Notation: $\hat{\Gamma} \vdash \delta_1, ..., \delta_n$ means $\hat{\Gamma} \vdash \delta_1$ and $\hat{\Gamma} \vdash \delta_2$ and ... and $\hat{\Gamma} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Narrowing 1 (Subtypes)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$. The tricky cases are S-TypePoly and S-TypeVar; the others follow by routine application of the inductive hypothesis to subderivations.

Case: S-Reflexive. Then $\hat{\tau}_1 = \hat{\tau}_2$, and $\hat{\tau}_1 <: \hat{\tau}_2$ holds in any context, including $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}$.

Case: S-Transitive. Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_C$. By inversion, there is some $\hat{\tau}_B$ such that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption, we get the judgements $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_C$, which is the same as $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

Case: S-ResourceSet. Follows immediately, since the premises of this rule have nothing to do with the context. That is, if $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$, then by inversion, $r \in \overline{r_1} \implies r \in \overline{r_2}$. Then by S-ResourceSet, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$.

Case: S-Arrow. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \to_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon} \hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$ and $\varepsilon' \subseteq \varepsilon$. To these first two judgements, apply the inductive assumption, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$. Then by S-Arrow, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A \to_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon} \hat{\tau}'_B$.

Case: S-TypePoly. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$. By inversion, we have the following two judgements:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Using (1) and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis can be used to obtain (3).

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

Let $\Delta' = \Delta, Y <: \hat{\tau}'_A$. With this, and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, we shall apply the inductive hypothesis to obtain (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Expanding the definition of Δ', we get (5),

5. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

From (3) and (5), we can use S-TypePoly to obtain (6), which is the theorem conclusion.

6. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$

Case: S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash X <: \hat{\tau}$. It is also true that (1) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}'$, by use of S-TypeVar. The assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ can be widened to (2) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$. Then by (1) and (2), we can apply S-Transitive to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}$.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$, so the judgement can be strengthened to $\hat{\Gamma}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. Then the judgement can be weakened to $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash Y <: \hat{\tau}_B$.

Lemma 2 (Narrowing 2 (Effects)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$*, then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Lemma 3 (Narrowing 3 (Types)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$. ε-Abs, ε-PolyTypeAbs, ε-PolyTypeApp, ε-PolyFxAbs, ε-PolyFxApp are the tricky cases; they require the use of the inductive hypothesis in a slightly more tricky way. The other cases follow by routine induction.

$\boxed{\textit{Case: } \varepsilon\text{-Var.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash x : \hat{\tau}_A \texttt{ with } \varnothing$, where $\hat{e} = x$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash x : \hat{\tau}_A \texttt{ with } \varnothing$. Then by weakening, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash x : \hat{\tau}_A \texttt{ with } \varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-Resource.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash r : \{\overline{r}\} \texttt{ with } \varnothing$, where $\hat{e} = r$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash r : \{r\} \texttt{ with } \varnothing$. Then by weakening, $\hat{\Gamma}, x <: \hat{\tau}', \hat{\Delta} \vdash r : \{r\} \texttt{ with } \varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-OperCall.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$, and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\} \texttt{ with } \varepsilon_1$. To this second judgement we apply the inductive hypothesis, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\} \texttt{ with } \varepsilon_1$. With this new judgement, apply ε-OperCall to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$.

$\boxed{\textit{Case: } \varepsilon\text{-Subsume.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B, \varepsilon \subseteq \varepsilon'$. By applying Narrowing Lemma 1 to the first judgement, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau} <: \hat{\tau}'$. By applying the Narrowing Lemma for effects[1], $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$. With these two judgements, ε-Subsume can be used to obtain the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$.

$\boxed{\textit{Case: } \varepsilon\text{-Abs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.\hat{e}_2 : \hat{\tau}_1 \to_{\varepsilon_2} \hat{\tau}_2 \texttt{ with } \varnothing$, where $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$. By letting $\hat{\Delta}' = \hat{\Delta}, x : \hat{\tau}_1$, this second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$.

   Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, apply the inductive hypothesis to obtain (2),

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$.

   Using the definition of $\hat{\Delta}'$, this can be simplified,

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$.

   Then with (3) we can use ε-Abs to get (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.e_2 : \hat{\tau}_1 \to_{\varepsilon_2} \hat{\tau}_2 \texttt{ with } \varnothing$

$\boxed{\textit{Case: } \varepsilon\text{-App.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3 \texttt{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$, where the following judgements are true from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \to_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varepsilon_1$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$

   By applying the inductive assumption to (1) and (2), we get (3) and (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \to_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varepsilon_1$

[1] This has yet to be proven

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Then by ε-App, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_2 `with` $\varnothing$. From inversion, we have $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2. By letting $\Delta' = \Delta, Y <: \hat{\tau}_1$, the second judgement can be rewritten,

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

By applying the inductive hypothesis to (1), we get judgement (2), which further simplifies to (3) by simplifying $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Then by ε-PolyTypeAbs, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_2 `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon.\hat{e}_1 : \forall\phi \subseteq \varepsilon.\hat{\tau}_1$ `caps` ε_1 `with` $\varnothing$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. By letting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon$, the second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1

Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis gives judgement (2), which further simplifies to (3) by expanding the definition of $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1

Then from (2), we can apply ε-PolyFxAbs, giving the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon.\hat{e}_1 : \forall\phi \subseteq \varepsilon.\hat{\tau}_1$ `caps` ε_1 `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2$ `with` $[\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$, where the following judgements are from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to get (3). With the same assumption and (2), we can apply Narrowing Lemma 1 (Subtypes) to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

From (3) and (4), ε-PolyTypeApp gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2$ `with` $[\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2$ `with` $[\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$, where the following are true by inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall\phi \subseteq \varepsilon.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to obtain (3). With the same assumption and (2), we can apply the Narrowing Lemma for Effect Judgements[2] to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall\phi \subseteq \varepsilon.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

With (3) and (4) we can apply ε-PolyFxApp to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2$ `with` $[\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-Import.}}$ (We prove for a single import). Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash$ `import`$(\varepsilon_s)\ x_1 = \hat{e}_1$ `in` e : `annot` (τ, ε_s) `with` $\varepsilon_s \cup \varepsilon_1$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. By inductive hypothesis, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. This, together with the other premises obtained by inversion, gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash$ `import`$(\varepsilon_s)\ x_1 = \hat{e}_1$ `in` e : `annot` (τ, ε_s) `with` $\varepsilon_s \cup \varepsilon_1$.

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[2] Doesn't actually exist yet