Econ 897 (math camp). Part I

Problem set №5. Solutions

July 16

Problem 1. Prove that a closed subset of a sequential compact is sequentially compact (in any topological space).

Solution. The limit exists because the larger set is sequential compact. Because the subset is closed, the limit belongs to it.

Problem 2. Prove that the image of a sequential compact under a continuous mapping is a sequential compact (for any topological spaces).

Solution. Suppose $f : X \to Y$ and we have a sequence $y_n \to y$ in $f(X)$. Take any $x_n \in f^{-1}(y_n)$. We have some sequence in X, and it has a converging subsequence. The image of this sequence, of course, converges in Y.

Problem 3. Prove that the direct product of two sequential compacts is sequentially compact (for any topological spaces).

Solution. This is the nice one, because so much easier than the proof for compacts. Suppose $(x_n, y_n) \in X \times Y$, where both X, Y are sequentially compact. Then (x_n) is a sequence in X with a convergent subsequence with limit x. Take the corresponding indexes and get a sequence in Y. It has a converging subsequence in Y, with limit y. Get the corresponding indexes and obtain a converging subsequence of (x_n, y_n) in $X \times Y$, where the limit is (x, y).

Problem 4. Which of the following sets are closed? open? compact?

(i) $\{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2 + 1\}$;

(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 > 1\}$;

(iii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 > 1, x^2 + y^2 \leq 1000\}$;

(iv) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$;

(v) $\left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + \sin^2(z) = 1\right\}$;

Solution.

(i) inverse image of closed set $\{1\}$ under continuous $(x, y) \mapsto x^2 - y^2$, and thus closed; unbounded, so it is not a compact[1];

(ii) inverse image of open set, open; unbounded, so it is not a compact;

(iii) neither closed, nor open; not a compact (because any compact subspace of Haussdorf space is closed);

(iv) inverse image of closed set, closed; bounded, so it is compact;

[1] For an unbounded set we can create a cover with open balls $B_n(0)$ for $n \in \mathbb{N}$. Such cover will do not have a finite subcover, though it will cover all $\mathbb{R}$.

(v) inverse image of closed set, closed; unbounded, so it is not a compact.

Problem 5. Prove that $C([a,b])$ (with sup metric) is not compact using the fact that every continuous function on compact attains it maximum.

Solution. $G\colon f \mapsto f(a)$ is a continuous function from $C([a,b])$ to $\mathbb{R}$ but it has no maximum.

Problem 6 (*)**.** If $f : M \to N$ is continuous and M is compact, then f is uniformly continuous. Prove this result without using sequences.

Hint. Use balls.

Solution. For each $x \in M$ there is $\epsilon(x)$ such $d_N(f(x), f(y)) < \delta/2$ for all $y \in B_{\epsilon(x)}(x)$. We have a cover of X, and thus we have a finite subcover by $B_{\epsilon(x_i)/2}(x_i)$ for some $i = 1, \ldots, n$. If we take

$$\epsilon = \min\left\{\frac{\epsilon(x_1)}{2}, \ldots, \frac{\epsilon(x_1)}{2}\right\}$$

it satisfies the definition of uniform continuity. Suppose $d_M(x,y) < \epsilon$. Then $x \in B_{\epsilon(x_i)}$ for some i, and $d(y, x_i) \leqslant d(y,x) + d(x, x_i) \leqslant \epsilon + \epsilon(x_i)/2 \leqslant \epsilon(x_i)$, and thus $y \in B_{\epsilon(x_i)}(x_i)$ and

$$d_N(f(x), f(y)) \leqslant d_N(f(x), f(x_i)) + d_N(f(x_i), f(y_i)) < \delta/2 + \delta/2 = \delta$$

Problem 7. Prove that an image of Cauchy sequence under an uniformly continuous mapping is a Cauchy sequence. Is it true for a continuous mapping?

Solution. Assume $f\colon M \to N$ and x_n is a Cauchy sequence in M. For any $\delta > 0$ there exists $\varepsilon > 0$ such that for all x, y in M if $d_M(x,y) < \varepsilon$ then $d_N(x,y) < \delta$. x_n is a Cauchy sequence, do $\exists N \forall m, n > N$ $d_M(x_m, x_n) < \varepsilon$. Then combining gives that $\forall \delta > 0$ $\exists N \forall m, n > N$ $d_M(f(x_m), f(x_n)) < \varepsilon$ and $f(x_n)$ is Cauchy.

For continuous mapping it's not true (consider $x_n = n$ and $y_n = \arctan n$, the first one is not Cauchy and the second one is Cauchy).

Problem 8. Prove that an image of a totally bounded set under an uniformly continuous mapping is totally bounded. Is it true for a continuous mapping?

Solution. Assume $f\colon M \to N$ and M is totally bounded. For any $\delta > 0$ take corresponding ε in M. M is totally bounded, so it can be covered by closed balls of radius ε. The image of every such ball is in the closed ball of radius δ.

For continuous mapping it's not true (consider continuous mapping between $(0,1)$ and $\mathbb{R}$).