Week 1 - Basic Control System Theory and Inital Simulations

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1 Introduction

This report will highlight the fundamentals of control theory and systems and will be used to document these systems simulations using SIMULINK. The main topics of discussion will be First and Second Order systems, stability analysis, and open-loop and closed-loop systems.

2 First Order Systems

A first order system is one that can be described with the use of a first-order ordinary differential equation (e.g. RC circuits, spring-mass system).

2.1 Theory Concepts

This first-order ODE is usually written in the form:

$$\frac{dy(t)}{dt} + \frac{1}{\tau}y(t) = x(t) \tag{1}$$

where:

- x: input of the system;
- τ: the time constant;
- y: output of the system.

The time constant characterizes how fast the system will respond to an input.

These system's transfer function can be written as:

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1} \tag{2}$$

where:

- X: Laplace transform of the input of the system;

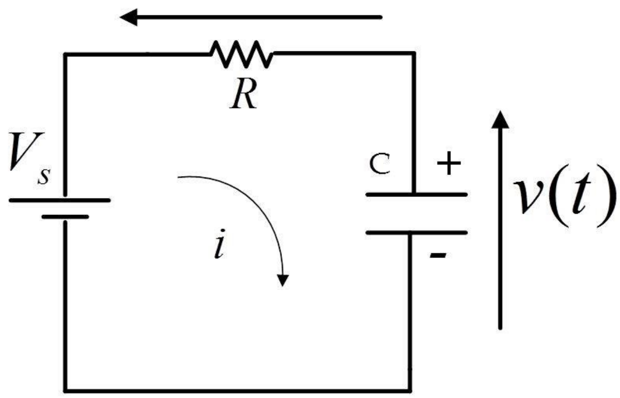


Figure 1: Schematic of an RC circuit

- K: DC gain of the system
- Y: Laplace transform of the output of the system.

2.1.1 RC circuit

As stated previously, an example of a first-order system is an RC circuit, that is, a circuit that combines a resistor and a capacitor in series, as seen in Fig. 1.

By using Kirchoff's voltage law, we can write:

$$v_s(t) = v_c(t) + RC\frac{dv_c(t)}{dt} \tag{3}$$

Where:

- v_s: input voltage
- v_c: capacitor voltage
- R: resistance
- C: capacitance

Assuming that initial conditions are all set to 0, taking the Laplace transform of both sides, we end up with the following:

$$V_s(s) = V_c(s) + RCsV_c(s) \tag{4}$$

Rearranging the terms, we can write:

$$\frac{V_c}{V_s} = \frac{1}{RCs + 1} \tag{5}$$

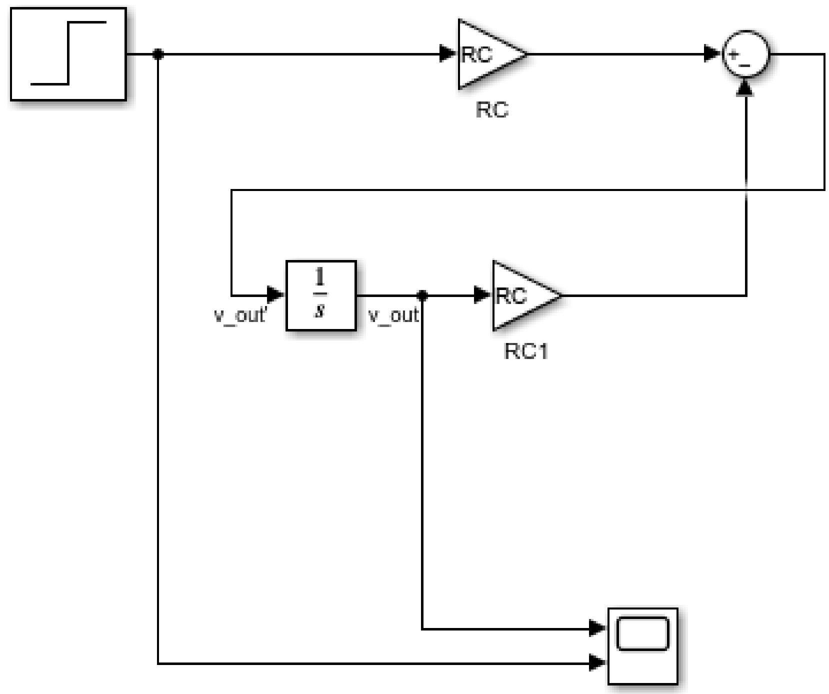


Figure 2: Time domain RC circuit ODE block diagram

2.1.2 Time domain simulation

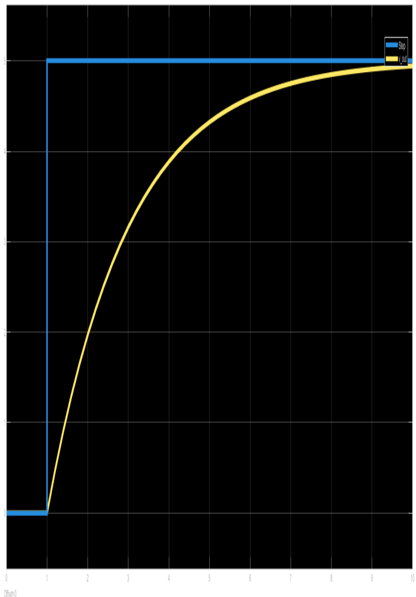
In this section, we analyze the step response of the system with different time constant(RC) values.

A SIMULINK block diagram of the ODE in the time domain that describes this system can be seen in figure 2

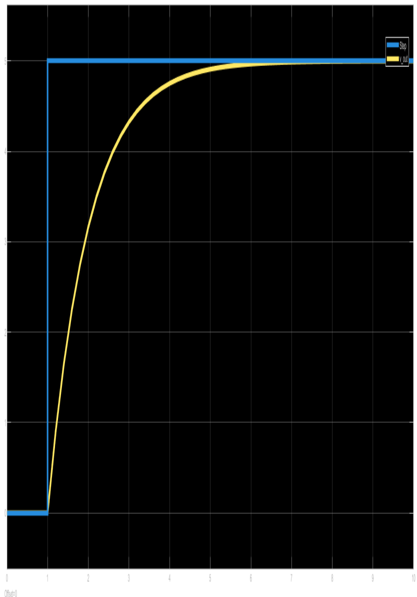
Four values of RC were used in this simulation, and the results are displayed in Fig. 6. The value of v_s was set to 5 V and the simulation runtime was 10 s.

The simulation metric results are displayed on table 1. We can see how both the rise time (the time it takes for the system to go from 10 % to 90 % of its maximum output) and the settling time (the time it takes for the system's output to reach a range within 2 % of its maximum value) both decrease as τ increases.

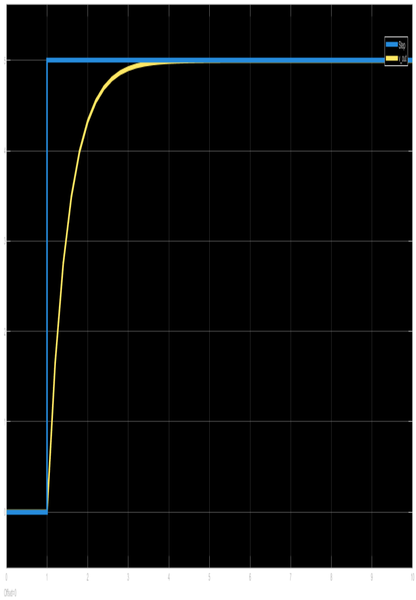
Additionally, a system is stable if and only if all poles of its transfer function have a negative real part. Since the only pole of equation 5 is $-\frac{1}{RC}$, this means that either the value of R or C would have to be negative for the system to be unstable, which is not physically possible.



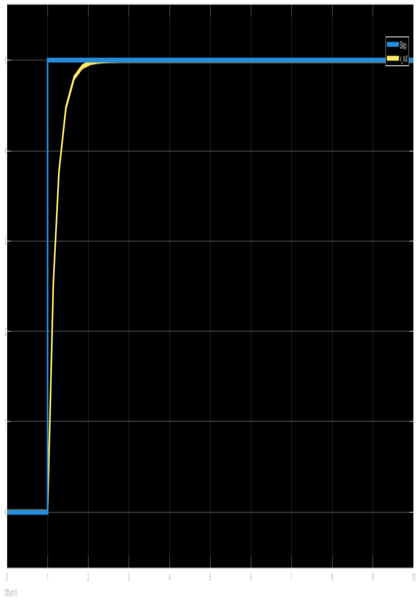
(a) Time domain analysis, $\tau = 0.5$ s



(b) Time domain analysis, $\tau = 1$ s



(c) Time domain analysis, $\tau = 2$ s



(d) Time domain analysis, $\tau = 5$ s

Figure 3: Time domain analysis plots

Settling Time [s]	Rise Time [s]	Time Constant(RC) [s]
7.9566	4.2066	0.5
4.9110	2.1961	1.0
2.9626	1.0977	2.0
1.8039	0.4376	5.0

Table 1: Simulation results

3 Second Order Systems

A second order system can usually be described by the following ODE:

$$\frac{1}{\omega^2}\frac{d^2y(t)}{dt^2} + \frac{2\zeta}{\omega}\frac{dy(t)}{dt} + y(t) = x(t) \tag{6}$$

Where:

- ω: natural frequency
- ζ: damping ratio

Taking the Laplace transform of both sides (initial conditions are null) we have:

$$\frac{1}{\omega^2}s^2Y(s) + \frac{2\zeta}{\omega}sY(s) + Y(s) = X(s) \tag{7}$$

Solving for $\frac{Y(s)}{X(s)}$ we have:

$$\frac{Y(s)}{X(s)} = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} \tag{8}$$

In order to figure out the values of the poles of the transfer function to perform stability analysis, we need to solve the following equation:

$$s^2 + 2\zeta\omega s + \omega^2 = 0 \tag{9}$$

which has the following solutions:

$$S_{1,2} = -\zeta\omega \pm \omega\sqrt{\zeta - 1} \tag{10}$$

By varying the values of ζ we can obtain the following possibilities:

- $\zeta = 0$: the roots of the equation (poles) will be complex conjugates and therefore the system will oscillate permanently
- $0 < \zeta < 1$: the roots will have a negative real part and a complex component making the system stable but underdamped, meaning it will oscillate for a certain time before reaching steady state
- $\zeta = 1$: the roots will have a negative real part making the system critically damped, meaning it stabilizes without oscillating
- $\zeta > 1$: the roots will a have negative real part and depending how close the roots are to the origin the slower the system will react

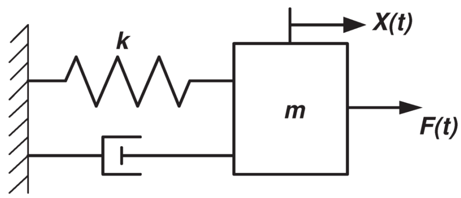


Figure 4: Spring-mass-damper system diagram

3.1 Spring-mass-damper System

One well-known example of a second order system is a spring-mass-damper system, (seen on Fig. 4). This system is composed of a mass that slides on a frictionless plane and is attached to a string and a damper.

We can describe the movement of the mass by using Newton's second law of motion $F = ma$, by which, we can reach the following ODE:

$$-kx(t) - c\frac{dx}{dt}(t) + f(t) = m\frac{d^2x}{dt^2}(t) \tag{11}$$

Where:

- k: spring constant
- c: damping coefficient
- x: mass displacement (output)
- f: driving force (input)

Rearranging the terms, we can write:

$$\frac{m}{k}\frac{d^2x}{dt} + \frac{c}{k}\frac{dx}{dt} + x = f \tag{12}$$

which lets us know that: $\omega = \sqrt{\frac{k}{m}}$ and $\zeta = \frac{c}{2\sqrt{km}}$

This allows us to write the transfer function of this system:

$$\frac{X(s)}{F(s)} = \frac{\frac{k}{m}}{s^2 + \frac{c}{\sqrt{km}}s + \frac{k}{m}} \tag{13}$$

3.2 Simulation results

In order to study the effects of changing the value of the damping constant on the system's response to a step input, a simulation environment on SIMULINK has been developed. Starting off, it is possible to see the block diagram that describes the ODE of the system in Fig. 5.

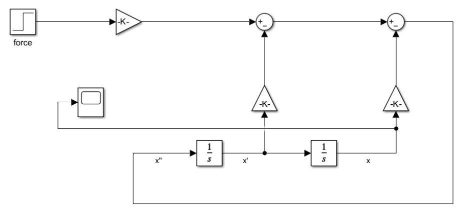


Figure 5: Block diagram of spring-mass-damper system

The time responses of this system to the same step input (1 N) are plotted in Figs. The results meet the expectations considering the values of *zeta* and the conditions mentioned in the beginning of the chapter. In underdamping conditions, we see that rise time increases for larger values of *zeta* but settling time decreases. However for overdamping conditions, both rise time and settling time increase as the damping ratio increases.

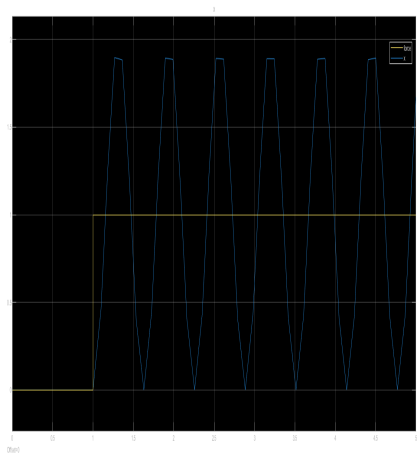
Damping Ratio	Rise Time [s]	Settling Time [s]
0	0.1800	4.9962
0.3	0.1487	2.1241
0.7	0.2298	1.5982
1	0.3440	1.5934
2	0.8247	2.4890
5	2.0302	4.2628

Table 2: System Response Metrics for Different Damping Ratios

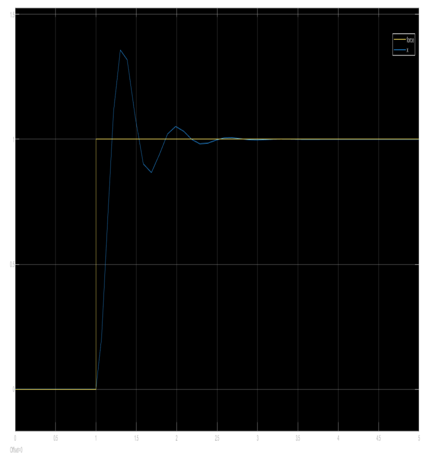
4 Open and Closed Loop Systems

In an open loop system (see Fig. 7 a controller receives an input and based on that input, it performs a control action on what is the called the plant/process in order to produce an output. These system are usually simpler, construction and design wise, and are also cheaper than the alternative. However, since the control action is totally independent of the output of the system, they end up being poorly equipped to handle disturbances and end up being unreliable in most situations.

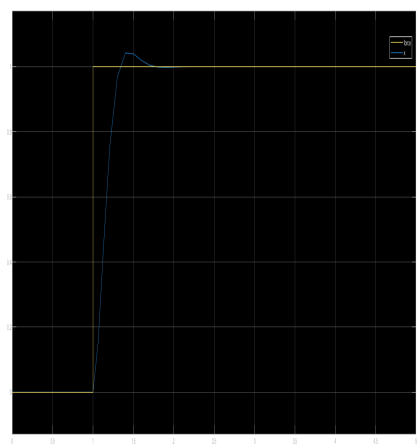
We can solve the open loop systems' weakness by considering a closed loop version of the same system in which a feedback path is fed to the controller along with the input. This is represented in figure 8.



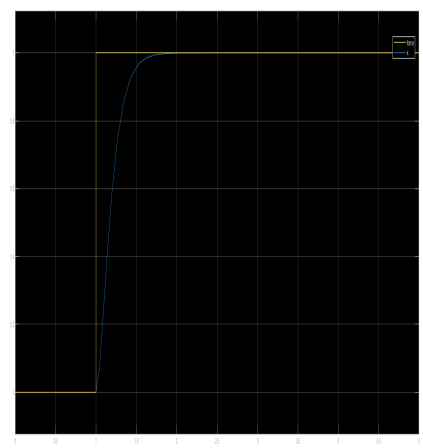
(a) Time domain analysis, $\zeta = 0$



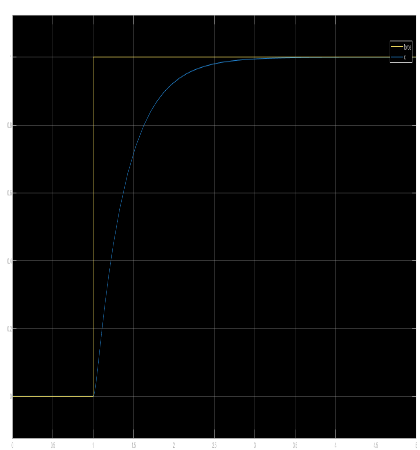
(b) Time domain analysis, $\zeta = 0.3$



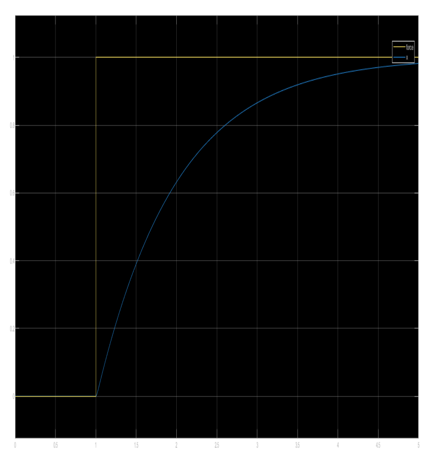
(c) Time domain analysis, $\zeta = 0.7$



(d) Time domain analysis, $\zeta = 1$



(e) Time domain analysis, $\zeta = 2$



(f) Time domain analysis, $\zeta = 5$

Figure 6: Time domain analysis plots for spring-mass-damper system

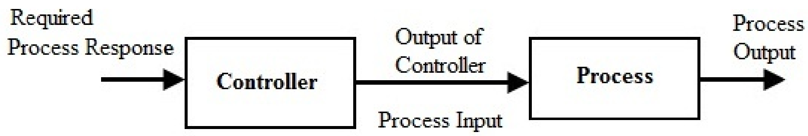


Figure 7: Block diagram of an open loop system

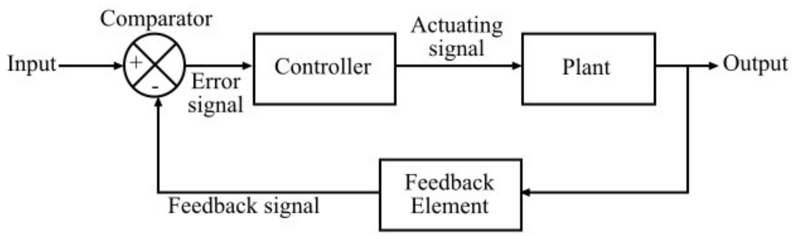


Figure 8: Block diagram of an closed loop system

4.1 Simulation Comparison - Thermal System

In order to test and compare the performance of these 2 types of systems, a simulation comparing a open loop version of an heating system with its closed loop version was conducted.

Beginning by modeling a thermal system is used to heat up a room, the ODE that describes this system is given by:

$$\frac{dT}{dt} = \frac{1}{C}(Q - U(T - T_a)) \tag{14}$$

Where:

- T: temperature of the room
- T_a: ambient temperature
- Q heat input from the heater
- U heat transfer coefficient
- C heat capacity of the room

The open loop version of this system will have no feedback and therefore the heat input Q will stay constant. However, for the closed loop system, the heater will turn off if the temperature of the room is greater or equal than a set temperature.

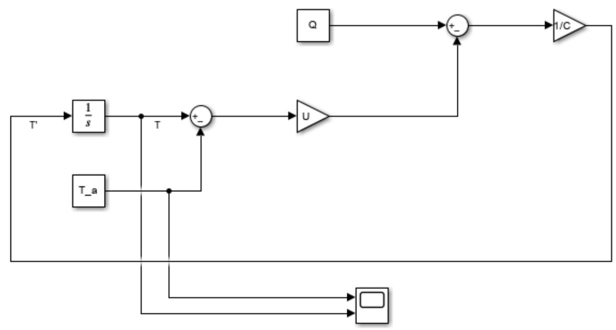


Figure 9: Block diagram for an open loop thermal system

To determine the value of the constants C, Q and U we need to define dimensions and conditions of the room. Heat capacity can be approximated by:

$$C_{air} = \rho V c \tag{15}$$

where:

- ρ: density of air (1.225 kg/m^3)
- V: volume of the room. In this case, the considered dimensions of the room are 5 m $\times$ 5 m $\times$ 3 m ($V = 75$ m^3)
- : c: specific heat capacity of air (1012 J/(kg·K))

So, in this case, $C \approx 92000$J/K. The average heating power of a common residential heater is 1500 W and, considering the room has proper insulation ($u = 0.5$ W/m^2/K), to get the total heat transfer coefficient we multiply this value by the total area of the room (85 m^2) giving us $U = 42.5$ W/K. Since a heater is usually used in the winter months, we consider $T_a = 10\,°$C

The simulation block diagram for the open loop system can be seen on figure 9.

The plot of time domain analysis is seen on the

For comparison, a close loop version of this system was designed, where a on-off controller receives the current temperature which changes the value of Q to 0 in cases where the temperature is currently equal to the user defined temperature setting $T_{setpoint} = 22\,°$C. Its block diagram is present on Fig. 11

The time-domain response plot of this system is represented on figure 12. At first glance, we see a perfect temperature stabilization characterized by the overlap between the setpoint and the temperature signals. However if we zoom in, as seen on Fig. 13. We see the oscillation on the temperature signal.

This analysis highlights the main differences between open loop and closed loop systems. The open loop version is completely independent from the output

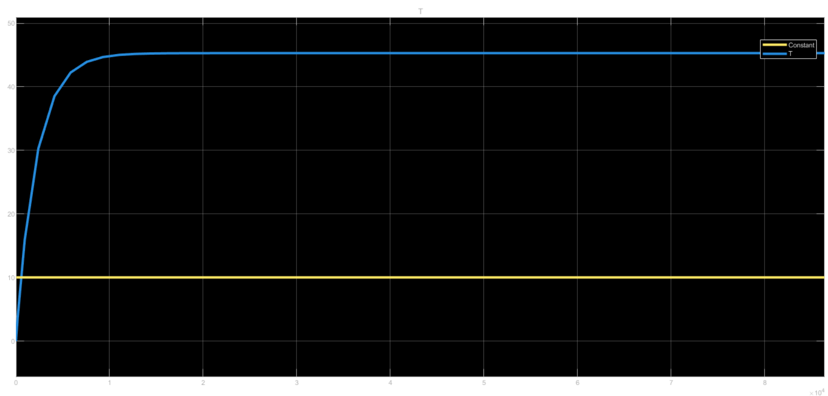


Figure 10: Caption

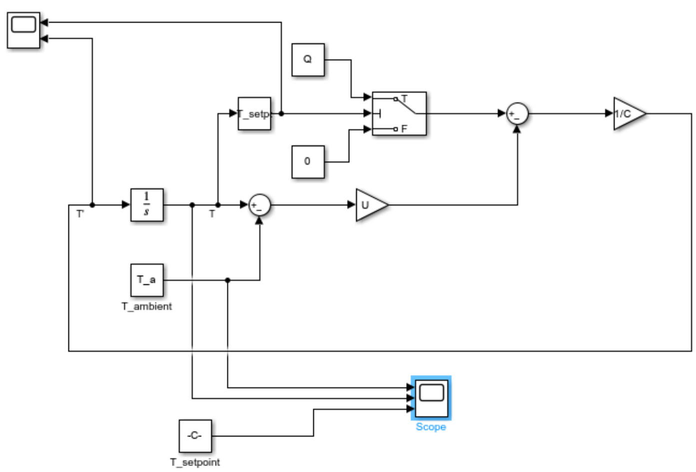


Figure 11: Closed loop thermal system block diagram

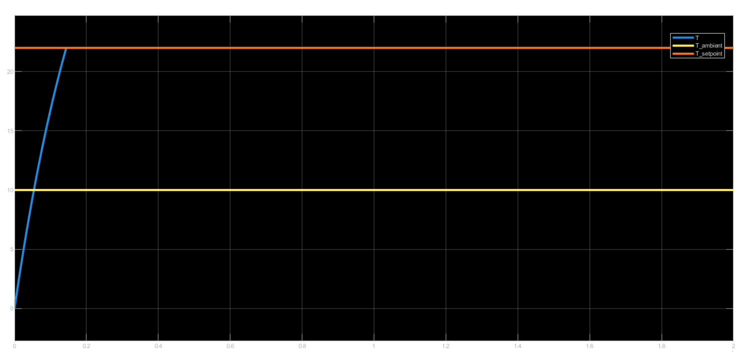


Figure 12: Closed loop thermal system time domain plot

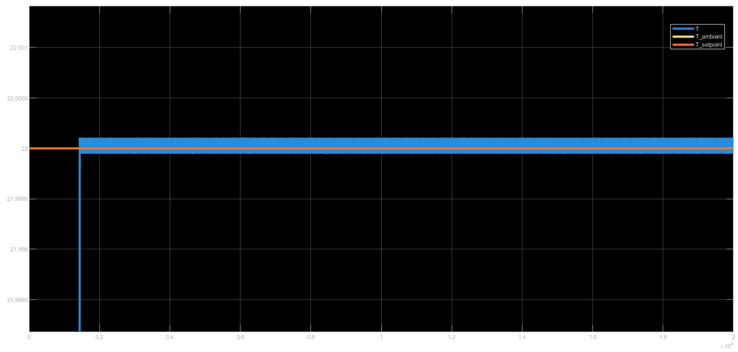


Figure 13: Closed loop thermal system time domain plot

hence it just stabilizes when $t \to +\infty$ and the closed loop version is able, through feedback of the output, perform control actions and stabilize itself on a desired setpoint.