# A quickly convergent series for $\pi$, based on an Edo-Japan series

Hans-Christian Meynert

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*In Edo times (1600–1868) the connection and communication of Japan and Europe was not permitted by the government of the shogun. It was not allowed for European ships to anchor in Japan. So Japanese mathematics (called wasan) was only possible on the base of ancient Chinese mathematical sources. Nevertheless, the Japanese mathematicians found many of the results the Europeans arrived at by the same time, determinants, integrals, spherical volume and surface, to name but the most prominent ones.*

*The series presented here is not taken from wasan as such. But it only makes use of prerequisites of that time. It makes use of another series, namely, a series for the square root function $f(x) = \sqrt{1-x}$ which had been found by the samurai Takebe Katahiro (1664–1739) who was an advisor of the shogun. He deduced, within this, the coefficients of the series by conjecture, i. e. by improper induction. He calculated, say, the first ten or fifteen coefficients, and from these, he inferred a general law for all of the summands. This method he called "tetsujutsu" "combining the series" (see [1], sections 8.3, 8.3.1, and for the series pp. 286-287, section 8.5.2). Another fictional samurai could have possibly then deduced the series presented here, as the integral of the monomial had been known to Edo mathematicians. The samurai Seki Takakazu, the teacher of Takebe calculated the integral of the monomial in his book Katsuyou Sanpou (see [1], chapter 5, note 56).*

*I now give the detailed proof of the here presented series formula for $\pi$ in modern notation.*

**1. Lemma:** Let

$$f(x) = \sqrt{1-x}$$

Then, we have

$$f^{(n)}(x) = \begin{cases} (1-x)^{\frac{1}{2}} & \text{for } n = 0 \\ -\frac{1}{2}(1-x)^{-\frac{1}{2}} & \text{for } n = 1 \\ -\frac{(2n-3)!!}{2^n}(1-x)^{-\frac{2n-1}{2}} & \text{for } n \geq 2 \end{cases}$$

and you will have the corresponding Taylor series for $f$, that is, for any square root.

Proof: The second derivative of $f$ will be

$$f''(x) = -\frac{1}{2} \cdot \frac{1}{2} \cdot (1-x)^{-\frac{3}{2}}$$

This, obviously, is the beginning of the induction. The $(n+1)$'th derivative of $f$ will be

$$f^{(n+1)}(x) = -\frac{(2n-3)!!}{2^n}\left(-\frac{2n-1}{2}\right) \cdot (-1) \cdot (1-x)^{-\frac{2n-1}{2}-1}$$

$$= -\frac{(2(n+1)-3)!!}{2^{n+1}}(1-x)^{-\frac{2(n+1)-1}{2}}$$

Q. E. D.

**2. Theorem:** (Square root series expansion[1], Takebe Katahiro, Edo-Japan, around 1720) If

$$|x| < 1$$

then

$$\sqrt{1-x} = 1 - \frac{1}{2}x - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!}x^n$$

is a convergent series for the square root.

[1]Taken from [1], pp. 286-287, section 8.5.2: "Summary of the Method of the *Enri Kohaijutsu*".

Proof: This follows at once from Lemma (1.), if you insert this in a Taylor series and consider

$$2^n n! = (2n)!!$$

Q. E. D.

**3. Definition:** Denote by $A$ a sector of $\frac{1}{12}$ of a unit circle. Then, you have

$$\pi = 12A$$

**4. Definition:** Denote by $B$ the following integral:

$$B = \int_0^{\frac{1}{2}} \sqrt{1-x^2}dx$$

**5. Lemma:** $B$ can be calculated as follows:

$$B = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{3} \left(\frac{1}{2}\right)^3 - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!} \left(\frac{1}{2}\right)^{2n+1} \frac{1}{2n+1}$$

Proof: This follows, if you insert the square root series from Theorem (2.) and integrate the monomials from 0 to $\frac{1}{2}$. Q. E. D.

**6. Lemma:** $A$ can be expressed as follows:

$$A = B - \sqrt{\frac{3}{4}} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

Proof: This follows, if you subtract from $B$ a triangle of base $\frac{1}{2}$ and height $\sqrt{\frac{3}{4}}$. Q. E. D.

**7. Theorem:** The number $\pi$ can then be expressed as follows:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \binom{2n}{n} \frac{3}{2n+1}$$

Proof: For we have

$$\pi = 12A = 12\left(B - \sqrt{\frac{3}{4}} \cdot \frac{1}{4}\right)$$

$$= 6 - 3\sqrt{\frac{3}{4}} - 2\left(\frac{1}{2}\right)^3 - 12\sum_{n=2}^{\infty} \frac{(2n-3)!!}{(2n)!!} \left(\frac{1}{2}\right)^{2n+1} \frac{1}{2n+1}$$

$$
\begin{aligned}
&= 6 - 3\sqrt{1-\frac{1}{4}} - 2\left(\frac{1}{2}\right)^3 - 12\sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left(\frac{1}{2}\right)^{2n+1}\frac{1}{2n+1} \\
&\underset{*}{=} 3 + \frac{3}{8} - 2\left(\frac{1}{2}\right)^3 + 3\sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left(\frac{1}{4}\right)^{n} \\
&\quad -12\sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left(\frac{1}{2}\right)^{2n+1}\frac{1}{2n+1} \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left[3\left(\frac{1}{4}\right)^{n} - 12\left(\frac{1}{2}\right)^{2n+1}\frac{1}{2n+1}\right] \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left[3\left(\frac{1}{4}\right)^{n} - 6\left(\frac{1}{4}\right)^{n}\frac{1}{2n+1}\right] \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\cdot\left(\frac{1}{4}\right)^{n}\cdot\frac{3\cdot(2n+1)-6}{2n+1} \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n-3)!!}{(2n)!!}\cdot\left(\frac{1}{4}\right)^{n}\cdot\frac{3\cdot(2n-1)}{2n+1} \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\left(\frac{1}{4}\right)^{n}\cdot\frac{3}{2n+1} \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n)!}{(2n)!!\cdot(2n)!!}\cdot\left(\frac{1}{4}\right)^{n}\cdot\frac{3}{2n+1} \\
&= 3 + \frac{1}{8} + \sum_{n=2}^{\infty}\frac{(2n)!}{n!\cdot n!}\cdot\left(\frac{1}{16}\right)^{n}\cdot\frac{3}{2n+1} \\
&= \sum_{n=0}^{\infty}\frac{1}{16^n}\binom{2n}{n}\frac{3}{2n+1}
\end{aligned}
$$

where in step (*) the expansion from Theorem (2.) has been used again. Q. E. D.

Comment: The error of $\pi$ decreases, with each summand, by a factor less than $\frac{1}{4}$.

# References

[1] Annick Horiuchi. *Japanese Mathematics in the Edo Period (1600-1868).* Birkhäuser, 2010.