

# Lecture 6: Continuous and discrete-continuous decision problems

Fedor Iskhakov, Australian National University  
Anders Munk-Nielsen, University of Copenhagen  
John Rust, Georgetown University  
Bertel Schjerning, University of Copenhagen

Empirical IO: Dynamic Structural Models  
Department of Economics (NHH)  
29 Nov - 3 Dec., 2021

# Discrete and continuous choice?

In economics discrete and continuous choice co-exist

- How much to work + when to retire/become an entrepreneur
- How much to save + when to buy a house/car/durables
- Which car to buy + how much to drive

Often modeled separately using traditional solution methods:

- Discrete choice → optimization over finite set
- Continuous choice → first order conditions + concavity(?)
- Dynamic → dynamic programming (VFI,policy,time iterations)

# Discrete and continuous choice?

In discrete-continuous choice models:

- Intrinsic non-concavity
- First order conditions not sufficient
- Kinks in value functions + discontinuities in policy functions

Traditional methods are not ideal

- Need global optimizer in each point of the state space
- Need to locate and keep track of kinks and discontinuities
- Need special numerical procedures for non-smooth objects

⇒ **Endogenous grid point methods**

# Plan for the lecture

- ➊ Original EGM for continuous choice **only**  
*Only for particular (yet interesting and important) models  
(stochastic growth models, consumption-savings (buffer stock)  
models)*
- ➋ DC-EGM for discrete-continuous choice **without taste shocks**  
*For models with one continuous and additional discrete choices  
**Nasty and scary***
- ➌ DC-EGM for discrete-continuous choice **with taste shocks**  
*For models with one continuous and additional discrete choices  
Structural taste shocks or logit smoothing  
**Much better, possible to work with***
- ➍ Some words on multi-dimensional extensions and occasionally  
binding constraints

# What is EGM?

The **M**ethod of **E**ndogenous **G**ridpoints — fast method for solving dynamic stochastic consumption/savings problems

- ❶ finite and infinite horizon
- ❷ Strictly concave monotone and differentiable utility function
- ❸ one continuous state variable (*wealth*) and  
one continuous choice (*consumption*)
- ❹ particular structure of the law of motion for state variables  
(*intertemporal budget constraint*)
- ❺ very well accommodate potentially binding borrowing constraints

# DC-EGM for Discrete-Continuous problems

Expand the class of problems to be solved:

- ① A1. Strictly concave monotone and differentiable utility function
- ② Continuous state  $M_t$  with a particular motion rule
- ③ Additional (discrete) state variables  $st_t$
- A2. Transition probabilities of  $st_t$  are independent of  $M_t$
- ④ One continuous ( $c_t$ ) and one<sup>\*</sup> discrete choice variable  $d_t$

Two flavors:

- ① **Without taste shocks:** DC-EGM iterates on value function and policy function, produces exact solutions for the optimal thresholds for discrete decisions (discrete policy)
- ② **With taste shocks:** DC-EGM iterates on **discrete choice specific** value and policy functions, produces choice probabilities for discrete alternatives

# Learning outcomes = points to remember

- ➊ If your model has one continuous (consumption) choice and additional discrete choices → Use DC-EGM
- ➋ In regular cases DC-EGM avoids all root-finding operations
- ➌ If utility is separable in continuous and discrete choices, DC-EGM deals very easily with credit constraints
- ➍ Extreme value taste shocks → solution is much better behaved
- ➎ Faster and more accurate than traditional approaches

# EGM

# Simple consumption/savings model (Phelps)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} [u(c) + \beta E V_{t+1} (\tilde{R}(M_t - c))]$$

$M_t$  cash-in-hand, all resources available at period  $t$

$A_t = M_t - c_t$  assets at the end of period  $t$  (savings)

$\tilde{R}$  deterministic or stochastic return on savings

$u(c)$  utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \xrightarrow{\rho \rightarrow 0} \log(c)$$

# Analytic solution (Hakansson, 1970, Phelps, 1962)

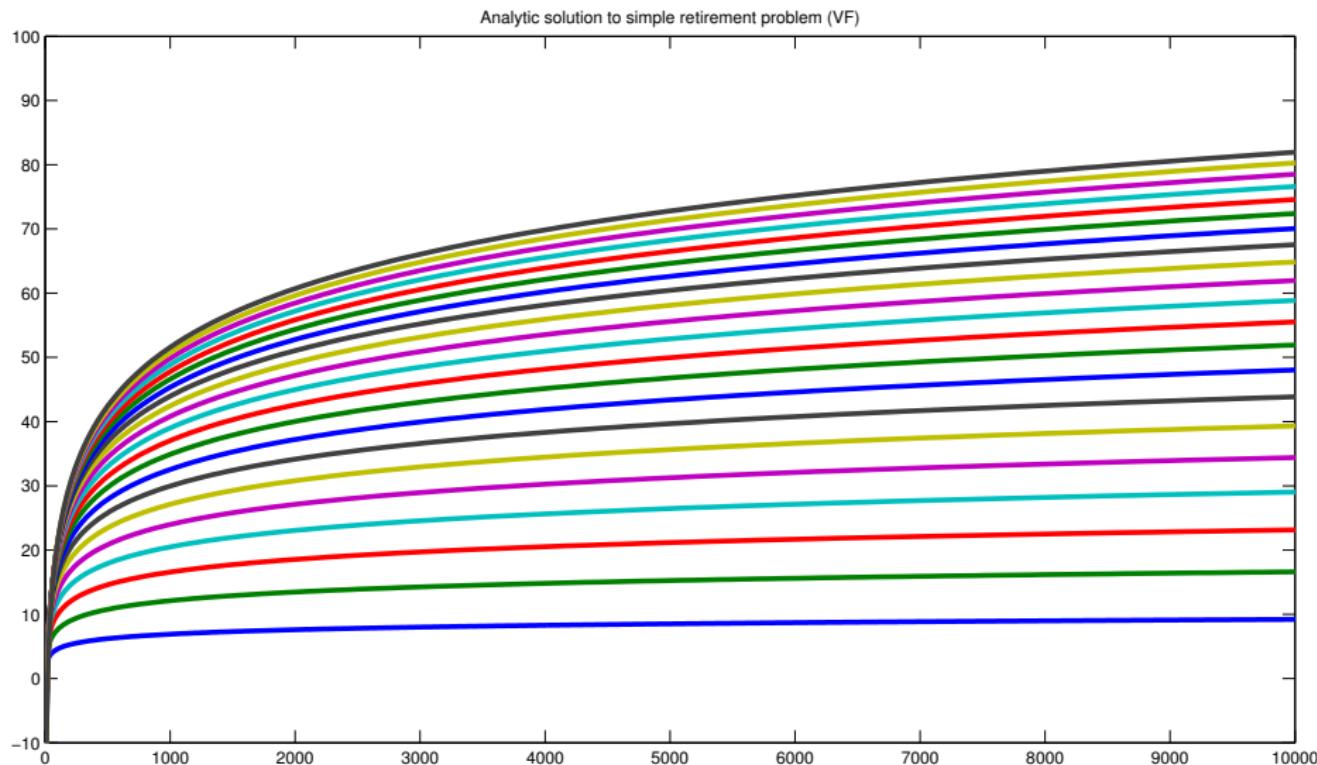
$$V_{T-t}(M) = \left[ \frac{M^\rho}{\rho} \right] \left( \sum_{i=0}^t K^i \right)^{(1-\rho)} - \frac{1}{\rho} \left( \sum_{i=0}^t \beta^i \right)$$

$$V_{T-t}(M) \xrightarrow{\rho \rightarrow 0} \log(M) \left( \sum_{i=0}^t \beta^i \right) + K_t$$

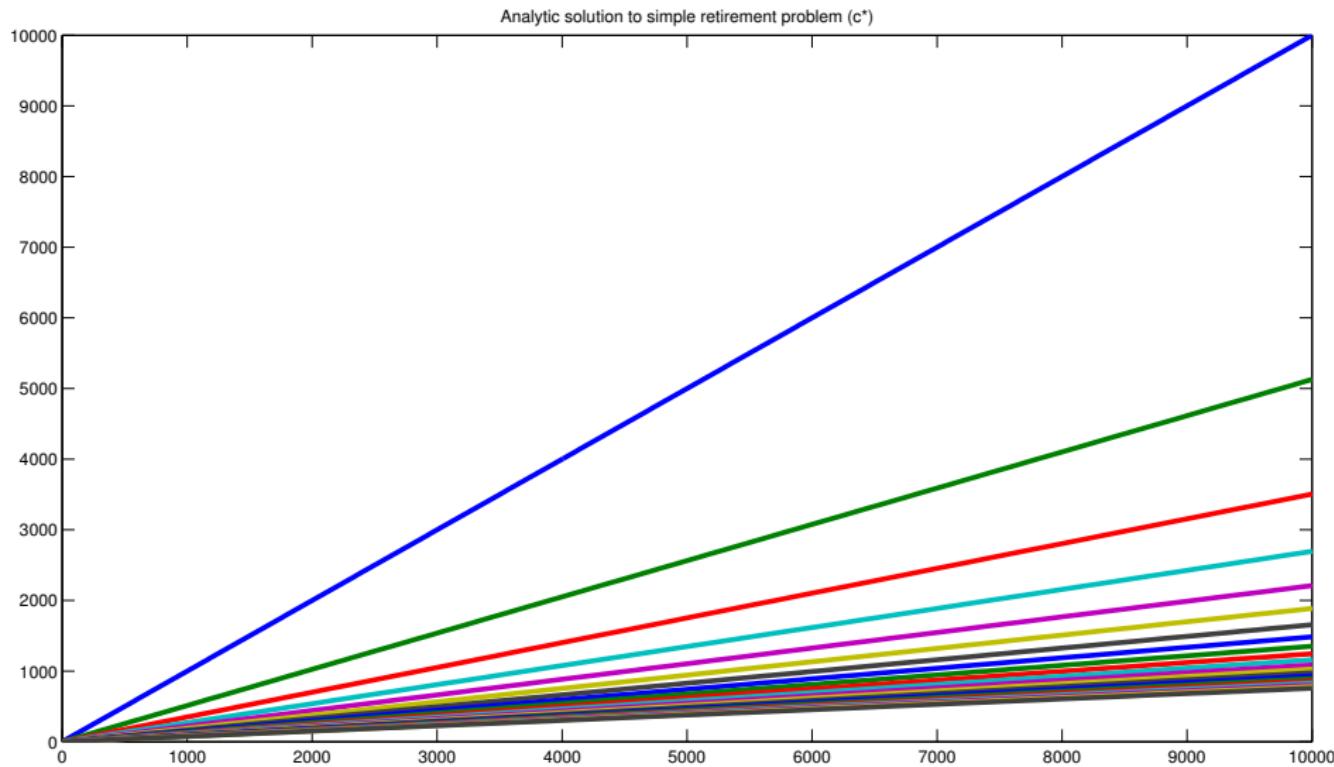
$$c_{T-t}(M) = M \left( \sum_{i=0}^t K^i \right)^{-1}$$

$K$  and  $K_t$  are functions of primitives,  $K \xrightarrow{\rho \rightarrow 0} \beta$

# Analytic solution : value functions



# Analytic solution : consumption rule



# Simple consumption/savings model (Deaton)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} [u(c) + \beta E V_{t+1}(R(M_t - c) + \tilde{y})]$$

- $M_t$  cash-in-hand, all resources available at period  $t$   
 $A_t = M_t - c_t$  assets at the end of period  $t$  (savings)  
 $R$  deterministic return on savings  
 $\tilde{y}$  stochastic income  
 $u(c)$  utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \xrightarrow{\rho \rightarrow 0} \log(c)$$

No analytical solution!

# Traditional approach : value function iterations

① Fix grid over  $M_t$ . For every point on this grid:

② In the terminal period calculate

$$V_T(M_T) = \max_{0 \leq c_T \leq M_T} \{u(c_T)\} \text{ and}$$

$$c_T^* = \operatorname{argmax}_{0 \leq c_T \leq M_T} \{u(c_T)\}$$

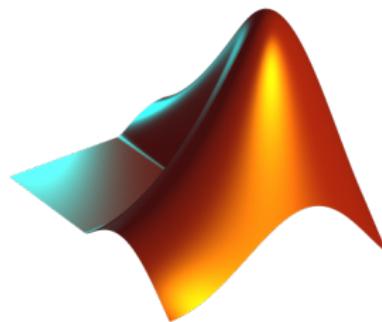
③ With  $t + 1$  value function at hand, proceed backward to period  $t$  and calculate

$$V_t(M_t) = \max_{0 \leq c_t \leq M_t} \left\{ u(c_t) + \beta EV_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right\}$$

and

$$c_t^* = \operatorname{argmax}_{0 \leq c_t \leq M_t} \left\{ u(c_t) + \beta EV_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right\}$$

using Bellman equation



- ➊ Introduction to the code
- ➋ Phelps and Deaton models
- ➌ Run VFI solver

# Euler equation

Bellman equation:  $V_t(M_t) = \max_{0 \leq c_t \leq M_t} [u(c_t) + \beta E V_{t+1} (\tilde{R}(M_t - c_t))]$

F.O.C. for Bellman equation:  $u'(c_t) = \beta E \left[ \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \tilde{R} \right]$

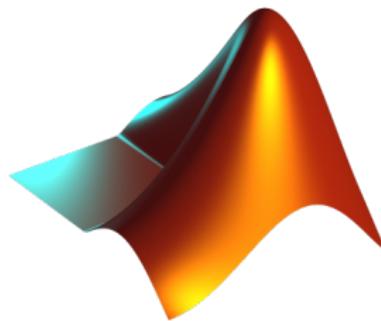
Envelope theorem:

$$\begin{aligned} \frac{\partial V_t(M_t)}{\partial M_t} &= \beta E \left[ \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \tilde{R} \right] \Rightarrow \frac{\partial V_t(M_t)}{\partial M_t} = u'(c_t) \Rightarrow \\ &\Rightarrow \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} = u'(c_{t+1}) \end{aligned}$$

Euler equation to characterize the **interior solutions**:  $u'(c_t) = \beta E [u'(c_{t+1}) \tilde{R}]$

# Traditional approach : solving Euler equation

- ➊ Fix grid over  $M_t$ . For every point on this grid:
- ➋ In the terminal period calculate  
 $c_T^* = \underset{0 \leq c_T \leq M_T}{\operatorname{argmax}} \{u(c_T)\}$
- ➌ With  $t + 1$  optimal consumption rule  $c_{t+1}^*(M_{t+1})$  at hand, proceed backward to period  $t$  and calculate  
 $c_t$  from **equation**  
$$u'(c_t) = \beta E \left[ u' \left( c_{t+1}^* \left( \tilde{R}(M_t - c_t) \right) \right) \tilde{R} \right]$$
to recover  $c_t^*(M_t)$
- ➍ When  $M_t$  is small enough so credit constraint binds, the Euler equation does not hold, and special provisions are necessary



### Exercise 1:

- ① Code up the Euler equation solver
- ② Verify solution against VFI solver

# What if no root-finding is necessary?

## With numerical optimization

- Relatively slow: iterative numerical optimization in each point of state space!
- Hard to find global optimum in non-convex problems
- Loss of accuracy due to the absence of the point where credit constraint starts to bind on the fixed grid

Even when using state-of-the-art solvers!

## Without numerical optimiation

- Much faster: no iterative methods in each point of the state space
- More accurate: using analytical structure of the problem

# Endogenous gridpoint method (EGM)



Carroll 2006 *Economics Letters*

The method of endogenous gridpoints for solving dynamic stochastic optimization problems.

## Idea

- Instead of searching for optimal decision in each point of the state space (traditional approaches)
- Look for the state variable (level of assets) where arbitrary chosen decision (consumption → savings) would be optimal (EGM)

# EGM algorithm

Start with  $c_T^* = M_T$ . In each period  $t = T, T - 1, \dots, 1$ :

## EGM step

- ① Take a guess  $A$  = current period savings ( $= M_t - c_t$ )  
(from fixed or adaptive list/grid)
- ② Intertemporal budget constraint:  $A \rightarrow M_{t+1}$   
 $M_{t+1} = \tilde{R}(M_t - c_t) = \tilde{R} \cdot A$
- ③ Policy function at period  $t + 1$ :  $M_{t+1} \rightarrow c_{t+1}$   
 $c_{t+1} = c_{t+1}^*(M_{t+1})$
- ④ Inverted Euler equation:  $c_{t+1} \rightarrow c_t$   
 $c_t = (u')^{-1} \left( \beta E \left[ \tilde{R} \cdot u' (c_{t+1}^*(M_{t+1})) | A \right] \right)$
- ⑤ Intratemporal budget constraint:  $c_t + A = M_t \rightarrow c_t(M_t)$   
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# EGM algorithm

Start with  $c_T^* = M_T$ . In each period  $t = T, T-1, \dots, 1$ :

## EGM step

- ➊ Take a guess  $A = \text{current period savings} (= M_t - c_t)$  (from fixed or adaptive list/grid)
- ➋ Intertemporal budget constraint:  $A \rightarrow M_{t+1}$   
 $M_{t+1} = \tilde{R}(M_t - c_t) = \tilde{R} \cdot A$
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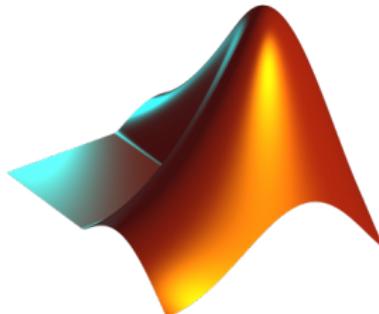
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# Matlab implementation (minimal.m)

```
o [quadp quadw]=quadpoints(EXPN,0,1);
quadstnrm=norminv(quadp,0,1);
sgrid=linspace(0,MMAX,NM);
policy{TBAR}.w=[0 MMAX];
policy{TBAR}.c=[0 MMAX];
5 for it=TBAR-1:-1:1
    w1=Y+exp(quadstnrm*SIGMA)*(1+R)*sgrid;
    c1=interp1(policy{it+1}.w,policy{it+1}.c,w1,'linear',
    rhs=quadw'*(1./c1);
    policy{it}.c=[0 1./(DF*(1+R)*rhs)];
10   policy{it}.w=[0 sgrid+policy{it}.c(2:end)];
end
```

# Accuracy and speed of EGM

	Traditional Euler	EGM
Running time	37 sec.	0.11 sec.
Max abs error, $c_t^*$	5e-9	4e-14
Mean abs error, $c_t^*$	1.4e-12	1.5e-14
Max abs error, $V_t(M, \mathbb{R})$	39.466	15.163
Mean abs error, $V_t(M, \mathbb{R})$	2.5e-02	3.2e-02



- ➊ Compare speed of VFI and EGM solvers
- ➋ Look at the simulator code
- ➌ Simulate flat consumption path using VFI and EGM solutions

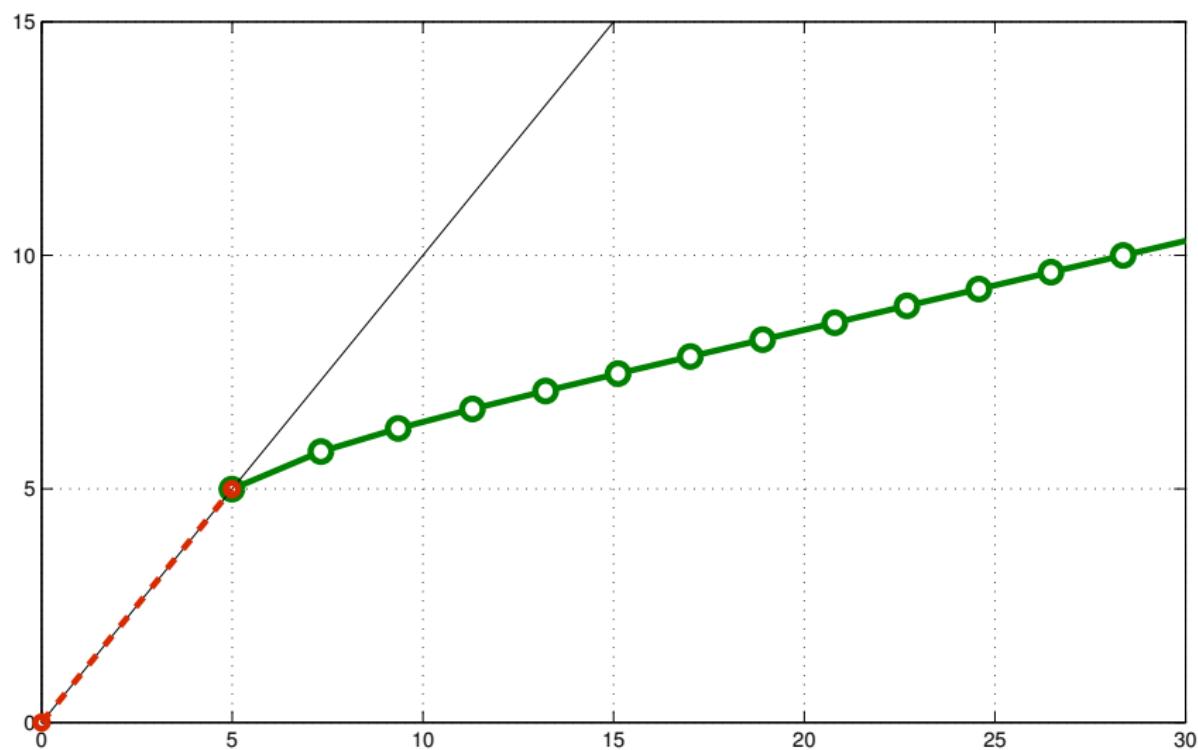
# EGM and credit constraint

Theorem: Monotonicity of savings

Monotone and concave utility function  $\Rightarrow$   
end-of-period assets  $A_t = M_t - c_t$  are non-decreasing in  $M_t$

- With  $A = 0$  the EGM loop recovers the value of cash-in-hand  $M_t^{cc}$  that bounds the credit constrained region
- For all  $M_t < M_t^{cc}$  credit constrained binds  $\Rightarrow c_t = M_t$
- Consumption rule in the credit constrained region is  $45^\circ$  line between  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$
- As simple as “connect the dots”  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$

# EGM and credit constraint



# Credit constraints and value function

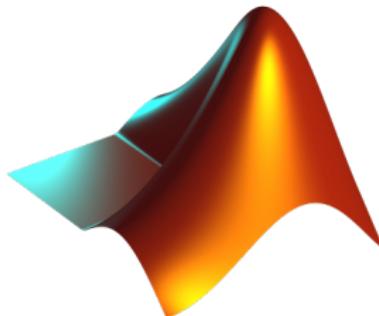
- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Inevitable when value functions have to be computed, but..

$$M_t < M_t^{cc}$$

$$V_t(M) = u(M) + \beta EV_{t+1}(0)$$

$EV_{t+1}(0)$  — expected value of ending period  $t$  with  $A_t = 0$

- Value function has analytic form for  $M_t < M_t^{cc}$ !



- ➊ How to evaluate value function with EGM

### Exercise 2:

- ➋ Code up an EGM solver for infinite horizon problem

# DC-EGM

# Generalization of EGM



Iskhakov, Jørgensen, Rust, Schjerning, QE forthcoming  
The Endogenous Grid Method for Discrete-Continuous Dynamic  
Choice Models with (or without) Taste Shocks

- The DC-EGM paper
- Two flavors: with and without EV taste shocks
- Solution method made for empirical applications



Giulio Fella, RED 2014  
A Generalized Endogenous Grid Method for Non-Smooth and  
Non-Concave Problems

- Identify the regions of the problem where Euler equation is not sufficient for optimality
- Use global optimization methods inside (VFI) and EGM outside
- Similar to DC-EGM without taste shocks

# Simple retirement model

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} \max_{0 \leq c \leq M_t} u(c, \mathbb{R}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \\ \max_{0 \leq c \leq M_t} u(c, \mathbb{W}) + \beta EV_{t+1} \left( \tilde{R}(M_t + y - c), \mathbb{W} \right) \end{array} \right\}$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c, \mathbb{R}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

$\mathbb{R}, \mathbb{W}$  retirement and working **states  $st_t$**  that evolve according to **discrete choices  $d \in \{\mathbb{R}, \mathbb{W}\}$**

$y$  deterministic wage income

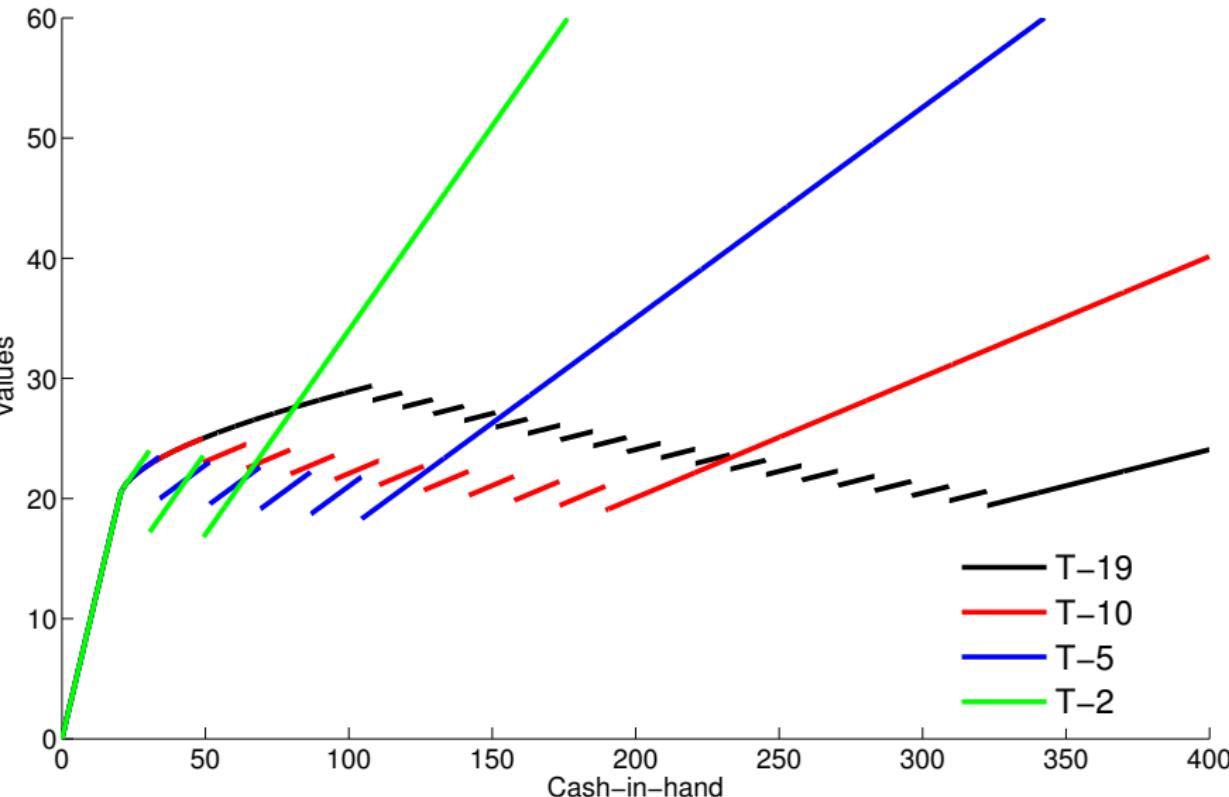
$$u(c, d) = \frac{c^\rho - 1}{\rho} - 1(d = \mathbb{W}) \xrightarrow{\rho \rightarrow 0} \log(c) - 1(d = \mathbb{W})$$

# Analytic solution

$$u(c) = \log(c), R = 1 \Rightarrow c_{T-t}^*(M, \mathbb{W}) =$$

$$\left\{ \begin{array}{ll} M & \text{if } M \leq y/\beta \\ (y+M)/(1+\beta) & \text{if } y/\beta \leq M \leq \overline{M}_{T-t}^{l_1} \\ (2y+M)/(1+\beta+\beta^2) & \text{if } \overline{M}_{T-t}^{l_1} \leq M \leq \overline{M}_{T-t}^{l_2} \\ \dots & \dots \\ ((t-1)y+M) \left( \sum_{i=0}^{t-1} \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{l_{t-2}} \leq M \leq \overline{M}_{T-t}^{l_{t-1}} \\ (ty+M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{l_{t-1}} \leq M \leq \overline{M}_{T-t}^{r_1} \\ [(t-1)y+M] \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_1} \leq M \leq \overline{M}_{T-t}^{r_2} \\ \dots & \dots \\ (2y+M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_{t-2}} \leq M \leq \overline{M}_{T-t}^{r_{t-1}} \\ (y+M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_{t-1}} \leq M \leq \overline{M}_{T-t} \\ M \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t} < M \end{array} \right.$$

## Analytic solution



# How to approach discrete/continuous choice

The goal:

- Avoid root finding
- Keep efficient treatment of credit constraints

## DC-EGM ver. 1.0

- ❶ EGM step for each discrete choice  $d$  and every state  $st$
- ❷ Compute  $d$ -specific value functions and consumption rules
- ❸ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- ❹ Reconstruct overall consumption rule and value function from optimal switching points

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- No root finding!
- Efficient treatment of credit constraints (to be shown)
- Need to compute value functions
- Need to compute upper envelope

# Is Euler equation still a necessary condition?

## DC-EGM ver. 1.0

- ① EGM step for each discrete choice  $d$  and every state  $st$
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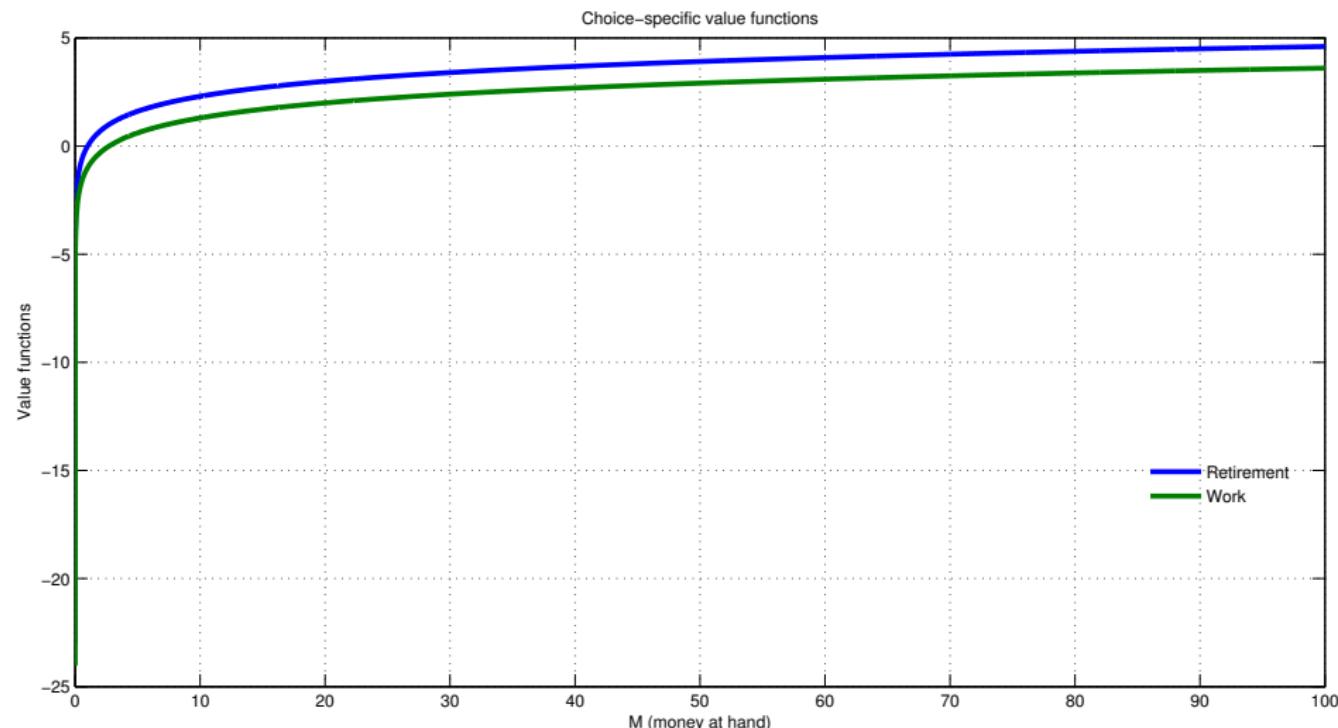


Clausen & Strub, 2010-2013

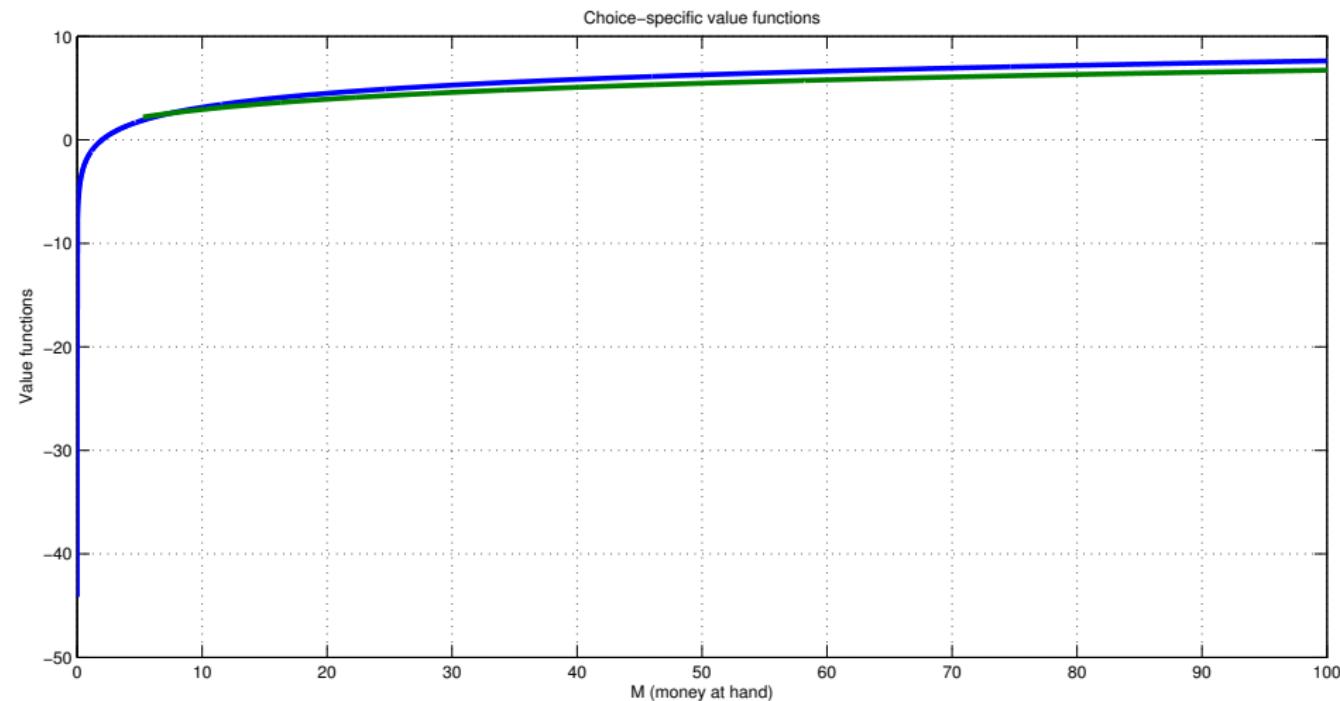
A General and Intuitive Envelope Theorem.

Show that Euler equation remains a necessary condition for the optimal continuous consumption.

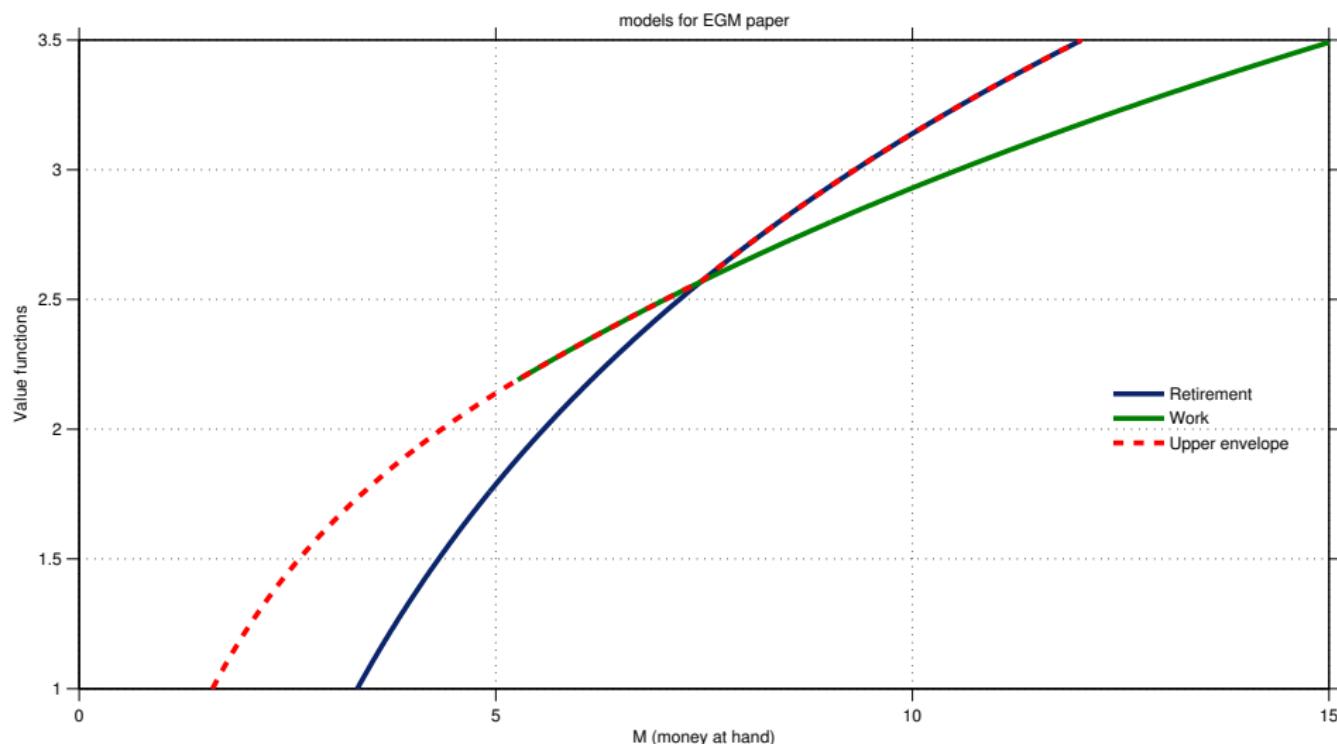
# Period $T$ : choice specific value functions



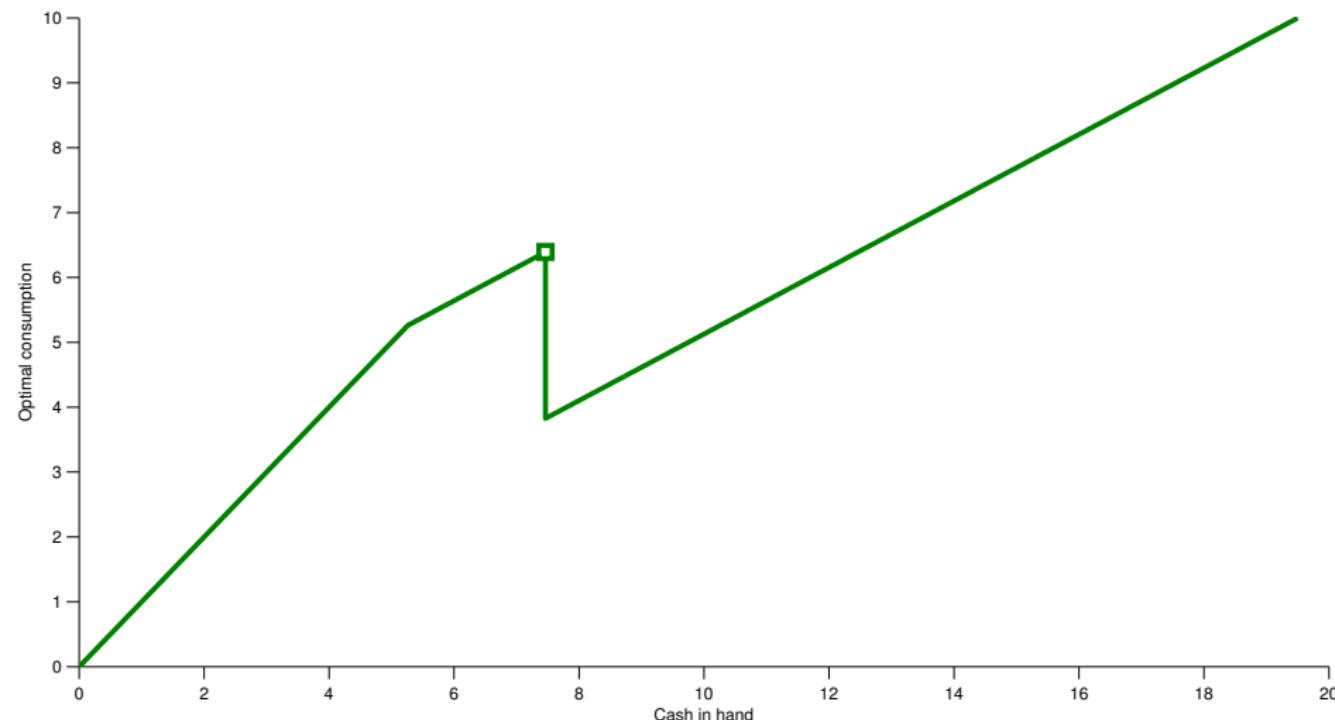
# Period $T - 1$ : Choice specific VF



# Period $T - 1$ : Choice specific VF



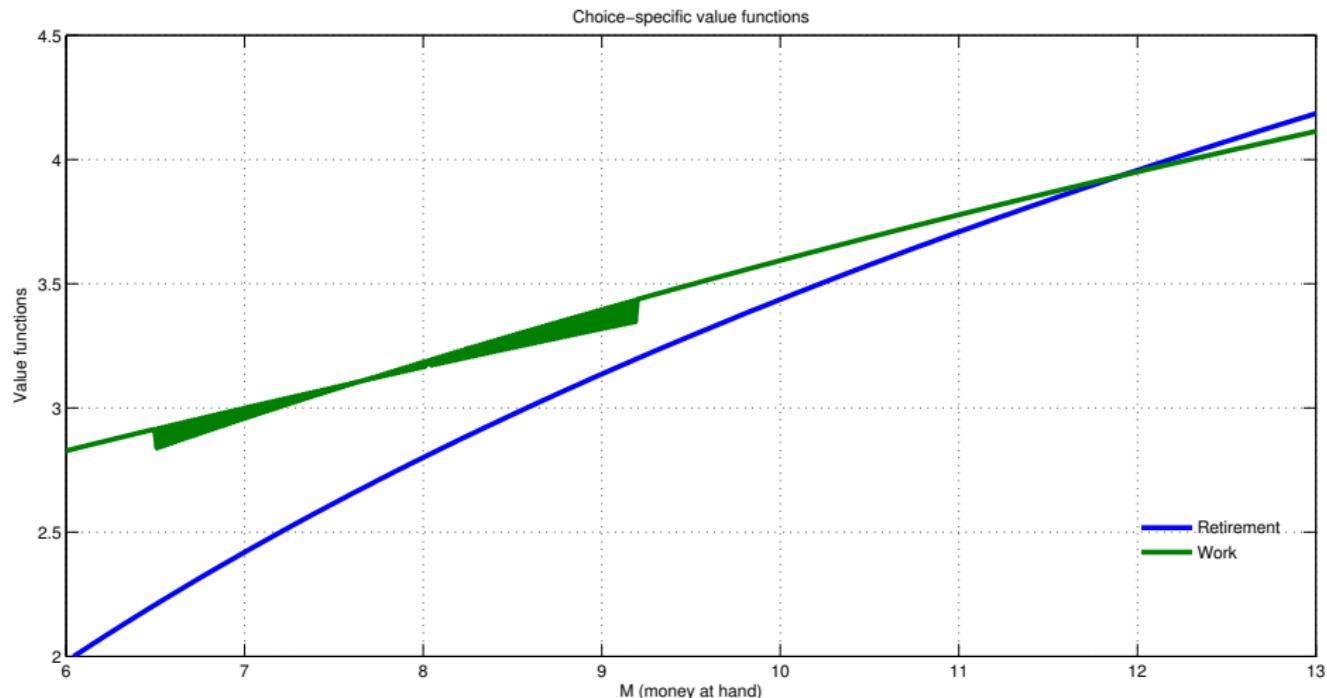
# Period $T - 1$ : Optimal consumption



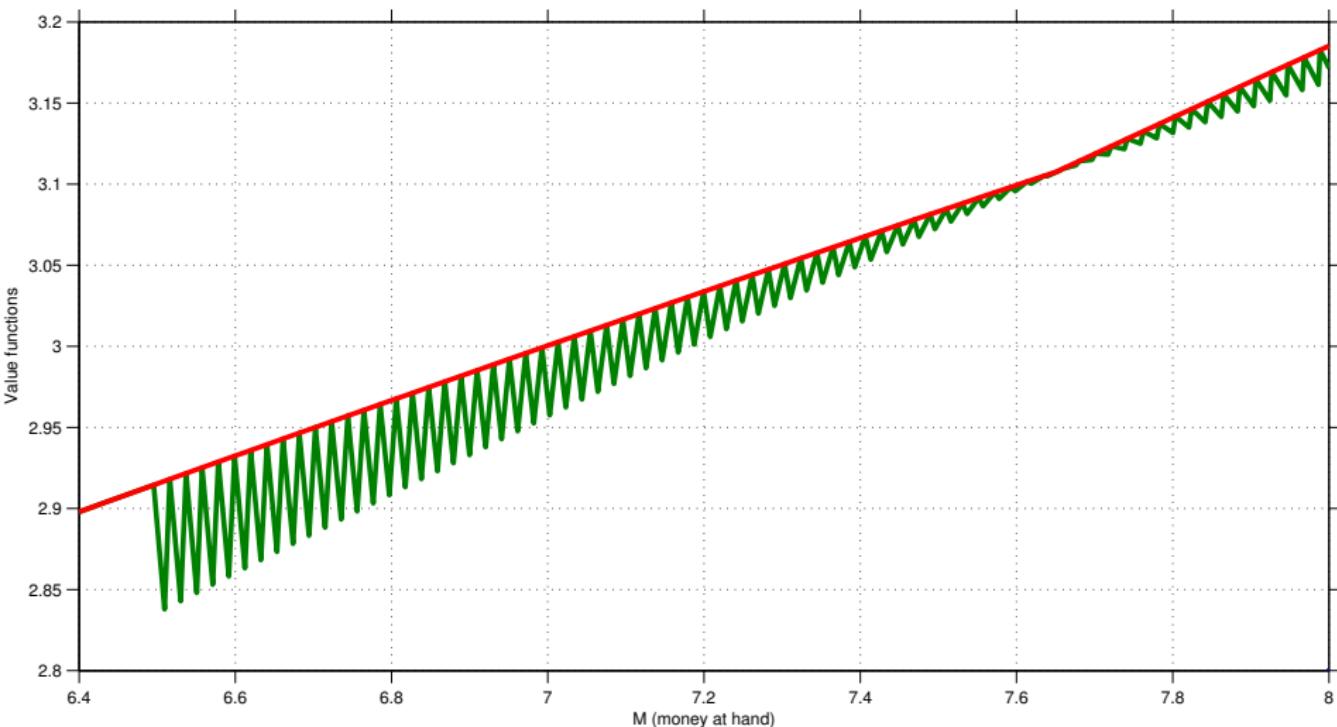
# So, what is going on

- ➊  $d$ -specific value functions intersect  
(due to trade-off between income and disutility of work)  
↓
- ➋ The **upper envelope** of the value functions has a kink  
and combined consumption function has a discontinuity

# Period $T - 2$ : Choice specific VF



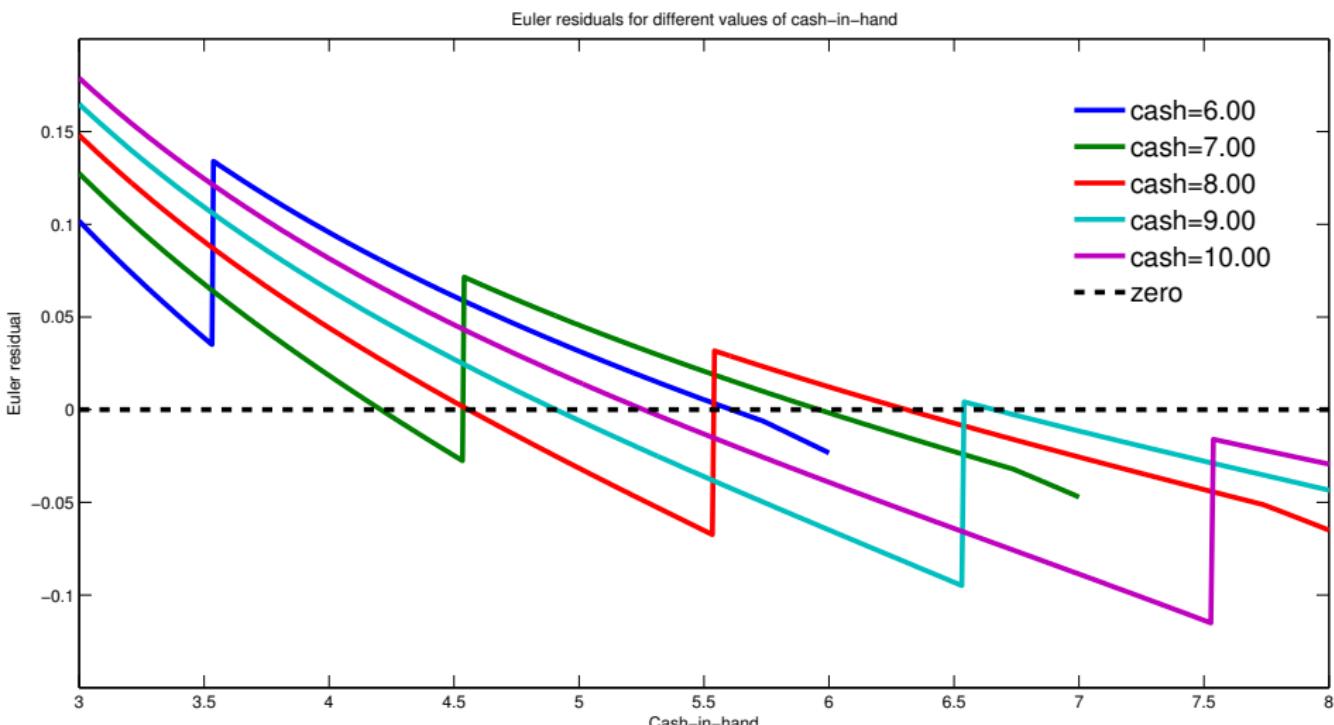
## Period $T - 2$ : Secondary upper envelope



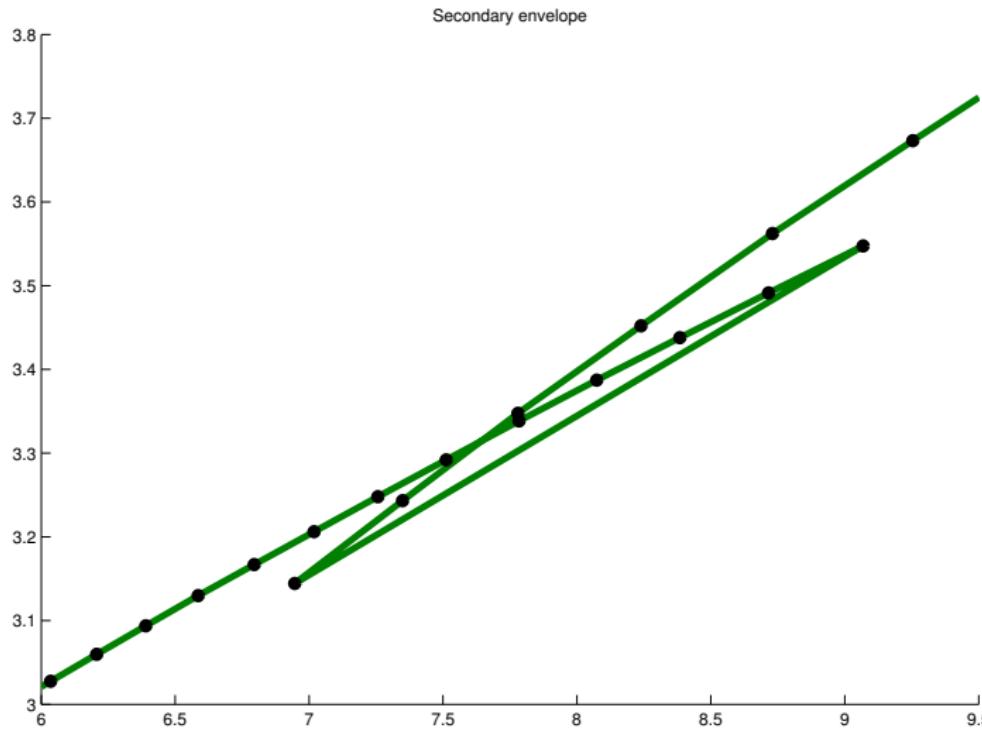
# So, what is going on

- ➊  $d$ -specific value functions intersect  
(due to trade-off between income and disutility of work)  
↓
- ➋ The **upper envelope** of the value functions has a kink  
and combined consumption function has a discontinuity  
↓
- ➌ Derivative of the value function has a discontinuity  
at the kink  
↓
- ➍ For some values of wealth (on endogenous grid) Euler equation has  
two solutions!  
If endogenous grid points are sorted → **zigzag region**

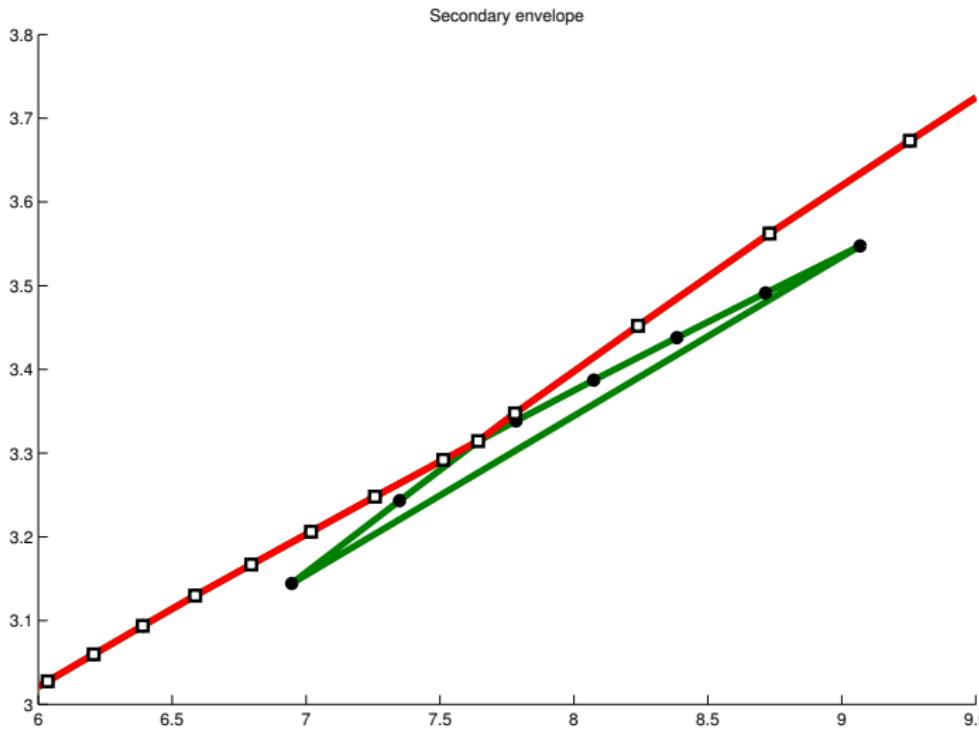
# Multiple zeros of Euler residuals



# Period $T - 2$ : Secondary upper envelope: detect



# Period $T - 2$ : Secondary upper envelope: result



# How to algorithmically detect “zigzag” regions?

## Theorem: monotonicity

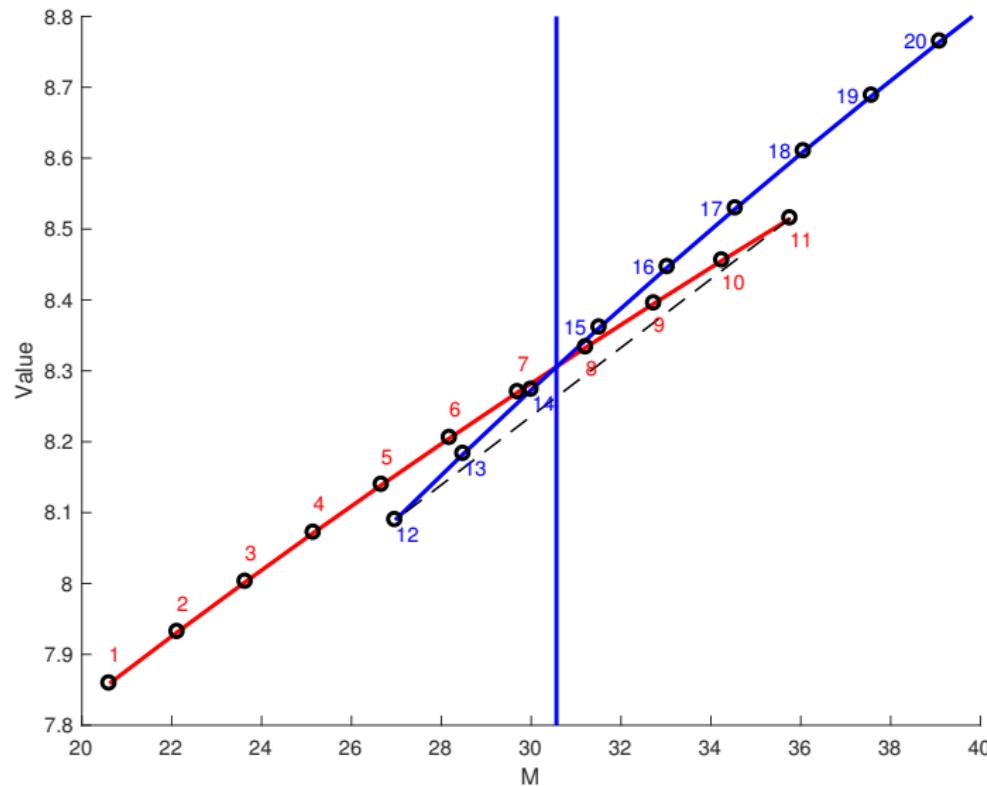
Under weak regularity conditions on the utility function and intertemporal budget constraint, savings function is weakly increasing.

$$A_t(M'_t) \geq A_t(M''_t) \text{ for every } M'_t \geq M''_t \text{ for all } t.$$

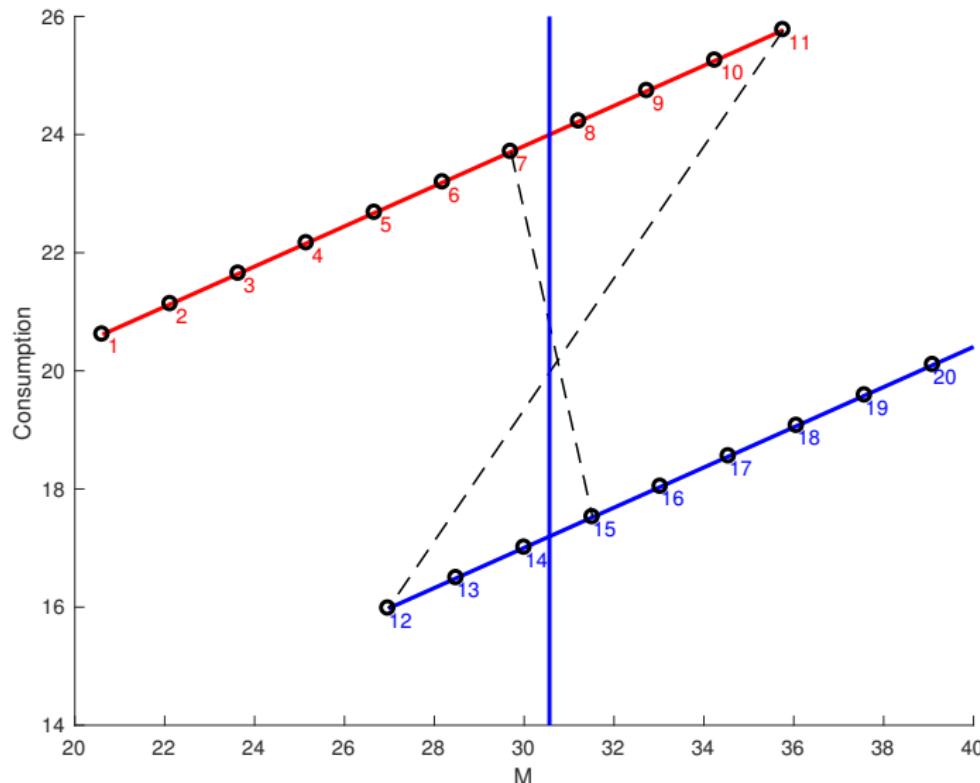
Note: savings function may still have “upward” jumps

- ① Sort the exogenous grid over  $A$  in **ascending order**
- ② Then the sequence of endogenous grid points over  $M$  has to be in **ascending order as well** as long as Euler equation is sufficient
- ③ Every time the endogenous grid **“bends back”** the endogenous grid is separated into subsets of points
- ④ Calculate the **Upper envelope** on the segments over the subsets
- ⑤ **Delete suboptimal endogenous points**
- ⑥ Find and add a kink point to the endogenous grid

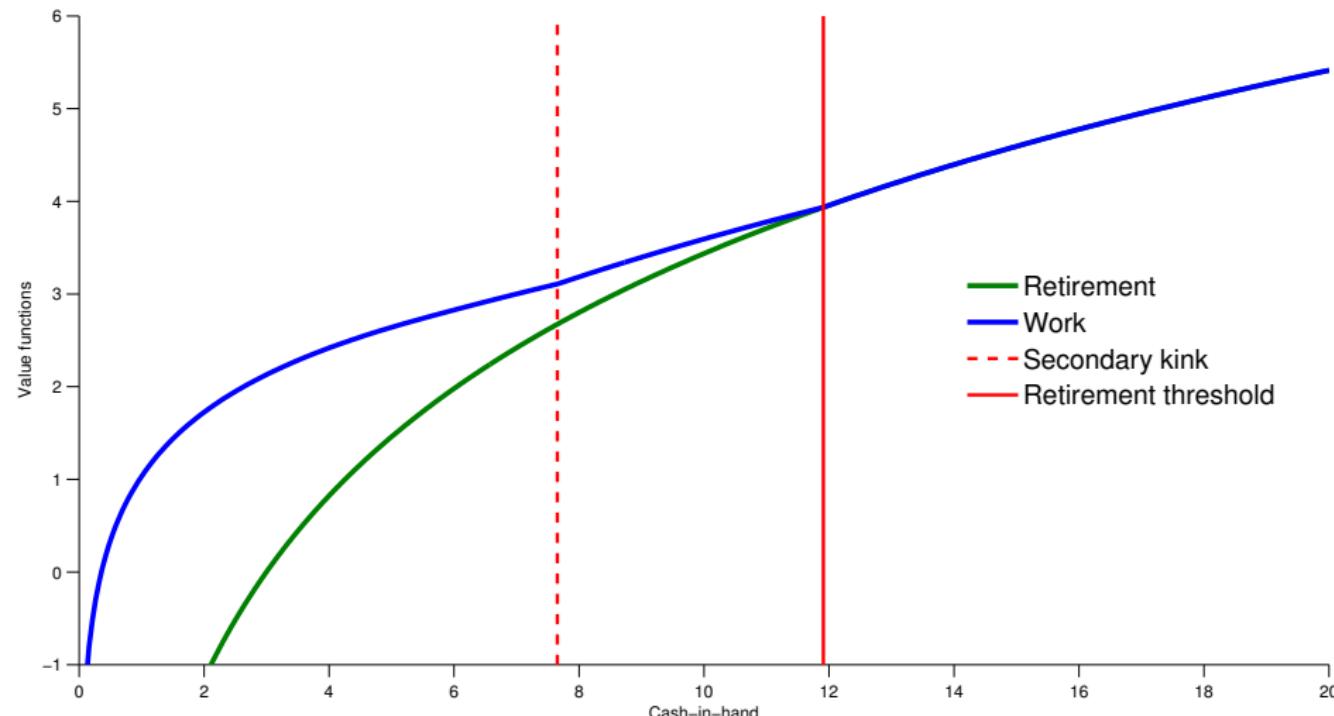
# What happens to optimal consumption?



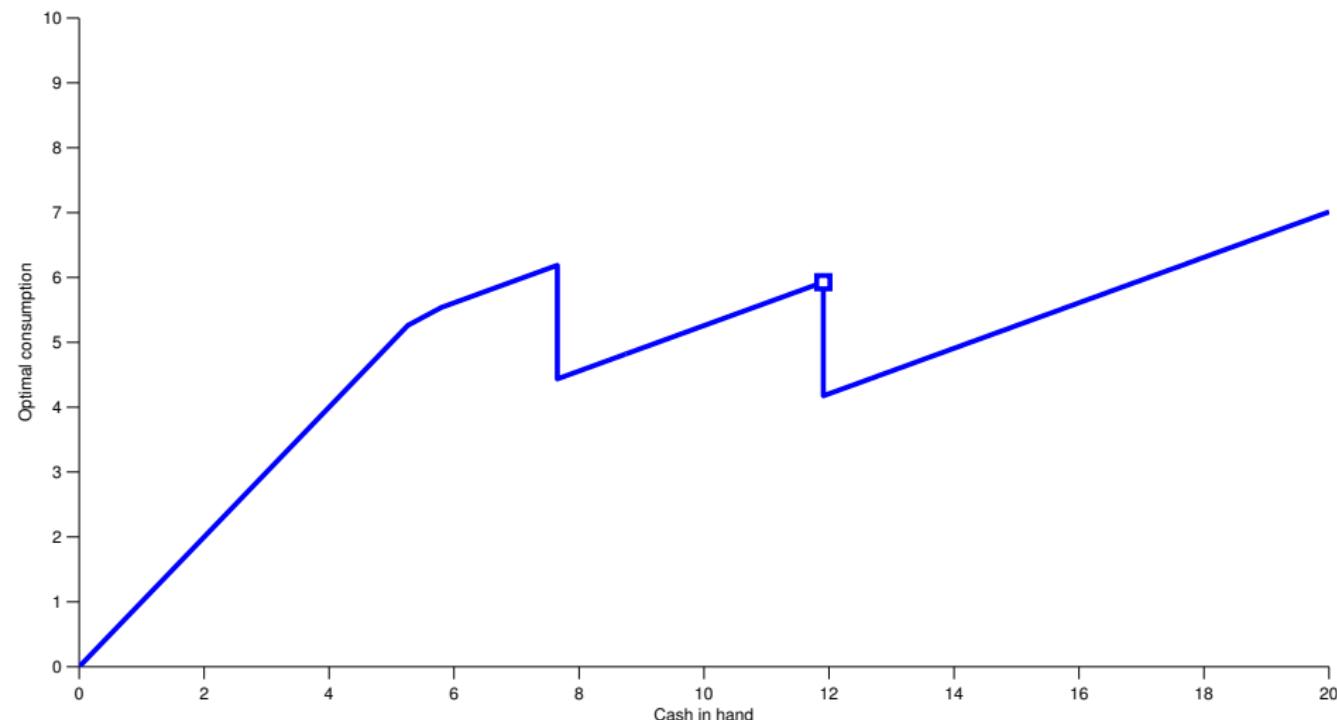
# What happens to optimal consumption?



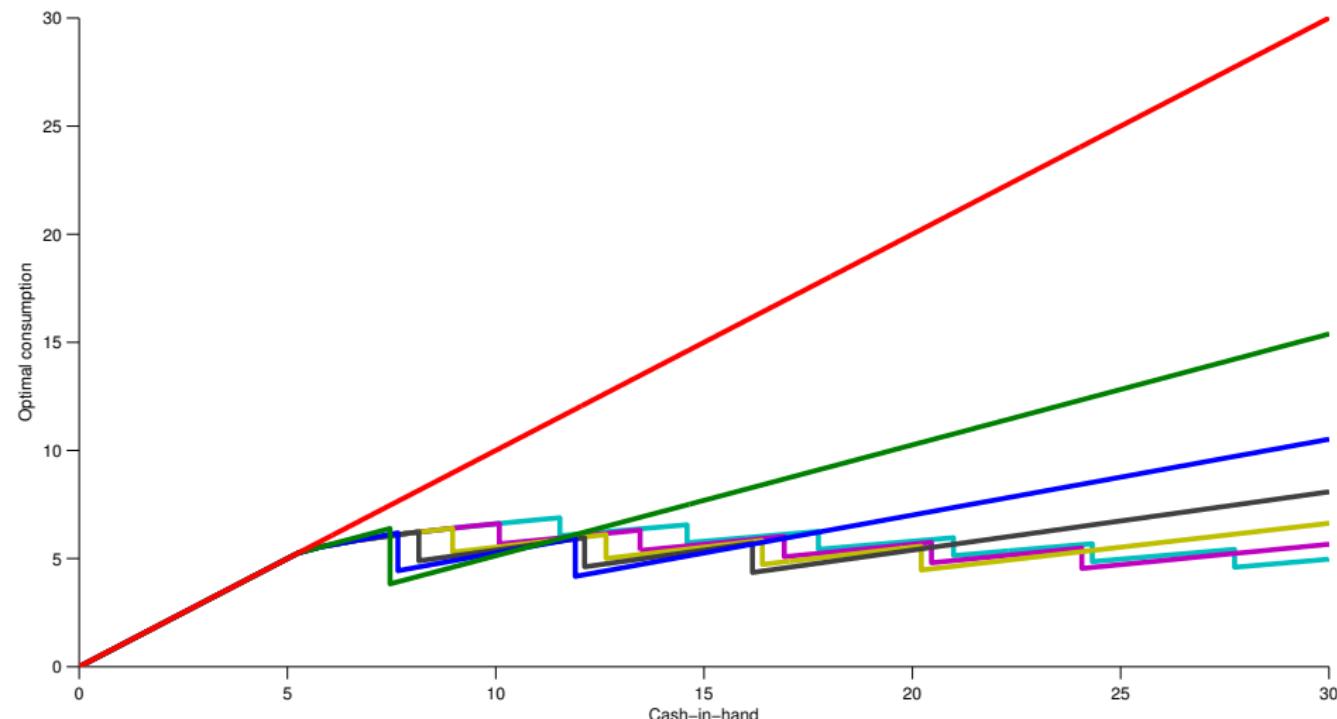
## Period $T - 2$ : VF, primary and secondary kinks



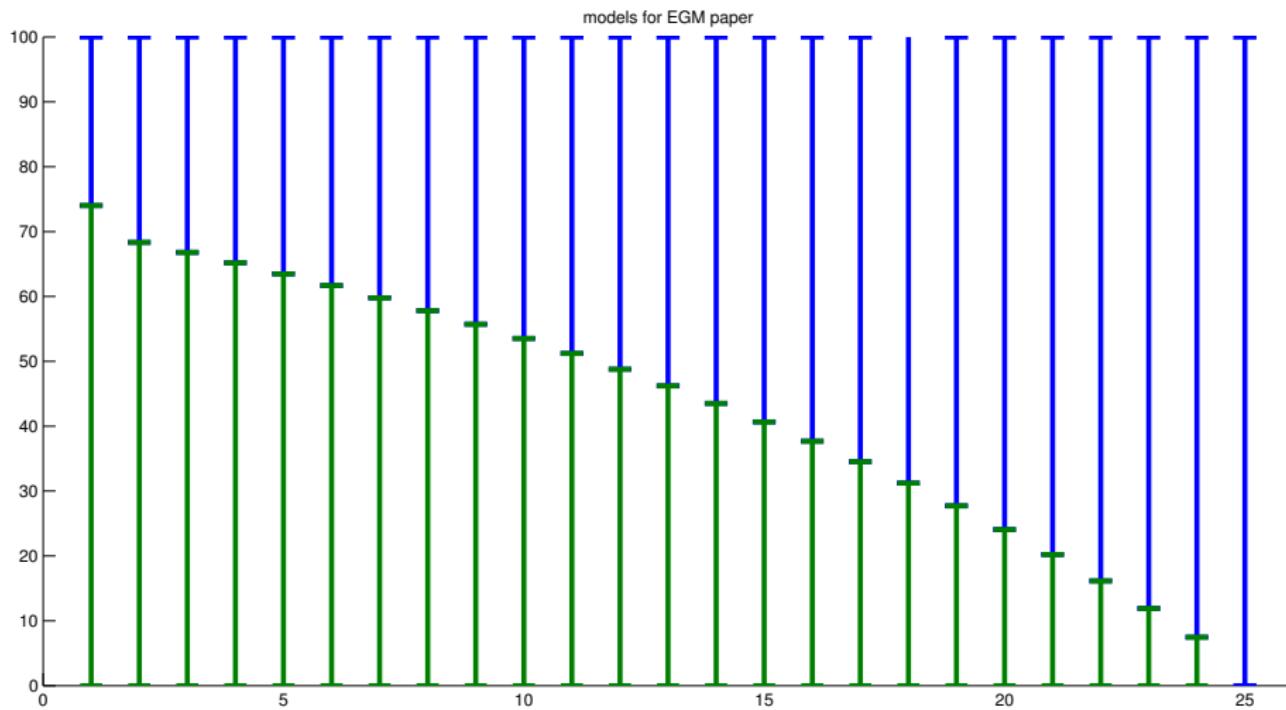
# Period $T - 2$ : Optimal consumption



# Optimal consumption (many periods)



# Optimal retirement (many periods)



# DC-EGM full algorithm

## DC-EGM ver. 2.0

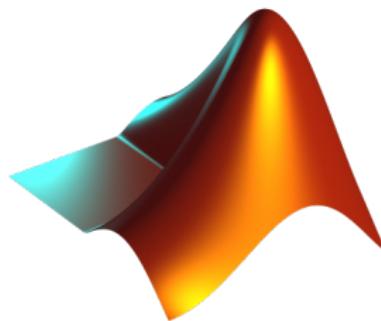
- ❶ Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
- ❷ EGM step for each discrete choice  $d$  and every state  $st$
- ❸ Compute  $d$ -specific value functions and consumption rules
- ❹ Compute the “secondary” upper envelope over the “zig-zag” regions of the  $d$ -specific value functions and update the corresponding consumption rules
- ❺ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- ❻ Reconstruct overall consumption rule and value function from optimal switching points

# Properties of the full solution

- ➊ Value functions are non-concave and have **kinks**
- ➋ Consumption functions have **discontinuities**
- ➌ Discontinuities/kinks **propagate** through time and **accumulate**

This properties are attributes of the model itself.  
Any solution method has to deal with these complexities.

DC-EGM matches the analytical solution perfectly!



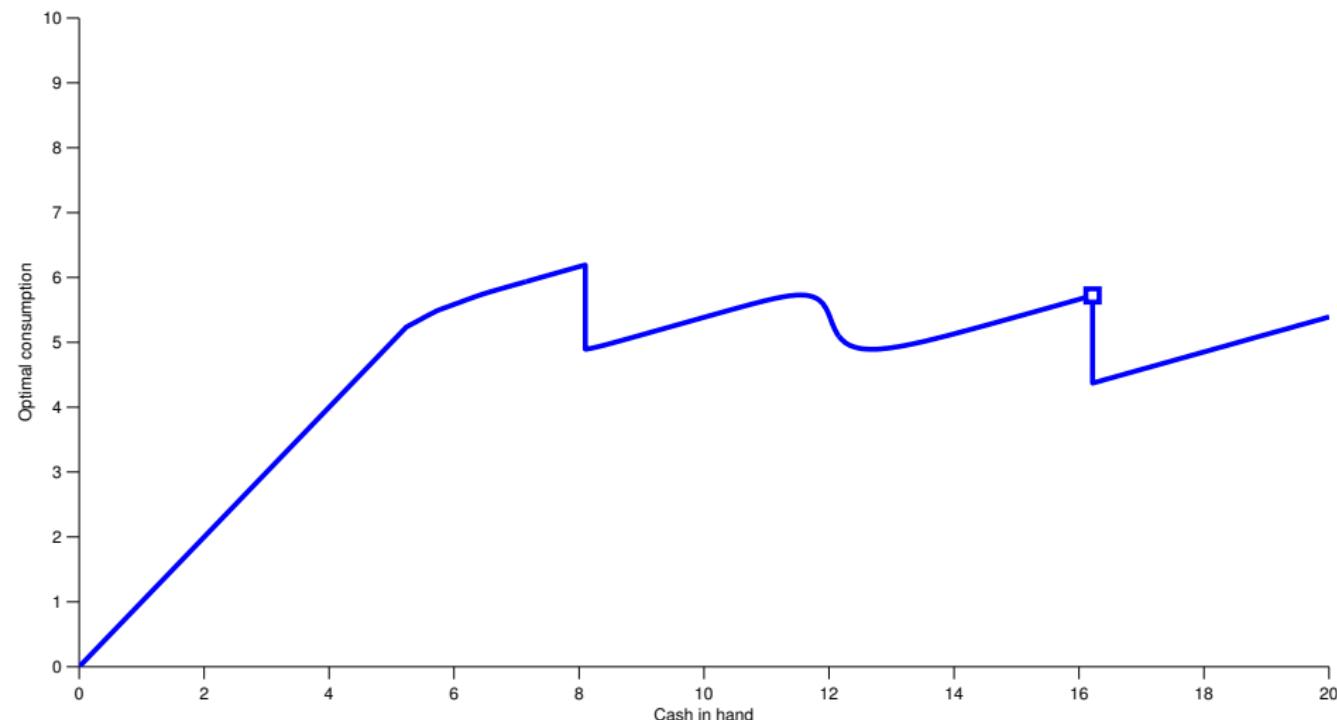
- ➊ Replicate the solution using `model_retirement.m`
- ➋ Simulate the consumption path for  $\beta R = 1$  and discuss the accuracy of the solutions

# Random returns $\tilde{R}$

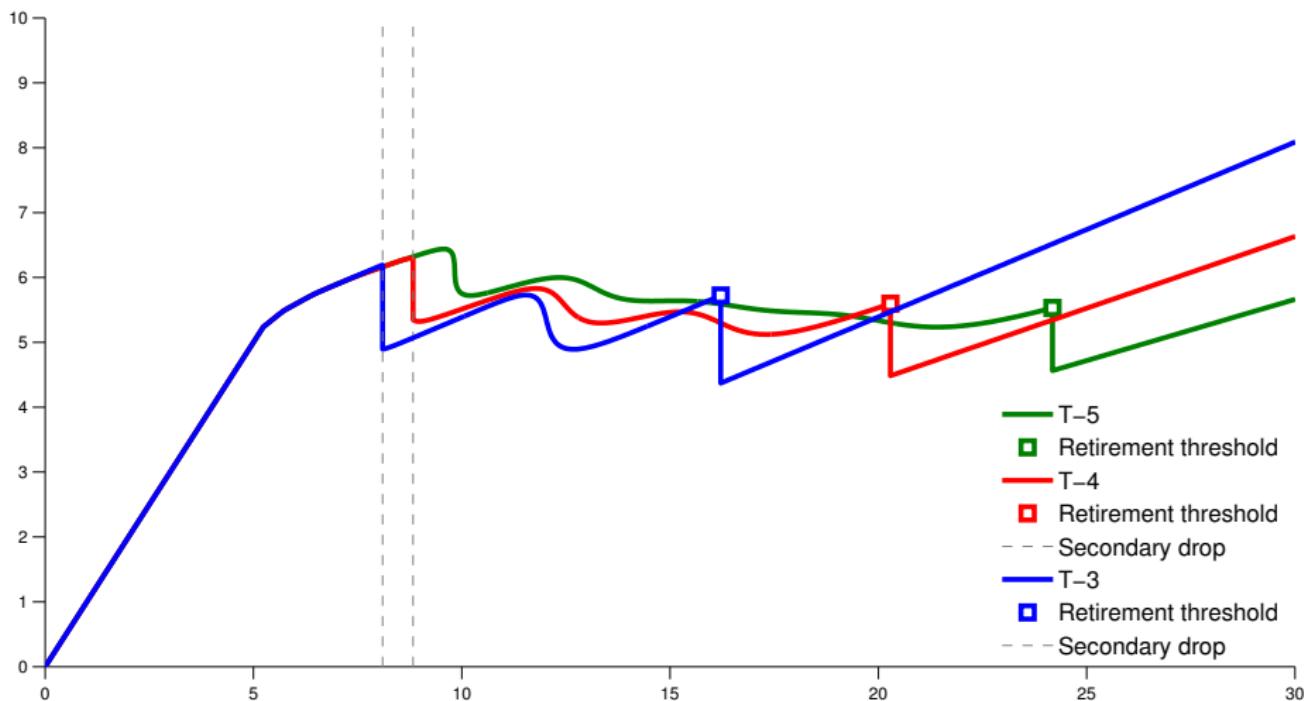
Random shocks do help, however:

- Smooth out secondary kinks only
- Primary kinks (switching between discrete options) remain
- May not smooth out all kinks: continuous but sharp declines in optimal consumption at  $t$  may lead to a discontinuity/kink at  $t - 1$
- Expectations in Euler equation have to be taken over discontinuous functions
  - More kinks/discontinuities from sloppy computation
  - Need to integrate over “continuous” intervals separately

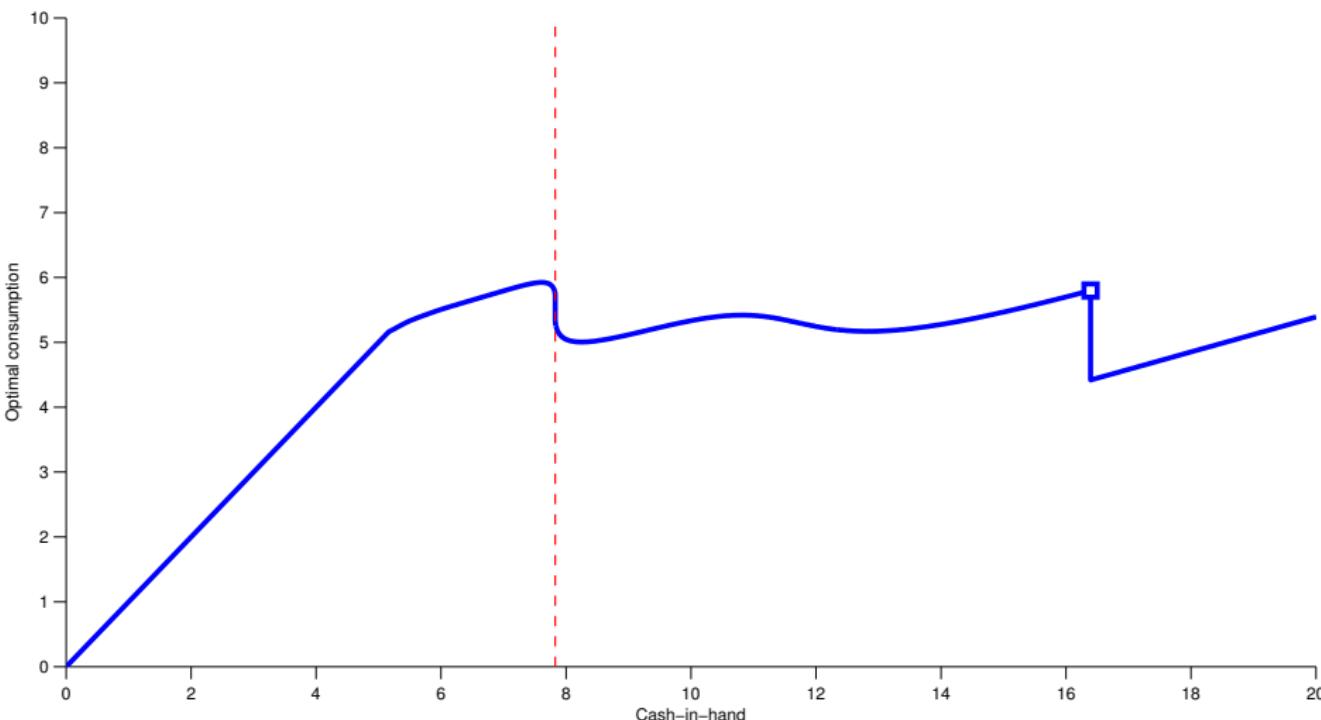
# Period $T - 3$ : Optimal consumption with $\sigma = .1$



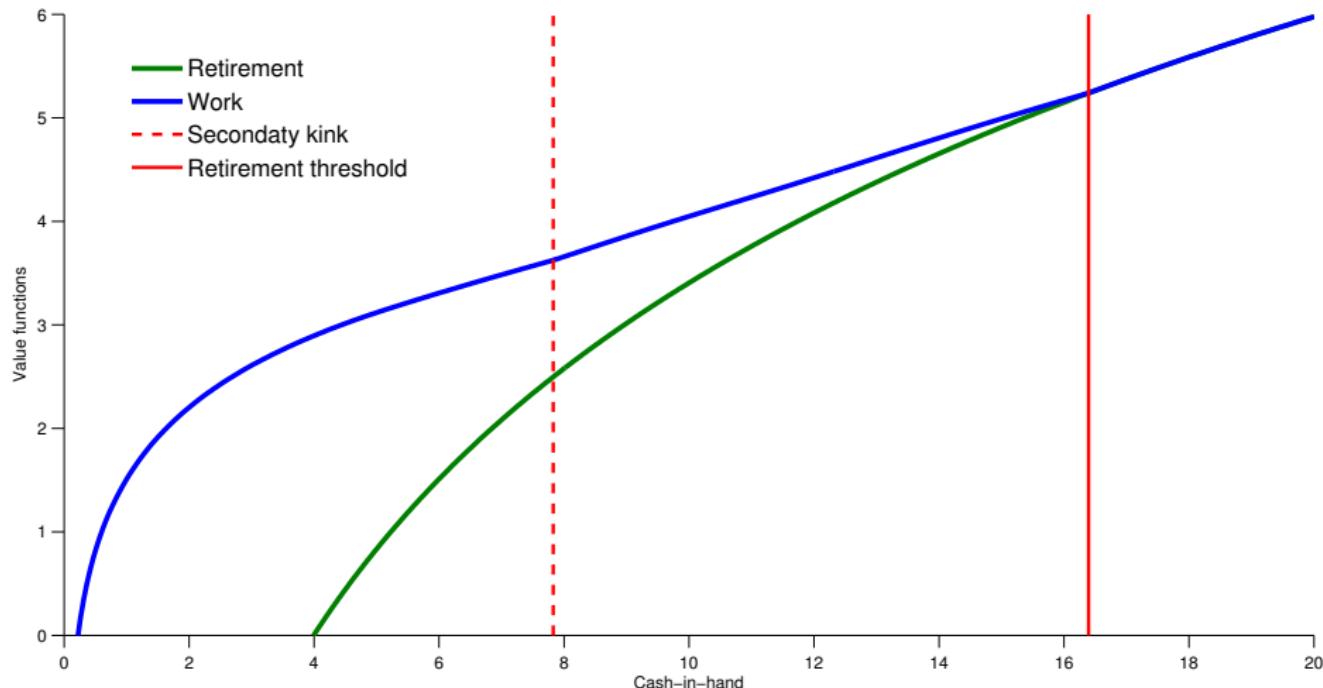
# Before $T - 3$ : Optimal consumption with $\sigma = .1$



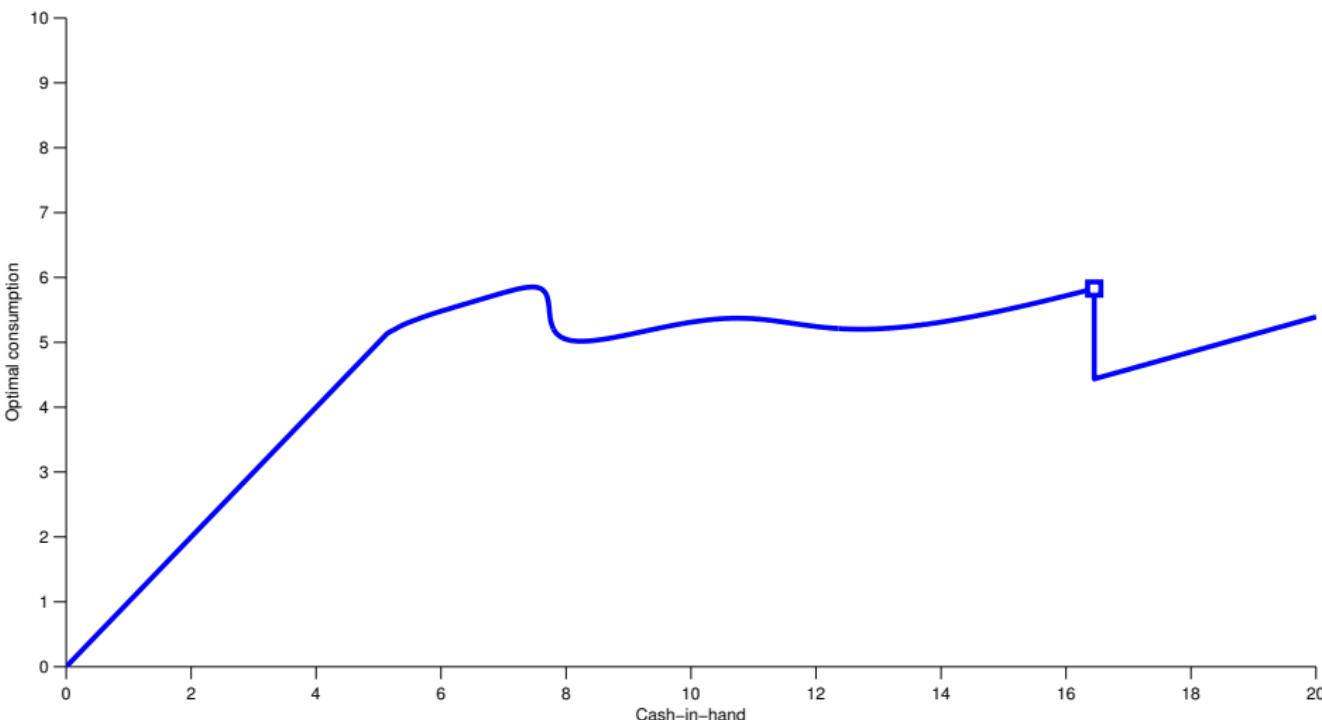
# Period $T - 3$ : Optimal consumption with $\sigma = .2$



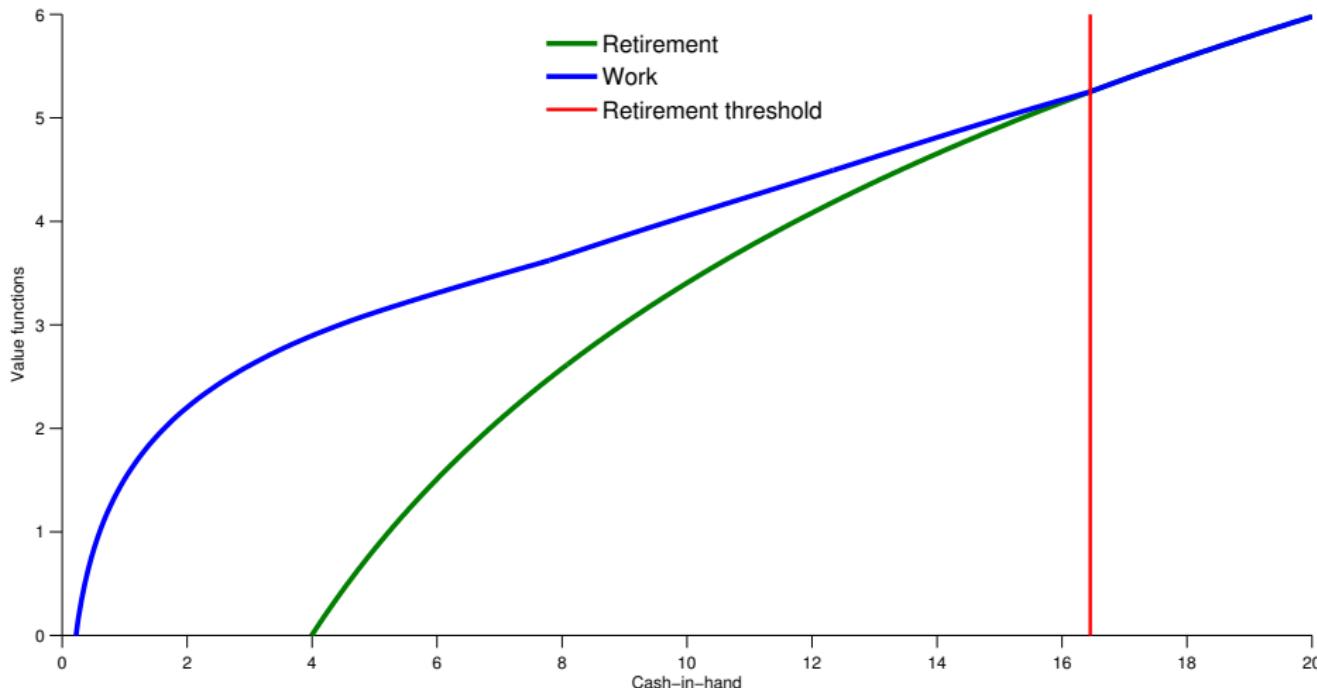
Period  $T - 3$  : VF with  $\sigma = .2$



# Period $T - 3$ : Optimal consumption with $\sigma = .22$

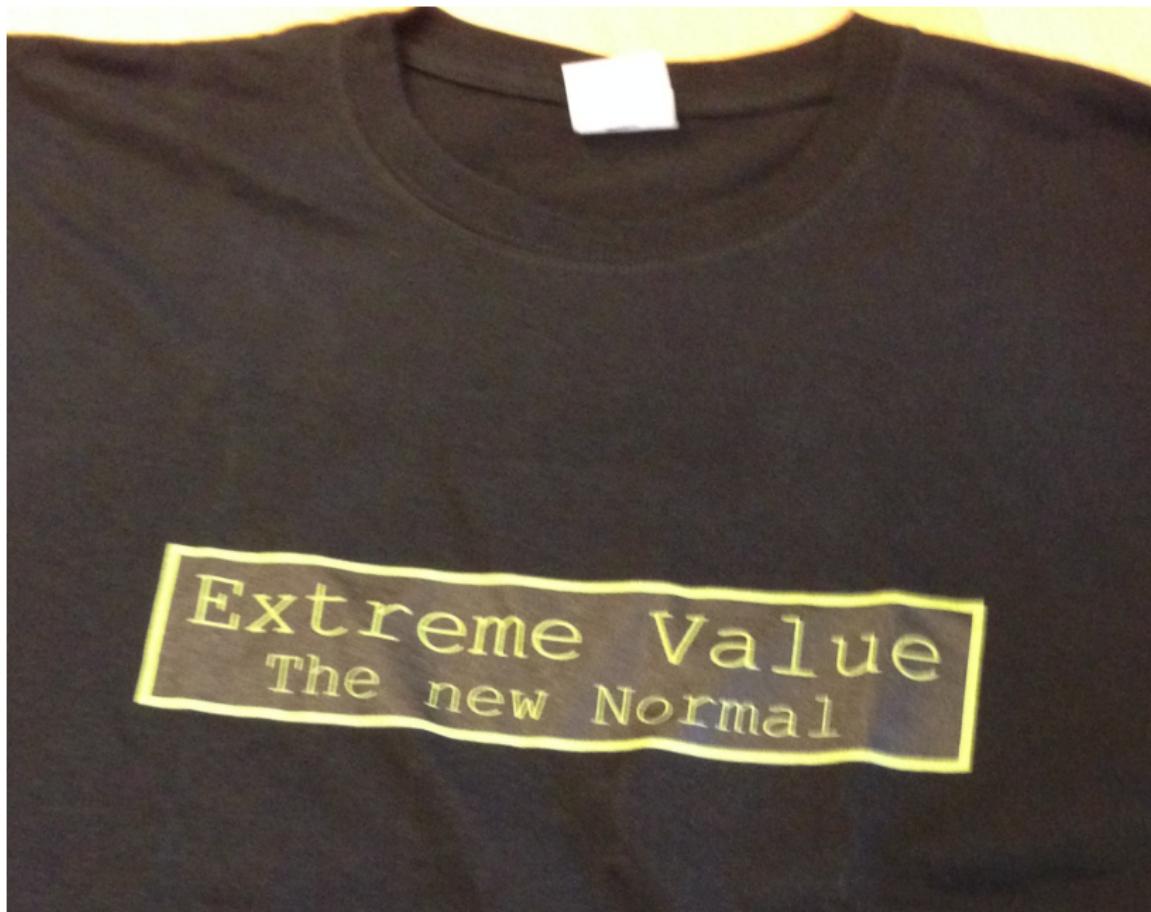


Period  $T - 3$  : VF with  $\sigma = .22$



# Extreme value distributed taste shocks

- Smooth out **primary kinks**
- Extreme value distribution – closed form expectations and standard in empirical applications
- Two interchangeable interpretations
  - Structural: unobserved state variables
  - Logit smoothing: to streamline the solution
- Work together with other shocks in the model
  - EV taste shocks smooth out primary kinks
  - Random returns smooth out secondary kinks
- Complete smoothing is not guaranteed in general: secondary kinks may persist



# Retirement problem with taste shocks

Re-formulate in terms of **choice specific** value functions

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} v_t(M_t, \mathbb{W}, \mathbb{R}) + \sigma \epsilon_{\mathbb{R}} \\ v_t(M_t, \mathbb{W}, \mathbb{W}) + \sigma \epsilon_{\mathbb{W}} \end{array} \right\}$$

$$\begin{aligned} v_t(M_t, \mathbb{W}, \mathbb{W}) &= \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t + y - c), \mathbb{W} \right) \right] \\ v_t(M_t, \mathbb{W}, \mathbb{R}) &= \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right] \end{aligned}$$

$$EV_{t+1}(x, \mathbb{W}) = \sigma \log \left[ \exp \frac{v_{t+1}(x, \mathbb{W}, \mathbb{W})}{\sigma} + \exp \frac{v_{t+1}(x, \mathbb{W}, \mathbb{R})}{\sigma} \right]$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

# Smoothed Euler equation

Without taste shocks – “**discontinuous**” Euler equation:

$$u'(c_t) = \beta E \left[ u' \left( c_{t+1}(\mathbb{W}/\mathbb{R}) \right) \tilde{R} \right]$$

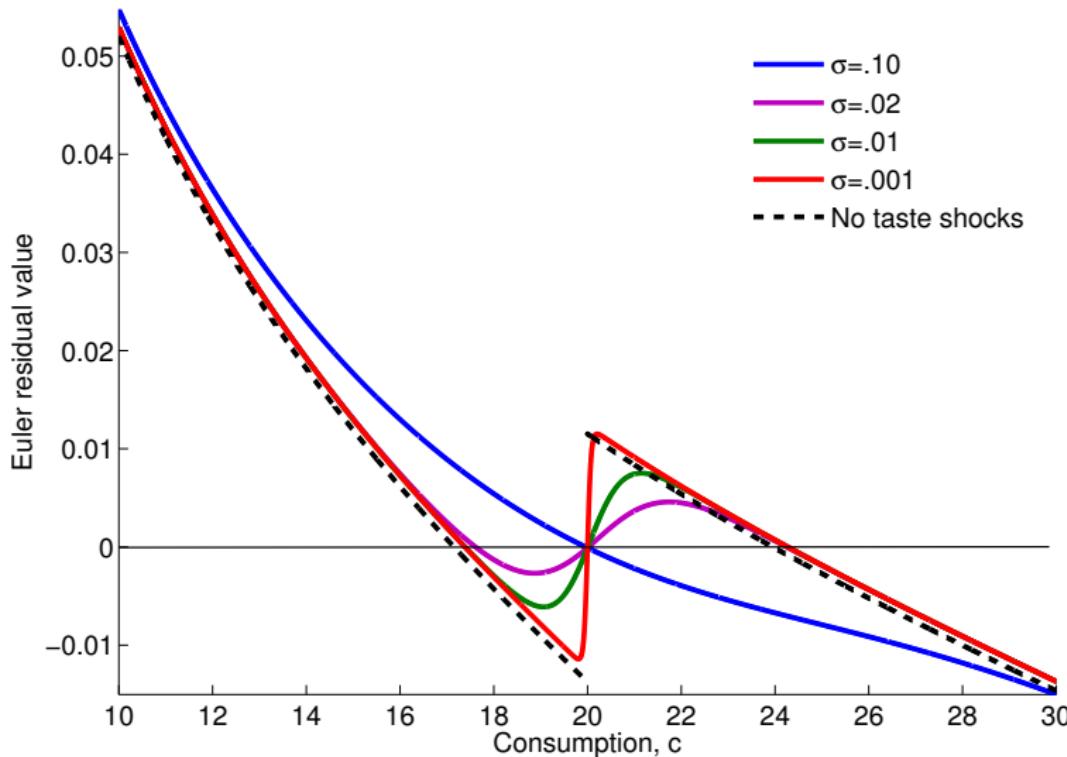
With EV taste shocks – **smoothed** Euler equation:

$$u'(c_t) = \beta E \left[ P_{t+1}(\mathbb{W}) u' \left( c_{t+1}(\mathbb{W}) \right) \tilde{R} + P_{t+1}(\mathbb{R}) u' \left( c_{t+1}(\mathbb{R}) \right) \tilde{R} \right]$$

Choice probability

$$P_{t+1}(\mathbb{W}) = \frac{\exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{W})}{\sigma}}{\exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{W})}{\sigma} + \exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{R})}{\sigma}}$$

# Smoothed Euler equation



# DC-EGM with taste shocks

## DC-EGM ver. 3.0

- ① Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
- ② EGM step for each discrete choice  $d$  and every state  $st$
- ③ Compute  $d$ -specific value functions and consumption rules
- ④ Compute the “secondary” upper envelope over the “zig-zag” regions of the  $d$ -specific value functions and update the corresponding consumption rules
- ⑤ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- ⑥ Reconstruct overall consumption rule and value function from optimal switching points

# DC-EGM with taste shocks

- ➊ With EV taste shocks DC-EGM becomes **simpler**
- ➋ The problem is re-formulated in terms of **choice specific value functions**
- ➌ Calculation of *primary* upper envelope is replaced by calculation of **logsum**
- ➍ Easier computation of expectations (due to less discontinuities)
- ➎ More memory is required to store choice specific value functions

# Extreme value Homotopy

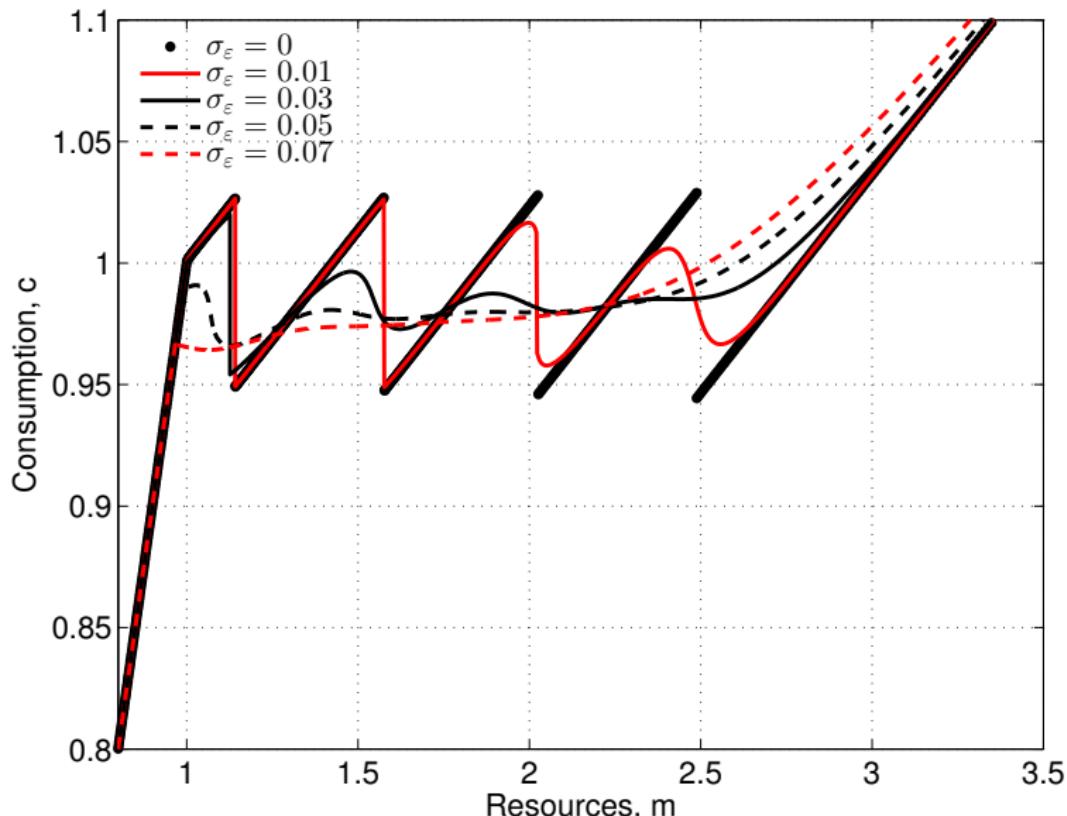
Theorem: approximation with logit smoother

Let  $\sigma$  index scale of Type 1 extreme value taste shocks for the discrete choices in a DC-DP problem with  $D$  choices. Then we have the following bound

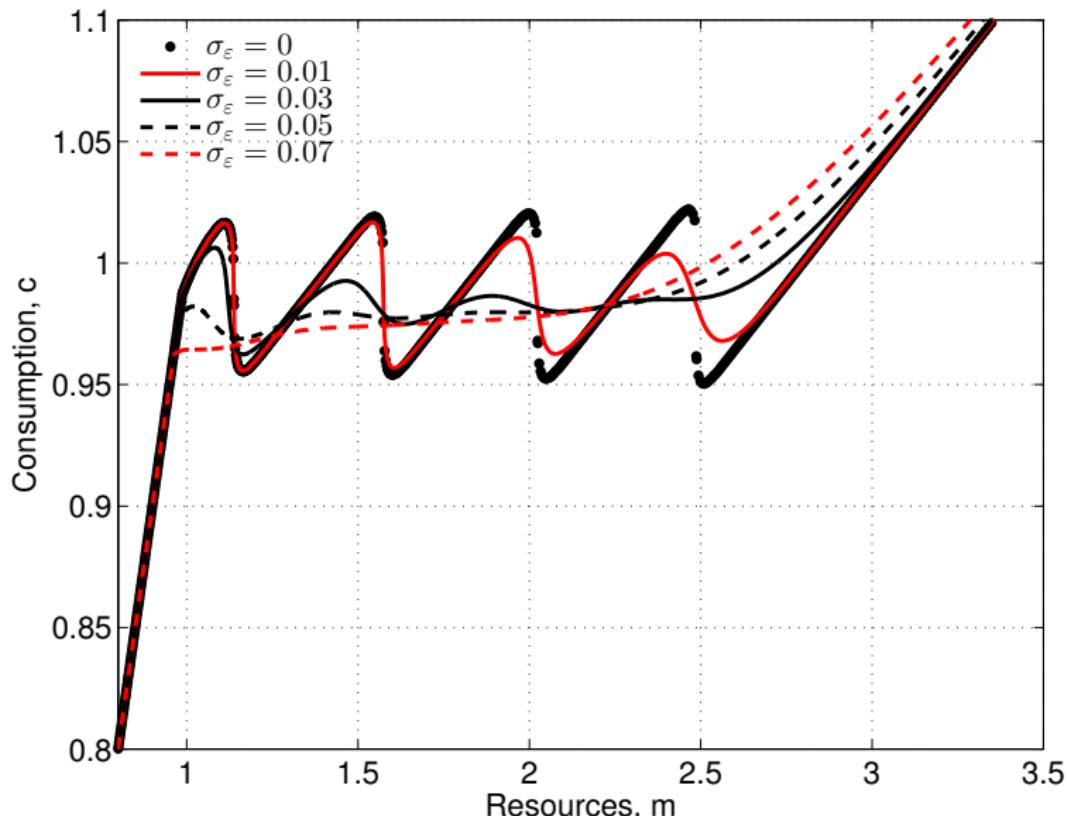
$$|EV_{\sigma,t}(s) - V_t(s)| \leq \sigma \left[ \sum_{j=0}^{T-t} \beta^j \right] \log(D)$$

This implies that the extreme-value perturbed policy functions  $c_{\sigma,t}(s, \epsilon)$  and  $\delta_{\sigma,t}(s, \epsilon)$  converge pointwise to  $c_t(s)$  and  $\delta_t(s)$ , the optimal continuous and discrete decision rules to a DP problem without any taste shocks as  $\sigma \rightarrow 0$ .

## Optimal consumption with taste shocks only



# Optimal consumption with random returns



# Credit constraints

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Instead we “connect the dots”  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$

$M_t^{cc}$  — level of wealth corresponding to  $A_t = 0$

- Inevitable when value functions have to be computed
- If utility is additively separable in consumption and discrete choices (AS), the problem can be avoided entirely!

# Credit constraints

## Dealing with credit constraints

- ❶ For each  $d_t$  compute  $M_{t,d_t}^{cc}$  correspond to zero savings  
EGM loop can be started from  $A = 0$

$$M_{t,d_t}^{cc} : \forall M < M_{t,d_t}^{cc} \quad c_t^* = M$$

- ❷ Value function for  $M < M_{t,d_t}^{cc}$  has analytic form

$$V_t^{d_t}(M) = u(M, d_t) + \beta EV_{t+1}^0(d_t)$$

$EV_{t+1}^0(d_t)$  — expected value of ending period  $t$  with  $A_t = 0$

- ❸ (AS)  $\Rightarrow V_t^{d_t}(M) = u(M) + v(d_t) + \beta EV_{t+1}^0(d_t)$

- ❹  $V_t^{d_t}(M)$  do not intersect when  $M < \min_{d_t} \{M_{t,d_t}^{cc}\}$

$\Rightarrow$  No need to compute utility of nearly zero consumption

# Credit constraints

## Dealing with credit constraints

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 $EV_{t+1}^0(d_t)$  — expected value of ending period  $t$  with  $A_t = 0$
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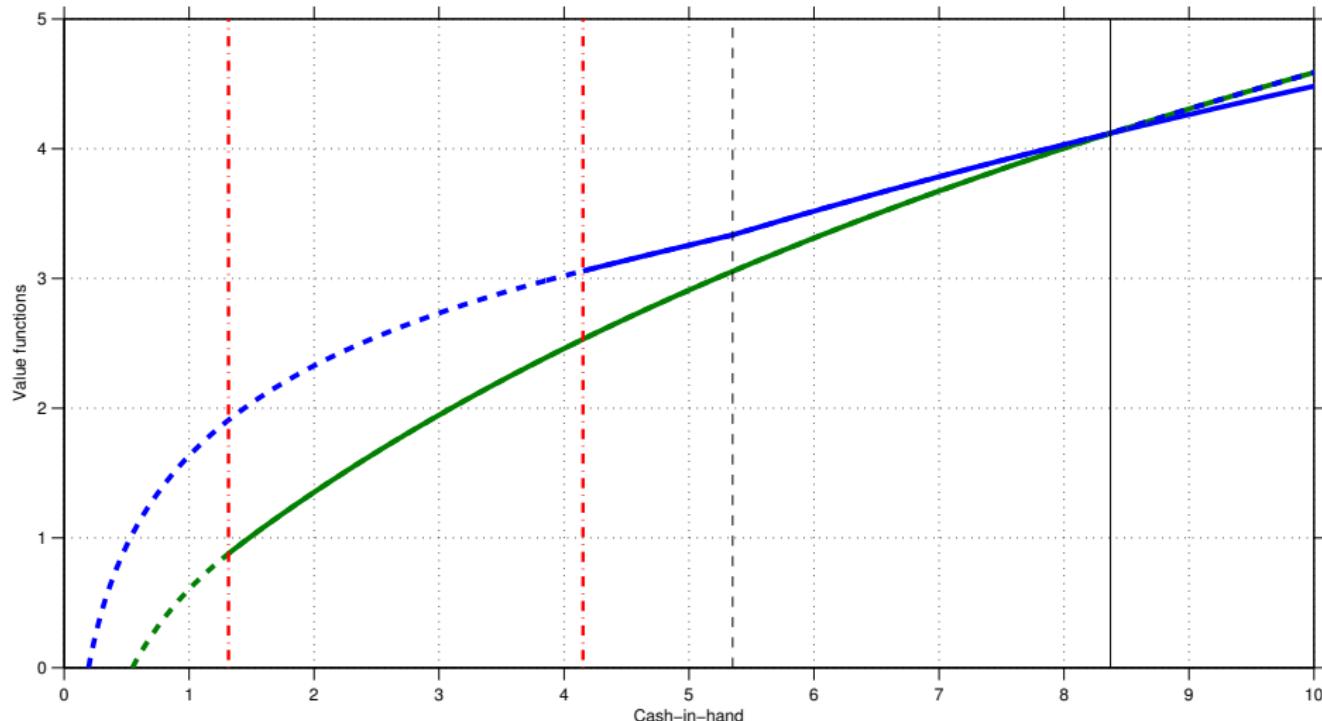
$EV_{t+1}^0(d_t)$  — expected value of ending period  $t$  with  $A_t = 0$

- ③ (AS)  $\Rightarrow V_t^{d_t}(M) = u(M) + v(d_t) + \beta EV_{t+1}^0(d_t)$

- ④  $V_t^{d_t}(M)$  do not intersect when  $M < \min_{d_t} \{ M_{t,d_t}^{cc} \}$

$\Rightarrow$  No need to compute utility of nearly zero consumption

## Pension benefit .25y



# Multi-dimensional generalizations

# EGM + VFI



Barillas & Fernandez-Villaverde, JEDC 2007

A Generalization of the Endogenous Grid Method

- ➊ Run EGM w.r.t. *one* choice keeping other controls fixed
- ➋ Perform a VFI w.r.t. the rest of decision variables

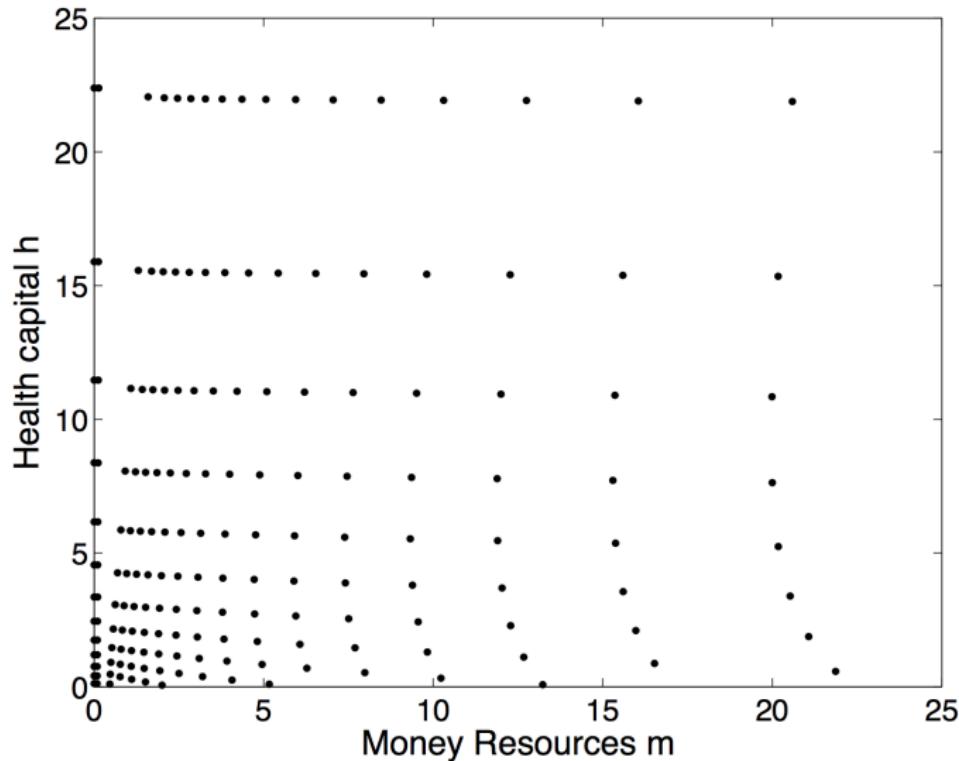


Ludwig & Schön, Computational Economics, 2018

Endogenous Grids in Higher Dimensions: Delaunay Interpolation and Hybrid Methods

- ➊ Solve the model of human capital investment + consumption/savings
- ➋ Compare three approaches which differ by the interpolation method
- ➌ Need to interpolate on irregular multidimensional grid

## Multidimensional endogenous grid



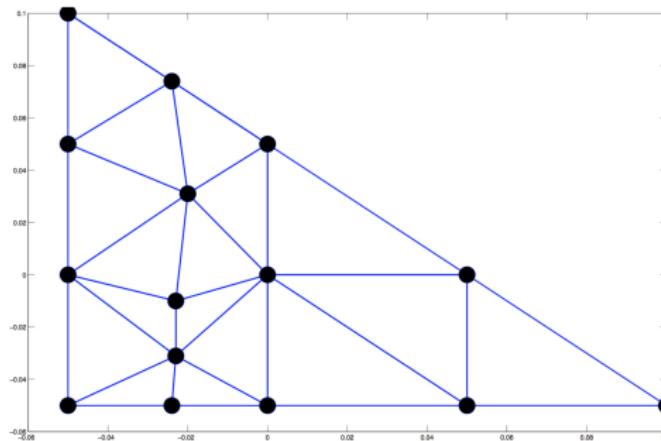
# Interpolation on the irregular grid



Johannes Brumm, Michael Grill, JEDC 2014

Computing equilibria in dynamic models with occasionally binding constraints

- Delaunay triangulation based interpolation



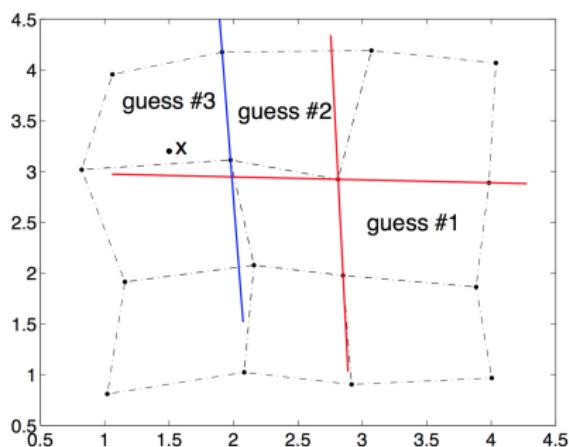
# Interpolation on the irregular grid



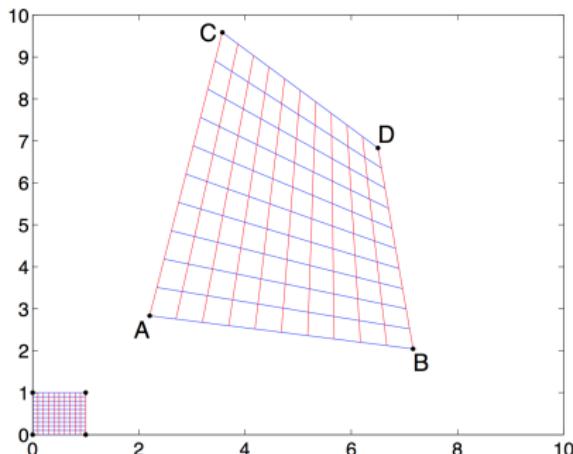
Matthew White, JEDC 2015

The Method of Endogenous Gridpoints in Theory and Practice

- Focus on general theory of multidimensional EGM
- Map non-linear rectangles into regular ones



(a) Identifying the sector by visibility walk



(b) Identifying relative coordinates

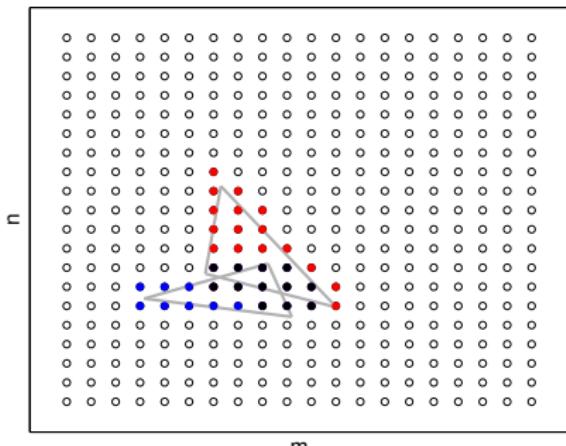
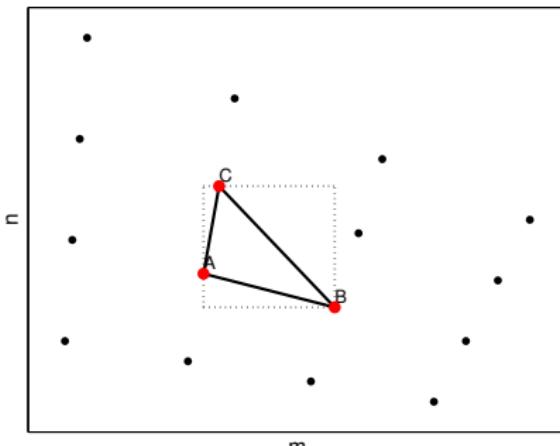
# Interpolation on the irregular grid



Jeppe Druedahl, Thomas Jørgensen, JEDC 2017

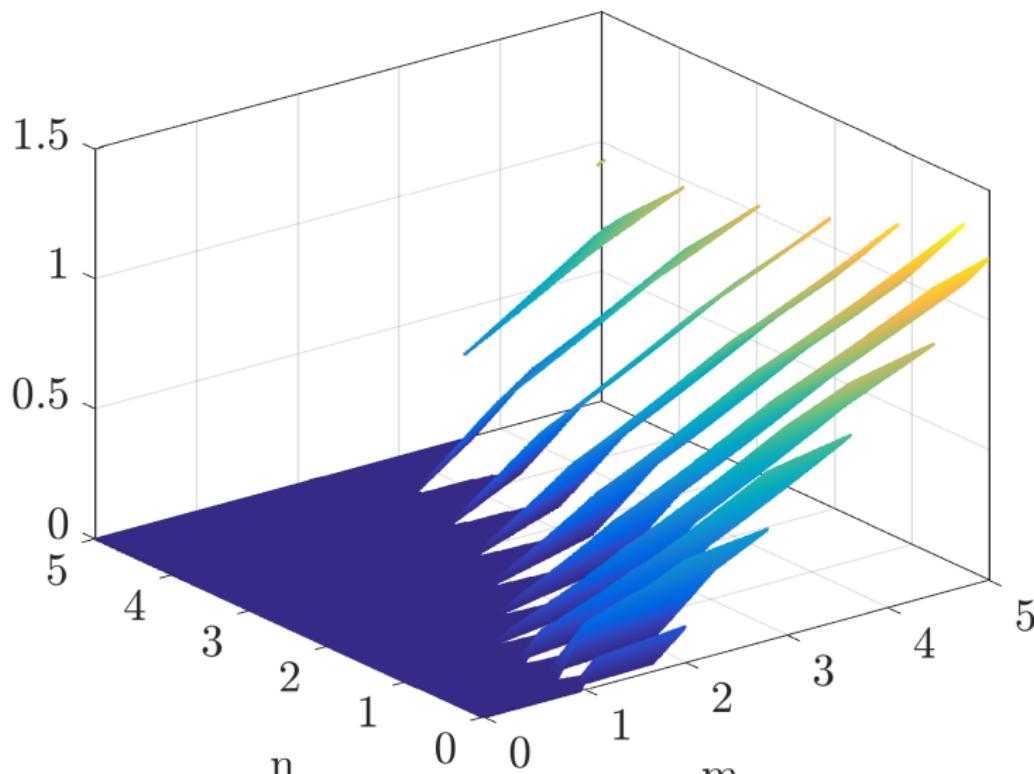
A General Endogenous Grid Method for Multi-Dimensional Models  
with Non-Convexities and Constraints

- Add occasionally binding constraints and allow for non-convexities
- Re-interpolate on regular grid while performing upper envelope



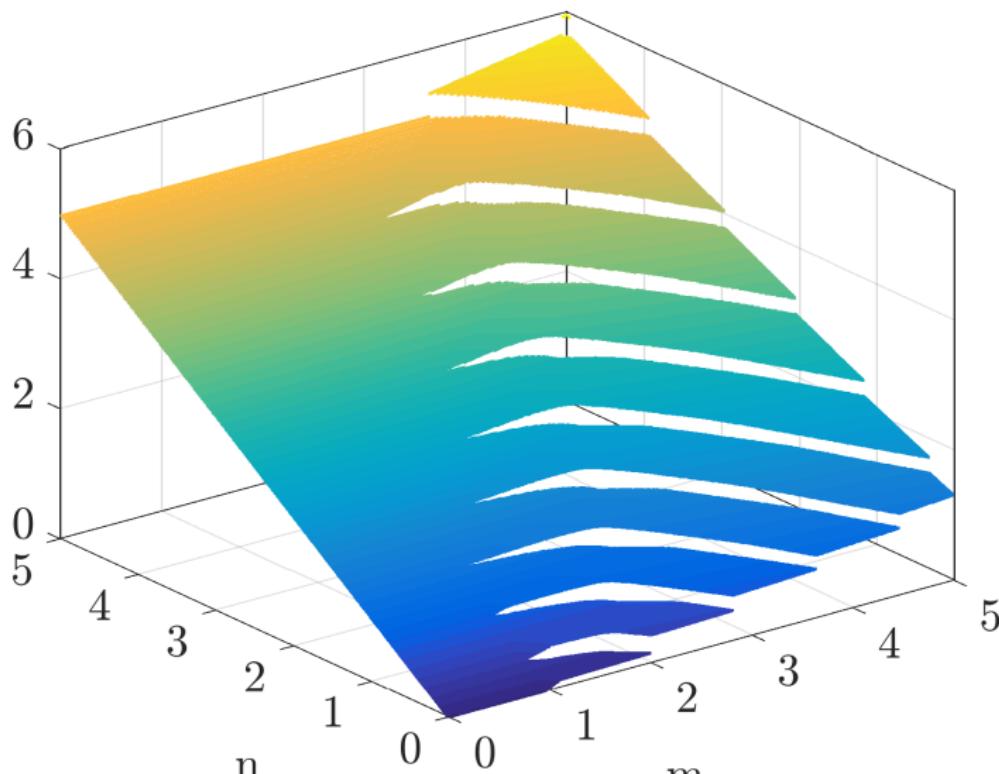
# Consumption + pension contributions model

## Pension fund contributions policy function



# Consumption + pension contributions model

Next period pension wealth  $n$



# General theory on multidimensional EGM



Matthew White, JEDC 2015

The Method of Endogenous Gridpoints in Theory and Practice

- Invertibility condition for the system of non-linear equations



Jeppe Druedahl, Thomas Jørgensen, JEDC 2017

A General Endogenous Grid Method for Multi-Dimensional Models with Non-Convexities and Constraints

- Formulate the sufficient condition,  
i.e. particular mapping has to be an injection



Iskhakov, Econ Letters 2015

Multidimensional endogenous gridpoint method: solving triangular dynamic stochastic optimization problems without root-finding operations + Corrigendum

- Focus on analytical invertibility to avoid root-finding operations

# Sufficient conditions for EGM to be applicable

- ① Concave utility function
- ② Post decision states ( $A_t$ ) form a set of sufficient statistics for the states and decisions in period  $t$
- ③ State variables can be analytically computed from post decision states ( $M_t = A_t + c_t$ )
- ④ The Hessian of the utility function can be converted to lower-triangular by permuting its rows and relabeling the variables

Then the dynamic problem can be solved (for interior solution) without root-finding operations by multidimensional EGM

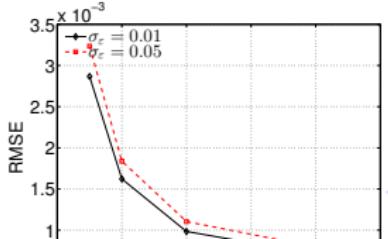
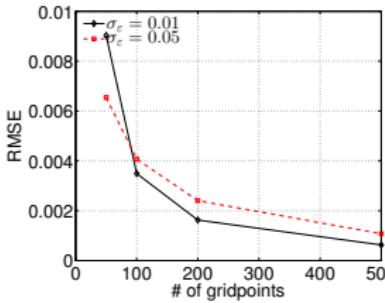
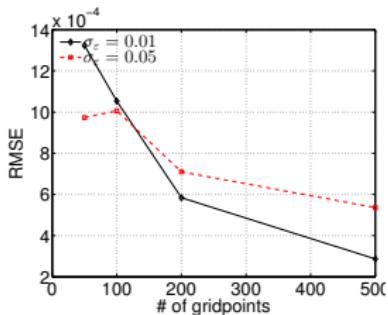
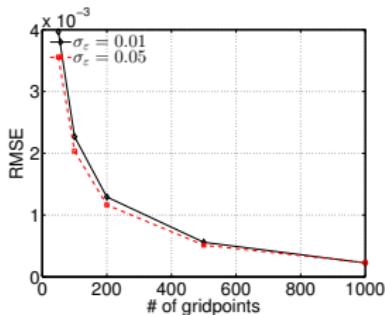
# Estimating life cycle models using endogenous gridpoint methods

# What to do with EGM methods

We can solve **many** problems of this type ⇒

- ➊ Fast solver for important problems with discrete/continuous choice  
→
  - calibration
  - structural estimation with your favourite method
  - NFXP: inner loop to solve the model, outer loop to optimize the objective function
- ➋ Use the solver repeatedly in some “outer loop” →
  - individual heterogeneity : solve the model for each individual in the sample
  - unobserved heterogeneity : random effects
  - flexibility of distributional assumptions

# Monte Carlo experiments



# EGM vs. MPEC



Jørgensen, 2012 *Economics Letters*

Structural Estimation of Continuous Choice Models: Evaluating EGM and MPEC.

Deaton consumption/savings model in infinite horizon, MC experiment with ML on synthetic data, 1 structural parameter

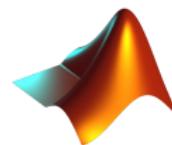
$\beta$		RMSE	Time
.70	EGM	0.002	0.1 sec.
	MPEC	0.049	112.4 sec.
.95	EGM	0.006	1.9 sec.
	MPEC	0.009	93.7 sec.
.99	EGM	0.000	5.0 sec.
	MPEC	0.000	30.9 sec.

## Points to take home

- ➊ EGM and DC-EGM is fast and accurate solution methods
- ➋ No root-finding operations in regular case
- ➌ Efficient with credit constraint
- ➍ Deterministic discrete-continuous problems are hard:
- ➎ Kinks in value functions, discontinuous policy functions
- ➏ Snowball effect in the accumulation of kinks over time
- ➐ With EV taste shocks the problem is alleviated
- ➑ EV taste shocks can be structural or added for smoothing
- ➒ Facilitate estimation using discrete choice data

[github.com/fediskhakov/dcegm](https://github.com/fediskhakov/dcegm)

# Exercises



- ❶ Code up the Euler equation solver in `model_phelps.m` and `model_deaton.m` and verify solution against VFI solver
- ❷ Code up an EGM solver for infinite horizon version of the problems in `model_phelps.m` and `model_deaton.m`
- ❸
  - (a) Replicate the solution to consumption/retirement problem using `model_retirement.m`;
  - (b) Investigate how the variance of shocks to income and scale of the taste shock effect the solution;
  - (c) Simulate the consumption path for  $\beta R = 1$  and discuss the accuracy of the solutions
- ❹ Add education to the retirement model so that wage incomes varies by education, discuss the differences in labor supply decisions
- ❺ Add part time work decision in the retirement model and simulate the case of phased retirement