δ for every

$$|\alpha_0 - \alpha| < \frac{\delta}{\|x-y\|}$$

(or $\delta \in B_{\frac{\delta}{\|x-y\|}}(\alpha_0)$).

Problem 6. Consider X, which is a connected topological space, and a continuous function $f : X \to Y$. Show that that the graph of f is connected. Show that a continuous correspondence with connected domain might have graph that is not connected.

Solution. We have two continuous functions $id : X \to X, f : X \to Y$. The function $X \to X \times Y, (x,x) \mapsto (x, f(x))$ is also continuous, and its image is exactly Gr_f. The image of a connected set is connected.

Consider $\phi : [0,1] \rightrightarrows [0,1], \phi(x) = \{0,1\}$ for any x. It is constant, and thus continuous. But the image consists of two line segments and is not connected.

Problem 7. Are the following correspondences $\phi\colon S \rightrightarrows S$ where $S = [0,2]$ upper hemicontinous? lower hemicontinuous?

(i) $\phi(x) = \begin{cases} \{1\} \text{ for } x < 1 \\ S \text{ for } x \ge 1 \end{cases};$

(ii) $\phi(x) = \begin{cases} \{1\} \text{ for } x \leq 1 \\ S \text{ for } x > 1 \end{cases};$

Solution. Both correspondences are uhc and lhc everywhere except 1. The first one is not lhc at 1, the second one is not uhc at 1.

Problem 8. Check if the following correspondences satisfy the hypotheses of the Kakutani fixed-point theorem. In each case, find a fixed point of the correspondence, if it exists. All correspondences are defined on $S = [0, 2]$.

(i) $\phi(s) = \begin{cases} \left[\frac{3}{2}, 2\right] \text{ for } s < 1; \\ \left[\frac{3}{2}, 2\right] \cup \left[0, \frac{1}{2}\right] \text{ for } s = 1; \\ \left[0, \frac{1}{2}\right] \text{ for } s > 1. \end{cases}$

(ii) $\phi(s) = \begin{cases} \left[\frac{3}{2}, 2\right] \text{ for } s < 1; \\ \left[\frac{1}{2}, 2\right] \text{ for } s = 1; \\ \left[0, \frac{1}{2}\right] \text{ for } s > 1. \end{cases}$

(iii) $\phi(s) = \begin{cases} \left[\frac{3}{2}, 2\right] \text{ for } s < 1; \\ [0, 2] \text{ for } s = 1; \\ \left[0, \frac{1}{2}\right] \text{ for } s > 1. \end{cases}$

(iv) $\phi(s) = \begin{cases} \left[\frac{3}{2}, \frac{3}{2} + \frac{s}{4}\right] \text{ for } s < 1; \\ [0, 2] \text{ for } s = 1; \\ \left[1 - \frac{s}{2}, \frac{1}{2}\right] \text{ for } s > 1. \end{cases}$

(v) $\phi(s) = \begin{cases} (s, 2) \text{ for } s < 2; \\ (0, 2) \text{ for } s = 2. \end{cases}$

Solution. (i) not convex-valued; no fixed points.

(ii) not closed graph; $s = 1$ is a fixed point.

(iii) satisfies, $s = 1$ is a fixed point.

(iv) satisfies, $s = 1$ is a fixed point.

(v) not closed grapg, no fixed points.