# Econ 897 (math camp). Part I

## Problem set №3. Solutions

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**Problem 1.** Prove that following statements are equivalent

(i) $x_0$ is an isolated point of $(X, d)$;

(ii) there exist $\varepsilon$-neighborhood of $x_0$ that contains finite number of elements of $X$.

*Solution.* If $x_0$ is not a limited point, there exists a punctured $\varepsilon$-neighborhood that doesn't contain any elements of $X$. Hence corresponding $\varepsilon$-neighborhood will contain finite number of elements – one element ($x_0$).

In other direction: there exist $\varepsilon$-neighborhood of $x_0$ that contains finite number $\{x_i\}_{i=1}^n$ of elements of $X$, let's take $\bar{\varepsilon} = \min_{i=1}^b d(x_0, x_n)$. Punctured $\bar{\varepsilon}$-neighborhood will be empty, that means $x_0$ is an isolated point.

**Problem 2.** Show that any contraction is continuous.

*Solution.* It's not hard to see, that for any $x_0$ image of $U_\delta(x_0)$ will entirely lie in $U_{\lambda\delta}(f(x_0))$. So in definition of continuity we can take any $\delta \leq \frac{\varepsilon}{\lambda}$.

**Problem 3** (*)**.** Let $f$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Prove that if there is no $x$ such that $f(x) = x$, then there is no $x$ such that $f(f(x)) = x$.

*Hint.* Try to look at the intermediate value theorem as a sufficient condition for function taking certain values.

*Solution.* If there is no $x$ such that $f(x) = x$, then function $g(x) = f(x) - x$ is never equal to 0. That mean there is no interval $[a, b]$ such that $g(x)$ is negative at $a$ and positive at $b$ or vice versa – otherwise by intermediate value theorem $g$ will be equal to 0 somewhere on this interval. That means that for every pair of $a, b \in \mathbb{R}$ $g(a)$ and $g(b)$ have the same sign. And then $g$ is either strictly positive on $\mathbb{R}$, either strictly negative. WLOG let's assume it is positive. Then $f(x) > x$ for all $x$, and then $f(f(x)) > f(x) > x$ for all $x$.

**Problem 4.** Is it true that if $\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = c$, then $\lim_{x\to a} g(f(x)) = c$?

*Solution.* No. Consider the following counterexample. $g(x) = x^2$ if $x \neq 2$ and $g(x) = 5$ if $x = 2$. Then $\lim_{x\to 2} g(x) = 4$. Then define $f(x) = 2$ constantly. Then in any point $\lim_{x\to a} g(f(x)) = 5$.

**Problem 5.** Not-constant continuous function maps $I \subset \mathbb{R}$ into $\mathbb{R}$. What can $f(I)$ be if $I$ is:

(i) open interval $(a, b)$;

(ii) closed interval $[a, b]$;

(iii) $\mathbb{R}$?

*Solution.* (i) open interval, closed interval or real line, open or closed ray;

(ii) closed interval;

(iii) open interval, closed interval or real line, open or closed ray.

**Problem 6.** Prove that any infinite bounded subset of $\mathbb{R}$ has at least one limit point.

*Solution.* Let's denote this set as $X$. It entirely belongs to some interval $[a_0, b_0]$. If we will split it in two halves, at least one of them would contain infinitely many elements of $X$. Let's call it $[a_1, b_1]$. Repeating this procedure, we will obtain two sequences $(a_n)$ and $(b_n)$. They converge (they are monotone and bounded) and they converge to one point. This point will be the limit point of $X$. Indeed any $\varepsilon$-neighborhood will contain $[a_n, b_n]$ for some $n$ and hence infinitey many points of $X$.

**Problem 7.** Prove that the following two statements about the set $X \subset \mathbb{R}$ are equivalent:

(i) $X$ is open;

(ii) $\mathbb{R} \setminus X$ contains all it's limit points.

Here limit point doesn't necessarily have to lie within set $\mathbb{R} \setminus X$.

*Solution.* Assume $X$ is open and $\mathbb{R} \setminus X$ doesn't contain all limit points. That mean there exists sequence $(x_n) \in \mathbb{R} \setminus X$ that converges to point in $X$. Then every open ball around $x$ contains infinitely many elements of this sequence. Hence $X$ is not open. Contradiction.

Now assume $\mathbb{R} \setminus X$ contains all it's limit points and $X$ is not open. That means there exists a point in $X$ ($x_0$) such that in every neighborhood of $x_0$ there at least one point from $\mathbb{R} \setminus X$. Decreasing $\varepsilon$, we will construct sequence in $\mathbb{R} \setminus X$ that converges to $x_0$. That contradicts the fact that $\mathbb{R} \setminus X$ contains all it's limit points.