

Dynamical Processes over Networks

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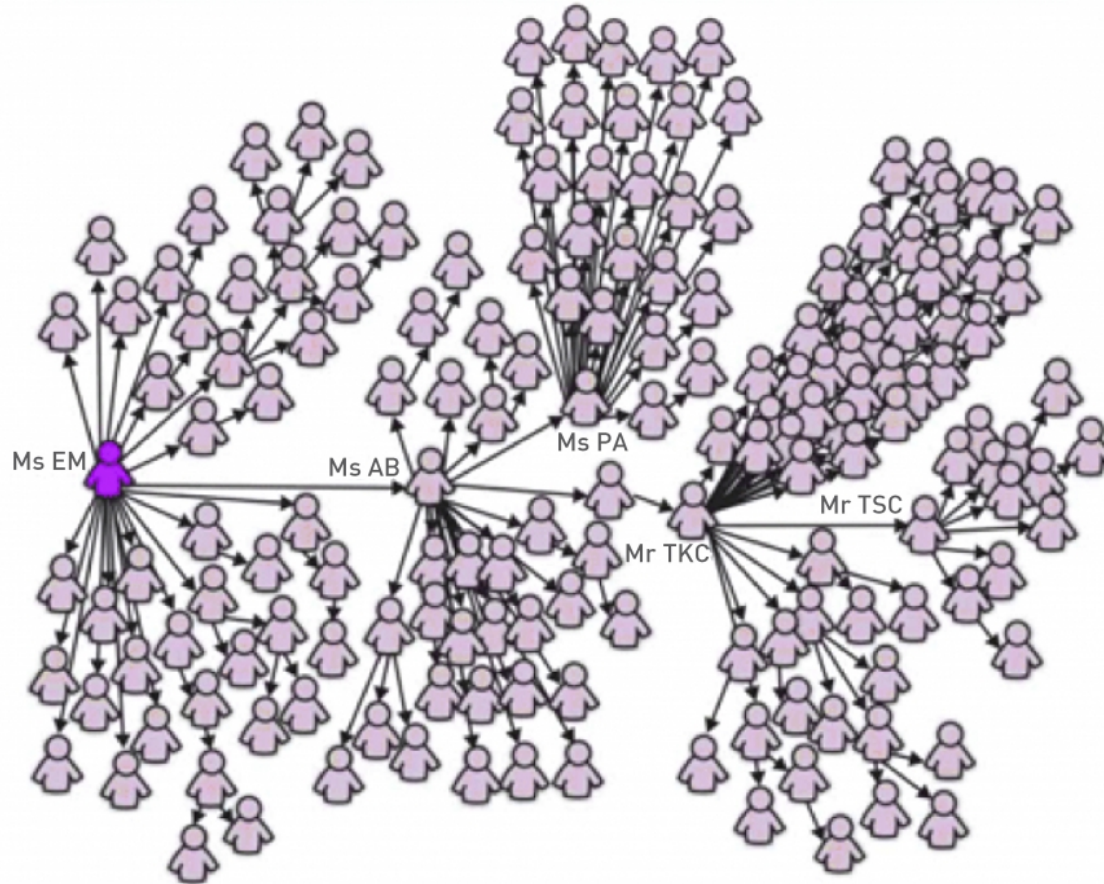
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Modeling epidemics over Networks

First analysis of
dynamical processes
over networks

Will let us exercise
some of the ways of
thinking about these
processes



Modeling epidemics over Networks

Questions

Are there network properties that predict spread of infection?

Are certain network types more resilient to infection than others?

If we can intervene (vaccinate) are there nodes in the network that are more effective to vaccinate?

Modeling epidemics over Networks

Questions

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We'll start by looking at spread over non-networked populations

Susceptibility and infection (SI model)

Individuals in the population can be in two states

An infected individual can infect *any* susceptible individual they are in contact with

If we start ($t = 0$) with some number of infected individuals (i_0).
How many infected individuals are there at time t ?



SI model

$$\frac{d}{dt}I(t) = \beta \langle k \rangle \frac{S(t)}{N} I(t)$$

$\langle k \rangle$, average number of contacts per individual in one time step

β , "rate" probability an I infects an S upon contact

SI model

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$$\frac{di}{dt} = \beta \langle k \rangle i(1 - i)$$

SI model

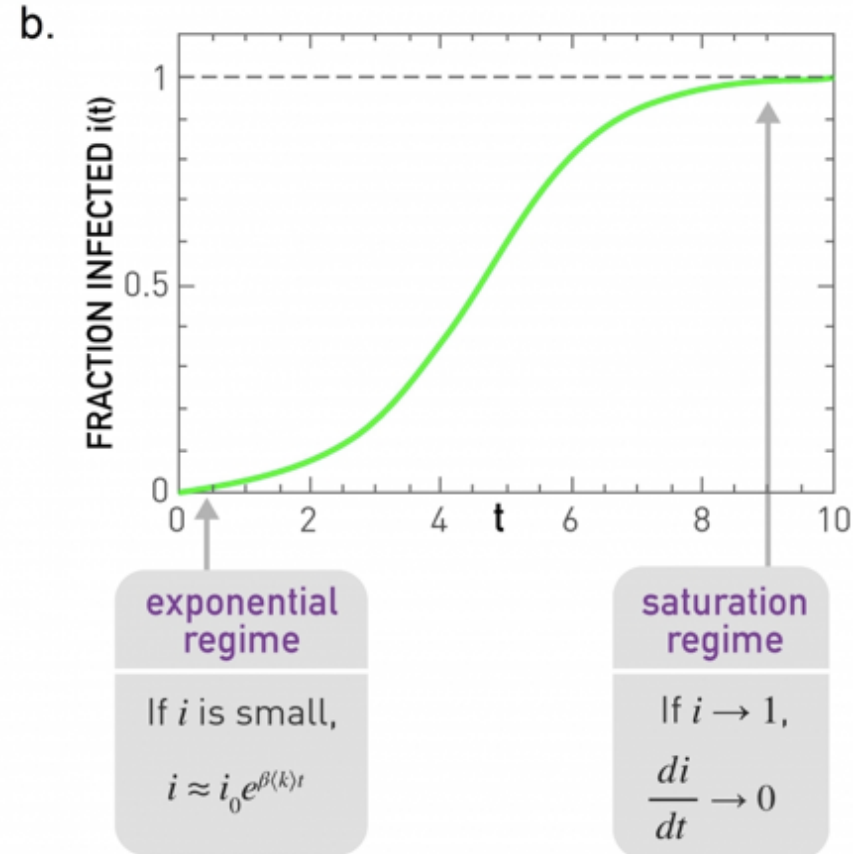
Fraction of infected individuals in population

$$i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$$

Characteristic time (t s.t.

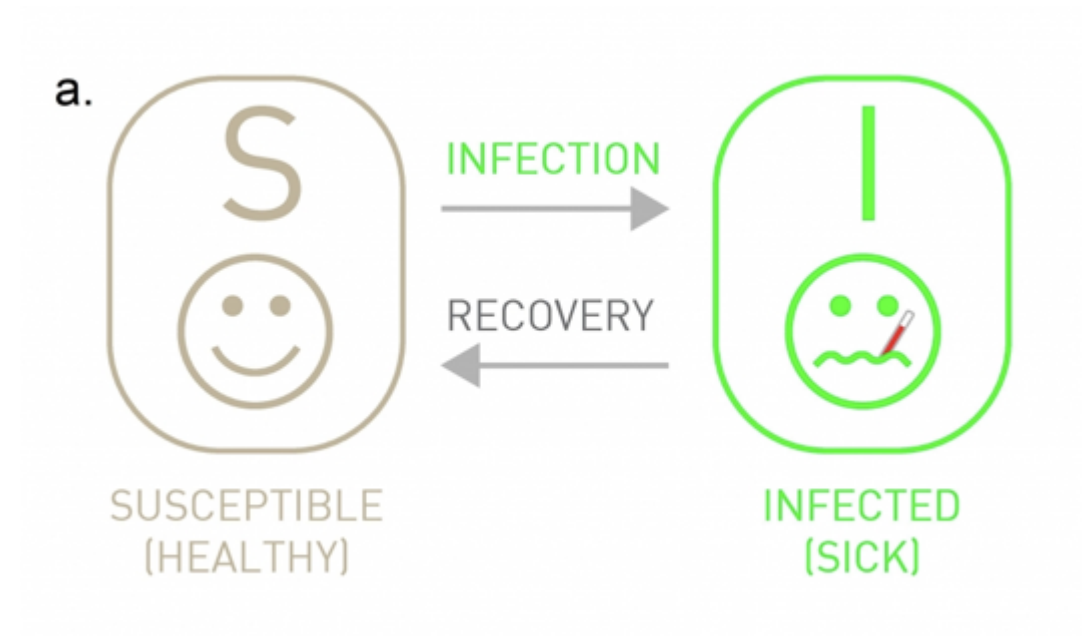
$$i(t) = 1/e \approx .36)$$

$$\tau = \frac{1}{\beta \langle k \rangle}$$



SIS model

Infection ends (recovery), individual becomes susceptible again



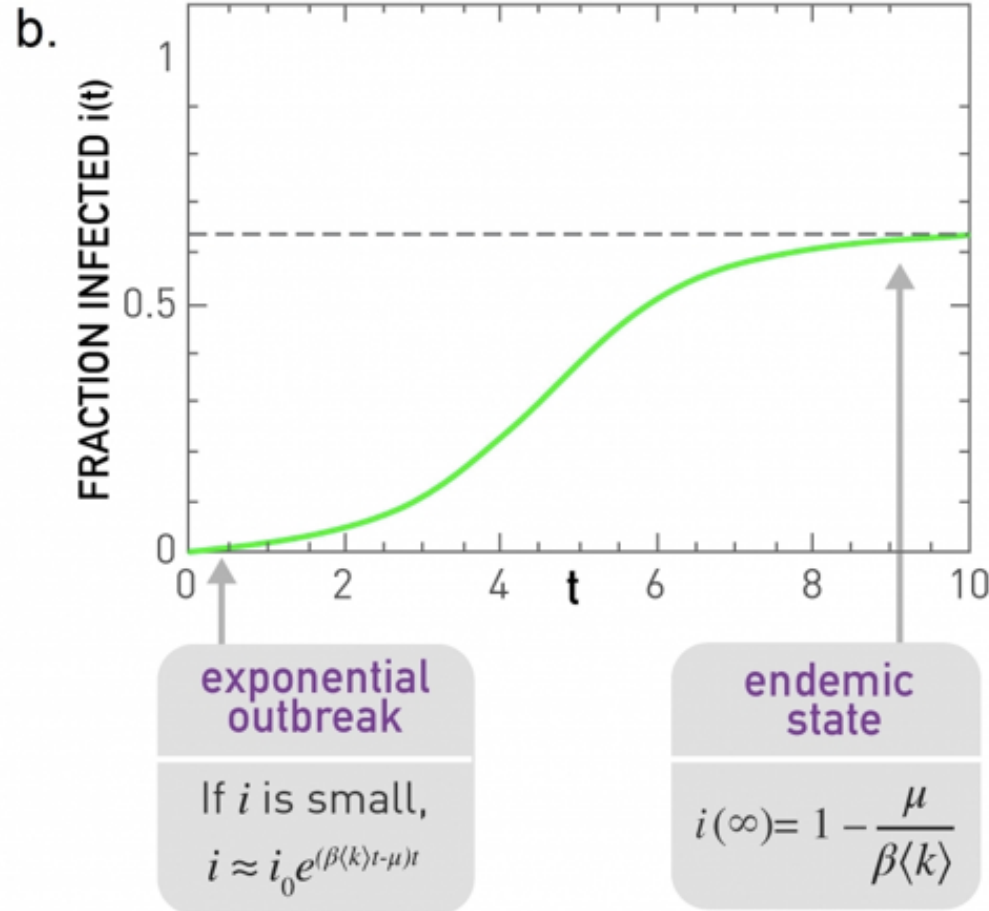
SIS model

$$\frac{di}{dt} = \beta \langle k \rangle i(1 - i) - \mu i$$

μ - recovery rate

SIS model

$$i(t) = \left(1 - \frac{\mu}{\beta\langle k \rangle}\right) \times \frac{C e^{(\beta\langle k \rangle - \mu)t}}{1 + C e^{(\beta\langle k \rangle - \mu)t}}$$



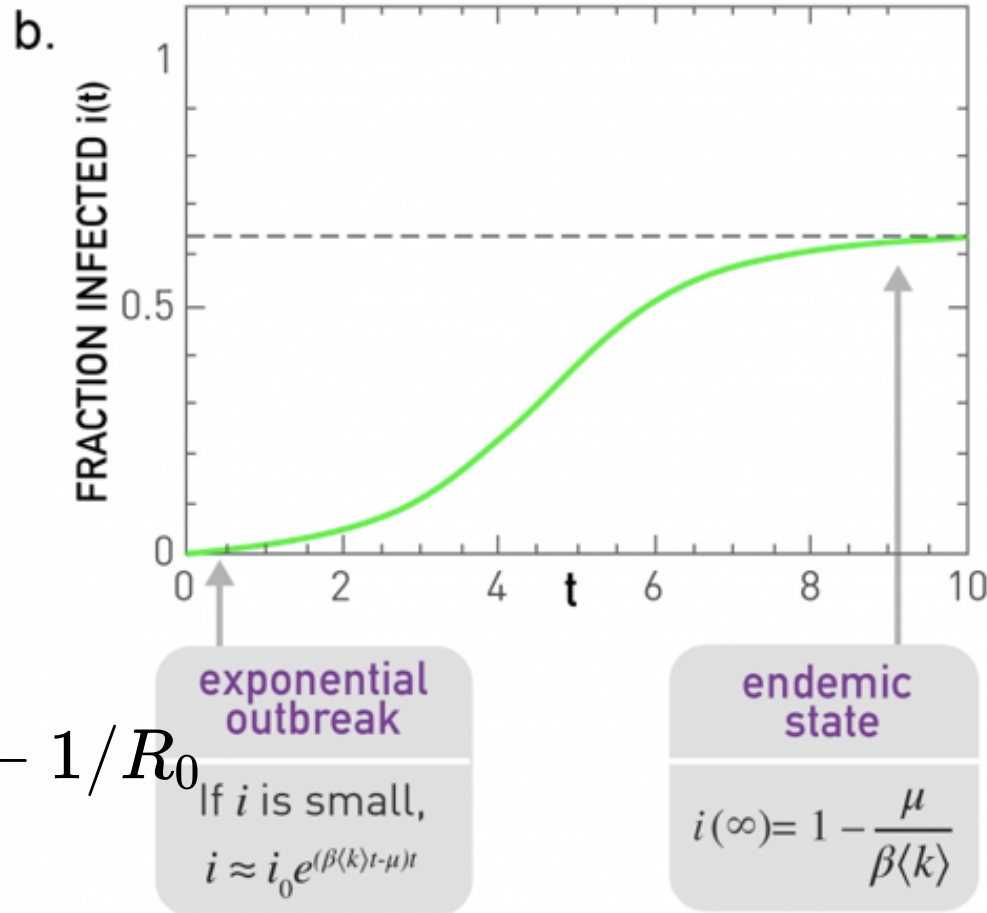
SIS model

Endemic State

Pathogen persists in population after saturation

$$R_0 = \frac{\beta \langle k \rangle}{\mu} > 1$$

$$i(\infty) = 1 - \frac{\mu}{\beta \langle k \rangle} = 1 - 1/R_0$$



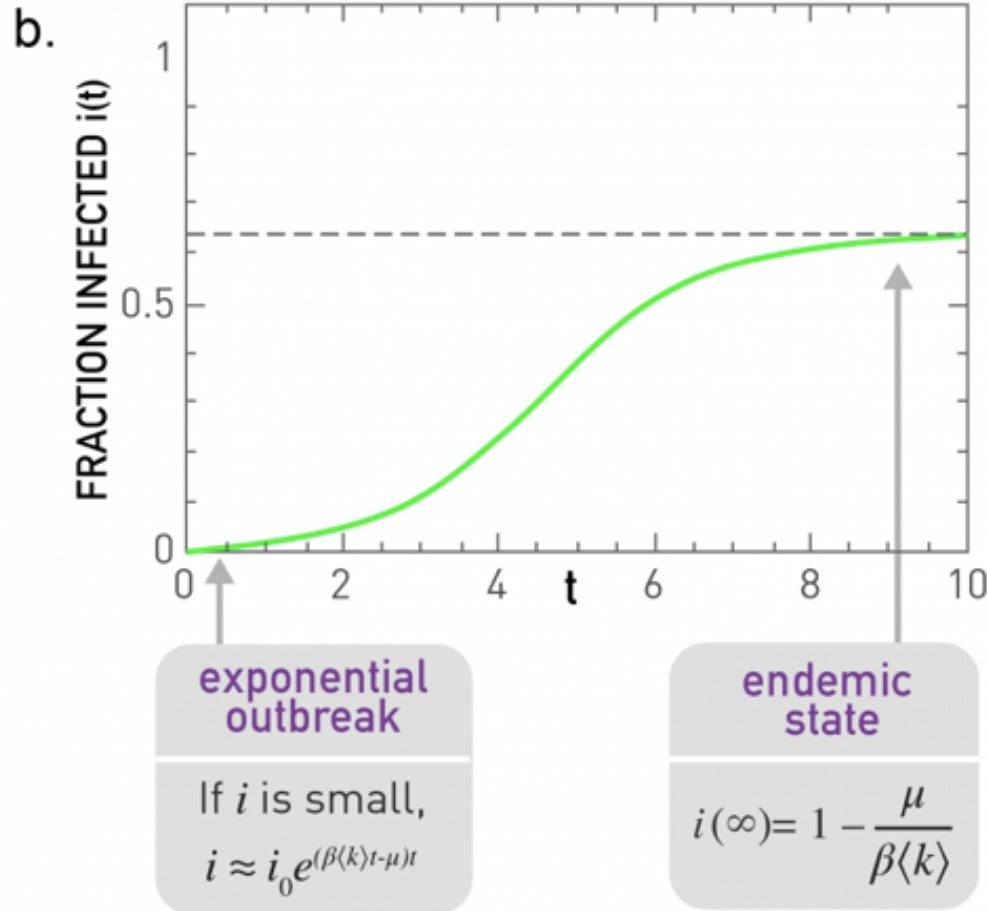
SIS model

Disease-free State

Pathogen disappears
from population

$$R_0 = \frac{\beta \langle k \rangle}{\mu} < 1$$

$$i(\infty) = 0$$



SIS model

Basic Reproductive Number

$$R_0 = \frac{\beta \langle k \rangle}{\mu}$$

Characteristic Time

$$\tau = \frac{1}{\mu(R_0 - 1)}$$

SIR model

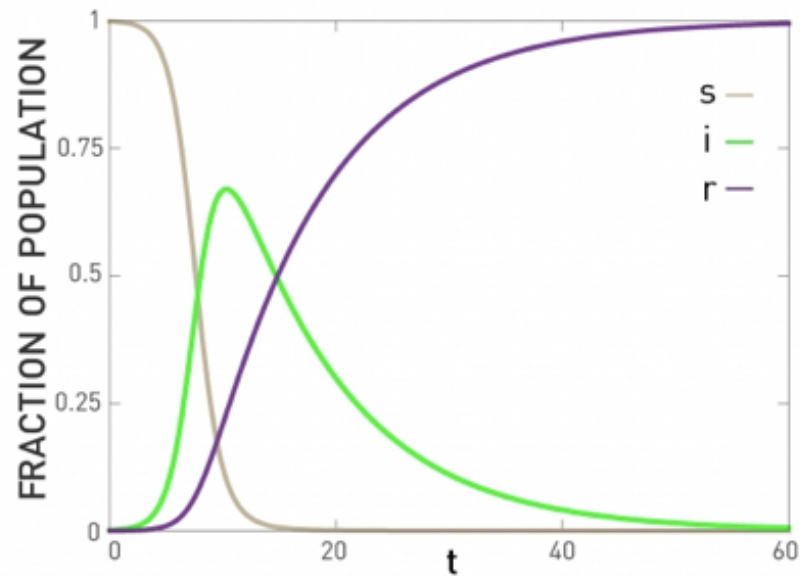
Individuals **removed** after infection (either death or immunity)



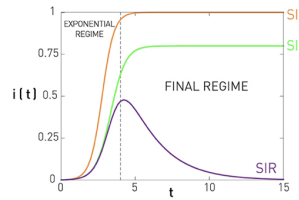
SIR model

c.

$$\begin{aligned}\frac{di}{dt} &= \beta \langle k \rangle i(1 - r - i) - \mu i \\ \frac{dr}{dt} &= \mu i \\ \frac{ds}{dt} &= -\beta \langle k \rangle i(1 - r - i)\end{aligned}$$



Summary



	SI	SIS	SIR
Exponential Regime: Number of infected individuals grows exponentially	$i = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$	$i = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$	No closed solution
Final Regime: Saturation at $t \rightarrow \infty$	$i(\infty) = 1$	$i(\infty) = 1 - \frac{\mu}{\beta \langle k \rangle}$	$i(\infty) = 0$
Epidemic Threshold: Disease does not always spread	No threshold	$R_0 = 1$	$R_0 = 1$

Epidemic processes over networks (SI)

Consider node i in network:

$s_i(t)$ average probability node i is *susceptible* at time t

$x_i(t)$ average probability node i is *infected* at time t

Epidemic processes over networks (SI)

$$\frac{ds_i}{dt} = -s_i\beta \sum_{j=1}^N a_{ij}x_j$$

$$\frac{dx_i}{dt} = s_i\beta \sum_{j=1}^N a_{ij}x_j$$

Epidemic processes over networks (SI)

$$\frac{ds_i}{dt} = -s_i\beta \sum_{j=1}^N a_{ij}x_j$$

$$\frac{dx_i}{dt} = s_i\beta \sum_{j=1}^N a_{ij}x_j$$

For large N , and early t

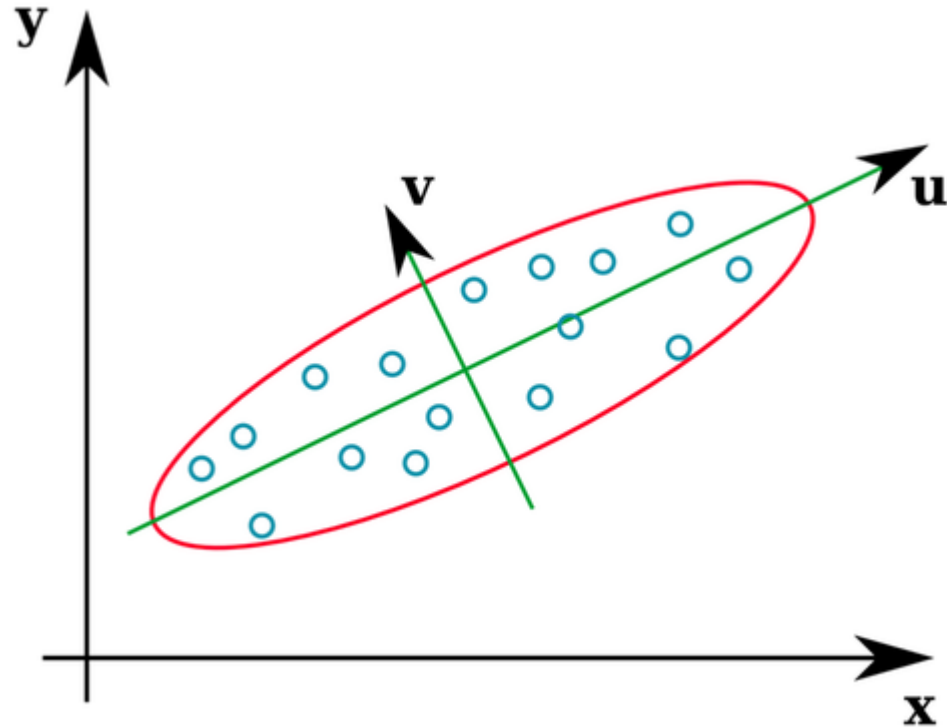
$$\frac{dx}{dt} = \beta Ax$$

Eigenvalue decomposition of A

$$A = V^T \Lambda V$$

Any quantity x_i over nodes in the graph can be written as

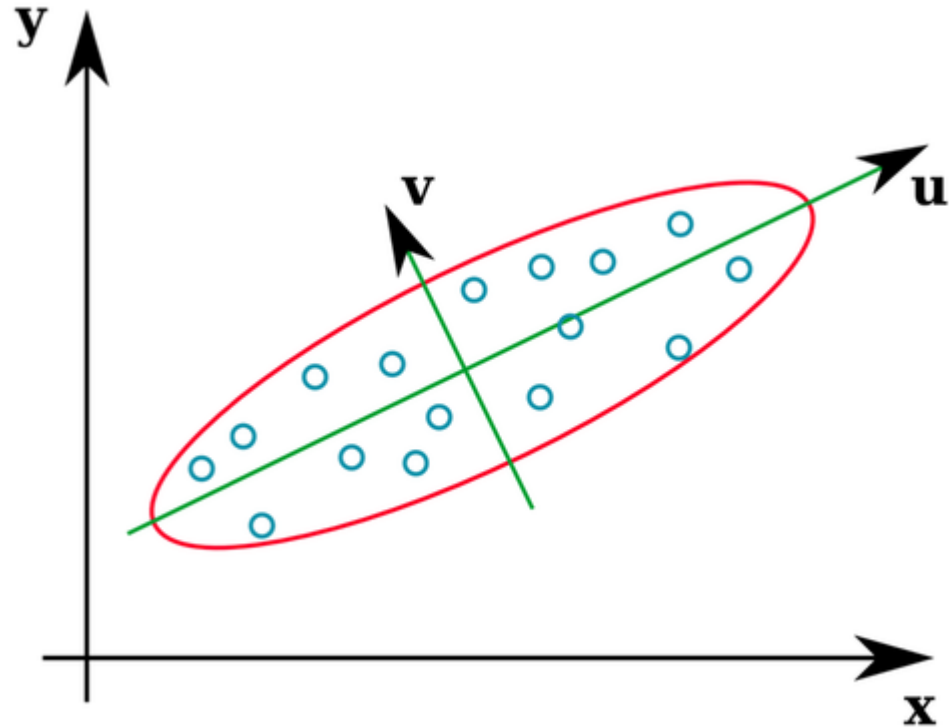
$$\mathbf{x} = \sum_{r=1}^N c_r \mathbf{v}_r$$



Eigenvalue decomposition of A

$$A\mathbf{v}_r = \lambda_r \mathbf{v}_r$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$$



Eigenvalue decomposition of A

Let's revisit centrality

$$\mathbf{x}(t) = A^t \mathbf{x}(0) = \sum_{r=1}^N c_r A^t \mathbf{v}_r$$

Then

$$\mathbf{x}(t) = \sum_{r=1}^N c_r \lambda_r^t \mathbf{v}_r = \lambda_1^t \sum_{i=1}^N c_r \left(\frac{\lambda_r}{\lambda_1} \right)^t \mathbf{v}_r$$

Eigenvalue decomposition of A

As t grows, first term dominates

$$\mathbf{x}(t) = c_1 \lambda_1^t \mathbf{v}_1$$

So set centrality \mathbf{x} to be proportional to first *eigenvector* \mathbf{v}_1

Eigenvalue decomposition of A

As t grows, first term dominates

$$\mathbf{x}(t) = c_1 \lambda_1^t \mathbf{v}_1$$

So set centrality \mathbf{x} to be proportional to first *eigenvector* \mathbf{v}_1

In which case $x = Ax$ if $\mathbf{x} = \frac{1}{\lambda_1} \mathbf{v}_1$ as desired

Back to epidemics (SI)

$\mathbf{x}(t)$ average probability each node is infected at time t

$$\frac{d\mathbf{x}}{dt} = \beta A\mathbf{x}$$

Can write as

$$\mathbf{x}(t) = \sum_{r=1}^N c_r(t) \mathbf{v}_r$$

Back to epidemics (SI)

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \sum_{r=1}^N \frac{dc_r}{dt} \mathbf{v}_r \\ &= \beta A \sum_{r=1}^N c_r(t) \mathbf{v}_r = \beta \sum_{r=1}^N \lambda_r c_r(t) \mathbf{v}_r\end{aligned}$$

Back to epidemics (SI)

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Implying

$$\frac{dc_r}{dt} = \beta \lambda_r c_r$$

Back to epidemics (SI)

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \sum_{r=1}^N \frac{dc_r}{dt} \mathbf{v}_r \\ &= \beta A \sum_{r=1}^N c_r(t) \mathbf{v}_r = \beta \sum_{r=1}^N \lambda_r c_r(t) \mathbf{v}_r\end{aligned}$$

Implying

$$\frac{dc_r}{dt} = \beta \lambda_r c_r$$

With solution

$$c_r(t) = c_r(0) e^{\beta \lambda_r t}$$

Back to epidemics (SI)

As before, first term dominates so

$$\mathbf{x}(t) \sim e^{\beta\lambda_1 t} \mathbf{v}_1$$

Eigen-centrality!

Back to epidemics (SIR)

$$\frac{d\mathbf{x}}{dt} = \beta A\mathbf{x} - \mu\mathbf{x}$$

Similarly

$$\mathbf{x}(t) \sim e^{(\beta\lambda_1 - \mu)t}$$

Back to epidemics (SIR)

$$\frac{d\mathbf{x}}{dt} = \beta A\mathbf{x} - \mu\mathbf{x}$$

Similarly

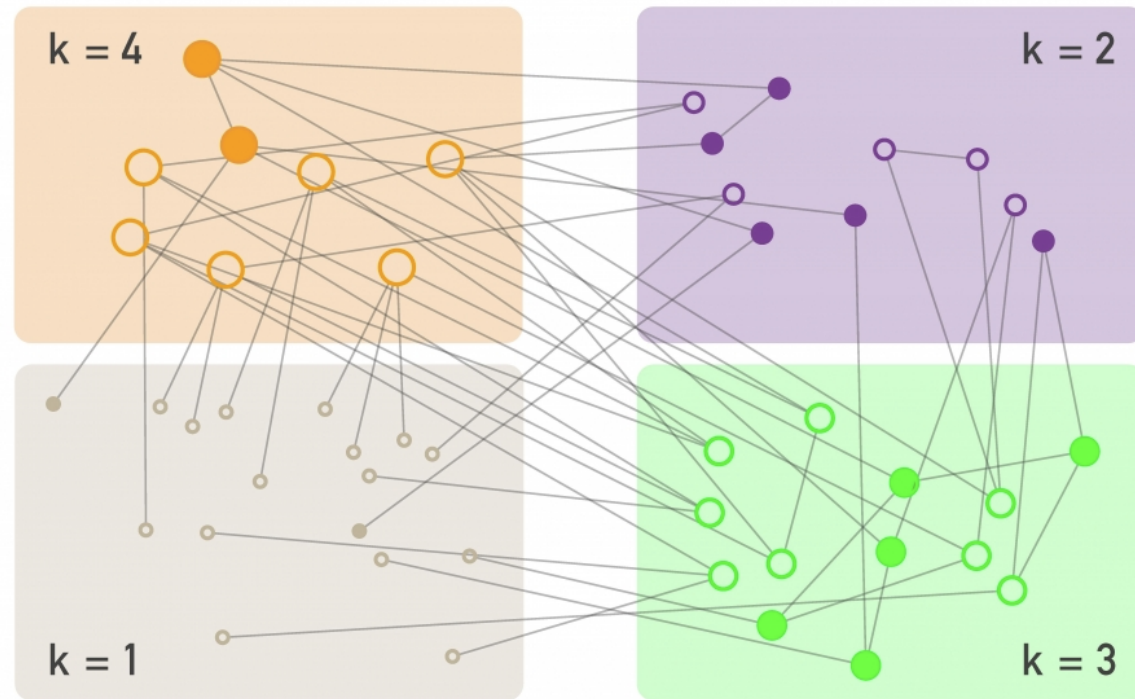
$$\mathbf{x}(t) \sim e^{(\beta\lambda_1 - \mu)t}$$

Is there an epidemic? Not if $R_0 = \frac{\beta}{\mu} = \frac{1}{\lambda_1}$

Degree distributions and epidemics

Degree Block Approximation

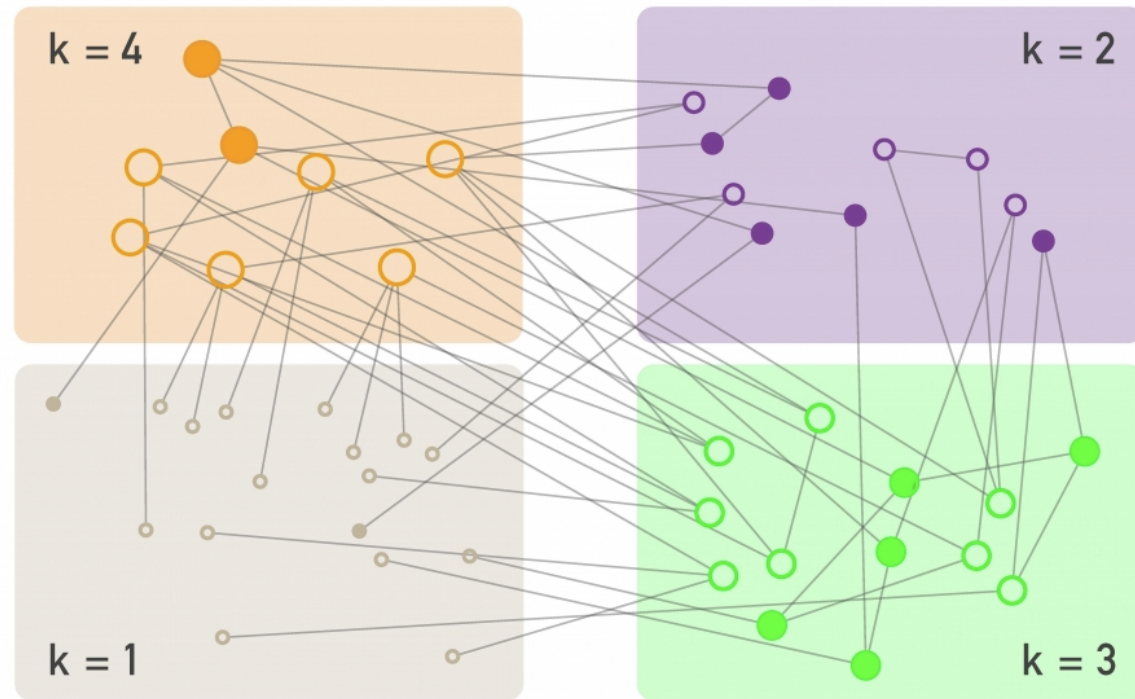
Assume that all nodes of the same degree are statistically equivalent



Degree distributions and epidemics

Degree Block Approximation

Assume that all nodes of the same degree are statistically equivalent



Degree distributions and epidemics

SI model

Fraction of nodes of degree k that are infected

$$i_k = \frac{I_k}{N_k}$$

$$\frac{di_k}{dt} = \beta(1 - i_k)k\Theta_k$$

With Θ_k the fraction of infected neighbors for a node of degree k

Degree distributions and epidemics

For early time and assuming no degree correlation

$$\frac{di_k}{dt} = \beta k i_0 \frac{\langle k \rangle - 1}{\langle k \rangle} e^{t/\tau^{SI}}$$

with

$$\tau^{SI} = \frac{\langle k \rangle}{\beta(\langle k^2 \rangle - \langle k \rangle)}$$

Degree distributions and epidemics

Characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta(\langle k^2 \rangle - \langle k \rangle)}$$

For random networks $\langle k^2 \rangle = \langle k \rangle(\langle k \rangle + 1)$

$$\tau^{SI} = \frac{1}{\beta \langle k \rangle}$$

Degree distributions and epidemics

Characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta(\langle k^2 \rangle - \langle k \rangle)}$$

For power law network $\gamma \geq 3$ $\langle k^2 \rangle$ is finite, characteristic time is finite

Degree distributions and epidemics

Characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta(\langle k^2 \rangle - \langle k \rangle)}$$

For power law network $\gamma \geq 3$ $\langle k^2 \rangle$ is finite, characteristic time is finite

For power law $\gamma < 3$, $\langle k^2 \rangle$ does not converge as $N \rightarrow \infty$ so characteristic time goes to 0

Degree distributions and epidemics

Model	Continuum Equation	τ	λ_c
SI	$\frac{di_k}{dt} = \beta [1 - i_k] k \theta_k$	$\frac{\langle k \rangle}{\beta(\langle k^2 \rangle - \langle k \rangle)}$	0
SIS	$\frac{di_k}{dt} = \beta [1 - i_k] k \theta_k - \mu i_k$	$\frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$	$\frac{\langle k \rangle}{\langle k^2 \rangle}$
SIR	$\frac{di_k}{dt} = \beta s_k \theta_k - \mu i_k$ $s_k = 1 - i_k - r_k$	$\frac{\langle k \rangle}{\beta \langle k^2 \rangle - (\mu + \beta) \langle k \rangle}$	$\frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$

Immunization

Suppose a fraction g of nodes is immunized (i.e. resistant)

Rate of infection in SIR model changes from

$$\lambda = \frac{\beta}{\mu}$$

to

$$\lambda(1 - g)$$

Immunization

We can choose a fraction g_c such that rate of infection is below epidemic threshold

For a random network

$$g_c = 1 - \frac{\mu}{\beta} \frac{1}{\langle k \rangle + 1}$$

Immunization

For a power-law network

$$g_c = 1 - \frac{\mu}{\beta} \frac{\langle k \rangle}{\langle k^2 \rangle}$$

For high $\langle k^2 \rangle$ need to immunize almost the entire population

Immunization

Immunization can be more effective if performed selectively.

In power law contact networks, what is the effect of immunizing high-degree nodes?

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In power law contact networks, what is the effect of immunizing high-degree nodes?

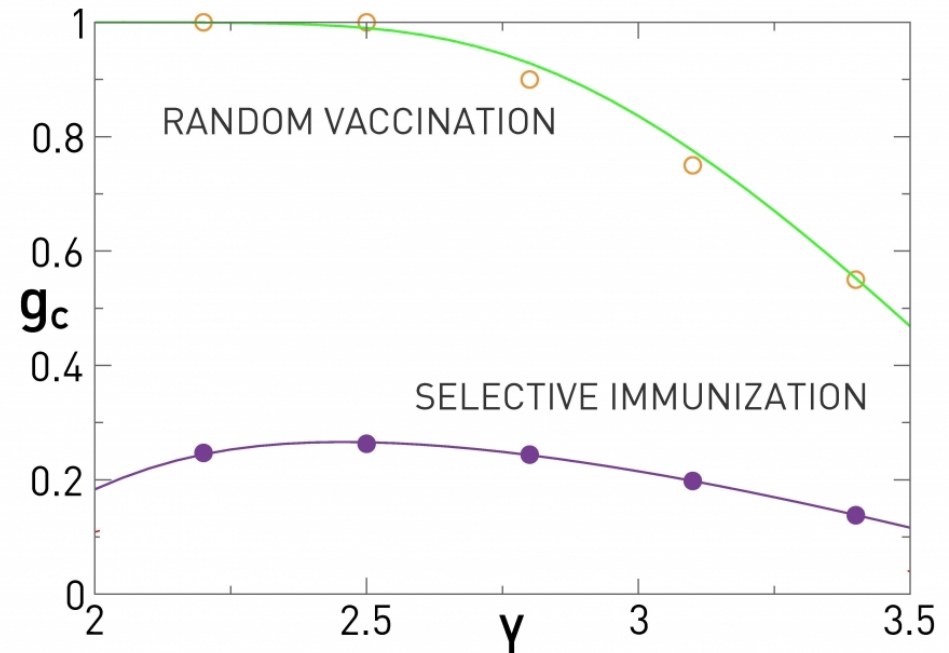
First, how do you find them?

Idea: choose individuals at random, ask them to nominate a neighbor in contact graph

Immunization

What is the expected degree of the nominated neighbor?

$$\propto kp_k$$



Summary

- Epidemic as first example of dynamical process over network
- Role of eigenvalue property in understanding epidemic spread
- Role of degree distribution (specifically scale) in understanding spread
- Role of high-degree nodes in robustness of networks to epidemics (immunization)