Econ 897 (math camp). Part I

Problem set №1

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Problem 1. If formal statement has two different quantifiers for different variables, are they interchangeable? More precisely, are

$$\exists\, y \in Y \;\forall\, x \in X \;\; p(x, y)$$

and

$$\forall\, x \in X \;\exists\, y \in Y \;\; p(x, y)$$

identical? Prove or provide counterexample.

Solution. They are not interchangeable. For example statement

$$\exists y \in \mathbb{R} \; \forall x \in \mathbb{R} \;\; x + y = 5$$

is false and statement

$$\forall x \in \mathbb{R} \; \exists y \in \mathbb{R} \;\; x + y = 5$$

is true.

Problem 2. Using mathematical induction, prove

$$1 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Solution. First let's prove the base of induction. For $n = 1$ the LHS is equal to 1 and the RHS is equal to $2^2/4 = 1$, so they are equal.

Now let's prove step of induction. Assume that

$$1 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Then

$$1 + \cdots + n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2 \frac{n^2 + 4n + 4}{4} = \\ = \frac{(n+1)^2(n+2)^2}{4}$$

That finalizes the proof.

Problem 3. Provide an example of binary relation on $\mathbb{R}$ that is

(i) symmetric, but not transitive or reflexive;

(ii) reflexive, but not symmetric or transitive;

(iii) transitive, but not reflexive or symmetric;

Solution. (i) $xRy \Leftrightarrow x \neq y$;

(ii) $xRy \Leftrightarrow x < y + 1$;

(iii) $xRy \Leftrightarrow x > y$.

Problem 4. Recalling that binary relation on set A is a subset of $A \times A$, show that:

(i) intersection of two transitive binary relations is transitive;

(ii) intersection of two symmetric relations is symmetric;

(iii) intersection of two reflexive relations is reflexive;

Does it mean that intersection of two equivalence relations is an equivalence relation?

Solution. Clearly, if some condition is satisfied in both sets, than it is satisfied in their intersection.

Problem 5. Let's define a binary relation R on the set of all functions on R in a following way:

$$fRg \text{ if } \forall\, x \in \mathbb{R}\ f(x) \leqslant g(x)$$

Is this relation a total order, a partial order or not order at all?

Solution. This relation is a partial order. Clearly it is reflexive and transitive. It is also antisymmetric: if for every point x in $\mathbb{R}$ $f(x) \leqslant g(x)$ **and** $g(x) \leqslant f(x)$, then for every point x in $\mathbb{R}$ $f(x) = g(x)$. That means this functions coincide. It is not total order though: some functions are incomparable, because they intersect. For example, $f(x) = x^2$ and $g(x) = 5$.

Problem 6. Assume there are mappings $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f = \mathbf{id}_X$$

where $\mathbf{id}_X$ is a mapping from X onto X such that $\forall x \in X \mathbf{id}_X(x) = x$ (the identity).

Is it true that $g = f^{-1}$?

Solution. We know that g is inverse if and only if

$$g \circ f = \mathbf{id}_X$$

and

$$f \circ g = \mathbf{id}_Y$$

Let's take $X = \{1, 2\}$ and $Y = \{1, 2, 3\}$ with $f(x) = x$ for all $x \in X$, and $g(1) = 1$, $g(2) = 2$, $g(3) = 2$. Then it's easy to see that $g \circ f$ is indeed identity, but $f \circ g(3) = 2$ so it is not identity.

Problem 7. For mappings $f : X \to Y$ and $g : Y \to Z$ show that

(i) if both of them are injections, their composition $g \circ f$ is also an injection;

(ii) if both of them are surjections, their composition $g \circ f$ is also a surjection.

Solution. A mapping is an injection if and only if preimage of every element is either empty either consists of only one element. A mapping is surjection if and only if preimage of every element is non-empty. Using this definitions makes the rest of proof trivial.

Problem 8. Recall that on lecture we showed that $\mathbb{Q}$ and $\mathbb{N}$ have equal cardinality. Is it true that $\mathbb{Q} \times \mathbb{N}$ and $\mathbb{N}$ have equal cardinality?

Solution. $\mathbb{Q}$ and $\mathbb{N}$ have equal cardinality that means exist bijection between $\mathbb{Q}$ and $\mathbb{N}$. Combining this bijection with $\mathbf{id}_{\mathbb{N}}$ gives us bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{N}$. To prove that $\mathbb{N} \times \mathbb{N}$ and and $\mathbb{N}$ have equal cardinality use the following bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (assuming $0 \in \mathbb{N}$):

$$f(m, n) = \frac{(m+n)(m+n+1)}{2} + m$$

Problem 9. Which of the following sets are countable?

(i) $\mathbb{Q} \times \mathbb{Z}$;

(ii) $\mathbb{Q} \times \mathbb{N} \times Z$;

(iii) $\mathbb{R} \times \mathbb{N}$;

(iv) $2^{\{1,6,9,57\}}$;

(v) set of all finite sequences of natural numbers;

(vi) any set of non-intersecting intervals on the real line.

Solution. Sets are

(i) denumerable;

(ii) denumerable;

(iii) $\sim \mathbb{R}$;

(iv) finite;

(v) denumerable;

(vi) denumerable.

Problem 10. Show (using Cantor–Schröder–Bernstein theorem) that any open and closed interval on real line has the same cardinality.

Solution. It is sufficient to create any injection of one onto each other and vice versa.

Problem 11. What cardinality does the set of all possible bijections from one countable set to another have?

Solution. It has cardinality of the continuum. To prove it we can construct an injection $f : 2^{\mathbb{N}} \to X$. Here X is a set of all bijections of $\mathbb{N}$ onto itself. For $A \subset \mathbb{N}$ and $n \in N$ let's define $f(A)$ as a bijection that leaves set A in place and deranges $\mathbb{N} \setminus A$.

Now we need to construct an injection $\bar{f} : X \to 2^{\mathbb{N}}$. To do this just replace $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$.

Problem 12. Does there exist a set A, such that 2^A has the same cardinality as $\mathbb{N}$?

Solution. No. We showed that every finite set has finite power set and every infinite set can't have less cardinality that $\mathbb{N}$. Then for every infinite set it's power set will have cardinality of at least continuum.

Problem 13. Show that for any countable X the following is true:

$$\#\mathbb{R} = \#(\mathbb{R} \cup X)$$

Solution. Let's replace X with $\mathbb{N}$. Then knowing that there exist a bijection between $\mathbb{N} \cup \mathbb{N}$ and $\mathbb{N}$ we can rewrite the problem as following:

$$\#\left((\mathbb{R} \setminus \mathbb{N}) \cup \mathbb{N}\right) = \#((\mathbb{R} \setminus \mathbb{N}) \cup \mathbb{N} \cup \mathbb{N})$$

to make it obvious.