

1 Grammar

$e ::= x$	<i>exprs.</i>	
r		
$\lambda x : \tau. e$		
$e e$		$\varepsilon ::= \{\bar{r}.\pi\}$ <i>effects</i>
$e.\pi$		
$\hat{e} ::= x$	<i>labelled exprs.</i>	$\tau ::= \{\bar{r}\}$ <i>types</i>
r		$\tau \rightarrow \tau$
$\lambda x : \hat{\tau}.\hat{e}$		$\hat{\tau} ::= \{\bar{r}\}$ <i>labelled types</i>
$\hat{e} \hat{e}$		$\hat{\tau} \rightarrow_{\varepsilon} \hat{\tau}$
$\hat{e}.\pi$		$\Gamma ::= \emptyset$ <i>type ctx.</i>
import (ε) $x = \hat{e}$ in e		$\Gamma, x : \tau$
$v ::= r$	<i>values.</i>	$\hat{\Gamma} ::= \emptyset$ <i>labelled type ctx.</i>
$\lambda x : \tau. e$		$\hat{\Gamma}, x : \hat{\tau}$
$\hat{v} ::= r$	<i>labelled values</i>	
$\lambda x : \hat{\tau}.\hat{e}$		

2 Functions

Definition ($\text{annot} :: \tau \times \varepsilon \rightarrow \hat{\tau}$)

1. $\text{annot}(\{\bar{r}\}, _) = \{\bar{r}\}$
2. $\text{annot}(\tau_1 \rightarrow \tau_2, \varepsilon) = \text{annot}(\tau_1, \varepsilon) \rightarrow_{\varepsilon} \text{annot}(\tau_2, \varepsilon)$

Definition ($\text{annot} :: e \times \varepsilon \rightarrow \hat{e}$)

1. $\text{annot}(x, _) = x$
2. $\text{annot}(r, _) = r$
3. $\text{annot}(e_1 e_2, \varepsilon) = \text{annot}(e_1) \text{annot}(e_2)$
4. $\text{annot}(e.\pi, \varepsilon) = \text{annot}(e).\pi$
5. $\text{annot}(\lambda x : \tau. e, \varepsilon) = \lambda x : \text{annot}(\tau, \varepsilon). \text{annot}(e, \varepsilon)$

Definition ($\text{annot} :: \Gamma \times \varepsilon \rightarrow \hat{\Gamma}$)

1. $\text{annot}(\emptyset, _) = \emptyset$
2. $\text{annot}((\Gamma, x : \tau), \varepsilon) = \text{annot}(\Gamma, \varepsilon), x : \text{annot}(\tau, \varepsilon)$

Definition ($\text{erase} :: \hat{\tau} \rightarrow \tau$)

1. $\text{erase}(\{\bar{r}\}, _) = \{\bar{r}\}$
2. $\text{erase}(\hat{\tau}_1 \rightarrow_{\varepsilon} \hat{\tau}_2) = \text{erase}(\hat{\tau}_1) \rightarrow \text{erase}(\hat{\tau}_2)$

Definition ($\text{erase} :: \hat{e} \rightarrow e$)

1. $\text{erase}(x) = x$
2. $\text{erase}(r) = r$
3. $\text{erase}(e_1 e_2) = \text{erase}(e_1) \text{erase}(e_2)$
4. $\text{erase}(e.\pi) = \text{erase}(e).\pi$
5. $\text{erase}(\lambda x : \hat{\tau}.\hat{e}) = \lambda x : \text{erase}(\hat{\tau}). \text{erase}(\hat{e})$

Definition ($\text{effects} :: \tau \rightarrow \varepsilon$)

1. $\text{effects}(\{\bar{r}\}) = \{r.\pi \mid r \in \bar{r}, \pi \in \Pi\}$
2. $\text{effects}(\hat{\tau}_1 \rightarrow_\varepsilon \hat{\tau}_2) = \text{ho-effects}(\hat{\tau}_1) \cup \varepsilon \cup \text{effects}(\hat{\tau}_2)$

This function computes those effects:

- Directly invoked by a function of type $\hat{\tau}$.
- Captured by functions created/returned by $\hat{\tau}$.

Definition ($\text{ho-effects} :: \tau \rightarrow \varepsilon$)

1. $\text{ho-effects}(\{\bar{r}\}) = \emptyset$
2. $\text{ho-effects}(\hat{\tau}_1 \rightarrow_\varepsilon \hat{\tau}_2) = \text{effects}(\hat{\tau}_1) \cup \text{ho-effects}(\hat{\tau}_2)$

This function computes those effects:

- Captured by functions passed into $\hat{\tau}$.

Examples

Suppose a is a base type that captures no effects.

Consider $\hat{\tau} = (a \rightarrow_b (a \rightarrow_c a)) \rightarrow_d (a \rightarrow_e a)$.

$$\begin{aligned} \text{effects}(\hat{\tau}) &= \{d, e\} \\ \text{ho-effects}(\hat{\tau}) &= \{b, c\} \end{aligned}$$

Consider $\hat{\tau} = ((a \rightarrow_b a) \rightarrow_c (a \rightarrow_d a)) \rightarrow_e ((a \rightarrow_f a) \rightarrow_g (a \rightarrow_h a))$

$$\begin{aligned} \text{effects}(\hat{\tau}) &= \{e, b, g, h\} \\ \text{ho-effects}(\hat{\tau}) &= \{c, d, f\} \end{aligned}$$

3 Static Rules

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{}{\Gamma, x : \tau \vdash x : \tau} \text{ (T-VAR)} \quad \frac{}{\Gamma, r : \{r\} \vdash r : \{r\}} \text{ (T-RESOURCE)} \quad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ (T-ABS)} \\[10pt] \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau_3 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau_3} \text{ (T-APP)} \quad \frac{\Gamma \vdash e : \{\bar{r}\} \quad \forall r \in \bar{r} \mid r \in R \quad \pi \in \Pi}{\Gamma \vdash e.\pi : \text{Unit}} \text{ (T-OPERCALL)} \end{array}$$

$$\boxed{\hat{I} \vdash \hat{e} : \hat{\tau} \text{ with } \varepsilon}$$

$$\begin{array}{c} \frac{}{\hat{I}, x : \tau \vdash x : \tau \text{ with } \emptyset} \text{ (\varepsilon-VAR)} \quad \frac{}{\hat{I}, r : \{r\} \vdash r : \{r\} \text{ with } \emptyset} \text{ (\varepsilon-RESOURCE)} \\[10pt] \frac{\hat{I}, x : \hat{\tau}_2 \vdash \hat{e} : \hat{\tau}_3 \text{ with } \varepsilon_3}{\hat{I} \vdash \lambda x : \tau_2. \hat{e} : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3 \text{ with } \emptyset} \text{ (\varepsilon-ABS)} \quad \frac{\hat{I} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_\varepsilon \hat{\tau}_3 \text{ with } \varepsilon_1 \quad \hat{I} \vdash \hat{e}_2 : \hat{\tau}_2 \text{ with } \varepsilon_2}{\hat{I} \vdash \hat{e}_1 \hat{e}_2 : \hat{\tau}_3 \text{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon} \text{ (\varepsilon-APP)} \\[10pt] \frac{\hat{I} \vdash \hat{e} : \{\bar{r}\} \quad \forall r \in \bar{r} \mid r : \{r\} \in \Gamma \quad \pi \in \Pi}{\hat{I} \vdash \hat{e}.\pi : \text{Unit with } \{\bar{r}.\pi\}} \text{ (\varepsilon-OPERCALL)} \quad \frac{\hat{I} \vdash e : \tau \text{ with } \varepsilon \quad \tau' <: \tau \quad \varepsilon' \subseteq \varepsilon}{\hat{I} \vdash e : \tau' \text{ with } \varepsilon'} \text{ (\varepsilon-SUBSUME)} \\[10pt] \frac{\hat{I} \vdash \hat{e} : \hat{\tau} \text{ with } \varepsilon_1 \quad \varepsilon = \text{effects}(\hat{\tau}) \quad \text{ho-safe}(\hat{\tau}, \varepsilon) \quad x : \text{erase}(\hat{\tau}) \vdash e : \tau}{\hat{I} \vdash \text{import}(\varepsilon) x = \hat{e} \text{ in } e : \text{annot}(\tau, \varepsilon) \text{ with } \varepsilon \cup \varepsilon_1} \text{ (\varepsilon-MODULE)} \end{array}$$

$\text{safe}(\tau, \varepsilon)$

$$\frac{}{\text{safe}(\{\bar{r}\}, \varepsilon)} \text{ (SAFE-RESOURCE)} \quad \frac{}{\text{safe}(\text{Unit}, \varepsilon)} \text{ (SAFE-UNIT)}$$

$$\frac{\varepsilon \subseteq \varepsilon' \quad \text{ho-safe}(\hat{\tau}_1, \varepsilon) \quad \text{safe}(\hat{\tau}_2, \varepsilon)}{\text{safe}(\hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2, \varepsilon)} \text{ (SAFE-ARROW)}$$

$\text{ho-safe}(\hat{\tau}, \varepsilon)$

$$\frac{}{\text{ho-safe}(\{\bar{r}\}, \varepsilon)} \text{ (HOSAFE-RESOURCE)} \quad \frac{}{\text{ho-safe}(\text{Unit}, \varepsilon)} \text{ (HOSAFE-UNIT)}$$

$$\frac{\text{safe}(\hat{\tau}_1, \varepsilon) \quad \text{ho-safe}(\hat{\tau}_2, \varepsilon)}{\text{ho-safe}(\hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2, \varepsilon)} \text{ (HOSAFE-ARROW)}$$

$\hat{\tau} <: \hat{\tau}$

$$\frac{\varepsilon \subseteq \varepsilon' \quad \hat{\tau}_2 <: \hat{\tau}_2' \quad \hat{\tau}_1' <: \hat{\tau}_1}{\hat{\tau}_1 \rightarrow_{\varepsilon} \hat{\tau}_2 <: \hat{\tau}_1' \rightarrow_{\varepsilon'} \hat{\tau}_2'} \text{ (S-EFFECTS)}$$

4 Dynamic Rules

$\hat{e} \longrightarrow \hat{e} \mid \varepsilon$

$$\frac{\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon}{\hat{e}_1 \hat{e}_2 \longrightarrow \hat{e}_1' \hat{e}_2 \mid \varepsilon} \text{ (E-APP1)} \quad \frac{\hat{e}_2 \longrightarrow \hat{e}_2' \mid \varepsilon}{\hat{v}_1 \hat{e}_2 \longrightarrow \hat{v}_1 \hat{e}_2' \mid \varepsilon} \text{ (E-APP2)} \quad \frac{}{(\lambda x : \hat{\tau}. \hat{e}) \hat{v}_2 \longrightarrow [\hat{v}_2/x] \hat{e} \mid \emptyset} \text{ (E-APP3)}$$

$$\frac{\hat{e} \rightarrow \hat{e}' \mid \varepsilon}{\hat{e}. \pi \longrightarrow \hat{e}'. \pi \mid \varepsilon} \text{ (E-OPERCALL1)} \quad \frac{r \in R \quad \pi \in \Pi}{r. \pi \longrightarrow \text{unit} \mid \{r. \pi\}} \text{ (E-OPERCALL2)}$$

$$\frac{\hat{e} \longrightarrow \hat{e}' \mid \varepsilon'}{\text{import}(\varepsilon) \ x = \hat{e} \text{ in } e \longrightarrow \text{import}(\varepsilon) \ x = \hat{e}' \text{ in } e \mid \varepsilon'} \text{ (E-MODULE1)}$$

$$\frac{}{\text{import}(\varepsilon) \ x = \hat{v} \text{ in } e \longrightarrow [\hat{v}/x] \text{annot}(e, \varepsilon) \mid \emptyset} \text{ (E-MODULE2)}$$

5 Encodings

5.1 \perp

We can define the bottom type as $\perp = \{\}$, because there is no empty-set literal.

5.2 `unit`, `Unit`

Define `unit` = $\lambda x : \{\}.x$, i.e. the function which takes an empty set of resources and returns it. We shall refer to its type, which is $\{\} \rightarrow_{\emptyset} \{\}$, as `Unit`. It has various properties befitting `unit`.

1. `unit` cannot be invoked, as $\{\}$ is uninhabited.
2. `unit` is a value.
3. The only term with type `Unit` is `unit`.
4. $\vdash \text{unit} : \text{Unit}$, by using ε -ABS and ε -VAR.
5. $\text{effects}(\text{Unit}) = \text{ho-effects}(\text{Unit}) = \emptyset$
6. $\text{safe}(\text{Unit}, \varepsilon)$ and $\text{ho-safe}(\text{Unit}, \varepsilon)$

6 Proofs

Theorem 1 (Progress). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ with ε_A then \hat{e}_A is a value or $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.*

Proof. By induction on $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ with ε_A .

Case: ε -RESOURCE, ε -UNIT, ε -ABS Then \hat{e}_A is a value.

Case: ε -SUBSUME Then $\hat{\Gamma} \vdash e : \tau'$ with ε' , and $\hat{\Gamma} \vdash e : \tau$ with ε , where $\tau' <: \tau$ and $\varepsilon' \subseteq \varepsilon$ are subderivations. The theorem conclusion holds by inductive assumption applied to $\hat{\Gamma} \vdash e : \tau$ with ε .

Case: ε -APP Then $\hat{e}_A = \hat{e}_1 \hat{e}_2$. We consider the cases in which \hat{e}_1 and \hat{e}_2 are values.

If \hat{e}_1 is not a value then by inductive assumption there is a reduction $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$. Then $\hat{e}_1 \hat{e}_2$ reduces by the rule E-APP1, giving $\hat{e}_1 \hat{e}_2 \longrightarrow \hat{e}'_1 \hat{e}_2 \mid \varepsilon$.

If \hat{e}_2 is not a value then WLOG \hat{e}_1 is a value. By inductive assumption $\hat{e}_2 \longrightarrow \hat{e}'_2 \mid \varepsilon$. Then $\hat{e}_1 \hat{e}_2$ reduces by the rule E-APP2, giving $\hat{e}_1 \hat{e}_2 \longrightarrow \hat{e}_1 \hat{e}'_2 \mid \varepsilon$.

If \hat{e}_1 and \hat{e}_2 are both values then by canonical forms $\hat{e}_1 = \hat{v}_1 = \lambda x : \tau_2.e$. Then $\hat{e}_1 \hat{e}_2$ reduces by the rule E-APP3, giving $\hat{e}_1 \hat{e}_2 \longrightarrow [\hat{v}_2/x]\hat{e} \mid \emptyset$.

Case: ε -OPERCALL Then $\hat{e}_A = \hat{e}_1.\pi$. We consider whether \hat{e}_1 is a value.

If \hat{e}_1 is not a value then by inductive assumption there is a reduction $\hat{e}_1 \longrightarrow \hat{e}'_1 \mid \varepsilon$. Then $\hat{e}_1.\pi$ reduces by the rule E-OPERCALL1, giving $\hat{e}_1.\pi \longrightarrow \hat{e}'_1.\pi \mid \varepsilon$.

If \hat{e}_1 is a value then $\hat{e}_1 = r$ by canonical forms. By the assumption that $r.\pi$ is closed under Γ , we know $r \in R$ and $\pi \in \Pi$. Then $\hat{e}_1.\pi$ reduces by the rule E-OPERCALL2, giving $r.\pi \longrightarrow \text{unit} \mid \varepsilon$.

Case: ε -MODULE Then $e_A = \text{import}(\varepsilon) x = \hat{e} \text{ in } e$. If \hat{e} is an expression then it can be reduced, so $\hat{e} \longrightarrow \hat{e}' \mid \varepsilon'$, and so by E-MODULE1 we get $\text{import}(\varepsilon) x = \hat{e} \text{ in } e \longrightarrow \text{import}(\varepsilon) x = \hat{e}' \text{ in } e \mid \varepsilon'$. Otherwise $\hat{e} = \hat{v}$ is a value. Then by E-MODULE2 we get $\text{import}(\varepsilon) x = \hat{v} \longrightarrow [\hat{v}/x]\text{annot}(e, \varepsilon) \mid \emptyset$.

Lemma 1 (Substitution). *If $\hat{\Gamma}, x : \hat{\tau}' \vdash e : \hat{\tau}$ with ε and $\hat{\Gamma} \vdash v : \hat{\tau}'$ with \emptyset then $\hat{\Gamma} \vdash [v/x]e : \hat{\tau}$ with ε .*

Lemma 2. *If $\text{effects}(\hat{\tau}) \subseteq \varepsilon$ and $\text{ho-safe}(\hat{\tau}, \varepsilon)$ then $\hat{\tau} <: \text{annot}(\text{erase}(\hat{\tau}), \varepsilon)$.*

Lemma 3. *If $\text{ho-effects}(\hat{\tau}) \subseteq \varepsilon$ and $\text{safe}(\hat{\tau}, \varepsilon)$ then $\text{annot}(\text{erase}(\hat{\tau}), \varepsilon) <: \hat{\tau}$.*

Proof. By simultaneous induction.

Case: $\hat{\tau} = \{\bar{r}\}$ Then $\hat{\tau} = \text{annot}(\text{erase}(\hat{\tau}), \varepsilon)$ and the results for both lemmas hold immediately.

Case: $\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$, $\text{effects}(\hat{\tau}) \subseteq \varepsilon$, $\text{ho-safe}(\hat{\tau}, \varepsilon)$ It is sufficient to show $\hat{\tau}_2 <: \text{annot}(\text{erase}(\hat{\tau}_2), \varepsilon)$ and $\text{annot}(\text{erase}(\hat{\tau}_1), \varepsilon) <: \hat{\tau}_1$, because the result will hold by S-EFFECTS. To achieve this we shall inductively apply **lemma 2** to $\hat{\tau}_2$ and **lemma 3** to $\hat{\tau}_1$.

From $\text{effects}(\hat{\tau}) \subseteq \varepsilon$ we have $\text{ho-effects}(\hat{\tau}_1) \cup \varepsilon' \cup \text{effects}(\hat{\tau}_2) \subseteq \varepsilon$ and therefore $\text{effects}(\hat{\tau}_2) \subseteq \varepsilon$. From $\text{ho-safe}(\hat{\tau}, \varepsilon)$ we have $\text{ho-safe}(\hat{\tau}_2, \varepsilon)$. Therefore we can apply **lemma 2** to $\hat{\tau}_2$.

From $\text{effects}(\hat{\tau}) \subseteq \varepsilon$ we have $\text{ho-effects}(\hat{\tau}_1) \cup \varepsilon' \cup \text{effects}(\hat{\tau}_2) \subseteq \varepsilon$ and therefore $\text{ho-effects}(\hat{\tau}_1) \subseteq \varepsilon$. From $\text{ho-safe}(\hat{\tau}, \varepsilon)$ we have $\text{safe}(\hat{\tau}_1, \varepsilon)$. Therefore we can apply **lemma 3** to $\hat{\tau}_1$.

Case: $\hat{\tau} = \hat{\tau}_1 \rightarrow_{\varepsilon'} \hat{\tau}_2$, $\text{ho-effects}(\hat{\tau}) \subseteq \varepsilon$, $\text{safe}(\hat{\tau}, \varepsilon)$ It is sufficient to show $\text{annot}(\text{erase}(\hat{\tau}_2), \varepsilon) <: \hat{\tau}_2$ and $\hat{\tau}_1 <: \text{annot}(\text{erase}(\hat{\tau}_1), \varepsilon)$, because the result will hold by S-EFFECTS. To achieve this we shall inductively apply **lemma 3** to $\hat{\tau}_2$ and **lemma 2** to $\hat{\tau}_1$.

From $\text{ho-effects}(\hat{\tau}) \subseteq \varepsilon$ we have $\text{effects}(\hat{\tau}_1) \cup \text{ho-effects}(\hat{\tau}_2) \subseteq \varepsilon$ and therefore $\text{ho-effects}(\hat{\tau}_2) \subseteq \varepsilon$. From $\text{safe}(\hat{\tau}, \varepsilon)$ we have $\text{safe}(\hat{\tau}_2, \varepsilon)$. Therefore we can apply **lemma 3** to $\hat{\tau}_2$.

From $\text{ho-effects}(\hat{\tau}) \subseteq \varepsilon$ we have $\text{effects}(\hat{\tau}_1) \cup \text{ho-effects}(\hat{\tau}_2) \subseteq \varepsilon$ and therefore $\text{effects}(\hat{\tau}_1) \subseteq \varepsilon$. From $\text{safe}(\hat{\tau}, \varepsilon)$ we have $\text{ho-safe}(\hat{\tau}_1, \varepsilon)$. Therefore we can apply **lemma 2** to $\hat{\tau}_1$.

Theorem 2 (Preservation). *If $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A$ with ε_A and $e_A \longrightarrow e_B \mid \varepsilon_C$, then $\hat{\Gamma} \vdash e_B : \tau_B$ with ε_B , where $e_B < e_B$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$.*

Proof. By induction on $\hat{\Gamma} \vdash \hat{e}_A : \tau_A$ with ε_A , and then on $e_A \longrightarrow e_B \mid \varepsilon$.

$\boxed{\varepsilon\text{-VAR}, \varepsilon\text{-RESOURCE}, \varepsilon\text{-UNIT}, \varepsilon\text{-ABS}}$ Then e_A cannot be reduced and so the theorem statement vacuously holds.

$\boxed{\varepsilon\text{-APP}}$ Then $e_A = \hat{e}_1 \hat{e}_2$ and $\hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon} \hat{\tau}_3$ with ε_1 and $\hat{\Gamma} \vdash \hat{e}_2 : \hat{\tau}_2$ with ε_2 . If the reduction rule used was E-APP1 or E-APP2, then the result follows by applying the inductive hypothesis to \hat{e}_1 and \hat{e}_2 respectively.

Otherwise the rule used was E-APP3. Then $(\lambda x : \hat{\tau}_2. \hat{e}) \hat{v}_2 \longrightarrow [\hat{v}_2/x] \hat{e} \mid \emptyset$. By inversion on the typing rule for $\lambda x : \hat{\tau}_2. \hat{e}$ we know $\hat{\Gamma}, x : \hat{\tau}_2 \vdash \hat{e} : \hat{\tau}_3$ with ε_3 . By canonical forms, $\varepsilon_2 = \emptyset$ because $\hat{e}_2 = \hat{v}_2$ is a value. Then by the substitution lemma, $\hat{\Gamma} \vdash [\hat{v}_2/x] \hat{e} : \hat{\tau}_3$ with ε_3 . By canonical forms, $\varepsilon_1 = \varepsilon_2 = \emptyset = \varepsilon_C$. Therefore $\varepsilon_A = \varepsilon_3 = \varepsilon_B \cup \varepsilon_C$.

$\boxed{\varepsilon\text{-OperCall}}$ Then $e_A = e_1. \pi$ and $\hat{\Gamma} \vdash e_1 : \{\bar{r}\}$ with ε_1 . If the reduction rule used was E-OPERCALL1 then the result follows by applying the inductive hypothesis to \hat{e}_1 .

Otherwise the reduction rule used was E-OPERCALL2 and $v_1. \pi \longrightarrow \text{unit} \mid \{r. \pi\}$. By canonical forms, $\hat{\Gamma} \vdash v_1 : \text{unit}$ with $\{r. \pi\}$. Also, $\hat{\Gamma} \vdash \text{unit} : \text{Unit}$ with \emptyset . Then $\tau_B = \tau_A$. Also, $\varepsilon_C \cup \varepsilon_B = \{r. \pi\} = \varepsilon_A$.

$\boxed{\varepsilon\text{-Module}}$ Then $e_A = \text{import}(\varepsilon) x = \hat{e}$ in e . If the reduction rule used was E-MODULECALL1 then the result follows by applying the inductive hypothesis to \hat{e} .

Otherwise \hat{e} is a value and the reduction used was E-MODULECALL2. The following are true:

1. $e_A = \text{import}(\varepsilon) x = \hat{v}$ in e
2. $\hat{\Gamma} \vdash e_A : \text{annot}(\tau, \varepsilon)$ with $\varepsilon \cup \varepsilon_1$
3. $\text{import}(\varepsilon) x = \hat{v}$ in $e \longrightarrow [\hat{v}/x] \text{annot}(e, \varepsilon) \mid \emptyset$
4. $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}$ with \emptyset
5. $\varepsilon = \text{effects}(\hat{\tau})$
6. $\text{ho-safe}(\hat{\tau}, \varepsilon)$
7. $x : \text{erase}(\hat{\tau}) \vdash e : \tau$

Apply the **annotation lemma** with $\Gamma = \emptyset$ to get $\hat{\Gamma}, x : \hat{\tau} \vdash \text{annot}(e, \varepsilon) : \text{annot}(\tau, \varepsilon)$ with ε .

By **4.** we have $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}$ with \emptyset .

By **substitution lemma**, $\hat{\Gamma} \vdash [\hat{v}/x] \text{annot}(e, \varepsilon) : \text{annot}(\tau, \varepsilon)$ with ε .

By **canonical forms**, $\varepsilon_1 = \varepsilon_C = \emptyset$. Then $\varepsilon_B = \varepsilon = \varepsilon_A \cup \varepsilon_C$. By examination, $\tau_A = \tau_B = \text{annot}(\tau, \varepsilon)$.

Lemma 4 (Annotation). *If the following are true:*

- $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}$ with \emptyset
- $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e : \tau$
- $\varepsilon = \text{effects}(\hat{\tau})$
- $\text{ho-safe}(\hat{\tau}, \varepsilon)$

Then $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(e, \varepsilon) : \text{annot}(\tau, \varepsilon)$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon))$.

Proof. By induction on $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e : \tau$.

Case: T-VAR Then $e = x$ and $\Gamma, y : \text{erase}(\hat{\tau}) \vdash x : \tau$. There are two cases: $x = y$ or $x \neq y$.

Subcase 1: $x = y$. Then by ε -VAR we get $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash x : \hat{\tau}$ with \emptyset . First note that $\text{annot}(x, \varepsilon) = x$ in this case. Therefore $\Gamma, y : \text{erase}(\hat{\tau}) \vdash \text{annot}(\text{erase}(x), \varepsilon) : \hat{\tau}$ with \emptyset . We know by assumption that $\text{effects}(\hat{\tau}) = \varepsilon$ and $\text{ho-safe}(\hat{\tau}, \varepsilon)$. Applying **Lemma 2** we know $\hat{\tau} <: \text{annot}(\text{erase}(\hat{\tau}), \varepsilon)$. Lastly, by ε -SUBSUME we have $\Gamma, y : \text{erase}(\hat{\tau}) \vdash \text{annot}(\text{erase}(x), \varepsilon) : \text{annot}(\text{erase}(x), \varepsilon)$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon))$.

Subcase 2: $x \neq y$. Then $x : \tau \in \Gamma$. Together with the definition $\text{annot}(x, \varepsilon) = x$, we know $x : \text{annot}(\tau, \varepsilon) \in \text{annot}(\Gamma, \varepsilon)$. By ε -VAR we have $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(x, \varepsilon) : \text{annot}(\tau, \varepsilon)$ with \emptyset . Lastly, by ε -SUBSUME we have $\Gamma, y : \text{erase}(\hat{\tau}) \vdash \text{annot}(\text{erase}(x), \varepsilon) : \text{annot}(\text{erase}(x), \varepsilon)$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon))$.

Case: T-RESOURCE Then $\Gamma, y : \text{erase}(\hat{\tau}) \vdash r : \{r\}$. By definition, $\text{annot}(r, \varepsilon) = r$ and $\text{annot}(\{r\}, \varepsilon)$. By ε -RESOURCE $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash r : \{r\}$ with \emptyset . By ε -SUBSUME, $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash r : \{r\}$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon))$.

Case: T-ABS Then $\Gamma, y : \text{erase}(\hat{\tau}) \vdash \lambda x : \tau_1. e_{\text{body}} : \tau_1 \rightarrow \tau_2$.

By inversion, we get the sub-derivation $\Gamma, y : \text{erase}(\hat{\tau}), x : \tau_1 \vdash e_2 : \tau_2$. By definition, $\text{annot}(e, \varepsilon) = \text{annot}(\lambda x : \tau_1. e_2, \varepsilon) = \lambda x : \text{annot}(\tau_1, \varepsilon). \text{annot}(e_2, \varepsilon)$ and $\text{annot}(\tau, \varepsilon) = \text{annot}(\tau_1 \rightarrow \tau_2, \varepsilon) = \text{annot}(\tau_1, \varepsilon) \rightarrow_{\varepsilon} \text{annot}(\tau_2, \varepsilon)$.

To apply the inductive assumption to e_2 we use the unlabelled context $\Gamma, x : \tau_1$. The inductive assumption tells us $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau}, x : \text{annot}(\tau_1, \varepsilon) \vdash \text{annot}(e_2, \varepsilon) : \text{annot}(\tau_2, \varepsilon)$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon)) \cup \text{effects}(\text{annot}(\tau_1, \varepsilon))$. Call this last effect-set ε' .

By ε -ABS, we get $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \lambda x : \text{annot}(\tau_1, \varepsilon). \text{annot}(e_2, \varepsilon) : \text{annot}(\hat{\tau}_1) \rightarrow_{\varepsilon'} \text{annot}(\hat{\tau}_2)$ with \emptyset .

By ε -SUBSUME, we get $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(e, \varepsilon) : \text{annot}(\hat{\tau}_1) \rightarrow_{\varepsilon} \text{annot}(\hat{\tau}_2)$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon))$.

Case: T-APP Then $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e_1 \ e_2 : \tau_3$, where $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e_1 : \tau_2 \rightarrow \tau_3$ and $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e_2 : \tau_2$.

By applying the inductive assumption to e_1 and e_2 , we get $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(e_1, \varepsilon) : \text{annot}(\tau_1, \varepsilon)$ with ε and $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(e_2, \varepsilon) : \text{annot}(\tau_2, \varepsilon)$ with ε .

By simplifying: $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(e_1, \varepsilon) : \text{annot}(\tau_2, \varepsilon) \rightarrow_{\varepsilon} \text{annot}(\tau_3, \varepsilon)$ with ε .

By ε -APP, we get $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash \text{annot}(e_1 \ e_2, \varepsilon) : \text{annot}(\tau_3, \varepsilon)$ with ε .

Case: T-OPERCALL Then $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e_1. \pi : \text{Unit}$.

By inversion we get the sub-derivation $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e_1 : \{\bar{r}\}$.

By definition, $\text{annot}(\{\bar{r}\}, \varepsilon) = \{\bar{r}\}$.

By inductive assumption, $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash e_1 : \{\bar{r}\}$ with $\varepsilon \cup \text{effects}(\text{annot}(\Gamma, \varepsilon))$.

By ε -OPERCALL, $\hat{\Gamma}, \text{annot}(\Gamma, \varepsilon), y : \hat{\tau} \vdash e_1. \pi : \{\bar{r}\}$ with $\varepsilon \cup \{\bar{r}. \pi\}$.

It remains to show $\{\bar{r}. \pi\} \subseteq \varepsilon$. We shall do this by considering where r must have come from (which subcontext left of the turnstile).

Subcase 1. $r = \hat{\tau}$. As $\varepsilon = \text{effects}(\hat{\tau})$, then $r. \pi \in \text{effects}(\hat{\tau})$.

Subcase 2. $r : \{r\} \in \Gamma$. As $\text{annot}(r, \varepsilon) = r$, then $r.\pi \in \text{annot}(\Gamma, \varepsilon)$.

Subcase 3. $r : \{r\} \in \hat{\Gamma}$. Then because $\Gamma, y : \text{erase}(\hat{\tau}) \vdash e_1 : \{\bar{r}\}$, then $r \in \Gamma$ or $r = \text{erase}(\hat{\tau}) = \hat{\tau}$ and one of the above subcases must also hold.