

Notation: $\hat{F} \vdash \delta_1, \dots, \delta_n$ means $\hat{F} \vdash \delta_1$ and $\hat{F} \vdash \delta_2$ and ... and $\hat{F} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Substitution (Values)). *If $\hat{F}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ with ε and $\hat{F} \vdash \hat{v} : \hat{\tau}'$ with \emptyset , then $\hat{F} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}$ with ε*

Proof. By induction on the derivation of $\hat{F}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ with ε . We show for those extra cases in polymorphic CC.

Case: ε -POLYTYPEABS. Then $\hat{e} = \lambda X <: \hat{\tau}_1. \hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1. [\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in \hat{F} can be typed the same as \hat{e}_1 in $\hat{F}, x : \hat{\tau}'$. Then by applying ε -POLYTYPEABS, we get the conclusion.

Case: ε -POLYFXABS. Then $\hat{e} = \lambda \phi \subseteq \varepsilon_1. \hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda \phi \subseteq \varepsilon_1. [\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in \hat{F} can be typed the same as \hat{e}_1 in $\hat{F}, x : \hat{\tau}'$. Then by applying ε -POLYFXABS, we get the conclusion.

Case: ε -POLYTYPEAPP. Then $\hat{e} = \hat{e}_1 \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1 \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in \hat{F} can be typed the same as \hat{e}_1 in $\hat{F}, x : \hat{\tau}'$. Then by applying ε -POLYTYPEAPP, we get the conclusion.

Case: ε -POLYFXAPP. Then $\hat{e} = \hat{e}_1 \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1 \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in \hat{F} can be typed the same as \hat{e}_1 in $\hat{F}, x : \hat{\tau}'$. Then by applying ε -POLYFXAPP, we get the conclusion.

Lemma 2 (Type Substitution Preserves Subsetting). *If $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ and $\hat{F} \vdash \hat{\tau}' <: \hat{\tau}$ then $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$*

Proof. By induction on the derivation of $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Case: ε -FXSET. Trivial.

Case: ε -FXVAR. Then $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \phi \subseteq \varepsilon_2$, and either (1) $\phi \subseteq \varepsilon_2 \in \hat{F}$ or (2) $\phi \subseteq \varepsilon_2 \in \hat{\Delta}$. If (1) then $\hat{F} \vdash \phi \subseteq \varepsilon_2$, so by widening $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$. Otherwise (2), in which case $\phi \subseteq \varepsilon_2 \in [\hat{\tau}'/X]\hat{\Delta}$ by the definition of type-variable substitution on a context, so $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$.

Lemma 3 (Type Substitution Preserves Subtyping). *If $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ and $\hat{F} \vdash \hat{\tau}' <: \hat{\tau}$ then $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$*

Proof. By induction on the derivation of $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

Case: S-REFLEXIVE. Then $\hat{\tau}_1 = \hat{\tau}_2$, so $\hat{F} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ by S-REFLEXIVE. Then by widening, $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

Case: S-TRANSITIVE. Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_B$. By inversion, $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption to these judgements, we get $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$ and $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$. Then by S-TRANSITIVE, $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$.

Case: S-RESOURCESET. Sets of resources are unchanged by type-variable substitution, so $[\hat{\tau}'/X]\{\bar{r}_1\} = \{\bar{r}_1\}$ and $[\hat{\tau}'/X]\{\bar{r}_2\} = \{\bar{r}_2\}$. Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

Case: S-ARROW. Then the subtyping judgement from the assumption is $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B$. By inversion we have judgements (1-3),

1. $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_B <: \hat{\tau}_B$
3. $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

By applying the inductive hypothesis to (1) and (2), we get (4) and (5),

4. $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$
5. $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}_B$

By inspection, type-variable bindings do not affect judgements of the form $\hat{F} \vdash \varepsilon \subseteq \varepsilon$. Furthermore, the types in a context do not affect judgements of this form. Therefore, we can rewrite (3) as (6),

7. $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

From (4-6), we may apply S-ARROW to get $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A \rightarrow_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \rightarrow_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$. By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}'/X](\hat{\tau}_A \rightarrow_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B)$, which is the same as $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TYPEPOLY. Then $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A. \hat{\tau}_B$ and $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A. \hat{\tau}'_B$. By inversion, $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{F}, X <: \hat{\tau}, \hat{\Delta}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}_B$. Applying the inductive assumption to both these judgements, $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{F}, [\hat{\tau}'/X]\hat{\Delta}, Z <: [\hat{\tau}'/X]\hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}_B$. Then by S-TYPEPOLY, $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash (\forall Y <: [\hat{\tau}'/X]\hat{\tau}_A. [\hat{\tau}'/X]\hat{\tau}_B) <: (\forall Z <: [\hat{\tau}'/X]\hat{\tau}'_A. [\hat{\tau}'/X]\hat{\tau}'_B)$, which is the same as $\hat{F}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TYPEVAR. Then $\hat{F}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{F}, X <: \hat{\tau} \vdash X <: \hat{\tau}$. We want to show (1) $\hat{F}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]X <: [\hat{\tau}'/X]\hat{\tau}$. Firstly, $[\hat{\tau}'/X]X = \hat{\tau}'$. Secondly, because $\text{WF}(\hat{F}, X <: \hat{\tau})$ then $X \notin \text{free-vars}(\hat{\tau})$, so $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}$. Therefore, judgement (1) is the same as $\hat{F}, X <: \hat{\tau} \vdash \hat{\tau}' <: \hat{\tau}$, which is true by assumption.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation, so $\hat{F}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$ is true by widening the context in the judgement $\hat{F} \vdash Y <: \hat{\tau}_2$ ¹. Then $\hat{F} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$ by inductive assumption. By widening, $\hat{F}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$.

Lemma 4 (Type Substitution Preserves Typing). *If $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ with ε and $\hat{F} \vdash \hat{\tau}'' <: \hat{\tau}'$, then $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$ with ε*

Proof. By induction on the derivation of $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ with ε .

Case: ε -VAR, ε -RESOURCE. Then $\hat{e} = [\hat{\tau}''/X]\hat{e}$, so the typing judgement in the consequent can be the one from the antecedent.

Case: ε -OPERCALL. Then $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1. \pi : \text{Unit}$ with $\varepsilon_1 \cup \{r. \pi \mid r \in \bar{r}\}$. By inversion we have (1). Noting that $[\hat{\tau}''/X]\{\bar{r}\} = \{\bar{r}\}$, we can apply the inductive hypothesis to get (2),

1. $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\bar{r}\}$ with ε_1
2. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : \{\bar{r}\}$ with ε_1

Then from (2), we can apply ε -OPERCALL to get $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1. \pi) : \text{Unit}$ with $\varepsilon_1 \cup \{r. \pi \mid r \in \bar{r}\}$. Since $[\hat{\tau}''/X]\text{Unit} = \text{Unit}$, we're done.

Case: ε -SUBSUME. Then $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ with ε . By inversion, (1) and (2) are true.

1. $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_2 <: \hat{\tau}$

¹ Note there is no explicit widening rule; be careful with this one.

2. $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
3. $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_2 \text{ with } \varepsilon_2$

By a previous lemma, type substitution preserves subtyping. Applying this to (1) yields (4). On the other hand, only effect-variable bindings in a context will affect subsetting judgements. Based on this, we can delete the binding $X <: \hat{\tau}$ and perform the substitution $[\hat{\tau}''/X]\hat{\Delta}$, neither of which will change any effect-variable bindings, and in doing so obtain judgement (5). Lastly, we can apply the inductive hypothesis to (3), obtaining (6).

5. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}_2 <: [\hat{\tau}''/X]\hat{\tau}$
6. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
7. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_2 \text{ with } \varepsilon_2$

From (4-6) we can apply ε -SUBSUME to get $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau} \text{ with } \varepsilon_2$.

Case: ε -ABS. Then $\hat{F}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda y : \hat{\tau}_2.\hat{e}_3 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \text{ with } \emptyset$. By inversion, we have (1). By setting $\hat{\Delta}' = \hat{\Delta}, y : \hat{\tau}_2$, this can be rewritten as (2). From inductive hypothesis we get (3). Then by simplifying $\hat{\Delta}'$, this simplifies to (4).

1. $\hat{F}, X <: \hat{\tau}', \hat{\Delta}, y : \hat{\tau}_2 \vdash \hat{e}_3 : \hat{\tau}_3 \text{ with } \varepsilon_3$
2. $\hat{F}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_3 : \hat{\tau}_3 \text{ with } \varepsilon_3$
3. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \varepsilon_3$
4. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}, y : [\hat{\tau}''/X]\hat{\tau}_2 \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \varepsilon_3$

From (4) we can apply ε -ABS to get $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda y : [\hat{\tau}''/X]\hat{\tau}_2.[\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_2 \rightarrow_{\varepsilon_3} [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \emptyset$. This can be rewritten as $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda y : \hat{\tau}_2.\hat{e}_3) : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \text{ with } \emptyset$.

Case: ε -APP. Then $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 \hat{e}_2 : \hat{\tau}_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$. By inversion, we have:

1. $\hat{F}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \text{ with } \varepsilon_1$
2. $\hat{F}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2 \text{ with } \varepsilon_2$

Applying inductive hypothesis to (1) and (2) gives (3) and (4),

3. $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3) \text{ with } \varepsilon_1$
4. $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_2 : [\hat{\tau}''/X]\hat{\tau}_2 \text{ with } \varepsilon_2$

Then from (3) and (4) we can apply ε -APP to get $\hat{F}, \hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1 \hat{e}_2) : [\hat{\tau}''/X]\hat{\tau}_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

Case: ε -POLYTYPEABS, Then $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A \text{ cap } \varepsilon_A \text{ with } \emptyset$. By inversion, we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}_B$, we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite this as (4).

1. $\hat{F}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_B \vdash \hat{e}_A : \hat{\tau}_A \text{ with } \varepsilon_A$
2. $\hat{F}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_A : \hat{\tau}_A \text{ with } \varepsilon_A$
3. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A \text{ with } \varepsilon_A$
4. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta}, Y <: [\hat{\tau}''/X]\hat{\tau}_B \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A \text{ with } \varepsilon_A$

From (4) we can apply ε -POLYTYPEABS, giving (5), which can be rewritten as (6).

5. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{e}_A : \forall Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{\tau}_A \text{ cap } \varepsilon_A \text{ with } \emptyset$
6. $\hat{F}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A \text{ cap } \varepsilon_A) \text{ with } \emptyset$

Case: ε -POLYFXABS. Then $\hat{F}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon_A.\hat{e}_B : \forall \phi \subseteq \varepsilon_A.\hat{\tau}_B \text{ cap } \varepsilon_B \text{ with } \emptyset$. By inversion we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon_A$, this can be rewritten as (2). The inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite that as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash \hat{e}_B : \hat{\tau}_B \text{ with } \varepsilon_B$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_B : \hat{\tau}_B \text{ with } \varepsilon_B$
3. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta}' \vdash [\hat{\tau}''/X] \hat{e}_B : [\hat{\tau}''/X] \hat{\tau}_B \text{ with } \varepsilon_B$
4. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash [\hat{\tau}''/X] \hat{e}_B : [\hat{\tau}''/X] \hat{\tau}_B \text{ with } \varepsilon_B$

From (4) we can apply ε -POLYFXABS, giving (5), which can be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon_A. [\hat{\tau}''/X] \hat{e}_B : \forall \phi \subseteq \varepsilon_A. [\hat{\tau}''/X] \hat{\tau}_B \text{ cap } \varepsilon_B \text{ with } \emptyset$
6. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] (\lambda \phi \subseteq \varepsilon_A. \hat{e}_B) : [\hat{\tau}''/X] (\forall \phi \subseteq \varepsilon_A. \hat{\tau}_B \text{ cap } \varepsilon_B) \text{ with } \emptyset$

Case: ε -POLYTYPEAPP. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 \hat{\tau}'_A : [\hat{\tau}'_A/Y] \hat{\tau}_B \text{ with } [\hat{\tau}'_A/Y] \varepsilon_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall Y <: \hat{\tau}_A. \hat{\tau}_B \text{ caps } \varepsilon_B \text{ with } \varepsilon_C$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

By inductive hypothesis on (1) we get (3). By a previous lemma, type substitution preserves subtyping, so from (2) we obtain (4).

3. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] \hat{e}_1 : [\hat{\tau}''/X] (\forall Y <: \hat{\tau}_A. \hat{\tau}_B \text{ caps } \varepsilon_B) \text{ with } \varepsilon_C$
4. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] \hat{\tau}'_A <: [\hat{\tau}''/X] \hat{\tau}_A$

From (3-4), applying ε -POLYTYPEAPP gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] (\hat{e}_1 \hat{\tau}'_A) : [\hat{\tau}''/X] ([\hat{\tau}'_A/Y] \hat{\tau}_B) \text{ with } [\hat{\tau}'_A/Y] \varepsilon_B \cup \varepsilon_C$

Case: ε -POLYFXAPP Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 \varepsilon'_A : [\varepsilon'_A/\phi] \hat{\tau}_B \text{ with } [\varepsilon'_A/\phi] \varepsilon_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall \phi \subseteq \varepsilon_A. \hat{\tau}_B \text{ caps } \varepsilon_B \text{ with } \varepsilon_C$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

By inductive hypothesis on (1) we get (3). Applying the lemma that type substitution preserves subsetting, we obtain (4) from (2).

3. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] \hat{e}_1 : [\hat{\tau}''/X] (\forall \phi \subseteq \varepsilon_A. \hat{\tau}_B \text{ caps } \varepsilon_B) \text{ with } \varepsilon_C$
4. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

From (3-4), applying ε -POLYFXAPP gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X] \hat{\Delta} \vdash [\hat{\tau}''/X] (\hat{e}_1 \varepsilon'_A) : [\hat{\tau}''/X] ([\varepsilon'_A/\phi] \hat{\tau}_B) \text{ with } [\varepsilon'_A/\phi] \varepsilon_B \cup \varepsilon_C$

Case: ε -Import TODO

Lemma 5 (Effect Substitution Preserves Subsetting). *If $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ and $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ then $\hat{\Gamma}, [\varepsilon''/\phi] \hat{\Delta} \vdash [\varepsilon''/\phi] \varepsilon_1 \subseteq [\varepsilon''/\phi] \varepsilon_2$*

Proof. By induction on the derivation of $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

ε -FXSET. By ε -FXSET, $\hat{\Gamma}, [\varepsilon''/\phi] \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$. Because ε_1 and ε_2 are concrete sets of effects, then $[\varepsilon''/\phi] \varepsilon_1 = \varepsilon_1$ and $[\varepsilon''/\phi] \varepsilon_2 = \varepsilon_2$, so we are done.

ε -FXVAR. Then $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \Phi \subseteq \varepsilon''$. We know that $\Phi \subseteq \varepsilon''$ occurs in the context somewhere, so consider case-by-case which part.

Subcase: $\Phi = \phi$. Then $[\varepsilon''/\phi] \varepsilon_1 = \varepsilon''$. By well-formedness, $\phi \notin \text{freevars}(\varepsilon_2)$, so $[\varepsilon''/\phi] \varepsilon_2 = \varepsilon_2$. By inversion on the rule, $\varepsilon_2 = \varepsilon'$. We already know by assumption that $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$, so by widening, $\hat{\Gamma}, [\varepsilon''/X] \hat{\Delta} \vdash \varepsilon'' \subseteq \varepsilon'$.

Lemma 6 (Effect Substitution Preserves Subtyping). *If $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ and $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ then $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$*

Proof. By induction on derivations of $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

S-REFLEXIVE. Use S-REFLEXIVE to get the desired judgement directly.

S-TRANSITIVE. By inversion we have (1) and (2). Applying the inductive assumption to these yields (3) and (4), which can be used to apply S-TRANSITIVE, giving judgement (5).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_C$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_C <: \hat{\tau}_2$
3. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_C$
4. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_C <: [\varepsilon''/\phi]\hat{\tau}_2$
5. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$

S-RESOURCESET. Substitution on a resource set leaves it unchanged, so the judgement in the antecedent can be used for the judgement in the consequent.

S-ARROW. Then we have (1). By inversion, we also have (2-4).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon_C} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon'_C} \hat{\tau}'_B$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
3. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
4. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_C \subseteq \varepsilon'_C$

Applying the inductive assumption to (2) and (3) yields (5) and (6). By a previous lemma, we know that effect substitution preserves subsetting. Applying this lemma to (4) yields (7).

5. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_A <: [\varepsilon''/\phi]\hat{\tau}_A$
6. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_B$
7. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\varepsilon_C \subseteq [\varepsilon''/\phi]\varepsilon'_C$

With (5-7) we can apply S-ARROW, giving (8), which is the same as (9).

8. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_A \rightarrow_{[\varepsilon''/\phi]\varepsilon'_C} [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_A \rightarrow_{[\varepsilon''/\phi]\varepsilon_C} [\varepsilon''/\phi]\hat{\tau}'_B$
9. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\hat{\tau}_A \rightarrow_{\varepsilon_C} \hat{\tau}_B) <: [\varepsilon''/\phi](\hat{\tau}'_A \rightarrow_{\varepsilon'_C} \hat{\tau}'_B)$

S-TYPEPOLY. Then we have (1). By inversion, we also have (2-3).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash (\forall X <: \hat{\tau}_1. \hat{\tau}_2) <: (\forall Y <: \hat{\tau}'_1. \hat{\tau}'_2)$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_1 <: \hat{\tau}_1$
3. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}, Y <: \hat{\tau}'_1 \vdash \hat{\tau}_2 <: \hat{\tau}'_2$

By applying the inductive hypothesis to (2), we obtain (4).

4. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_1 <: [\varepsilon''/\phi]\hat{\tau}_1$

Now, let $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}'_1$. Then we can rewrite (3) as (5), and apply the inductive assumption to get (6). By simplifying $\hat{\Delta}'$, we get (7).

5. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
6. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$
7. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta}, Y <: [\varepsilon''/\phi]\hat{\tau}'_1 \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$

From (2) and (7) we can apply S-TYPEPOLY to get (8), which can be rewritten as the more readable (9).

8. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash (\forall X <: [\varepsilon''/\phi]\hat{\tau}_1. [\varepsilon''/\phi]\hat{\tau}_2) <: (\forall Y <: [\varepsilon''/\phi]\hat{\tau}'_1. [\varepsilon''/\phi]\hat{\tau}'_2)$
9. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\forall X <: \hat{\tau}_1. \hat{\tau}_2) <: [\varepsilon''/\phi](\forall Y <: \hat{\tau}'_1. \hat{\tau}'_2)$

S-TYPEVAR. Then $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash X <: \hat{\tau}$. By inversion, there is a binding $X <: \hat{\tau}$ in the context, so consider case-by-case where it is.

Subcase: $X <: \hat{\tau} \in \hat{\Delta}$. Then $X <: [\varepsilon''/\phi]\hat{\tau} \in [\varepsilon''/\phi]\hat{\Delta}$, so $[\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$. By widening, $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$.

Subcase: $X <: \hat{\tau} \in \hat{\Gamma}$. TODO

Lemma 7 (Effect Substitution Preserves Types and Effects). *If $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_A : \hat{\tau}_A$ with ε_A and $\hat{\Gamma} \vdash \phi \subseteq \varepsilon'$ then $\hat{\Gamma}, [\varepsilon'/\phi]\hat{\Delta} \vdash [\varepsilon'/\phi]\hat{e} : [\varepsilon'/\phi]\hat{\tau}$ with $[\varepsilon'/\phi]\varepsilon$*

Proof. TODO

Definition 1 (Domain). *For any $\hat{\Gamma}$, define $\text{dom}(\hat{\Gamma})$ as follows:*

- $\text{dom}(\emptyset) = \emptyset$
- $\text{dom}(\hat{\Gamma}, x : \hat{\tau}) = \text{dom}(\hat{\Gamma}) \cup \{x\}$
- $\text{dom}(\hat{\Gamma}, X <: \hat{\tau}) = \text{dom}(\hat{\Gamma}) \cup \{X\}$
- $\text{dom}(\hat{\Gamma}, \Phi \subseteq \varepsilon) = \text{dom}(\hat{\Gamma}) \cup \{\Phi\}$

Lemma 8 (Reverse Narrowing 1). *If $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$ and $\Phi \notin \text{dom}(\hat{\Gamma})$ then $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$.

S-Reflex. Then $\varepsilon_1 = \varepsilon_2$. By S-REFLEX, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_1$.

S-Trans. By inversion and inductive assumption, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2, \varepsilon_2 \subseteq \varepsilon_3$. By S-TRANS, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_3$.

S-FxSet. Concrete sets of effects are invariant under substitution, so $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$.

S-FxVar. Then $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon_2 \vdash [\varepsilon/X](\Phi_2 \subseteq \varepsilon_2)$. Because $\Phi \notin \text{dom}(\hat{\Gamma})$, then $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon_2, \Phi \subseteq \varepsilon \vdash \Phi_2 \subseteq \varepsilon_2$ by S-FXVAR.

Lemma 9 (Reverse Narrowing 2). *If $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$ and $\Phi \notin \text{dom}(\hat{\Gamma})$ then $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$.

S-Reflex. Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_1)$, and $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_1$ by S-REFLEX.

S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash [\varepsilon/\Phi](X <: \hat{\tau})$. Because Φ is an effect-variable, and not a type-variable, then $\hat{\Gamma}, X <: \hat{\tau}, \Phi \subseteq \varepsilon \vdash X <: \hat{\tau}$ by S-TYPEVAR.

S-ResourceSet. By S-RESOURCESET, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \{\bar{r}_1\} <: \{\bar{r}_2\}$.

S-Trans. Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_3)$. By inversion and induction, we have $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2, \hat{\tau}_2 <: \hat{\tau}_3$. Then by S-TRANS, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_3$.

S-Arrow. Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}'_1 \rightarrow_{\varepsilon'_3} \hat{\tau}'_2)$. By inversion, we know (1-3):

1. $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}'_1 <: \hat{\tau}_1)$
2. $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}'_2 <: \hat{\tau}'_2)$
3. $\hat{I} \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon'_3)$

By applying the inductive assumption to (1-2) we get (4-5). By applying Reverse Narrowing 1 to 3, we get 6.

4. $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}'_1 <: \hat{\tau}_1$
5. $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}'_2 <: \hat{\tau}'_2$
6. $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon'_3$

From (4-6), we can use S-ARROW to get the judgement $\hat{I}, \Phi \subseteq \varepsilon \vdash (\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}'_1 \rightarrow_{\varepsilon'_3} \hat{\tau}'_2)$.

S-PolyType. Then $\hat{I} \vdash [\varepsilon/\Phi](\forall X <: \hat{\tau}_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$. By inversion, we know (1-3):

1. $\hat{I} \vdash [\varepsilon/\Phi](\hat{\tau}'_1 <: \hat{\tau})$
2. $\hat{I}, Y <: \hat{\tau}'_1 \vdash [\varepsilon/\Phi](\hat{\tau}_2 <: \hat{\tau}'_2)$
3. $\hat{I}, Y <: \hat{\tau}'_1 \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon'_3)$

By applying the inductive assumption to (1-3), we get (4-6).

4. $\hat{I}, \Phi \subseteq \varepsilon \vdash \hat{\tau}'_1 <: \hat{\tau}$
5. $\hat{I}, Y <: \hat{\tau}'_1, \Phi \subseteq \varepsilon \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
6. $\hat{I}, Y <: \hat{\tau}'_1, \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon'_3$

5 can be rewritten as 7, and 6 as 8.

7. $\hat{I}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}'_1 \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
8. $\hat{I}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}'_1 \vdash \varepsilon_3 \subseteq \varepsilon'_3$

From (4,7,8) we can apply S-POLYTYPE to get $\hat{I}, \Phi \subseteq \varepsilon \vdash (\forall X <: \hat{\tau}_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$.

S-PolyFx. Then $\hat{I} \vdash [\varepsilon/\Phi](\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall \Phi'_1 \subseteq \varepsilon'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$. By inversion we know (1-3):

1. $\hat{I} \vdash [\varepsilon/\Phi](\varepsilon'_1 \subseteq \varepsilon_1)$
2. $\hat{I}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}'_1)$
3. $\hat{I}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon'_3)$

By applying the Reverse Narrowing Lemma 1 to (1), we get (3). By applying the inductive assumption to (2-3), we get (5-6).

4. $\hat{I}, \Phi \subseteq \varepsilon \vdash \varepsilon'_1 \subseteq \varepsilon_1$
5. $\hat{I}, \Phi_2 \subseteq \varepsilon', \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}'_1$
6. $\hat{I}, \Phi_2 \subseteq \varepsilon', \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon'_3$

5 can be rewritten as 7, and 6 as 8.

7. $\hat{I}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \hat{\tau}_1 <: \hat{\tau}'_1$
8. $\hat{I}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \varepsilon_3 \subseteq \varepsilon'_3$

With (4,7,8), we can apply S-POLYFX to get $\hat{I}, \Phi \subseteq \varepsilon \vdash (\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \text{ caps } \varepsilon_3) <: (\forall \Phi'_1 \subseteq \varepsilon'_1.\hat{\tau}'_2 \text{ caps } \varepsilon'_3)$.

Lemma 10 (Approximation 1). *If $\hat{I} \vdash \hat{e} : \hat{\tau}$ with ε and $\text{effects}(\hat{\tau}) \subseteq \varepsilon_s$ and $\text{ho-safe}(\hat{\tau}, \varepsilon_s)$ then $\hat{\tau} <: \text{annot}(\text{erase}(\hat{\tau}), \varepsilon_s)$.*

Lemma 11 (Approximation 2). *If $\hat{I} \vdash \hat{e} : \hat{\tau}$ with ε and $\text{ho-effects}(\hat{\tau}) \subseteq \varepsilon_s$ and $\text{safe}(\hat{\tau}, \varepsilon_s)$ then $\text{annot}(\text{erase}(\hat{\tau}), \varepsilon_s) <: \hat{\tau}$.*

Proof. By simultaneous induction on derivations of **safe** and **ho-safe**, and then on derivations of $\hat{I} \vdash \hat{e} : \hat{\tau}$ with ε .

ε -POLYFXABS. Then \hat{e} has the form given in (1). Suppose $\text{ho-safe}(\hat{\tau}, \varepsilon)$. By inversion on **ho-safe** we know (2-3). By inversion on ε -POLYFXABS, we also know (4). By definition of **effects**, we have (5).

1. $\hat{e} = \lambda\Phi \subseteq \varepsilon_1. \hat{e}_2$
2. $\hat{I} \vdash \varepsilon_1 \subseteq \varepsilon_s$
3. $\text{ho-safe}([\emptyset/\Phi]\hat{\tau}_2, \varepsilon_s)$
4. $\hat{I}, \Phi \subseteq \varepsilon_1 \vdash \hat{e}_2 : \hat{\tau}_2$ with $\hat{\tau}_2$ with ε_2
5. $\text{effects}(\hat{\tau}) = \varepsilon_2 \cup \text{effects}([\emptyset/\Phi]\hat{\tau}_2)$

To show $\hat{\tau} <: \text{annot}(\text{erase}(\hat{\tau}), \varepsilon)$, it is sufficient to prove that $\hat{I}, \Phi \subseteq \varepsilon_1 \vdash \hat{\tau}_2 <: \text{annot}(\text{erase}(\hat{\tau}_2), \varepsilon)$. From (3, 5) we can apply the inductive hypothesis, giving (6). Because $\text{effects}([\emptyset/\Phi]\hat{\tau}_2) \subseteq$ and $\text{ho-safe}([\emptyset/\Phi]\hat{\tau}_2, \varepsilon)$ are true by inversion, then by inductive hypothesis we get (6). Because erasing and annotating deletes all type variables, this can be rewritten as (7).

6. $\hat{I} \vdash [\emptyset/\Phi]\hat{\tau}_2 <: \text{annot}(\text{erase}([\emptyset/\Phi]\hat{\tau}_2), \varepsilon)$.
7. $\hat{I} \vdash [\emptyset/\Phi](\hat{\tau}_2 <: \text{annot}(\text{erase}([\emptyset/\Phi]\hat{\tau}_2), \varepsilon))$

Because of the convention of α -conversion, at the point where a binding for Φ is introduced, Φ will be a never-before-seen variable name, so $\Phi \notin \text{dom}(\hat{I})$. By applying the Reverse Narrowing Lemma, we obtain (8). By widening the bound on Φ , we get (9). Then by an application of S-POLYFXABS, we get the theorem conclusion.

1. $\hat{I}, \Phi \subseteq \emptyset \vdash \hat{\tau}_2 <: \text{annot}(\text{erase}([\emptyset/\Phi]\hat{\tau}_2), \varepsilon)$
2. $\hat{I}, \Phi \subseteq \varepsilon_1 \vdash \hat{\tau}_2 <: \text{annot}(\text{erase}([\emptyset/\Phi]\hat{\tau}_2), \varepsilon)$

Theorem 1 (Progress). *If $\hat{I} \vdash \hat{e} : \hat{\tau}$ with ε and \hat{e} is not a value, then $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$, for some \hat{e}', ε .*

Proof. By induction on the derivation of $\hat{I} \vdash \hat{e} : \hat{\tau}$ with ε .

Case: ε -POLYTYPEABS. Trivial; \hat{e} is a value.

Case: ε -POLYFXABS. Trivial; \hat{e} is a value.

Case: ε -POLYTYPEAPP. Then $\hat{e} = \hat{e}_1 \hat{\tau}'$. If \hat{e}_1 is not a value then $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$ by inductive hypothesis, and applying E-POLYTYPEAPP1 gives the reduction $\hat{e}_1 \hat{\tau}' \longrightarrow \hat{e}''_1 \hat{\tau}' \mid \varepsilon$. Otherwise, \hat{e}_1 is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1. \hat{e}_2$, and applying E-POLYTYPEAPP2 gives the reduction $(\lambda X <: \hat{\tau}_1. \hat{e}_2) \hat{\tau}' \longrightarrow [\hat{\tau}'/X] \hat{e}_2 \mid \emptyset$.

Case: ε -POLYFXAPP. Then $\hat{e} = \hat{e}_1 \varepsilon'$. If \hat{e}_1 is not a value then $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$ by inductive hypothesis, and applying E-POLYFXAPP1 gives the reduction $\hat{e}_1 \varepsilon' \longrightarrow \hat{e}'_1 \varepsilon' \mid \varepsilon$. Otherwise, \hat{e}_1 is a value, so $\hat{e} = \lambda\phi \subseteq \varepsilon_1. \hat{e}_2$, and applying E-POLYFXAPP2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1. \hat{e}_2) \varepsilon' \longrightarrow [\varepsilon'/\phi] \hat{e}_2$.

Theorem 2 (Preservation). *If $\hat{I} \vdash \hat{e}_A : \hat{\tau}_A$ with ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$, then $\hat{I} \vdash \hat{e}_B : \hat{\tau}_B$ with ε_B , where $\hat{\tau}_B <: \hat{\tau}_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$.*

Proof. By induction on the derivations of $\hat{I} \vdash \hat{e}_A : \hat{\tau}_A$ with ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

ε -POLYTYPEABS. Trivial; \hat{e} is a value.

ε -POLYFXABS. Trivial; \hat{e} is a value.

ε -POLYTYPEAPP. Then $\hat{e} = \hat{e}_1 \hat{\tau}'$. The typing rule from the judgement can be rewritten as (1). From inversion, we also have (2) and (3).

1. $\hat{I} \vdash \hat{e}_1 \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2 \text{ with } \varepsilon_1 \cup \varepsilon_2$
2. $\hat{I} \vdash \hat{e}_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2 \text{ caps } \varepsilon_1 \text{ with } \varepsilon_2$
3. $\hat{I} \vdash \hat{\tau}' <: \hat{\tau}_1$

Now consider which reduction rule was used.

Subcase: E-POLYTYPEAPP1. Then $\hat{e}_1 \hat{\tau}' \longrightarrow \hat{e}'_1 \hat{\tau}' \mid \varepsilon$. By inversion on the reduction rule, $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$. With (2), we can apply the inductive assumption and ε -SUBSUME to get (4). With (4) and (3), we can then apply ε -POLYTYPEAPP to get (5). Then by comparing (1) and (6), we see $\hat{\tau}_B = \hat{\tau}_A$ and $\hat{\varepsilon} = \hat{\varepsilon}_A = \hat{\varepsilon}_B$.

4. $\hat{I} \vdash \hat{e}'_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2 \text{ caps } \varepsilon_1 \text{ with } \varepsilon_2$
5. $\hat{I} \vdash \hat{e}'_1 \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2 \text{ with } \varepsilon_1 \cup \varepsilon_2$

Subcase: E-POLYTYPEAPP2. Then $(\lambda X <: \hat{\tau}_1.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \emptyset$. Because of the form of \hat{e}_1 in this subcase, the only rule which could have been applied to obtain judgement (2) is ε -TYPEABS. By inversion on this rule we get (4). From (4) and (3), we can apply the lemma that type-and-effect judgements are preserved under type variable substitution to obtain (5). Finally, by comparing (1) and (5) we see $\hat{\tau}_A = [\hat{\tau}'/X]\hat{\tau}_2 = \hat{\tau}_B$, and $\varepsilon_B \cup \varepsilon = \varepsilon_1 \subseteq \varepsilon_1 \cup \varepsilon_2 = \varepsilon_A$.

4. $\hat{I}, X <: \hat{\tau}_1 \vdash \hat{e}' : \hat{\tau}_2 \text{ with } \varepsilon_1$
5. $\hat{I} \vdash [\hat{\tau}'/X]\hat{e}' : [\hat{\tau}'/X]\hat{\tau}_2 \text{ with } \varepsilon_1$

Case: ε -POLYFXAPP. Then $\hat{e} = \hat{e}_1 \varepsilon'$. Consider which reduction rule was used.

Subcase: E-POLYFXAPP1. Then $\hat{e}_1 \varepsilon' \longrightarrow \hat{e}'_1 \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$. With the inductive hypothesis and subsumption, \hat{e}'_1 can be typed in \hat{I} the same as \hat{e}_1 . Then by ε -POLYFXAPP, $\hat{I} \vdash \hat{e}'_1 \varepsilon' : \hat{\tau}_A \text{ with } \varepsilon_A$. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-POLYFXAPP2. Then $(\lambda \phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \emptyset$. **The result follows by the substitution lemma.**