Chapter 1

Suggested Solutions

1.1 Introduction to Differentiation

1. Let $f(x) = \begin{cases} x^\alpha \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$. For what values of α is $f(x)$ differentiable at $x = 0$?

Solution. The function is not well defined some some non-integer values of α. For example, if $\alpha = 0.5$, $x^{\alpha-1} = 1/\sqrt{x}$. Therefore, I will restrict this proof to integer values of α.

$$S(x) = \frac{f(x) - f(0)}{x - 0} = \frac{x^\alpha \sin(1/x)}{x} = x^{\alpha-1}\sin(1/x)$$

- If $\alpha = 1$, $S(x) = \sin(1/x)$, which oscillates around for x close to 0.
- If $\alpha < 1$ then $x^{\alpha-1}$ is not defined for some values of α. For example the sequence $x_n = 1/(2\pi n)$ has the property that $S(x_n) = 0$ and if $x_n = 1/(2\pi n + (\pi/2))$, then $S(x_n) = (2\pi n + (\pi/2))^{\alpha-1} \to \infty$. Therefore, it doesn't satisfy the sequential definition of convergence.
- If $\alpha > 1$ it does converge:
$$-x^{\alpha-1} \leq x^{\alpha-1}\sin(1/x) \leq x^{\alpha-1}$$
Since $\lim_{x\to 0} x^{\alpha-1} = 0$, then $\lim_{x\to 0} S(x) = 0$. Then the derivative exists for integer values of α strictly greater than one but not for other integer values of α.

□

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Let $y_0 = g(x_0)$ for some $x_0 \in \mathbb{R}$. Find examples for the following cases when:

   (a) g is differentiable at x_0 and f is not differentiable at y_0;

   (b) g is not differentiable at x_0 and f is differentiable at y_0;

   (c) g is not differentiable at x_0 and f is not differentiable at y_0,

   but $f \circ g(x)$ is differentiable.

Solution. (a) Consider $f(y) = |y|$, $g(x) = x^2$. Consider $x_0 = 0$ and $y_0 = 0$. Then $f \circ g(x) \equiv x^2$, hence differentiable at x_0.

(b) Consider $f(y) = y^2$, $g(x) = |x|$ and $x_0 = 0$.

(c) Consider
$$f(x) = g(x) = \begin{cases} \frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$
Then neither f nor g is continuous at 0. But $f \circ g(x) \equiv x$ which is differentiable.

□

3. (Exercise 11 on page 186, Pugh) Assume that $f : (-1, 1) \to \mathbb{R}$ and $f'(0)$ exists. If $\alpha_n, \beta_n \to 0$ as $n \to \infty$, define the different quotient

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

(a) Prove that $\lim_{n\to\infty} D_n = f'(0)$ under each of the following conditions (Hint: First rewrite this expression in terms of $\frac{f(\beta_n)-f(0)}{\beta_n}$ and $\frac{f(\alpha_n)-f(0)}{\alpha_n}$ and use the sequential definition of the limit.

i. $\alpha_n < 0 < \beta_n$.

Solution. Rewrite

$$\begin{aligned} D_n &= \frac{f(\beta_n)-f(0)}{\beta_n}\frac{\beta_n}{\beta_n-\alpha_n} + \frac{f(\alpha_n)-f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n-\alpha_n} \\ &= \frac{f(\beta_n)-f(0)}{\beta_n} + \Big(\frac{f(\alpha_n)-f(0)}{\alpha_n} - \frac{f(\beta_n)-f(0)}{\beta_n}\Big)\frac{-\alpha_n}{\beta_n-\alpha_n}. \end{aligned}$$

Because $0 \le \frac{-\alpha_n}{\beta_n-\alpha_n} \le 1$, as $n\to\infty$, the right hand side tends to $f'(0)$. □

ii. $0 < \alpha_n < \beta_n$ and $\frac{\beta_n}{\beta_n-\alpha_n} \le M$.

Solution. The proof is similar to previous one. Rewrite

$$\begin{aligned} D_n &= \frac{f(\beta_n)-f(0)}{\beta_n}\frac{\beta_n}{\beta_n-\alpha_n} + \frac{f(\alpha_n)-f(0)}{\alpha_n}\frac{-\alpha_n}{\beta_n-\alpha_n} \\ &= \frac{f(\alpha_n)-f(0)}{\alpha_n} + \Big(\frac{f(\beta_n)-f(0)}{\beta_n} - \frac{f(\alpha_n)-f(0)}{\alpha_n}\Big)\frac{\beta_n}{\beta_n-\alpha_n}. \end{aligned}$$

Because $\frac{\beta_n}{\beta_n-\alpha_n}$ is bounded, the limit exists and is equal to $f'(0)$. □

iii. $f'(x)$ exists and is continuous for all $x \in (-1,1)$.

Solution. For each n, the mean value theorem implies that there exists $\theta_n \in (0,1)$ such that

$$D_n = f'\big(\alpha_n + \theta_n(\beta_n-\alpha_n)\big).$$

Taking limits on both sides, the continuity of f' implies $\lim D_n = f'(0)$. □

(b) Set $f(x) = x^2\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Observe that f is differentiable everywhere in $(-1,1)$ and $f'(0) = 0$. Find α_n and β_n that tend to 0 in such a way that D_n converges to a limit unequal to $f'(0)$.

Solution. Let $\beta_n = \frac{1}{n} + \frac{1}{n^2}$ and $\alpha_n = \frac{1}{n}$. □

1.2 Mean Value Theorems

1. In the auctions example.

   (a) Assume in addition that $\sigma(v)$ is a function such that $\forall v \in [0,1], b^*(v) = \sigma(v)$ (there is a symmetric equilibrium). Use Equation **??** to show that:

   $$\sigma(v) = v - \sigma'(v)\frac{F(v)}{F'(v)}$$

   The right hand side is called the virtual value.

   Solution. Substituting $b(v) = \sigma(v)$, then $\sigma^{-1}(b) = v$. The equation simplifies to:

   $$(v - \sigma(v))F'(\sigma^{-1}(\sigma(v)))\frac{1}{\sigma'(\sigma^{-1}(\sigma(v)))} - F(\sigma^{-1}(\sigma(v))) = 0$$

   $$(v - \sigma(v))F'(v)\frac{1}{\sigma'(v)} - F(\sigma^{-1}(\sigma(v))) = 0$$

   $$(v - \sigma(v))F'(v)\frac{1}{\sigma'(v)} - F(v) = 0$$

   Rearranging the equation,

   $$\sigma(v) = v - \sigma'(v)\frac{F(v)}{F'(v)}$$

   □

   (b) Using the above equation and the signs of the derivatives, show that if $\forall v \in [0,1], b^*(v) = \sigma(v)$ then $\forall v \in [0,1], \sigma(v) \leq v$ (this show that in a symmetric equilibrium everyone bids weakly below their valuation).

   Solution. Rearrange the above formula:

   $$\sigma(v) = v - \sigma'(v)\frac{F(v)}{F'(v)}$$

   .

   Since the second term is negative, then $\sigma(v) \leq v$.

   □

2. Assume f function is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Suppose $f(0) = 0$ and f' is increasing on $(0,\infty)$. Prove

   $$g(x) = \frac{f(x)}{x}$$

   is increasing on $(0,\infty)$.

   Solution. Consider $x_2 > x_1 > 0$. Then by the mean value theorem, there exists $\xi_1 \in (0, x_1)$ and $\xi_2 \in (x_1, x_2)$ such that

   $$f(x_1) = f'(\xi_1)(x_1 - 0) + f(0) = f'(\xi_1)x_1$$

and

$$f(x_2) = f'(\xi_2)(x_2 - x_1) + f(x_1) = f'(\xi_2)(x_2 - x_1) + f'(\xi_1)x_1 \geq f'(\xi_1)x_2,$$

where the inequality comes from the fact that $\xi_2 > \xi_1$ and f' is increasing. Therefore

$$\frac{f(x_2)}{x_2} \geq \frac{f(x_1)}{x_1}.$$

□

1.3 Taylor Expansion

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable. Assume $f(0) > 0$, $f'(0) < 0$ and $f''(x) < 0$ for all $x \in \mathbb{R}$. Prove there exists $\xi \in \left(0, -\frac{f(0)}{f'(0)}\right)$ such that $f(\xi) = 0$.

   Solution. By Taylor's theorem, we have

   $$f(x) = f(0) + f'(0)x + \frac{f''(\eta)}{2}x^2 \quad \text{for some } \eta \text{ between 0 and} x.$$

   Then

   $$f\big(-\frac{f(0)}{f'(0)}\big) = \frac{f''(\eta)}{2}\big(-\frac{f(0)}{f'(0)}\big)^2 < 0.$$

   Because $f(0) > 0$, there exists $\xi \in \left(0, -\frac{f(0)}{f'(0)}\right)$ such that $f(\xi) = 0$. □

2. Assume $f : [a, b] \to \mathbb{R}$ is twice differentiable and $f'(a) = f'(b) = 0$. Prove there exists $\xi \in (a, b)$ such that

   $$\big|f''(\xi)\big| \geq \frac{4}{(b-a)^2}\big|f(b) - f(a)\big|.$$

   (Hint: expand $f\big(\frac{a+b}{2}\big)$ at a and b respectively)

   Solution. By Taylor's theorem, we have

   $$f\big(\frac{a+b}{2}\big) = f(a) + f'(a)\frac{b-a}{2} + \frac{1}{2}f''(\xi_1)\big(\frac{b-a}{2}\big)^2 \quad \text{for some } \xi_1 \in \big(a, \frac{a+b}{2}\big),$$

   and

   $$f\big(\frac{a+b}{2}\big) = f(b) - f'(b)\frac{b-a}{2} + \frac{1}{2}f''(\xi_2)\big(\frac{b-a}{2}\big)^2 \quad \text{for some } \xi_2 \in \big(\frac{a+b}{2}, b\big).$$

   Then

   $$\frac{4}{(b-a)^2}\big|f(b) - f(a)\big| = \frac{1}{2}\big|f''(\xi_1) - f''(\xi_2)\big| \leq \frac{1}{2}\big(\big|f''(\xi_1)\big| + \big|f''(\xi_2)\big|\big) \leq \max\big\{\big|f''(\xi_1)\big|, \big|f''(\xi_2)\big|\big\}.$$

   □

3. Let $f : [a, b] \to \mathbb{R}$ be twice differentiable. Assume $\sup_{x\in[a,b]} |f''(x)| \leq M$ for some constant M. Assume also f achieves its global maximum at some point x^* in (a, b). Prove

   $$\big|f'(a)\big| + \big|f'(b)\big| \leq M(b-a).$$

   Solution. Because $x^* \in (a, b)$, we know $f'(x^*) = 0$. Now apply the mean value theorem to f': there exists $\xi_1 \in (a, x^*)$ and $\xi_2 \in (x^*, b)$ such that

   $$f'(a) = f'(x^*) + f''(\xi_1)(a - x^*),$$

   and

   $$f'(b) = f'(x^*) + f''(\xi_2)(b - x^*).$$

Hence

$$f'(a) = f''(\xi_1)(a - x^*) \quad \text{and} \quad f'(b) = \frac{f''(\xi_2)}{2}(b - x^*).$$

Thus,

$$|f'(a)| + |f'(b)| \leq |f''(\xi_1)|(x^* - a) + |f''(\xi_2)|(b - x^*) \leq M(b - a).$$

$\square$

1.4 First-Order Differentiation in $\mathbb{R}^n$

1. (Euler's Equations) Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable. Fix $(x, y) \in \mathbb{R}^2$. Define $g(t) = f(tx, ty)$ for all $t > 0$. Show g is differentiable and

$$g'(t) = x\frac{\partial f}{\partial x}(tx, ty) + y\frac{\partial f}{\partial y}(tx, ty).$$

Assume in addition, there exists $\alpha > 0$ such that

$$f(tx, ty) = t^\alpha f(x, y) \quad \forall t > 0 \quad \text{and} \quad \forall (x, y) \in \mathrm{R}^2. \tag{1.1}$$

Show for all $(x, y) \in \mathbb{R}^2$,

$$x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = \alpha f(x, y). \tag{1.2}$$

A function with the property (1.1) is said to be homogeneous of degree α. The equation (1.2) is called Euler's formula.

Proof. Fix $(x, y) \in \mathbb{R}^2$. Let $h : \mathbb{R} \to \mathbb{R}^2$ be the linear mapping $t \mapsto t \begin{pmatrix} x \\ y \end{pmatrix}$. So $g(t) = f(h(t))$. Since both f and h are differentiable, we know g is differentiable. By chain rule,

$$g'(t) = Dg(t) = Df(h(t))Dh(t) = (\frac{\partial f}{\partial x}(tx, ty), \frac{\partial f}{\partial y}(tx, ty)) \begin{pmatrix} x \\ y \end{pmatrix}.$$

If in addition, (1.1) holds, then we know $g'(t) = \alpha t^{\alpha-1} f(x, y)$ for all t. This implies

$$x\frac{\partial f}{\partial x}(tx, ty) + y\frac{\partial f}{\partial y}(tx, ty) = \alpha t^{\alpha-1} f(x, y), \quad \forall t.$$

Evaluating both sides at $t = 1$ yields the desired result. □

2. (Exercise 16 on page 347, Pugh) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f = (x, y, z)$ and $g = w$ where

$$\begin{aligned} w &= w(x, y, z) = xy + yz + zx \\ x &= x(s, t) = st \quad y = y(s, t) = s\cos t \quad z = z(s, t) = s\sin t \end{aligned}$$

(a) Find the matrices that represent the linear transformations $(Df)_p$ and $(Dg)_q$ where $p = (s_0, t_0) = (0, 1)$ and $q = f(p)$.

Proof. The representation matrix of $(Df)_p$ is

$$\left.\begin{pmatrix} t & s \\ \cos t & -s\sin t \\ \sin t & s\cos t \end{pmatrix}\right|_{(s,t)=(0,1)} = \begin{pmatrix} 1 & 0 \\ \cos 1 & 0 \\ \sin 1 & 0 \end{pmatrix}.$$

The representation matrix of $(Dg)_q$ is

$$(y+z, x+z, x+y)|_{(x,y,z)=(0,0,0)} = (0,0,0).$$

$\square$

(b) Use the Chain rule to calculate the 1×2 matrix $[\partial w/\partial s, \partial w/\partial t]$ that represents $(D(g \circ f))_p$.

Proof. It is simply

$$(0,0,0)\begin{pmatrix} 1 & 0 \\ \cos 1 & 0 \\ \sin 1 & 0 \end{pmatrix} = (0,0).$$

$\square$

(c) Plug the functions $x = x(s,t)$, $y = y(s,t)$ and $z = z(s,t)$ directly into $w = w(x,y,z)$ and recalculate $[\partial w/\partial s, \partial w/\partial t]$, verifying the answer given in (b).

Proof. Plugging x, y, z into w yields

$$w(s,t) = s^2 t \cos t + s^2 \cos t \sin t + s^2 t \sin t.$$

Hence

$$\begin{aligned} & (\frac{\partial w}{\partial s}, \frac{\partial w}{\partial t})|_{(s,t)=(0,1)} \\ = \; & (2st\cos t + 2s\cos t \sin t + 2st\sin t, s^2\cos t - s^2 t \sin t - s^2\sin^2 t + s^2\cos^2 t + s^2 \sin t + s^2 t\cos t)|_{(0,1)} \\ = \; & (0,0). \end{aligned}$$

$\square$

1.5 Second-Order Differentiation in $\mathbb{R}^n$

1. We showed that a matrix representation exists for a linear map. Why does it have to be unique?

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1^3 + p_2^3 \end{pmatrix}.$$

Prove for any $p \in \mathbb{R}^2$, the matrix that represents $(D^2 f)_p$ is

$$\begin{pmatrix} 6p_1 & 0 \\ 0 & 6p_2 \end{pmatrix}.$$

Proof. We will take it as given that we know $(Df)_p$ is represented by

$$\begin{pmatrix} 3p_1^2 & 3p_2^2 \end{pmatrix}.$$

Therefore

$$\begin{aligned} ||R(v)(u)|| &= \Big|\Big|(Df)_{p+v}(u) - (Df)_p(u) - T(v,u)\Big|\Big| \\ &= \Big|\Big| \begin{bmatrix} 3(p_1+v_1)^2 & 3(p_2+v_2)^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 3p_1^2 & 3p_2^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{pmatrix} 6p_1 & 0 \\ 0 & 6p_2 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Big|\Big| \\ &= \begin{bmatrix} 3v_1^2 & 3v_2^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 3v_1^2 u_1 + 3v_2^2 u_2 \end{aligned}$$

Notice that residual is linear in the second argument by construction. In this case it is possible to directly compute the operator norm $||R(v)(\cdot)||$,

$$||R(v)(\cdot)|| = \sup_{u \in \mathbb{R}^n, ||u||=1} ||R(v)(u)|| = \sup_{u \in \mathbb{R}^n, ||u||=1} \{3v_1^2 u_1 + 3v_2^2 u_2\}$$

where $u_1^2 + u_2^2 = 1$. To compute the sup, it is without loss to consider $u_1, u_2 \geq 0$. Note that then $u_2 = \sqrt{1 - u_1^2} \leq 1 - u_1$, where the equality holds when $u_1 = 0$ or $u_1 = 1$. Thus we have

$$3v_1^2 u_1 + 3v_2^2 u_2 \leq 3v_1^2 u_1 + 3v_2^2(1 - u_1) \leq \max\{3v_1^2, 3v_2^2\}$$

The equality could be achieved bt taking $u_1 = 1, u_2 = 0$ if $v_1^2 \geq v_2^2$, and $u_1 = 0, u_2 = 1$ if $v_1^2 < v_2^2$. Therefore, we have shown that $||R(v, \cdot)|| = \max\{3v_1^2, 3v_2^2\}$ and hence

$$\frac{||R(v)(\cdot)||}{||v||} = \frac{\max\{3v_1^2, 3v_2^2\}}{||v||} = \max\{3(v_1^2/||v||), 3(v_2^2/||v||)\}.$$

$|v_i| \leq ||v||$ is bounded by construction. Therefore,

$$\lim_{v \to 0_{2\times 1}} v_1^2/||v|| = 0$$

$$\lim_{v \to 0_{2\times 1}} v_2^2/||v|| = 0$$

Therefore, $R(v)(\cdot)$ is sublinear and therefore we have show that our candidate matrix is second derivative.

$\square$

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(x) = x^T A^T Ax$$

where A is an $n \times n$ matrix. Calculate the matrices that represent $(Df)_x$.

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}$ be such that $g(y) = y^T y = \sum_{i=1}^m y_i^2$. By calculating the first order partials, it is easy to see

$$2(y_1, \cdots, y_m) = 2y^T$$

represents $(Dg)_y$. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be such that $h(x) = Ax$. Then A represents $(Dh)_x$. Since $f(x) = g(h(x))$, by chain rule $(Df)_x = (Dg)_{h(x)} \circ (Dh)_x$, and

$$2h(x)^T A = 2x^T A^T A$$

is the representation matrix.

$\square$

4. Assume that X is an $n \times k$ full rank matrix and that $Y \in \mathbb{R}^n$. Show that $\hat{\beta} = (X^t X)^{-1} X^t Y$ is the solution to the least squares criterion function by computing the first order conditions of

$$(Y - X\beta)^t (Y - X\beta)$$

1.6 Comparative Statics

1. Consider the Auctions Example in previous chapters. Show that $b^*(v)$ is increasing in v. [Hint: Use the implicit function theorem].

   Solution. Because this $b^*(v)$ is an interior maximum, $\frac{\partial^2}{\partial b^2}U_v(b^*) < 0$. Furthermore, since σ is strictly increasing then σ^{-1} is also strictly increasing (by using the inverse mapping theorem we can further show that $\sigma' > 0$).

   For applying the implicit function theorem, let $H(b, v) = \frac{\partial U_v}{\partial b} = 0$. Then

$$B = \frac{\partial H}{\partial b} = \frac{\partial^2}{\partial b^2}U_v(b^*) < 0. \text{ Then } B^{-1} < 0$$

$$A = \frac{\partial H}{\partial v} = \frac{\partial^2}{\partial b \partial v}U_v(b^*) = F'(\sigma^{-1}(b))\frac{\partial}{\partial b}\sigma^{-1}(b) > 0$$

   Using the implicit function theorem:

$$\frac{\partial b^*(v)}{\partial v} = -B^{-1}A > 0$$

   Therefore $b^*(v)$ is an increasing function of v.

   □

2. Consider the following Keynesian IS-LM model. Suppose

$$\begin{aligned} Y &= C(Y - T) + I(r) + G \\ M &= L(Y, r) \end{aligned}$$

   where Y is GDP, T is taxes, r is interest rate, G is government spending and M is money supply. The functions $C(\cdot)$, $I(\cdot)$ and $L(\cdot, \cdot)$ are consumption function, investment function and money supply function respectively. Assume they are continuously differentiable and

$$0 < C'(x) < 1, \quad I'(r) < 0, \quad \frac{\partial L}{\partial Y} > 0, \quad \text{and} \quad \frac{\partial L}{\partial r} < 0.$$

   Suppose G, M and T are independent variables which can be controlled, Y and r are dependent variables determined by G, M and T. Analyze the relationships between $\{Y, r\}$ and $\{G, M, T\}$.

   Solution. Define

$$\begin{aligned} f(T, G, M, Y, r) &= Y - C(Y - T) - I(r) - G \\ h(T, G, M, Y, r) &= L(Y, r) - M. \end{aligned}$$

   Then

$$\begin{pmatrix} \frac{\partial f}{\partial Y} & \frac{\partial f}{\partial r} \\ \frac{\partial h}{\partial Y} & \frac{\partial h}{\partial Y} \end{pmatrix} = \begin{pmatrix} 1 - C'(Y - T) & -I'(r) \\ \frac{\partial L}{\partial Y} & \frac{\partial L}{\partial r} \end{pmatrix}.$$

This matrix is invertible because its determinant $\Delta = (1 - C'(Y-T))\frac{\partial L}{\partial r} + I'(r)\frac{\partial L}{\partial Y} < 0$. Therefore

$$
\begin{aligned}
\begin{pmatrix} \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial G} & \frac{\partial Y}{\partial M} \\ \frac{\partial r}{\partial T} & \frac{\partial r}{\partial G} & \frac{\partial r}{\partial M} \end{pmatrix} &= -\begin{pmatrix} 1 - C'(Y-T) & -I'(r) \\ \frac{\partial L}{\partial Y} & \frac{\partial L}{\partial r} \end{pmatrix}^{-1} \begin{pmatrix} C'(Y-T) & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= -\frac{1}{\Delta} \begin{pmatrix} \frac{\partial L}{\partial r} & I'(r) \\ -\frac{\partial L}{\partial Y} & 1 - C'(Y-T) \end{pmatrix} \begin{pmatrix} C'(Y-T) & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
&= -\frac{1}{\Delta} \begin{pmatrix} \frac{\partial L}{\partial r} C'(Y-T) & -\frac{\partial L}{\partial r} & -I'(r) \\ -\frac{\partial L}{\partial Y} C'(Y-T) & \frac{\partial L}{\partial Y} & -1 + C'(Y-T) \end{pmatrix}
\end{aligned}
$$

Therefore $\frac{\partial Y}{\partial T} < 0$, $\frac{\partial Y}{\partial G} > 0$, $\frac{\partial Y}{\partial M} > 0$, $\frac{\partial r}{\partial T} < 0$, $\frac{\partial r}{\partial G} > 0$ and $\frac{\partial r}{\partial M} < 0$. $\square$

Bibliography

Greene, W. H. (2012). *Econometric analysis.* Pearson Education India.

Naiman, D. Q. and E. R. Scheinerman (2017). Arbitrage and geometry. *arXiv preprint arXiv:1709.07446*.

Rudin, W. et al. (1964). *Principles of mathematical analysis*, Volume 3. McGraw-hill New York.