Notation: $\hat{\Gamma} \vdash \delta_1, ..., \delta_n$ means $\hat{\Gamma} \vdash \delta_1$ and $\hat{\Gamma} \vdash \delta_2$ and ... and $\hat{\Gamma} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Substitution (Values)). *If* $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ `with` ε *and* $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}'$ `with` $\varnothing$*, then* $\hat{\Gamma} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}$ `with` ε

Proof. By induction on the derivation of $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ `with` ε. We show for those extra cases in polymorphic `CC`.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1.[\hat{v}/y]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{e} = \lambda \phi \subseteq \varepsilon_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda \phi \subseteq \varepsilon_1.[\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{e} = \hat{e}_1\ \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeApp, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{e} = \hat{e}_1\ \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxApp, we get the conclusion.

Lemma 2 (Type Substitution Preserves Subsetting). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-FxSet.}}$ Trivial.

$\boxed{\textit{Case: } \varepsilon\text{-FxVar.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \phi \subseteq \varepsilon_2$, and either (1) $\phi \subseteq \varepsilon_2 \in \hat{\Gamma}$ or (2) $\phi \subseteq \varepsilon_2 \in \hat{\Delta}$. If (1) then $\hat{\Gamma} \vdash \phi \subseteq \varepsilon_2$, so by widening $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$. Otherwise (2), in which case $\phi \subseteq \varepsilon_2 \in [\hat{\tau}'/X]\hat{\Delta}$ by the definition of type-variable substitution on a context, so $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$.

Lemma 3 (Type Substitution Preserves Subtyping). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

$\boxed{\textit{Case: } \text{S-Reflexive.}}$ Then $\hat{\tau}_1 = \hat{\tau}_2$, so $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ by S-Reflexive. Then by widening, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

$\boxed{\textit{Case: } \text{S-Transitive.}}$ Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption to these judgements, we get $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$ and $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$.

$\boxed{\textit{Case: } \text{S-ResourceSet.}}$ Sets of resources are unchanged by type-variable substitution, so $[\hat{\tau}'/X]\{\overline{r_1}\} = \{\overline{r_1}\}$ and $[\hat{\tau}'/X]\{\overline{r_2}\} = \{\overline{r_2}\}$. Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

$\boxed{\textit{Case: } \text{S-Arrow.}}$ Then the subtyping judgement from the assumption is $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \to_{\varepsilon} \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B$. By inversion we have judgements (1-3),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
3. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

By applying the inductive hypothesis to (1) and (2), we get (4) and (5),

4. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$
5. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_B$

By inspection, type-variable bindings do not affect judgements of the form $\hat{\Gamma} \vdash \varepsilon \subseteq \varepsilon$. Furthermore, the types in a context do not affect judgements of this form. Therefore, we can rewrite (3) as (6),

7. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

From (4-6), we may apply S-ARROW to get $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A \to_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \to_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$. By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}'/X](\hat{\tau}_A \to_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B)$, which is the same as $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TYPEPOLY. Then $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ and $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A.\hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}'_A$. Applying the inductive assumption to both these judgements, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta}, Z <: [\hat{\tau}'/X]\hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}'_A$. Then by S-TYPEPOLY, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash (\forall Y <: [\hat{\tau}'/X]\hat{\tau}_A.[\hat{\tau}'/X]\hat{\tau}_B) <: (\forall Z <: [\hat{\tau}'/X]\hat{\tau}'_A.[\hat{\tau}'/X]\hat{\tau}'_B)$, which is the same as $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TYPEVAR. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash X <: \hat{\tau}$. We want to show (1) $\hat{\Gamma}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]X <: [\hat{\tau}'/X]\hat{\tau}$. Firstly, $[\hat{\tau}'/X]X = \hat{\tau}'$. Secondly, because $\texttt{WF}(\hat{\Gamma}, X <: \hat{\tau})$ then $X \notin \texttt{free-vars}(\hat{\tau})$, so $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}$. Therefore, judgement (1) is the same as $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}' <: \hat{\tau}$, which is true by assumption.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation, so $\hat{\Gamma}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$ is true by widening the context in the judgement $\hat{\Gamma} \vdash Y <: \hat{\tau}_2$[1]. Then $\hat{\Gamma} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$ by inductive assumption. By widening, $\hat{\Gamma}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$.

Lemma 4 (Type Substitution Preserves Typing). *If* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\hat{\Gamma} \vdash \hat{\tau}'' <: \hat{\tau}'$, *then* $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$ $\texttt{with}$ ε

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.

Case: ε-VAR, ε-RESOURCE. Then $\hat{e} = [\hat{\tau}''/X]\hat{e}$, so the typing judgement in the consequent can be the one from the antecedent.

Case: ε-OPERCALL. Then $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with}$ $\varepsilon_1 \cup \{r.\pi \mid r \in \bar{r}\}$. By inversion we have (1). Noting that $[\hat{\tau}''/X]\{\bar{r}\} = \{\bar{r}\}$, we can apply the inductive hypothesis to get (2),

1. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\bar{r}\}$ $\texttt{with}$ ε_1
2. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : \{\bar{r}\}$ $\texttt{with}$ ε_1

Then from (2), we can apply ε-OPERCALL to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1.\pi) : \texttt{Unit with}$ $\varepsilon_1 \cup \{r.\pi \mid r \in \bar{r}\}$. Since $[\hat{\tau}''/X]\texttt{Unit} = \texttt{Unit}$, we're done.

Case: ε-SUBSUME. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε. By inversion, (1) and (2) are true.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_2 <: \hat{\tau}$

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[1] Note there is no explicit widening rule; be careful with this one.

2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
3. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_2 \texttt{ with } \varepsilon_2$

By a previous lemma, type substitution preserves subtyping. Applying this to (1) yields (4). On the other hand, only effect-variable bindings in a context will affect subsetting judgements. Based on this, we can delete the binding $X <: \hat{\tau}$ and perform the substitution $[\hat{\tau}''/X]\hat{\Delta}$, neither of which will change any effect-variable bindings, and in doing so obtain judgement (5). Lastly, we can apply the inductive hypothesis to (3), obtaining (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}_2 <: [\hat{\tau}''/X]\hat{\tau}$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
7. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_2 \texttt{ with } \varepsilon_2$

From (4-6) we can apply ε-Subsume to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau} \texttt{ with } \varepsilon_2$.

Case: ε-Abs. Then $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda y : \hat{\tau}_2.\hat{e}_3 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varnothing$. By inversion, we have (1). By setting $\hat{\Delta}' = \hat{\Delta}, y : \hat{\tau}_2$, this can be rewritten as (2). From inductive hypothesis we get (3). Then by simplifying $\hat{\Delta}'$, this simplifies to (4).

1. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, y : \hat{\tau}_2 \vdash \hat{e}_3 : \hat{\tau}_3 \texttt{ with } \varepsilon_3$
2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_3 : \hat{\tau}_3 \texttt{ with } \varepsilon_3$
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3 \texttt{ with } \varepsilon_3$
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, y : [\hat{\tau}''/X]\hat{\tau}_2 \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3 \texttt{ with } \varepsilon_3$

From (4) we can apply ε-Abs to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda y : [\hat{\tau}''/X]\hat{\tau}_2.[\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_2 \rightarrow_{\varepsilon_3} [\hat{\tau}''/X]\hat{\tau}_3 \texttt{ with } \varnothing$. This can be rewritten as $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda y : \hat{\tau}_2.\hat{e}_3) : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \texttt{ with } \varnothing$.

Case: ε-App. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3 \texttt{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$. By inversion, we have:

1. $\hat{\Gamma}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varepsilon_1$
2. $\hat{\Gamma}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$

Applying inductive hypothesis to (1) and (2) gives (3) and (4),

3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \texttt{ with } \varepsilon_1$
4. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_2 : [\hat{\tau}''/X]\hat{\tau}_2 \texttt{ with } \varepsilon_2$

Then from (3) and (4) we can apply ε-App to get $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \hat{e}_2) : [\hat{\tau}''/X]\hat{\tau}_3 \texttt{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

Case: ε-PolyTypeAbs, Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A \texttt{ cap } \varepsilon_A \texttt{ with } \varnothing$. By inversion, we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}_B$, we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite this as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_B \vdash \hat{e}_A : \hat{\tau}_A \texttt{ with } \varepsilon_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_A : \hat{\tau}_A \texttt{ with } \varepsilon_A$
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A \texttt{ with } \varepsilon_A$
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, Y <: [\hat{\tau}''/X]\hat{\tau}_B \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A \texttt{ with } \varepsilon_A$

From (4) we can apply ε-PolyTypeAbs, giving (5), which can be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{e}_A : \forall Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{\tau}_A \texttt{ cap } \varepsilon_A \texttt{ with } \varnothing$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\Delta \vdash [\hat{\tau}''/X](\lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A \texttt{ cap } \varepsilon_A) \texttt{ with } \varnothing$

Case: ε-PolyFxAbs. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon_A.\hat{e}_B : \forall \phi \subseteq \varepsilon_A.\hat{\tau}_B \texttt{ cap } \varepsilon_B \texttt{ with } \varnothing$. By inversion we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon_A$, this can be rewritten as (2). The inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite that as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_B : [\hat{\tau}''/X]\hat{\tau}_B$ `with` ε_B
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, \phi \subseteq \varepsilon_A \vdash [\hat{\tau}''/X]\hat{e}_B : [\hat{\tau}''/X]\hat{\tau}_B$ `with` ε_B

From (4) we can apply ε-PolyFxAbs, giving (5), whichc an be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon_A.[\hat{\tau}''/X]\hat{e}_B : \forall\phi \subseteq \varepsilon_A.[\hat{\tau}''/X]\hat{\tau}_B$ `cap` ε_B `with` $\varnothing$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda\phi \subseteq \varepsilon_A.\hat{e}_B) : [\hat{\tau}''/X](\forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `cap` $\varepsilon_B)$ `with` $\varnothing$

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{\tau}'_A : [\hat{\tau}'_A/Y]\hat{\tau}_B$ `with` $[\hat{\tau}'_A/Y]\varepsilon_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ `caps` ε_B `with` ε_C
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

By inductive hypothesis on (1) we get (3). By a previous lemma, type substitution preserves subtyping, so from (2) we obtain (4).

3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\forall Y <: \hat{\tau}_A.\hat{\tau}_B$ `caps` $\varepsilon_B)$ `with` ε_C
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}'_A <: [\hat{\tau}''/X]\hat{\tau}_A$

From (3-4), applying ε-PolyTypeApp gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \hat{\tau}'_A) : [\hat{\tau}''/X]([\hat{\tau}'_A/Y]\hat{\tau}_B)$ `with` $[\hat{\tau}'_A/Y]\varepsilon_B \cup \varepsilon_C$

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \varepsilon'_A : [\varepsilon'_A/\phi]\hat{\tau}_B$ `with` $[\varepsilon'_A/\phi]\hat{\varepsilon}_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `caps` ε_B `with` ε_C
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

By inductive hypothesis on (1) we get (3). Applying the lemma that type substitution preserves subsetting, we obtain (4) from (2).

3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `caps` $\varepsilon_B)$ `with` ε_C
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

From (3-4), applying ε-PolyFxApp gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \varepsilon'_A) : [\hat{\tau}''/X]([\varepsilon'_A/\phi]\hat{\tau}_B)$ `with` $[\varepsilon'_A/\phi]\hat{\varepsilon}_B \cup \varepsilon_C$

$\boxed{\textit{Case: } \varepsilon\text{-Import}}$ TODO

Lemma 5 (Effect Substitution Preserves Subsetting). *If* $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ *then* $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\varepsilon_1 \subseteq [\varepsilon''/\phi]\varepsilon_2$

Proof. By induction on the derivation of $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

$\boxed{\varepsilon\text{-FxSet.}}$ By ε-FxSet, $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$. Because ε_1 and ε_2 are concrete sets of effects, then $[\varepsilon''/\phi]\varepsilon_1 = \varepsilon_1$ and $[\varepsilon''/\phi]\varepsilon_2 = \varepsilon_2$, so we are done.

$\boxed{\varepsilon\text{-FxVar.}}$ Then $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \Phi \subseteq \varepsilon''$. We know that $\Phi \subseteq \varepsilon''$ occurs in the context somewhere, so consider case-by-case which part.

Subcase: $\Phi = \phi$. Then $[\varepsilon''/\phi]\varepsilon_1 = \varepsilon''$. By well-formedness, $\phi \notin$ `freevars`(ε_2), so $[\varepsilon''/\phi]\varepsilon_2 = \varepsilon_2$. By inversion on the rule, $\varepsilon_2 = \varepsilon'$. We already know by assumption that $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$, so by widening, $\hat{\Gamma}, [\varepsilon''/X]\hat{\Delta} \vdash \varepsilon'' \subseteq \varepsilon'$.

Lemma 6 (Effect Substitution Preserves Subtyping). *If* $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ *then* $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$

Proof. By induction on derivations of $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

$\boxed{\text{S-Reflexive.}}$ Use S-Reflexive to get the desired judgement directly.

$\boxed{\text{S-Transitive.}}$ By inversion we have (1) and (2). Applying the inductive assumption to these yields (3) and (4), which can be used to apply S-Transitive, giving judgement (5).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_C$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_C <: \hat{\tau}_2$
3. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_C$
4. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_C <: [\varepsilon''/\phi]\hat{\tau}_2$
5. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$

$\boxed{\text{S-ResourceSet.}}$ Substitution on a resource set leaves it unchanged, so the judgement in the antecedent can be used for the judgement in the consequent.

$\boxed{\text{S-Arrow.}}$ Then we have (1). By inversion, we also have (2-4).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon_C} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon'_C} \hat{\tau}'_B$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
3. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
4. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_C \subseteq \varepsilon'_C$

Applying the inductve assumption to (2) and (3) yields (5) and (6). By a previous lemma, we know that effect substitution preserves subsetting. Applying this lemma to (4) yields (7).

5. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_A <: [\varepsilon''/\phi]\hat{\tau}_A$
6. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_B$
7. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\varepsilon_C \subseteq [\varepsilon''/\phi]\varepsilon'_C$

With (5-7) we can apply S-Arrow, giving (8), which is the same as (9).

8. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_A \rightarrow_{[\varepsilon''/\phi]\varepsilon'_C} [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_A \rightarrow_{[\varepsilon''/\phi]\varepsilon_C} [\varepsilon''/\phi]\hat{\tau}'_B$
9. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\hat{\tau}_A \rightarrow_{\varepsilon_C} \hat{\tau}_B) <: [\varepsilon''/\phi](\hat{\tau}'_A \rightarrow_{\varepsilon'_C} \hat{\tau}'_B)$

$\boxed{\text{S-TypePoly.}}$ Then we have (1). By inversion, we also have (2-3).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash (\forall X <: \hat{\tau}_1.\hat{\tau}_2) <: (\forall Y <: \hat{\tau}'_1.\hat{\tau}'_2)$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_1 <: \hat{\tau}_1$
3. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}, Y <: \hat{\tau}'_1 \vdash \hat{\tau}_2 <: \hat{\tau}'_2$

By applying the inductive hypothesis to (2), we obtain (4).

4. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_1 <: [\varepsilon''/\phi]\hat{\tau}_1$

Now, let $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}'_1$. Then we can rewrite (3) as (5), and apply the inductive assumption to get (6). By simplifying $\hat{\Delta}'$, we get (7).

5. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
6. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$
7. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta}, Y <: [\varepsilon''/\phi]\hat{\tau}'_1 \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$

From (2) and (7) we can apply S-TypePoly to get (8), which can be rewritten as the more readable (9).

8. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash (\forall X <: [\varepsilon''/\phi]\hat{\tau}_1.[\varepsilon''/\phi]\hat{\tau}_2) <: (\forall Y <: [\varepsilon''/\phi]\hat{\tau}_1'.[\varepsilon''/\phi]\hat{\tau}_2')$
9. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\forall X <: \hat{\tau}_1.\hat{\tau}_2) <: [\varepsilon''/\phi](\forall Y <: \hat{\tau}_1'.\hat{\tau}_2')$

$\boxed{\text{S-TypeVar.}}$ Then $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash X <: \hat{\tau}$. By inversion, there is a binding $X <: \hat{\tau}$ in the context, so consider case-by-case where it is.

Subcase: $X <: \hat{\tau} \in \hat{\Delta}$. Then $X <: [\varepsilon''/\phi]\hat{\tau} \in [\varepsilon''/\phi]\hat{\Delta}$, so $[\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$. By widening, $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$.

Subcase: $X <: \hat{\tau} \in \hat{\Gamma}$. TODO

Lemma 7 (Effect Substitution Preserves Types and Effects). *If* $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A *and* $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ *then* $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]e : [\varepsilon''/\Phi]\hat{\tau}$ `with` $[\varepsilon''/\Phi]\varepsilon$

Proof. By induction on the derivation of $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ `with` ε.

$\boxed{\varepsilon\text{-Var, }\varepsilon\text{-Resource.}}$ Then $\hat{e} = [\varepsilon''/\Phi]\hat{e}$, so the typing judgement in the consequent can be the one from the antecedent.

$\boxed{\varepsilon\text{-OperCall.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_1.\pi :$ `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}, \pi \in \Pi\}$. By inversion we have (1). Noting that $[\varepsilon''/\Phi]\{\overline{r}\} = \{\overline{r}\}$, we can apply the inductive hypothesis to get (2).

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\}$ `with` ε_1
2. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{e}_1 : \{\overline{r}\}$ `with` ε_1

Then from (2), we can apply ε-OperCall to get $\hat{\Gamma}, [\varepsilon''/X]\hat{\Delta} \vdash [\varepsilon''/X](\hat{e}_1.\pi) :$ `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}, \pi \in \Pi\}$.

$\boxed{\varepsilon\text{-Subsume.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ `with` ε. By inversion, (1-3) are true.

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
2. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_2 <: \hat{\tau}$
3. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_2$ `with` ε_2

By previous lemmas, substitution of effect-variables preserves subsetting and subtyping judgements. Applying these to (1-2) gives (4-5). By applying the inductive hypothesis to (3), we get (6).

1. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\varepsilon_2 \subseteq \varepsilon)$
2. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{\tau}_2 <: \hat{\tau})$
3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]\hat{e} : [\varepsilon''/\Phi]\hat{\tau}_2$ `with` $[\varepsilon''/\Phi]\varepsilon_2$

From (4-6) we can apply ε-Subsume, giving $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]\hat{e} : [\varepsilon''/\Phi]\hat{\tau}$ `with` $[\varepsilon''/\Phi]\varepsilon$.

$\boxed{\varepsilon\text{-Abs.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta} \vdash \lambda y : \hat{\tau}_2.\hat{e}_3 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` $\varnothing$. We inversion we have (1). By setting $\hat{\Delta}' = \hat{\Delta}, y : \hat{\tau}_2$, this can be rewritten as (2). From inductive hypothesis, we get (3). Simplifying $\hat{\Delta}'$ gives (4).

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta}, y : \hat{\tau}_2 \vdash \hat{e}_3 : \hat{\tau}_3$ `with` ε_3
5. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta}' \vdash \hat{e}_3 : \hat{\tau}_3$ `with` ε_3
6. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta}' \vdash [\varepsilon''/\Phi]\hat{e}_3 : [\varepsilon''/\Phi]\hat{\tau}_3$ `with` $[\varepsilon''/\Phi]\varepsilon_3$
7. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta}, y : [\varepsilon''/\Phi]\hat{\tau}_2 \vdash [\varepsilon''/\Phi]\hat{e}_3 : [\varepsilon''/\Phi]\hat{\tau}_3$ `with` $[\varepsilon''/\Phi]\varepsilon_3$

Then from (4) we can apply ε-Abs to get $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash \lambda y : [\varepsilon''/\Phi]\hat{\tau}_2.[\varepsilon''/\Phi]\hat{e}_3 : [\varepsilon''/\Phi]\hat{\tau}_2 \rightarrow_{[\varepsilon''/\Phi]\varepsilon_3} [\varepsilon''/\Phi]\hat{\tau}_3$ `with` $\varnothing$. This can be rewritten as $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\lambda y : \hat{\tau}_2.\hat{e}_3) : [\varepsilon''/\Phi](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3)$ `with` $[\varepsilon''/\Phi]\varnothing$.

$\boxed{\varepsilon\text{-App.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$. By inversion, we have:

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varepsilon_1$
2. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$

By applying the inductive hypothesis to (1) and (2) gives (3) and (4),

3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]\hat{e}_1 : [\varepsilon''/\Phi](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \texttt{ with } [\varepsilon''/\Phi]\varepsilon_1$
4. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]\hat{e}_2 : [\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ with } [\varepsilon''/\Phi]\varepsilon_2$

Then from (3) and (4) we can apply ε-App to get $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{e}_1\ \hat{e}_2) : [\varepsilon''/\Phi]\hat{\tau}_3 \texttt{ with } [\varepsilon''/\Phi](\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3)$.

$\boxed{\varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \lambda X <: \hat{\tau}_1.\hat{e}_2 : \forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2 \texttt{ with } \varnothing$. By inversion, we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, X <: \hat{\tau}_1$, we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite this as (4).

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta}, X <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$
2. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$
3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta}' \vdash [\varepsilon''/\Phi]\hat{e}_2 : [\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ with } [\varepsilon''/\Phi]\varepsilon_2$
4. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta}, X <: [\varepsilon''/\Phi]\hat{\tau}_1 \vdash [\varepsilon''/\Phi]\hat{e}_2 : [\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ with } [\varepsilon''/\Phi]\varepsilon_2$

Applying ε-PolyTypeAbs to (4) gives (5), which can be rewritten as (6).

5. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash \lambda X <: [\varepsilon''/\Phi]\hat{\tau}_1.[\varepsilon''/\Phi]\hat{e}_2 : \forall X <: [\varepsilon''/\Phi]\hat{\tau}_1.[\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ caps } [\varepsilon''\Phi]\varepsilon_2 \texttt{ with } \varnothing$
6. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\lambda X <: \hat{\tau}_1.\hat{e}_2) : [\varepsilon''/\Phi](\forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2) \texttt{ with } [\varepsilon''/\Phi]\varnothing$

$\boxed{\varepsilon\text{-PolyFxAbs.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \lambda\Phi' <: \varepsilon_1.\hat{e}_2 : \forall\Phi' \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2 \texttt{ with } \varnothing$. By inversion, we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, \Phi' \subseteq \varepsilon_1$, we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite this as (4).

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta}, \Phi' \subseteq \varepsilon_1 \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$
2. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$
3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta}' \vdash [\varepsilon''/\Phi]\hat{e}_2 : [\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ with } [\varepsilon''/\Phi]\varepsilon_2$
4. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta}, \Phi' \subseteq [\varepsilon''/\Phi]\varepsilon_1 \vdash [\varepsilon''/\Phi]\hat{e}_2 : [\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ with } [\varepsilon''/\Phi]\varepsilon_2$

Applying ε-PolyFxAbs to (4) gives (5), which can be rewritten as (6).

5. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash \lambda\Phi' \subseteq [\varepsilon''/\Phi]\varepsilon_1.\hat{e}_2 : \forall\Phi' \subseteq [\varepsilon''/\Phi]\varepsilon_1.[\varepsilon''/\Phi]\hat{\tau}_2 \texttt{ caps } [\varepsilon''/\Phi]\varepsilon_2 \texttt{ with } \varnothing$
6. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\lambda\Phi' \subseteq \varepsilon_1.\hat{e}_2) : [\varepsilon''/\Phi](\forall\Phi' \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2) \texttt{ with } [\varepsilon''/\Phi]\varnothing$

$\boxed{\varepsilon\text{-PolyTypeApp.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_1\ \hat{\tau}_1' : [\hat{\tau}_1'/X]\hat{\tau}_B \texttt{ with } [\hat{\tau}_1'/X]\varepsilon_2 \cup \varepsilon_3$, where we get (1) and (2) from inversion,

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2 \texttt{ with } \varepsilon_3$
2. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

By applying the inductive hypothesis to (1) and (2), we get (3) and (4),

3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]\hat{e}_1 : [\varepsilon''/\Phi](\forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2) \texttt{ with } [\varepsilon''/\Phi]\varepsilon_3$
4. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{\tau}_1' <: \hat{\tau}_1)$

From (3) and (4), applying ε-PolyTypeApp gives (5), which can be rewritten as (6). [2]

5. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{e}_1\ \hat{\tau}_1') : [\varepsilon''/\Phi, \hat{\tau}_1'/X]\hat{\tau}_2 \texttt{ with } [\varepsilon''/\Phi, \hat{\tau}_1'/X]\varepsilon_2 \cup [\varepsilon''/\Phi]\varepsilon_3$
6. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{e}_1\ \hat{\tau}_1') : [\varepsilon''/\Phi]([\hat{\tau}_1'/X]\hat{\tau}_2) \texttt{ with } [\varepsilon''/\Phi]([\hat{\tau}_1'/X]\varepsilon_2 \cup \varepsilon_3)$

$\boxed{\varepsilon\text{-PolyFxApp.}}$ Then $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_1\ \hat{\varepsilon}_1' : [\hat{\varepsilon}_1'/\Phi']\hat{\tau}_2 \texttt{ with } [\hat{\varepsilon}_1'/\Phi']\varepsilon_2 \cup \varepsilon_3$, where we get (1) and (2) by inversion,

[2] Isn't there actually a subsumption step hidden in here? When moving from $\hat{\tau}_1'$ to $\hat{\tau}_1$

1. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_1 : \forall \Phi' \subseteq \varepsilon_1.\hat{\tau}_2$ `caps` ε_2 `with` ε_3
2. $\hat{\Gamma}, \Phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1' \subseteq \varepsilon_1$

By applying the inductive hypothesis to (1) and (2), we get (3) and (4),

3. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi]\hat{e}_1 : [\varepsilon''/\Phi](\forall \Phi' \subseteq \varepsilon_1.\hat{\tau}_2$ `caps` $\varepsilon_2)$ `with` $[\varepsilon''/\Phi]\varepsilon_3$
4. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\varepsilon_1' \subseteq \varepsilon_1)$

From (3) and (4), applying ε-PolyFxApp gives (5), which can be rewritten as (6).[3]

5. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{e}_1\ \varepsilon_1') : [\varepsilon''/\Phi, \varepsilon_1'/\Phi']\hat{\tau}_2$ `with` $[\varepsilon''/\Phi, \varepsilon_1'/\Phi']\varepsilon_2 \cup [\varepsilon''/\Phi]\varepsilon_3$
6. $\hat{\Gamma}, [\varepsilon''/\Phi]\hat{\Delta} \vdash [\varepsilon''/\Phi](\hat{e}_1\ \varepsilon_1') : [\varepsilon''/\Phi]([\varepsilon_1'/\Phi']\hat{\tau}_2)$ `with` $[\varepsilon''/\Phi]([\varepsilon_1'/\Phi']\varepsilon_2 \cup \varepsilon_3)$

$\boxed{\varepsilon\text{-Import.}}$ TODO

Definition 1 (Domain). *For any $\hat{\Gamma}$, define* $\mathtt{dom}(\hat{\Gamma})$ *as follows:*

- $\mathtt{dom}(\varnothing) = \varnothing$
- $\mathtt{dom}(\hat{\Gamma}, x : \hat{\tau}) = \mathtt{dom}(\hat{\Gamma}) \cup \{x\}$
- $\mathtt{dom}(\hat{\Gamma}, X <: \hat{\tau}) = \mathtt{dom}(\hat{\Gamma}) \cup \{X\}$
- $\mathtt{dom}(\hat{\Gamma}, \Phi \subseteq \varepsilon) = \mathtt{dom}(\hat{\Gamma}) \cup \{\Phi\}$

Lemma 8 (Abstraction (Subsetting)). *If $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$ and $\Phi \notin \mathtt{dom}(\hat{\Gamma})$ then $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$.

$\boxed{\text{S-Reflex.}}$ Then $\varepsilon_1 = \varepsilon_2$. By S-Reflex, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_1$.

$\boxed{\text{S-Trans.}}$ By inversion and inductive assumption, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2, \varepsilon_2 \subseteq \varepsilon_3$. By S-Trans, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_3$.

$\boxed{\text{S-FxSet.}}$ Concrete sets of effects are invariant under substitution, so $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$.

$\boxed{\text{S-FxVar.}}$ Then $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon_2 \vdash [\varepsilon/X](\Phi_2 \subseteq \varepsilon_2)$. Because $\Phi \notin \mathtt{dom}(\hat{\Gamma})$, then $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon_2, \Phi \subseteq \varepsilon \vdash \Phi_2 \subseteq \varepsilon_2$ by S-FxVar.

Lemma 9 (Abstraction (Subtyping)). *If $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$ and $\Phi \notin \mathtt{dom}(\hat{\Gamma})$ then $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$.

$\boxed{\text{S-Reflex.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_1)$, and $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_1$ by S-Reflex.

$\boxed{\text{S-TypeVar.}}$ Then $\hat{\Gamma}, X <: \hat{\tau} \vdash [\varepsilon/\Phi](X <: \hat{\tau})$. Because Φ is an effect-variable, and not a type-variable, then $\hat{\Gamma}, X <: \hat{\tau}, \Phi \subseteq \varepsilon \vdash X <: \hat{\tau}$ byS-TypeVar.

$\boxed{\text{S-ResourceSet.}}$ By S-ResourceSet, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$.

$\boxed{\text{S-Trans.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_3)$. By inversion and induction, we have $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2, \hat{\tau}_2 <: \hat{\tau}_3$. Then by S-Trans, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_3$.

$\boxed{\text{S-Arrow.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi]((\hat{\tau}_1 \to_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}_1' \to_{\varepsilon_3'} \hat{\tau}_2'))$. By inversion, we know (1-3):

[3] As in the previous case, isn't there a hidden subsumption step when moving from ε_1' to ε_1?

1. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1' <: \hat{\tau}_1)$
2. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_2 <: \hat{\tau}_2')$
3. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon_3')$

By applying the inductive assumption to (1-2) we get (4-5). By applying Reverse Narrowing 1 to 3, we get 6.

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1' <: \hat{\tau}_1$
5. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_2 <: \hat{\tau}_2'$
6. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon_3'$

From (4-6), we can use S-Arrow to get the judgement $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash (\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}_1' \rightarrow_{\varepsilon_3'} \hat{\tau}_2')$.

$\boxed{\text{S-PolyType.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi]((\forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3'))$. By inversion, we know (1-3):

1. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1' <: \hat{\tau})$
2. $\hat{\Gamma}, Y <: \hat{\tau}_1' \vdash [\varepsilon/\Phi](\hat{\tau}_2 <: \hat{\tau}_2')$
3. $\hat{\Gamma}, Y <: \hat{\tau}_1' \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon_3')$

By applying the inductive assumption to (1-3), we get (4-6).

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1' <: \hat{\tau}$
5. $\hat{\Gamma}, Y <: \hat{\tau}_1', \Phi \subseteq \varepsilon \vdash \hat{\tau}_2 <: \hat{\tau}_2'$
6. $\hat{\Gamma}, Y <: \hat{\tau}_1', \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon_3'$

5 can be rewritten as 7, and 6 as 8.

7. $\hat{\Gamma}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}_1' \vdash \hat{\tau}_2 <: \hat{\tau}_2'$
8. $\hat{\Gamma}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}_1' \vdash \varepsilon_3 \subseteq \varepsilon_3'$

From (4,7,8) we can apply S-PolyType to get $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash (\forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3')$.

$\boxed{\text{S-PolyFx.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi]((\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall \Phi_1' \subseteq \varepsilon_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3'))$. By inversion we know (1-3):

1. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1' \subseteq \varepsilon_1)$
2. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_1')$
3. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon_3')$

By applying the Reverse Narrowing Lemma 1 to (1), we get (3). By applying the inductive assumption to (2-3), we get (5-6).

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1' \subseteq \varepsilon_1$
5. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon', \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_1'$
6. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon'.\Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon_3'$

5 can be rewritten as 7, and 6 as 8.

7. $\hat{\Gamma}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \hat{\tau}_1 <: \hat{\tau}_1'$
8. $\hat{\Gamma}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \varepsilon_3 \subseteq \varepsilon_3'$

With (4,7,8), we can apply S-PolyFx to get $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash (\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall \Phi_1' \subseteq \varepsilon_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3')$.

Lemma 10 (Abstraction (Typing)). *If* $\hat{\Gamma} \vdash [\varepsilon'/\Phi]\hat{e} : [\varepsilon'/\Phi]\hat{\tau} \texttt{ with } [\varepsilon'/\Phi]\varepsilon$ *and* $\Phi \notin \texttt{dom}\ (\hat{\Gamma})$ *then* $\hat{\Gamma}, \Phi \subseteq \varepsilon' \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon$.

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon'/\Phi]\hat{e} : [\varepsilon'/\Phi]\hat{\tau}$ `with` $[\varepsilon'/\Phi]\varepsilon$.

$\boxed{\varepsilon\text{-Var.}}$ Then $\hat{\Gamma}, x : [\varepsilon'/\Phi]\hat{\tau} \vdash [\varepsilon'/\Phi]x : [\varepsilon'/\Phi]\hat{\tau}$ `with` $[\varepsilon'/\Phi]\varnothing$. Then $\hat{\Gamma}, x : \hat{\tau} \vdash x : \hat{\tau}$ `with` $\varnothing$ by ε-Var.

$\boxed{\varepsilon\text{-Resource.}}$ Then $\hat{\Gamma} \vdash [\varepsilon'/\Phi]r : [\varepsilon'/\Phi]\{\overline{r}\}$ `with` $[\varepsilon'/\Phi]\varnothing$. By ε-Resource, $\hat{\Gamma}, \Phi \subseteq \varepsilon' \vdash r : \{\overline{r}\}$ `with` $\varnothing$.

$\boxed{\varepsilon\text{-OperCall.}}$ Then $\hat{\Gamma} \vdash [\varepsilon'/\Phi](\hat{e}_1.\pi) : [\varepsilon'/\Phi]$`Unit with` $[\varepsilon'/\Phi](\varepsilon_1 \cup \{r.\pi \mid r \in \{\overline{r}\}, \pi \in \Pi\})$. By inversion, $\hat{\Gamma} \vdash [\varepsilon'/\Phi]\hat{e}_1 : [\varepsilon'/\Phi]\{\overline{r}\}$ `with` $[\varepsilon'/\Phi]\varepsilon_1$. Applying the inductive assumption, we get $\hat{\Gamma}, \Phi \subseteq \varepsilon' \vdash \hat{e}_1 : \{\overline{r}\}$ `with` ε_1. Then by ε-OperCall, $\hat{\Gamma}, \Phi \subseteq \varepsilon' \vdash \hat{e}_1.\pi :$ `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \{\overline{r}\}, \pi \in \Pi\}$.

$\boxed{\varepsilon\text{-Subsume.}}$ Then $\hat{\Gamma} \vdash [\varepsilon'/\Phi]\hat{e} : [\varepsilon'/\Phi]\hat{\tau}$ `with` $[\varepsilon'/\Phi]\varepsilon$. By inversion, $\hat{\Gamma} \vdash [\varepsilon'/\Phi]\hat{e} : \hat{\tau}_B$ `with` ε_B, where $\hat{\Gamma} \vdash \hat{\tau}_B <: [\varepsilon'/\Phi]\hat{\tau}$ and $\hat{\Gamma} \vdash \varepsilon_B \subseteq [\varepsilon'/\Phi]\varepsilon$.

$\boxed{\varepsilon\text{-Abs.}}$
$\boxed{\varepsilon\text{-App.}}$
$\boxed{\varepsilon\text{-PolyTypeAbs.}}$
$\boxed{\varepsilon\text{-PolyFxAbs.}}$
$\boxed{\varepsilon\text{-PolyTypeApp.}}$
$\boxed{\varepsilon\text{-PolyFxApp.}}$
$\boxed{\varepsilon\text{-Import.}}$

Lemma 11. $[\varnothing/\Phi]\varepsilon \subseteq \varepsilon$.

Proof. If $\varepsilon \neq \Phi$ then $[\varnothing/\Phi]\varepsilon = \varepsilon$. Otherwise, $[\varnothing/\Phi]\varepsilon = \varnothing \subseteq \varepsilon$.

Lemma 12. *For any* $\hat{\tau}$, `effects`$([\varnothing/\Phi]\hat{\tau}) \subseteq$ `effects`$(\hat{\tau})$ *and* `ho-effects`$([\varnothing/\Phi]\hat{\tau}) \subseteq$ `ho-effects`$(\hat{\tau})$.

Proof. By simultaneous induction on the form of $\hat{\tau}$. First, consider `effects`$([\varnothing/\Phi]\hat{\tau}) \subseteq$ `effects`$(\hat{\tau})$.

$\boxed{\hat{\tau} = \{\overline{r}\}.}$ Then $[\varnothing/\Phi]\hat{\tau} = \hat{\tau}$, so the result is trivial.

$\boxed{\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2.}$ Then $[\varnothing/\Phi]\hat{\tau} = [\varnothing/\Phi]\hat{\tau}_1 \rightarrow_{[\varnothing/\Phi]\varepsilon_3} [\varnothing/\Phi]\hat{\tau}_2$. By definition, `effects`$(\hat{\tau}) =$ `ho-effects`$(\hat{\tau}_1) \cup \varepsilon_3 \cup$ `effects`$(\hat{\tau}_2)$. By inductive hypothesis, we know `ho-effects`$([\varnothing/\Phi]\hat{\tau}_1) \subseteq$ `ho-effects`$(\hat{\tau}_1)$ and `effects`$([\varnothing/\Phi]\hat{\tau}_2) \subseteq$ `effects`$(\hat{\tau}_2)$. We also have $[\varnothing/\Phi]\varepsilon_3 \subseteq \varepsilon_3$ by the previous lemma.

$\boxed{\hat{\tau} = \forall\Phi \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2.}$ Then `effects`$(\hat{\tau}) = \varepsilon_2 \cup [\varnothing/\Phi]\hat{\tau}_2$. By the previous lemma, $[\varnothing/\Phi]\varepsilon_2 \subseteq \varepsilon_2$, and it is trivial that `effects`$([\varnothing/\Phi]\hat{\tau}_2) \subseteq$ `effects`$([\varnothing/\Phi]\hat{\tau}_2)$.

$\boxed{\hat{\tau} = \forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2}$ TODO

Now consider `ho-effects`$([\varnothing/\Phi]\hat{\tau}) \subseteq$ `ho-effects`$(\hat{\tau})$.

$\boxed{\hat{\tau} = \{\overline{r}\}.}$ Same as above; trivial.

$\boxed{\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2.}$ Then $[\varnothing/\Phi]\hat{\tau} = [\varnothing/\Phi]\hat{\tau}_1 \rightarrow_{[\varnothing/\Phi]\varepsilon_3} [\varnothing/\Phi]\hat{\tau}_2$. By definition, `ho-effects`$(\hat{\tau}) =$ `effects`$(\hat{\tau}_1) \cup$ `ho-effects`$(\hat{\tau}_2)$. By inductive hypothesis, we know `effects`$([\varnothing/\Phi]\hat{\tau}_1) \subseteq$ `effects`$(\hat{\tau}_1)$ and `ho-effects`$([\varnothing/\Phi]\hat{\tau}_2) \subseteq$ `ho-effects`$(\hat{\tau}_2)$.

$\boxed{\hat{\tau} = \forall\Phi \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2.}$ Then `ho-effects`$(\hat{\tau}) = \varepsilon_1 \cup [\varnothing/\Phi]\hat{\tau}_2$. By the previous lemma, we know $[\varnothing/\Phi]\varepsilon_1 \subseteq \varepsilon_1$. It is trivial that `ho-effects`$([\varnothing/\Phi]\hat{\tau}_2) \subseteq$ `ho-effects`$([\varnothing/\Phi]\hat{\tau}_2)$.

$\boxed{\hat{\tau} = \forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2}$ TODO

Lemma 13 (Approximation 1). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε *and* `effects`$(\hat{\tau}) \subseteq \varepsilon_s$ *and* `ho-safe`$(\hat{\tau}, \varepsilon_s)$ *then* $\hat{\tau} <:$ `annot`(`erase`$(\hat{\tau}), \varepsilon_s$).

Lemma 14 (Approximation 2). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\texttt{ho-effects}(\hat{\tau}) \subseteq \varepsilon_s$ *and* $\texttt{safe}(\hat{\tau}, \varepsilon_s)$ *then* $\texttt{annot}(\texttt{erase}(\hat{\tau}), \varepsilon_s) <: \hat{\tau}$.

Proof. By simultaneous induction on derivations of $\texttt{safe}$ and $\texttt{ho-safe}$, and then on derivations of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.

$\boxed{\varepsilon\text{-PolyFxAbs.}}$ Then $\hat{e}$ has the form given in (1). The definition of $\texttt{effects}$ is (2), and the definition of $\texttt{ho-safe}(\hat{\tau}, \varepsilon_s)$ is (3).

1. $\hat{e} = \forall \Phi \subseteq \varepsilon_1.\hat{\tau}_2$ $\texttt{caps}$ ε_2
2. $\texttt{effects}(\hat{\tau}) = \varepsilon_2 \cup \texttt{effects}([\varnothing/\Phi]\hat{\tau}_2) \subseteq \varepsilon_s$
3. $\texttt{ho-safe}(\forall \Phi \subseteq \varepsilon_1.\hat{\tau}_2$ $\texttt{caps}$ $\varepsilon_2, \varepsilon_s) = \varepsilon_1 \subseteq \varepsilon_s \wedge \texttt{ho-safe}([\varnothing/\Phi]\hat{\tau}_2, \varepsilon_s)$

By inversion on the typing judgement, we know (4). By the substitution lemma for effect variables, we have (5). With this typing judgement and (2) and (3), we can apply the inductive assumption to $[\varnothing/\Phi]\hat{e}_2$ to obtain (6).

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon_1 \vdash \hat{e}_2 : \hat{\tau}_2$ $\texttt{with}$ ε_2
5. $\hat{\Gamma} \vdash [\varnothing/\Phi]\hat{e}_2 : [\varnothing/\Phi]\hat{\tau}_2$ $\texttt{with}$ $[\varnothing/\Phi]\varepsilon_2$
6. $\hat{\Gamma} \vdash [\varnothing/\Phi]\hat{\tau}_2 <: \texttt{annot}(\texttt{erase}([\varnothing/\Phi]\hat{\tau}_2), \varepsilon_s)$

Now clearly, $\texttt{erase}([\varnothing/\Phi]\hat{\tau}_2) = \texttt{erase}(\hat{\tau}_2)$, so (6) can be rewritten as (7).

7. $\hat{\Gamma} \vdash [\varnothing/\Phi]\hat{\tau}_2 <: \texttt{annot}(\texttt{erase}(\hat{\tau}_2), \varepsilon_s)$

Because annotating replaces every annotation on the type, then $\texttt{freevars}(\texttt{annot}(\texttt{erase}(\hat{\tau}_2), \varepsilon_s)) = \varnothing$, so substituting $\varnothing$ for Φ in that type will leave it unchanged. Therefore, we can rewrite (7) as (8), which further simplifies to (9).

8. $\hat{\Gamma} \vdash [\varnothing/\Phi]\hat{\tau}_2 <: [\varnothing/\Phi]\texttt{annot}(\texttt{erase}(\hat{\tau}_2), \varepsilon_s)$
9. $\hat{\Gamma} \vdash [\varnothing/\Phi](\hat{\tau}_2 <: \texttt{annot}(\texttt{erase}(\hat{\tau}_2), \varepsilon_s)$

By the convention of α-conversion, it is assumed that abstractions only introduce previously unseen variables. Therefore, the original binding for Φ is the first occurrence of Φ seen so far. Any subderivation can therefore be performed without a bniding for Φ, so we can assume without loss of generality that $\Phi \notin \texttt{dom}(\hat{\Gamma})$. By applying the abstraction lemma to (9), we obtain (10).

10. $\hat{\Gamma}, \Phi \subseteq \varnothing \vdash \hat{\tau}_2 <: \texttt{annot}(\texttt{erase}(\hat{\tau}_2), \varepsilon_s)$

Theorem 1 (Progress). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\hat{e}$ *is not a value, then* $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$*, for some* $\hat{e}', \varepsilon$.

Proof. By induction on the derivation of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}''\hat{\tau}' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$.

Theorem 2 (Preservation). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$, then $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B, where $\hat{\tau}_B <: \hat{\tau}_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$.*

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. The typing rule from the judgement can be rewritten as (1). From inversion, we also have (2) and (3).

1. $\hat{\Gamma} \vdash \hat{e}_1\ \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2$ `with` $\varepsilon_1 \cup \varepsilon_2$
2. $\hat{\Gamma} \vdash \hat{e}_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
3. $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}_1$

Now consider which reduction rule was used.

Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion on the reduction rule, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With (2), we can apply the inductive assumption and ε-Subsume to get (4). With (4) and (3), we can then apply ε-PolyTypeApp to get (5). Then by comparing (1) and (6), we see $\hat{\tau}_B = \hat{\tau}_A$ and $\hat{\varepsilon} = \hat{\varepsilon}_A = \hat{\varepsilon}_B$.

4. $\hat{\Gamma} \vdash \hat{e}_1' : \forall X <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
5. $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2$ `with` $\varepsilon_1 \cup \varepsilon_2$

Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_1.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$. Because of the form of $\hat{e}_1$ in this subcase, the only rule which could have been applied to obtain judgement (2) is ε-TypeAbs. By inversion on this rule we get (4). From (4) and (3), we can apply the lemma that type-and-effect judgements are preserved under type variable substitution to obtain (5). Finally, by comparing (1) and (5) we see $\hat{\tau}_A = [\hat{\tau}'/X]\hat{\tau}_2 = \hat{\tau}_B$, and $\varepsilon_B \cup \varepsilon = \varepsilon_1 \subseteq \varepsilon_1 \cup \varepsilon_2 = \varepsilon_A$.

4. $\hat{\Gamma}, X <: \hat{\tau}_1 \vdash \hat{e}' : \hat{\tau}_2$ `with` ε_1
5. $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{e}' : [\hat{\tau}'/X]\hat{\tau}_2$ `with` ε_1

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.

Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-PolyFxApp2. Then $(\lambda \phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.