Probabilistic Watershed: Sampling all spanning forests for seeded segmentation and semi-supervised learning

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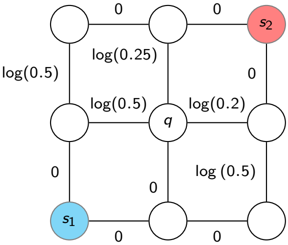
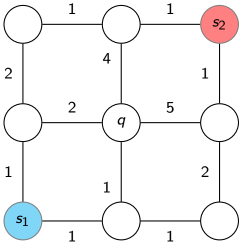
Notation: edge cost vs edge weight

$$w(e) = \exp(-\mu c(e)), \ \mu \geq 0$$



(a) Graph's edge-costs



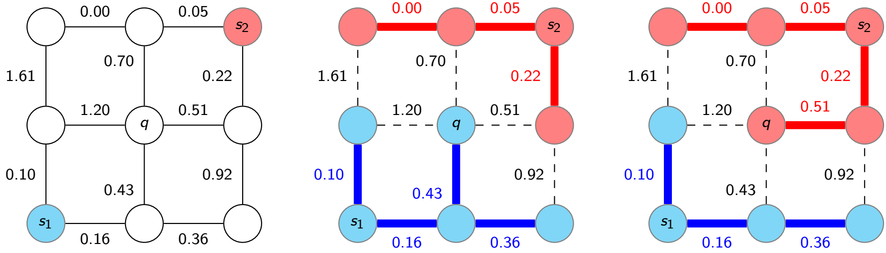
(b) Graph's edge-weights, $\mu = 1$

$$c(G) = \sum_{e \in E_G} c(e)$$

$$w(G) = \prod_{e \in E_G} w(e)$$

Watershed computes a minimum cost spanning forest (mSF)

The assignment of q is doubtful



(a) Graph with two seeds (b) mSF: Cost=1.32 (c) SF: Weight=1.4

Figure: (2a) Graph with edge-costs and two seeds. (2b)The mSF (Watershed segmentation) assigns q to seed s_1. (2c) Low cost, but not minimum, spanning forest connecting q to s_2. The Watershed has a winner-takes-all behaviour. It only considers the mSF while ignores the rest of seed separating spanning forests

Question to solve

What is the probability of sampling a forest such that a node of interest is assigned to a certain seed?

Gibbs Distribution of $\mathcal{F}^{s_2}$

$$\mathcal{F}_{s_1}^{s_2} := \{\text{2-forests separating } s_1 \text{ and } s_2\}$$

Definition: probability distribution over the forests

Given $J \in \mathbb{R}_{\geq 0}$

$$\Pr^* = \arg\min_{\Pr} \sum_{f \in \mathcal{F}_{s_1}^{s_2}} \Pr(f) c(f), \quad \text{s.t.} \quad \sum_{f \in \mathcal{F}_{s_1}^{s_2}} \Pr(f) = 1 \text{ and } \mathcal{H}(\Pr) = J, \tag{1}$$

where $\mathcal{H}(\Pr)$= entropy Pr

Solution: Gibbs Probability distribution

$$\Pr^*(f) = \frac{\exp(-\mu c(f))}{\sum_{f' \in \mathcal{F}_{s_1}^{s_2}} \exp(-\mu c(f'))} = \frac{w(f)}{\sum_{f' \in \mathcal{F}_{s_1}^{s_2}} w(f')}$$

Probabilistic Watershed

$$\mathcal{F}_{s_1}^{s_2} := \{\text{2-forests separating } s_1 \text{ and } s_2\}$$
$$\mathcal{F}_{s_1,q}^{s_2} := \{f \in \mathcal{F}_{s_1}^{s_2} \; : \; s_1 \text{ and } q \text{ are connected}\}$$

Probability q is connected with s_1

$$\Pr(q \sim s_1) := \frac{w\left(\mathcal{F}_{s_1,q}^{s_2}\right)}{w\left(\mathcal{F}_{s_1}^{s_2}\right)} = \frac{\sum\limits_{f \in \mathcal{F}_{s_1,q}^{s_2}} w(f)}{\sum\limits_{f \in \mathcal{F}_{s_1}^{s_2}} w(f)}.$$

Example

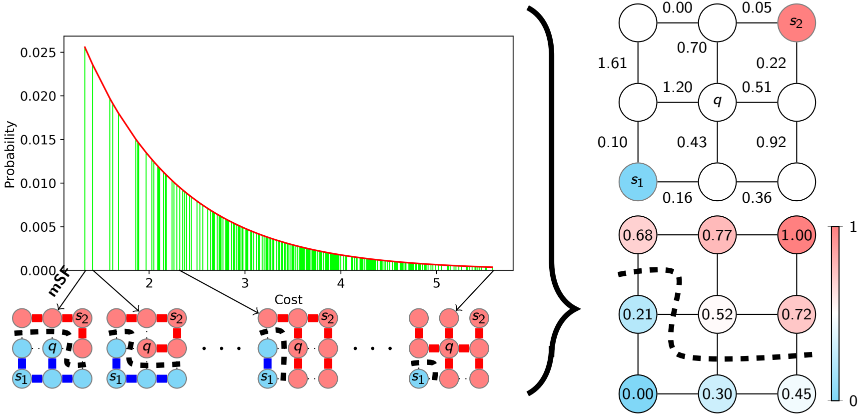


Figure: (**Top left**) Gibbs distribution over all spanning forests. (**Bottom right**) Probabilistic Watershed probabilities for assigning a node to s_2. The Probabilistic Watershed computes the expected seed assignment of every node for a Gibbs distribution over all exponentially many spanning forests in closed-form. It thus avoids the winner-takes-all behaviour of the Watershed.

Probabilistic Watershed

$$\mathcal{F}_{s_1}^{s_2} := \{\text{2-forests separating } s_1 \text{ and } s_2\}$$
$$\mathcal{F}_{s_1,q}^{s_2} := \{f \in \mathcal{F}_{s_1}^{s_2} \ : \ s_1 \text{ and } q \text{ are connected}\}$$

Probability q is connected to s_1

$$\Pr(q \sim s_1) := \frac{w\left(\mathcal{F}_{s_1,q}^{s_2}\right)}{w\left(\mathcal{F}_{s_1}^{s_2}\right)} = \frac{\sum\limits_{f \in \mathcal{F}_{s_1,q}^{s_2}} w(f)}{\sum\limits_{f \in \mathcal{F}_{s_1}^{s_2}} w(f)}.$$

Question: How do we compute $\mathcal{F}_{s_1}^{s_2}$ and $\mathcal{F}_{s_1,q}^{s_2}$?
Answer: Matrix Tree Theorem (MTT).

Matrix Tree Theorem

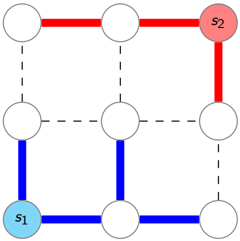
Matrix Tree Theorem (MTT)

For any weighted multigraph G, the sum of the weights of the spanning trees of G, $w(\mathcal{T})$, is equal to

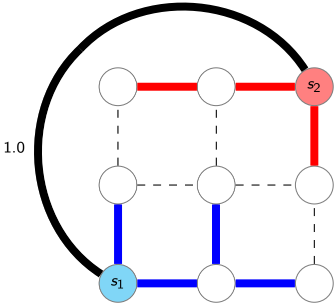
$$w(\mathcal{T}) := \sum_{t \in \mathcal{T}} \prod_{e \in E_t} w(e) = \frac{1}{|V|} \det(L + \frac{1}{|V|} \mathbb{1}\mathbb{1}^\top) = \det(L^{[v]}),$$

where $\mathbb{1}$ is the column full of 1. $L^{[v]}$ is the matrix obtained after removing the row and column corresponding to an arbitrary node v.

The relation between forests and trees allows the use of the MTT.



(a) Forest.



(b) Tree

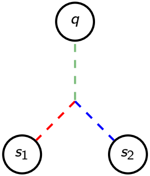
Computation of $w(\mathcal{F}_{s_1}^{s_2})$

r_{uv}^{eff} = effective resistance distance between the nodes u and v.

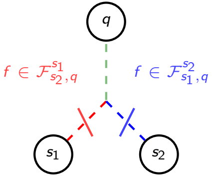
Theorem (Consequence of the MTT)

$$w(\mathcal{F}_{s_1}^{s_2}) = w(\mathcal{T}) r_{s_1 s_2}^{\text{eff}} \propto r_{s_1 s_2}^{\text{eff}}.$$

Computation of $w(\mathcal{F}^{q}_{s_1,s_2})$ and $w(\mathcal{F}^{q}_{s_2,s_1})$



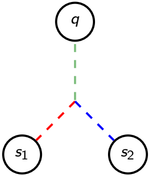
(a) spanning tree $t \in \mathcal{T}$



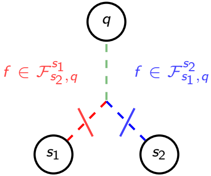
(b) forest $f \in \mathcal{F}^{s_2}_{s_1}$

Figure: $\mathcal{F}^{s_2}_{s_1,q}$ and $\mathcal{F}^{s_1}_{s_2,q}$ form a partition of $\mathcal{F}^{s_2}_{s_1}$ since q must be connected either to s_1 or s_2, but not to both. Given an spanning tree (dashed lines represent all possible spanning trees), to form a forest in $\mathcal{F}^{s_2}_{s_1}$ we will have to remove an edge from the red part, forming a forest in $\mathcal{F}^{s_1}_{s_2,q}$, or from the blue part, forming a forest in $\mathcal{F}^{s_2}_{s_1,q}$.

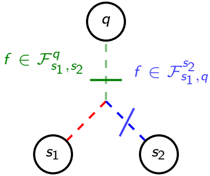
Computation of $w(\mathcal{F}^q_{s_1,s_2})$ and $w(\mathcal{F}^q_{s_2,s_1})$



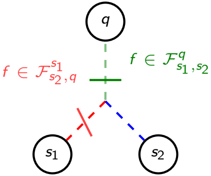
(a) spanning tree $t \in \mathcal{T}$



(b) forest $f \in \mathcal{F}^{s_2}_{s_1}$



(c) forest $f \in \mathcal{F}^{q}_{s_2}$



(d) forest $f \in \mathcal{F}^{q}_{s_1}$

Figure: Analogously $\mathcal{F}^q_{s_1,s_2}$ and $\mathcal{F}^{s_1}_{q,s_2}$ form a partition of $\mathcal{F}^q_{s_1}$, and $\mathcal{F}^q_{s_2,s_1}$ and $\mathcal{F}^{s_2}_{q,s_1}$ form a partition of $\mathcal{F}^q_{s_2}$ respectively.

Computation $w(\mathcal{F}^{q}_{s_1,s_2}), w(\mathcal{F}^{q}_{s_2,s_1})$

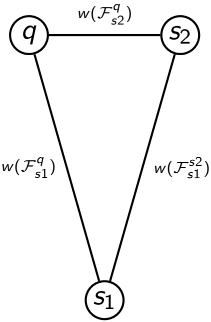
Linear system with 3 unknowns and 3 equations.

$$
\begin{aligned}
w(\mathcal{F}^{s_2}_{s_1,q}) + w(\mathcal{F}^{q}_{s_1,s_2}) &= w(\mathcal{F}^{q}_{s_2}) \\
w(\mathcal{F}^{q}_{s_1,s_2}) + w(\mathcal{F}^{s_1}_{s_2,q}) &= w(\mathcal{F}^{q}_{s_1}) \\
w(\mathcal{F}^{s_2}_{s_1,q}) + w(\mathcal{F}^{s_1}_{s_2,q}) &= w(\mathcal{F}^{s_2}_{s_1}).
\end{aligned}
$$

Probabilistic Watershed corresponds to the effective resistance distance

Probability measures the gap of the triangle inequality

$$\Pr(q \sim s_1) = \frac{w(\mathcal{F}_{s_2}^{q}) + w(\mathcal{F}_{s_1}^{s_2}) - w(\mathcal{F}_{s_1}^{q})}{2w(\mathcal{F}_{s_1}^{s_2})} = \frac{r_{s_2 q}^{\text{eff}} + r_{s_2 s_1}^{\text{eff}} - r_{s_1 q}^{\text{eff}}}{2 r_{s_2 s_1}^{\text{eff}}}. \tag{2}$$



$$\Pr(q \sim s_1) \geq \Pr(q \sim s_2) \iff r_{s_1 q}^{\text{eff}} \leq r_{s_2 q}^{\text{eff}}.$$

Random Walker = Probabilistic Watershed

Theorem

The probability, $x_q^{s_2}$, that a Random Walker starting at node q reaches first s_2 before reaching s_1 ([Grady, 2006]) is equal to the probability defined by the Probabilistic Watershed.

$$x_q^{s_2} = P(q \sim s_2).$$

Power Watershed [Couprie et al., 2011]

Power Watershed minimization problem

$$x^* = \arg\min_x \sum_{e=(v,u)\in E} (w(e))^\alpha \left(|x_v - x_u|\right)^\beta, \text{ s.t. } x_{s_1} = 1,\ x_{s_2} = 0, \quad (3)$$

Random Walker: $\alpha = 1,\ \beta = 2$
Power Watershed: $\alpha \to \infty,\ \beta = 2$

α and μ are equivalent.

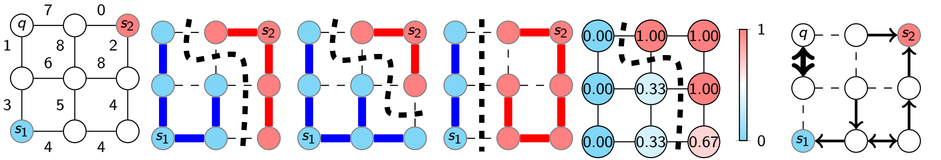
$$w(e) = \exp(-\mu c(e)) \to (w(e))^\alpha = \exp(-\mu\alpha c(e)) = \exp(-\mu_\alpha c(e))$$

Power Watershed [Couprie et al., 2011]

Power Watershed count minimum cost/maximum weight spanning forests

Given $S = \{s_1, s_2\}$ a set of seeds, let us denote the potential of node q being assigned to seed s_1 by the Power Watershed with $\beta = 2$ as $x_q^{\mathrm{PW}_1}$. Let further $w_{\max}$ be $\max_{f \in \mathcal{F}_{s_1}^{s_2}} w(f)$. Then

$$x_q^{\mathrm{PW}_1} = \Pr\nolimits_\infty(q \sim s_1) := \frac{\left|\{f \in \mathcal{F}_{s_1,q}^{s_2} \ : \ w(f) = w_{\max}\}\right|}{\left|\{f \in \mathcal{F}_{s_1}^{s_2} \ : \ w(f) = w_{\max}\}\right|}.$$



(a) Graph (b) mSF1 (c) mSF2 (d) mSF3 (e) $P_\infty(\text{node} \sim s_2)$ (f) RW reachability

Figure: Forest-interpretation of Power Watershed. The Power Watershed computes the ratio between the mSFs connecting a node to s_2 and all possible mSFs. (7f) indicates the allowed Random Walker transitions when $\mu \to \infty$ with headed arrows. The Random Walker interpretation of the Power Watershed breaks down in the limit case since a Random Walker starting at node q does not reach any seed, but oscillates along the bold arrow.

Conclusion

- Probabilistic Watershed = Random Walker
- Random Walker/Probabilistic Watershed probabilities are proportional to the triangle inequality gap.

$$\Pr(q \sim s_1) \propto r^{\text{eff}}_{s_2 q} + r^{\text{eff}}_{s_2 s_1} - r^{\text{eff}}_{s_1 q}$$

- Power Watershed counts minimum cost spanning forests

References

Couprie, C., Grady, L., Najman, L., and Talbot, H. (2011).
Power watershed: A unifying graph-based optimization framework.
IEEE Transactions on Pattern Analysis and Machine Intelligence, 33:1384–1399.

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