Probabilistic Watershed: Sampling all spanning forests for seeded segmentation and semi-supervised learning

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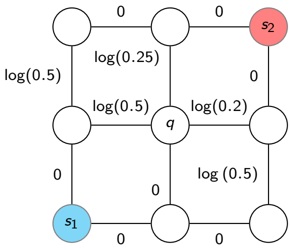
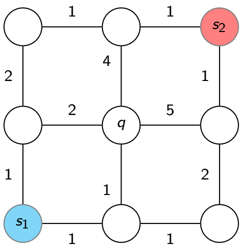
Notation: edge cost vs edge weight

$$w(e) = \exp(-\mu c(e)),\ \ \mu \geq 0$$



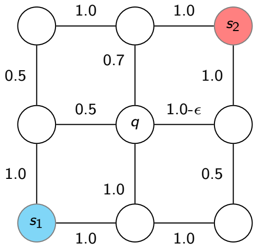
(a) Graph's edge-costs



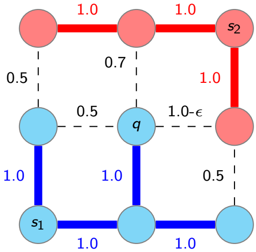
(b) Graph's edge-weights, $\mu = 1$

$$c(G) = \sum_{e \in E_G} c(e)$$

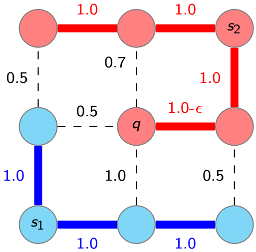
$$w(G) = \prod_{e \in E_G} w(e)$$



(a) Graph with two seeds



(b) MSF: Weight=1



(c) SF: Weight=1-ϵ

Figure: (2a) Weighted graph with two seeds and an edge depending on $\epsilon \in (0, 1)$. (2b)The MSF assigns q to seed s_1. (2c) spanning forest connecting q to s_2 with a weight close to the MSF when $\epsilon \to 0$.

Question to solve

What is the probability of sampling a forest such that a node of interest is assigned to a certain seed?

Gibbs Distribution of $\mathcal{F}^{s_2}$

Definition: probability distribution over the forests

Given $J \in \mathbb{R}_{\geq 0}$

$$\Pr^* = \arg\min_{\Pr} \sum_{f \in \mathcal{F}_{s_1}^{s_2}} \Pr(f) c(f), \quad \text{s.t.} \quad \sum_{f \in \mathcal{F}_{s_1}^{s_2}} \Pr(f) = 1 \text{ and } \mathcal{H}(\Pr) = J, \tag{1}$$

where $\mathcal{H}(\Pr)$= entropy Pr

Solution:

$$\Pr^*(f) = \frac{\exp(-\mu c(f))}{\sum_{f' \in \mathcal{F}_{s_1}^{s_2}} \exp(-\mu c(f'))} = \frac{w(f)}{\sum_{f' \in \mathcal{F}_{s_1}^{s_2}} w(f')}$$

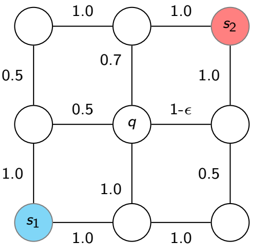
Probabilistic Watershed

$$\mathcal{F}_{s_1}^{s_2} := \{\text{2-forests separating } s_1 \text{ and } s_2\}$$
$$\mathcal{F}_{s_1,q}^{s_2} := \{f \in \mathcal{F}_{s_1}^{s_2} \ : \ s_1 \text{ and } q \text{ are connected}\}$$

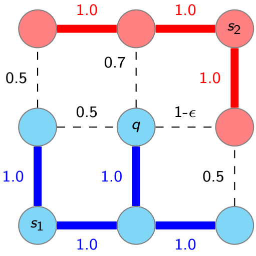
Probability q is connected with s_1

$$\Pr(q \sim s_1) := \frac{w\left(\mathcal{F}_{s_1,q}^{s_2}\right)}{w\left(\mathcal{F}_{s_1}^{s_2}\right)} = \frac{\sum_{f \in \mathcal{F}_{s_1,q}^{s_2}} w(f)}{\sum_{f \in \mathcal{F}_{s_1}^{s_2}} w(f)}.$$

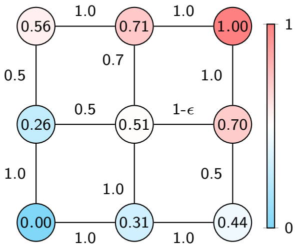
Example



(a) Graph with two seeds



(b) MSF: Weight=1



(c) $P(\text{node} \sim s_2), \epsilon = 0.1$

Figure: (a) Graph with two seeds and an edge depending on $\epsilon \in (0, 1)$. (b)The MSF assigns q to seed s_1. (c) In average it is more likely to sample a forest connecting q and s_2.

Probabilistic Watershed

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$$\Pr(q \sim s_1) := \frac{w\left(\mathcal{F}_{s_1,q}^{s_2}\right)}{w\left(\mathcal{F}_{s_1}^{s_2}\right)} = \frac{\sum\limits_{f \in \mathcal{F}_{s_1,q}^{s_2}} w(f)}{\sum\limits_{f \in \mathcal{F}_{s_1}^{s_2}} w(f)}.$$

Matrix Tree Theorem

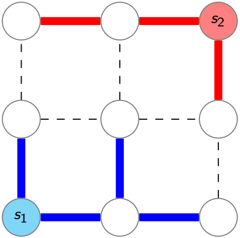
Matrix Tree Theorem (MTT)

For any weighted multigraph G, the sum of the weights of the spanning trees of G, $w(\mathcal{T})$, is equal to

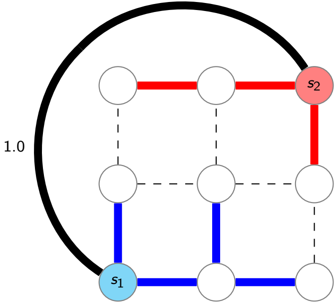
$$w(\mathcal{T}) := \sum_{t \in \mathcal{T}} \prod_{e \in E_t} w(e) = \frac{1}{|V|} \det(L + \frac{1}{|V|} \mathbb{1}\mathbb{1}^\top) = \det(L^{[v]}),$$

where $\mathbb{1}$ is the column full of 1. $L^{[v]}$ is the matrix obtained after removing the row and column corresponding to an arbitrary node v.

Computation of $w(\mathcal{F}_{s_1}^{s_2})$



(a) Forest.



(b) Tree

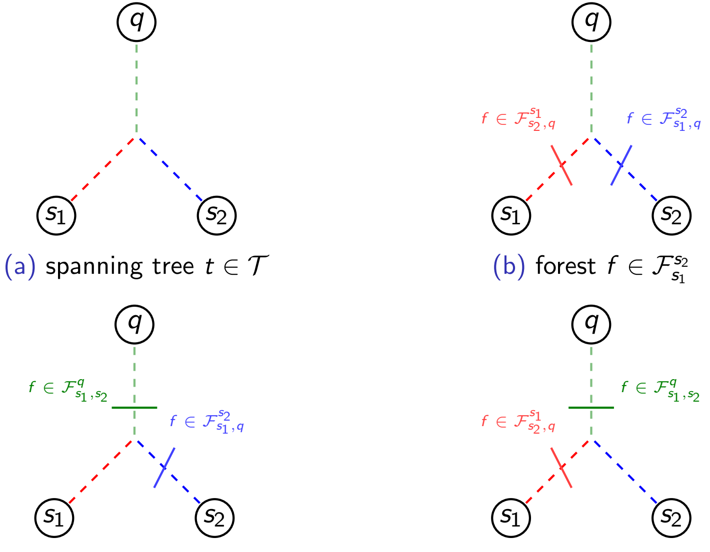
Computation of $w(\mathcal{F}_{s_1}^{s_2})$

r_{uv}^{eff} = effective resistance distance between the nodes u and v.

Theorem

$$w(\mathcal{F}_{s_1}^{s_2}) = w(\mathcal{T})r_{s_1 s_2}^{\text{eff}} \propto r_{s_1 s_2}^{\text{eff}}.$$

Computation of $w(\mathcal{F}^q_{s_1,s_2})$ and $w(\mathcal{F}^q_{s_2,s_1})$



(a) spanning tree $t \in \mathcal{T}$

(b) forest $f \in \mathcal{F}^{s_2}_{s_1}$

(c) forest $f \in \mathcal{F}^{q}_{s_2}$

(d) forest $f \in \mathcal{F}^{q}_{s_1}$

Computation $w(\mathcal{F}^q_{s_1,s_2}), w(\mathcal{F}^q_{s_2,s_1})$

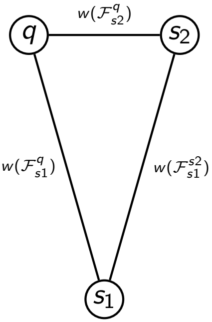
Linear system with 3 unknowns and 3 equations.

$$
\begin{aligned}
w(\mathcal{F}^{s_2}_{s_1,q}) + w(\mathcal{F}^{q}_{s_1,s_2}) &= w(\mathcal{F}^{q}_{s_2}) \\
w(\mathcal{F}^{q}_{s_1,s_2}) + w(\mathcal{F}^{s_1}_{s_2,q}) &= w(\mathcal{F}^{q}_{s_1}) \\
w(\mathcal{F}^{s_2}_{s_1,q}) + w(\mathcal{F}^{s_1}_{s_2,q}) &= w(\mathcal{F}^{s_2}_{s_1}).
\end{aligned}
$$

Probabilistic Watershed corresponds to the effective resistance distance

Probability measures the gap of the triangle inequality

$$\Pr(q \sim s_1) = \frac{w(\mathcal{F}_{s_2}^{q}) + w(\mathcal{F}_{s_1}^{s_2}) - w(\mathcal{F}_{s_1}^{q})}{2w(\mathcal{F}_{s_1}^{s_2})} = \frac{r_{s_2 q}^{\text{eff}} + r_{s_2 s_1}^{\text{eff}} - r_{s_1 q}^{\text{eff}}}{2r_{s_2 s_1}^{\text{eff}}}. \qquad (2)$$



$$\Pr(q \sim s_1) \geq \Pr(q \sim s_2) \iff r_{s_1 q}^{\text{eff}} \leq r_{s_2 q}^{\text{eff}}.$$

Random Walker = Probabilistic Watershed

Theorem

The probability, $x_q^{s_2}$, that a Random Walker starting at node q reaches first s_2 before reaching s_1 ([**?**]) is equal to the probability defined by the Probabilistic Watershed.

$$x_q^{s_2} = P(q \sim s_2).$$

Power Watershed [?]

Power Watershed minimization problem

$$x^* = \arg\min_x \sum_{e=(v,u)\in E} (w(e))^\alpha \left(|x_v - x_u|\right)^\beta, \text{ s.t. } x_{s_1} = 1,\ x_{s_2} = 0, \quad (3)$$

$$w(e) = \exp(-\mu c(e)) \rightarrow (w(e))^\alpha = \exp(-\mu\alpha c(e)) = \exp(-\mu_\alpha c(e))$$

Power Watershed [?]

Power Watershed count MSFs

Given $S = \{s_1, s_2\}$ a set of seeds, let us denote the potential of node q being assigned to seed s_1 by the Power Watershed with $\beta = 2$ as $x_q^{\mathrm{PW}_1}$. Let further $w_{\max}$ be $\max_{f \in \mathcal{F}_{s_1}^{s_2}} w(f)$. Then

$$x_q^{\mathrm{PW}_1} = \Pr_\infty(q \sim s_1) := \frac{\left|\{f \in \mathcal{F}_{s_1,q}^{s_2} \;:\; w(f) = w_{\max}\}\right|}{\left|\{f \in \mathcal{F}_{s_1}^{s_2} \;:\; w(f) = w_{\max}\}\right|}.$$

Conclusion & Future Work

- **Probabilistic Watershed = Random Walker**$\rightarrow$ Directed Probabilistic Watershed $\stackrel{?}{=}$ Directed Random Walker?
- **Power Watershed counts minimum cost spanning forests**

References

The End