Econ 897 (math camp). Part I

Exam

August 22

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You should work on the exam independently. The exam is closed book, so you are not allowed to use any material. This part is supposed to take one hour. If you are not able to solve a problem, skip it and move to the next one. If problem has several questions, each question has the same weight. You can use results from previous questions even if you have not solved them. **You don't have to solve all problems**. Good luck!

Problem 1 (8 points)**.** Let's denote the set of all polynomials with rational coefficients as $\mathbb{Q}[x]$. Then let's denote the set of all roots of this polynomials as $\mathbb{A}$:

$$\mathbb{A} = \{a \mid \exists f \in \mathbb{Q}[x] \colon f(a) = 0\}$$

What is the cardinality of $\mathbb{A}$?

Hint. What is the cardinality of $\mathbb{Q}[x]$?

Problem 2 (10 points)**.** Assume we have a sequence (a_n) (in $\mathbb{R}$) such that

$$a_0 \geqslant a_1 \geqslant \cdots \geqslant a_n \geqslant \cdots \geqslant 0$$

and $a_n \to 0$. Sequence (b_n) is defined in the following way

$$b_n = \sum_{i=0}^{n} (-1)^i a_i$$

Prove that it converges.

Problem 3 (10 points)**.** Let's define metric space (X, d) where $X \subset \mathbb{R}^\infty$ is a set of all sequences $x = (x_1, x_2, \cdots)$ such that $\forall i \ \ |x_i| < 1$ and

$$d(x, y) = \sup_i |x_i - y_i|$$

Consider topology τ on X induced by this metric. Let Y be a set of all sequences $x = (x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} |x_i| < 1$. Naturally Y is a subset of X.

(i) Is X bounded?

(ii) Is X totally bounded?

(iii) Is Y closed?

(iv) Is Y compact?

(v) Is X compact?

Problem 4 (12 points)**.** Recall that $\bar{A}$ is a closure of set A (smallest closed set that contains A). Let f be a function from X to Y where X and Y are topological spaces.

(i) Prove that if for every subset $A \subset X$

$$f(\bar{A}) \subset \overline{f(A)}$$

than f is continuous.

(ii) Prove that if f is continuous than for every subset $A \subset X$

$$f(\bar{A}) \subset \overline{f(A)}$$

Upenn Math Camp Part II

Final

August 22, 2022

If you are not able to answer an item of a question you can continue to the next assuming that the previous item holds.

1. **(20 points)**This part focus on log concavity, an assumption satisfied by many demand functions. A function $f : \mathbb{R} \to (0, +\infty)$ is log concave if function $g(x) = log f(x)$ is concave

   (a) (2 points) Write down the definition of concave function for $g(x)$ and rewrite it using $f(x)$ to get the definition of log concavity.

   (b) (4 points) Show that if f is log concave and twice differentiable, then $f(x)f''(x) - f'(x)^2 \leq 0$. Which means $\frac{f'(x)}{f(x)}$ is decreasing in x. This is another definition of log concavity.

   (c) (3 points) Show that if $h : \mathbb{R} \to (0, +\infty)$ is concave, twice differentiable, then h is log concave.

   In the following two questions, we have all the regular assumptions for demand function: $p > 0$; $q \geq 0$; q is decreasing in p.

   (d) (5 points) Another assumption of demand function is decreasing marginal revenue: denote the inverse demand function $p(q)$, a demand function $q(p)$ exhibits decreasing marginal revenue if the marginal revenue $MR(q) = p(q) + qp'(q)$ is decreasing in q. Show that if the demand function is log concave and twice differentiable, then it is decreasing marginal revenue.

   (e) (6 points) Point out whether the following demand functions are concave? If not, log concave? If not, decreasing marginal revenue?

   - Linear: $q(p) = a - bp$
   - Logit: $q(p) = \frac{e^{-\beta p}}{1+e^{-\beta p}}$

- CES: $q(q) = p^{\alpha} \quad \alpha < -1$

2. **(10 points)** This part focuses on Implicit function Theorem. Consider an economy with exogeneous capital K. The firm's problem is:

$$\max_{L} f(K, L) - wL$$

The market clearing condition is:

$$L = g(w)$$

Suppose $f_L, f_K > 0; f_{LL}, f_{KK} < 0; f_{LK} > 0; g' > 0$.

(a) (3 points) Write down firm's first order condition.

(b) (7 points) If K increases, what will happen to L, w. Suppose $Y = f(K, L)$, what will happen to Y.

3. **(10 points)** Choose one to answer. If you answer two, I will just grade the first one.

(a) Prove that a symmetric matrix is positive definite if and only if all the eigenvalues are greater than 0, i.e., $x^t Ax > 0, \quad \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}_{n\times 1}\} \iff \lambda_i > 0, \quad \forall i \in \{1, 2, ..., n\}$

(b) A is an $m \times n$ matrix, show that $Othog(A) = Ker(A^t)$.

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Final Exam

Instructions

1. This exam is closed book.
2. Please restraint yourself from commenting the solutions with other students.
3. In all True or False questions, if the statement is True, provide a sketch proof. Else, provide a counterexample.
4. Unless stated otherwise, every function involved in each problem is twice differentiable.
5. This exam consists of three questions, each one being of 50 points. Please read the instructions at the beginning of each question.

Optimization (50 Points)

Consider the following optimization problem:

$$\max \ x^\alpha + y^\alpha \quad \text{subject to}$$

$$x + y \leq 1,$$

$$0 \leq x,$$

$$0 \leq y.$$

Consider that $\alpha > 0$.

1. (10 Points) For which values of α we can guarantee that the Strong Duality result will hold? Justify your answer.
2. (10 Points) Compute the Lagrange Dual function $\mathcal{L}^\star(\mu)$ considering that the value of α satisfies the Strong Duality theorem conditions.
3. (15 Points) Using the Lagrange Dual function, solve the problem for $\alpha = 1/3$. Make a graph that shows the feasible region as well as the solution of the problem.
4. (15 Points) Using a graph (and if you want some theory), show that $x(\alpha)$, the optimal value of x as a function of α, is a singleton for all $\alpha < 1$ and that $x(\alpha)$ cannot be zero for any $0 < \alpha < 1$. Now, show that $x(\alpha)$ is not a singleton and it can be zero for $\alpha > 1$. Is this a violation of the Maximum Theorem, which states that $x(\alpha)$ must be upper hemi-continuous?[1] Plot $x(\alpha)$ as a function of α.

[1] Remember that the Maximum Theorem has some assumptions in it, and one of them may not hold in this problem.

Measure Theory (50 Points)

Consider a measure space $(X, \mathcal{S}, \mu)$.

1. (10 Points) Let us consider three sequences of functions f_n, g_n, h_n such that each element of each sequence is integrable and $0 \leq f_n \leq g_n \leq h_n$. Suppose that $f_n \to f, g_n \to g$, and $h_n \to h$. State an assumption(s) and a theorem(s) that will allow us to conclude that $f, g, h \in \mathcal{L}_1$.

2. (20 Points) Consider the sequence f_n given by:

$$f_n(x) = \begin{cases} x - \frac{1}{n} & \text{if} \quad x \in \left[0, 1 - \frac{1}{n}\right] \\ 0 & \text{otherwise.} \end{cases}$$

   Considering this sequence:

   - Draw f_n for $n \in \{1, 2, 3\}$.
   - Compute the limit function of this sequence.
   - True or False: For this sequence, we can be sure that:

$$\int \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} \int f_n d\mu.$$

3. (20 Points) Consider a sequence f_n given by:

$$f_n(x) = \frac{n sin(x)}{1 + nx^{3/2}}.$$

   Use the Dominated Convergence Theorem to show that the limit function of this sequence is integrable in $X = [1, \infty)$ with the Lebesgue measure.[2]

[2]Hint: $sin(y) \leq 1$ for all $y \in \mathbb{R}$. Also:

$$\int_a^b z^m d\lambda = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}.$$

Miscellaneous Problems (50 Points)

Answer FIVE of the following quick problems. If you answer more, I will take 50 points off the exam, for not reading this instruction.

1. (10 Points) Consider a linear program. Show that if $x \in \mathcal{D}_p$ and $y \in \mathcal{D}_d$ are such that $c^T x = b^T y$ then x solves the primal problem while y solves the dual problem.

2. (10 Points) Vivian wants to approximate a function $f : [a, b] \to \mathbb{R}$. She wishes to make the "best" linear approximation according to the $p \in [1, \infty)$ norm, i.e., she wishes to find $\alpha, \beta \in \mathbb{R}$ such that they solve:

$$\min \int_{[a,b]} |f(x) - \alpha - \beta x|^p \, d\lambda.$$

   Vivian knows that this problem is convex, and hence the first order conditions are sufficient. Which assumptions does Vivian need to impose on her problem in order to be able to use this first-order conditions approach?

3. (10 Points) Consider a linear program. Show that if the primal is unbounded then the dual feasible region is empty.

4. (10 Points) Consider that X, Y are two independent random variables, and let $M_Z(t)$ be the moment generating function of Z. Show that:

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

5. (10 Points) Consider the following optimization problem:

$$\max \ \frac{1}{2} x^T A x \quad \text{subject to}$$

$$Bx \leq c,$$

$$x \geq 0,$$

   where A, B are matrices and c is a vector with positive entries. Which conditions do A, B need to satisfy in order for this problem to be concave? Justify your answer.

6. (10 Points) Briefly explain why defining the CDF of a random variable X is enough for us to be able to compute the probability that $X \in B$ as long as B is Borelean.

7. (10 Points) True or False: Consider the following maximization problem:

$$\max \quad f(x) \quad \text{subject to:}$$

$$g(x) \geq 0.$$

   If we assume that f is concave then the set of maxima $\mathcal{D}^\star$ is convex.