

**Notation:**  $\hat{F} \vdash \delta_1, \dots, \delta_n$  means  $\hat{F} \vdash \delta_1$  and  $\hat{F} \vdash \delta_2$  and ... and  $\hat{F} \vdash \delta_n$ , where each  $\delta_i$  is a judgement.

**Lemma 1 (Substitution (Values)).** *If  $\hat{F}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$  and  $\hat{F} \vdash \hat{v} : \hat{\tau}'$  with  $\emptyset$ , then  $\hat{F} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}$  with  $\varepsilon$*

*Proof.* By induction on the derivation of  $\hat{F}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$ . We show for those extra cases in polymorphic CC.

**Case:  $\varepsilon$ -POLYTYPEABS.** Then  $\hat{e} = \lambda X <: \hat{\tau}_1. \hat{e}_1$ , and  $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1. [\hat{v}/x]\hat{e}_1$ . By inversion and inductive hypothesis,  $[\hat{v}/x]\hat{e}_1$  in  $\hat{F}$  can be typed the same as  $\hat{e}_1$  in  $\hat{F}, x : \hat{\tau}'$ . Then by applying  $\varepsilon$ -POLYTYPEABS, we get the conclusion.

**Case:  $\varepsilon$ -POLYFXABS.** Then  $\hat{e} = \lambda \phi \subseteq \varepsilon_1. \hat{e}_1$ , and  $[\hat{v}/x]\hat{e} = \lambda \phi \subseteq \varepsilon_1. [\hat{v}/x]\hat{e}_1$ . By inversion and inductive hypothesis,  $[\hat{v}/x]\hat{e}_1$  in  $\hat{F}$  can be typed the same as  $\hat{e}_1$  in  $\hat{F}, x : \hat{\tau}'$ . Then by applying  $\varepsilon$ -POLYFXABS, we get the conclusion.

**Case:  $\varepsilon$ -POLYTYPEAPP.** Then  $\hat{e} = \hat{e}_1 \hat{\tau}_1$ , and  $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1 \hat{\tau}_1$ . By inductive hypothesis,  $[\hat{v}/x]\hat{e}_1$  in  $\hat{F}$  can be typed the same as  $\hat{e}_1$  in  $\hat{F}, x : \hat{\tau}'$ . Then by applying  $\varepsilon$ -POLYTYPEAPP, we get the conclusion.

**Case:  $\varepsilon$ -POLYFXAPP.** Then  $\hat{e} = \hat{e}_1 \varepsilon$ , and  $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1 \varepsilon$ . By inductive hypothesis,  $[\hat{v}/x]\hat{e}_1$  in  $\hat{F}$  can be typed the same as  $\hat{e}_1$  in  $\hat{F}, x : \hat{\tau}'$ . Then by applying  $\varepsilon$ -POLYFXAPP, we get the conclusion.

**Lemma 2 (Type Substitution Preserves Subsetting).** *If  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$  and  $\hat{F} \vdash \hat{\tau}' <: \hat{\tau}$  then  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$*

*Proof.* By induction on the derivation of  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ .

**Case:  $\varepsilon$ -FXSET.** Trivial.

**Case:  $\varepsilon$ -FXVAR.** Then  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \phi \subseteq \varepsilon_2$ , and either (1)  $\phi \subseteq \varepsilon_2 \in \hat{F}$  or (2)  $\phi \subseteq \varepsilon_2 \in \hat{\Delta}$ . If (1) then  $\hat{F} \vdash \phi \subseteq \varepsilon_2$ , so by widening  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$ . Otherwise (2), in which case  $\phi \subseteq \varepsilon_2 \in [\hat{\tau}'/X]\hat{\Delta}$  by the definition of type-variable substitution on a context, so  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$ .

**Lemma 3 (Type Substitution Preserves Subtyping).** *If  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$  and  $\hat{F} \vdash \hat{\tau}' <: \hat{\tau}$  then  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$*

*Proof.* By induction on the derivation of  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ .

**Case: S-REFLEXIVE.** Then  $\hat{\tau}_1 = \hat{\tau}_2$ , so  $\hat{F} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$  by S-REFLEXIVE. Then by widening,  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ .

**Case: S-TRANSITIVE.** Let  $\hat{\tau}_1 = \hat{\tau}_A$  and  $\hat{\tau}_2 = \hat{\tau}_B$ . By inversion,  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$  and  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$ . Applying the inductive assumption to these judgements, we get  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$  and  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$ . Then by S-TRANSITIVE,  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$ .

**Case: S-RESOURCESET.** Sets of resources are unchanged by type-variable substitution, so  $[\hat{\tau}'/X]\{\bar{r}_1\} = \{\bar{r}_1\}$  and  $[\hat{\tau}'/X]\{\bar{r}_2\} = \{\bar{r}_2\}$ . Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

**Case: S-ARROW.** Then the subtyping judgement from the assumption is  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B$ . By inversion we have judgements (1-3),

1.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_B <: \hat{\tau}_B$
3.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

By applying the inductive hypothesis to (1) and (2), we get (4) and (5),

4.  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$
5.  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}_B$

By inspection, type-variable bindings do not affect judgements of the form  $\hat{F} \vdash \varepsilon \subseteq \varepsilon$ . Furthermore, the types in a context do not affect judgements of this form. Therefore, we can rewrite (3) as (6),

7.  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

From (4-6), we may apply S-ARROW to get  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A \rightarrow_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \rightarrow_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$ . By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as  $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}'/X](\hat{\tau}_A \rightarrow_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B)$ , which is the same as  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ .

**Case: S-TYPEPOLY.** Then  $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A. \hat{\tau}_B$  and  $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A. \hat{\tau}'_B$ . By inversion,  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$  and  $\hat{F}, X <: \hat{\tau}, \hat{\Delta}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}_B$ . Applying the inductive assumption to both these judgements,  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$  and  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta}, Z <: [\hat{\tau}'/X]\hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}_B$ . Then by S-TYPEPOLY,  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash (\forall Y <: [\hat{\tau}'/X]\hat{\tau}_A. [\hat{\tau}'/X]\hat{\tau}_B) <: (\forall Z <: [\hat{\tau}'/X]\hat{\tau}'_A. [\hat{\tau}'/X]\hat{\tau}'_B)$ , which is the same as  $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ .

**Case: S-TYPEVAR.** Then  $\hat{F}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$ . There are two cases, depending on whether  $X = Y$ .

**Subcase 1.**  $X = Y$ . Then  $\hat{F}, X <: \hat{\tau} \vdash X <: \hat{\tau}$ . We want to show (1)  $\hat{F}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]X <: [\hat{\tau}'/X]\hat{\tau}$ . Firstly,  $[\hat{\tau}'/X]X = \hat{\tau}'$ . Secondly, because  $\text{WF}(\hat{F}, X <: \hat{\tau})$  then  $X \notin \text{free-vars}(\hat{\tau})$ , so  $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}$ . Therefore, judgement (1) is the same as  $\hat{F}, X <: \hat{\tau} \vdash \hat{\tau}' <: \hat{\tau}$ , which is true by assumption.

**Subcase 2.**  $X \neq Y$ . Then  $X <: \hat{\tau}$  is not used in the derivation, so  $\hat{F}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$  is true by widening the context in the judgement  $\hat{F} \vdash Y <: \hat{\tau}_2$ <sup>1</sup>. Then  $\hat{F} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$  by inductive assumption. By widening,  $\hat{F}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$ .

**Lemma 4 (Type Substitution Preserves Typing).** *If  $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$  and  $\hat{F} \vdash \hat{\tau}'' <: \hat{\tau}'$ , then  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$  with  $\varepsilon$*

*Proof.* By induction on the derivation of  $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$ .

**Case:  $\varepsilon$ -VAR,  $\varepsilon$ -RESOURCE.** Then  $\hat{e} = [\hat{\tau}''/X]\hat{e}$ , so the typing judgement in the consequent can be the one from the antecedent.

**Case:  $\varepsilon$ -OPERCALL.** Then  $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1. \pi : \text{Unit}$  with  $\varepsilon_1 \cup \{r. \pi \mid r \in \bar{r}\}$ . By inversion we have (1). Noting that  $[\hat{\tau}''/X]\{\bar{r}\} = \{\bar{r}\}$ , we can apply the inductive hypothesis to get (2),

1.  $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\bar{r}\}$  with  $\varepsilon_1$
2.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : \{\bar{r}\}$  with  $\varepsilon_1$

Then from (2), we can apply  $\varepsilon$ -OPERCALL to get  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1. \pi) : \text{Unit}$  with  $\varepsilon_1 \cup \{r. \pi \mid r \in \bar{r}\}$ . Since  $[\hat{\tau}''/X]\text{Unit} = \text{Unit}$ , we're done.

**Case:  $\varepsilon$ -SUBSUME.** Then  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$ . By inversion, (1) and (2) are true.

1.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_2 <: \hat{\tau}$

<sup>1</sup> Note there is no explicit widening rule; be careful with this one.

2.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
3.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_2 \text{ with } \varepsilon_2$

By a previous lemma, type substitution preserves subtyping. Applying this to (1) yields (4). On the other hand, only effect-variable bindings in a context will affect subsetting judgements. Based on this, we can delete the binding  $X <: \hat{\tau}$  and perform the substitution  $[\hat{\tau}''/X]\hat{\Delta}$ , neither of which will change any effect-variable bindings, and in doing so obtain judgement (5). Lastly, we can apply the inductive hypothesis to (3), obtaining (6).

5.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}_2 <: [\hat{\tau}''/X]\hat{\tau}$
6.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
7.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_2 \text{ with } \varepsilon_2$

From (4-6) we can apply  $\varepsilon$ -SUBSUME to get  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau} \text{ with } \varepsilon_2$ .

**Case:  $\varepsilon$ -ABS.** Then  $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda y : \hat{\tau}_2.\hat{e}_3 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \text{ with } \emptyset$ . By inversion, we have (1). By setting  $\hat{\Delta}' = \hat{\Delta}, y : \hat{\tau}_2$ , this can be rewritten as (2). From inductive hypothesis we get (3). Then by simplifying  $\hat{\Delta}'$ , this simplifies to (4).

1.  $\hat{F}, X <: \hat{\tau}', \hat{\Delta}, y : \hat{\tau}_2 \vdash \hat{e}_3 : \hat{\tau}_3 \text{ with } \varepsilon_3$
2.  $\hat{F}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_3 : \hat{\tau}_3 \text{ with } \varepsilon_3$
3.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \varepsilon_3$
4.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}, y : [\hat{\tau}''/X]\hat{\tau}_2 \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \varepsilon_3$

From (4) we can apply  $\varepsilon$ -ABS to get  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda y : [\hat{\tau}''/X]\hat{\tau}_2.[\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_2 \rightarrow_{\varepsilon_3} [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \emptyset$ . This can be rewritten as  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda y : \hat{\tau}_2.\hat{e}_3) : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \text{ with } \emptyset$ .

**Case:  $\varepsilon$ -APP.** Then  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 \hat{e}_2 : \hat{\tau}_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$ . By inversion, we have:

1.  $\hat{F}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \text{ with } \varepsilon_1$
2.  $\hat{F}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \text{ with } \varepsilon_2$

Applying inductive hypothesis to (1) and (2) gives (3) and (4),

3.  $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \text{ with } \varepsilon_1$
4.  $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_2 : [\hat{\tau}''/X]\hat{\tau}_2 \text{ with } \varepsilon_2$

Then from (3) and (4) we can apply  $\varepsilon$ -APP to get  $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1 \hat{e}_2) : [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$ .

**Case:  $\varepsilon$ -POLYTYPEABS,** Then  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A \text{ cap } \varepsilon_A \text{ with } \emptyset$ . By inversion, we have (1). Setting  $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}_B$ , we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding  $\hat{\Delta}'$  lets us rewrite this as (4).

1.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_B \vdash \hat{e}_A : \hat{\tau}_A \text{ with } \varepsilon_A$
2.  $\hat{F}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_A : \hat{\tau}_A \text{ with } \varepsilon_A$
3.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A \text{ with } \varepsilon_A$
4.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}, Y <: [\hat{\tau}''/X]\hat{\tau}_B \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A \text{ with } \varepsilon_A$

From (4) we can apply  $\varepsilon$ -POLYTYPEABS, giving (5), which can be rewritten as (6).

5.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{e}_A : \forall Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{\tau}_A \text{ cap } \varepsilon_A \text{ with } \emptyset$
6.  $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A \text{ cap } \varepsilon_A) \text{ with } \emptyset$

**Case:  $\varepsilon$ -POLYFXABS.** Then  $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon_A.\hat{e}_B : \forall \phi \subseteq \varepsilon_A.\hat{\tau}_B \text{ cap } \varepsilon_B \text{ with } \emptyset$ . By inversion we have (1). Setting  $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon_A$ , this can be rewritten as (2). The inductive hypothesis gives us (3). Expanding  $\hat{\Delta}'$  lets us rewrite that as (4).

1.  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash \hat{e}_B : \hat{\tau}_B \text{ with } \varepsilon_B$
2.  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_B : \hat{\tau}_B \text{ with } \varepsilon_B$
3.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta}' \vdash [\hat{\tau}''/X] \hat{e}_B : [\hat{\tau}''/X] \hat{\tau}_B \text{ with } \varepsilon_B$
4.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash [\hat{\tau}''/X] \hat{e}_B : [\hat{\tau}''/X] \hat{\tau}_B \text{ with } \varepsilon_B$

From (4) we can apply  $\varepsilon$ -POLYFXABS, giving (5), which can be rewritten as (6).

5.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon_A. [\hat{\tau}''/X] \hat{e}_B : \forall \phi \subseteq \varepsilon_A. [\hat{\tau}''/X] \hat{\tau}_B \text{ cap } \varepsilon_B \text{ with } \emptyset$
6.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] (\lambda \phi \subseteq \varepsilon_A. \hat{e}_B) : [\hat{\tau}''/X] (\forall \phi \subseteq \varepsilon_A. \hat{\tau}_B \text{ cap } \varepsilon_B) \text{ with } \emptyset$

Case:  $\varepsilon$ -POLYTYPEAPP. Then  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}'_A : [\hat{\tau}'_A/Y] \hat{\tau}_B \text{ with } [\hat{\tau}'_A/Y] \varepsilon_B \cup \varepsilon_C$ , where we get (1) and (2) from inversion.

1.  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall Y <: \hat{\tau}_A. \hat{\tau}_B \text{ caps } \varepsilon_B \text{ with } \varepsilon_C$
2.  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

By inductive hypothesis on (1) we get (3). By a previous lemma, type substitution preserves subtyping, so from (2) we obtain (4).

3.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] \hat{e}_1 : [\hat{\tau}''/X] (\forall Y <: \hat{\tau}_A. \hat{\tau}_B \text{ caps } \varepsilon_B) \text{ with } \varepsilon_C$
4.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] \hat{\tau}'_A <: [\hat{\tau}''/X] \hat{\tau}_A$

From (3-4), applying  $\varepsilon$ -POLYTYPEAPP gives (5).

5.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] (\hat{e}_1 : \hat{\tau}'_A) : [\hat{\tau}''/X] ([\hat{\tau}'_A/Y] \hat{\tau}_B) \text{ with } [\hat{\tau}'_A/Y] \varepsilon_B \cup \varepsilon_C$

Case:  $\varepsilon$ -POLYFXAPP Then  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \varepsilon'_A : [\varepsilon'_A/\phi] \hat{\tau}_B \text{ with } [\varepsilon'_A/\phi] \varepsilon_B \cup \varepsilon_C$ , where we get (1) and (2) from inversion.

1.  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall \phi \subseteq \varepsilon_A. \hat{\tau}_B \text{ caps } \varepsilon_B \text{ with } \varepsilon_C$
2.  $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

By inductive hypothesis on (1) we get (3). Applying the lemma that type substitution preserves subsetting, we obtain (4) from (2).

3.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] \hat{e}_1 : [\hat{\tau}''/X] (\forall \phi \subseteq \varepsilon_A. \hat{\tau}_B \text{ caps } \varepsilon_B) \text{ with } \varepsilon_C$
4.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

From (3-4), applying  $\varepsilon$ -POLYFXAPP gives (5).

5.  $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] (\hat{e}_1 : \varepsilon'_A) : [\hat{\tau}''/X] ([\varepsilon'_A/\phi] \hat{\tau}_B) \text{ with } [\varepsilon'_A/\phi] \varepsilon_B \cup \varepsilon_C$

Case:  $\varepsilon$ -Import TODO

**Lemma 5 (Effect Substitution Preserves Subsetting).** *If  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$  and  $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$  then  $\hat{\Gamma}, [\varepsilon''/\phi] \hat{\Delta} \vdash [\varepsilon''/\phi] \varepsilon_1 \subseteq [\varepsilon''/\phi] \varepsilon_2$*

*Proof.* By induction on the derivation of  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ .

$\varepsilon$ -FXSET. By  $\varepsilon$ -FXSET,  $\hat{\Gamma}, [\varepsilon''/\phi] \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ . Because  $\varepsilon_1$  and  $\varepsilon_2$  are concrete sets of effects, then  $[\varepsilon''/\phi] \varepsilon_1 = \varepsilon_1$  and  $[\varepsilon''/\phi] \varepsilon_2 = \varepsilon_2$ , so we are done.

$\varepsilon$ -FXVAR. Then  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \Phi \subseteq \varepsilon''$ . We know that  $\Phi \subseteq \varepsilon''$  occurs in the context somewhere, so consider case-by-case which part.

**Subcase:**  $\Phi = \phi$ . Then  $[\varepsilon''/\phi] \varepsilon_1 = \varepsilon''$ . By well-formedness,  $\phi \notin \text{freevars}(\varepsilon_2)$ , so  $[\varepsilon''/\phi] \varepsilon_2 = \varepsilon_2$ . By inversion on the rule,  $\varepsilon_2 = \varepsilon'$ . We already know by assumption that  $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ , so by widening,  $\hat{\Gamma}, [\varepsilon''/X] \hat{\Delta} \vdash \varepsilon'' \subseteq \varepsilon'$ .

**Lemma 6 (Effect Substitution Preserves Subtyping).** *If  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$  and  $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$  then  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$*

*Proof.* By induction on derivations of  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ .

**S-REFLEXIVE.** Use S-REFLEXIVE to get the desired judgement directly.

**S-TRANSITIVE.** By inversion we have (1) and (2). Applying the inductive assumption to these yields (3) and (4), which can be used to apply S-TRANSITIVE, giving judgement (5).

1.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_C$
2.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_C <: \hat{\tau}_2$
3.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_C$
4.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_C <: [\varepsilon''/\phi]\hat{\tau}_2$
5.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$

**S-RESOURCESET.** Substitution on a resource set leaves it unchanged, so the judgement in the antecedent can be used for the judgement in the consequent.

**S-ARROW.** Then we have (1). By inversion, we also have (2-4).

1.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon_C} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon'_C} \hat{\tau}'_B$
2.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
3.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
4.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_C \subseteq \varepsilon'_C$

Applying the inductive assumption to (2) and (3) yields (5) and (6). By a previous lemma, we know that effect substitution preserves subsetting. Applying this lemma to (4) yields (7).

5.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_A <: [\varepsilon''/\phi]\hat{\tau}_A$
6.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_B$
7.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\varepsilon_C \subseteq [\varepsilon''/\phi]\varepsilon'_C$

With (5-7) we can apply S-ARROW, giving (8), which is the same as (9).

8.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_A \rightarrow_{[\varepsilon''/\phi]\varepsilon'_C} [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_A \rightarrow_{[\varepsilon''/\phi]\varepsilon_C} [\varepsilon''/\phi]\hat{\tau}'_B$
9.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\hat{\tau}_A \rightarrow_{\varepsilon_C} \hat{\tau}_B) <: [\varepsilon''/\phi](\hat{\tau}'_A \rightarrow_{\varepsilon'_C} \hat{\tau}'_B)$

**S-TYPEPOLY.** Then we have (1). By inversion, we also have (2-3).

1.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash (\forall X <: \hat{\tau}_1. \hat{\tau}_2) <: (\forall Y <: \hat{\tau}'_1. \hat{\tau}'_2)$
2.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_1 <: \hat{\tau}_1$
3.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}, Y <: \hat{\tau}'_1 \vdash \hat{\tau}_2 <: \hat{\tau}'_2$

By applying the inductive hypothesis to (2), we obtain (4).

4.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_1 <: [\varepsilon''/\phi]\hat{\tau}_1$

Now, let  $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}'_1$ . Then we can rewrite (3) as (5), and apply the inductive assumption to get (6). By simplifying  $\hat{\Delta}'$ , we get (7).

5.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
6.  $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$
7.  $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta}, Y <: [\varepsilon''/\phi]\hat{\tau}'_1 \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$

From (2) and (7) we can apply S-TYPEPOLY to get (8), which can be rewritten as the more readable (9).

8.  $\hat{I}, [\varepsilon''/\phi]\hat{\Delta} \vdash (\forall X <: [\varepsilon''/\phi]\hat{\tau}_1. [\varepsilon''/\phi]\hat{\tau}_2) <: (\forall Y <: [\varepsilon''/\phi]\hat{\tau}'_1. [\varepsilon''/\phi]\hat{\tau}'_2)$
9.  $\hat{I}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\forall X <: \hat{\tau}_1. \hat{\tau}_2) <: [\varepsilon''/\phi](\forall Y <: \hat{\tau}'_1. \hat{\tau}'_2)$

**S-TYPEVAR.** Then  $\hat{I}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash X <: \hat{\tau}$ . By inversion, there is a binding  $X <: \hat{\tau}$  in the context, so consider case-by-case where it is.

**Subcase:**  $X <: \hat{\tau} \in \hat{\Delta}$ . Then  $X <: [\varepsilon''/\phi]\hat{\tau} \in [\varepsilon''/\phi]\hat{\Delta}$ , so  $[\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$ . By widening,  $\hat{I}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$ .

**Subcase:**  $X <: \hat{\tau} \in \hat{I}$ . TODO

**Lemma 7 (Effect Substitution Preserves Types and Effects).** *If  $\hat{I}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_A : \hat{\tau}_A$  with  $\varepsilon_A$  and  $\hat{I} \vdash \phi \subseteq \varepsilon'$  then  $\hat{I}, [\varepsilon'/\phi]\hat{\Delta} \vdash [\varepsilon'/\phi]\hat{e} : [\varepsilon'/\phi]\hat{\tau}$  with  $[\varepsilon'/\phi]\varepsilon$*

*Proof.* TODO

**Definition 1 (Domain).** *For any  $\hat{I}$ , define  $\text{dom}(\hat{I})$  as follows:*

- $\text{dom}(\emptyset) = \emptyset$
- $\text{dom}(\hat{I}, x : \hat{\tau}) = \text{dom}(\hat{I}) \cup \{x\}$
- $\text{dom}(\hat{I}, X <: \hat{\tau}) = \text{dom}(\hat{I}) \cup \{X\}$
- $\text{dom}(\hat{I}, \Phi \subseteq \varepsilon) = \text{dom}(\hat{I}) \cup \{\Phi\}$

**Lemma 8 (Reverse Narrowing 1).** *If  $\hat{I} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$  and  $\Phi \notin \text{dom}(\hat{I})$  then  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$ .*

*Proof.* By induction on the derivation of  $\hat{I} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$ .

**S-Reflex.** Then  $\varepsilon_1 = \varepsilon_2$ . By S-REFLEX,  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_1$ .

**S-Trans.** By inversion and inductive assumption,  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2, \varepsilon_2 \subseteq \varepsilon_3$ . By S-TRANS,  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_3$ .

**S-FxSet.** Concrete sets of effects are invariant under substitution, so  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$ .

**S-FxVar.** Then  $\hat{I}, \Phi_2 \subseteq \varepsilon_2 \vdash [\varepsilon/X](\Phi_2 \subseteq \varepsilon_2)$ . Because  $\Phi \notin \text{dom}(\hat{I})$ , then  $\hat{I}, \Phi_2 \subseteq \varepsilon_2, \Phi \subseteq \varepsilon \vdash \Phi_2 \subseteq \varepsilon_2$  by S-FXVAR.

**Lemma 9 (Reverse Narrowing 2).** *If  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$  and  $\Phi \notin \text{dom}(\hat{I})$  then  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2$ .*

*Proof.* By induction on the derivation of  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$ .

**S-Reflex.** Then  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_1)$ , and  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_1$  by S-REFLEX.

**S-TypeVar.** Then  $\hat{I}, X <: \hat{\tau} \vdash [\varepsilon/\Phi](X <: \hat{\tau})$ . Because  $\Phi$  is an effect-variable, and not a type-variable, then  $\hat{I}, X <: \hat{\tau}, \Phi \subseteq \varepsilon \vdash X <: \hat{\tau}$  by S-TYPEVAR.

**S-ResourceSet.** By S-RESOURCESET,  $\hat{I}, \Phi \subseteq \varepsilon \vdash \{\bar{r}_1\} <: \{\bar{r}_2\}$ .

**S-Trans.** Then  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_3)$ . By inversion and induction, we have  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2, \hat{\tau}_2 <: \hat{\tau}_3$ . Then by S-TRANS,  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_3$ .

**S-Arrow.** Then  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2 <: (\hat{\tau}'_1 \rightarrow_{\varepsilon'_3} \hat{\tau}'_2))$ . By inversion, we know (1-3):

1.  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}'_1 <: \hat{\tau}_1)$
2.  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}'_2 <: \hat{\tau}'_2)$
3.  $\hat{I} \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon'_3)$

By applying the inductive assumption to (1-2) we get (4-5). By applying Reverse Narrowing 1 to 3, we get 6.

4.  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}'_1 <: \hat{\tau}_1$
5.  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}'_2 <: \hat{\tau}'_2$
6.  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon'_3$

From (4-6), we can use S-ARROW to get the judgement  $\hat{I}, \Phi \subseteq \varepsilon \vdash (\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}'_1 \rightarrow_{\varepsilon'_3} \hat{\tau}'_2)$ .

S-PolyType. Then  $\hat{I} \vdash [\varepsilon/\Phi](\forall X <: \hat{\tau}_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$ . By inversion, we know (1-3):

1.  $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}'_1 <: \hat{\tau})$
2.  $\hat{I}, Y <: \hat{\tau}'_1 \vdash [\varepsilon/\Phi](\hat{\tau}_2 <: \hat{\tau}'_2)$
3.  $\hat{I}, Y <: \hat{\tau}'_1 \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon'_3)$

By applying the inductive assumption to (1-3), we get (4-6).

4.  $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}'_1 <: \hat{\tau}$
5.  $\hat{I}, Y <: \hat{\tau}'_1, \Phi \subseteq \varepsilon \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
6.  $\hat{I}, Y <: \hat{\tau}'_1, \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon'_3$

**5 can be rewritten as 7, and 6 as 8.**

7.  $\hat{I}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}'_1 \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
8.  $\hat{I}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}'_1 \vdash \varepsilon_3 \subseteq \varepsilon'_3$

From (4,7,8) we can apply S-POLYTYPE to get  $\hat{I}, \Phi \subseteq \varepsilon \vdash (\forall X <: \hat{\tau}_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$ .

S-PolyFx. Then  $\hat{I} \vdash [\varepsilon/\Phi](\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall \Phi'_1 \subseteq \varepsilon'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$ . By inversion we know (1-3):

1.  $\hat{I} \vdash [\varepsilon/\Phi](\varepsilon'_1 \subseteq \varepsilon_1)$
2.  $\hat{I}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}'_1)$
3.  $\hat{I}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon'_3)$

By applying the Reverse Narrowing Lemma 1 to (1), we get (3). By applying the inductive assumption to (2-3), we get (5-6).

4.  $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon'_1 \subseteq \varepsilon_1$
5.  $\hat{I}, \Phi_2 \subseteq \varepsilon', \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}'_1$
6.  $\hat{I}, \Phi_2 \subseteq \varepsilon', \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon'_3$

**5 can be rewritten as 7, and 6 as 8.**

7.  $\hat{I}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \hat{\tau}_1 <: \hat{\tau}'_1$
8.  $\hat{I}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \varepsilon_3 \subseteq \varepsilon'_3$

With (4,7,8), we can apply S-POLYFX to get  $\hat{I}, \Phi \subseteq \varepsilon \vdash (\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall \Phi'_1 \subseteq \varepsilon'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$ .

**Lemma 10.**  $[\emptyset/\Phi]\varepsilon \subseteq \varepsilon$ .

*Proof.* If  $\varepsilon \neq \Phi$  then  $[\emptyset/\Phi]\varepsilon = \varepsilon$ . Otherwise,  $[\emptyset/\Phi]\varepsilon = \emptyset \subseteq \varepsilon$ .

**Lemma 11.** For any  $\hat{\tau}$ ,  $\text{effects}([\emptyset/\Phi]\hat{\tau}) \subseteq \text{effects}(\hat{\tau})$  and  $\text{ho-effects}([\emptyset/\Phi]\hat{\tau}) \subseteq \text{ho-effects}(\hat{\tau})$ .

*Proof.* By simultaneous induction on the form of  $\hat{\tau}$ . First, consider  $\mathbf{effects}([\emptyset/\Phi]\hat{\tau}) \subseteq \mathbf{effects}(\hat{\tau})$ .

$\hat{\tau} = \{\bar{r}\}$ . Then  $[\emptyset/\Phi]\hat{\tau} = \hat{\tau}$ , so the result is trivial.

$\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2$ . Then  $[\emptyset/\Phi]\hat{\tau} = [\emptyset/\Phi]\hat{\tau}_1 \rightarrow_{[\emptyset/\Phi]\varepsilon_3} [\emptyset/\Phi]\hat{\tau}_2$ . By definition,  $\mathbf{effects}(\hat{\tau}) = \mathbf{ho-effects}(\hat{\tau}_1) \cup \varepsilon_3 \cup \mathbf{effects}(\hat{\tau}_2)$ . By inductive hypothesis, we know  $\mathbf{ho-effects}([\emptyset/\Phi]\hat{\tau}_1) \subseteq \mathbf{ho-effects}(\hat{\tau}_1)$  and  $\mathbf{effects}([\emptyset/\Phi]\hat{\tau}_2) \subseteq \mathbf{effects}(\hat{\tau}_2)$ . We also have  $[\emptyset/\Phi]\varepsilon_3 \subseteq \varepsilon_3$  by the previous lemma.

$\hat{\tau} = \forall \Phi \subseteq \varepsilon_1. \hat{\tau}_2 \text{ caps } \varepsilon_2$ . Then  $\mathbf{effects}(\hat{\tau}) = \varepsilon_2 \cup [\emptyset/\Phi]\hat{\tau}_2$ . By the previous lemma,  $[\emptyset/\Phi]\varepsilon_2 \subseteq \varepsilon_2$ , and it is trivial that  $\mathbf{effects}([\emptyset/\Phi]\hat{\tau}_2) \subseteq \mathbf{effects}([\emptyset/\Phi]\hat{\tau}_2)$ .

□

Now consider  $\mathbf{ho-effects}([\emptyset/\Phi]\hat{\tau}) \subseteq \mathbf{ho-effects}(\hat{\tau})$ .

$\hat{\tau} = \{\bar{r}\}$ . Same as above; trivial.

$\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2$ . Then  $[\emptyset/\Phi]\hat{\tau} = [\emptyset/\Phi]\hat{\tau}_1 \rightarrow_{[\emptyset/\Phi]\varepsilon_3} [\emptyset/\Phi]\hat{\tau}_2$ . By definition,  $\mathbf{ho-effects}(\hat{\tau}) = \mathbf{effects}(\hat{\tau}_1) \cup \mathbf{ho-effects}(\hat{\tau}_2)$ . By inductive hypothesis, we know  $\mathbf{effects}([\emptyset/\Phi]\hat{\tau}_1) \subseteq \mathbf{effects}(\hat{\tau}_1)$  and  $\mathbf{ho-effects}([\emptyset/\Phi]\hat{\tau}_2) \subseteq \mathbf{ho-effects}(\hat{\tau}_2)$ .

$\hat{\tau} = \forall \Phi \subseteq \varepsilon_1. \hat{\tau}_2 \text{ caps } \varepsilon_2$ . Then  $\mathbf{ho-effects}(\hat{\tau}) = \varepsilon_1 \cup [\emptyset/\Phi]\hat{\tau}_2$ . By the previous lemma, we know  $[\emptyset/\Phi]\varepsilon_1 \subseteq \varepsilon_1$ . It is trivial that  $\mathbf{ho-effects}([\emptyset/\Phi]\hat{\tau}_2) \subseteq \mathbf{ho-effects}([\emptyset/\Phi]\hat{\tau}_2)$ .

□

**Lemma 12.**  $\mathbf{safe}([\emptyset/\Phi]\hat{\tau}, \varepsilon_s) \implies \mathbf{safe}(\hat{\tau}, \varepsilon_s)$  and  $\mathbf{ho-safe}([\emptyset/\Phi]\hat{\tau}, \varepsilon_s) \implies \mathbf{ho-safe}(\hat{\tau}, \varepsilon_s)$ .

*Proof.* TODO

**Lemma 13 (Approximation 1).** If  $\hat{I} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$  and  $\mathbf{effects}(\hat{\tau}) \subseteq \varepsilon_s$  and  $\mathbf{ho-safe}(\hat{\tau}, \varepsilon_s)$  then  $\hat{\tau} <: \mathbf{annot}(\mathbf{erase}(\hat{\tau}), \varepsilon_s)$ .

**Lemma 14 (Approximation 2).** If  $\hat{I} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$  and  $\mathbf{ho-effects}(\hat{\tau}) \subseteq \varepsilon_s$  and  $\mathbf{safe}(\hat{\tau}, \varepsilon_s)$  then  $\mathbf{annot}(\mathbf{erase}(\hat{\tau}), \varepsilon_s) <: \hat{\tau}$ .

*Proof.* By simultaneous induction on derivations of **safe** and **ho-safe**, and then on derivations of  $\hat{I} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$ .

$\varepsilon$ -POLYFXABS. Then  $\hat{e}$  has the form given in (1). The definition of **effects** is (2). From a previous lemma,  $\mathbf{ho-safe}(\hat{\tau}_2, \varepsilon_s) \implies \mathbf{ho-safe}([\emptyset/\Phi]\hat{\tau}_2, \varepsilon_s)$ , so we know (3). With (2-3) we can apply the inductive hypothesis to  $[\emptyset/\Phi]\hat{\tau}_2$ , giving (4).

1.  $\hat{e} = \forall \Phi \subseteq \varepsilon_1. \hat{\tau}_2 \text{ caps } \varepsilon_2$
2.  $\mathbf{effects}(\hat{\tau}) = \varepsilon_2 \cup \mathbf{effects}([\emptyset/\Phi]\hat{\tau}_2) \subseteq \varepsilon_s$
3.  $\mathbf{ho-safe}([\emptyset/\Phi]\hat{\tau}_2, \varepsilon_s)$
4.  $\hat{I} \vdash [\emptyset/\Phi]\hat{\tau}_2 <: \mathbf{annot}(\mathbf{erase}([\emptyset/\Phi]\hat{\tau}_2), \varepsilon_s)$

But we want  $\hat{\tau}_2 <: \mathbf{annot}(\mathbf{erase}(\hat{\tau}_2), \varepsilon_s)$ .

**Theorem 1 (Progress).** If  $\hat{I} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$  and  $\hat{e}$  is not a value, then  $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$ , for some  $\hat{e}', \varepsilon$ .

*Proof.* By induction on the derivation of  $\hat{I} \vdash \hat{e} : \hat{\tau}$  with  $\varepsilon$ .

*Case:*  $\varepsilon$ -POLYTYPEABS. Trivial;  $\hat{e}$  is a value.

*Case:*  $\varepsilon$ -POLYFXABS. Trivial;  $\hat{e}$  is a value.

*Case:*  $\varepsilon$ -POLYTYPEAPP. Then  $\hat{e} = \hat{e}_1 \hat{\tau}'$ . If  $\hat{e}_1$  is not a value then  $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$  by inductive hypothesis, and applying E-POLYTYPEAPP1 gives the reduction  $\hat{e}_1 \hat{\tau}' \longrightarrow \hat{e}'' \hat{\tau}' \mid \varepsilon$ . Otherwise,  $\hat{e}_1$  is a value, so  $\hat{e} = \lambda X <: \hat{\tau}_1. \hat{e}_2$ ,



and applying E-POLYTYPEAPP2 gives the reduction  $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \emptyset$ .

*Case:  $\varepsilon$ -POLYFXAPP.* Then  $\hat{e} = \hat{e}_1 \ \varepsilon'$ . If  $\hat{e}_1$  is not a value then  $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$  by inductive hypothesis, and applying E-POLYFXAPP1 gives the reduction  $\hat{e}_1 \ \varepsilon' \longrightarrow \hat{e}'_1 \ \varepsilon' \mid \varepsilon$ . Otherwise,  $\hat{e}$  is a value, so  $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_2$ , and applying E-POLYFXAPP2 gives the reduction  $(\lambda\phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$ .

**Theorem 2 (Preservation).** *If  $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$  with  $\varepsilon_A$  and  $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$ , then  $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$  with  $\varepsilon_B$ , where  $\hat{\tau}_B <: \hat{\tau}_A$  and  $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ , for some  $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$ .*

*Proof.* By induction on the derivations of  $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$  with  $\varepsilon_A$  and  $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$ .

*Case:  $\varepsilon$ -POLYTYPEABS.* Trivial;  $\hat{e}$  is a value.

*Case:  $\varepsilon$ -POLYFXABS.* Trivial;  $\hat{e}$  is a value.

*Case:  $\varepsilon$ -POLYTYPEAPP.* Then  $\hat{e} = \hat{e}_1 \ \hat{\tau}'$ . The typing rule from the judgement can be rewritten as (1). From inversion, we also have (2) and (3).

1.  $\hat{\Gamma} \vdash \hat{e}_1 \ \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2$  with  $\varepsilon_1 \cup \varepsilon_2$
2.  $\hat{\Gamma} \vdash \hat{e}_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2$  caps  $\varepsilon_1$  with  $\varepsilon_2$
3.  $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}_1$

Now consider which reduction rule was used.

**Subcase: E-POLYTYPEAPP1.** Then  $\hat{e}_1 \ \hat{\tau}' \longrightarrow \hat{e}'_1 \ \hat{\tau}' \mid \varepsilon$ . By inversion on the reduction rule,  $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$ . With (2), we can apply the inductive assumption and  $\varepsilon$ -SUBSUME to get (4). With (4) and (3), we can then apply  $\varepsilon$ -POLYTYPEAPP to get (5). Then by comparing (1) and (6), we see  $\hat{\tau}_B = \hat{\tau}_A$  and  $\hat{\varepsilon} = \hat{\varepsilon}_A = \hat{\varepsilon}_B$ .

4.  $\hat{\Gamma} \vdash \hat{e}'_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2$  caps  $\varepsilon_1$  with  $\varepsilon_2$
5.  $\hat{\Gamma} \vdash \hat{e}'_1 \ \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2$  with  $\varepsilon_1 \cup \varepsilon_2$

**Subcase: E-POLYTYPEAPP2.** Then  $(\lambda X <: \hat{\tau}_1.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \emptyset$ . Because of the form of  $\hat{e}_1$  in this subcase, the only rule which could have been applied to obtain judgement (2) is  $\varepsilon$ -TYPEABS. By inversion on this rule we get (4). From (4) and (3), we can apply the lemma that type-and-effect judgements are preserved under type variable substitution to obtain (5). Finally, by comparing (1) and (5) we see  $\hat{\tau}_A = [\hat{\tau}'/X]\hat{\tau}_2 = \hat{\tau}_B$ , and  $\varepsilon_B \cup \varepsilon = \varepsilon_1 \subseteq \varepsilon_1 \cup \varepsilon_2 = \varepsilon_A$ .

4.  $\hat{\Gamma}, X <: \hat{\tau}_1 \vdash \hat{e}' : \hat{\tau}_2$  with  $\varepsilon_1$
5.  $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{e}' : [\hat{\tau}'/X]\hat{\tau}_2$  with  $\varepsilon_1$

*Case:  $\varepsilon$ -POLYFXAPP.* Then  $\hat{e} = \hat{e}_1 \ \varepsilon'$ . Consider which reduction rule was used.

**Subcase: E-POLYFXAPP1.** Then  $\hat{e}_1 \ \varepsilon' \longrightarrow \hat{e}'_1 \ \varepsilon' \mid \varepsilon$ . By inversion,  $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$ . With the inductive hypothesis and subsumption,  $\hat{e}'_1$  can be typed in  $\hat{\Gamma}$  the same as  $\hat{e}_1$ . Then by  $\varepsilon$ -POLYFXAPP,  $\hat{\Gamma} \vdash \hat{e}'_1 \ \varepsilon' : \hat{\tau}_A$  with  $\varepsilon_A$ . That  $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$  follows by inductive hypothesis.

**Subcase: E-POLYFXAPP2.** Then  $(\lambda\phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \emptyset$ . **The result follows by the substitution lemma.**