Math Camp

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1 Day 1

Important notes

- Due to housing issue, I might have to miss the first day.
- Due to this problem, I will be manually studying for about 5 to 6 subsections using the notes given by the instructor.

Short notes in class

- Part 1: real calculus.
- Part 2: linear algebra.
- Part 3: probability and optimization.
- Each part lasts two weeks.
- HW assignment every week but not graded.
- Friday: Quiz (2 hours).
- Only thing that matters is math camp exam: Mostly everyone passes if one takes classes.
- Communication done through canvas.
- Part 1 deals with mathematical logic, spaces (metrics, topology), correspondence.

Some notation

$\mathbb{N}$: Natural numbers.
$\mathbb{Z}$: Integer numbers.
$\mathbb{Q}$: Rational numbers.
$\mathbb{R}$: Real numbers.

Binary relation

Binary relation R on set X is a subset of $X \times X$.

1. R is reflexive if: $\forall x \in X$, xRx

2. R is symmetric if: $\forall x, y \in X$, $xRy \implies yRx$
3. R is transitive if: $\forall x, y \in X$, $xRy, yRz \implies xRz$

Def: A binary relation is an equivalence relation if it is reflexive, symmetric and transitive. We denote this as $x \sim y$.

Def: R is antisymmetric if $\forall x, y \in X, xRy$ and $yRx \implies x = y$.

Def: R is partial order on X if it is reflexive, antisymmetric and transitive.

Def: $x, y \in X$ are comparable if xRy or yRx.

Def: A partial order R is a total order if every two elements are comparable.

Def: $f : X \to Y$ is an injection if $\forall a, b \in X, a \neq b \implies f(a) \neq f(b)$.

Def: $f : X \to Y$ is an surjection if $Y = f(X) = \{y \in Y | \exists x \in X : f(x) = y\}$.

Def: $f : X \to Y$ is called bijection if it is both an injection and surjection.

Fact: $f : X \to Y$ is an injection if $\forall y \in Y$, $f^{-1}(\{y\})$ has no more than one element.

Fact: $f : X \to Y$ is an surjection if $\forall y \in Y$, $f^{-1}(\{y\})$ has at least one element.

Fact: $f : X \to Y$ is an bijection if $\forall y \in Y$, $\#f^{-1}(\{y\}) = 1$.

Fact: f have inverse $\iff$ f is a bijection.

Proof. i. f^{-1} exists $\implies$ f is a bijection: $f^{-1}(\{y\}) = \{f^{-1}(y)\}$
. Assume not $\implies$ $\exists a \in f^{-1}(\{y\})\ s.t.a \neq f^{-1}(y)$. But $f(a) = y \implies f^{-1}(f(a)) = f^{-1}(y)$.

ii. f is a bijection $\implies$ f^{-1} exists. Construct g s.t. $g(y)$ is the only element of $f^{-1}(\{y\})$. $f \cdot g : f(g(y)) = y$; $g \cdot f : g(f(x)) = y$. □

Fact: Bijection between two finite sets exists only if $\#A = \#B$.

Def: A and B have equal cardinality if there is a bijection $f : A \to B$.

Fact: Having the same cardinality is an equivalence relation.

Proof. i. reflexivity: id_A is a bijection.

ii. symmetry: $f : A \to B$ is a bijection $\implies$ f^{-1} is also a bijection.

iii. transitivity: $f : A \to B$ and $g : B \to C$ is bijection, then $g \cdot f$ is also bijection. □

Def: A is denumerable if $\#A = \#\mathbb{N}$. (countably infinite) A is countable if it is finite or denumerable. A is uncountable if it is not countable.

Ex: Show $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof. (0,0), (0,1) ...
(1,0), (1,1) ...
Just go diagonal. Then it will get you a mapping that is bijection.

$$f(m,n) = \frac{(m+n)(m+n+1)}{2} + m.$$ □

Corr: $\forall k > 0$, $\mathbb{N}^k$ is denumerable.

Proof. later. □

Prob: Let $A \subset B$. A is infinite and B is denumerable. Then A is denumerable.

Proof. $f : \mathbb{N} \to B$. $f^{-1}(A) \subset \mathbb{N}$ Later solve it. □

Corr: Let $f : A \to B$ be a surjection. If A is denumerable and B is infinite, then B is denumerable.

Proof. Later. □

Def: We say $\#A \leq \#B$ if there is an injection $f : A \to B$. If there is no bijection from A to B, we can say that the cardinality of A is strictly smaller than that of B.

Prob: Prove that $\{0,1\}^{\mathbb{N}}$ is uncountable.

Proof. Later. But just use the fact that if denumerable, its element can be written as a sequence. □

Def: The set of all subsets of A is called power set of A and is noted as 2^A.

Prob: If A is finite $\implies \#2^A = 2^{\#A}$.

Proof. Later. □

Theorem For every $A, \#2^A > \#A$.

Proof. Injection: $x \to \{x\}$ from A to 2^A.
Suppose there is a bijection $f : A \to 2^A$. $B = \{x \in A | x not \in f(x)\} = 2^A$. $\implies \exists y \in A s.t. f(y) = B$.
1. $y \in B \implies y not \in f(y) = B$ by def of B.
2. $y not \in B \implies y \in f(y) = B$ by def of B.
This is contradiction. Thus done. □

Prob: $\#\mathbb{R} = \#2^{\mathbb{N}}$.

Proof. We show that $[0,1] = 2^{\mathbb{N}}$ in cardinality. □

Prob: $\forall n \in \mathbb{N}, \mathbb{R}^n \sim \mathbb{R}$.

Proof. Later. □

Theorem If $\#A \leq \#B$ and vice versa, then the cardinality for both set is same.

Some Exercises

Exercise 1.3.11: Why such partition exists only for equivalence relation? Are all three properties required for existence of such partition?

Proof. car $\square$

Exercise 1.3.20: Let's define a binary relation R on $\mathbb{N}$ in a following way: xRy if y is divisible by x. Is this relation a partial order, a total order or it is not order at all?

Proof. car $\square$

Exercise 1.4.10: Define a bijection in a similar way (using preimages).

Proof. car $\square$

Exercise 1.4.11: Show that $f : X \to Y$ and $g : Y \to Z$ both are bijections, then $g \cdot f : X \to Z$ is a bijection too.

Proof. car $\square$

Exercise 1.4.15: $f : X \to Y$ has an inverse if and only if it is a bijection.

Proof. car $\square$