Econ 897 (math camp). Part I

Friday Quiz №2. Solutions

July 21

You should work on the quiz independently. You are not allowed to use any material. You have 2 hours. Questions are not allowed. If you are not sure about something, explain and make an assumption. If you are not able to solve a problem, skip it and move to the next one. You don't have to solve all problems. If you are doing it not in class, please submit your solution on Canvas by Saturday 6PM or send it to me by e-mail (korolkov@sas.upenn.edu). Good luck!

Problem 1 (8 points)**.** Which of the following sets is open? closed? compact? connected? convex[1]?

(i) $\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < 5\}$;

(ii) $\{(x, y) \in \mathbb{R}^2 \mid y > \frac{1}{x} \text{ and } x + y \geqslant 5 \text{ and } x > 0\}$;

(iii) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqslant 100 \text{ and } x^2 + y^2 + z^2 \geqslant 1\}$;

(iv) $\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leqslant 5 \text{ or } \max(|x|, |y|) \leqslant 4\}$;

Solution.

(i) open as an inverse image of $(-\infty, 5)$ under continuous mapping $f(x, y) = |x|+|y|$; not compact because not closed, connected because path-connected and convex (clear from picture);

(ii) not open (point $(2, 3)$ lies directly on the border and in the set), not closed (point $(5, 1/5)$ lies on the border and not in the set), convex and so path-connected and connected;

(iii) closed (an inverse image of $[1, 100]$ under continuous mapping $f(x, y, z) = x^2 + y^2 + z^2$), compact (closed and bounded), connected (path-connected), not convex (anything between $(0, 0, 1)$ and $(0, 0, -1)$ is not in the set);

(iv) closed (intersection of inverse images of closed sets), compact (closed and bounded), connected (path-connected), not convex (everything between points $(4, 4)$ and $(5, 0)$ is not in the set).

Problem 2 (6 points)**.** Set A in the metric space is said to be bounded if $\forall\, x, y \in A$ and some $L \in \mathbb{R}$ $d(x, y) < L$ – or, in other words, A entirely lies in some open ball of radius L. Provide an example of a bounded closed set A in a metric space such that A is not totally bounded.

Solution. Consider the following subset of $\mathcal{C}([0, 1])$:

$$A = \{f \in \mathcal{C}[(0, 1)] \mid sup|f| \leqslant 1\}.$$

It is closed and bounded. It is not totally bounded. Indeed, assume it is covered by $\{B_{1/2}(f_i)\}_{i=1}^n$. Then take $\{(x_i, y_i) \in \mathbb{R}^2\}_{i=1}^n$ such that $x_i = \frac{i}{n+1}$ and for all i $|f(x_i) - y_i| > \frac{1}{2}$. Continuous functions that go through all this points will not lie in any of the balls.

Problem 3 (6 points)**.** Show that $\bar{E}$ is the set of all points $x \in E$ such that any open neighborhood of x has a non-empty intersection with E.

[1]For path-connectedness and convexity correct picture would be enough.

Solution. If there is an open neighborhood U of x that has empty intersection with E, then $E \subset C = X \backslash U$ and $x \notin C$, and thus x cannot be in

Now suppose any open neighborhood of x has a non-empty intersection with E. We need to show it is in the closure. Suppose it is not. Then there is some $C \supset E$ such that it is closed but does not contain x. But then $X \backslash C$ is a neighborhood of x has an empty intersection with E. This is a contradiction.

Problem 4 (10 points)**.** Let's define metric space (X, d) where $X = [-1,1]^\infty \subset \mathbb{R}^\infty$ is a set of all sequences $x = (x_1, x_2, \cdots)$ such that $\forall i \ \ |x_i| \leqslant 1$ and

$$d(x, y) = \sup_i |x_i - y_i|$$

Consider topology τ on X induced by this metric. Let Y be a set of all sequences $x = (x_1, x_2, \cdots)$ such that $\sum_{i=1}^\infty |x_i| < 1$. Naturally Y is a subset of X.

(i) Is X bounded?

(ii) Is X complete?

(iii) Is Y open?

(iv) Is Y closed?

(v) Is X compact?

Solution.

(i) It is (all elements of X are in $B_{1+\varepsilon}((0, 0, \cdots))$.

(ii) Yes, it is. Here sequences converge only if they converge component-wise. If sequence is Cauchy then it is Cauchy for every component. Then it converges starting from some N (one for all components) for every component because $[-1, 1]$ is complete. That means that it converges overall.

(iii) No, it's not open. Consider any point y in Y. Increase the absolute value of every component by $\varepsilon/2$ (so if it's positive, add $\varepsilon/2$ and if it's negative substract $\varepsilon/2$). The resulting sequence is in ε-neighborhood of y, but the sum $\sum_{i=1}^\infty |y_i|$ is not defined.

(iv) No, it's not closed also. Any sequence such that $\sum_{i=1}^\infty |x_i| = 1$ is a limit point of Y and doesn't lie in it.

(v) No, because it's not totally bounded (we discussed in class why it is not totally bounded). The another interesting way to prove it is to take sequence x_n, where x_n has 0 everywhere except n-th place and 1 there. For every $m \neq n$ $d(x_m, x_n) = 1$, so this sequence won't have converging subsequence. That means X is not sequentially compact and hence not compact.

Problem 5 (8 points)**.** Use connectedness to show that $[0, 1]$ is not homeomorphic to the unit circle

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}.$$

Solution. Assume they are homeomorphic with f being the homeomorphism from $[0, 1]$ to S^1. That will still be homeomorphism if we will restrict it to $[0, 1] \setminus \left\{\frac{1}{2}\right\}$. But $[0, 1] \setminus \left\{\frac{1}{2}\right\}$ is not connected and S^1 without any point is connected. Contradiction.

Problem 6 (6 points)**.** For each of the points in $[0,1]$, describe whether $\phi : [0,1] \rightrightarrows [0,10]$, defined as

$$\phi(x) = \begin{cases} 5, x \leqslant 0.1 \\ \{1,9\}, 0.1 < x < 0.3 \\ [1,9], x = 0.3 \\ 5, 0.3 < x < 0.5 \\ [5,9], x = 0.5 \\ \{10x-4, 14-10x\}, 0.5 < x \leqslant 0.9 \\ 5, 0.9 < x \leqslant 1 \end{cases}$$

is uhc and lhc in it. Does it have closed graph? Plot it. Is $\phi([0,1])$ compact?

Solution.

(i) For $x \in [0,0.1) \cup (0.1,0.3) \cup (0.3,0.5) \cup (0.9,1]$ the correspondence is locally constant and thus continuous.

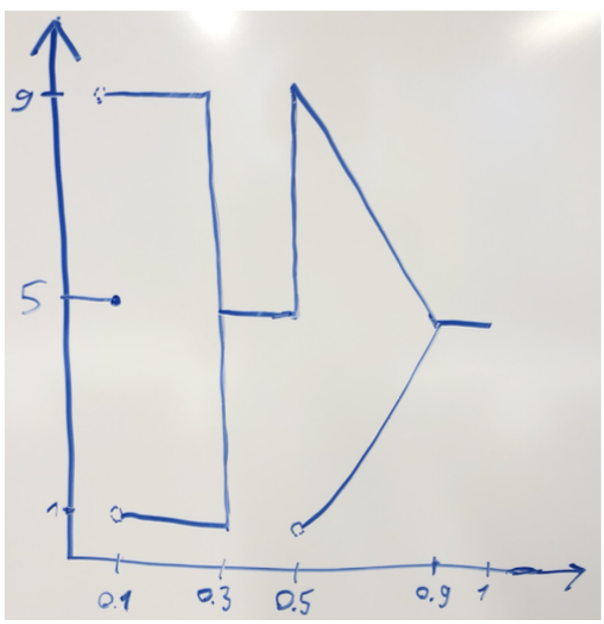
(ii) At $x = 0.1$, it is not uhc and not lhc (and thus not continuous). A small neighborhood around 5 does not intersect/contain points in $\phi(0.1+\epsilon)$.

(iii) At $x = 0.3$, it is uhc but not lhc (and thus not continuous). A small neighborhood around 9 does not intersect points in $\phi(0.3+\epsilon)$.

(iv) At $x = 0.5$, it is not uhc and not lhc (and thus not continuous). A small neighborhood around 5 does not intersect points in $\phi(0.5+\epsilon)$. And a neighborhood $(5-\delta, 9+\delta) \supset \phi(0.5)$ does not contain $10x-4$ for $x = 0.5+\epsilon$.

(v) At $x \in (0.5,0.9]$, it is both uhc and lhc (and thus continuous).

The graph is not closed. The image is compact as it equals $[1,9]$, but this is not ensured by any theorem we covered. See the plot below:



Problem 7 (6 points)**.** Let ϕ be nonempty valued correspondence that maps $[0, 2]$ into $[0, 2]$. Suppose that $\phi(1)$ is a convex set and that ϕ is upper hemicontinuous at $x = 1$. Assume further that $\phi(x) \subset [1, 2]$ for $x < 1$ and $\phi(x) \subset [0, 1]$ for $x > 1$. Show that $x = 1$ is a fixed point of the correspondence.

Hint. This does not follow from Kakutani's theorem. The proof is easy if you understand the concepts.

Solution. Assume $\phi(1)$ is some set A. Then by uhc any open neighborhood V of A should contain some points in $[1, 2]$ (because images of $x - \epsilon$ lies there for any $\epsilon > 0$) and $[0, 1]$ (because images of $x + \epsilon$ lies there for any $\epsilon > 0$). If any open neighborhood of A contain this points, then A also should contain some points in $[1, 2]$ and $[0, 1]$. Then by convexity A contains 1. That means that $1 \in \phi(1)$, so 1 is indeed a fixed point.