Notation: $\hat{\Gamma} \vdash \delta_1, ..., \delta_n$ means $\hat{\Gamma} \vdash \delta_1$ and $\hat{\Gamma} \vdash \delta_2$ and ... and $\hat{\Gamma} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Narrowing 1 (Subtypes)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$. The tricky cases are S-TypePoly and S-TypeVar; the others follow by routine application of the inductive hypothesis to subderivations.

Case: S-Reflexive. Then $\hat{\tau}_1 = \hat{\tau}_2$, and $\hat{\tau}_1 <: \hat{\tau}_2$ holds in any context, including $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}$.

Case: S-Transitive. Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_C$. By inversion, there is some $\hat{\tau}_B$ such that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption, we get the judgements $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_C$, which is the same as $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

Case: S-ResourceSet. Follows immediately, since the premises of this rule have nothing to do with the context. That is, if $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$, then by inversion, $r \in \overline{r_1} \implies r \in \overline{r_2}$. Then by S-ResourceSet, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$.

Case: S-Arrow. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon} \hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$ and $\varepsilon' \subseteq \varepsilon$. To these first two judgements, apply the inductive assumption, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$. Then by S-Arrow, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon} \hat{\tau}'_B$.

Case: S-TypePoly. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$. By inversion, we have the following two judgements:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Using (1) and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis can be used to obtain (3).

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

Let $\Delta' = \Delta, Y <: \hat{\tau}'_A$. With this, and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, we shall apply the inductive hypothesis to obtain (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Expanding the definition of Δ', we get (5),

5. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

From (3) and (5), we can use S-TypePoly to obtain (6), which is the theorem conclusion.

6. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$

Case: S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash X <: \hat{\tau}$. It is also true that (1) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}'$, by use of S-TypeVar. The assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ can be widened to (2) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$. Then by (1) and (2), we can apply S-Transitive to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}$.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$, so the judgement can be strengthened to $\hat{\Gamma}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. Then the judgement can be weakened to $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash Y <: \hat{\tau}_B$.

Lemma 2 (Narrowing 2 (Effects)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$*, then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$*.*

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Lemma 3 (Narrowing 3 (Types)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A. ε-ABS, ε-POLYTYPEABS, ε-POLYTYPEAPP, ε-POLYFXABS, ε-POLYFXAPP are the tricky cases; they require the use of the inductive hypothesis in a slightly more tricky way. The other cases follow by routine induction.

$\boxed{\textit{Case: } \varepsilon\text{-VAR.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash x : \hat{\tau}_A$ `with` $\varnothing$, where $\hat{e} = x$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash x : \hat{\tau}_A$ `with` $\varnothing$. Then by weakening, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash x : \hat{\tau}_A$ `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-RESOURCE.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash r : \{\overline{r}\}$ `with` $\varnothing$, where $\hat{e} = r$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash r : \{r\}$ `with` $\varnothing$. Then by weakening, $\hat{\Gamma}, x <: \hat{\tau}', \hat{\Delta} \vdash r : \{r\}$ `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-OPERCALL.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1.\pi :$ `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$, and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\}$ `with` ε_1. To this second judgement we apply the inductive hypothesis, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\}$ `with` ε_1. With this new judgement, apply ε-OPERCALL to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi :$ `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$.

$\boxed{\textit{Case: } \varepsilon\text{-SUBSUME.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B, \varepsilon \subseteq \varepsilon'$. By applying Narrowing Lemma 1 to the first judgement, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau} <: \hat{\tau}'$. By applying the Narrowing Lemma for effects[1], $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$. With these two judgements, ε-SUBSUME can be used to obtain the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A.

$\boxed{\textit{Case: } \varepsilon\text{-ABS.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.\hat{e}_2 : \hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2$ `with` $\varnothing$, where $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2. By letting $\hat{\Delta}' = \hat{\Delta}, x : \hat{\tau}_1$, this second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2.

   Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, apply the inductive hypothesis to obtain (2),

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2.

   Using the definition of $\hat{\Delta}'$, this can be simplified,

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2.

   Then with (3) we can use ε-ABS to get (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.e_2 : \hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2$ `with` $\varnothing$

$\boxed{\textit{Case: } \varepsilon\text{-APP.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$, where the following judgements are true from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` ε_1
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

   By applying the inductive assumption to (1) and (2), we get (3) and (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` ε_1

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[1] This has yet to be proven

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Then by ε-App, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

Case: ε-PolyTypeAbs. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_2 `with` $\varnothing$. From inversion, we have $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2. By letting $\Delta' = \Delta, Y <: \hat{\tau}_1$, the second judgement can be rewritten,

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

By applying the inductive hypothesis to (1), we get judgement (2), which further simplifies to (3) by simplifying $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Then by ε-PolyTypeAbs, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_2 `with` $\varnothing$.

Case: ε-PolyFxAbs. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon.\hat{e}_1 : \forall \phi \subseteq \varepsilon.\hat{\tau}_1$ `caps` ε_1 `with` $\varnothing$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. By letting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon$, the second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1

Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis gives judgement (2), which further simplifies to (3) by expanding the definition of $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1

Then from (2), we can apply ε-PolyFxAbs, giving the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon.\hat{e}_1 : \forall \phi \subseteq \varepsilon.\hat{\tau}_1$ `caps` ε_1 `with` $\varnothing$.

Case: ε-PolyTypeApp. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2$ `with` $[\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$, where the following judgements are from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to get (3). With the same assumption and (2), we can apply Narrowing Lemma 1 (Subtypes) to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

From (3) and (4), ε-PolyTypeApp gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2$ `with` $[\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$.

Case: ε-PolyFxApp. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2$ `with` $[\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$, where the following are true by inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall \phi \subseteq \varepsilon.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to obtain (3). With the same assumption and (2), we can apply the Narrowing Lemma for Effect Judgements[2] to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall \phi \subseteq \varepsilon.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

[2] Doesn't actually exist yet

With (3) and (4) we can apply ε-PolyFxApp to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2$ $\texttt{with}$ $[\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-Import.}}$ (We prove for a single import). Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \texttt{import}(\varepsilon_s)\ x_1 = \hat{e}_1\ \texttt{in}\ e : \texttt{annot}\ (\tau, \varepsilon_s)\ \texttt{with}\ \varepsilon_s \cup \varepsilon_1$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$. By inductive hypothesis, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1\ \texttt{with}\ \varepsilon_1$. This, together with the other premises obtained by inversion, gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \texttt{import}(\varepsilon_s)\ x_1 = \hat{e}_1\ \texttt{in}\ e : \texttt{annot}\ (\tau, \varepsilon_s)\ \texttt{with}\ \varepsilon_s \cup \varepsilon_1$.

Lemma 4 (Substitution (Values)). *If $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$ and $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}'\ \texttt{with}\ \varnothing$, then $\hat{\Gamma} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$*

Proof. By induction on the derivation of $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}\ \texttt{with}\ \varepsilon$. We show for those extra cases in polymorphic CC.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1.[\hat{v}/y]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda\phi \subseteq \varepsilon_1.[\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{e} = \hat{e}_1\ \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeApp, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{e} = \hat{e}_1\ \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxApp, we get the conclusion.

Lemma 5 (Type Substitution Preserves Subtyping). *If $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ and $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ then $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$*

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

$\boxed{\textit{Case: } \text{S-Reflexive.}}$ Then $\hat{\tau}_1 = \hat{\tau}_2$, so $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ by S-Reflexive. Then by widening, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$

$\boxed{\textit{Case: } \text{S-Transitive.}}$ Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption to these judgements, we get $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$ and $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$.

$\boxed{\textit{Case: } \text{S-ResourceSet.}}$ Sets of resources are unchanged by type-variable substitution, so $[\hat{\tau}'/X]\{\overline{r_1}\} = \{\overline{r_1}\}$ and $[\hat{\tau}'/X]\{\overline{r_2}\} = \{\overline{r_2}\}$. Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

$\boxed{\textit{Case: } \text{S-Arrow.}}$ Then the subtyping judgement from the assumption is $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \to_\varepsilon \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B$. By inversion we have judgements (1-3),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$
3. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

By applying the inductive hypothesis to (1) and (2), we get (4) and (5),

4. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$

5. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_B$

By inspection, type-variable bindings do not affect judgements of the form $\hat{\Gamma} \vdash \varepsilon \subseteq \varepsilon$. Furthermore, the types in a context do not affect judgements of this form. Therefore, we can rewrite (3) as (6),

7. $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$

From (4-6), we may apply S-Arrow to get $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_A \rightarrow_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \rightarrow_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$. By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}'/X](\hat{\tau}_A \rightarrow_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \rightarrow_{\varepsilon'} \hat{\tau}'_B)$, which is the same as $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TypePoly. Then $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ and $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A.\hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}'_A$. Applying the inductive assumption to both these judgements, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta}, Z <: [\hat{\tau}'/X]\hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B <: [\hat{\tau}'/X]\hat{\tau}'_A$. Then by S-TypePoly, $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash (\forall Y <: [\hat{\tau}'/X]\hat{\tau}_A.[\hat{\tau}'/X]\hat{\tau}_B) <: (\forall Z <: [\hat{\tau}'/X]\hat{\tau}'_A.[\hat{\tau}'/X]\hat{\tau}'_B)$, which is the same as $\hat{\Gamma}, [\hat{\tau}'/X]\hat{\Delta} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

Case: S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau} \vdash X <: \hat{\tau}$. We want to show (1) $\hat{\Gamma}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]X <: [\hat{\tau}'/X]\hat{\tau}$. Firstly, $[\hat{\tau}'/X]X = \hat{\tau}'$. Secondly, because $\texttt{WF}(\hat{\Gamma}, X <: \hat{\tau})$ then $X \notin \texttt{free-vars}(\hat{\tau})$, so $[\hat{\tau}'/X]\hat{\tau} = \hat{\tau}$. Therefore, judgement (1) is the same as $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}' <: \hat{\tau}$, which is true by assumption.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation, so $\hat{\Gamma}, X <: \hat{\tau} \vdash Y <: \hat{\tau}_2$ is true by widening the context in the judgement $\hat{\Gamma} \vdash Y <: \hat{\tau}_2$[3]. Then $\hat{\Gamma} \vdash [\hat{\tau}'/X]Y <: [\hat{\tau}'/X]\hat{\tau}_2$ by inductive assumption. By widening, $\hat{\Gamma}, X <: \hat{\tau} \vdash [\hat{\tau}'/X]Y <: \hat{\tau}'/X]\hat{\tau}_2$.

Lemma 6 (Type Substitution Preserves Typing). *If* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ `with` ε *and* $\hat{\Gamma} \vdash \hat{\tau}'' <: \hat{\tau}'$, *then* $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$ `with` ε

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ `with` ε.

Case: ε-Var, ε-Resource. Then $\hat{e} = [\hat{\tau}''/X]\hat{e}$, so the typing judgement in the consequent can be the one from the antecedent.

Case: ε-OperCall. Then $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \bar{r}\}$. By inversion we have (1). Noting that $[\hat{\tau}''/X]\{\bar{r}\} = \{\bar{r}\}$, we can apply the inductive hypothesis to get (2),

1. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\bar{r}\}$ `with` ε_1
2. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : \{\bar{r}\}$ `with` ε_1

Then from (2), we can apply ε-OperCall to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1.\pi) : \texttt{Unit with } \varepsilon_1 \cup \{r.\pi \mid r \in \bar{r}\}$. Since $[\hat{\tau}''/X]\texttt{Unit} = \texttt{Unit}$, we're done.

Case: ε-Subsume. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}$ `with` ε. By inversion, (1) and (2) are true.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_2 <: \hat{\tau}$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
3. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_2$ `with` ε_2

By a previous lemma, type substitution preserves subtyping. Applying this to (1) yields (4). On the other hand, only effect-variable bindings in a context will affect subsetting judgements. Based on this, we can delete the binding $X <: \hat{\tau}$ and perform the substitution $[\hat{\tau}''/X]\hat{\Delta}$, neither of which will change any effect-variable bindings, and in doing so obtain judgement (5). Lastly, we can apply the inductive hypothesis to (3), obtaining (6).

[3] Note there is no explicit widening rule; be careful with this one.

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}_2 <: [\hat{\tau}''/X]\hat{\tau}$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon_2 \subseteq \varepsilon$
7. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}_2$ `with` ε_2

From (4-6) we can apply ε-Subsume to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e} : [\hat{\tau}''/X]\hat{\tau}$ `with` ε_2.

Case: ε-Abs. Then $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda y : \hat{\tau}_2.\hat{e}_3 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` $\varnothing$. By inversion, we have (1). By setting $\hat{\Delta}' = \hat{\Delta}, y : \hat{\tau}_2$, this can be rewritten as (2). From inductive hypothesis we get (3). Then by simplifying $\hat{\Delta}'$, this simplifies to (4).

1. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, y : \hat{\tau}_2 \vdash \hat{e}_3 : \hat{\tau}_3$ `with` ε_3
2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_3 : \hat{\tau}_3$ `with` ε_3
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3$ `with` ε_3
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, y : [\hat{\tau}''/X]\hat{\tau}_2 \vdash [\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_3$ `with` ε_3

From (4) we can apply ε-Abs to get $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda y : [\hat{\tau}''/X]\hat{\tau}_2.[\hat{\tau}''/X]\hat{e}_3 : [\hat{\tau}''/X]\hat{\tau}_2 \rightarrow_{\varepsilon_3} [\hat{\tau}''/X]\hat{\tau}_3$ `with` $\varnothing$. This can be rewritten as $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda y : \hat{\tau}_2.\hat{e}_3) : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3)$ `with` $\varnothing$.

Case: ε-App. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$. By inversion, we have:

1. $\hat{\Gamma}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` ε_1
2. $\hat{\Gamma}, X <: \hat{\tau}_1, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Applying inductive hypothesis to (1) and (2) gives (3) and (4),

3. $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3)$ `with` ε_1
4. $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_2 : [\hat{\tau}''/X]\hat{\tau}_2$ `with` ε_2

Then from (3) and (4) we can apply ε-App to get $\hat{\Gamma}, \hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \hat{e}_2) : [\hat{\tau}''/X]\hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

Case: ε-PolyTypeAbs, Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A$ `cap` ε_A `with` $\varnothing$. By inversion, we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, Y <: \hat{\tau}_B$, we can rewrite it as (2). Inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite this as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_B \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A$ `with` ε_A
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, Y <: [\hat{\tau}''/X]\hat{\tau}_B \vdash [\hat{\tau}''/X]\hat{e}_A : [\hat{\tau}''/X]\hat{\tau}_A$ `with` ε_A

From (4) we can apply ε-PolyTypeAbs, giving (5), which can be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{e}_A : \forall Y <: [\hat{\tau}''/X]\hat{\tau}_B.[\hat{\tau}''/X]\hat{\tau}_A$ `cap` ε_A `with` $\varnothing$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\Delta \vdash [\hat{\tau}''/X](\lambda Y <: \hat{\tau}_B.\hat{e}_A : \forall Y <: \hat{\tau}_B.\hat{\tau}_A$ `cap` $\varepsilon_A)$ `with` $\varnothing$

Case: ε-PolyFxAbs. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon_A.\hat{e}_B : \forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `cap` ε_B `with` $\varnothing$. By inversion we have (1). Setting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon_A$, this can be rewritten as (2). The inductive hypothesis gives us (3). Expanding $\hat{\Delta}'$ lets us rewrite that as (4).

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon_A \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B
3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}' \vdash [\hat{\tau}''/X]\hat{e}_B : [\hat{\tau}''/X]\hat{\tau}_B$ `with` ε_B
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta}, \phi \subseteq \varepsilon_A \vdash [\hat{\tau}''/X]\hat{e}_B : [\hat{\tau}''/X]\hat{\tau}_B$ `with` ε_B

From (4) we can apply ε-PolyFxAbs, giving (5), whichc an be rewritten as (6).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \lambda\phi \subseteq \varepsilon_A.[\hat{\tau}''/X]\hat{e}_B : \forall\phi \subseteq \varepsilon_A.[\hat{\tau}''/X]\hat{\tau}_B$ `cap` ε_B `with` $\varnothing$
6. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\lambda\phi \subseteq \varepsilon_A.\hat{e}_B) : [\hat{\tau}''/X](\forall\phi \subseteq \varepsilon_A.\hat{\tau}_B$ `cap` $\varepsilon_B)$ `with` $\varnothing$

Case: ε-PolyTypeApp. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{\tau}'_A : [\hat{\tau}'_A/Y]\hat{\tau}_B$ `with` $[\hat{\tau}'_A/Y]\varepsilon_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ `caps` ε_B `with` ε_C
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

By inductive hypothesis on (1) we get (3). By a previous lemma, type substitution preserves subtyping, so from (2) we obtain (4).

3. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\forall Y <: \hat{\tau}_A.\hat{\tau}_B$ `caps` $\varepsilon_B)$ `with` ε_C
4. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{\tau}'_A <: [\hat{\tau}''/X]\hat{\tau}_A$

From (3-4), applying ε-PolyTypeApp gives (5).

5. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \hat{\tau}'_A) : [\hat{\tau}''/X]([\hat{\tau}'_A/Y]\hat{\tau}_B)$ `with` $[\hat{\tau}'_A/Y]\varepsilon_B \cup \varepsilon_C$

Case: ε-PolyFxApp Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \varepsilon'_A : [\varepsilon'_A/\phi]\hat{\tau}_B$ `with` $[\varepsilon'_A/\phi]\hat{\varepsilon}_B \cup \varepsilon_C$, where we get (1) and (2) from inversion.

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \forall \phi \subseteq \varepsilon_A.\hat{\tau}_B$ `caps` ε_B `with` ε_C
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

By inductive hypothesis on (1) we get (3). By a previous lemma, type substitution preserves subsetting[4]. Using this, and the knowledge that type-variable substitution on an effect-set does nothing, we obtain (4) from (2).

1. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X]\hat{e}_1 : [\hat{\tau}''/X](\forall \phi \subseteq \varepsilon_A.\hat{\tau}_B$ `caps` $\varepsilon_B)$ `with` ε_C
2. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash \varepsilon'_A \subseteq \varepsilon_A$

From (3-4), applying ε-PolyFxApp gives (5).

1. $\hat{\Gamma}, [\hat{\tau}''/X]\hat{\Delta} \vdash [\hat{\tau}''/X](\hat{e}_1\ \varepsilon'_A) : [\hat{\tau}''/X]([\varepsilon'_A/\phi]\hat{\tau}_B)$ `with` $[\varepsilon'_A/\phi]\hat{\varepsilon}_B \cup \varepsilon_C$

Case: ε-Import

Theorem 1 (Progress). *If $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε and $\hat{e}$ is not a value, then $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon$, for some $\hat{e}', \varepsilon$.*

Proof. By induction on the derivation of $\hat{\Gamma} \vdash \hat{e} : \hat{\tau}$ `with` ε.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}''\hat{\tau}' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat{\tau}_1.\hat{e}_2)\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. If $\hat{e}_1$ is not a value then $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}'_1\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat{e}$ is a value, so $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1.\hat{e}_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat{e}_2$.

Theorem 2 (Preservation). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$, then $\hat{\Gamma} \vdash \hat{e}_B : \hat{\tau}_B$ `with` ε_B, where $\hat{e}_B <: \hat{e}_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat{e}_B, \varepsilon, \hat{\tau}_B, \varepsilon_B$.*

[4] Haven't stated and proved this yet

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ `with` ε_A and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. Consider which reduction rule was used.
Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyTypeApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.
Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_3.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$.
The result follows by the substitution lemma.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.
Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A$ `with` ε_A. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.
Subcase: E-PolyFxApp2. Then $(\lambda\phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.