Lemma 1 (Substitution of Values). *If* $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}'$ $\texttt{with}$ $\varnothing$*, then* $\hat{\Gamma} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}$ $\texttt{with}$ ε

Proof. By induction on the derivation of $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.

Case: ε-PolyTypeAbs. Then $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1.[\hat{v}/y]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeAbs, we get the conclusion.

Case: ε-PolyFxAbs. Then $\hat{e} = \lambda \phi \subseteq \varepsilon_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda \phi \subseteq \varepsilon_1.[\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxAbs, we get the conclusion.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeApp, we get the conclusion.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxApp, we get the conclusion.

Lemma 2 (Substitution of Types). *If* $\hat{\Gamma}, X <: \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\hat{\Gamma} \vdash \hat{\tau}'' <: \hat{\tau}'$*, then* $\hat{\Gamma} \vdash [\hat{\tau}''/X]\hat{e} : \hat{\tau}$ $\texttt{with}$ ε

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.'

Case: ε-Var, ε-Resource. Then $\hat{e} = [\hat{\tau}''/X]\hat{e}$, so the typing judgement in the antecedent and consequent can be the same.

Case: ε-Abs. Then $\hat{e} = \lambda x : \hat{\tau}_1.\hat{e}_2$, and $[\hat{\tau}''/X]\hat{e}_2 = \lambda x : [\hat{\tau}''/X]\hat{\tau}_1.[\hat{\tau}''/X]\hat{e}_2$. WLOG assume that $\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$. By inductive assumption and inversion, $[\hat{\tau}''/X]\hat{e}_2$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_2$ in $\hat{\Gamma}, X <: \hat{\tau}'$. By ε-Abs, $\hat{\Gamma} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$.
But now we have to establish that this new type we just derived is a subtype of $\hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$. To do that requires us to show that $\hat{\tau}_1 <: [\hat{\tau}''/X]\hat{\tau}_1$, because function types are contravariant in their input type under the subtyping relation. However, the substitution should intuitively be making the type more precise, so the subtyping is going the wrong way.

Theorem 1 (Progress). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\hat{e}$ *is not a value, then* $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$*, for some* $\hat{e}', \varepsilon$.

Proof. By induction on the derivation of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}''\hat{\tau}' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda \phi \subseteq \varepsilon_1.\hat{e}_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda \phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$.

Theorem 2 (Preservation). *If* $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ $\texttt{with}$ ε_A *and* $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$*, then* $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$ $\texttt{with}$ ε_B*, where* $\hat{e}_B <: \hat{e}_A$ *and* $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$*, for some* $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$*.*

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ $\texttt{with}$ ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. Consider which reduction rule was used.
Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyTypeApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : \hat{\tau}_A$ $\texttt{with}$ ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.
Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_3.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$.
The result follows by the substitution lemma.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.
Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A$ $\texttt{with}$ ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.
Subcase: E-PolyFxApp2. Then $(\lambda \phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.