Notation: $\hat{\Gamma} \vdash \delta_1, ..., \delta_n$ means $\hat{\Gamma} \vdash \delta_1$ and $\hat{\Gamma} \vdash \delta_2$ and ... and $\hat{\Gamma} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Substitution (Values)). *If* $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$ *and* $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}'\ \texttt{with}\ \varnothing$*, then* $\hat{\Gamma} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$

Proof. By induction on the derivation of $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$. We show for those extra cases in polymorphic CC.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1.[\hat{v}/y]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda\phi \subseteq \varepsilon_1.[\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{e} = \hat{e}_1\ \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeApp, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{e} = \hat{e}_1\ \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxApp, we get the conclusion.

Lemma 2 (Type Substitution Preserves Subsetting). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-FxSet.}}$ Trivial.

$\boxed{\textit{Case: } \varepsilon\text{-FxVar.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \phi \subseteq \varepsilon_2$, and either (1) $\phi \subseteq \varepsilon_2 \in \hat{\Gamma}$ or (2) $\phi \subseteq \varepsilon_2 \in \hat{\Delta}$. If (1) then $\hat{\Gamma} \vdash \phi \subseteq \varepsilon_2$, so by widening $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$. Otherwise (2), in which case $\phi \subseteq \varepsilon_2 \in [\hat{\tau}'/X]\hat{\Delta}$ by the definition of type-variable substitution on a context, so $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \phi \subseteq \varepsilon_2$.

Lemma 3 (Type Substitution Preserves Subtyping). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

$\boxed{\textit{Case: } \text{S-Reflexive.}}$ Then $\hat{\tau}_1 = \hat{\tau}_2$, so $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ by S-Reflexive. Then by widening, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

$\boxed{\textit{Case: } \text{S-Transitive.}}$ Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption to these judgements, we get $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$ and $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$.

$\boxed{\textit{Case: } \text{S-ResourceSet.}}$ Sets of resources are unchanged by type-variable substitution, so $[\hat{\tau}'/X]\{\overline{r_1}\} = \{\overline{r_1}\}$ and $[\hat{\tau}'/X]\{\overline{r_2}\} = \{\overline{r_2}\}$. Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

$\boxed{\textit{Case: } \text{S-Arrow.}}$ Then the subtyping judgement from the assumption is $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_\varepsilon \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B$. By inversion we have judgements (1-3),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
3. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

By applying the inductive hypothesis to (1) and (2), we get (4) and (5),

4. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$
5. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_B$

By inspection, type-variable bindings do not affect judgements of the form $\hat{\Gamma} \vdash \varepsilon \subseteq \varepsilon$. Furthermore, the types in a context do not affect judgements of this form. Therefore, we can rewrite (3) as (6),

7. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

From (4-6), we may apply S-Arrow to get $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A \rightarrow_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \rightarrow_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$. By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}'/X](\hat{\tau}_A \rightarrow_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B)$, which is the same as $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TypePoly. Then $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ and $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A.\hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}'_A$. Applying the inductive assumption to both these judgements, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta}, Z <: [\hat{\tau}'/X]\hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}'_A$. Then by S-TypePoly, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash (\forall Y <: [\hat{\tau}'/X]\hat{\tau}_A.[\hat{\tau}'/X]\hat{\tau}_B) <: (\forall Z <: [\hat{\tau}'/X]\hat{\tau}'_A.[\hat{\tau}'/X]\hat{\tau}'_B)$, which is the same as $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash X <: \hat{\tau}$. We want to show (1) $\hat{\Gamma}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]X <: [\hat{\tau}'/X]\hat{\tau}$. Firstly, $[\hat{\tau}'/X]X = \hat{\tau}'$. Secondly, because $\texttt{WF}(\hat{\Gamma}, X <: \hat{\tau})$ then $X \notin \texttt{free-vars}(\hat{\tau})$, so $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}$. Therefore, judgement (1) is the same as $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}' <: \hat{\tau}$, which is true by assumption.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation, so $\hat{\Gamma}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$ is true by widening the context in the judgement $\hat{\Gamma} \vdash Y <: \hat{\tau}_2$[1]. Then $\hat{\Gamma} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$ by inductive assumption. By widening, $\hat{\Gamma}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$.

Lemma 4 (Type Substitution Preserves Typing). *If* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε *and* $\hat{\Gamma} \vdash \hat{\tau}'' <: \hat{\tau}'$, *then* $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$ $\texttt{with}$ ε

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε.

Case: ε-Var, ε-Resource. Then $\hat{e} = [\hat{\tau}''/X]\hat{e}$, so the typing judgement in the consequent can be the one from the antecedent.

Case: ε-OperCall. Then $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \bar{r}\}$. By inversion we have (1). Noting that $[\hat{\tau}''/X]\{\bar{r}\} = \{\bar{r}\}$, we can apply the inductive hypothesis to get (2),

1. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\bar{r}\}$ $\texttt{with}$ ε_1
2. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : \{\bar{r}\}$ $\texttt{with}$ ε_1

Then from (2), we can apply ε-OperCall to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1.\pi) : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \bar{r}\}$. Since $[\hat{\tau}''/X]\texttt{Unit} = \texttt{Unit}$, we're done.

Case: ε-Subsume. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ $\texttt{with}$ ε. By inversion, (1) and (2) are true.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_2 <: \hat{\tau}$

[1] Note there is no explicit widening rule; be careful with this one.

2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
3. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_2$ `with` ε_2

By a previous lemma, type substitution preserves subtyping. Applying this to (1) yields (4). On the other hand, only effect-variable bindings in a context will affect subsetting judgements. Based on this, we can delete the binding $X <: \hat{\tau}$ and perform the substitution $[\hat{\tau}''/X]\hat{\Delta}$, neither of which will change any effect-variable bindings, and in doing so obtain judgement (5). Lastly, we can apply the inductive hypothesis to (3), obtaining (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}_2 <: [\hat{\tau}''/X]\hat{\tau}$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
7. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_2$ `with` ε_2

From (4-6) we can apply ε-Subsume to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$ `with` ε_2.

Case: ε-Abs. Then $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda y : \hat{\tau}_2.\hat{e}_3 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` $\varnothing$. By inversion, we have (1). By setting $\hat{\Delta}' = \hat{\Delta}, y : \hat{\tau}_2$, this can be rewritten as (2). From inductive hypothesis we get (3). Then by simplifying $\hat{\Delta}'$, this simplifies to (4).

1. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, y : \hat{\tau}_2 \vdash \hat{e}_3 : \hat{\tau}_3$ `with` ε_3
2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_3 : \hat{\tau}_3$ `with` ε_3
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3$ `with` ε_3
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, y : [\hat{\tau}''/X]\hat{\tau}_2 \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3$ `with` ε_3

From (4) we can apply ε-Abs to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda y : [\hat{\tau}''/X]\hat{\tau}_2.[\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_2 \rightarrow_{\varepsilon_3} [\hat{\tau}''/X]\hat{\tau}_3$ `with` $\varnothing$. This can be rewritten as $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda y : \hat{\tau}_2.\hat{e}_3) : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3)$ `with` $\varnothing$.

Case: ε-App. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$. By inversion, we have:

1. $\hat{\Gamma}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` ε_1
2. $\hat{\Gamma}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Applying inductive hypothesis to (1) and (2) gives (3) and (4),

3. $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3)$ `with` ε_1
4. $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_2 : [\hat{\tau}''/X]\hat{\tau}_2$ `with` ε_2

Then from (3) and (4) we can apply ε-App to get $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \hat{e}_2) : [\hat{\tau}''/X]\hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

Case: ε-PolyTypeAbs, Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A$ `cap` ε_A `with` $\varnothing$. By inversion, we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}_B$, we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite this as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_B \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A$ `with` ε_A
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, Y <: [\hat{\tau}''/X]\hat{\tau}_B \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A$ `with` ε_A

From (4) we can apply ε-PolyTypeAbs, giving (5), which can be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{e}_A : \forall Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{\tau}_A$ `cap` ε_A `with` $\varnothing$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\Delta \vdash [\hat{\tau}''/X](\lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A$ `cap` $\varepsilon_A)$ `with` $\varnothing$

Case: ε-PolyFxAbs. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon_A.\hat{e}_B : \forall \phi \subseteq \varepsilon_A.\hat{\tau}_B$ `cap` ε_B `with` $\varnothing$. By inversion we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon_A$, this can be rewritten as (2). The inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite that as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_B : [\hat{\tau}''/X]\hat{\tau}_B$ `with` ε_B
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, \phi \subseteq \varepsilon_A \vdash [\hat{\tau}''/X]\hat{e}_B : [\hat{\tau}''/X]\hat{\tau}_B$ `with` ε_B

From (4) we can apply ε-PolyFxAbs, giving (5), whichc an be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon_A.[\hat{\tau}''/X]\hat{e}_B : \forall\phi \subseteq \varepsilon_A.[\hat{\tau}''/X]\hat{\tau}_B$ `cap` ε_B `with` $\varnothing$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda\phi \subseteq \varepsilon_A.\hat{e}_B) : [\hat{\tau}''/X](\forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `cap` $\varepsilon_B)$ `with` $\varnothing$

Case: ε-PolyTypeApp. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{\tau}'_A : [\hat{\tau}'_A/Y]\hat{\tau}_B$ `with` $[\hat{\tau}'_A/Y]\varepsilon_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ `caps` ε_B `with` ε_C
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

By inductive hypothesis on (1) we get (3). By a previous lemma, type substitution preserves subtyping, so from (2) we obtain (4).

3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\forall Y <: \hat{\tau}_A.\hat{\tau}_B$ `caps` $\varepsilon_B)$ `with` ε_C
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}'_A <: [\hat{\tau}''/X]\hat{\tau}_A$

From (3-4), applying ε-PolyTypeApp gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \hat{\tau}'_A) : [\hat{\tau}''/X]([\hat{\tau}'_A/Y]\hat{\tau}_B)$ `with` $[\hat{\tau}'_A/Y]\varepsilon_B \cup \varepsilon_C$

Case: ε-PolyFxApp Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \varepsilon'_A : [\varepsilon'_A/\phi]\hat{\tau}_B$ `with` $[\varepsilon'_A/\phi]\hat{\varepsilon}_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `caps` ε_B `with` ε_C
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

By inductive hypothesis on (1) we get (3). Applying the lemma that type substitution preserves subsetting, we obtain (4) from (2).

3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `caps` $\varepsilon_B)$ `with` ε_C
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

From (3-4), applying ε-PolyFxApp gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \varepsilon'_A) : [\hat{\tau}''/X]([\varepsilon'_A/\phi]\hat{\tau}_B)$ `with` $[\varepsilon'_A/\phi]\hat{\varepsilon}_B \cup \varepsilon_C$

Case: ε-Import TODO

Lemma 5 (Effect Substitution Preserves Subsetting). *If* $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ *then* $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\varepsilon_1 \subseteq [\varepsilon''/\phi]\varepsilon_2$

Proof. By induction on the derivation of $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

ε-FxSet. By ε-FxSet, $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$. Because ε_1 and ε_2 are concrete sets of effects, then $[\varepsilon''/\phi]\varepsilon_1 = \varepsilon_1$ and $[\varepsilon''/\phi]\varepsilon_2 = \varepsilon_2$, so we are done.

ε-FxVar. Then $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \Phi \subseteq \varepsilon''$. We know that $\Phi \subseteq \varepsilon''$ occurs in the context somewhere, so consider case-by-case which part.

Subcase: $\Phi = \phi$. Then $[\varepsilon''/\phi]\varepsilon_1 = \varepsilon''$. By well-formedness, $\phi \notin$ `freevars`(ε_2), so $[\varepsilon''/\phi]\varepsilon_2 = \varepsilon_2$. By inversion on the rule, $\varepsilon_2 = \varepsilon'$. We already know by assumption that $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$, so by widening, $\hat{\Gamma}, [\varepsilon''/X]\hat{\Delta} \vdash \varepsilon'' \subseteq \varepsilon'$.

Lemma 6 (Effect Substitution Preserves Subtyping). *If* $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \varepsilon'' \subseteq \varepsilon'$ *then* $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$

Proof. By induction on derivations of $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

$\boxed{\text{S-REFLEXIVE.}}$ Use S-REFLEXIVE to get the desired judgement directly.

$\boxed{\text{S-TRANSITIVE.}}$ By inversion we have (1) and (2). Applying the inductive assumption to these yields (3) and (4), which can be used to apply S-TRANSITIVE, giving judgement (5).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_C$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_C <: \hat{\tau}_2$
3. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_C$
4. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_C <: [\varepsilon''/\phi]\hat{\tau}_2$
5. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_1 <: [\varepsilon''/\phi]\hat{\tau}_2$

$\boxed{\text{S-RESOURCESET.}}$ Substitution on a resource set leaves it unchanged, so the judgement in the antecedent can be used for the judgement in the consequent.

$\boxed{\text{S-ARROW.}}$ Then we have (1). By inversion, we also have (2-4).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_A \to_{\varepsilon_C} \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon'_C} \hat{\tau}'_B$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
3. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
4. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \varepsilon_C \subseteq \varepsilon'_C$

Applying the inductve assumption to (2) and (3) yields (5) and (6). By a previous lemma, we know that effect substitution preserves subsetting. Applying this lemma to (4) yields (7).

5. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_A <: [\varepsilon''/\phi]\hat{\tau}_A$
6. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_B$
7. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\varepsilon_C \subseteq [\varepsilon''/\phi]\varepsilon'_C$

With (5-7) we can apply S-ARROW, giving (8), which is the same as (9).

8. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}_A \to_{[\varepsilon''/\phi]\varepsilon'_C} [\varepsilon''/\phi]\hat{\tau}_B <: [\varepsilon''/\phi]\hat{\tau}'_A \to_{[\varepsilon''/\phi]\varepsilon_C} [\varepsilon''/\phi]\hat{\tau}'_B$
9. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\hat{\tau}_A \to_{\varepsilon_C} \hat{\tau}_B) <: [\varepsilon''/\phi](\hat{\tau}'_A \to_{\varepsilon'_C} \hat{\tau}'_B)$

$\boxed{\text{S-TYPEPOLY.}}$ Then we have (1). By inversion, we also have (2-3).

1. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash (\forall X <: \hat{\tau}_1.\hat{\tau}_2) <: (\forall Y <: \hat{\tau}'_1.\hat{\tau}'_2)$
2. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{\tau}'_1 <: \hat{\tau}_1$
3. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}, Y <: \hat{\tau}'_1 \vdash \hat{\tau}_2 <: \hat{\tau}'_2$

By applying the inductive hypothesis to (2), we obtain (4).

4. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]\hat{\tau}'_1 <: [\varepsilon''/\phi]\hat{\tau}_1$

Now, let $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}'_1$. Then we can rewrite (3) as (5), and apply the inductive assumption to get (6). By simplifying $\hat{\Delta}'$, we get (7).

5. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash \hat{\tau}_2 <: \hat{\tau}'_2$
6. $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta}' \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$
7. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta}, Y <: [\varepsilon''/\phi]\hat{\tau}'_1 \vdash [\varepsilon''/\phi]\hat{\tau}_2 <: [\varepsilon''/\phi]\hat{\tau}'_2$

From (2) and (7) we can apply S-TYPEPOLY to get (8), which can be rewritten as the more readable (9).

8. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash (\forall X <: [\varepsilon''/\phi]\hat{\tau}_1.[\varepsilon''/\phi]\hat{\tau}_2) <: (\forall Y <: [\varepsilon''/\phi]\hat{\tau}_1'.[\varepsilon''/\phi]\hat{\tau}_2')$
9. $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi](\forall X <: \hat{\tau}_1.\hat{\tau}_2) <: [\varepsilon''/\phi](\forall Y <: \hat{\tau}_1'.\hat{\tau}_2')$

$\boxed{\text{S-TypeVar.}}$ Then $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash X <: \hat{\tau}$. By inversion, there is a binding $X <: \hat{\tau}$ in the context, so consider case-by-case where it is.

Subcase: $X <: \hat{\tau} \in \hat{\Delta}$. Then $X <: [\varepsilon''/\phi]\hat{\tau} \in [\varepsilon''/\phi]\hat{\Delta}$, so $[\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$. By widening, $\hat{\Gamma}, [\varepsilon''/\phi]\hat{\Delta} \vdash [\varepsilon''/\phi]X <: [\varepsilon''/\phi]\hat{\tau}$.

Subcase: $X <: \hat{\tau} \in \hat{\Gamma}$. TODO

Lemma 7 (Effect Substitution Preserves Types and Effects). *If* $\hat{\Gamma}, \phi \subseteq \varepsilon', \hat{\Delta} \vdash \hat{e}_A : \hat{\tau}_A \texttt{ with } \varepsilon_A$ *and* $\hat{\Gamma} \vdash \phi \subseteq \varepsilon'$ *then* $\hat{\Gamma}, [\varepsilon'/\phi]\hat{\Delta} \vdash [\varepsilon'/\phi]e : [\varepsilon'/\phi]\hat{\tau} \texttt{ with } [\varepsilon'/\phi]\varepsilon$

Proof. TODO

Definition 1 (Domain). *For any* $\hat{\Gamma}$*, define* $\texttt{dom}(\hat{\Gamma})$ *as follows:*

- $\texttt{dom}(\varnothing) = \varnothing$
- $\texttt{dom}(\hat{\Gamma}, x : \hat{\tau}) = \texttt{dom}(\hat{\Gamma}) \cup \{x\}$
- $\texttt{dom}(\hat{\Gamma}, X <: \hat{\tau}) = \texttt{dom}(\hat{\Gamma}) \cup \{X\}$
- $\texttt{dom}(\hat{\Gamma}, \Phi \subseteq \varepsilon) = \texttt{dom}(\hat{\Gamma}) \cup \{\Phi\}$

Lemma 8 (Reverse Narrowing 1). *If* $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$ *and* $\Phi \notin \texttt{dom}(\hat{\Gamma})$ *then* $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$*.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1 \subseteq \varepsilon_2)$.

$\boxed{\text{S-Reflex.}}$ Then $\varepsilon_1 = \varepsilon_2$. By S-Reflex, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_1$.

$\boxed{\text{S-Trans.}}$ By inversion and inductive assumption, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2, \varepsilon_2 \subseteq \varepsilon_3$. By S-Trans, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_3$.

$\boxed{\text{S-FxSet.}}$ Concrete sets of effects are invariant under substitution, so $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1 \subseteq \varepsilon_2$.

$\boxed{\text{S-FxVar.}}$ Then $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon_2 \vdash [\varepsilon/X](\Phi_2 \subseteq \varepsilon_2)$. Because $\Phi \notin \texttt{dom}(\hat{\Gamma})$, then $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon_2, \Phi \subseteq \varepsilon \vdash \Phi_2 \subseteq \varepsilon_2$ by S-FxVar.

Lemma 9 (Reverse Narrowing 2). *If* $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$ *and* $\Phi \notin \texttt{dom}(\hat{\Gamma})$ *then* $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2$*.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_2)$.

$\boxed{\text{S-Reflex.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_1)$, and $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_1$ by S-Reflex.

$\boxed{\text{S-TypeVar.}}$ Then $\hat{\Gamma}, X <: \hat{\tau} \vdash [\varepsilon/\Phi](X <: \hat{\tau})$. Because Φ is an effect-variable, and not a type-variable, then $\hat{\Gamma}, X <: \hat{\tau}, \Phi \subseteq \varepsilon \vdash X <: \hat{\tau}$ byS-TypeVar.

$\boxed{\text{S-ResourceSet.}}$ By S-ResourceSet, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$.

$\boxed{\text{S-Trans.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_3)$. By inversion and induction, we have $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_2, \hat{\tau}_2 <: \hat{\tau}_3$. Then by S-Trans, $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_3$.

$\boxed{\text{S-Arrow.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi]((\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}_1' \rightarrow_{\varepsilon_3'} \hat{\tau}_2'))$. By inversion, we know (1-3):

1. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1' <: \hat{\tau}_1)$
2. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_2 <: \hat{\tau}_2')$
3. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon_3')$

By applying the inductive assumption to (1-2) we get (4-5). By applying Reverse Narrowing 1 to 3, we get 6.

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1' <: \hat{\tau}_1$
5. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_2 <: \hat{\tau}_2'$
6. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon_3'$

From (4-6), we can use S-Arrow to get the judgement $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash (\hat{\tau}_1 \rightarrow_{\varepsilon_3} \hat{\tau}_2) <: (\hat{\tau}_1' \rightarrow_{\varepsilon_3'} \hat{\tau}_2')$.

$\boxed{\text{S-PolyType.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi]((\forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3'))$. By inversion, we know (1-3):

1. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\hat{\tau}_1' <: \hat{\tau})$
2. $\hat{\Gamma}, Y <: \hat{\tau}_1' \vdash [\varepsilon/\Phi](\hat{\tau}_2 <: \hat{\tau}_2')$
3. $\hat{\Gamma}, Y <: \hat{\tau}_1' \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon_3')$

By applying the inductive assumption to (1-3), we get (4-6).

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \hat{\tau}_1' <: \hat{\tau}$
5. $\hat{\Gamma}, Y <: \hat{\tau}_1', \Phi \subseteq \varepsilon \vdash \hat{\tau}_2 <: \hat{\tau}_2'$
6. $\hat{\Gamma}, Y <: \hat{\tau}_1', \Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon_3'$

5 can be rewritten as 7, and 6 as 8.

7. $\hat{\Gamma}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}_1' \vdash \hat{\tau}_2 <: \hat{\tau}_2'$
8. $\hat{\Gamma}, \Phi \subseteq \varepsilon, Y <: \hat{\tau}_1' \vdash \varepsilon_3 \subseteq \varepsilon_3'$

From (4,7,8) we can apply S-PolyType to get $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash (\forall X <: \hat{\tau}_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall Y <: \hat{\tau}_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3')$.

$\boxed{\text{S-PolyFx.}}$ Then $\hat{\Gamma} \vdash [\varepsilon/\Phi]((\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall \Phi_1' \subseteq \varepsilon_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3'))$. By inversion we know (1-3):

1. $\hat{\Gamma} \vdash [\varepsilon/\Phi](\varepsilon_1' \subseteq \varepsilon_1)$
2. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\hat{\tau}_1 <: \hat{\tau}_1')$
3. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon' \vdash [\varepsilon/\Phi](\varepsilon_3 \subseteq \varepsilon_3')$

By applying the Reverse Narrowing Lemma 1 to (1), we get (3). By applying the inductive assumption to (2-3), we get (5-6).

4. $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash \varepsilon_1' \subseteq \varepsilon_1$
5. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon', \Phi \subseteq \varepsilon \vdash \hat{\tau}_1 <: \hat{\tau}_1'$
6. $\hat{\Gamma}, \Phi_2 \subseteq \varepsilon'.\Phi \subseteq \varepsilon \vdash \varepsilon_3 \subseteq \varepsilon_3'$

5 can be rewritten as 7, and 6 as 8.

7. $\hat{\Gamma}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \hat{\tau}_1 <: \hat{\tau}_1'$
8. $\hat{\Gamma}, \Phi \subseteq \varepsilon, \Phi_2 \subseteq \varepsilon' \vdash \varepsilon_3 \subseteq \varepsilon_3'$

With (4,7,8), we can apply S-PolyFx to get $\hat{\Gamma}, \Phi \subseteq \varepsilon \vdash (\forall \Phi_1 \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_3) <: (\forall \Phi_1' \subseteq \varepsilon_1'.\hat{\tau}_2' \texttt{ caps } \varepsilon_3')$.

Lemma 10. $[\varnothing/\Phi]\varepsilon \subseteq \varepsilon$.

Proof. If $\varepsilon \neq \Phi$ then $[\varnothing/\Phi]\varepsilon = \varepsilon$. Otherwise, $[\varnothing/\Phi]\varepsilon = \varnothing \subseteq \varepsilon$.

Lemma 11. *For any* $\hat{\tau}$, $\texttt{effects}([\varnothing/\Phi]\hat{\tau}) \subseteq \texttt{effects}(\hat{\tau})$ *and* $\texttt{ho-effects}([\varnothing/\Phi]\hat{\tau}) \subseteq \texttt{ho-effects}(\hat{\tau})$.

Proof. By simultaneous induction on the form of $\hat{\tau}$. First, consider $\texttt{effects}([\varnothing/\Phi]\hat{\tau}) \subseteq \texttt{effects}(\hat{\tau})$.

$\boxed{\hat{\tau} = \{\bar{r}\}.}$ Then $[\varnothing/\Phi]\hat{\tau} = \hat{\tau}$, so the result is trivial.

$\boxed{\hat{\tau} = \hat{\tau}_1 \to_{\varepsilon_3} \hat{\tau}_2.}$ Then $[\varnothing/\Phi]\hat{\tau} = [\varnothing/\Phi]\hat{\tau}_1 \to_{[\varnothing/\Phi]\varepsilon_3} [\varnothing/\Phi]\hat{\tau}_2$. By definition, $\texttt{effects}(\hat{\tau}) = \texttt{ho-effects}(\hat{\tau}_1) \cup \varepsilon_3 \cup \texttt{effects}(\hat{\tau}_2)$. By inductive hypothesis, we know $\texttt{ho-effects}([\varnothing/\Phi]\hat{\tau}_1) \subseteq \texttt{ho-effects}(\hat{\tau}_1)$ and $\texttt{effects}([\varnothing/\Phi]\hat{\tau}_2) \subseteq \texttt{effects}(\hat{\tau}_2)$. We also have $[\varnothing/\Phi]\varepsilon_3 \subseteq \varepsilon_3$ by the previous lemma.

$\boxed{\hat{\tau} = \forall\Phi \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2.}$ Then $\texttt{effects}(\hat{\tau}) = \varepsilon_2 \cup [\varnothing/\Phi]\hat{\tau}_2$. By the previous lemma, $[\varnothing/\Phi]\varepsilon_2 \subseteq \varepsilon_2$, and it is trivial that $\texttt{effects}([\varnothing/\Phi]\hat{\tau}_2) \subseteq \texttt{effects}([\varnothing/\Phi]\hat{\tau}_2)$.

$\boxed{\cdot}$

Now consider $\texttt{ho-effects}([\varnothing/\Phi]\hat{\tau}) \subseteq \texttt{ho-effects}(\hat{\tau})$.

$\boxed{\hat{\tau} = \{\bar{r}\}.}$ Same as above; trivial.

$\boxed{\hat{\tau} = \hat{\tau}_1 \to_{\varepsilon_3} \hat{\tau}_2.}$ Then $[\varnothing/\Phi]\hat{\tau} = [\varnothing/\Phi]\hat{\tau}_1 \to_{[\varnothing/\Phi]\varepsilon_3} [\varnothing/\Phi]\hat{\tau}_2$. By definition, $\texttt{ho-effects}(\hat{\tau}) = \texttt{effects}(\hat{\tau}_1) \cup \texttt{ho-effects}(\hat{\tau}_2)$. By inductive hypothesis, we know $\texttt{effects}([\varnothing/\Phi]\hat{\tau}_1) \subseteq \texttt{effects}(\hat{\tau}_1)$ and $\texttt{ho-effects}([\varnothing/\Phi]\hat{\tau}_2) \subseteq \texttt{ho-effects}(\hat{\tau}_2)$.

$\boxed{\hat{\tau} = \forall\Phi \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2.}$ Then $\texttt{ho-effects}(\hat{\tau}) = \varepsilon_1 \cup [\varnothing/\Phi]\hat{\tau}_2$. By the previous lemma, we know $[\varnothing/\Phi]\varepsilon_1 \subseteq \varepsilon_1$. It is trivial that $\texttt{ho-effects}([\varnothing/\Phi]\hat{\tau}_2) \subseteq \texttt{ho-effects}([\varnothing/\Phi]\hat{\tau}_2)$.

$\boxed{\cdot}$

Lemma 12. $\texttt{safe}([\varnothing/\Phi]\hat{\tau}, \varepsilon_s) \implies \texttt{safe}(\hat{\tau}, \varepsilon_s)$ *and* $\texttt{ho-safe}([\varnothing/\Phi]\hat{\tau}, \varepsilon_s) \implies \texttt{ho-safe}(\hat{\tau}, \varepsilon_s)$.

Proof. TODO

Lemma 13 (Approximation 1). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon$ *and* $\texttt{effects}(\hat{\tau}) \subseteq \varepsilon_s$ *and* $\texttt{ho-safe}(\hat{\tau}, \varepsilon_s)$ *then* $\hat{\tau} <: \texttt{annot}(\texttt{erase}(\hat{\tau}), \varepsilon_s)$.

Lemma 14 (Approximation 2). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon$ *and* $\texttt{ho-effects}(\hat{\tau}) \subseteq \varepsilon_s$ *and* $\texttt{safe}(\hat{\tau}, \varepsilon_s)$ *then* $\texttt{annot}(\texttt{erase}(\hat{\tau}), \varepsilon_s) <: \hat{\tau}$.

Proof. By simultaneous induction on derivations of $\texttt{safe}$ and $\texttt{ho-safe}$, and then on derivations of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon$.

$\boxed{\varepsilon\text{-PolyFxAbs.}}$ Then $\hat{e}$ has the form given in (1). The definition of $\texttt{effects}$ is (2). From a previous lemma, $\texttt{ho-safe}(\hat{\tau}_2, \varepsilon_s) \implies \texttt{ho-safe}([\varnothing/\Phi]\hat{\tau}_2, \varepsilon_s)$, so we know (3). With (2-3) we can apply the inductive hypothesis to $[\varnothing/\Phi]\hat{\tau}_2$, giving (4).

1. $\hat{e} = \forall\Phi \subseteq \varepsilon_1.\hat{\tau}_2 \texttt{ caps } \varepsilon_2$
2. $\texttt{effects}(\hat{\tau}) = \varepsilon_2 \cup \texttt{effects}([\varnothing/\Phi]\hat{\tau}_2) \subseteq \varepsilon_s$
3. $\texttt{ho-safe}([\varnothing/\Phi]\hat{\tau}_2, \varepsilon_s)$
4. $\hat{\Gamma} \vdash [\varnothing/\Phi]\hat{\tau}_2 <: \texttt{annot}(\texttt{erase}([\varnothing/\Phi]\hat{\tau}_2), \varepsilon_s)$

But we want $\hat{\tau}_2 <: \texttt{annot}(\texttt{erase}(\hat{\tau}_2), \varepsilon_s)$.

Theorem 1 (Progress). *If* $\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon$ *and* $\hat{e}$ *is not a value, then* $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$*, for some* $\hat{e}', \varepsilon$.

Proof. By induction on the derivation of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}''\hat{\tau}' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$.

Theorem 2 (Preservation). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$, then $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B, where $\hat{\tau}_B <: \hat{\tau}_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$.*

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Trivial; $\hat{e}$ is a value.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Trivial; $\hat{e}$ is a value.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. The typing rule from the judgement can be rewritten as (1). From inversion, we also have (2) and (3).

1. $\hat{\Gamma} \vdash \hat{e}_1\ \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2$ `with` $\varepsilon_1 \cup \varepsilon_2$
2. $\hat{\Gamma} \vdash \hat{e}_1 : \forall X <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
3. $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}_1$

Now consider which reduction rule was used.

Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion on the reduction rule, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With (2), we can apply the inductive assumption and ε-Subsume to get (4). With (4) and (3), we can then apply ε-PolyTypeApp to get (5). Then by comparing (1) and (6), we see $\hat{\tau}_B = \hat{\tau}_A$ and $\hat{\varepsilon} = \hat{\varepsilon}_A = \hat{\varepsilon}_B$.

4. $\hat{\Gamma} \vdash \hat{e}_1' : \forall X <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
5. $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : [\hat{\tau}'/X]\hat{\tau}_2$ `with` $\varepsilon_1 \cup \varepsilon_2$

Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_1.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$. Because of the form of $\hat{e}_1$ in this subcase, the only rule which could have been applied to obtain judgement (2) is ε-TypeAbs. By inversion on this rule we get (4). From (4) and (3), we can apply the lemma that type-and-effect judgements are preserved under type variable substitution to obtain (5). Finally, by comparing (1) and (5) we see $\hat{\tau}_A = [\hat{\tau}'/X]\hat{\tau}_2 = \hat{\tau}_B$, and $\varepsilon_B \cup \varepsilon = \varepsilon_1 \subseteq \varepsilon_1 \cup \varepsilon_2 = \varepsilon_A$.

4. $\hat{\Gamma}, X <: \hat{\tau}_1 \vdash \hat{e}' : \hat{\tau}_2$ `with` ε_1
5. $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{e}' : [\hat{\tau}'/X]\hat{\tau}_2$ `with` ε_1

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.

Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-PolyFxApp2. Then $(\lambda\phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.