Notation: $\hat{\Gamma} \vdash \delta_1, ..., \delta_n$ means $\hat{\Gamma} \vdash \delta_1$ and $\hat{\Gamma} \vdash \delta_2$ and ... and $\hat{\Gamma} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Narrowing 1 (Subtypes)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$. The tricky cases are S-TypePoly and S-TypeVar; the others follow by routine application of the inductive hypothesis to subderivations.

Case: S-Reflexive. Then $\hat{\tau}_1 = \hat{\tau}_2$, and $\hat{\tau}_1 <: \hat{\tau}_2$ holds in any context, including $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}$.

Case: S-Transitive. Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_C$. By inversion, there is some $\hat{\tau}_B$ such that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption, we get the judgements $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_C$, which is the same as $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

Case: S-ResourceSet. Follows immediately, since the premises of this rule have nothing to do with the context. That is, if $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$, then by inversion, $r \in \overline{r_1} \implies r \in \overline{r_2}$. Then by S-ResourceSet, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$.

Case: S-Arrow. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \to_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon} \hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$ and $\varepsilon' \subseteq \varepsilon$. To these first two judgements, apply the inductive assumption, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$. Then by S-Arrow, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A \to_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon} \hat{\tau}'_B$.

Case: S-TypePoly. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$. By inversion, we have the following two judgements:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Using (1) and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis can be used to obtain (3).

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

Let $\Delta' = \Delta, Y <: \hat{\tau}'_A$. With this, and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, we shall apply the inductive hypothesis to obtain (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Expanding the definition of Δ', we get (5),

5. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

From (3) and (5), we can use S-TypePoly to obtain (6), which is the theorem conclusion.

6. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$

Case: S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash X <: \hat{\tau}$. It is also true that (1) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}'$, by use of S-TypeVar. The assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ can be widened to (2) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$. Then by (1) and (2), we can apply S-Transitive to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}$.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$, so the judgement can be strengthened to $\hat{\Gamma}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. Then the judgement can be weakened to $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash Y <: \hat{\tau}_B$.

Lemma 2 (Narrowing 2 (Effects)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Lemma 3 (Narrowing 3 (Types)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$. ε-Abs, ε-PolyTypeAbs, ε-PolyTypeApp, ε-PolyFxAbs, ε-PolyFxApp are the tricky cases; they require the use of the inductive hypothesis in a slightly more tricky way. The other cases follow by routine induction.

$\boxed{\textit{Case: } \varepsilon\text{-Var.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash x : \hat{\tau}_A \texttt{ with } \varnothing$, where $\hat{e} = x$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash x : \hat{\tau}_A \texttt{ with } \varnothing$. Then by weakening, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash x : \hat{\tau}_A \texttt{ with } \varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-Resource.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash r : \{\overline{r}\} \texttt{ with } \varnothing$, where $\hat{e} = r$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash r : \{r\} \texttt{ with } \varnothing$. Then by weakening, $\hat{\Gamma}, x <: \hat{\tau}', \hat{\Delta} \vdash r : \{r\} \texttt{ with } \varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-OperCall.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$, and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\} \texttt{ with } \varepsilon_1$. To this second judgement we apply the inductive hypothesis, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\} \texttt{ with } \varepsilon_1$. With this new judgement, apply ε-OperCall to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$.

$\boxed{\textit{Case: } \varepsilon\text{-Subsume.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B, \varepsilon \subseteq \varepsilon'$. By applying Narrowing Lemma 1 to the first judgement, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau} <: \hat{\tau}'$. By applying the Narrowing Lemma for effects[1], $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$. With these two judgements, ε-Subsume can be used to obtain the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A \texttt{ with } \varepsilon_A$.

$\boxed{\textit{Case: } \varepsilon\text{-Abs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.\hat{e}_2 : \hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2 \texttt{ with } \varnothing$, where $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$. By letting $\hat{\Delta}' = \hat{\Delta}, x : \hat{\tau}_1$, this second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$.

   Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, apply the inductive hypothesis to obtain (2),

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$.

   Using the definition of $\hat{\Delta}'$, this can be simplified,

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$.

   Then with (3) we can use ε-Abs to get (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.e_2 : \hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2 \texttt{ with } \varnothing$

$\boxed{\textit{Case: } \varepsilon\text{-App.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3 \texttt{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$, where the following judgements are true from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varepsilon_1$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$

   By applying the inductive assumption to (1) and (2), we get (3) and (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \texttt{ with } \varepsilon_1$
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \texttt{ with } \varepsilon_2$

[1] This has yet to be proven

Then by ε-App, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3\ \texttt{with}\ \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2\ \texttt{caps}\ \varepsilon_2\ \texttt{with}\ \varnothing$. From inversion, we have $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2\ \texttt{with}\ \varepsilon_2$. By letting $\Delta' = \Delta, Y <: \hat{\tau}_1$, the second judgement can be rewritten,

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2\ \texttt{with}\ \varepsilon_2$

By applying the inductive hypothesis to (1), we get judgement (2), which further simplifies to (3) by simplifying $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2\ \texttt{with}\ \varepsilon_2$
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2\ \texttt{with}\ \varepsilon_2$

Then by ε-PolyTypeAbs, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2\ \texttt{caps}\ \varepsilon_2\ \texttt{with}\ \varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon.\hat{e}_1 : \forall\phi \subseteq \varepsilon.\hat{\tau}_1\ \texttt{caps}\ \varepsilon_1\ \texttt{with}\ \varnothing$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$. By letting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon$, the second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$

Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis gives judgement (2), which further simplifies to (3) by expanding the definition of $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$

Then from (2), we can apply ε-PolyFxAbs, giving the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon.\hat{e}_1 : \forall\phi \subseteq \varepsilon.\hat{\tau}_1\ \texttt{caps}\ \varepsilon_1\ \texttt{with}\ \varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2\ \texttt{with}\ [\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$, where the following judgements are from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2\ \texttt{caps}\ \varepsilon_1\ \texttt{with}\ \varepsilon_2$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to get (3). With the same assumption and (2), we can apply Narrowing Lemma 1 (Subtypes) to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2\ \texttt{caps}\ \varepsilon_1\ \texttt{with}\ \varepsilon_2$
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

From (3) and (4), ε-PolyTypeApp gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2\ \texttt{with}\ [\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2\ \texttt{with}\ [\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$, where the following are true by inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall\phi \subseteq \varepsilon.\hat{\tau}_2\ \texttt{caps}\ \varepsilon_1\ \texttt{with}\ \varepsilon_2$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to obtain (3). With the same assumption and (2), we can apply the Narrowing Lemma for Effect Judgements[2] to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall\phi \subseteq \varepsilon.\hat{\tau}_2\ \texttt{caps}\ \varepsilon_1\ \texttt{with}\ \varepsilon_2$
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

[2] Doesn't actually exist yet

With (3) and (4) we can apply ε-PolyFxApp to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2\ \texttt{with}\ [\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-Import.}}$ (We prove for a single import). Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \texttt{import}(\varepsilon_s)\ x_1 = \hat{e}_1\ \texttt{in}\ e : \texttt{annot}\ (\tau, \varepsilon_s)\ \texttt{with}\ \varepsilon_s \cup \varepsilon_1$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$. By inductive hypothesis, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$. This, together with the other premises obtained by inversion, gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \texttt{import}(\varepsilon_s)\ x_1 = \hat{e}_1\ \texttt{in}\ e : \texttt{annot}\ (\tau, \varepsilon_s)\ \texttt{with}\ \varepsilon_s \cup \varepsilon_1$.

Lemma 4 (Substitution (Values)). *If* $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$ *and* $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}'\ \texttt{with}\ \varnothing$*, then* $\hat{\Gamma} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$

Proof. By induction on the derivation of $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$. We show for those extra cases in polymorphic `CC`.

Case: ε-PolyTypeAbs. Then $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1.[\hat{v}/y]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeAbs, we get the conclusion.

Case: ε-PolyFxAbs. Then $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda\phi \subseteq \varepsilon_1.[\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxAbs, we get the conclusion.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeApp, we get the conclusion.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxApp, we get the conclusion.

Lemma 5 (Subtyping Preserved Under Substitution). *If* $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

Case: S-Reflexive. Then $\hat{\tau}_1 = \hat{\tau}_2$, so $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ by S-Reflexive.

Case: S-Transitive. Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption to these judgements, we get $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$ and $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$. By S-Transitive, $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$.

Case: S-ResourceSet. Sets of resources are unchanged by type-variable substitution, so $[\hat{\tau}'/X]\{\overline{r_1}\} = \{\overline{r_1}\}$ and $[\hat{\tau}'/X]\{\overline{r_2}\} = \{\overline{r_2}\}$. Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

Case: S-Arrow. Then the subtyping judgement from the assumption is $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_A \to_\varepsilon \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B$. By inversion and inductive assumption, we get the judgements $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_B$ and $\varepsilon \subseteq \varepsilon'$. Then by S-Arrow, we have $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_A \to_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \to_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$. By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as $\hat{\Gamma} \vdash [\hat{\tau}'/X](\hat{\tau}_1 \to_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B)$. This is the same as $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TypePoly. Then $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ and $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A.\hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}'_A$. Applying the inductive assumption to both these judgements, $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma}, Z <: \hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B < [\hat{\tau}'/X]\hat{\tau}'_A$. Then by S-TypePoly, $\hat{\Gamma} \vdash \forall Y <: [\hat{\tau}'/X]\hat{\tau}_A.[\hat{\tau}'/X]\hat{\tau}_B <: \forall Z <: [\hat{\tau}'/X]\hat{\tau}'_A.[\hat{\tau}'/X]\hat{\tau}'_B$, which is the same as $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TypeVar. Then $\hat{\tau}_1 = X$ and $\hat{\tau}_2 = \hat{\tau}$. Therefore, $[\hat{\tau}'/X]\hat{\tau}_1 = \hat{\tau}'$. Our goal now is to show $\hat{\Gamma} \vdash \hat{\tau}' <: [\hat{\tau}'/X]\hat{\tau}$. To do this, consider the form of $\hat{\tau}$.

1. If $\hat{\tau} = X$, then $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}'$. Then $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}'$ by S-Reflexive.
2. If $\hat{\tau} = \{\overline{r}\}$, then $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}$. The needed judgement, which is $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, is an assumption.
3. If $\hat{\tau} = \hat{\tau}_A \rightarrow_\varepsilon \hat{\tau}_B$, then $[\hat{\tau}'/X]\hat{\tau} = [\hat{\tau}'/X]\hat{\tau}_A \rightarrow_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B$. Because $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$..... **STUCK HERE, USE NARROWING LEMMAS**

Lemma 6 (Substitution (Types)). *If $\hat{\Gamma}, X <: \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ `with` ε and $\hat{\Gamma} \vdash \hat{\tau}'' <: \hat{\tau}'$, then $\hat{\Gamma} \vdash [\hat{\tau}''/X]\hat{e} : \hat{\tau}$ `with` ε*

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ `with` ε.'

Case: ε-Var, ε-Resource. Then $\hat{e} = [\hat{\tau}''/X]\hat{e}$, so the typing judgement in the antecedent and consequent can be the same.

Case: ε-Abs. Then $\hat{e} = \lambda x : \hat{\tau}_1.\hat{e}_2$, and $[\hat{\tau}''/X]\hat{e}_2 = \lambda x : [\hat{\tau}''/X]\hat{\tau}_1.[\hat{\tau}''/X]\hat{e}_2$. WLOG assume that $\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$. By inductive assumption and inversion, $[\hat{\tau}''/X]\hat{e}_2$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_2$ in $\hat{\Gamma}, X <: \hat{\tau}'$. By ε-Abs, $\hat{\Gamma} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$.

But now we have to establish that this new type we just derived is a subtype of $\hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$. To do that requires us to show that $\hat{\tau}_1 <: [\hat{\tau}''/X]\hat{\tau}_1$, because function types are contravariant in their input type under the subtyping relation. However, the substitution should intuitively be making the type more precise, so the subtyping is going the wrong way.

Theorem 1 (Progress). *If $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε and $\hat{e}$ is not a value, then $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$, for some $\hat{e}', \varepsilon$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}''\hat{\tau}' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$.

Theorem 2 (Preservation). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$, then $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B, where $\hat{e}_B <: \hat{e}_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$.*

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. Consider which reduction rule was used.

Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyTypeApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_3.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$.
The result follows by the substitution lemma.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.

Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-PolyFxApp2. Then $(\lambda\phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.