Theorem 1 (Progress). *If $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε and $\hat{e}$ is not a value, then $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$, for some $\hat{e}', \varepsilon$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}''\hat{\tau}' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$.

Theorem 2 (Preservation). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$, then $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B, where $\hat{e}_B <: \hat{e}_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$.*

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. Consider which reduction rule was used.
Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyTypeApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.
Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_3.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.
Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.
Subcase: E-PolyFxApp2. Then $(\lambda\phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.