Notation: $\hat{\Gamma} \vdash \delta_1, ..., \delta_n$ means $\hat{\Gamma} \vdash \delta_1$ and $\hat{\Gamma} \vdash \delta_2$ and ... and $\hat{\Gamma} \vdash \delta_n$, where each δ_i is a judgement.

Lemma 1 (Narrowing 1 (Subtypes)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$. The tricky cases are S-TypePoly and S-TypeVar; the others follow by routine application of the inductive hypothesis to subderivations.

Case: S-Reflexive. Then $\hat{\tau}_1 = \hat{\tau}_2$, and $\hat{\tau}_1 <: \hat{\tau}_2$ holds in any context, including $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}$.

Case: S-Transitive. Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_C$. By inversion, there is some $\hat{\tau}_B$ such that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption, we get the judgements $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Then by S-Transitive, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_C$, which is the same as $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

Case: S-ResourceSet. Follows immediately, since the premises of this rule have nothing to do with the context. That is, if $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$, then by inversion, $r \in \overline{r_1} \implies r \in \overline{r_2}$. Then by S-ResourceSet, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \{\overline{r_1}\} <: \{\overline{r_2}\}$.

Case: S-Arrow. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon} \hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$ and $\varepsilon' \subseteq \varepsilon$. To these first two judgements, apply the inductive assumption, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_B <: \hat{\tau}'_B$. Then by S-Arrow, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_A \rightarrow_{\varepsilon'} \hat{\tau}_B <: \hat{\tau}'_A \rightarrow_{\varepsilon} \hat{\tau}'_B$.

Case: S-TypePoly. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$. By inversion, we have the following two judgements:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Using (1) and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis can be used to obtain (3).

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}'_A <: \hat{\tau}_A$

Let $\Delta' = \Delta, Y <: \hat{\tau}'_A$. With this, and the assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, we shall apply the inductive hypothesis to obtain (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{\tau}_B <: \hat{\tau}'_B$

Expanding the definition of Δ', we get (5),

5. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}'_A \vdash \hat{\tau}_B <: \hat{\tau}'_B$

From (3) and (5), we can use S-TypePoly to obtain (6), which is the theorem conclusion.

6. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash (\forall Y <: \hat{\tau}_A.\hat{\tau}_B) <: (\forall Z <: \hat{\tau}'_A.\hat{\tau}'_B)$

Case: S-TypeVar. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash X <: \hat{\tau}$. It is also true that (1) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}'$, by use of S-TypeVar. The assumption $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ can be widened to (2) $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$. Then by (1) and (2), we can apply S-Transitive to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash X <: \hat{\tau}$.

Subcase 2. $X \neq Y$. Then $X <: \hat{\tau}$ is not used in the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$, so the judgement can be strengthened to $\hat{\Gamma}, \hat{\Delta} \vdash Y <: \hat{\tau}_B$. Then the judgement can be weakened to $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash Y <: \hat{\tau}_B$.

Lemma 2 (Narrowing 2 (Effects)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$ *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$*, then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$*.*

Proof. By induction on $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon_1 \subseteq \varepsilon_2$.

Lemma 3 (Narrowing 3 (Types)). *If* $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε *and* $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ *then* $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A. ε-Abs, ε-PolyTypeAbs, ε-PolyTypeApp, ε-PolyFxAbs, ε-PolyFxApp are the tricky cases; they require the use of the inductive hypothesis in a slightly more tricky way. The other cases follow by routine induction.

$\boxed{\textit{Case: } \varepsilon\text{-Var.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash x : \hat{\tau}_A$ `with` $\varnothing$, where $\hat{e} = x$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash x : \hat{\tau}_A$ `with` $\varnothing$. Then by weakening, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash x : \hat{\tau}_A$ `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-Resource.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash r : \{\overline{r}\}$ `with` $\varnothing$, where $\hat{e} = r$. Since $X <: \hat{\tau}$ is not used in the derivation, we can strengthen the context to get $\hat{\Gamma}, \hat{\Delta} \vdash r : \{r\}$ `with` $\varnothing$. Then by weakening, $\hat{\Gamma}, x <: \hat{\tau}', \hat{\Delta} \vdash r : \{r\}$ `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-OperCall.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1.\pi$: `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$, and $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\}$ `with` ε_1. To this second judgement we apply the inductive hypothesis, giving $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \{\overline{r}\}$ `with` ε_1. With this new judgement, apply ε-OperCall to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1.\pi$: `Unit with` $\varepsilon_1 \cup \{r.\pi \mid r \in \overline{r}\}$.

$\boxed{\textit{Case: } \varepsilon\text{-Subsume.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_A <: \hat{\tau}_B, \varepsilon \subseteq \varepsilon'$. By applying Narrowing Lemma 1 to the first judgement, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau} <: \hat{\tau}'$. By applying the Narrowing Lemma for effects[1], $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon \subseteq \varepsilon'$. With these two judgements, ε-Subsume can be used to obtain the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e} : \hat{\tau}_A$ `with` ε_A.

$\boxed{\textit{Case: } \varepsilon\text{-Abs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.\hat{e}_2 : \hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2$ `with` $\varnothing$, where $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2. By letting $\hat{\Delta}' = \hat{\Delta}, x : \hat{\tau}_1$, this second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2.

   Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, apply the inductive hypothesis to obtain (2),

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2.

   Using the definition of $\hat{\Delta}'$, this can be simplified,

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, x : \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2.

   Then with (3) we can use ε-Abs to get (4),

4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda x : \hat{\tau}_1.e_2 : \hat{\tau}_1 \rightarrow_{\varepsilon_2} \hat{\tau}_2$ `with` $\varnothing$

$\boxed{\textit{Case: } \varepsilon\text{-App.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$, where the following judgements are true from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` ε_1
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

   By applying the inductive assumption to (1) and (2), we get (3) and (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_2 \rightarrow_{\varepsilon_3} \hat{\tau}_3$ `with` ε_1
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

[1] This has yet to be proven

Then by ε-App, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1\ \hat{e}_2 : \hat{\tau}_3$ `with` $\varepsilon_1 \cup \varepsilon_2 \cup \varepsilon_3$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_2 `with` $\varnothing$. From inversion, we have $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2. By letting $\Delta' = \Delta, Y <: \hat{\tau}_1$, the second judgement can be rewritten,

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

By applying the inductive hypothesis to (1), we get judgement (2), which further simplifies to (3) by simplifying $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, Y <: \hat{\tau}_1 \vdash \hat{e}_2 : \hat{\tau}_2$ `with` ε_2

Then by ε-PolyTypeAbs, we get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda Y <: \hat{\tau}_1.\hat{e}_2 : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_2 `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon.\hat{e}_1 : \forall \phi \subseteq \varepsilon.\hat{\tau}_1$ `caps` ε_1 `with` $\varnothing$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. By letting $\hat{\Delta}' = \hat{\Delta}, \phi \subseteq \varepsilon$, the second judgement can be rewritten as (1),

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1

Using (1) and the assumption that $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$, the inductive hypothesis gives judgement (2), which further simplifies to (3) by expanding the definition of $\hat{\Delta}'$,

2. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}' \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1
3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta}, \phi \subseteq \varepsilon \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1

Then from (2), we can apply ε-PolyFxAbs, giving the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \lambda \phi \subseteq \varepsilon.\hat{e}_1 : \forall \phi \subseteq \varepsilon.\hat{\tau}_1$ `caps` ε_1 `with` $\varnothing$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2$ `with` $[\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$, where the following judgements are from inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to get (3). With the same assumption and (2), we can apply Narrowing Lemma 1 (Subtypes) to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall Y <: \hat{\tau}_1.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{\tau}_1' <: \hat{\tau}_1$

From (3) and (4), ε-PolyTypeApp gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \hat{\tau}_1' : [\hat{\tau}_1'/Y]\hat{\tau}_2$ `with` $[\hat{\tau}_1'/Y]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2$ `with` $[\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$, where the following are true by inversion:

1. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_A : \forall \phi \subseteq \varepsilon.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
2. $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

With the assumption that $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{\tau}' <: \hat{\tau}$ and (1), we can apply the inductive hypothesis to obtain (3). With the same assumption and (2), we can apply the Narrowing Lemma for Effect Judgements[2] to get (4),

3. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A : \forall \phi \subseteq \varepsilon.\hat{\tau}_2$ `caps` ε_1 `with` ε_2
4. $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \varepsilon' \subseteq \varepsilon$

[2] Doesn't actually exist yet

With (3) and (4) we can apply ε-PolyFxApp to get $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_A\ \varepsilon' : [\varepsilon'/\phi]\hat{\tau}_2$ `with` $[\varepsilon'/\phi]\varepsilon_1 \cup \varepsilon_2$.

$\boxed{\textit{Case: } \varepsilon\text{-Import.}}$ (We prove for a single import). Then $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash$ `import`$(\varepsilon_s)\ x_1 = \hat{e}_1$ `in` e : `annot` (τ, ε_s) `with` $\varepsilon_s \cup \varepsilon_1$. By inversion, $\hat{\Gamma}, X <: \hat{\tau}, \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. By inductive hypothesis, $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash \hat{e}_1 : \hat{\tau}_1$ `with` ε_1. This, together with the other premises obtained by inversion, gives the judgement $\hat{\Gamma}, X <: \hat{\tau}', \hat{\Delta} \vdash$ `import`$(\varepsilon_s)\ x_1 = \hat{e}_1$ `in` e : `annot` (τ, ε_s) `with` $\varepsilon_s \cup \varepsilon_1$.

Lemma 4 (Substitution (Values)). *If $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ `with` ε and $\hat{\Gamma} \vdash \hat{v} : \hat{\tau}'$ `with` $\varnothing$, then $\hat{\Gamma} \vdash [\hat{v}/x]\hat{e} : \hat{\tau}$ `with` ε*

Proof. By induction on the derivation of $\hat{\Gamma}, x : \hat{\tau}' \vdash \hat{e} : \hat{\tau}$ `with` ε. We show for those extra cases in polymorphic `CC`.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeAbs.}}$ Then $\hat{e} = \lambda X <: \hat{\tau}_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda X <: \hat{\tau}_1.[\hat{v}/y]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxAbs.}}$ Then $\hat{e} = \lambda\phi \subseteq \varepsilon_1.\hat{e}_1$, and $[\hat{v}/x]\hat{e} = \lambda\phi \subseteq \varepsilon_1.[\hat{v}/x]\hat{e}_1$. By inversion and inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxAbs, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyTypeApp.}}$ Then $\hat{e} = \hat{e}_1\ \hat{\tau}_1$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \hat{\tau}_1$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyTypeApp, we get the conclusion.

$\boxed{\textit{Case: } \varepsilon\text{-PolyFxApp.}}$ Then $\hat{e} = \hat{e}_1\ \varepsilon$, and $[\hat{v}/x]\hat{e} = [\hat{v}/x]\hat{e}_1\ \varepsilon$. By inductive hypothesis, $[\hat{v}/x]\hat{e}_1$ in $\hat{\Gamma}$ can be typed the same as $\hat{e}_1$ in $\hat{\Gamma}, x : \hat{\tau}'$. Then by applying ε-PolyFxApp, we get the conclusion.

Lemma 5 (Subtyping Preserved Under Substitution). *If $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_1 <: \hat{\tau}_2$ and $\hat{\Gamma} \vdash \hat{\tau}' <: \hat{\tau}$ then $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$*

Proof. By induction on the derivation of $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_1 <: \hat{\tau}_2$.

$\boxed{\textit{Case: } \text{S-Reflexive.}}$ Then $\hat{\tau}_1 = \hat{\tau}_2$, so $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$ by S-Reflexive.

$\boxed{\textit{Case: } \text{S-Transitive.}}$ Let $\hat{\tau}_1 = \hat{\tau}_A$ and $\hat{\tau}_2 = \hat{\tau}_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_A <: \hat{\tau}_B$ and $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_B <: \hat{\tau}_C$. Applying the inductive assumption to these judgements, we get $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_B$ and $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}_C$. By S-Transitive, $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_A <: [\hat{\tau}'/X]\hat{\tau}_C$.

$\boxed{\textit{Case: } \text{S-ResourceSet.}}$ Sets of resources are unchanged by type-variable substitution, so $[\hat{\tau}'/X]\{\overline{r_1}\} = \{\overline{r_1}\}$ and $[\hat{\tau}'/X]\{\overline{r_2}\} = \{\overline{r_2}\}$. Then the subtyping judgement in the conclusion of the theorem can be the original one from the assumption.

$\boxed{\textit{Case: } \text{S-Arrow.}}$ Then the subtyping judgement from the assumption is $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}_A \to_\varepsilon \hat{\tau}_B <: \hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B$. By inversion and inductive assumption, we get the judgements $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_B$ and $\varepsilon \subseteq \varepsilon'$. Then by S-Arrow, we have $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_A \to_\varepsilon [\hat{\tau}'/X]\hat{\tau}_B <: [\hat{\tau}'/X]\hat{\tau}'_A \to_{\varepsilon'} [\hat{\tau}'/X]\hat{\tau}'_B$. By applying the definition of substitution on an arrow type in reverse, we can rewrite this judgement as $\hat{\Gamma} \vdash [\hat{\tau}'/X](\hat{\tau}_1 \to_\varepsilon \hat{\tau}_B) <: [\hat{\tau}'/X](\hat{\tau}'_A \to_{\varepsilon'} \hat{\tau}'_B)$, which is the same as $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}_1 <: [\hat{\tau}'/X]\hat{\tau}_2$.

$\boxed{\textit{Case: } \text{S-TypePoly.}}$s Then $\hat{\tau}_1 = \forall Y <: \hat{\tau}_A.\hat{\tau}_B$ and $\hat{\tau}_2 = \forall Z <: \hat{\tau}'_A.\hat{\tau}'_B$. By inversion, $\hat{\Gamma}, X <: \hat{\tau} \vdash \hat{\tau}'_A <: \hat{\tau}_A$ and $\hat{\Gamma}, X <: \hat{\tau}, Z <: \hat{\tau}'_A \vdash \hat{\tau}'_B <: \hat{\tau}'_A$. Applying the inductive assumption to both these judgements, $\hat{\Gamma} \vdash [\hat{\tau}'/X]\hat{\tau}'_A <: [\hat{\tau}'/X]\hat{\tau}_A$ and $\hat{\Gamma}, Z <: \hat{\tau}'_A \vdash [\hat{\tau}'/X]\hat{\tau}'_B < [\hat{\tau}'/X]\hat{\tau}'_A$. Then by S-TypePoly, $\hat{\Gamma} \vdash \forall Y <:$

$[\hat\tau'/X]\hat\tau_A.[\hat\tau'/X]\hat\tau_B <: \forall Z <: [\hat\tau'/X]\hat\tau'_A.[\hat\tau'/X]\hat\tau'_B$, which is the same as $\hat\Gamma \vdash [\hat\tau'/X]\hat\tau_1 <: [\hat\tau'/X]\hat\tau_2$.

$\boxed{\textit{Case:}\ \text{S-TypeVar.}}$ Then $\hat\Gamma, X <: \hat\tau \vdash Y <: \hat\tau_2$. There are two cases, depending on whether $X = Y$.

Subcase 1. $X = Y$. Then $\hat\Gamma, X <: \hat\tau \vdash X <: \hat\tau$. We want to show (1) $\hat\Gamma, X <: \hat\tau \vdash [\hat\tau'/X]X <: [\hat\tau'/X]\hat\tau$. Firstly, $[\hat\tau'/X]X = \hat\tau'$. Secondly, because $\texttt{WF}(\hat\Gamma, X <: \hat\tau)$ then $X \notin \texttt{free-vars}(\hat\tau)$, so $[\hat\tau'/X]\hat\tau = \hat\tau$. Therefore, judgement (1) is the same as $\hat\Gamma, X <: \hat\tau \vdash \hat\tau' <: \hat\tau$, which is true by assumption.

Subcase 2. $X \neq Y$. Then $X <: \hat\tau$ is not used in the derivation, so $\hat\Gamma, X <: \hat\tau \vdash Y <: \hat\tau_2$ is true by widening the context in the judgement $\hat\Gamma \vdash Y <: \hat\tau_2$[3]. Then $\hat\Gamma \vdash [\hat\tau'/X]Y <: [\hat\tau'/X]\hat\tau_2$ by inductive assumption. By widening, $\hat\Gamma, X <: \hat\tau \vdash [\hat\tau'/X]Y <: \hat\tau'/X]\hat\tau_2$.

Lemma 6 (Substitution (Types)). *If $\hat\Gamma, X <: \hat\tau', \hat\Delta \vdash \hat e : \hat\tau$ $\texttt{with}$ ε and $\hat\Gamma \vdash \hat\tau'' <: \hat\tau'$, then $\hat\Gamma, \hat\Delta \vdash [\hat\tau''/X]\hat e : \hat\tau$ $\texttt{with}$ ε*

Proof. By induction on the derivation of $\hat\Gamma, X <: \hat\tau' \vdash \hat e : \hat\tau$ $\texttt{with}$ ε.'

$\boxed{\textit{Case:}\ \varepsilon\text{-Var},\ \varepsilon\text{-Resource.}}$ Then $\hat e = [\hat\tau''/X]\hat e$, so the typing judgement in the antecedent and consequent can be the same.

$\boxed{\textit{Case:}\ \varepsilon\text{-Abs.}}$ Then $\hat e = \lambda x : \hat\tau_1.\hat e_2$, and $[\hat\tau''/X]\hat e = \lambda x : [\hat\tau''/X]\hat\tau_1.[\hat\tau''/X]\hat e_2$. WLOG assume that $\hat\tau = \hat\tau_1 \to_{\varepsilon'} \hat\tau_2$. By inversion, we have (1). Letting $\hat\Delta = x : \hat\tau_1$, this can be rewritten as (2). By applying the inductive assumption to (2), we get (3), which simplifies to (4) by simplifying $\hat\Delta$ back into $x : \hat\tau_1$.

1. $\hat\Gamma, X <: \hat\tau', x : \hat\tau_1 \vdash \hat e_2 : \hat\tau_2$ $\texttt{with}$ ε'
2. $\hat\Gamma, X <: \hat\tau', \hat\Delta \vdash \hat e_2 : \hat\tau_2$ $\texttt{with}$ ε'
3. $\hat\Gamma, \hat\Delta \vdash [\hat\tau''/X]\hat e_2 : \hat\tau_2$ $\texttt{with}$ ε
4. $\hat\Gamma, x : \hat\tau_1 \vdash [\hat\tau''/X]\hat e_2 : \hat\tau_2$ $\texttt{with}$ ε'

From (4), we can apply ε-Abs to get $\hat\Gamma \vdash \lambda x : \hat\tau_1.[\hat\tau''/X]\hat e_2 : \hat\tau_2$ $\texttt{with}$ ε'.
What now?

Theorem 1 (Progress). *If $\hat\Gamma \vdash \hat e : \hat\tau$ $\texttt{with}$ ε and $\hat e$ is not a value, then $\hat e \longrightarrow \hat e' \mid \varepsilon$, for some $\hat e', \varepsilon$.*

Proof. By induction on the derivation of $\hat\Gamma \vdash \hat e : \hat\tau$ $\texttt{with}$ ε.

Case: ε-PolyTypeAbs. Trivial; $\hat e$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat e$ is a value.

Case: ε-PolyTypeApp. Then $\hat e = \hat e_1\ \hat\tau'$. If $\hat e_1$ is not a value then $\hat e_1 \longrightarrow \hat e_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyTypeApp1 gives the reduction $\hat e_1\ \hat\tau' \longrightarrow \hat e''\hat\tau' \mid \varepsilon$. Otherwise, $\hat e$ is a value, so $\hat e = \lambda X <: \hat\tau_1.\hat e_2$, and applying E-PolyTypeApp2 gives the reduction $(\lambda X <: \hat\tau_1.\hat e_2)\hat\tau' \longrightarrow [\hat\tau'/X]\hat e_2 \mid \varnothing$.

Case: ε-PolyFxApp. Then $\hat e = \hat e_1\ \varepsilon'$. If $\hat e_1$ is not a value then $\hat e_1 \longrightarrow \hat e_1' \mid \varepsilon$ by inductive hypothesis, and applying E-PolyFxApp1 gives the reduction $\hat e_1\ \varepsilon' \longrightarrow \hat e_1'\ \varepsilon' \mid \varepsilon$. Otherwise, $\hat e$ is a value, so $\hat e = \lambda\phi \subseteq \varepsilon_1.\hat e_2$, and applying E-PolyFxApp2 gives the reduction $(\lambda\phi \subseteq \varepsilon_1.\hat e_2)\varepsilon' \longrightarrow [\varepsilon'/\phi]\hat e_2$.

Theorem 2 (Preservation). *If $\hat\Gamma \vdash \hat e_A : \hat\tau_A$ $\texttt{with}$ ε_A and $\hat e_A \longrightarrow \hat e_B \mid \varepsilon$, then $\hat\Gamma \vdash \hat e_B : \hat\tau_B$ $\texttt{with}$ ε_B, where $\hat e_B <: \hat e_A$ and $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$, for some $\hat e_B, \varepsilon, \hat\tau_B, \varepsilon_B$.*

[3] Note there is no explicit widening rule; be careful with this one.

Proof. By induction on the derivations of $\hat{\Gamma} \vdash \hat{e}_A : \hat{\tau}_A \texttt{ with } \varepsilon_A$ and $\hat{e}_A \longrightarrow \hat{e}_B \mid \varepsilon$.

Case: ε-PolyTypeAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyFxAbs. Trivial; $\hat{e}$ is a value.

Case: ε-PolyTypeApp. Then $\hat{e} = \hat{e}_1\ \hat{\tau}'$. Consider which reduction rule was used.

Subcase: E-PolyTypeApp1. Then $\hat{e}_1\ \hat{\tau}' \longrightarrow \hat{e}_1'\ \hat{\tau}' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyTypeApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\tau}' : \hat{\tau}_A \texttt{ with } \varepsilon_A$. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-PolyTypeApp2. Then $(\lambda X <: \hat{\tau}_3.\hat{e}')\hat{\tau}' \longrightarrow [\hat{\tau}'/X]\hat{e}' \mid \varnothing$.

The result follows by the substitution lemma.

Case: ε-PolyFxApp. Then $\hat{e} = \hat{e}_1\ \varepsilon'$. Consider which reduction rule was used.

Subcase: E-PolyFxApp1. Then $\hat{e}_1\ \varepsilon' \longrightarrow \hat{e}_1'\ \varepsilon' \mid \varepsilon$. By inversion, $\hat{e}_1 \longrightarrow \hat{e}_1' \mid \varepsilon$. With the inductive hypothesis and subsumption, $\hat{e}_1'$ can be typed in $\hat{\Gamma}$ the same as $\hat{e}_1$. Then by ε-PolyFxApp, $\hat{\Gamma} \vdash \hat{e}_1'\ \hat{\varepsilon}' : \hat{\tau}_A \texttt{ with } \varepsilon_A$. That $\varepsilon \cup \varepsilon_B \subseteq \varepsilon_A$ follows by inductive hypothesis.

Subcase: E-PolyFxApp2. Then $(\lambda \phi \subseteq \varepsilon_3.\hat{e}')\varepsilon' \longrightarrow [\varepsilon'/X]\hat{e}' \mid \varnothing$. **The result follows by the substitution lemma**.