

# CS 348

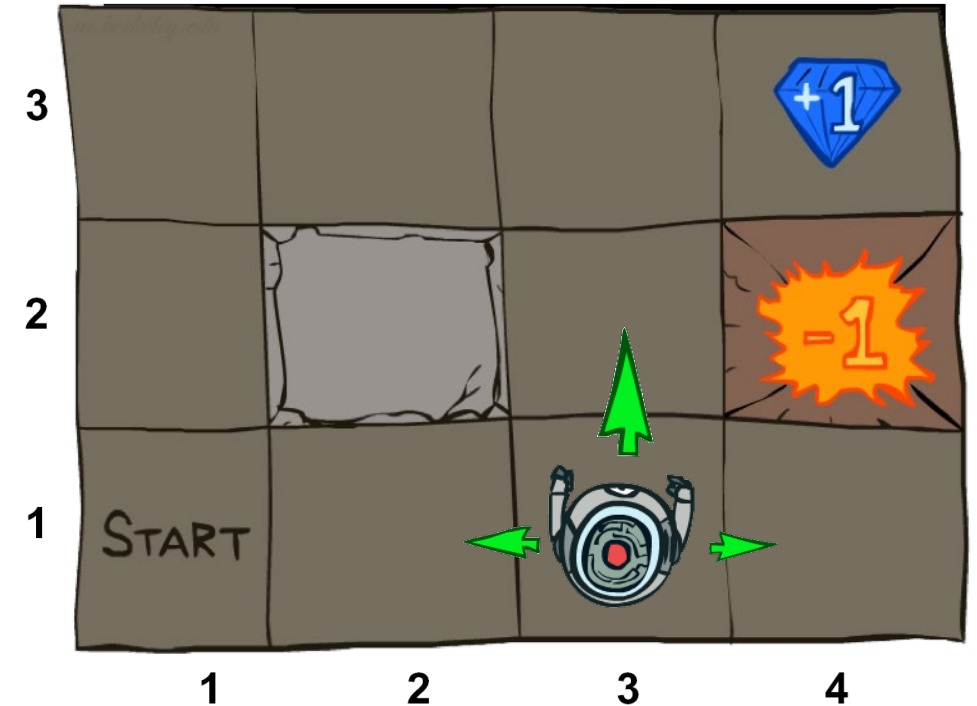
## Intro to Artificial Intelligence

### Markov Decision Processes II

- Class business
  - Anonymous feedback: <https://forms.gle/RJtdYQXCnpru3SeZ7>
  - Lab 3 (A\*) due Thursday
  - Upcoming labs:
    - Lab 4 (tic-tac-toe) due May 5, 7pm
    - Lab 5 (sudoku) due May 12, 7pm
    - Please start lab 5 early, probably hardest lab of the quarter.
  - Today: MDP's(17.1-17.3)
  - Discuss Lab 4 (Games/tic-tac-toe)

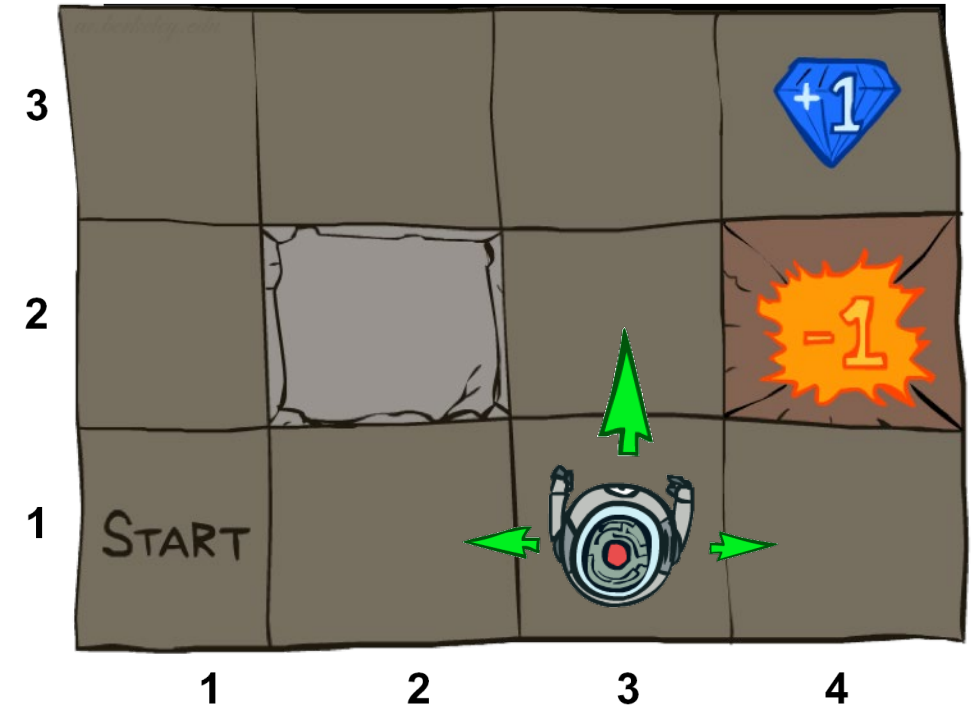
# Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards



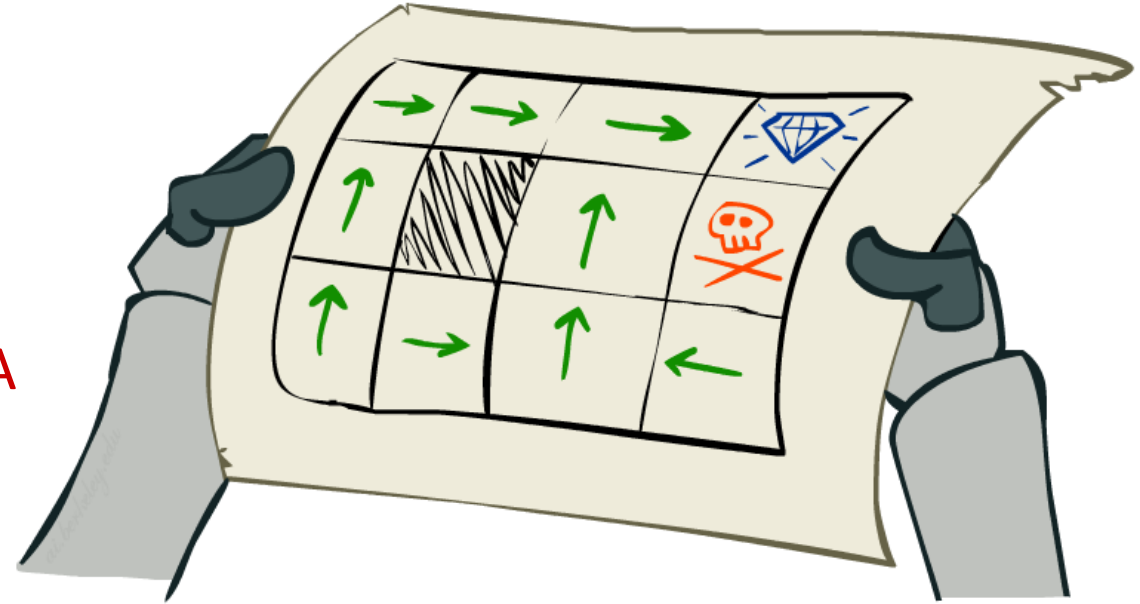
# Markov Decision Processes

- An MDP is defined by:
  - A **set of states**  $s \in S$
  - A **set of actions**  $a \in A$
  - A **transition function**  $T(s, a, s')$ 
    - Probability that  $a$  from  $s$  leads to  $s'$ , i.e.,  $P(s' | s, a)$
    - Also called the model or the dynamics
  - A **reward function**  $R(s, a, s')$ 
    - Sometimes just  $R(s)$  or  $R(s')$
  - discount  $\gamma$
  - A **start state**
  - Maybe a **terminal state**
- MDPs are non-deterministic search problems
  - One way to solve them is with expectimax search



# Goal: Policies

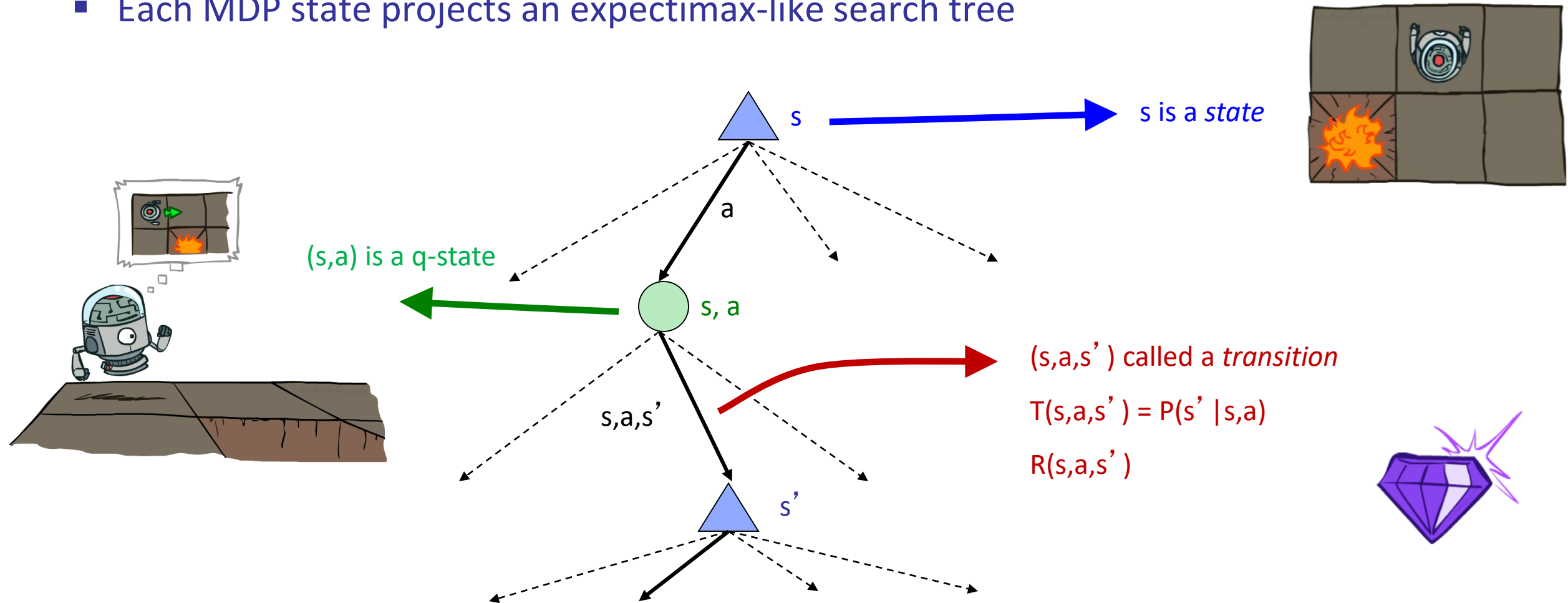
- In deterministic single-agent search problems, we wanted an optimal **plan**, or sequence of actions, from start to a goal
- For MDPs, we want an optimal **policy**  $\pi^*: S \rightarrow A$ 
  - A policy  $\pi$  gives an action for each state
  - An optimal policy is one that maximizes expected utility if followed
  - An explicit policy defines a reflex agent
- Expectimax didn't compute entire policies
  - It computed the action for a single state only



Optimal policy when  $R(s, a, s') = -0.03$   
for all non-terminals  $s$

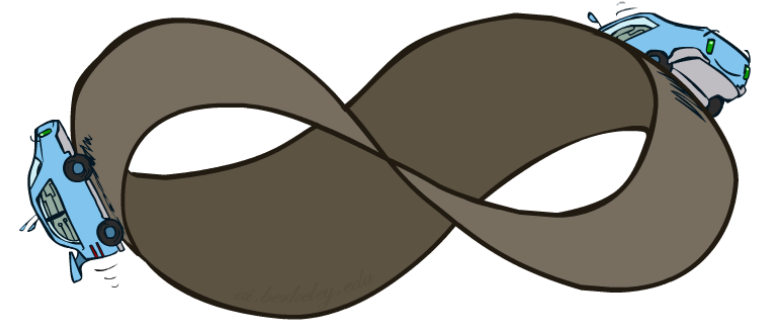
# MDP Search Trees

- Each MDP state projects an expectimax-like search tree

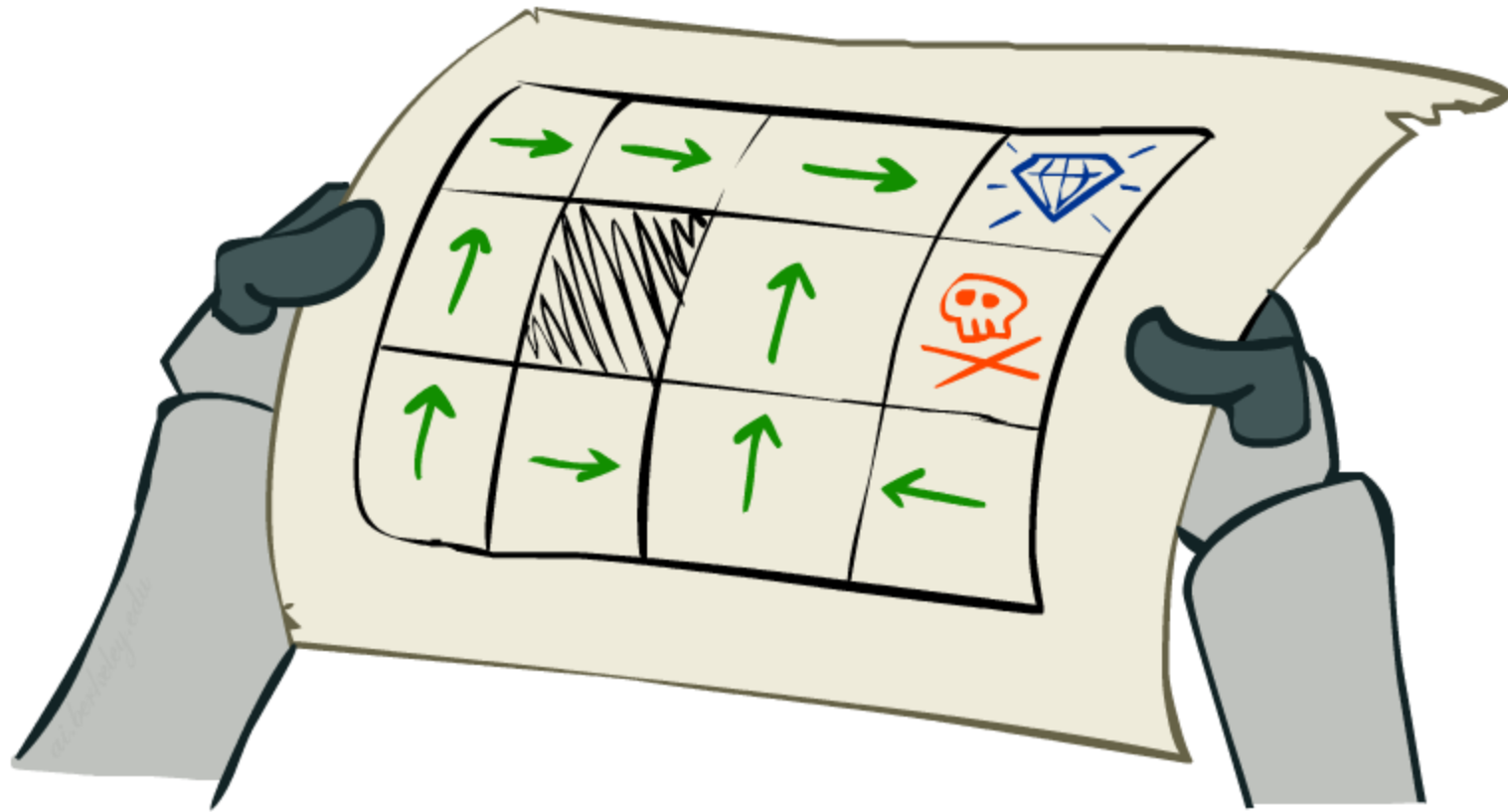


# Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?
- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed  $T$  steps (e.g. life)
  - Discounting: use  $0 < \gamma < 1$ 
    - Smaller  $\gamma$  means smaller “horizon” – shorter term focus

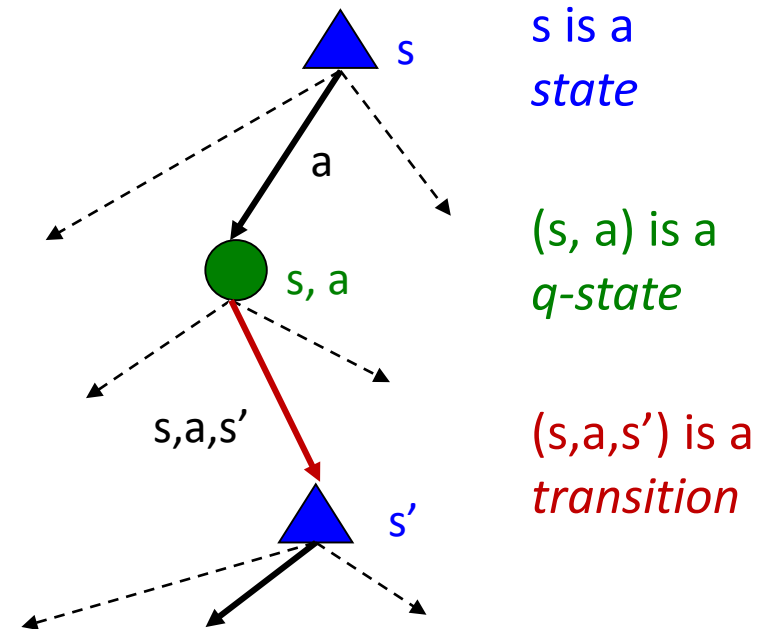


# Solving MDPs



# Optimal Quantities

- The value (utility) of a state  $s$ :  
 $V^*(s)$  = expected utility starting in  $s$  and acting optimally
- The value (utility) of a q-state  $(s,a)$ :  
 $Q^*(s,a)$  = expected utility starting out having taken action  $a$  from state  $s$  and (thereafter) acting optimally
- The optimal policy:  
 $\pi^*(s)$  = optimal action from state  $s$   
 $\pi^*(s) = \arg \max_a Q^*(s,a)$



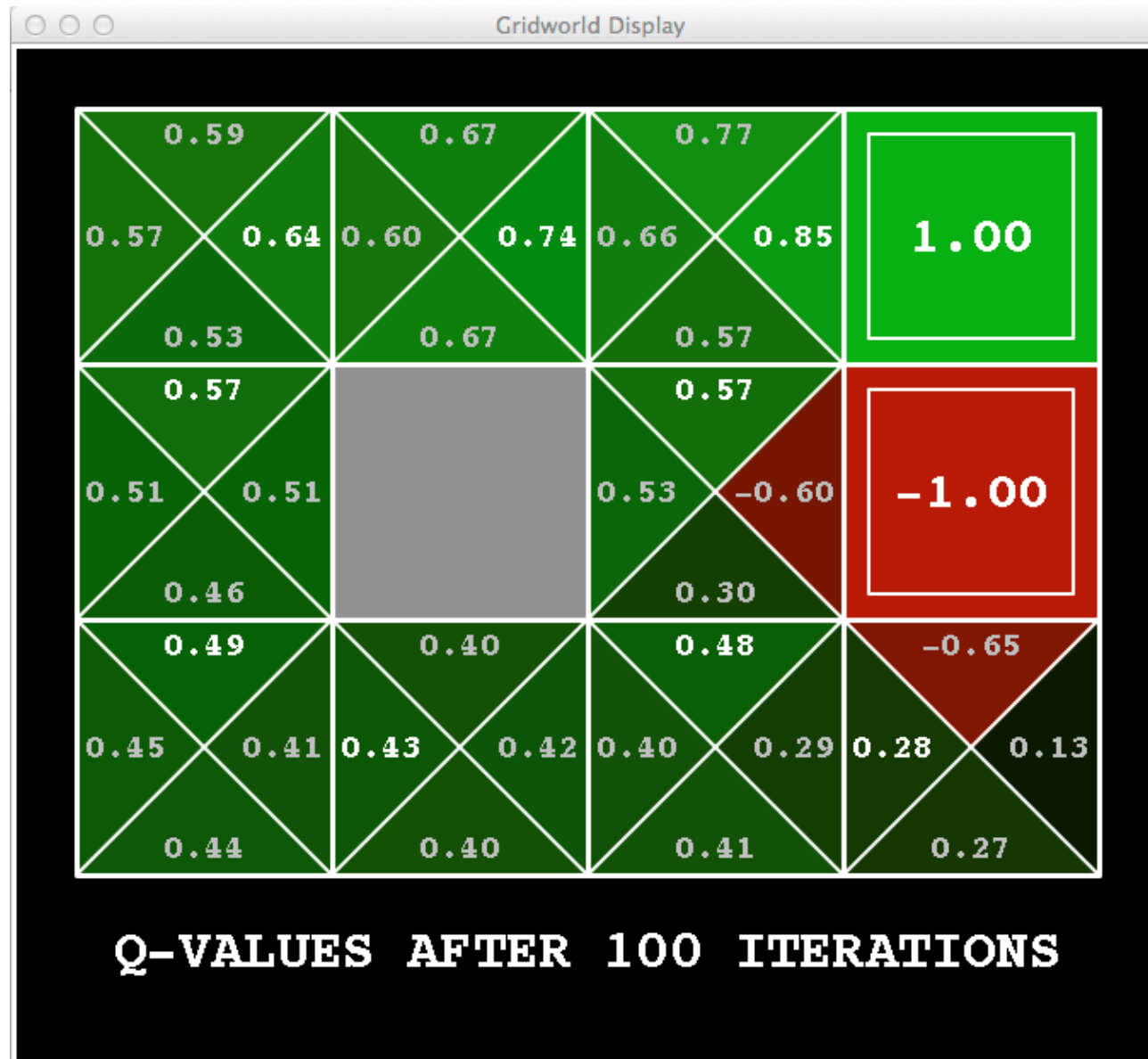


# Gridworld V Values



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# Gridworld Q Values



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# Values of States

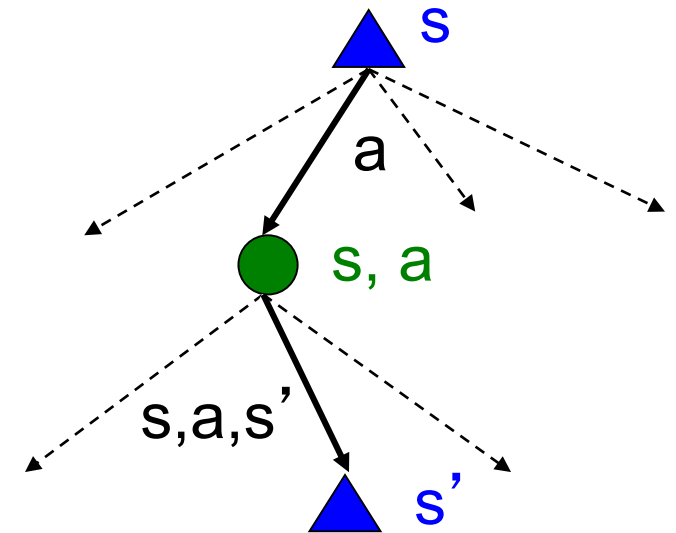
- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax computed!

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s, a)$$

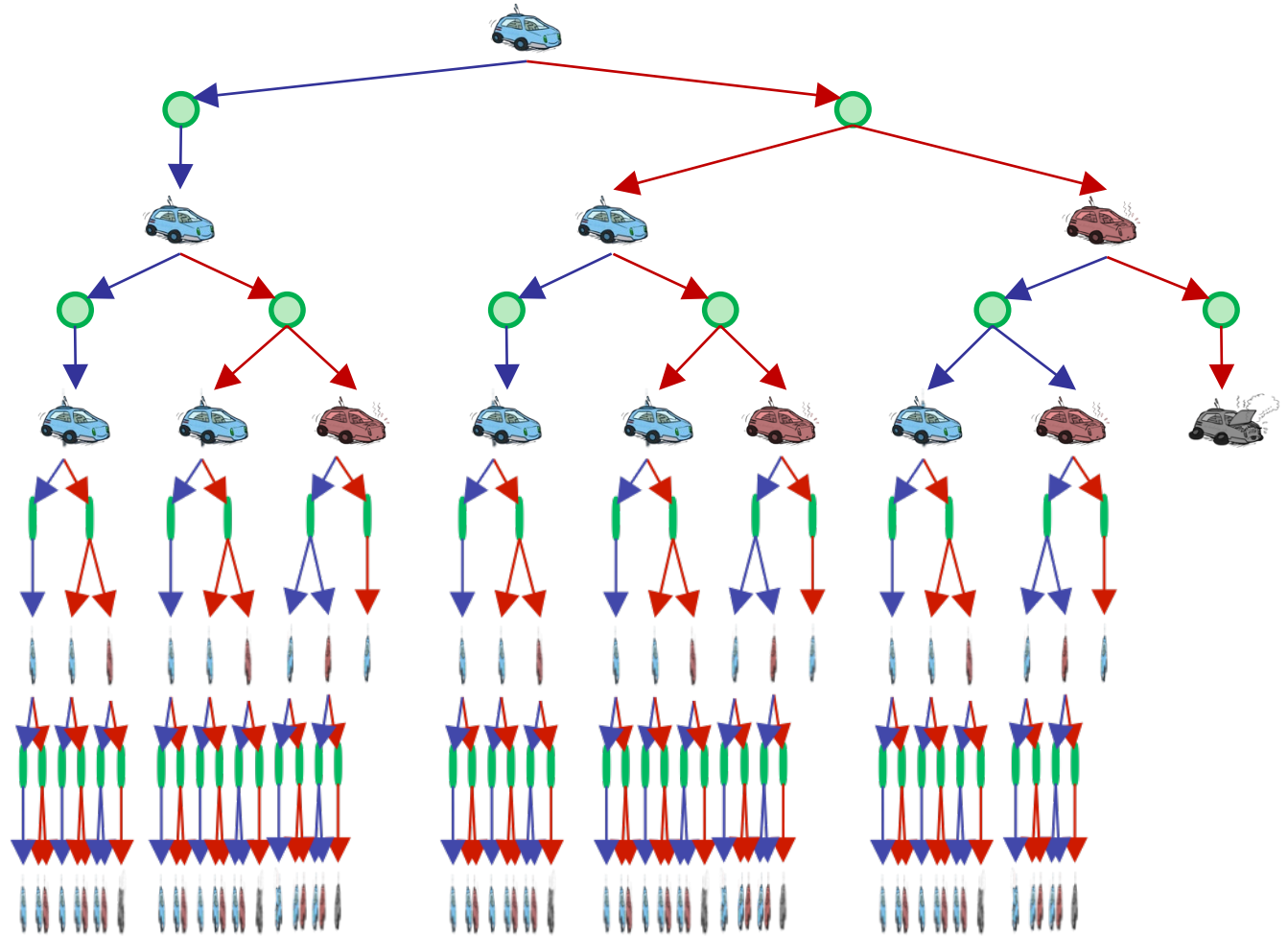
$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



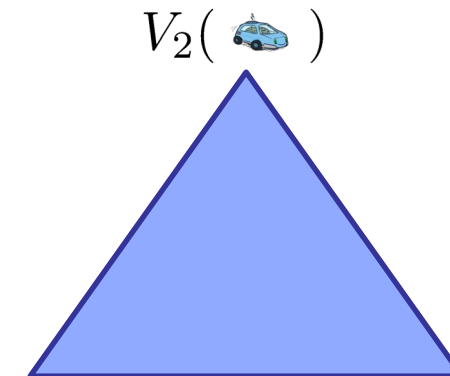
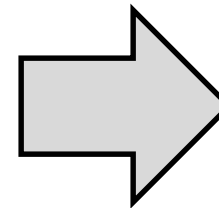
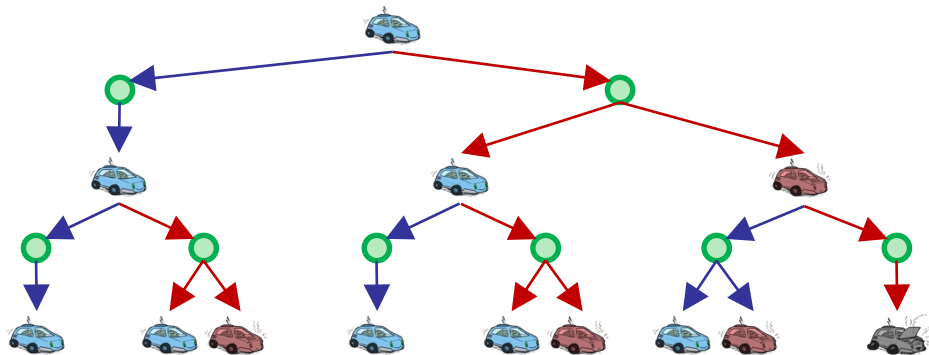
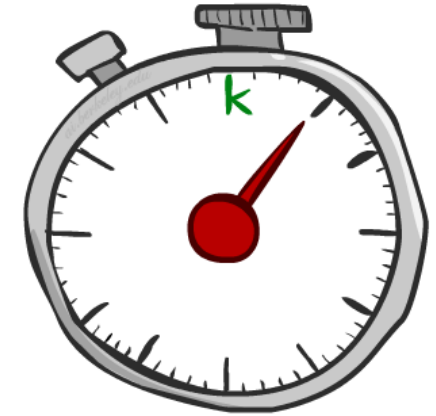
# Racing Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if  $\gamma < 1$



# Time-Limited Values

- Key idea: time-limited values
- Define  $V_k(s)$  to be the optimal value of  $s$  if the game ends in  $k$  more time steps
  - Equivalently, it's what a depth- $k$  expectimax would give from  $s$

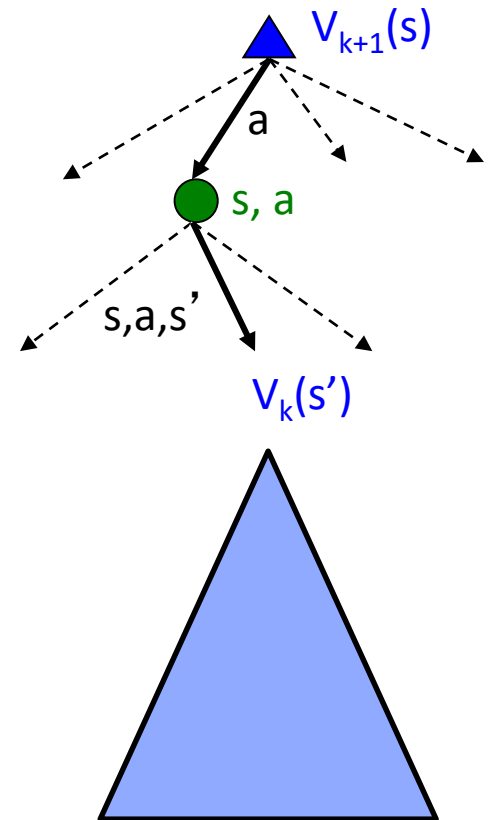


# Value Iteration

- Start with  $V_0(s) = 0$ : no time steps left means an expected reward sum of zero
- Given vector of  $V_k(s)$  values, do one ply of expectimax from each state:

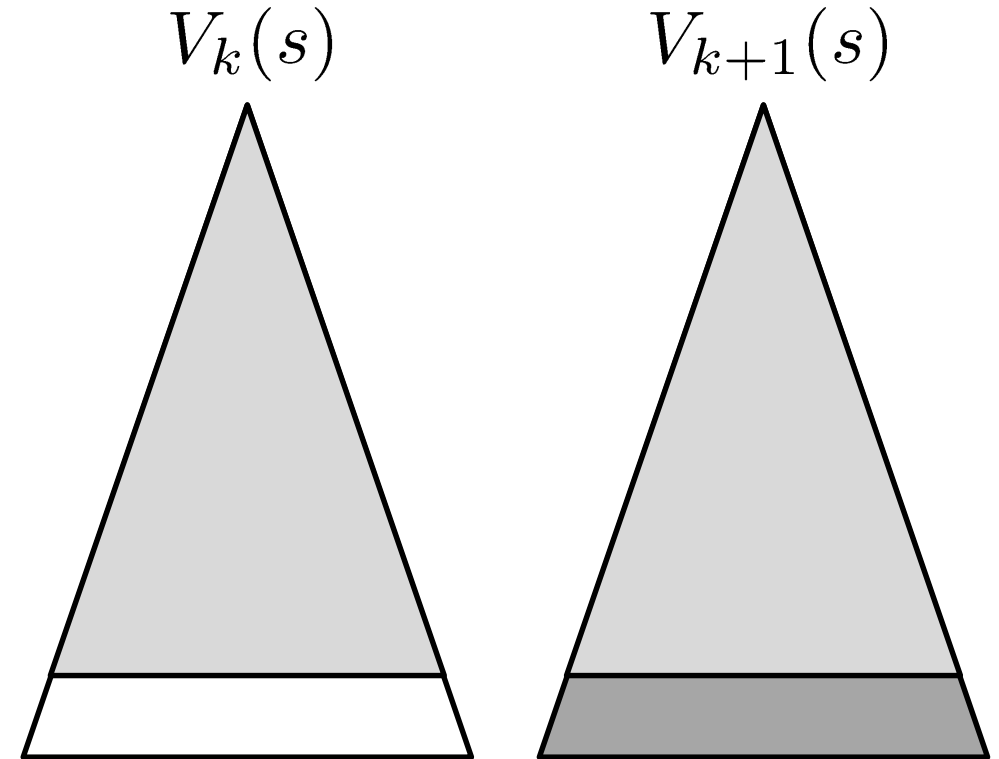
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Repeat until convergence
- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values

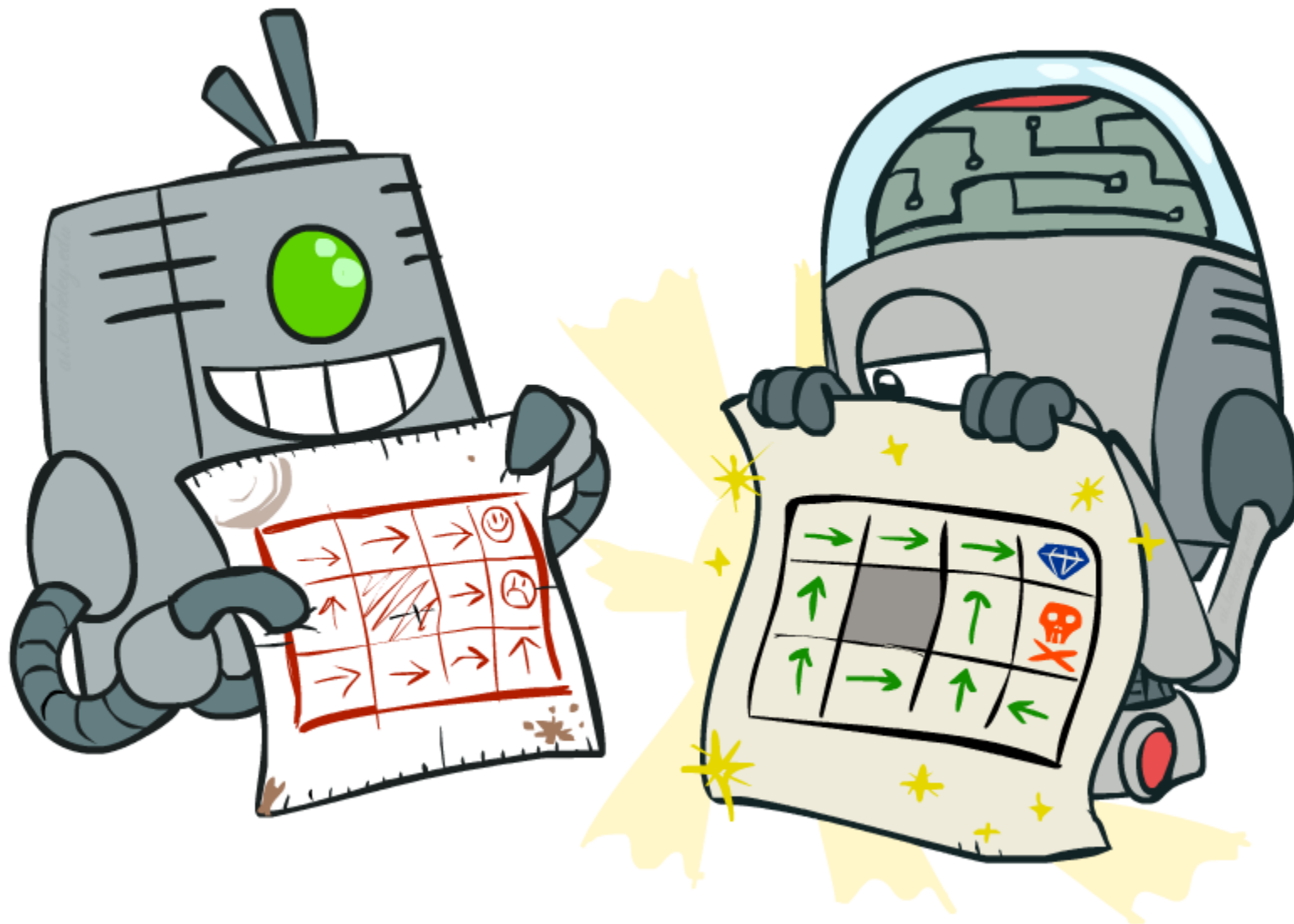


# Convergence

- How do we know the  $V_k$  vectors are going to converge?
- Case 1: If the tree has maximum depth  $M$ , then  $V_M$  holds the actual untruncated values
- Case 2: If the discount is less than 1
  - Sketch: For any state  $V_k$  and  $V_{k+1}$  can be viewed as depth  $k+1$  expectimax results in nearly identical search trees
  - The difference is that on the bottom layer,  $V_{k+1}$  has actual rewards while  $V_k$  has zeros
  - That last layer is at best all  $R_{\text{MAX}}$
  - It is at worst  $R_{\text{MIN}}$
  - But everything is discounted by  $\gamma^k$  that far out
  - So  $V_k$  and  $V_{k+1}$  are at most  $\gamma^k \max |R|$  different
  - So as  $k$  increases, the values converge



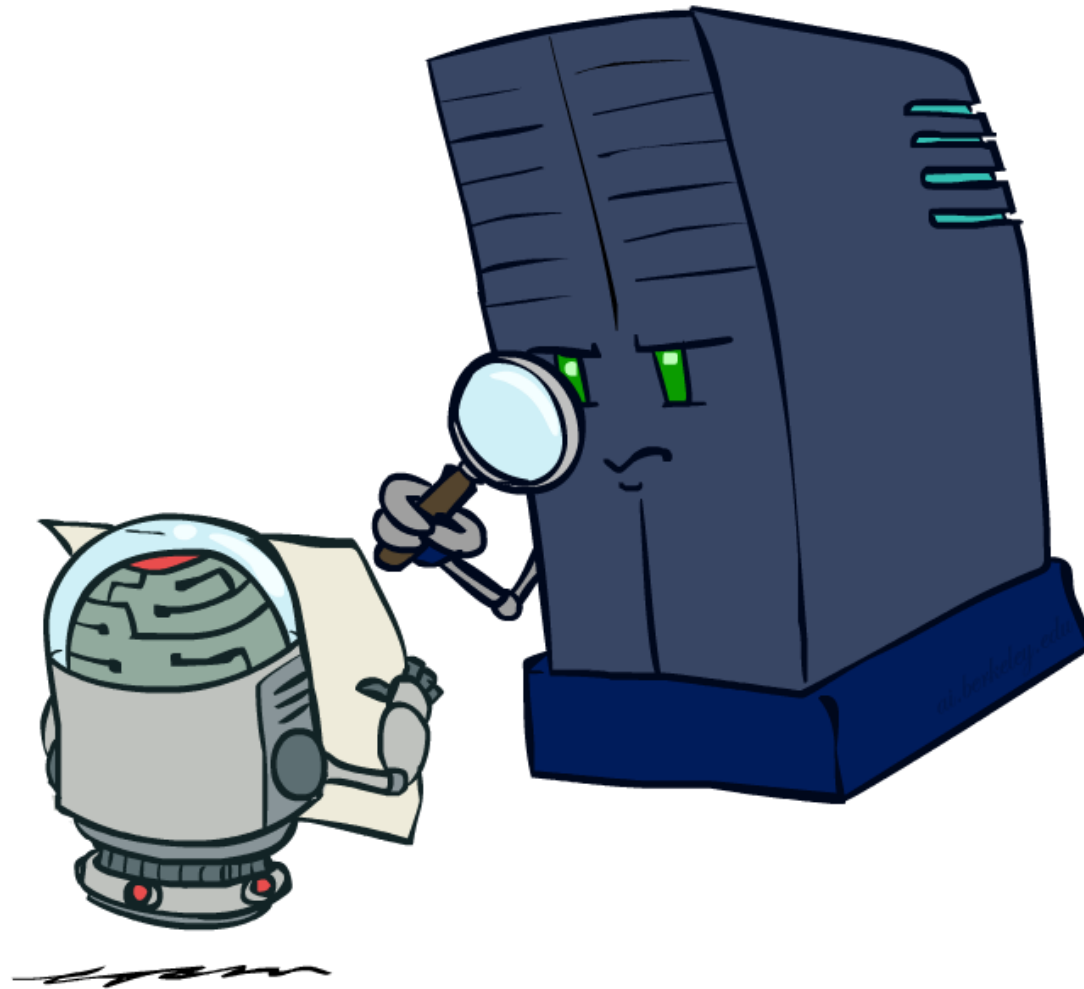
# Policy Methods





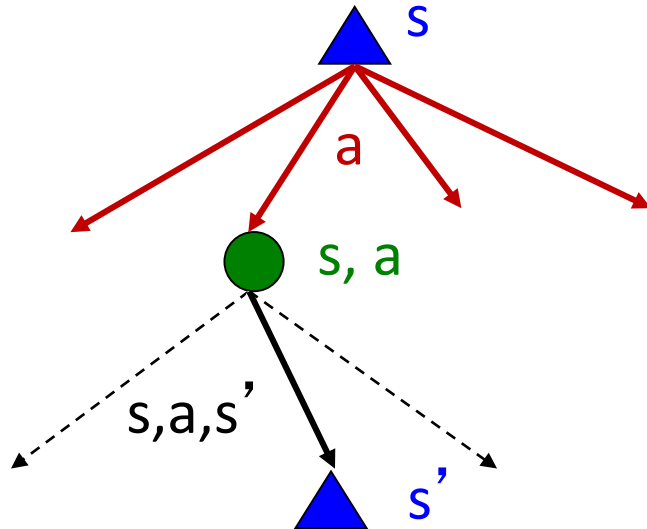
# Policy Evaluation

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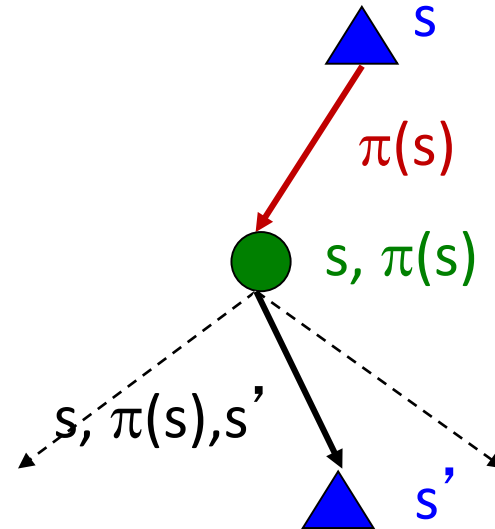


# Fixed Policies

Do the optimal action



Do what  $\pi$  says to do

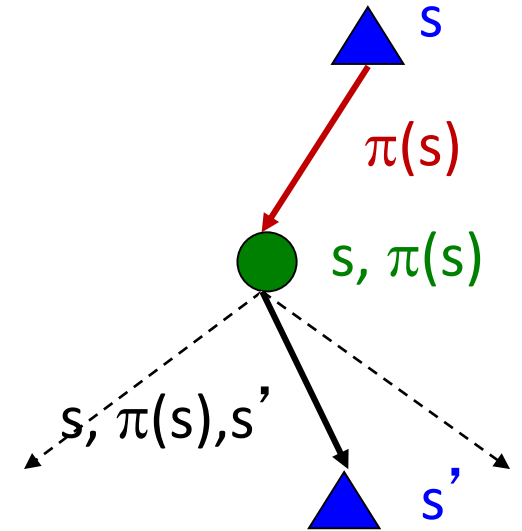


- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy  $\pi(s)$ , then the tree would be simpler – only one action per state
  - ... though the tree's value would depend on which policy we fixed

# Utilities for a Fixed Policy

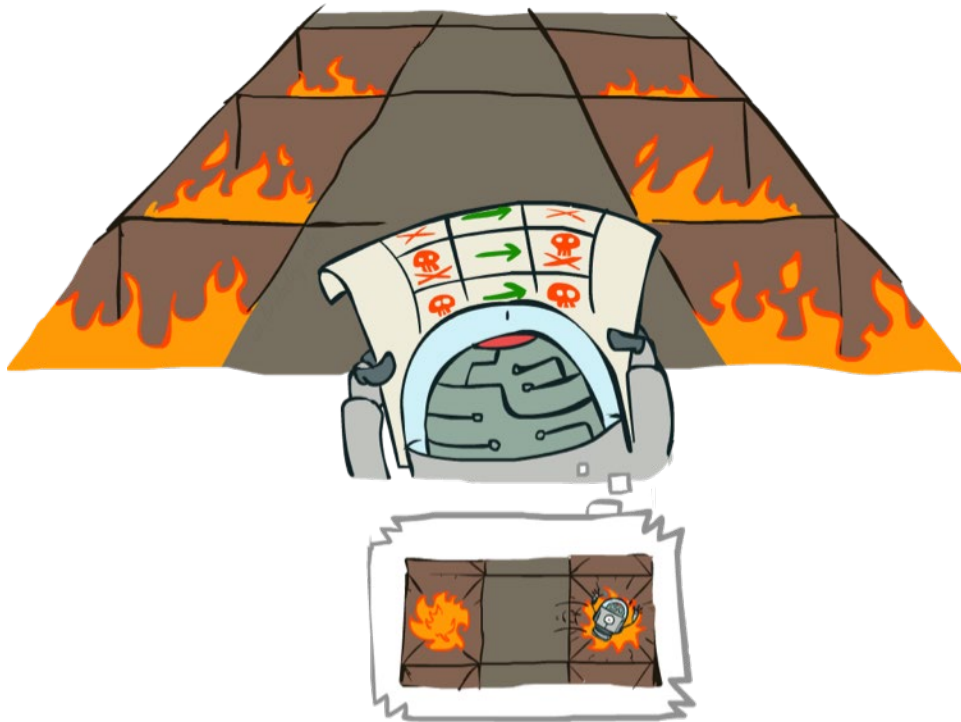
- Another basic operation: compute the utility of a state  $s$  under a fixed (generally non-optimal) policy
- Define the utility of a state  $s$ , under a fixed policy  $\pi$ :  
 $V^\pi(s)$  = expected total discounted rewards starting in  $s$  and following  $\pi$
- Recursive relation (one-step look-ahead):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')]$$

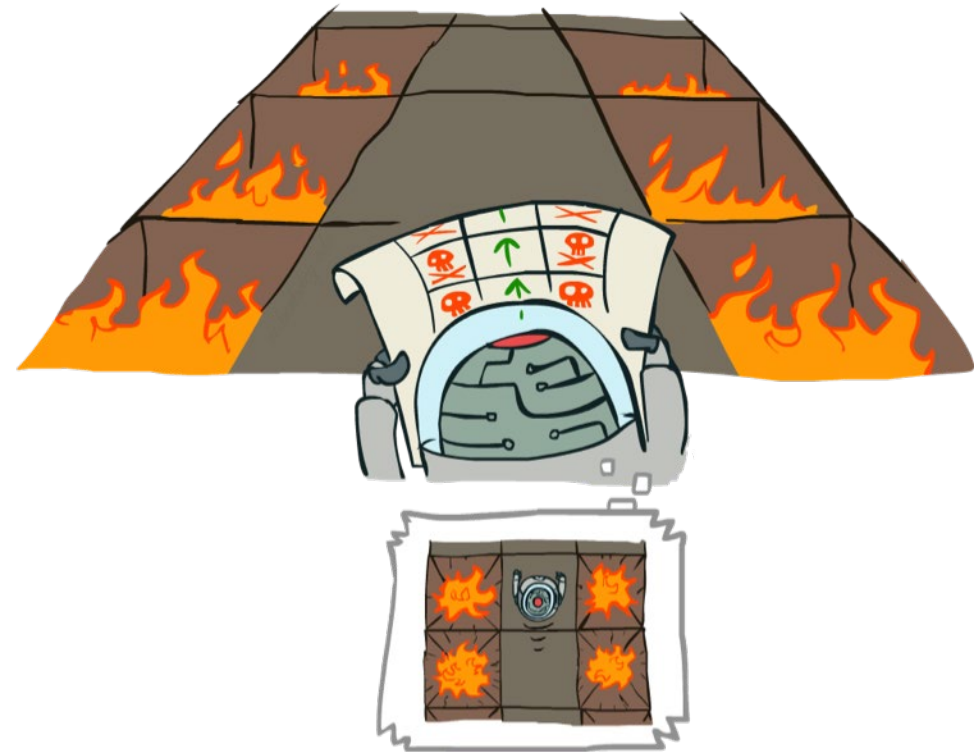


# Example: Policy Evaluation

Always Go Right



Always Go Forward



# Example: Policy Evaluation

Always Go Right



Always Go Forward

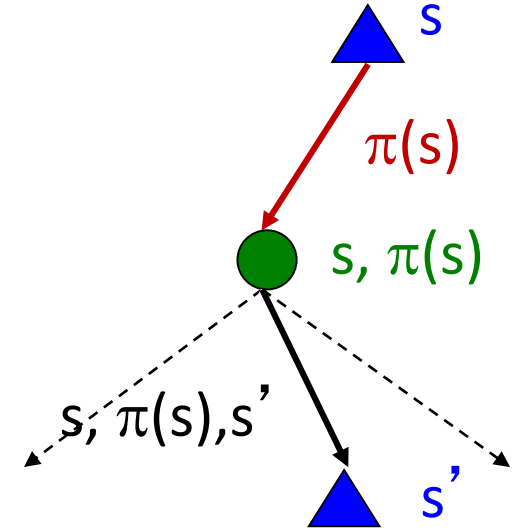


# Policy Evaluation

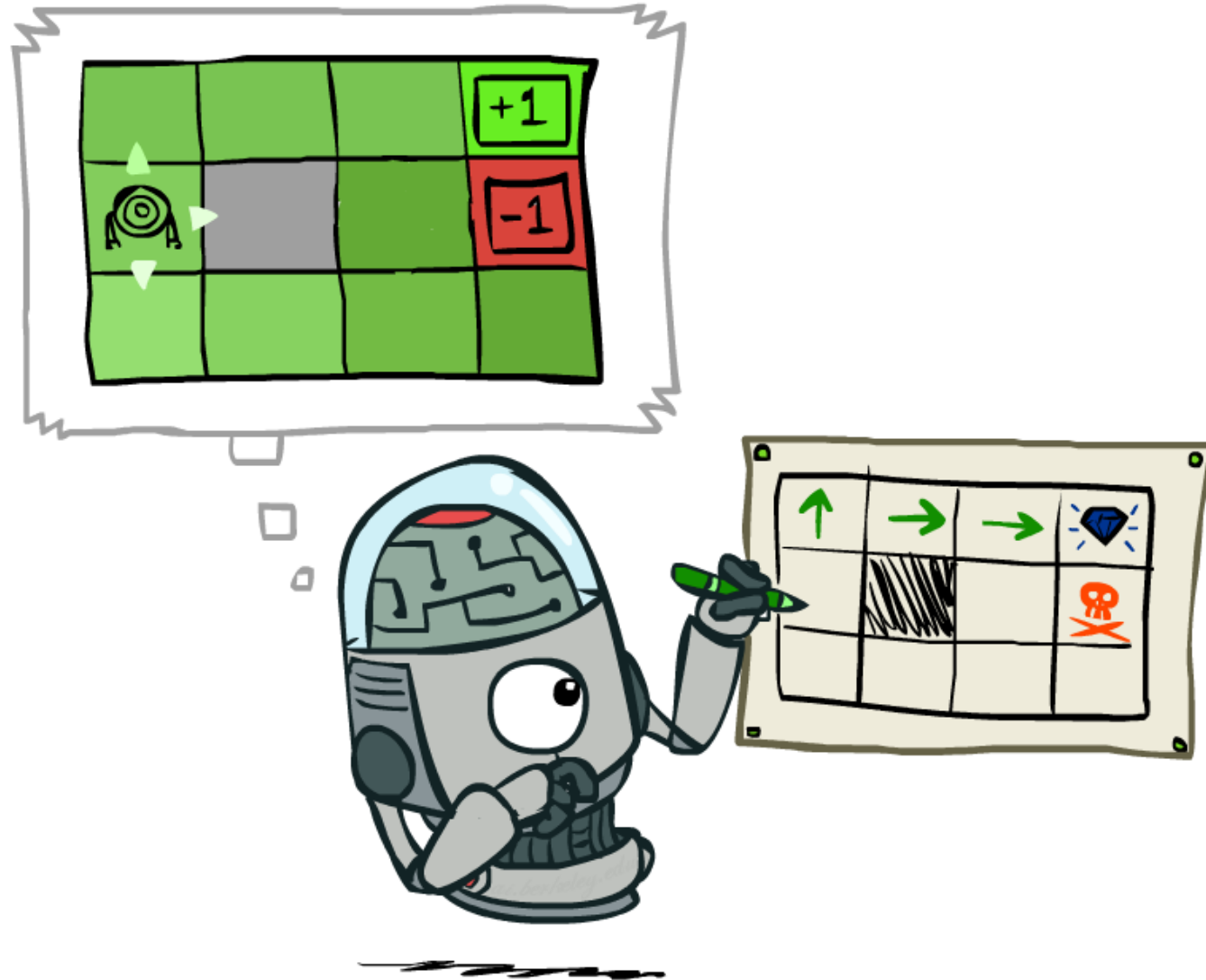
- How do we calculate the  $V$ 's for a fixed policy  $\pi$ ?
- Idea : Turn recursive equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$



# Policy Extraction



# Computing Actions from Values

- Let's imagine we have the optimal values  $V^*(s)$
- How should we act?
  - It's not obvious!
- We need to do a mini-expectimax (one step)



$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

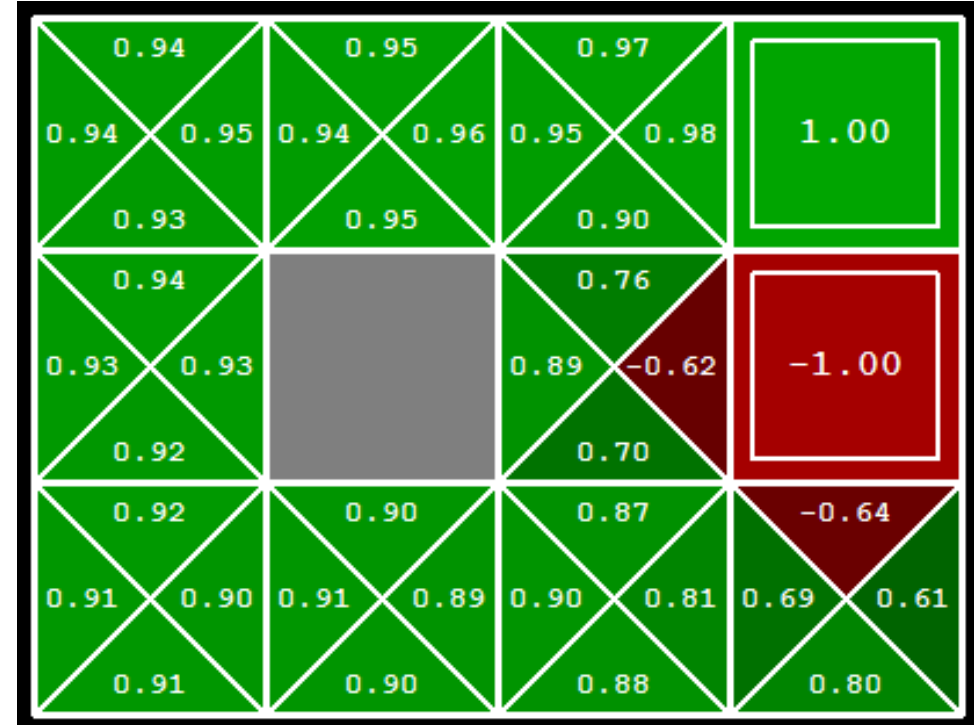
- This is called **policy extraction**, since it gets the policy implied by the values



# Computing Actions from Q-Values

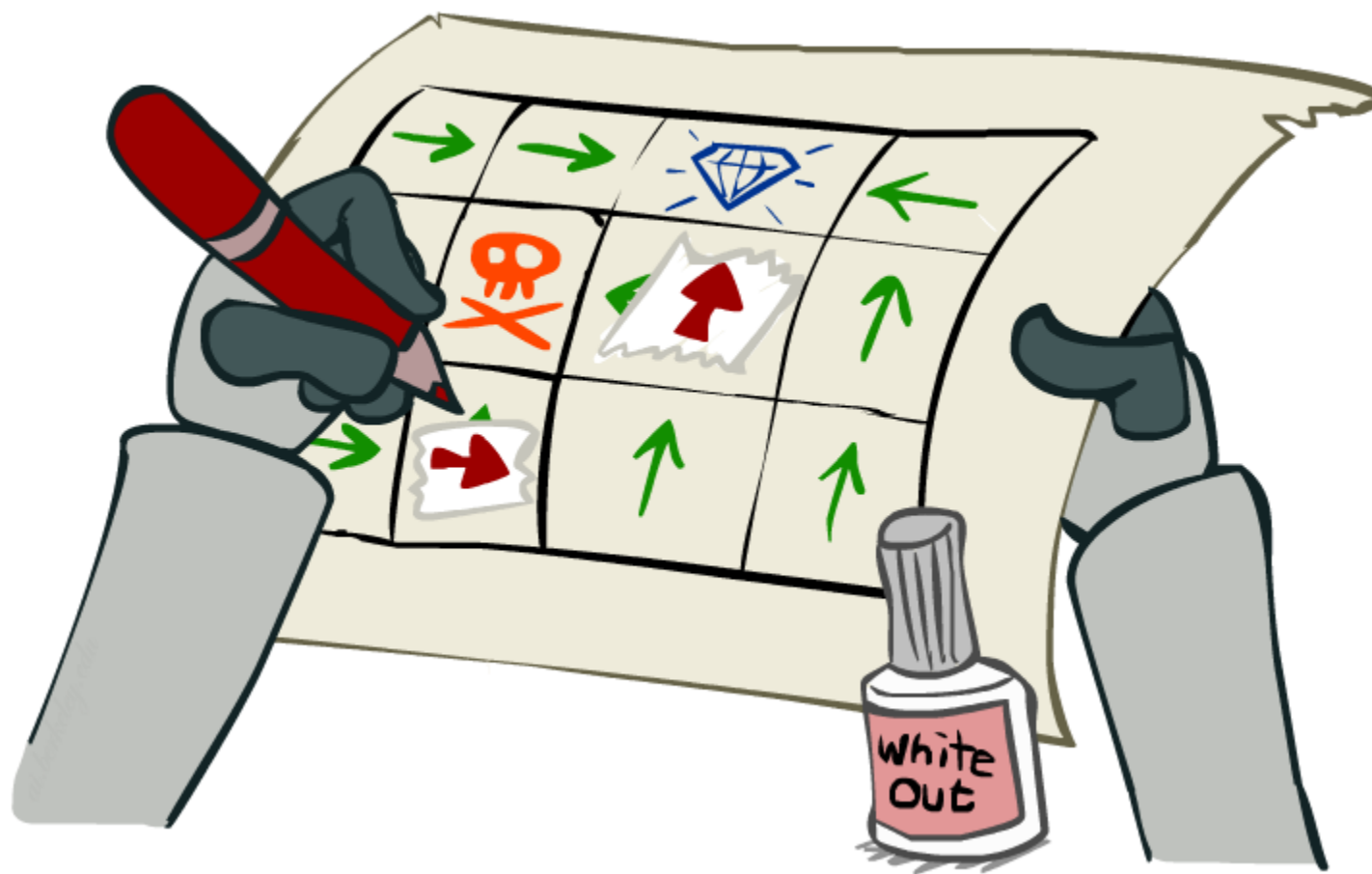
- Let's imagine we have the optimal q-values:
- How should we act?
  - Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$



- Important lesson: actions are easier to select from q-values than values!

# Policy Iteration

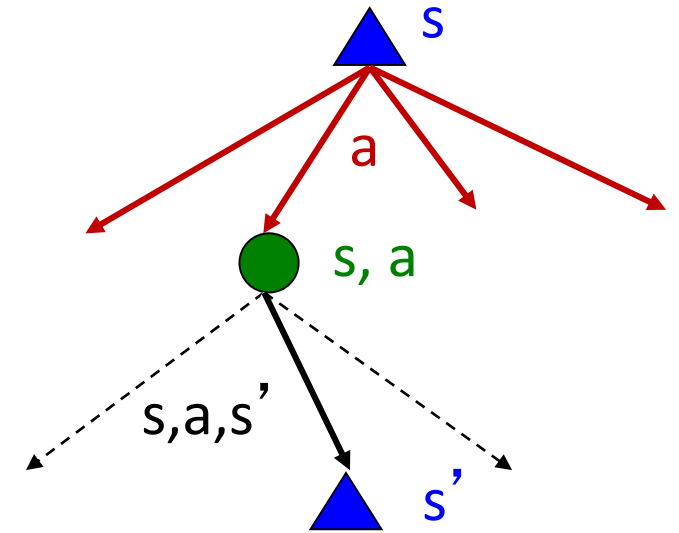


# Problems with Value Iteration

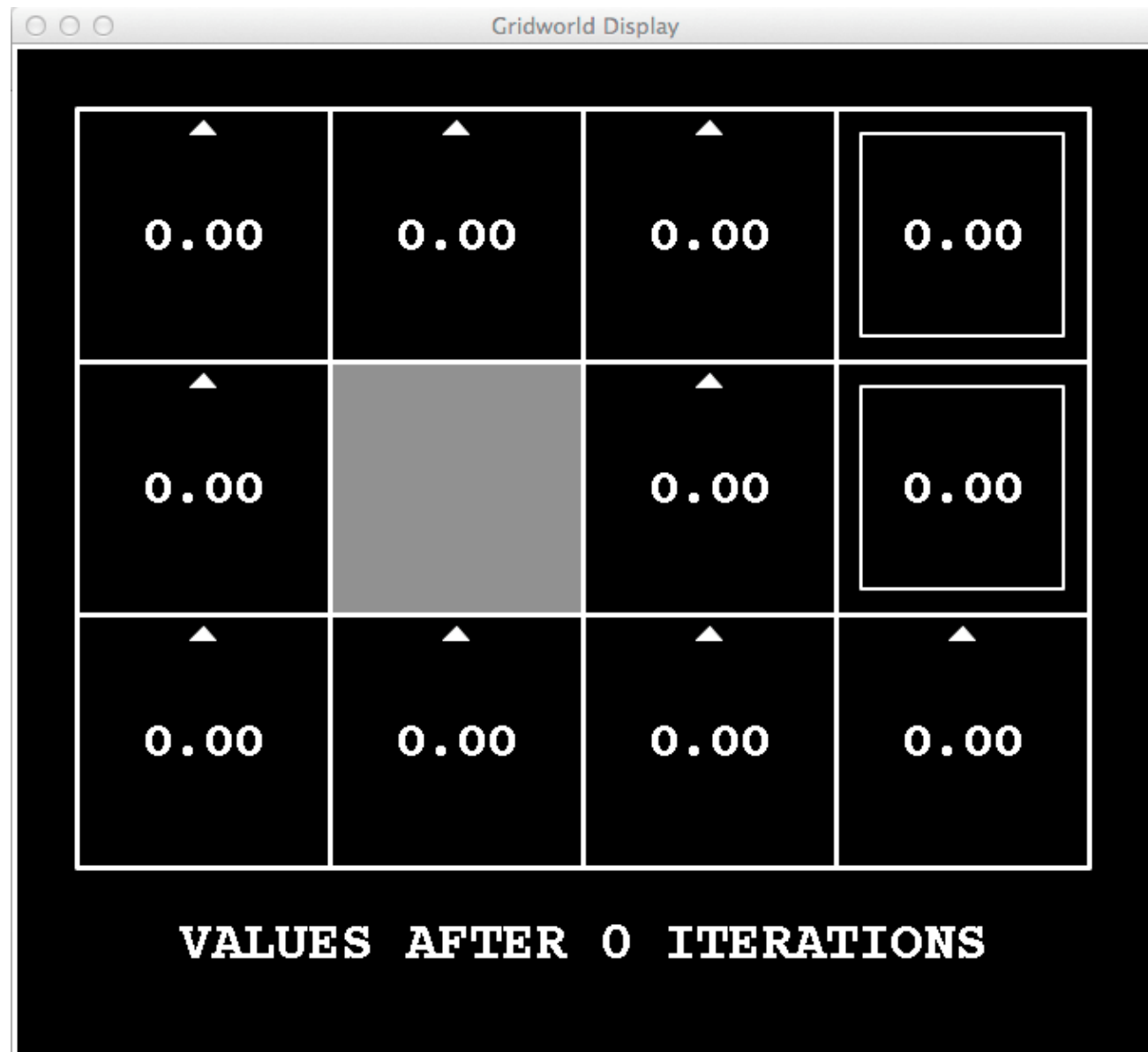
- Value iteration repeats the following updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow –  $O(S^2A)$  per iteration
- Observation 2: The “max” (optimal action) at each state rarely changes
- Observation 3: The policy often converges long before the values

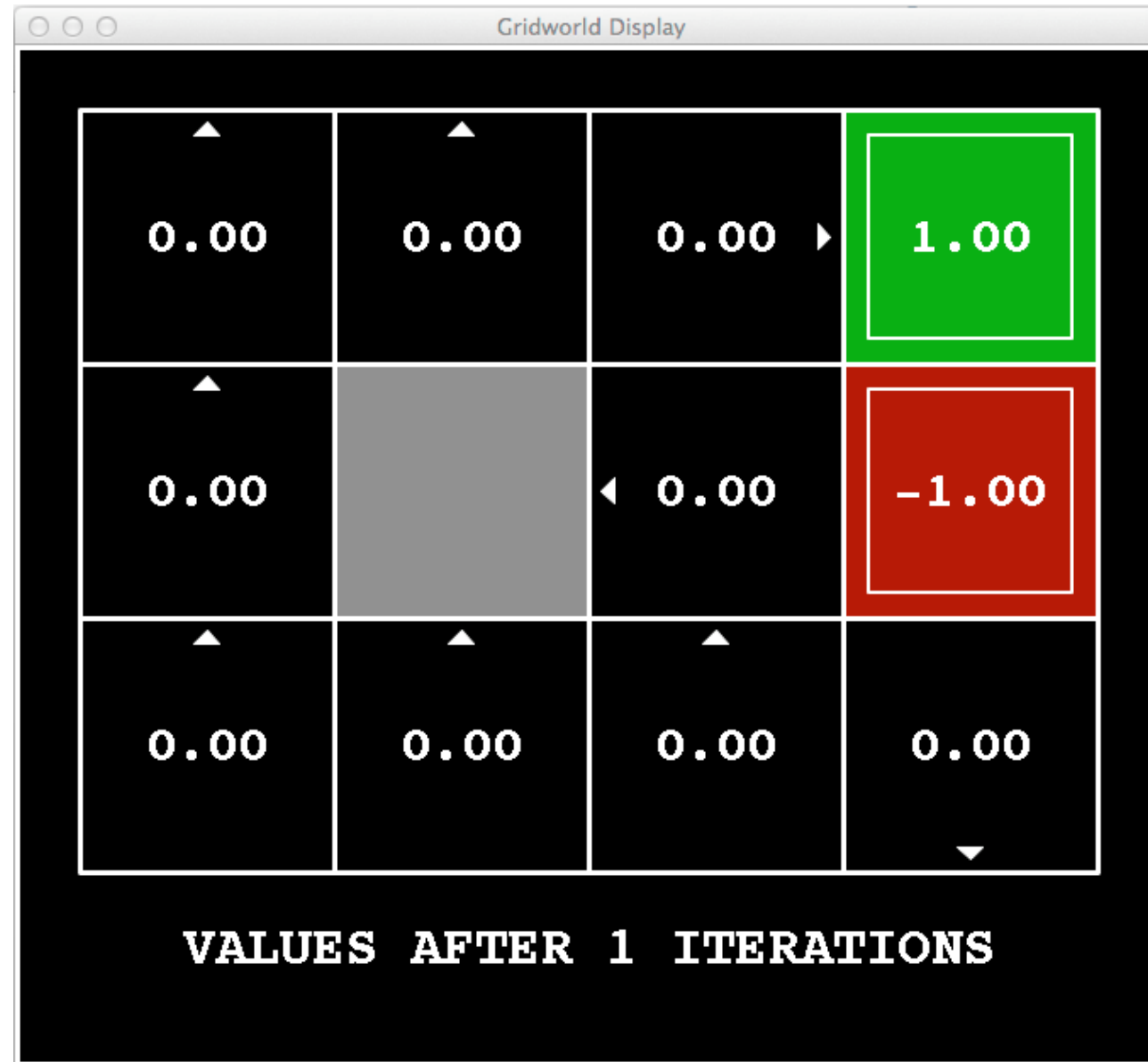


# k=0



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=1



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=2



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=3



Noise = 0.2  
Discount = 0.9  
Living reward = 0

$k=4$



Noise = 0.2  
Discount = 0.9  
Living reward = 0



k=5



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=6



Noise = 0.2  
Discount = 0.9  
Living reward = 0

$k=7$



Noise = 0.2  
Discount = 0.9  
Living reward = 0

k=8



Noise = 0.2  
Discount = 0.9  
Living reward = 0

k=9



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=10



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=11



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# k=12



Noise = 0.2  
Discount = 0.9  
Living reward = 0



# k=100



Noise = 0.2  
Discount = 0.9  
Living reward = 0

# Policy Iteration

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- Alternative approach for optimal values:
  - **Step 1: Policy evaluation:** calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - **Step 2: Policy improvement:** update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges
- This is **policy iteration**
  - It's still optimal!
  - Can converge (much) faster under some conditions

# Policy Iteration

- Evaluation: For fixed current policy  $\pi$ , find values with policy evaluation:
  - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') [R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')]$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

# Comparison

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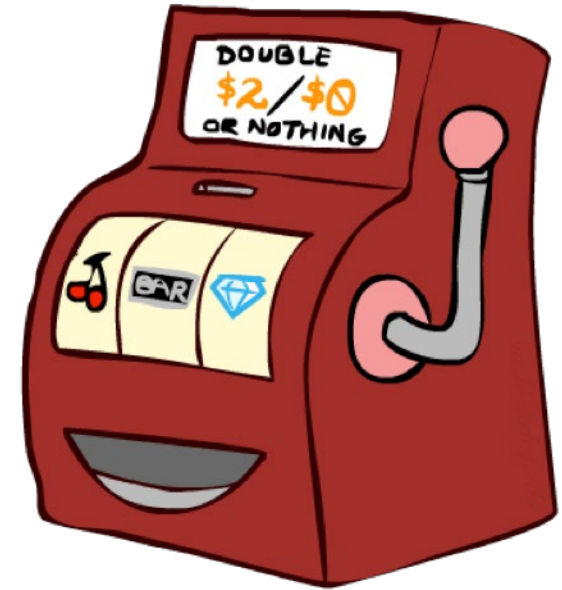
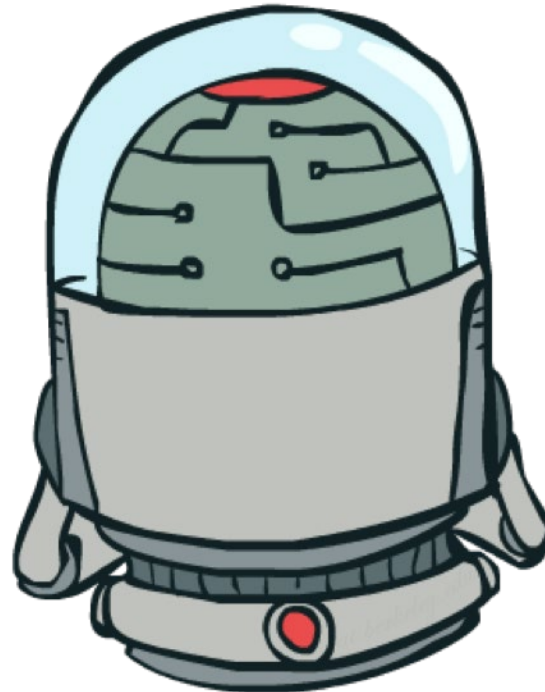
- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - The new policy will be better (or we're done)

# Summary: MDP Algorithms

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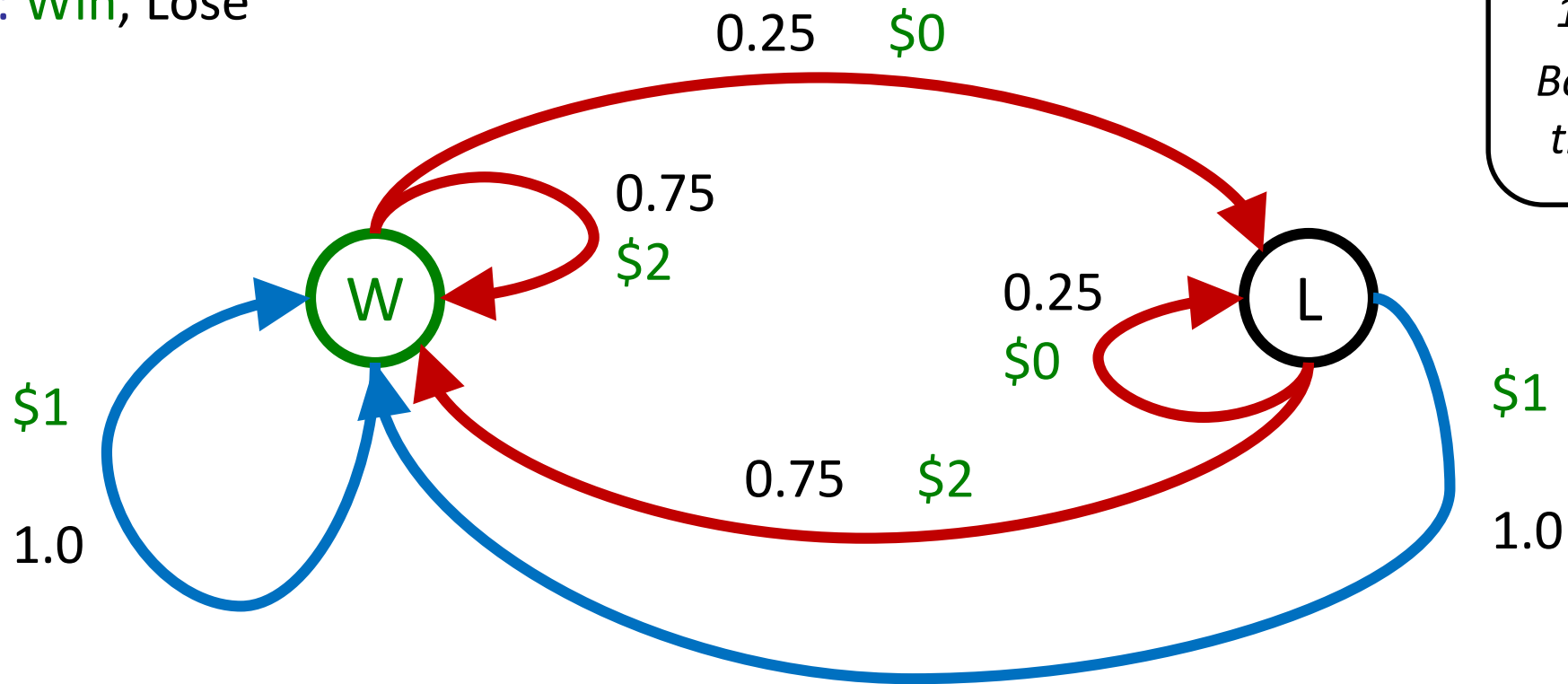
- So you want to....
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

# Double Bandits



# Double-Bandit MDP

- Actions: *Blue*, *Red*
- States: *Win*, Lose



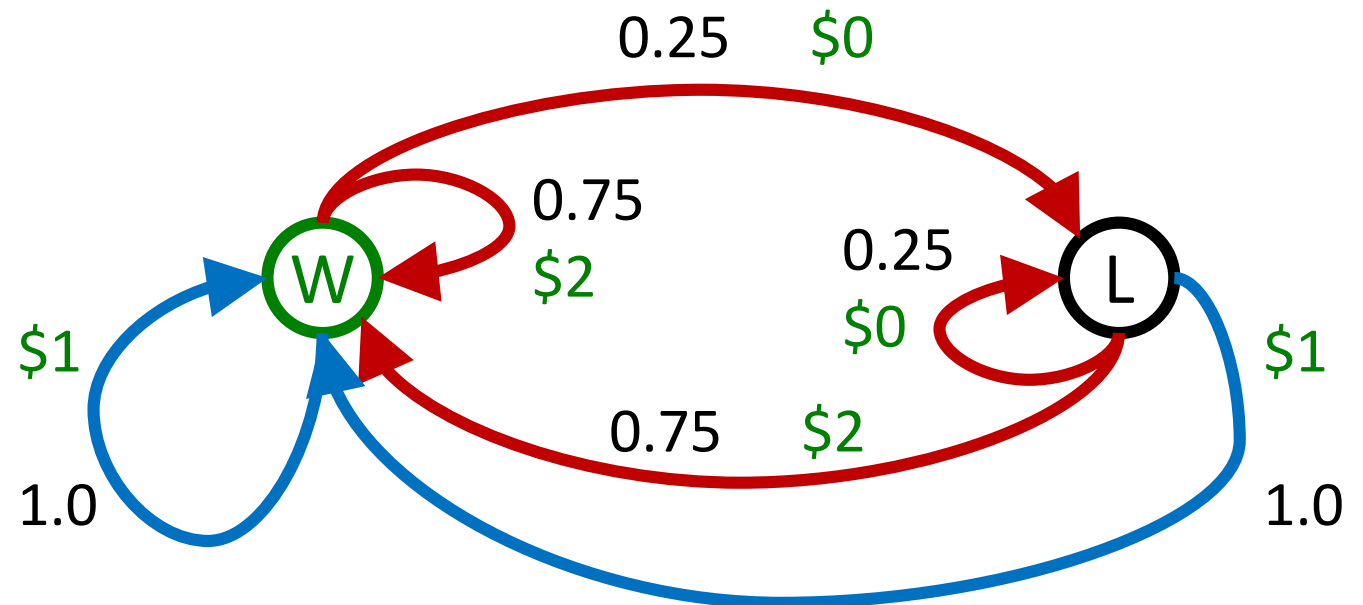
*No discount*  
*100 time steps*  
*Both states have the same value*

# Offline Planning

- Solving MDPs is offline planning
  - You determine all quantities through computation
  - You need to know the details of the MDP
  - You do not actually play the game!

*No discount*  
*100 time steps*  
*Both states have the same value*

	Value
Play Red	150
Play Blue	100





# Let's Play!

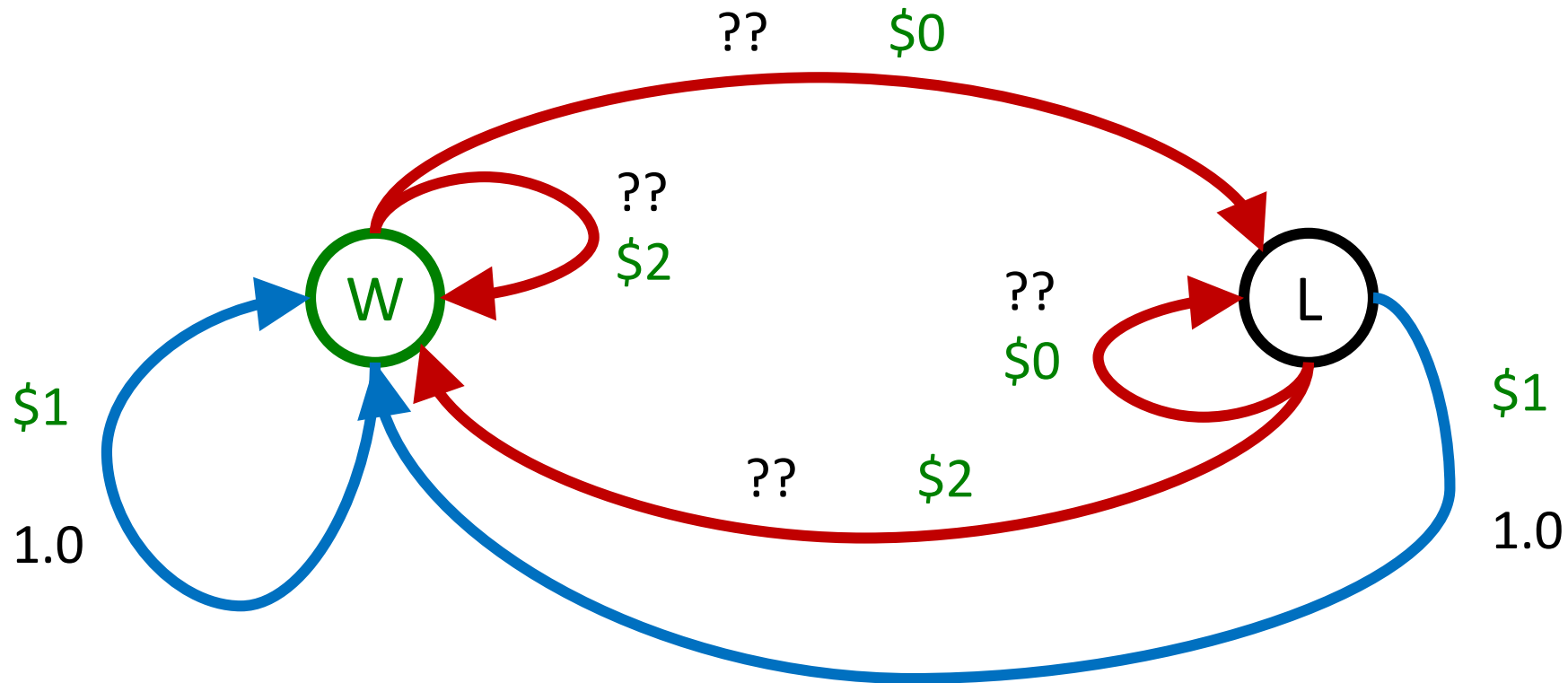


\$2 \$2 \$0 \$2 \$2

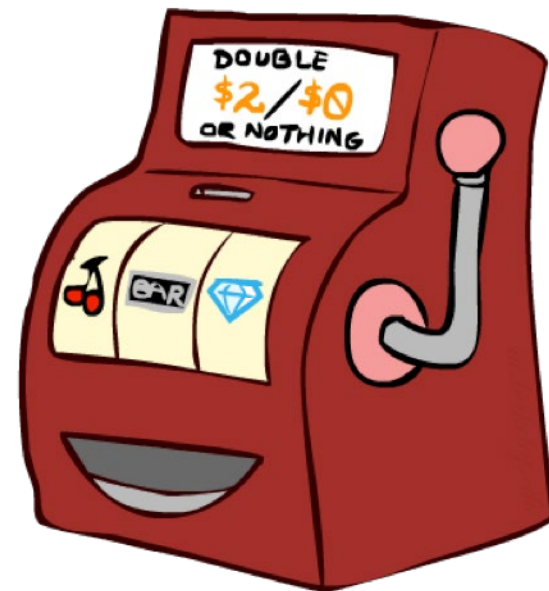
\$2 \$2 \$0 \$0 \$0

# Online Planning

- Rules changed! Red's win chance is different.



# Let's Play!



\$0 \$0 \$0 \$2 \$0  
\$2 \$0 \$0 \$0 \$0

# What Just Happened?

- That wasn't planning, it was learning!
  - Specifically, reinforcement learning
  - There was an MDP, but you couldn't solve it with just computation
  - You needed to actually act to figure it out
- Important ideas in reinforcement learning that came up
  - Exploration: you have to try unknown actions to get information
  - Exploitation: eventually, you have to use what you know
  - Regret: even if you learn intelligently, you make mistakes
  - Sampling: because of chance, you have to try things repeatedly
  - Difficulty: learning can be much harder than solving a known MDP



# Lab 4

- Create a TIC-TAC-TOE solver capable of predicting the result of a specific game when a board is provided.
- Complete 2 Functions
  - `minmax_tictactoe(board, turn)`
  - `abprun_tictactoe(board, turn)`
- Must use `game_status(board)` to check board
  - Use once per node in search tree

X	O	
	X	

board=[1,2,0,0,1,0,0,0,0]

# Lab4

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- AB pruning must use exact algorithm from slides
  - Don't use any additional information about game to improve it
- Helpful hint:
  - AB pruning code can be turned into mini-max with a very simple modification.
- Program in python3
- Don't use any additional modules other than the included common

# Alpha-Beta Implementation

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v = \max(v, \text{value}(\text{successor},$   
             $\alpha, \beta))$   
        if  $v \geq \beta$  return  $v$   
         $\alpha = \max(\alpha, v)$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    initialize  $v = +\infty$   
    for each successor of state:  
         $v = \min(v, \text{value}(\text{successor},$   
             $\alpha, \beta))$   
        if  $v \leq \alpha$  return  $v$   
         $\beta = \min(\beta, v)$   
    return  $v$ 
```







