THE DEHN TWIST ON $T^4 \# T^4$ IS NOT SMOOTHLY ISOTOPIC TO THE IDENTITY

HAOCHEN QIU

Abstract. In this project we construct an exotic diffeomorphism on a nonsimply connected manifold without the need for an exotic smooth structure. We use Pin(2)-equivariant family Bauer-Furuta invariant to show that, if $X1, X_2$ are two homology tori such that the determinants r_1, r_2 of them are odd. Then the Dehn twist along a 3-sphere in the neck of $X_1 \# X_2$ is not smoothly isotopic to the identity.

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1. Introduction

An exotic diffeomorphism is the a diffeomorphism that is continuously isotopic to the identity but In this project we construct an exotic diffeomorphism on a nonsimply connected manifold without the need for an exotic smooth structure.

A homology 4-torus is a smooth 4-manifold that has the same homology groups as a 4-dimensional torus T^4. The connected sum of two manifolds X_1 and X_2 can be written as

$$X_1 \# X_2 = (X_1 - D^4) \cup_{S^3} ([0,1] \times S^3) \cup_{S^3} (X_2 - D^4),$$

where $[0,1] \times S^3$ is called the neck of the connected sum. The Dehn twist along a 3-sphere in the neck is a diffeomorphism $d : X_1 \# X_2 \to X_1 \# X_2$ such that d is the identity outside the neck, and on the neck it has the form

$$\begin{aligned} [0,1] \times S^3 &\to [0,1] \times S^3 \\ (t,s) &\mapsto (t, \alpha_t(s)) \end{aligned}$$

where $\alpha \in \pi_1(SO(4), Id) = \mathbb{Z}/2$ is the nontrivial element. It looks like you rotate your head by 2π: your head and body are in the original position, and the only part that changes is your neck.

For a homology torus X, its cohomology groups are isomorphic to the ones of T^4, but the ring structure might be different. Let $\alpha_1, \cdots, \alpha_4$ be a basis of $H^1(X;\mathbb{Z})$, and define the determinant of X by

$$r := |\langle \alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4, [X]\rangle|$$

where $[X]$ is the fundamental class. The main theorem of this project is

Theorem 1.1. *If $X1, X_2$ are two homology tori such that the determinants r_1, r_2 of them are odd. Then the Dehn twist along a 3-sphere in the neck of $X_1 \# X_2$ is not smoothly isotopic to the identity.*

The main tool we use is the Bauer-Furuta invariant [BF02]. Its idea is to regard the Seiberg-Witten equation as an Pin(2)-equivariant map, and consider the property of the map. By a finite dimensional approximation, it is an equivariant stable mapping class for a spin manifold X:

$$\mathrm{BF}^{\mathrm{Pin}(2)}(X, \mathfrak{s}) \in \{TF_0, S^{n\mathbb{H}+b_2^+(X)\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}.$$

where $\mathfrak{s}$ is a Spin^c-structure of X, and TF_0 is the Thom space of a rank m quarternion bundle over $Pic^{\mathfrak{s}}(X) = T^{b_1(X)}$, such that

$$m - n = -\frac{\sigma(X)}{4}$$

where $\sigma(X)$ is the signature of X.

One can also forget the Pin(2)-action and define the nonequivariant Bauer-Furuta invariant by

$$\mathrm{BF}^{\{e\}}(X, \mathfrak{s}) := \mathrm{Res}^{\mathrm{Pin}(2)}_{\{e\}} \mathrm{BF}^{\mathrm{Pin}(2)}(X, \mathfrak{s}) \in \{TF_0, S^{4n+b_2^+(X)}\}.$$

Now one has a sequence of invariants that can detect exotic phenomena: the Seiberg-Witten invariant, the nonequivariant Bauer-Furuta invariant, and the Pin(2)-equivariant Bauer-Furuta invariant. They contain more and more infomation, but the computations get more and more complicated.

The main theorem comes from a sequence of results:

First, by a perturbation of the SW equation proposed by Ruberman-Strle[RS00], and a computation of the bundle TF_0 via the index theorem and the Steenrod square, we get

Theorem 1.2. *If X is a homology torus with odd determinant, and $\mathfrak{s}$ is the trivial structure, then*

$$BF^{\{e\}}(X, \mathfrak{s}) = (0, 0, 0, 0, 1) \in 4\mathbb{Z} \oplus \mathbb{Z}/2.$$

Actually $BF^{\{e\}}(X, \mathfrak{s})$ is the Hopf element η.

Second, we compute the nonequivariant family Bauer-Furuta invariant for the mapping torus of the Dehn twist $d : X_1 \# X_2 \to X_1 \# X_2$. It is denoted by

$$BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}.$$

We compute the bundle F by the index theorem, and prove that there exists a Hopf element ν in the stable CW structure of TF. Therefore, by Atiyah-Hirzebruch spectral sequence, $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau))$ must be trivial. This vanishing result is similar to the fact that, a 3-sphere can not be mapped to $\mathbf{C}P^2$ nontrivially, because the 4-cell in $\mathbf{C}P^2$ is attached to the 2-cell by the Hopf element η.

Finally, we compute the equivariant family Bauer-Furuta invariant $BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau))$. By a cofiber sequence we can throw away the fixed points in the equivariant map, and then apply the equivariant Hopf theorem to convert $BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau))$ to a nonequivariant stable mapping class. Now the dimension is changed and the Hopf invariant mentioned above has no effect. Hence we can apply the method of Kronheimer-Mrowka[KM20], and show that

Theorem 1.3. *$BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ is nontrivial.*

Compared with the previous results, there are some issues when computing the Bauer-Furuta invariant for T^4:

- The Bauer-Furuta invariant of T^4 is unkonwn. The map from the Bauer-Furuta invariant to the Seiberg-Witten invariant is not well-defined because T^4 doesn't satisfy the condition $b^+ - b_1 \geqslant 2$ in [BF02], and indeed, [RS00] shows that the perturbation used in the computation of the Seiberg-Witten invariant contains reducible solutions.
- Proposition 4.1 in [KM20] doen't work for T^4. The index of the twisted Dirac operator on T^4 is zero, which leads to a vanishing twist in the parameterized Bauer-Furuta invariant. Therefore, such invariant cannot distinguish the twisted spin structure and the product spin structure on a family of T^4.
- The (parameterized) Bauer-Furuta invariant of T^4 is a stable mapping class from T^4 to a sphere, which is hard to compute. The $K3$ surface considered in [KM20] and [Lin23], however, has $b_1 = 0$. Hence the (parameterized) Bauer-Furuta invariant of it is an element in the stable homotopy group of spheres, which is well known in low dimension (1, 2 and 3), and moreover, the equivariant version can be computed by algebraic topology.

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2. The family Bauer-Furata invariant for nonsimply connected manifolds

In this section, we introduce the definition of the Pin(2)-equivariant Bauer-Furuta invariant for spin families of nonsimply connected manifolds, and a refinement of this invariant.

In [BF02], Bauer-Furuta introduce a finite dimensional approximation of the Pin(2)-equivariant Seiberg-Witten map. It is an equivariant stable mapping class for a spin manifold, and people call it the Bauer-Furuta invariant. In Baraglia-Konno[BK22], Kronheimer-Mrowka[KM20] and Lin[Lin23], the family Bauer-Furuta invariant is introduced, but only for simply connected manifolds. For nonsimply connected manifolds, the original definition of Bauer-Furuta has to be generalized.

Suppose X is a closed spin 4-manifold and B is another closed manifold works as the parameter space. Suppose E_X is a smooth family of X over B, that is, a smooth bundle with fiber X and base B.

We work in the following settings:

- A spin structure $\mathfrak{s}$ on E_X, which is a double cover of the vertical frame bundle on E_X, suth that it restricts to the double cover $Spin(4) \to SO(4)$ on each fiber.
- A family of metrics $g : B \to \mathrm{Met}(X)$.
- Two quaternion bundles over E_X given by $\mathfrak{s}$ and g:

$$S^{\pm} := \bigsqcup_{b \in B} S_b^{\pm},$$

where $S_b^{\pm}$ are positive and negative spinor bundles on X_b given by g_b. Denote the space of spinors (sections of $S_b^{\pm}$) by $\Gamma(S_b^{\pm})$. Denote the parameterized Dirac operator over B by

$$\not{\mathfrak{D}}_A(X_b) : \Gamma(S_b^+) \to \Gamma(S_b^-)$$

for $b \in B$ and spin^c-connection A on X_b.

- A family of base points $* : B \to X$.
- Define the action of the gauge group $\mathcal{G}(X_b) = \mathrm{Map}(X_b, S^1)$ by letting $u \in \mathcal{G}(X_b)$ send $\Psi \in \Gamma(S_b^{\pm})$ to $u\Psi \in \Gamma(S_b^{\pm})$, and add udu to the connection 1-forms. Denote the based gauge group by

$$\mathcal{G}_0(*_b) := \{u \in \mathcal{G}(X_b) \mid u(*_b) = 1\}.$$

- Define

$$\mathcal{C}(\mathfrak{s}, g_b, *_b) := (L^{k,2}(\mathscr{A}(\mathfrak{s})) \oplus L^{k,2}(\Gamma(S_b^+)))/\mathcal{G}_0(*_b)$$
$$\mathcal{D}(\mathfrak{s}, g_b, *_b) := (L^{k-1,2}(i\Omega^0(X)/\mathbb{R}) \oplus L^{k-1,2}(i\Omega^+(X)) \oplus L^{k-1,2}(\Gamma(S_b^-)))/\mathcal{G}_0(*_b),$$

where $\mathscr{A}(\mathfrak{s})$ is the space of $U(1)$-connections of the determinant line bundle of the spin structure $\mathfrak{s}$. The Seiberg-Witten map is

$$\mathcal{F}_{(\mathfrak{s},g_b,*_b)} : \mathcal{C}(\mathfrak{s}, g_b, *_b) \to \mathcal{D}(\mathfrak{s}, g_b, *_b) \tag{2.1}$$

$$\mathcal{F}_{(\mathfrak{s},g_b,*_b)} \begin{pmatrix} A \\ \Phi \end{pmatrix} = \begin{pmatrix} d^*(A - A_0) \\ F_A^{+g_b} - \rho^{-1}(\sigma(\Phi, \Phi)) \\ \not{\mathfrak{D}}_A \Phi \end{pmatrix} \tag{2.2}$$

where $\rho : \Omega^+(X) \to \mathfrak{su}(S_g^+)$ is the map defined by the Clifford multiplication, and σ is the quadratic form given by $\sigma(\Phi, \Phi) = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 id$, and $F_A^{+g_b}$ is the self dual part of F_A with respect to g_b.

- $\mathcal{U}_b^+$ is the completion of $\Omega^1(X_b)$;
- $\mathcal{U}_b^-$ is the completion of $\Omega^2_+(X_b) \oplus \Omega^0(X_b)/\mathbb{R}$.

Denote by X_b the fiber of E_X over $b \in B$. A spin structure on X_b gives two quaternion bundles $S_b^{\pm}$ over the base X_b. Then

$$S^{\pm} := \bigsqcup_{b \in B} S_b^{\pm}$$

are two quaternion bundles over E_X. Denote the space of spinors (sections of $S_b^{\pm}$) by $\Gamma(S_b^{\pm})$. Define the parameterized Dirac operator over B by

$$\not{\mathfrak{D}}_{A_b}(X_b) : \Gamma(S_b^+) \to \Gamma(S_b^-)$$

for $b \in B$ and spin^c-connection A_b on X_b.

Now consider the action of the gauge group $\mathcal{G}(X_b) = map(X_b, S^1)$.

Define four Hilbert bundles $\mathcal{U}^+, \mathcal{V}^+, \mathcal{U}^-, \mathcal{V}^-$ over B:

- $\mathcal{V}_b^{\pm}$ are Sobolev completions of $\Gamma(S_b^{\pm})$;
- $\mathcal{U}_b^+$ is the completion of $\Omega^1(X_b)$;
- $\mathcal{U}_b^-$ is the completion of $\Omega^2_+(X_b) \oplus \Omega^0(X_b)/\mathbb{R}$.

The group Pin(2) acts on $\mathcal{V}_b^{\pm}$ by the left quaternion multiplication, and acts on $\mathcal{U}_b^{\pm}$ by reversing the sign.

3. The Bauer-Furuta invariant of homology tori

Theorem 3.1. *Suppose X is a homology torus with*

$$\langle \alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4, X\rangle = d, \tag{3.1}$$

where $\{\alpha_i\}$ is a basis for $H^1(X;\mathbb{Z})$, and d is odd. Let $\mathfrak{s}$ be the spin^c structure on X with trivial determinent line. Then the nonequivariant Bauer-Furuta invariant $BF^{\{e\}}(X,\mathfrak{s})$ is the generator of $\mathbb{Z}/2\mathbb{Z}$ in a group $4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

Proof. First we compute the group that the invariant lives. Let F_0 be the bundle $\mathbb{H}\to F_0\to \mathcal{T}^4$ with $c_1(T_0)=0$ and $c_2(T_0)=d$. Let TF_0 be its Thom space. The equivariant Bauer-Furuta invariant is

$$\mathrm{BF}^{\mathrm{Pin}(2)}(X,\mathfrak{s})\in\{TF_0, S^{\mathbb{H}+3\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}.$$

Forget the Pin(2)-action then we get the non equivariant Bauer-Furuta invariant

$$\mathrm{BF}^{\{e\}}(X,\mathfrak{s}) := \mathrm{Res}^{\mathrm{Pin}(2)}_{\{e\}}\mathrm{BF}^{\mathrm{Pin}(2)}(X,\mathfrak{s})\in\{TF_0,S^7\}.$$

A sketch of the proof: TF_0 can be regarded as a CW complex with one 0-cell, one 4-cell (corresponds to $S^{\mathbb{H}}$), four 5-cells (obtained from the 1-cells of $\mathcal{T}^4$ by multiplying a 4-cell corresponds to $S^{\mathbb{H}}$), six 6-cells (obtained from the 2-cells of $\mathcal{T}^4$ similarly), four 7-cells (obtained from the 3-cells of $\mathcal{T}^4$), one 8-cell (obtained from the 4-cell of $\mathcal{T}^4$). By a cofiber sequence and the CW approximation theorem, there is an isomorphism

$$\{TF_0,S^7\}\cong\{TF_0/TF_0^{(5)},S^7\}$$

where $TF_0^{(5)}$ is the 5-th skeleton of TF_0. We want to show that all attaching maps of $TF_0/TF_0^{(5)}$ are trivial.

First by Thom isomorphism and the cohomology of $\mathcal{T}^4$, we deduce that all cells of $TF_0/TF_0^{(5)}$ survive in the cohomology group, so all adjacent attaching maps are trivial.

Now the only possible nontrivial attaching map is the one from the 8-cell to 6-cells. Since the only nontrival element of π_1 is the Hopf map η, which can be detected by the Steenrod square, it suffices to show that Sq^2 is trivial. Let u be the Thom class of F_0. Note that u is an element in $H^*(F_0,F_0-\mathcal{T}^4)$ represented by the zero section of F_0. The cup product of $H^*(F_0,F_0-\mathcal{T}^4)$ with $H^*(\mathcal{T}^4)$ still produces closed submanfolds, so $H^*(\mathcal{T}^4)$ acts on $H^*(F_0,F_0-\mathcal{T}^4)$. By Cartan formula we have for any $x\in H^*(\mathcal{T}^4)$

$$\begin{aligned} Sq^n(ux) &= \sum_{i+j=n} Sq^i(u)Sq^j(x)\\ &= \sum_{i+j=n} uw_i(F_0)Sq^j(x)\end{aligned}$$

where w_i is the i-th Stiefel–Whitney class. The cohomology of $\mathcal{T}^4$ is an exterior algebra generated by four 1-classes which the Steenrod algebra acts on trivially. By the Cartan formula and the Adem relations, the Steenrod algebra acts on $H^*(\mathcal{T}^4)$ trivially. Hence $Sq^j(x)\neq 0$ iff $j=0$. So

$$Sq^2(ux) = uw_2(F_0)Sq^0(x).$$

But $w_2(F_0)\equiv c_1(F_0) \mod 2$ and from the structure of F_0 we have $c_1(F_0)=0$. So the attaching maps from the 8-cell to 6-cells are trivial.

Now we conclude that $TF_0/TF_0^{(5)}$ is equivalent to $6\mathbb{S}^6\vee 4\mathbb{S}^7\vee\mathbb{S}^8$ in the stable category. Hence

$$\{TF_0,S^7\}\cong\{TF_0/TF_0^{(5)},S^7\}\cong[6\mathbb{S}^6\vee 4\mathbb{S}^7\vee\mathbb{S}^8,\mathbb{S}^7]\cong 4\mathbb{Z}\oplus\mathbb{Z}/2.$$

By Ruberman-Strle[RS00], the preimage of a genric point under $BF^{\{e\}}(X,\mathfrak{s})$ is a Lie framed circle in a fiber of TF_0. So the restriction $BF^{\{e\}}(X,\mathfrak{s})|_{\mathbb{S}^8}$ is a suspension of the Hopf map

$$\Sigma^5\eta : \mathbb{S}^8 \to \mathbb{S}^7.$$

And the restricitons of $BF^{\{e\}}(X,\mathfrak{s})$ on those 7-cells have degree 0, otherwise the preimage of a generic point would contain discrete points. Therefore,

$$BF^{\{e\}}(X,\mathfrak{s}) = (0,0,0,0,1) \in 4\mathbb{Z} \oplus \mathbb{Z}/2.$$

□

4. Family Bauer-Furuta invariant of the connected sum of two tori

Suppose X_1 and X_2 are homology tori with determinant r_1 and r_2. Let $X = X_1 \# X_2$ and $\mathfrak{T}^8 \cong H^1(X,S^1)$. Let $\mathfrak{s}$ be the spin^c structure on X with trivial determinent line. Denote the family of Dirac operators by $\mathfrak{D}\!\!\!/$. The index bundle $Ind(\mathfrak{D}\!\!\!/)$ is an $\mathbb{H}$-bundle over $\mathfrak{T}^8$ since X is spin. Let $\mathbf{L} \to X \times \mathfrak{T}^8$ be the universal line bundle. Since the $\hat{A}$-genus of X is zero, the Chern character of the index bundle is

$$\mathrm{ch}(Ind(\mathfrak{D}\!\!\!/)) = \int_X \mathrm{ch}(\mathbf{L})$$

by Atiyah-Singer.

Suppose $\mathbf{L}$ is equipped with a connection

$$\mathbf{A} = 2\pi i \sum_{k=1}^{8} t_k \alpha_k$$

where $\alpha_1, \dots, \alpha_4$ is a basis for $H^1(X_1,\mathbb{Z})$ and $\alpha_5, \dots, \alpha_8$ is a basis for $H^1(X_2,\mathbb{Z})$, and $t_k \mapsto 2\pi i t_k \alpha_k$ are coordinates on $\mathfrak{T}^8 \cong H^1(X, i\mathbb{R})/H^1(X, 2\pi i\mathbb{Z})$. Then the first Chern class of $\mathbf{L}$ is

$$\Omega = \sum_k \alpha_k \wedge dt_k.$$

By the dimension reason

$$\mathrm{ch}(\mathbf{L}) = 1 + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{6}\Omega^3 + \frac{1}{24}\Omega^4.$$

A term in $\mathrm{ch}(\mathbf{L})$ contains a volumn form of X if and only if it contains $\alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4$ or $\alpha_5 \smile \alpha_6 \smile \alpha_7 \smile \alpha_8$. Therefore,

$$\mathrm{ch}(Ind(\mathfrak{D}\!\!\!/)) = \pm r_1[vol_{\mathfrak{T}_1^4}] \pm r_2[vol_{\mathfrak{T}_2^4}]$$

where $\mathfrak{T}_i^4 \cong H^1(X_i,S^1)$ are submanifolds of $\mathfrak{T}^8$. Hence $c_2(Ind(\mathfrak{D}\!\!\!/)) = \pm r_1[vol_{\mathfrak{T}_1^4}] \pm r_2[vol_{\mathfrak{T}_2^4}]$ and $c_i(Ind(\mathfrak{D}\!\!\!/)) = 0$ for $i \neq 2$. Hence

$$\mathrm{BF}^{\{e\}}(X,\mathfrak{s}) \in \{TF, S^{\mathbb{H}+6\mathbb{R}}\}$$

where TF is the Thom space of the $\mathbb{H}$-bundle $\mathbb{H} \to F \to \mathfrak{T}^8$. Hence we have

Lemma 4.1. $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau})) \in \{S^{\mathbb{R}} \wedge TF, S^{\mathbb{H}+6\mathbb{R}}\} = \{S^{\mathbb{R}} \wedge TF, S^{10}\}$

This also follows from the gluing theorem of Bauer-Furuta invariant, which asserts that the domain of the stable cohomotopy element for a connected sum is an extenal product of the domains of two elements.

Theorem 4.2. *Suppose X_1 and X_2 are homology tori with odd determinant. Let $\mathfrak{s}_i$ be the spin^c structure on X_i with trivial determinent line bundle. Then the nonequivariant family Bauer-Furuta invariant $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau})) \in \{S^{\mathbb{R}} \wedge TF, S^{10}\}$ is trivial.*

Proof. First we compute the structure of TF: TF can be regarded as a CW complex with one 0-cell, one 4-cell (corresponds to $S^{\mathbb{H}}$), several n-cells ($n = 5, 6, 7, 8, 9, 10, 11$, obtained from the $(n-4)$-cells of $\mathcal{T}^8$ by multiplying a 4-cell corresponds to $S^{\mathbb{H}}$), and one 12-cell (obtained from the 8-cell of $\mathcal{T}^8$).

We claim that the attaching maps from the 12-cell to two of the 8-cells are Hopf element ν. Let u be the Thom class of F. Note that u is an element in $H^*(F, F - \mathcal{T}^8)$ represented by the zero section of F. The cup product of $H^*(F, F - \mathcal{T}^8)$ with $H^*(\mathcal{T}^8)$ still produces closed submanfolds, so $H^*(\mathcal{T}^8)$ acts on $H^*(F, F - \mathcal{T}^8)$. By Cartan formula we have for any $x \in H^*(\mathcal{T}^8)$

$$\begin{aligned} Sq^n(ux) &= \sum_{i+j=n} Sq^i(u) Sq^j(x) \\ &= \sum_{i+j=n} u w_i(F_0) Sq^j(x) \end{aligned}$$

where w_i is the i-th Stiefel–Whitney class. The cohomology of $\mathcal{T}^8$ is an exterior algebra generated by four 1-classes which the Steenrod algebra acts on trivially. By the Cartan formula and the Adem relations, the Steenrod algebra acts on $H^*(\mathcal{T}^8)$ trivially. Hence $Sq^j(x) \neq 0$ iff $j = 0$. So

$$Sq^4(ux) = uw_4(F) Sq^0(x).$$

But $w_4(F) \equiv c_2(F) \mod 2$ and from the structure of F we have $c_2(F) = \pm r_1 [vol_{\mathcal{T}_1^4}] \pm [vol_{\mathcal{T}_2^4}]$. Note that $[vol_{\mathcal{T}_1^4}] \cup [vol_{\mathcal{T}_2^4}] = [vol_{\mathcal{T}^8}]$ and $[vol_{\mathcal{T}_i^4}] \cup [vol_{\mathcal{T}_i^4}] = 0$. Hence if $\{i, j\} = \{1, 2\}$ and r_i is odd,

$$Sq^4(u[vol_{\mathcal{T}_j^4}]) = u[vol_{\mathcal{T}_i^4}] \cup [vol_{\mathcal{T}_j^4}] = u[vol_{\mathcal{T}^8}]$$

is dual to the 12-cell. Since the Hopf element ν is detected by Sq^4, the attaching map from the 12-cell (dual to $u[vol_{\mathcal{T}^8}]$) to a 8-cell (dual to $u[vol_{\mathcal{T}_j^4}]$) is ν.

Therefore, in $S^{\mathbb{R}} \wedge TF$, the attaching map from the 13-cell to a 9-cell is ν. Hence the generator of

$$\pi^{10}(\Sigma TF^{(13)}/\Sigma TF^{(12)}) = H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-3}(*))$$

in the E_2 page of the Atiyah-Hirzebruch spectral sequence:

$$\begin{tikzcd}
H^9(\Sigma TF^{(9)}/\Sigma TF^{(8)}; \pi^0(*)) \arrow[rrddd, dashed, bend right, "d_4"] & \cdots & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^0(*)) \\
 & & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-1}(*)) \\
 & & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-2}(*)) \\
 & & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-3}(*))
\end{tikzcd}$$

doesn't survive to the E_∞ page. So the 13-cell can only be mapped to S^{10} trivially.

From the observation of Kronheimer-Mrowka[KM20], the preimage of a genric point under $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ is η^2 in a fiber of TF smash a Lie framed circle in $\mathbb{S}^{\mathbb{R}}$. Hence for $n \neq 13$, any n-cell in $S^{\mathbb{R}} \wedge TF$ is mapped to S^{10} trivially by $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$. $\square$

5. Equivariant family Bauer-Furuta invariant of the connected sum of two tori

To address the dimension issue in the previous section, we can consider the S^1-equivariant Bauer-Furuta invariant. By the equivariant Hopf theorem, we can convert

$$BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1}$$

to a nonequivariant stable mapping class, if $\{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1}$ has no fixed points. However, the base of the bundle F is the Picard torus $\mathfrak{T}^8$ fixed by the S^1-action.

To address this issue we have to use a refinement of the Bauer-Furuta invariant:

Definition 5.1. For a spin manifold X, define its free Bauer-Furuta invariant of the Spin^c-structure $\mathfrak{s}$ to be:

$$\mathrm{BF}^{\mathrm{Pin}(2)}_{\mathrm{free}}(X, \mathfrak{s}) \in \{TF_0/Pic^{\mathfrak{s}}(X), S^{n\mathbb{H}+b_2^+(X)\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)},$$

where TF_0 is the Thom space of a rank m quarternion bundle over $Pic^{\mathfrak{s}}(X) = T^{b_1(X)}$, such that

$$m - n = \frac{\sigma(X)}{4}.$$

For family invariant we can similarly define an invariant with domain acted freely by a subgroup of Pin(2). For example:

Definition 5.2. Define the free S^1-equivariant Bauer-Furuta invariant Bauer-Furuta invariant of the Dehn twist on a sum of two homology tori to be:

$$BF^{\{S^1\}}_{\mathrm{free}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathfrak{T}^8), S^{\mathbb{H}+6\mathbb{R}}\}^{S^1}.$$

These invariants work as well as the ordinary BF invariant, because in the Seiberg-Witten equation, the Picard torus is always mapped to zero (while the kernel of the index bundle might be mapped to nonzero self-dual 2-form).

5.1. Computation of the free S^1-equivariant Bauer-Furuta invariant.

Now consider the domain of this invariant. Let $S(F)$ be the sphere bundle of F. The fiber of $S(F)$ is $S(\mathbb{H})$. The structure map and the S^1-action on $S(F)$ are indeuced by those on F. Then $TF/\mathfrak{T}^8$ is $\Sigma^u S(F)$, the unreduced suspension of $S(F)$.

Because the fiber of $TF/\mathfrak{T}^8$ is the unreduced suspension $\Sigma^u S(\mathbb{H})$ of $S(\mathbb{H})$, the fiber of

$$(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathfrak{T}^8)$$

is $\Sigma\Sigma^u S(\mathbb{H})$ with $S^{\mathbb{R}} = \Sigma S^0 \subset \Sigma\Sigma^u S(\mathbb{H})$ pinched to a point. This space is

$$(S(\mathbb{H}) \times D^2)/(S(\mathbb{H}) \times S^1).$$

To see this, note that $\Sigma^u S(\mathbb{H})$ is a $(3+1)$-dimensional sphere with north and south poles S^0 that come from the construction of the unreduced suspension. $\Sigma\Sigma^u S(\mathbb{H})$ is a $(3+1+1)$-dimensional sphere with two orthogonal spheres in it: $S(\mathbb{H})$ and ΣS^0. Now collapse ΣS^0.

From this we have that $(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathfrak{T}^8)$ is the Thom space of the bundle $S(F) \oplus 2\mathbb{R}$, where $\mathbb{R}$ denotes the trivial bundle:

Lemma 5.3.

$$(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathfrak{T}^8) = T(S(F) \oplus 2\mathbb{R}).$$

Now the S^1-action on $((S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)) \wedge S^{-\mathbb{H}}$ is still free away from the base point. The S^1-action on $S^{6\mathbb{R}}$ is trivial. Hence we have an isomorphism (see [Lin23] Fact 2.2 and Fact 2.3) from

$$\{((S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)) \wedge S^{-\mathbb{H}}, S^{6\mathbb{R}}\}^{S^1}$$

to a nonequivariant stable mapping class group

$$\{(T(S(F) \oplus 2\mathbb{R}) \wedge S^{-\mathbb{H}})/S^1, S^{6\mathbb{R}}\}.$$

Although S^1 acts on $S^{-\mathbb{H}}$ nontrivially, the orbit space of $T(S(F) \oplus 2\mathbb{R}) \wedge S^{-\mathbb{H}}$ is equivalent to the orbit space of $T(S(F) \oplus 2\mathbb{R})$, smash with $S^{-4\mathbb{R}}$. Let G be a quotient space of F under the S^1-action. Then

$$\{(T(S(F) \oplus 2\mathbb{R}) \wedge S^{-\mathbb{H}})/S^1, S^{6\mathbb{R}}\} \cong \{T(S(G) \oplus 2\mathbb{R}) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\}.$$

The fiber of G is $\mathbb{H}/S^1$. The fiber of $T(S(G) \oplus 2\mathbb{R})$ is $(S(\mathbb{H}/S^1) \times S^2)/(\{pt\} \times S^2)$, which by definition is $S^{2\mathbb{R}} \wedge (S(\mathbb{H}/S^1)_+)$. Hence

$$\{T(S(G) \oplus 2\mathbb{R}) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} \cong \{S^{2\mathbb{R}} \wedge (S(G)_+) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\}.$$

to analyze the CW structure of the domain, it's better to convert the domain to a Thom space. Notice that for any X, we have $S^{\mathbb{R}} \wedge X_+ = S^1 \vee \Sigma X$. We borrow a copy of $\mathbb{R}$ and get

$$\{S^{2\mathbb{R}} \wedge (S(G)_+) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} \cong \{S^{\mathbb{R}} \wedge (S^1 \vee TG) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\}.$$

By a cofiber sequence and the dimension reason, we have

$$\{S^{\mathbb{R}} \wedge (S^1 \vee TG) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} \cong \{S^{\mathbb{R}} \wedge TG \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\}.$$

Now move $S^{-4\mathbb{R}}$ back:

$$\{S^{\mathbb{R}} \wedge TG \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} \cong \{S^{\mathbb{R}} \wedge TG, S^{10\mathbb{R}}\}.$$

Let G be a quotient space of F under the S^1-action. Then

$$\begin{aligned}
&\{((S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)) \wedge S^{-\mathbb{H}}, S^{6\mathbb{R}}\}^{S^1} = \{(S^{2\mathbb{R}} \wedge (S(F)_+) \wedge S^{-\mathbb{H}})/S^1, S^{6\mathbb{R}}\} && (5.1)\\
&= \{S^{2\mathbb{R}} \wedge (S(F)_+)/S^1 \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} && (5.2)\\
&= \{S^{2\mathbb{R}} \wedge (S(G)_+) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} && (5.3)\\
&= \{S^{\mathbb{R}} \wedge (S^1 \vee TG) \wedge S^{-4\mathbb{R}}, S^{6\mathbb{R}}\} && (5.4)
\end{aligned}$$

Now it's easy to analyze the CW structure of the domain $S^{\mathbb{R}} \wedge TG$. We compute the dimension of the cells other than the based point:

$S^{\mathbb{R}}$	$\mathcal{T}^8$	$\mathbb{H}/S^1$
1	0	3
	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	

Again since the dimension of the target is 10, we only need to consider 9-, 10-, 11-, and 12-cells of the domain $S^{\mathbb{R}} \wedge TG$. They come from the 8-, 9-, 10-, and 11-cells of TG. The structure of G is induced from the structure of F, so all Stiefel–Whitney classes vanish, and therefore all Steenrod squares vanish. The Hopf elements η is detected by the Steenrod square, hence the only possible nontrivial attaching maps are the ones from the 11-cell to 8-cells.

In general such attaching maps can be η^2, but in this case, those cells are obtained from the Thom class and the attaching maps come from the structure of G. The top cell of TG is the product of the top cell σ^8 of $\mathcal{T}^8$ and the fiber $\mathbb{H}/S^1 = \mathbb{R}^3$, while 8-cells of TG correspond to products of 5-cells of $\mathcal{T}^8$ and the fiber $\mathbb{H}/S^1 = \mathbb{R}^3$.

Let $\sigma^5 \times \mathbb{R}^3$ be any one of the 8-cells of TG. Consider the attaching map from $\partial\sigma^8 \times \mathbb{R}^3$ to $\sigma^5 \times \mathbb{R}^3$. Pick any generic point x on σ^5. The unit sphere of the normal bundle at $x \in \sigma^5 \subset \mathcal{T}^8$ is a 2-sphere, but $\pi_2(SO(3))$ is trivial. This means that for any point y in the fiber over x, the preimage of y under the attaching map is a trivial framed sphere. Hence the attaching map is trivial.

Now each cell in $(S^{\mathbb{R}} \wedge TG)/(S^{\mathbb{R}} \wedge TG)^{(8)}$ has trivial attaching map stably. By the Atiyah-Hirzebruch spectral sequence,

$$\{S^{\mathbb{R}} \wedge TG, S^{10\mathbb{R}}\} \cong \binom{8}{0}\pi^{10}(S^{12}) \oplus \binom{8}{1}\pi^{10}(S^{11}) \oplus \binom{8}{2}\pi^{10}(S^{10}).$$

From the observation of Kronheimer-Mrowka[KM20], the preimage of a genric point under

$$BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$$

is $\eta \wedge \eta'$ in a fiber of TF smash a Lie framed circle η'' in $\mathbb{S}^{\mathbb{R}}$. The S^1-action acts on the torus $\eta \wedge \eta'$, and $(\eta \wedge \eta')/S^1$ is a Lie framed circle in a fiber of TG. We conclude that

$$BF^{\mathrm{Pin}(2)}_{\mathrm{free}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau})) = ((\eta \wedge \eta')/S^1) \wedge \eta''$$

is the generator of $\pi^{10}(S^{12}) \subset \{S^{\mathbb{R}} \wedge TG, S^{10\mathbb{R}}\}$.

Now it's easy to analyze the S^1-CW structure of $(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)$. The S^1-action on $(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)$ is free away from the base point. Hence we have only one fixed 0-cell. Because $S(\mathbb{H})/S^1$

has one 0-cell and one 2-cell, the fiber of $(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)$ has one free $(0+1+2)$-cell and one free $(2+1+2)$-cell.

where P is the quotient space of $((S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)) \wedge S^{-\mathbb{H}}$ by the S^1-action. We compute the dimension of the cells other than the based point:

$S(\mathbb{H})/S^1$	D^2	$\mathcal{T}^8$	$S^{-\mathbb{H}}$
0	2	0	-4
2		1	
		2	
		3	
		4	
		5	
		6	
		7	
		8	

As the transition map of $(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8) = T(S(F) \oplus 2\mathbb{R})$ is induced from the quaternion bundle F, the cell structure of P is

Lemma 5.4. *Let G be a rank 3 vector bundle on a*

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