Notation: if $\hat{\Gamma} \vdash \texttt{effects}(\hat{\tau}) = \varepsilon$, we write $\texttt{effects}(\hat{\Gamma}, \hat{\tau}) = \varepsilon$.

Lemma 1 (Effect-VarAbstraction). *If* $\hat{\Gamma} \vdash \texttt{effects}([\varnothing/\Phi]\hat{\tau}) = \varepsilon$*, then* $\hat{\Gamma}, \Phi \subseteq \varnothing \vdash \texttt{effects}(\hat{\tau}) = \varepsilon$*.*

Lemma 2 (Effect-VarWeakening). *If the following are true:*

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta} \vdash \texttt{effects}(\hat{\tau}) = \varepsilon_0$
2. $\Phi \notin \texttt{dom}(\hat{\Delta})$
3. $\hat{\Gamma} \vdash \varepsilon_1 \subseteq \varepsilon_2$

Then $\hat{\Gamma}, \Phi \subseteq \varepsilon_2, \hat{\Delta} \vdash \texttt{effects}(\hat{\tau}) = \varepsilon_0 \cup \varepsilon_0'$*, where* $\varepsilon_0' \subseteq \varepsilon_2 - \varepsilon_1$*.*

Lemma 3 (Approximation I). *If the following are true:*

1. $\hat{\Gamma} \vdash \hat{\tau}$
2. $\hat{\Gamma} \vdash \texttt{effects}(\hat{\Gamma}, \hat{\tau}) \subseteq \varepsilon_s$
3. $\hat{\Gamma} \vdash \texttt{ho-safe}(\hat{\tau}, \varepsilon_s)$

then $\hat{\Gamma} \vdash \hat{\tau} <: \texttt{reannot}(\hat{\tau}, \varepsilon_s)$

Proof. By mutual induction with the Approximation II lemma on the form of $\hat{\tau}$.

$\boxed{\text{Case: } \hat{\tau} = \{\overline{r}\}}$ Since $\texttt{reannot}(\{\overline{r}\}, \varepsilon_s) = \{\overline{r}\}$, we can obtain the theorem conclusion by applying S-Reflexive.

$\boxed{\text{Case: } \hat{\tau} = \forall X <: \hat{\tau}_1.\hat{\tau}_2 \; \texttt{caps} \; \varepsilon_3}$ The theorem conclusion can be written as $\hat{\Gamma} \vdash (\forall \Phi \subseteq \varepsilon_1.\hat{\tau}_2 \; \texttt{caps} \; \varepsilon_3) <: (\forall \Phi \subseteq \varepsilon_1.\texttt{reannot}(\hat{\tau}_2, \varepsilon_s) \; \texttt{caps} \; \varepsilon_3)$. To establish this, we use S-PolyFx, which requires us to establish the following premises:

4. $\hat{\Gamma} \vdash \varepsilon \subseteq \varepsilon$
5. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1 \vdash \varepsilon_3 \subseteq \varepsilon_3$
6. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1 \vdash \hat{\tau}_2 <: \texttt{reannot}(\hat{\tau}_2, \varepsilon_s)$

(4) and (5) are true by reflexivity. Therefore, to establish the theorem conclusion, it is sufficient to establish (6). By inversion on (3), we know (7, 8).

7. $\hat{\Gamma} \vdash \varepsilon_1 \subseteq \varepsilon_s$
8. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1 \vdash \texttt{ho-safe}(\hat{\tau}_2, \varepsilon_s)$

By inversion on (2), we know $\texttt{effects}(\hat{\Gamma}, \widehat{[\varnothing/\Phi]}\hat{\tau}_2) \subseteq \varepsilon_s$. By the Effect-VarAbstraction lemma, this is the same as $\texttt{effects}((\hat{\Gamma}, \Phi \subseteq \varnothing), \hat{\tau}_2) \subseteq \varepsilon_s$. By the Effect-VarWeakening lemma, we know that $\texttt{effects}((\hat{\Gamma}, \Phi \subseteq \varepsilon_1), \hat{\tau}) \subseteq \varepsilon_s \cup \varepsilon_1$. Because of (7), $\varepsilon_s \cup \varepsilon_1 = \varepsilon_s$. Therefore, we have (9):

9. $\hat{\Gamma} \vdash \texttt{effects}((\hat{\Gamma}, \Phi \subseteq \varepsilon_1), \hat{\tau}_2) \subseteq \varepsilon_s$

Also, since $\hat{\Gamma} \vdash \hat{\tau}$, we know that $\hat{\Gamma} \vdash \hat{\tau}_2$. Trivially, $\hat{\Gamma}, \Phi \subseteq \varepsilon_1 \vdash \hat{\tau}_2$. With this judgement, as well as (8, 9), we can apply the inductive assumption of Approximation I, giving judgement (6).