

1 Grammar

$e ::= x$	<i>expressions</i>
r	
$\mathbf{new}_\sigma x \Rightarrow \overline{\sigma} = \overline{e}$	
$\mathbf{new}_d x \Rightarrow \overline{d} = e$	
$e.m(e)$	
$e.\pi$	
$\tau ::= \{\overline{\sigma}\}$	<i>types</i>
$\{\overline{r}\}$	
$\{\overline{d}\}$	
$\{\overline{d} \text{ captures } \varepsilon\}$	
$\sigma ::= d \text{ with } \varepsilon$	<i>labeled decls.</i>
$d ::= \mathbf{def } m(x : \tau) : \tau$	<i>unlabeled decls.</i>

Notes:

- σ denotes a declaration with effect labels; d a declaration without effect labels.
- \mathbf{new}_σ is for creating annotated objects; \mathbf{new}_d for unannotated objects.
- $\{\overline{\sigma}\}$ is the type of an annotated object. $\{\overline{d}\}$ is the type of an unannotated object.
- $\{\overline{d} \text{ captures } \varepsilon\}$ is a special kind of type that doesn't appear in source programs but may be assigned by the new rules in this section. Intuitively, ε is an upper-bound on the effects captured by $\{\overline{d}\}$.

2 Semantics

2.1 Static Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \text{ (T-VAR)} \qquad \frac{}{\Gamma, r : \{\overline{r}\} \vdash r : \{\overline{r}\}} \text{ (T-RESOURCE)}$$

$$\frac{\Gamma \vdash e_1 : \{\overline{r}\}}{\Gamma \vdash e_1.\pi : \mathbf{Unit}} \text{ (T-OPERCALL)}$$

$$\frac{\Gamma \vdash e_1 : \{\overline{\sigma}\} \quad \mathbf{def } m(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3 \in \{\overline{\sigma}\} \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1.m(e_2) : \tau_3} \text{ (T-METHCALL}_\sigma\text{)}$$

$$\frac{\Gamma \vdash e_1 : \{\overline{d}\} \quad \mathbf{def } m(y : \tau_2) : \tau_3 \in \{\overline{d}\} \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1.m(e_2) : \tau_3} \text{ (T-METHCALL}_d\text{)}$$

$$\frac{\Gamma, x : \{\overline{\sigma}\} \vdash \overline{\sigma} = \overline{e} \text{ OK}}{\Gamma \vdash \mathbf{new}_\sigma x \Rightarrow \overline{\sigma} = \overline{e} : \{\overline{\sigma}\}} \text{ (T-NEW}_\sigma\text{)} \qquad \frac{\Gamma, x : \{\overline{d}\} \vdash \overline{d} = e \text{ OK}}{\Gamma \vdash \mathbf{new}_d x \Rightarrow \overline{d} = e : \{\overline{d}\}} \text{ (T-NEW}_d\text{)}$$

$$\boxed{\Gamma \vdash d = e \text{ OK}}$$

$$\frac{d = \mathbf{def } m(y : \tau_2) : \tau_3 \quad \Gamma, y : \tau_2 \vdash e : \tau_3}{\Gamma \vdash d = e \text{ OK}} \text{ (}\varepsilon\text{-VALIDIMPL}_d\text{)}$$

$$\boxed{\Gamma \vdash \sigma = e \text{ OK}}$$

$$\frac{\Gamma, y : \tau_2 \vdash e : \tau_3 \text{ with } \varepsilon_3 \quad \sigma = \text{def } m(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3}{\Gamma \vdash \sigma = e \text{ OK}} \quad (\varepsilon\text{-VALIDIMPL}_\sigma)$$

$$\boxed{\Gamma \vdash e : \tau \text{ with } \varepsilon}$$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau \text{ with } \emptyset} \quad (\varepsilon\text{-VAR})$$

$$\frac{}{\Gamma, r : \{\bar{r}\} \vdash r : \{\bar{r}\} \text{ with } \emptyset} \quad (\varepsilon\text{-RESOURCE})$$

$$\frac{\Gamma, x : \{\bar{\sigma}\} \vdash \bar{\sigma} = \bar{e} \text{ OK}}{\Gamma \vdash \text{new}_\sigma x \Rightarrow \bar{\sigma} = \bar{e} : \{\bar{\sigma}\} \text{ with } \emptyset} \quad (\varepsilon\text{-NEWOBJ}) \quad \frac{\Gamma \vdash e_1 : \{\bar{r}\} \text{ with } \varepsilon_1}{\Gamma \vdash e_1.\pi : \text{Unit with } \{\bar{r}.\pi\} \cup \varepsilon_1} \quad (\varepsilon\text{-OPERCALL})$$

$$\frac{\Gamma \vdash e_1 : \{\bar{\sigma}\} \text{ with } \varepsilon_1 \quad \Gamma \vdash e_2 : \tau_2 \text{ with } \varepsilon_2 \quad \sigma_i = \text{def } m_i(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3}{\Gamma \vdash e_1.m_i(e_2) : \tau_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3} \quad (\varepsilon\text{-METHCALL})$$

$$\frac{\varepsilon_c = \text{effects}(\Gamma') \quad \Gamma' \subseteq \Gamma \quad \Gamma', x : \{\bar{d} \text{ captures } \varepsilon_c\} \vdash \bar{d} = \bar{e} \text{ OK}}{\Gamma \vdash \text{new}_d x \Rightarrow \bar{d} = \bar{e} : \{\bar{d} \text{ captures } \varepsilon_c\} \text{ with } \emptyset} \quad (\text{C-NEWOBJ})$$

$$\frac{\Gamma \vdash e_1 : \{\bar{d} \text{ captures } \varepsilon_c\} \text{ with } \varepsilon_1 \quad \Gamma \vdash e_2 : \tau_2 \text{ with } \varepsilon_2 \quad d_i = \text{def } m_i(y : \tau_2) : \tau_3}{\Gamma \vdash e_1.m_i(e_2) : \tau_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \text{effects}(\tau_2) \cup \varepsilon_c} \quad (\text{C-METHCALL})$$

$$\frac{\Gamma' \subseteq \Gamma \quad \Gamma' \vdash e : \tau}{\Gamma \vdash e : \tau \text{ with effects}(\Gamma')} \quad (\text{C-INFERENCE})$$

Notes:

- This system includes all the rules from the fully-annotated system.
- The T rules do standard typing of objects, without any effect analysis. Their sole purpose is so $\varepsilon\text{-ValidImpl}_d$ can be applied. **We are assuming the T-rules on their own are sound.**
- In C-NEWOBJ, Γ' is intended to be some subcontext of the current Γ . The object is labelled as capturing the effects in Γ' (exact definition in the next section).
- In C-NEWOBJ we must add $\text{effects}(\tau_2)$ to the static effects of the object, because the method body will have access to the resources captured by τ_2 (the type of the argument passed into the method).
- A good choice of Γ' would be Γ restricted to the free variables in the object definition.
- The purpose of C-INFERENCE is to ascribe static effects to unannotated portions of code (for instance, the body of an unlabeled method).
- As a useful convention we'll often use ε_c to denote the output of the **effects** function.

2.2 effects Function

The **effects** function returns the set of effects captured in a particular context.

- $\text{effects}(\emptyset) = \emptyset$
- $\text{effects}(\Gamma, x : \tau) = \text{effects}(\Gamma) \cup \text{effects}(\tau)$
- $\text{effects}(\{\bar{r}\}) = \{(r, \pi) \mid r \in \bar{r}, \pi \in \Pi\}$
- $\text{effects}(\{\bar{\sigma}\}) = \bigcup_{\sigma \in \bar{\sigma}} \text{effects}(\sigma)$
- $\text{effects}(\{\bar{d}\}) = \bigcup_{d \in \bar{d}} \text{effects}(d)$

- $\text{effects}(d \text{ with } \varepsilon) = \varepsilon \cup \text{effects}(d)$
- $\text{effects}(\text{def } m(x : \tau_1) : \tau_2) = \text{effects}(\tau_2)$
- $\text{effects}(\{\bar{d} \text{ captures } \varepsilon_c\}) = \varepsilon_c$

Notes:

- Since a method can return a capability for a resource r we need to figure out what the return type of a method captures. This requires a recursive crawl through the definitions and types inside it.
- In the last case we don't want to recurse to sub-declarations because the effects have already been captured previously (this is ε_c) by a potentially different context.

2.3 Dynamic Semantics

$$\boxed{e \longrightarrow e \mid \varepsilon}$$

$$\frac{e_1 \longrightarrow e'_1 \mid \varepsilon}{e_1.m(e_2) \longrightarrow e'_1.m(e_2) \mid \varepsilon} \text{ (E-METHCALL1)}$$

$$\frac{v_1 = \text{new}_\sigma x \Rightarrow \overline{\sigma = e} \quad e_2 \longrightarrow e'_2 \mid \varepsilon}{v_1.m(e_2) \longrightarrow v_1.m(e'_2) \mid \varepsilon} \text{ (E-METHCALL2}_\sigma\text{)} \quad \frac{v_1 = \text{new}_d x \Rightarrow \overline{d = e} \quad e_2 \longrightarrow e'_2 \mid \varepsilon}{v_1.m(e_2) \longrightarrow v_1.m(e'_2) \mid \varepsilon} \text{ (E-METHCALL2}_d\text{)}$$

$$\frac{v_1 = \text{new}_\sigma x \Rightarrow \overline{\sigma = e} \quad \text{def } m(y : \tau_1) : \tau_2 \text{ with } \varepsilon = e \in \overline{\sigma = e}}{v_1.m(v_2) \longrightarrow [v_1/x, v_2/y]e \mid \emptyset} \text{ (E-METHCALL3}_\sigma\text{)}$$

$$\frac{v_1 = \text{new}_d x \Rightarrow \overline{d = e} \quad \text{def } m(y : \tau_1) : \tau_2 = e \in \overline{d = e}}{v_1.m(v_2) \longrightarrow [v_1/x, v_2/y]e \mid \emptyset} \text{ (E-METHCALL3}_d\text{)}$$

$$\frac{e_1 \longrightarrow e'_1 \mid \varepsilon}{e_1.\pi \longrightarrow e'_1.\pi \mid \varepsilon} \text{ (E-OPERCALL1)} \quad \frac{}{r.\pi \longrightarrow \text{unit} \mid \{r.\pi\}} \text{ (E-OPERCALL2)}$$

$$\boxed{e \longrightarrow_* e \mid \varepsilon}$$

$$\frac{}{e \longrightarrow_* e \mid \emptyset} \text{ (E-MULTISTEP1)} \quad \frac{e \longrightarrow e' \mid \varepsilon}{e \longrightarrow_* e' \mid \varepsilon} \text{ (E-MULTISTEP2)}$$

$$\frac{e \longrightarrow_* e' \mid \varepsilon_1 \quad e' \longrightarrow_* e'' \mid \varepsilon_2}{e \longrightarrow_* e'' \mid \varepsilon_1 \cup \varepsilon_2} \text{ (E-MULTISTEP3)}$$

Notes:

- E-METHCALL2_d and E-METHCALL2_σ are really doing the same thing, but one applies to labeled objects (the σ version) and the other on unlabeled objects. Same goes for E-METHCALL3_σ and E-METHCALL3_d.
- E-METHCALL1 can be used for both labeled and unlabeled objects.

2.4 Substitution Function

We extend our Substitution function from the previous system in a straightforward way by adding a new case for unlabeled objects.

- $[e'/z]z = e'$
- $[e'/z]y = y$, if $y \neq z$
- $[e'/z]r = r$
- $[e'/z](e_1.m(e_2)) = ([e'/z]e_1).m([e'/z]e_2)$
- $[e'/z](e_1.\pi) = ([e'/z]e_1).\pi$
- $[e'/z](\text{new}_d x \Rightarrow \overline{d=e}) = \text{new}_d x \Rightarrow \overline{\sigma = [e'/z]e}$, if $z \neq x$ and $z \notin \text{freevars}(e_i)$
- $[e'/z](\text{new}_\sigma x \Rightarrow \overline{\sigma=e}) = \text{new}_\sigma x \Rightarrow \overline{\sigma = [e'/z]e}$, if $z \neq x$ and $z \notin \text{freevars}(e_i)$

3 Proofs

Lemma 3.1. (Canonical Forms)

Statement. Suppose e is a value. The following are true:

- If $\Gamma \vdash e : \{\bar{r}\} \text{ with } \varepsilon$, then $e = r$ for some resource r .
- If $\Gamma \vdash e : \{\bar{\sigma}\} \text{ with } \varepsilon$, then $e = \text{new}_\sigma x \Rightarrow \overline{\sigma=e}$.
- If $\Gamma \vdash e : \{\bar{d} \text{ captures } \varepsilon_c\} \text{ with } \varepsilon$, then $e = \text{new}_d x \Rightarrow \overline{d=e}$.

Furthermore, $\varepsilon = \emptyset$ in each case.

Proof. These typing judgements each appear exactly once in the conclusion of different rules. The result follows by inversion of ε -RESOURCE, ε -NEWOBJ, and C-NEWOBJ respectively. \square

Lemma 3.2. (Substitution Lemma)

Statement. If $\Gamma, z : \tau' \vdash e : \tau \text{ with } \varepsilon$, and $\Gamma \vdash e' : \tau' \text{ with } \varepsilon'$, then $\Gamma \vdash [e'/z]e : \tau \text{ with } \varepsilon$.

Intuition If you substitute z for something of the same type, the type of the whole expression stays the same after substitution.

Proof. We've already proven the lemma by structural induction on the ε rules. The new case is defined on a form not in the grammar for the fully-annotated system. So all that remains is to induct on derivations of $\Gamma \vdash e : \tau \text{ with } \varepsilon$ using the new C rules.

Case. C-METHCALL.

Then $e = e_1.m(e_2)$ and $[e'/z]e = ([e'/z]e_1).m([e'/z]e_2)$. By inductive assumption we know that e_1 and $[e'/z]e_1$ have the same types, and that e_2 and $[e'/z]e_2$ have the same types. Since e and $[e'/z]e$ have the same syntactic struture, and their corresponding subexpressions have the same types, then Γ can use C-METHCALL to type $[e'/z]e$ the same as e .

Case. C-INFERENCE.

Then $\Gamma \vdash e : \tau \text{ with effects}(\Gamma')$, where $\Gamma' \subseteq \Gamma$. By inversion $\Gamma' \vdash e : \tau$. Applying the inductive hypothesis (and our assumption that the T rules are sound) $\Gamma' \vdash [e'/z]e : \tau$. Since $\Gamma' \subseteq \Gamma$ we have $\Gamma' \vdash [e'/z]e : \tau \text{ with effects}(\Gamma')$ under C-INFERENCE. Because $\Gamma' \subseteq \Gamma$ then $\Gamma \vdash [e'/z]e : \tau \text{ with effects}(\Gamma')$.

Case. C-NEWOBJ.

Then $e = \text{new}_d x \Rightarrow \overline{d=e}$. z appears in some method body e_i . By inversion we know $\Gamma, x : \{\bar{\sigma}\} \vdash \overline{d=e}$ OK. The only rule with this conclusion is ε -VALIDIMPL_d; by inversion on that we know for each i that:

- $d_i = \text{def } m_i(y : \tau_1) : \tau_2 \text{ with } \varepsilon$
- $\Gamma, y : \tau_1 \vdash e_i : \tau_2 \text{ with } \varepsilon$

If z appears in the body of e_i then $\Gamma, z : \tau \vdash d_i = e_i$ OK by inductive assumption. Then we can use ε -VALIDIMPL_d to conclude $\overline{d = [e'/z]e}$ OK. This tells us that the types and static effects of all the methods are unchanged under substitution. By choosing the same $\Gamma' \subseteq \Gamma$ used in the original application of C-NEWOBJ, we can apply C-NEWOBJ to the expression after substitution. The types and static effects the methods are the same, and the same Γ' has been chosen, so $[e'/z]e$ will be ascribed the same type as e .

□

Lemma 3.3. (Monotonicity of effects)

Statement. If $\Gamma_1 \subseteq \Gamma_2$ then $\mathbf{effects}(\Gamma_1) \subseteq \mathbf{effects}(\Gamma_2)$

Proof. Because $\mathbf{effects}(\Gamma_1)$ is the union of $\mathbf{effects}(\tau)$, for every $(x, \tau) \in \Gamma_1 \subseteq \Gamma_2$. Then $\mathbf{effects}(\Gamma_1) \subseteq \mathbf{effects}(\Gamma_2)$.

□

Lemma 3.4. (Use Principle)

Statement. If $\Gamma \vdash e_A : \tau_A$ with ε_A , and $e_A \longrightarrow_* e'_A \mid \varepsilon$, then $\forall r. \pi \in \varepsilon \mid (r, \{r\}) \in \Gamma$. Furthermore, $\varepsilon \subseteq \mathbf{effects}(\Gamma)$.

Proof. The only reduction that can add effects to ε is $r.\pi$. So at some point, an expression of the form $r.\pi$ must have been evaluated. In the source program it must have had the form $e.\pi$. Since the entire program typechecked under Γ , e must have been typed to $\{r\}$ at some point. Since resources cannot be dynamically created, $(r, \{r\}) \in \Gamma$. Since every resource with an operation called upon it is Γ , $\varepsilon \subseteq \mathbf{effects}(\Gamma)$ follows by the definition of $\mathbf{effects}$ for the case of a resource.

Intuition. If you typecheck e with Γ , if an effect can happen on r when executing e then r must be in Γ .

□

Lemma 3.5. (Tightening Lemma)

Statement. If $\Gamma \vdash e : \tau$ with ε then $\Gamma \cap \mathbf{freevars}(e) \vdash e : \tau$ with ε .

Proof. The typing judgements operate on the form of e , so don't consider any variables external to e .

□

Note. We'll use $\mathbf{freevars}(e) \cap \Gamma$ to mean Γ , where the pair (x, τ) is thrown out if $x \notin \mathbf{freevars}(e)$.

Intuition. If you can typecheck e in Γ , you can throw out the parts in Γ not relevant to e and still typecheck it.

Definition 3.6. (label)

Given a program containing unlabeled parts we can safely label those parts. This process is well-defined if $\Gamma \vdash e : \tau$; then we say the labeling of e is $\mathbf{label}(\Gamma, e) = \hat{e}$.

- $\mathbf{label}(r, \Gamma) = r$
- $\mathbf{label}(x, \Gamma) = x$
- $\mathbf{label}(e_1.m(e_2), \Gamma) = \mathbf{label}(e_1, \Gamma).m(\mathbf{label}(e_2), \Gamma)$
- $\mathbf{label}(e_1.\pi(e_2), \Gamma) = \mathbf{label}(e_1, \Gamma).\pi(\mathbf{label}(e_2), \Gamma)$
- $\mathbf{label}(\mathbf{new}_\sigma x \Rightarrow \overline{\sigma = e}, \Gamma) = \mathbf{new}_\sigma x \Rightarrow \mathbf{label_helper}(\overline{\sigma = e}, \Gamma)$
- $\mathbf{label}(\mathbf{new}_d x \Rightarrow \overline{d = e}, \Gamma) = \mathbf{new}_\sigma x \Rightarrow \mathbf{label_helper}(\overline{d = e}, \Gamma)$
- $\mathbf{label_helper}(\sigma = e, \Gamma) = \sigma = \mathbf{label}(e, \Gamma)$
- $\mathbf{label_helper}(\mathbf{def } m(y : \tau_2) : \tau_3 = e, \Gamma) = \mathbf{def } m(y : \tau_2) : \tau_3 \text{ with } \mathbf{effects}(\Gamma \cap \mathbf{freevars}(e)) = \mathbf{label}(e, \Gamma)$

Notes:

- $\Gamma \cap \mathbf{freevars}(e)$ is the set of pairs $x : \tau \in \Gamma$, such that $x \in \mathbf{freevars}(e)$.
- $\mathbf{label}(e, \Gamma)$ is read as: “the labeling of e in Γ ”.
- Often the Γ we use is obvious in context; in such cases we write $\mathbf{label}(e)$ instead of $\mathbf{label}(e, \Gamma)$.
- Beware of confusing notation: there are two types of equality in the above definitions. One is the equality which defines \mathbf{label} , and the other is the equality $\sigma = e$ of declarations in the programming language.
- The program after labeling will be fully-labeled, so typing it will be sound under the ε rules.

- `label` is defined on expressions; `label-helper` on declarations. Everywhere other than this section we'll only use `label`.
- Initially it seems like `label` on a `newσ` object should just be the identity function; but the body of the methods of such an object may instantiate unlabeled objects and/or call methods on unlabeled objects, so we must recursively label those.
- We may sometimes say `labels(e) = ê`, and from then on refer to the labeled version of e as \hat{e} . We'll use $\hat{\tau}$ and $\hat{\varepsilon}$ to refer to the type and static effects of the labeled version.

Observation 3.7.

Statement. `label(e)` is a value if and only if e is a value.

Proof. By inspection of the definition of `label`.

□

Property 3.8. (Commutativity Between `label` and `sub`)

Statement. Fix Γ and define `label(e) = label(e, Γ)`. Then `label([e'/z]e) = [label(e')/z](label(e))`

Intuition. If perform substitution and labeling on an expression, the order in which you do things doesn't matter.

Proof. Induction on the form of e . In each case, “left-hand side” refers to `label([e'/z]e)` while “right-hand side” refers to `[label(e')/z](label(e))`.

Case. $e = r$.

By definition, `label(r) = r` and $[e'/z]r = r$, for any e' . Both sides are equivalent to r because `sub` and `label` act like the identity function.

Case. $e = x$.

By definition, `label(x) = x`. $[e'/z]x$ has two definitions, depending on if $x = z$; consider each case.

Subcase. $x \neq z$. Then $[e'/z]x = x$. Both sides are equivalent to x because `sub` and `label` act like the identity function.

Subcase. $x = z$. Then $[e'/z]x = z$. On the left-hand side, `label([e'/z]x) = label(e')`. On the right-hand side, `[label(e')/z]x = label(e')`.

Case. $e = e_1.\pi$.

On the left-hand side.

$$\begin{aligned}
 & \text{label}([e'/z](e_1.\pi)) \\
 &= \text{label}([e'/z]e_1).\pi && \text{(definition of sub)} \\
 &= (\text{label}([e'/z]e_1)).\pi && \text{(definition of label)} \\
 &= ([\text{label}(e')/z](\text{label}(e_1))).\pi && \text{(inductive assumption on } e_1)
 \end{aligned}$$

On the right-hand side.

$$\begin{aligned}
 & [\text{label}(e')/z](\text{label}(e_1.\pi)) \\
 &= [\text{label}(e')/z](\text{label}(e_1).\pi) && \text{(definition of label)} \\
 &= ([\text{label}(e')/z](\text{label}(e_1))).\pi && \text{(definition of sub)}
 \end{aligned}$$

Case. $e = e_1.m(e_2)$.

On the left-hand side.

$$\begin{aligned}
& \text{label}([e'/z](e_1.m(e_2))) \\
&= \text{label}([e'/z]e_1).m([e'/z]e_2) && \text{(definition of sub)} \\
&= (\text{label}([e'/z]e_1)).m(\text{label}([e'/z]e_2)) && \text{(definition of label)} \\
&= ([\text{label}(e')/z](\text{label}(e_1))).m(\text{label}([e'/z]e_2)) && \text{(inductive assumption on } e_1) \\
&= ([\text{label}(e')/z](\text{label}(e_1)).m([\text{label}(e')/z](\text{label}(e_2)))) && \text{(inductive assumption on } e_2)
\end{aligned}$$

On the right-hand side.

$$\begin{aligned}
& [\text{label}(e')/z](\text{label}(e_1.m(e_2))) \\
&= [\text{label}(e')/z](\text{label}(e_1)).m(\text{label}(e_2)) && \text{(definition of label)} \\
&= ([\text{label}(e')/z](\text{label}(e_1))).m([\text{label}(e')/z](\text{label}(e_2))) && \text{(definition of sub)}
\end{aligned}$$

Case. $e = \text{new}_\sigma x \Rightarrow \overline{\sigma = e}$.

On the left-hand side.

$$\begin{aligned}
& \text{label}([e'/z](\text{new}_\sigma x \Rightarrow \overline{\sigma_i = e_i})) \\
&= \text{label}(\text{new}_\sigma x \Rightarrow \overline{\sigma_i = [e'/z]e_i}) && \text{(definition of sub)} \\
&= \text{new}_\sigma x \Rightarrow \text{label-helper}(\sigma_i = [e'/z]e_i) && \text{(definition of label)} \\
&= \text{new}_\sigma x \Rightarrow \sigma_i = \text{label}([e'/z]e_i) && \text{(definition of label-helper on each } \sigma_i = [e'/z]e_i)
\end{aligned}$$

On the right-hand side.

$$\begin{aligned}
& [\text{label}(e')/z](\text{label}(\text{new}_\sigma x \Rightarrow \overline{\sigma_i = e_i})) \\
&= [\text{label}(e')/z](\text{new}_\sigma x \Rightarrow \text{label-helper}(\overline{\sigma_i = e_i})) && \text{(definition of label)} \\
&= [\text{label}(e')/z](\text{new}_\sigma x \Rightarrow \overline{\sigma_i = \text{label}(e_i)}) && \text{(definition of label-helper on each } \sigma_i = e_i) \\
&= \text{new}_\sigma x \Rightarrow \overline{\sigma_i = [\text{label}(e')/z](\text{label}(e_i))} && \text{(definition of sub)} \\
&= \text{new}_\sigma x \Rightarrow \sigma_i = \text{label}([e'/z]e_i) && \text{(inductive assumption on each } e_i)
\end{aligned}$$

Case. $e = \text{new}_d x \Rightarrow \overline{d = e}$.

The proof of this is quite similar to previous case for labeled objects. The main difference is that when labeling an unlabeled object, each $d_i = e_i$ turns into a $\sigma_i = e_i$. For clarity we will define $\varepsilon_i = \text{effects}(\Gamma \cap \text{freevars}(e_i))$, and $\sigma_i = d_i$ with ε_i (these are from the definition of label-helper).

On the left-hand side.

$$\begin{aligned}
& \text{label}([e'/z](\text{new}_d x \Rightarrow \overline{d_i = e_i})) \\
&= \text{label}(\text{new}_d x \Rightarrow \overline{d_i = [e'/z]e_i}) && \text{(definition of sub)} \\
&= \text{new}_d x \Rightarrow \text{label-helper}(d_i = [e'/z]e_i) && \text{(definition of label)} \\
&= \text{new}_d x \Rightarrow \overline{d_i \text{ with } \varepsilon_i = \text{label}([e'/z]e_i)} && \text{(definition of label-helper)} \\
&= \text{new}_d x \Rightarrow \sigma_i = \text{label}([e'/z]e_i) && (\sigma_i = d_i \text{ with } \varepsilon_i)
\end{aligned}$$

On the right-hand side.

$$\begin{aligned}
& [\text{label}(e')/z](\text{label}(\text{new}_d x \Rightarrow \overline{d_i = e_i})) \\
&= [\text{label}(e')/z](\text{new}_d x \Rightarrow \text{label-helper}(\overline{d_i = e_i})) && \text{(definition of label)} \\
&= [\text{label}(e')/z](\text{new}_\sigma x \Rightarrow \overline{d_i \text{ with } \varepsilon_i = \text{label}(e_i)}) && \text{(definition of label-helper on each } d_i = e_i) \\
&= [\text{label}(e')/z](\text{new}_\sigma x \Rightarrow \overline{\sigma_i = \text{label}(e_i)}) && (\sigma_i = d_i \text{ with } \varepsilon_i) \\
&= \text{new}_\sigma x \Rightarrow \overline{\sigma_i = [\text{label}(e')/z](\text{label}(e_i))} && \text{(definition of sub)} \\
&= \text{new}_\sigma x \Rightarrow \sigma_i = \text{label}([e'/z]e_i) && \text{(inductive assumption on each } e_i)
\end{aligned}$$

□

Lemma 3.8. (Runtime Invariance Under label)

Statement. If the following are true:

- $\Gamma \vdash e_A : \tau_A$ with ε_A
- $e_A \longrightarrow e_B \mid \varepsilon$
- $\hat{e}_A = \text{label}(e_A, \Gamma)$

Then $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$ and $\hat{e}_B = \text{label}(e_B, \Gamma)$.

Proof. Induct on the form of e_A and then on the reduction rule $e_A \longrightarrow e_B \mid \varepsilon$. Throughout this proof there is only a single context Γ , so we'll write $\mathbf{label}(e)$ instead of $\mathbf{label}(e, \Gamma)$ as a notational short-hand.

Case. $e = r, e = x, e = \mathbf{new}_\sigma x \Rightarrow \overline{\sigma} = \overline{e}, e = \mathbf{new}_d x \Rightarrow \overline{d} = \overline{e}$.

Then e is a value and the theorem statement holds automatically.

Case. $e = e_1.\pi$.

The only typing rule which applies is ε -OPERCALL, which tells us:

- $\Gamma \vdash e_1 : \{r\} \text{ with } \varepsilon_1$
- $\Gamma \vdash e_1.\pi : \mathbf{Unit} \text{ with } \varepsilon_1 \cup \{r.\pi\}$

There are two possible reductions.

Subcase. E-OPERCALL1. We also know $e_1 \longrightarrow e'_1 \mid \varepsilon$, and $e_1.\pi \longrightarrow e'_1.\pi \mid \varepsilon$. By inductive assumption, $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$, and $\hat{e}'_1 = \mathbf{label}(e'_1)$. Applying definitions, $\hat{e}_A = \mathbf{label}(e_1.\pi) = (\mathbf{label}(e_1)).\pi = \hat{e}_1.\pi$. Because $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$, we may apply the reduction E-OPERCALL1 to obtain $\hat{e}_1.\pi \longrightarrow \hat{e}'_1.\pi \mid \varepsilon$. Lastly, $\hat{e}_B = \mathbf{label}(e'_1.\pi) = (\mathbf{label}(e'_1)).\pi$, which we know to be $\hat{e}'_1.\pi$ by inductive assumption.

Subcase. E-OPERCALL2. We also know $e_1 = r$ and $r.\pi \longrightarrow \mathbf{Unit} \mid \{r.\pi\}$. Applying definitions, $\hat{e}_A = \mathbf{label}(r.\pi) = (\mathbf{label}(r)).\pi = r.\pi = e_A$. The theorem holds immediately.

Case. $e = e_1.m_i(e_2)$.

There are five possible reductions.

Subcase. E-METHCALL1. We also know $e_1 \longrightarrow e'_1 \mid \varepsilon$ and $e_1.m_i(e_2) \longrightarrow e'_1.m_i(e_2) \mid \varepsilon$. By inductive assumption, $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$, and $\mathbf{label}(e'_1) = \hat{e}'_1$. Applying definitions $\hat{e}_A = \mathbf{label}(e_1.m_i(e_2)) = (\mathbf{label}(e_1)).m_i(\mathbf{label}(e_2)) = \hat{e}_1.m_i(\hat{e}_2)$. Because $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$, we may apply the reduction E-METHCALL1 to obtain $\hat{e}_1.m_i(\hat{e}_2) \longrightarrow \hat{e}'_1.m_i(\hat{e}_2) \mid \varepsilon$. Lastly, $\hat{e}_B = \mathbf{label}(e'_1.m_i(\hat{e}_2)) = (\mathbf{label}(e'_1)).m_i(\mathbf{label}(e_2))$, which we know to be $\hat{e}'_1.m_i(\hat{e}_2) = \hat{e}_B$ by assumptions.

Subcase. E-METHCALL2 $_\sigma$. We also know $e_1 = v_1 = \mathbf{new}_\sigma x \Rightarrow \overline{\sigma} = \overline{e}$, and $e_2 \longrightarrow e'_2 \mid \varepsilon$ and $v_1.m_i(e_2) \longrightarrow v_1.m_i(e'_2) \mid \varepsilon$. By inductive assumption, $\hat{e}_2 \longrightarrow \hat{e}'_2 \mid \varepsilon$, and $\mathbf{label}(e'_2) = \hat{e}'_2$. Applying definitions, $\hat{e}_A = \mathbf{label}(v_1.m_i(e_2)) = (\mathbf{label}(v_1)).m_i(\mathbf{label}(e_2)) = \hat{v}_1.m_i(\hat{e}_2)$. Because $\hat{e}_2 \longrightarrow \hat{e}'_2 \mid \varepsilon$, we may apply the reduction E-METHCALL $_\sigma$ to obtain $\hat{v}_1.m_i(\hat{e}_2) \longrightarrow \hat{v}_1.m_i(\hat{e}'_2)$. Lastly, $\hat{e}_B = \mathbf{label}(v_1.m_i(e'_2)) = (\mathbf{label}(v_1)).m_i(\mathbf{label}(e'_2))$, which we know to be $\hat{v}_1.m_i(\hat{e}'_2)$ by assumptions.

Subcase. E-METHCALL2 $_d$. Identical to the above subcase, but $e_1 = v_1 = \mathbf{new}_d x \Rightarrow \overline{d} = \overline{e}$, and we apply the reduction rule E-METHCALL $_d$ instead.

Subcase. E-METHCALL3 $_\sigma$. We also know the following:

- $e_1 = v_1 = \mathbf{new}_\sigma x \Rightarrow \overline{\sigma} = \overline{e}$
- $e_2 = v_2$
- $\mathbf{def } m_i(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3 = e_{body} \in \{\overline{\sigma}\}$
- $v_1.m_i(v_2) \longrightarrow [v_1/x, v_2/y]e_{body} \mid \emptyset$.

Applying definitions, $\mathbf{label}(v_1.m_i(v_2)) = (\mathbf{label}(v_1)).m_i(\mathbf{label}(v_2)) = \hat{v}_1.m_i(\hat{v}_2)$, where we define $\hat{v}_1 = \mathbf{label}(v_1)$ and $\hat{v}_2 = \mathbf{label}(v_2)$. Before labeling, the object v_1 has method m_i with body e_{body} . The labeled version, \hat{v}_1 , has method m_i with body $\mathbf{label}(e_{body}) = \hat{e}_{body}$. Because v_1 and v_2 are values, so are \hat{v}_1 and \hat{v}_2 . Therefore we can apply E-METHCALL3 $_\sigma$ to $\hat{v}_1.m_i(\hat{v}_2)$, giving us $\hat{v}_1.m_i(\hat{v}_2) \longrightarrow [\hat{v}_1/x, \hat{v}_2/y]\hat{e}_{body} \mid \emptyset$. Because \mathbf{label} and \mathbf{sub} commute, $\mathbf{label}(e_B) = \mathbf{label}([v_1/x, v_2/y]e_{body}) = [\mathbf{label}(v_1)/x, \mathbf{label}(v_2)/y](\mathbf{label}(e_{body}))$, which is $[\hat{v}_1/x, \hat{v}_2/y]\hat{e}_{body} = \hat{e}_B$, by how we defined \hat{v}_1 , \hat{v}_2 , and \hat{e}_{body} .

Subcase. E-METHCALL3 $_d$. This case is identical to the previous one, except $e_1 = v_1 = \mathbf{new}_d x \Rightarrow \overline{d} = \overline{e}$. The same reasoning applies though.

Theorem 3.9. (Extension Theorem)

Statement. If $\Gamma \vdash e : \tau$ and $\hat{e} = \text{label}(e, \Gamma)$ then one of the following is true:

- e is a value, and $\Gamma \vdash \hat{e} : \hat{\tau}$ **with** $\hat{\varepsilon}$, where $\tau = \hat{\tau}$ and $\hat{\varepsilon} = \emptyset$.
- e is an expression, and $e \longrightarrow e' \mid \varepsilon$, and $\Gamma \vdash \hat{e} : \hat{\tau}$ **with** $\hat{\varepsilon}$, where $\hat{\tau} = \tau$ and $\varepsilon \subseteq \hat{\varepsilon}$.

Intuition. If Γ can type e without an effect, there is a way to label e with \hat{e} which contains the possible runtime effects of e (so $\hat{\varepsilon}$ is an upper-bound). **(Also, $\text{effects}(\Gamma)$ is an upper bound on $\hat{\varepsilon}$ but we omit this from the proof (for now) to keep it simple.)**

Proof. Proceed by induction on $\Gamma \vdash e : \tau$ and then on the reduction $e \longrightarrow e' \mid \varepsilon$.

Case. T-VAR.

$e = x$ is a value, and $\text{label}(x) = x$. By assumption that the program is closed under Γ , we can apply ε -VAR to conclude $\Gamma \vdash x : \tau$ **with** \emptyset .

Case. T-RESOURCE.

$e = r$ is a value, and $\text{label}(r) = r$. By assumption that the program is closed under Γ , we can apply ε -RESOURCE to conclude $\Gamma \vdash r : \{r\}$ **with** \emptyset .

Case. T-NEW $_{\sigma}$.

We also know $e = \text{new}_{\sigma} x \Rightarrow \overline{\sigma = e}$ and $\Gamma \vdash \sigma_i = e_i$ OK. By applying the definition of label , define $\hat{e} = \text{label}(\text{new}_{\sigma} x \Rightarrow \overline{\sigma = e}) = \text{new}_{\sigma} x \Rightarrow \overline{\sigma = \text{label}(e)}$. To type \hat{e} we want to use ε -NEWOBJ; to do that we need to know $\Gamma, x : \{\bar{\sigma}\} \vdash \overline{\sigma = \text{label}(e)}$ OK.

Fix some i . By assumption, $\Gamma \vdash \sigma_i = e_i$ OK. By inversion on ε -VALIDIMPL $_{\sigma}$, we know $\Gamma, y : \tau_2 \vdash e_i : \tau_3$ **with** ε_3 . Consider $\hat{e}_i = \text{label}(e_i)$. By inductive assumption, $\Gamma, y : \tau_2 \vdash \hat{e}_i : \tau_3$ **with** $\hat{\varepsilon}$, and by application of ε -VALIDIMPL $_{\sigma}$ we know $\Gamma \vdash \sigma_i = \text{label}(e_i)$ OK. **(We're applying inductive assumption to something of the form $\Gamma \vdash e : \tau$ with ε , not $\Gamma \vdash e : \tau$ though.)**

i was arbitrary; therefore $\Gamma \vdash \overline{\sigma = \text{label}(e)}$ OK. Therefore $\Gamma \vdash \hat{e} : \{\bar{\sigma}\}$ **with** \emptyset by ε -NEWOBJ $_{\sigma}$.

Case. T-NEW $_d$.

We also know $e = \text{new}_d x \Rightarrow \overline{d = e}$ and $\Gamma \vdash d_i = e_i$ OK. To simplify things, let $\varepsilon_i = \text{freevars}(\Gamma) \cap e_i$ (this definition comes from the definition of label-helper) and define \hat{e} in the following way:

$$\begin{aligned}
 & \text{label}(e) \\
 &= \text{label}(\text{new}_d x \Rightarrow \overline{d = e}) && \text{(definition of } e) \\
 &= \text{new}_{\sigma} x \Rightarrow \overline{\text{label-helper}(d_i = e_i)} && \text{(definition of label)} \\
 &= \text{new}_{\sigma} x \Rightarrow \overline{d_i \text{ with } \varepsilon_i = e_i} && \text{(definition of label-helper)} \\
 &= \text{new}_{\sigma} x \Rightarrow \overline{\sigma_i = e_i} && \text{(defining } \sigma = d_i \text{ with } \varepsilon_i)
 \end{aligned}$$

To type \hat{e} we want to use ε -NEWOBJ; to do that we need to know $\Gamma, x : \{\sigma\} \vdash \overline{\sigma_i = e_i}$ OK, so fix some i . By assumption $\Gamma \vdash d_i = e_i$ OK. By inversion on ε -VALIDIMPL $_d$ we know $\Gamma, y : \tau_2 \vdash e_i : \tau_3$. By inductive assumption on this, $\Gamma, y : \tau_2 \vdash \hat{e}_i : \tau_3$ **with** $\hat{\varepsilon}$.

Fix some i . By assumption $\Gamma \vdash d_i = e_i$ OK. By inversion on ε -VALIDIMPL $_d$, we know $\Gamma, y : \tau_2 \vdash e_i : \tau_3$. By inductive assumption, $\Gamma, y : \tau_2 \vdash \hat{e}_i : \tau_3$ **with** $\hat{\varepsilon}$, and by application of ε -VALIDIMPL $_{\sigma}$ we know $\Gamma \vdash \sigma_i = \text{label}(e_i)$ OK.

i was arbitrary; therefore $\Gamma \vdash \overline{\sigma = \text{label}(e)}$ OK. Therefore $\Gamma \vdash \hat{e} : \{\bar{\sigma}\}$ **with** \emptyset by ε -NEWOBJ.

Case. T-OPERCALL.

Then the following are known:

- $e = e_1.\pi$
- $\Gamma \vdash e_1 : \{\bar{r}\}$

– $\Gamma \vdash e_1.\pi : \mathbf{Unit}$

There are two reduction rules which could be applied to $e_1.\pi$.

Subcase. E-OPERCALL1. Then we know $e_1.\pi \longrightarrow e'_1.\pi \mid \varepsilon$, and $e_1 \rightarrow e'_1 \mid \varepsilon$. Because $\Gamma \vdash e_1 : \{\bar{r}\}$ by assumption of the typing rule, we may apply the inductive assumption. Then $\Gamma \vdash \hat{e}_1 : \{\bar{r}\}$ with $\hat{\varepsilon}_1$, where $\varepsilon \subseteq \hat{\varepsilon}_1$ and $\hat{e}_1 = \mathbf{label}(\Gamma, e_1)$.

By definition $\hat{e} = \mathbf{label}(\Gamma, e) = \mathbf{label}(\Gamma, e_1.\pi) = (\mathbf{label}(\Gamma, e_1)).\pi = \hat{e}_1.\pi$. We just established $\Gamma \vdash \hat{e}_1 : \{\bar{r}\}$ with $\hat{\varepsilon}_1$, so fulfill the requirements of ε -OPERCALL and can type $\hat{e} = \hat{e}_1.\pi$ with the judgement $\Gamma \vdash \hat{e}_1.\pi : \mathbf{Unit}$ with $\{r.\pi\} \cup \hat{\varepsilon}_1$.

$\varepsilon \subseteq \hat{\varepsilon}_1$ is an inductive assumption; so $\varepsilon \subseteq \hat{\varepsilon}_1 \cup \{r.\pi\} = \hat{\varepsilon}$. Also, $\hat{\tau} = \mathbf{Unit} = \tau$.

Subcase. E-OPERCALL2. Then we know $e = r.\pi$ and $r.\pi \longrightarrow \mathbf{Unit} \mid \{r.\pi\}$. By definition $\hat{e} = \mathbf{label}(\Gamma, e) = (\mathbf{label}(\Gamma, r)).\pi = r.\pi = e$, so $\hat{e} = e$. Then $\hat{\tau} = \tau$ automatically. We need only show $\varepsilon = r.\pi \in \hat{\varepsilon}$.

By ε -RESOURCE, $\Gamma \vdash r : \{r\}$ with \emptyset and by ε -OPERCALL, $\Gamma \vdash r.\pi : \mathbf{Unit}$ with $\{r.\pi\}$. Since $\hat{e} = r.\pi$, then $\hat{\varepsilon} = r.\pi = \varepsilon$.

Case. T-METHCALL $_{\sigma}$.

Then the following are known:

- $e = e_1.m_i(e_2)$
- $\Gamma \vdash e_1 : \{\bar{\sigma}\}$
- $\Gamma \vdash e_2 : \tau_2$
- $\Gamma \vdash e_1.m_i(e_2) : \tau_3$
- **def** $m_i(y : \tau_2) : \tau_3$ with $\varepsilon_3 \in \{\bar{\sigma}\}$

There are three reduction rules which could be applied to $e_1.m_i(e_2)$.

Subcase. E-OPERCALL1. Then we know $e_1 \longrightarrow e'_1 \mid \varepsilon$ and $e_1.m_i(e_2) \longrightarrow e'_1.m_i(e_2) \mid \varepsilon$. Because $\Gamma \vdash e_1 : \{\bar{\sigma}\}$ by assumption of the typing rule, we may apply the inductive assumption. Then $\Gamma \vdash \hat{e}_1 : \{\bar{\sigma}\}$ with $\hat{\varepsilon}_1$, where $\varepsilon \subseteq \hat{\varepsilon}_1$ and $\hat{e}_1 = \mathbf{label}(\Gamma, e_1)$.

By definition, $\hat{e} = \mathbf{label}(e_1.m_i(e_2)) = (\mathbf{label}(e_1)).m_i(\mathbf{label}(e_2)) = \hat{e}_1.m_i(\hat{e}_2)$. We just established $\Gamma \vdash \hat{e}_1 : \{\bar{\sigma}\}$ with $\hat{\varepsilon}_1$.

If e_2 is a value then $\hat{e}_2 = \mathbf{label}(e_2) = e_2$.

Can we type \hat{e}_2 though? Why?

Case. T-METHCALL $_d$.

We also know the following.

TODO

□

Theorem 3.10. (Refinement Theorem)

Statement. If $\Gamma \vdash e : \tau$ with ε and $\mathbf{label}(e) = \hat{e}$, then $\Gamma \vdash \hat{e} : \hat{\tau}$ with $\hat{\varepsilon}$, where $\hat{\varepsilon} \subseteq \varepsilon$ and $\tau = \hat{\tau}$.

Intuition. Labels can only make the static effects more precise; never less precise. **Needs to be edited/proofread so it makes sense with the new Extension theorem.**

Proof. By induction on the judgement $\Gamma \vdash e : \tau$ with ε .

Case. ε -RESOURCE, ε -VAR.

If e is a resource or a variable then $e = \hat{e}$ so the statement is automatically fulfilled.

Case. ε -OPERCALL.

Then $e = e_1.\pi$ and we know:

- $\Gamma \vdash e : \mathbf{Unit} \text{ with } \{r.\pi\} \cup \varepsilon_1$
- $\Gamma \vdash e_1 : \{\bar{r}\} \text{ with } \varepsilon_1$

Applying definitions, $\hat{e} = \mathbf{label}(e_1.\pi) = (\mathbf{label}(e_1)).\pi = \hat{e}_1.\pi$. By inductive assumption, $\Gamma \vdash \hat{e}_1 : \{\bar{r}\} \text{ with } \hat{\varepsilon}_1$, where $\hat{\varepsilon}_1 \subseteq \varepsilon_1$. Then $\Gamma \vdash \hat{e} : \mathbf{Unit} \text{ with } \{r.\pi\} \cup \hat{\varepsilon}_1$ by ε -OPERCALL. Importantly, $\{r.\pi\} \cup \hat{\varepsilon}_1 \subseteq \{r.\pi\} \cup \varepsilon_1$ as claimed.

Case. ε -METHCALL.

Then $e = e_1.m_i(e_2)$ and we know:

- $\Gamma \vdash e : \tau_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$
- $\Gamma \vdash e_1 : \{\bar{\sigma}\} \text{ with } \varepsilon_1$
- $\Gamma \vdash e_2 : \tau_2 \text{ with } \varepsilon_2$
- $\sigma_i = \mathbf{def } m_i(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3$

Applying definitions, $\hat{e} = \mathbf{label}(e_1.m_i(e_2)) = (\mathbf{label}(e_1)).m_i(\mathbf{label}(e_2)) = \hat{e}_1.m_i(\hat{e}_2)$. By inductive assumption, $\Gamma \vdash \hat{e}_1 : \{\bar{\sigma}\} \text{ with } \hat{\varepsilon}_1$ and $\Gamma \vdash \hat{e}_2 : \tau_2 \text{ with } \hat{\varepsilon}_2$, where $\hat{\varepsilon}_1 \subseteq \varepsilon_1$ and $\hat{\varepsilon}_2 \subseteq \varepsilon_2$. Then $\Gamma \vdash \hat{e} : \tau_3 \text{ with } \hat{\varepsilon}_1 \cup \hat{\varepsilon}_2 \cup \varepsilon_3$ under ε -METHCALL. Importantly, $\hat{\varepsilon}_1 \cup \hat{\varepsilon}_2 \cup \varepsilon_3 \subseteq \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$ as claimed.

Case. C-METHCALL.

Then $e = e_1.m_i(e_2)$ and we know:

- $\Gamma \vdash e : \tau_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$
- $\Gamma \vdash e_1 : \{\bar{d} \text{ captures } \varepsilon_c\} \text{ with } \varepsilon_1$
- $\Gamma \vdash e_2 : \tau_2 \text{ with } \varepsilon_2$
- $d_i = \mathbf{def } m_i(y : \tau_2) : \tau_3$

The reasoning is the same as the above case, but use C-METHCALL instead of ε -METHCALL.

Case. C-INFERENCE.

We know:

- $\Gamma' \subseteq \Gamma$
- $\Gamma' \vdash e : \tau$
- $\Gamma \vdash e : \tau \text{ with effects}(\Gamma')$

There aren't any judgements of the form $e : \tau \text{ with } \varepsilon$ in the antecedent of this rule so we cannot use the induction hypothesis. We will instead do a case-by-case analysis of the form of e .

Subcase. $e = r$ or $e = x$. Then $e = \hat{e}$ so the statement holds immediately.

Subcase. $e = e_1.\pi$. Then $\hat{e} = (\hat{e}_1).\pi = \hat{e}_1.\pi$. As e_1 is a subexpression of e , and since Γ can type e_1 , we may conclude $\Gamma \vdash e_1 : \{r\}$. By an application of C-INFERENCE choosing the same Γ' , we know $\Gamma \vdash e_1 : \{r\} \text{ with effects}(\Gamma')$. By applying the inductive hypothesis to e_1 we know that $\Gamma \vdash \hat{e}_1 : \{r\} \text{ with } \hat{\varepsilon}_1$, where $\hat{\varepsilon}_1 \subseteq \mathbf{effects}(\Gamma')$. Therefore $\Gamma \vdash \hat{e}_1 : \tau_1$. By an application of T-OPERCALL we know that

This one's kind of interesting. There aren't any judgements of the form $e : \tau \text{ with } \varepsilon$ in the antecedent of this rule, so we can't use the induction hypothesis. We also don't know anything about e .

Case. ε -NEWOBJ.

Then $e = \mathbf{new}_\sigma x \Rightarrow \bar{\sigma} \equiv \bar{e}$ and we know:

- $\Gamma \vdash e : \{\bar{\sigma}\} \text{ with } \emptyset$
- $\Gamma, x : \{\bar{\sigma}\} \vdash \bar{\sigma} \equiv \bar{e} \text{ OK}$

For each i , $\sigma_i = e_i$ OK only matches ε -VALIDIMPL $_\sigma$. By inversion on that rule, $\Gamma, y : \tau_2 \vdash e : \tau_3 \text{ with } \varepsilon_3$ and $\sigma_i = \mathbf{def } m_i(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3$. Applying definitions, $\hat{e} = \mathbf{label}(\mathbf{new}_\sigma x \Rightarrow \bar{\sigma} \equiv \bar{e}) = \mathbf{new}_\sigma x \Rightarrow \mathbf{label_helper}(\bar{\sigma} \equiv \bar{e})$. Then for each i , $\mathbf{label_helper}(\sigma_i = e_i) = \sigma_i = \mathbf{label}(e_i)$. Let $\hat{e}_i = \mathbf{label}(e_i)$. Applying the inductive assumption we get $\Gamma \vdash \hat{e}_i : \tau_3 \text{ with } \hat{\varepsilon}_3$. Then $\Gamma \vdash \sigma_i = \mathbf{label}(e_i)$ OK by ε -VALIDIMPL $_\sigma$. This was for any i , so $\Gamma \vdash \sigma_i = \mathbf{label}(e_i)$ OK. Finally we can apply ε -NEWOBJ to the labeled object $\sigma_i = \mathbf{label}(e_i)$, which gives the judgement $\Gamma \vdash \hat{e} : \{\bar{\sigma}\} \text{ with } \emptyset$.

Case. C-NEWOBJ.

Then $e = \text{new}_d x \Rightarrow \overline{d = e}$ and we know:

- $\Gamma \vdash e_1.m_i(e_2) : \tau_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$
- $\Gamma' \subseteq \Gamma$
- $\varepsilon_c = \text{effects}(\Gamma') \text{ with } \emptyset$
- $\Gamma', x : \{\overline{d} \text{ captures } \varepsilon_c\} \vdash \overline{d = e} \text{ OK}$

(Similar to above). For each i , $d_i = e_i \text{ OK}$ only matches $\varepsilon\text{-VALIDIMPL}_d$. By inversion on that rule, $\Gamma, y : \tau_2 \vdash e : \tau_3$ and $d_i = \text{def } m(y : \tau_2) : \tau_3 \text{ with } \varepsilon_3$. Applying definitions, $\hat{e} = \text{label}(\text{new}_\sigma x \Rightarrow \overline{\sigma = e}) = \text{new}_d x \Rightarrow \text{label-helper}(\overline{d = e})$. Then for each i , $\text{label-helper}(\text{def } m(y : \tau_2) : \tau_3 = e) = \text{def } m(y : \tau_2) : \tau_3 \text{ with effects}(\Gamma \cap \text{freevars}(e_i)) = \text{label}(e_i)$. Let $\hat{e}_i = \text{label}(e_i)$. By inductive assumption, $\Gamma \vdash \hat{e}_i : \tau_3 \text{ with } \hat{\varepsilon}_3$. This was for any i , so if σ_i is the labeled version of d_i then $\Gamma \vdash \overline{\sigma_i = \text{label}(e_i)} \text{ OK}$. Finally we can apply $\varepsilon\text{-NEWOBJ}$ to the labeled object $\overline{d_i = \text{label}(e_i)}$, which gives the judgement $\Gamma \vdash \hat{e} : \{\overline{d}\} \text{ with } \emptyset$. \square

Theorem 3.11. (Soundness Theorem)

Statement. If $\Gamma \vdash e_A : \tau_A \text{ with } \varepsilon_A$ and $e_A \longrightarrow e_B \mid \varepsilon$ then $\Gamma \vdash e_B : \tau_B \text{ with } \varepsilon_B$, where $\tau_B = \tau_A$ and $\varepsilon \subseteq \varepsilon_A$.

Proof. Let $\hat{e}_A = \text{label}(e_A)$. By Refinement Theorem, $\Gamma \vdash \hat{e}_A : \hat{\tau}_A \text{ with } \hat{\varepsilon}_A$, where $\tau_A = \hat{\tau}_A$ and $\hat{\varepsilon}_A \subseteq \varepsilon_A$. By Invariance Of Runtime Under **label**, we know $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$. Since \hat{e}_A is a fully-labeled program, by Soundness of ε rules, we know $\Gamma \vdash \hat{e}_B : \hat{\tau}_A \text{ with } \hat{\varepsilon}_B$ where $\varepsilon \subseteq \hat{\varepsilon}_A$. By Runtime Invariance Under **label**, $\text{label}(e_B) = \hat{e}_B$. By Refinement Theorem, $\Gamma \vdash \hat{e}_B : \tau_B \text{ with } \hat{\varepsilon}_B$.

Putting this all together we know $\tau_A = \hat{\tau}_A = \tau_B$ and $\varepsilon \subseteq \hat{\varepsilon}_A \subseteq \varepsilon_A$. \square