1 Grammar

$$
\begin{array}{rll}
e & ::= x & \textit{exprs.} \\
 & \mid\ r & \\
 & \mid\ \lambda x : \tau . e & \\
 & \mid\ e\ e & \\
 & \mid\ e.\pi & \\
 & \mid\ \texttt{unit} &
\end{array}
$$

$$
\begin{array}{rll}
\hat{e} & ::= x & \textit{labelled exprs.} \\
 & \mid\ r & \\
 & \mid\ \lambda x : \hat{\tau} . \hat{e} & \\
 & \mid\ \hat{e}\ \hat{e} & \\
 & \mid\ \hat{e}.\pi & \\
 & \mid\ \texttt{unit} & \\
 & \mid\ \texttt{import}(\varepsilon)\ x = \hat{e}\ \texttt{in}\ e &
\end{array}
$$

$$
\begin{array}{rll}
v & ::= r & \textit{values.} \\
 & \mid\ \lambda x : \tau . e & \\
 & \mid\ \texttt{unit} &
\end{array}
$$

$$
\begin{array}{rll}
\hat{v} & ::= r & \textit{labelled values} \\
 & \mid\ \lambda x : \hat{\tau} . \hat{e} & \\
 & \mid\ \texttt{unit} &
\end{array}
$$

$$
\begin{array}{rll}
\varepsilon & ::= \{\overline{r.\pi}\} & \textit{effects}
\end{array}
$$

$$
\begin{array}{rll}
\tau & ::= \{\bar{r}\} & \textit{types} \\
 & \mid\ \tau \to \tau & \\
 & \mid\ \texttt{Unit} &
\end{array}
$$

$$
\begin{array}{rll}
\hat{\tau} & ::= \{\bar{r}\} & \textit{labelled types} \\
 & \mid\ \hat{\tau} \to_{\varepsilon} \hat{\tau} & \\
 & \mid\ \texttt{Unit} &
\end{array}
$$

$$
\begin{array}{rll}
\Gamma & ::= \varnothing & \textit{type ctx.} \\
 & \mid\ \Gamma, x : \tau &
\end{array}
$$

$$
\begin{array}{rll}
\hat{\Gamma} & ::= \varnothing & \textit{labelled type ctx.} \\
 & \mid\ \hat{\Gamma}, x : \hat{\tau} &
\end{array}
$$

2 Functions

Definition ($\texttt{annot} :: \boldsymbol{\tau} \times \boldsymbol{\varepsilon} \to \hat{\boldsymbol{\tau}}$)

1. $\texttt{annot}(\{\bar{r}\}, \_) = \{\bar{r}\}$
2. $\texttt{annot}(\texttt{Unit}, \_) = \texttt{Unit}$
3. $\texttt{annot}(\tau_1 \to \tau_2, \varepsilon) = \texttt{annot}(\tau_1, \varepsilon) \to_{\varepsilon} \texttt{annot}(\tau_2, \varepsilon)$

Definition ($\texttt{annot} :: \mathtt{e} \times \boldsymbol{\varepsilon} \to \hat{\mathtt{e}}$)

1. $\texttt{annot}(x, \_) = e$
2. $\texttt{annot}(r, \_) = r$
3. $\texttt{annot}(\texttt{unit}, \_) = \texttt{unit}$
4. $\texttt{annot}(e_1 e_2, \varepsilon) = \texttt{annot}(e_1)\texttt{annot}(e_2)$
5. $\texttt{annot}(e.\pi, \varepsilon) = \texttt{annot}(e).\pi$
6. $\texttt{annot}(\lambda x : \tau.e, \varepsilon) = \lambda x : \texttt{annot}(\tau, \varepsilon).\texttt{annot}(e, \varepsilon)$

Definition ($\texttt{annot} :: \Gamma \times \boldsymbol{\varepsilon} \to \hat{\Gamma}$)

1. $\texttt{annot}(\varnothing, \_) = \varnothing$
2. $\texttt{annot}((\Gamma, x : \tau), \varepsilon) = \texttt{annot}(\Gamma, \varepsilon), x : \texttt{annot}(\tau, \varepsilon)$

Definition ($\texttt{erase} :: \hat{\boldsymbol{\tau}} \to \boldsymbol{\tau}$)

1. $\texttt{erase}(\{\bar{r}\}, \_) = \{\bar{r}\}$
2. $\texttt{erase}(\texttt{Unit}, \_) = \texttt{Unit}$
3. $\texttt{erase}(\hat{\tau}_1 \to_{\varepsilon} \hat{\tau}_2) = \texttt{erase}(\hat{\tau}_1) \to \texttt{erase}(\hat{\tau}_2)$

Definition ($\texttt{erase} :: \hat{\texttt{e}} \to \texttt{e}$)

1. $\texttt{erase}(x) = x$
2. $\texttt{erase}(r) = r$
3. $\texttt{erase}(\texttt{unit}) = \texttt{unit}$
4. $\texttt{erase}(e_1 e_2) = \texttt{erase}(e_1)\texttt{erase}(e_2)$
5. $\texttt{erase}(e.\pi) = \texttt{erase}(e).\pi$
6. $\texttt{erase}(\lambda x : \hat{\tau}.\hat{e}) = \lambda x : \texttt{erase}(\hat{\tau}).\texttt{erase}(\hat{e})$

Definition ($\texttt{effects} :: \boldsymbol{\tau} \to \boldsymbol{\varepsilon}$)

1. $\texttt{effects}(\texttt{Unit}) = \varnothing$
2. $\texttt{effects}(\{\bar{r}\}) = \{r.\pi \mid r \in \bar{r}, \pi \in \Pi\}$
3. $\texttt{effects}(\hat{\tau}_1 \to_\varepsilon \hat{\tau}_2) = \texttt{ho-effects}(\hat{\tau}_1) \cup \varepsilon \cup \texttt{effects}(\hat{\tau}_2)$

Definition ($\texttt{ho-effects} :: \boldsymbol{\tau} \to \boldsymbol{\varepsilon}$)

1. $\texttt{ho-effects}(\texttt{Unit}) = \varnothing$
2. $\texttt{ho-effects}(\{\bar{r}\}) = \varnothing$
3. $\texttt{ho-effects}(\hat{\tau}_1 \to_\varepsilon \hat{\tau}_2) = \texttt{effects}(\tau_1) \cup \texttt{ho-effects}(\hat{\tau}_2)$

3 Static Rules

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ (\text{T-Var}) \qquad \frac{}{\Gamma, r : \{r\} \vdash r : \{r\}}\ (\text{T-Resource})$$

$$\frac{}{\Gamma \vdash \texttt{unit} : \texttt{Unit}}\ (\text{T-Unit}) \qquad \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1.e : \tau_1 \to \tau_2}\ (\text{T-Abs})$$

$$\frac{\Gamma \vdash e_1 : \tau_2 \to \tau_3 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1\ e_2 : \tau_3}\ (\text{T-App}) \qquad \frac{\Gamma \vdash e : \{\bar{r}\} \quad \forall r \in \bar{r} \mid r \in R \quad \pi \in \Pi}{\Gamma \vdash e.\pi : \texttt{Unit}}\ (\text{T-OperCall})$$

$\boxed{\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon}$

$$\frac{}{\Gamma, x : \tau \vdash x : \tau \texttt{ with } \varnothing}\ (\varepsilon\text{-Var}) \qquad \frac{}{\Gamma, r : \{r\} \vdash r : \{r\} \texttt{ with } \varnothing}\ (\varepsilon\text{-Resource})$$

$$\frac{}{\Gamma \vdash \texttt{unit} : \texttt{Unit with } \varnothing}\ (\varepsilon\text{-Unit}) \qquad \frac{\Gamma, x : \tau_2 \vdash \hat{e} : \tau_3 \texttt{ with } \varepsilon}{\Gamma \vdash \lambda x : \tau_2.\hat{e} : \tau_2 \to_\varepsilon \tau_3 \texttt{ with } \varnothing}\ (\varepsilon\text{-Abs})$$

$$\frac{\Gamma \vdash \hat{e}_1 : \tau_2 \to_\varepsilon \tau_3 \texttt{ with } \varepsilon_1 \quad \Gamma \vdash \hat{e}_2 : \tau_2 \texttt{ with } \varepsilon_2}{\Gamma \vdash \hat{e}_1 \hat{e}_2 : \tau_3 \texttt{ with } \varepsilon_1 \cup \varepsilon_2 \cup \varepsilon}\ (\varepsilon\text{-App}) \qquad \frac{\Gamma \vdash \hat{e} : \{\bar{r}\} \quad \forall r \in \bar{r} \mid r \in R \quad \pi \in \Pi}{\Gamma \vdash \hat{e}.\pi : \texttt{Unit with } \{\bar{r}.\pi\}}\ (\varepsilon\text{-OperCall})$$

$$\frac{\begin{array}{c}\hat{\Gamma} \vdash \hat{e} : \hat{\tau} \texttt{ with } \varepsilon_1 \qquad \varepsilon = \texttt{effects}(\hat{\tau}) \\ \texttt{ho-safe}(\hat{\tau}, \varepsilon) \qquad x : \texttt{erase}(\hat{\tau}) \vdash e : \tau\end{array}}{\hat{\Gamma} \vdash \texttt{import}(\varepsilon)\ x = \hat{e} \texttt{ in } e : \texttt{annot}(\tau, \varepsilon) \texttt{ with } \varepsilon \cup \varepsilon_1}\ (\varepsilon\text{-Module})$$

$\boxed{\texttt{safe}(\tau, \varepsilon)}$

$$\frac{}{\texttt{safe}(\{\bar{r}\}, \varepsilon)}\ (\textsc{Safe-Resource}) \qquad \frac{}{\texttt{safe}(\texttt{Unit}, \varepsilon)}\ (\textsc{Safe-Unit})$$

$$\frac{\varepsilon \subseteq \varepsilon_2 \quad \texttt{ho-safe}(\hat{\tau}_1, \varepsilon) \quad \texttt{safe}(\hat{\tau}_2, \varepsilon)}{\texttt{safe}(\hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2, \varepsilon)}\ (\textsc{Safe-Arrow})$$

$\boxed{\texttt{ho-safe}(\hat{\tau}, \varepsilon)}$

$$\frac{}{\texttt{ho-safe}(\{\bar{r}\}, \varepsilon)}\ (\textsc{HOSafe-Resource}) \qquad \frac{}{\texttt{ho-safe}(\texttt{Unit}, \varepsilon)}\ (\textsc{HOSafe-Unit})$$

$$\frac{\texttt{safe}(\hat{\tau}_1, \varepsilon) \quad \texttt{ho-safe}(\hat{\tau}_2, \varepsilon)}{\texttt{ho-safe}(\hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2, \varepsilon)}\ (\textsc{HOSafe-Arrow})$$

$\boxed{\hat{\tau} <: \hat{\tau}}$

$$\frac{\varepsilon \subseteq \varepsilon' \quad \hat{\tau}_2 <: \hat{\tau}_2' \quad \hat{\tau}_1' <: \hat{\tau}_1}{\hat{\tau}_1 \rightarrow_{\varepsilon} \hat{\tau}_2 <: \hat{\tau}_1' \rightarrow_{\varepsilon'} \hat{\tau}_2'}\ (\textsc{S-Effects})$$

4 Dynamic Rules

$\boxed{\hat{e} \longrightarrow \hat{e} \mid \varepsilon}$

$$\frac{\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon}{\hat{e}_1 \hat{e}_2 \longrightarrow \hat{e}_1' \hat{e}_2 \mid \varepsilon}\ (\textsc{E-App1}) \qquad \frac{\hat{e}_2 \longrightarrow \hat{e}_2' \mid \varepsilon}{\hat{v}_1 \hat{e}_2 \longrightarrow \hat{v}_1 \hat{e}_2' \mid \varepsilon}\ (\textsc{E-App2}) \qquad \frac{}{(\lambda x : \tau . \hat{e}) \hat{v}_2 \longrightarrow [\hat{v}_2/x]\hat{e} \mid \varnothing}\ (\textsc{E-App3})$$

$$\frac{\hat{e} \rightarrow \hat{e}' \mid \varepsilon}{\hat{e}.\pi \longrightarrow \hat{e}'.\pi \mid \varepsilon}\ (\textsc{E-OperCall1}) \qquad \frac{r \in R \quad \pi \in \Pi}{r.\pi \longrightarrow \texttt{unit} \mid \{r.\pi\}}\ (\textsc{E-OperCall2})$$

$$\frac{}{\texttt{import}(\varepsilon)\ x = \hat{v}\ \texttt{in}\ e \longrightarrow [\hat{v}/x]\texttt{annot}(e, \varepsilon) \mid \varnothing}\ (\textsc{E-Module2})$$

5 Proofs

Lemma 1. *If* $\varepsilon \subseteq \texttt{effects}(\hat{\tau})$ *and* $\texttt{ho-safe}(\hat{\tau}, \varepsilon)$ *then* $\hat{\tau} <: \texttt{annot}(\texttt{erase}(\hat{\tau}), \varepsilon)$*.*

Lemma 2. *If* $\varepsilon \subseteq \texttt{ho-effects}(\hat{\tau})$ *and* $\texttt{safe}(\hat{\tau}, \varepsilon)$ *then* $\texttt{annot}(\texttt{erase}(\hat{\tau}), \varepsilon) <: \hat{\tau}$*.*

Theorem 1. *If* $\Gamma, x : \texttt{erase}(\hat{\tau}) \vdash e : \tau$ *and* $\varepsilon = \texttt{effects}(\hat{\tau})$ *and* $\texttt{ho-safe}(\hat{\tau}, \varepsilon)$ *then* $\texttt{annot}(\Gamma), x : \hat{\tau} \vdash \texttt{annot}(e, \varepsilon) : \texttt{annot}(\tau, \varepsilon)\ \texttt{with}\ \varepsilon$*.*

Theorem 2 (Progress). *If* $\Gamma \vdash \hat{e}_A : \tau_A\ \texttt{with}\ \varepsilon_A$ *then* $\hat{e}_A$ *is a value or* $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$*.*

Proof. By induction on $\Gamma \vdash \hat{e}_A : \tau_A\ \texttt{with}\ \varepsilon_A$.

$\boxed{\text{Case: } \varepsilon\text{-Resource, } \varepsilon\text{-Unit, } \varepsilon\text{-Abs}}$ Then $\hat{e}_A$ is a value.

$\boxed{\text{Case: } \varepsilon\text{-App}}$ Then $\hat{e}_A = \hat{e}_1\ \hat{e}_2$. We consider the cases in which $\hat{e}_1$ and $\hat{e}_2$ are values.

If $\hat{e}_1$ is not a value then by inductive assumption there is a reduction $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. Then $\hat{e}_1\ \hat{e}_2$ reduces by the rule E-App1, giving $\hat{e}_1\ \hat{e}_2 \longrightarrow \hat{e}_1'\ \hat{e}_2 \mid \varepsilon$.

If $\hat{e}_2$ is not a value then WLOG $\hat{e}_1$ is a value. By inductive assumption $\hat{e}_2 \longrightarrow \hat{e}_2' \mid \varepsilon$. Then $\hat{v}_1\ \hat{e}_2$ reduces by the rule E-App2, giving $\hat{v}_1\ \hat{e}_2 \longrightarrow \hat{v}_1\ \hat{e}_2' \mid \varepsilon$.

If $\hat{e}_1$ and $\hat{e}_2$ are both values then by canonical forms $\hat{e}_1 = \hat{v}_1 = \lambda x : \tau_2.e$. Then $\hat{v}_1\ \hat{v}_2$ reduces by the rule E-App3, giving $\hat{v}_1\ \hat{v}_2 \longrightarrow [\hat{v}_2/x]\hat{e} \mid \varnothing$.

$\boxed{\text{Case: } \varepsilon\text{-OperCall}}$ Then $\hat{e}_A = \hat{e}_1.\pi$. We consider whether $\hat{e}_1$ is a value.

If $\hat{e}_1$ is not a value then by inductive assumption there is a reduction $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. Then $\hat{e}_1.\pi$ reduces by the rule E-OperCall1, giving $\hat{e}_1.\pi \longrightarrow \hat{e}_1'.\pi \mid \varepsilon$.

If $\hat{e}_1$ is a value then $\hat{e}_1 = r$ by canonical forms. By the assumption that $r.\pi$ is closed under Γ, we know $r \in R$ and $\pi \in \Pi$. Then $\hat{e}_1.\pi$ reduces by the rule E-OperCall2, giving $r.\pi \longrightarrow \texttt{unit} \mid \varepsilon$.

$\boxed{\text{Case: } \varepsilon\text{-Module}}$ Then $e_A = \texttt{import}(\varepsilon)\ x = \hat{e}\ \texttt{in}\ e$ which reduces by the rule E-Module, giving $\texttt{import}(\varepsilon)\ x = \hat{e}\ \texttt{in}\ e \longrightarrow [\hat{v}/x]\texttt{annot}(e, \varepsilon) \mid \varnothing$.