RESEARCH STATEMENT

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1. Introduction

My research interests are low dimensional topology. In particular, my works find new differences between the smooth world and the continuous world.

One famous example in this area is Milnor's exotic 7-sphere: there are manifolds homeomorphic but not diffeomorphic to S^7. These are called exotic smooth structures on S^7. As another example, there are diffeomorphisms continuously isotopic but not smoothly isotopic to the identity. These are called exotic diffeomorphisms. In general, for a smooth manifold X, we can consider the homotopy type of $\mathrm{Diff}(X)$, the space of all diffeomorphisms on X, along with a natural map to $\mathrm{Homeo}(X)$, the space of all homeomorphisms on X. Exotic diffeomorphisms correspond to the kernel of the morphism $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$. More generally, we can consider the behavior of the higher homotopy groups.

Such problems are mysterious when X is 4-dimensional, since there is not enough room to apply standard techniques that convert these problems to ones in algebraic topology. The tool I use is gauge theory, which comes from physics. In particular, my works develop new invariants from Seiberg-Witten theory ([SW94]), and they detect exotic phenomena on nonsimply connected 4-manifolds, irreducible 4-manifolds, and 4-manifolds with small b_2^+ (the positive index of the intersection form). These projects not only generalize old results by novel analytical and algebraic techniques, but also capture topological ideas behind those techniques.

2. Thesis Projects

Let γ be a loop in a closed smooth 4-manifold X with a trivialization of the normal bundle. A surgery along γ is removing a neighborhood of γ, and gluing back a copy of $D^2 \times S^2$. For example, a surgery along $S^1 \times \{pt\} \subset S^1 \times S^3$ would produce S^4, while a surgery along a trivial loop on S^4 may produce $S^2 \times S^2$ or $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$. So such surgery establishes relations between lots of 4-manifolds. The four projects in my thesis (draft available at https://hcqiu.github.io/surgery.pdf, the first part available at https://arxiv.org/abs/2409.02265) describe how a surgery can preserve or produce exotic phenomena.

The tool we use comes from the Seiberg-Witten equations, which depends on a metric and a self-dual 2-form. The input of the equation for X includes a Spin^c-structure (they are related to elements in $H^2(X;\mathbb{Z})$), a $U(1)$-connection, and a "spinor". The set of equivalence classes of $U(1)$-connections and spinors under the "gauge group" $\mathrm{Map}(X, S^1)$ is called the **configuration space** (denoted by $\mathcal{B}$), which is a fiber bundle with fiber $\mathbf{C}P^\infty$ and base a torus $T^{b_1(X)}$. A tuple consisting of a metric and a perturbing 2-form is called a **parameter**. The solution of this equation with a suitable parameter is a smooth compact manifold in the configuration space. This manifold is called the SW **moduli space** (denoted by $\mathfrak{M}$). Its dimension is computed by the Atiyah-Singer index theorem, and if it is even, we can integrate a poduct of $c_1(\mathbf{C}P^\infty)$ on the moduli space and get the so-called **SW invariant** (when the dimension is 0, the integral just counts the points with signs). This is an invariant under diffeomorphism. Many examples of exotic 4-manifolds were found by computing this invariant for two homeomorphic manifolds.

The family SW invariant (FSW), on the other hand, can detect higher dimensional exotic phenomena. Given a smooth family of X over a base B and a corresponding family of parameters, the union of the solutions is called the parameterized moduli space, and if its dimension is 0 then FSW is the signed counts of points with orientation. For each $k \geqslant 0$, Ruberman-Auckly construct a $(k+1)$-family of X such that the FSW for this family is an invariant of $\pi_k(\mathrm{Diff}(X))$.

In the following projects, we generalize SW and FSW to 1-dimensional moduli space, such that new invariants (we call them SW^Θ and FSW^Θ) can detect exotic phenomena. Then we prove several surgery formulas that show how a surgery changes SW, FSW, SW^Θ and FSW^Θ.

2.1. **Surgery formula for homologically nontrivial loop** [Qiu24]**.** For a 4-manifold X with

$$H^1(X;\mathbb{Z}) = \mathbb{Z},$$

suppose $\mathfrak{s}$ is a Spin^c-structure such that $\dim \mathfrak{M}(X,\mathfrak{s}) = 1$. The configuration space is homotopy equivalent to a bundle over S^1 with fiber $\mathbf{C}P^\infty$. Let Θ be the pullback of a generator of $H^1(S^1;\mathbb{Z})$. Define the cut-down Seiberg-Witten invariant $SW^\Theta(X,\mathfrak{s})$ be the integral of Θ on $\mathfrak{M}(X,\mathfrak{s})$. We prove that this invariant detects exotic smooth structures.

Let $\gamma \subset X$ be a loop that represents a generator of $H_1(X;\mathbb{Z})/\text{torsion} = \mathbb{Z}$. Suppose a surgery along γ produces X'. We show that any Spin^c-structure $\mathfrak{s}$ on X can be extended to a unique Spin^c-structure $\mathfrak{s}'$ on X'. Since the surgery kills the first cohomology group, $H^1(X';\mathbb{Z}) = 0$ and therefore $\dim \mathfrak{M}(X',\mathfrak{s}') = 0$. Hence $SW(X',\mathfrak{s}')$ is defined by counting points in $\mathfrak{M}(X',\mathfrak{s})$. The main theorem of this project is

Theorem 2.1. $SW^\Theta(X,\mathfrak{s}) = SW(X',\mathfrak{s}')$.

This is proved by applying the classical gluing result in Nicolaescu's book [Nic00] twice. Let $S^1 \times D^3$ be a neighborhood of γ, and let $X_0 = X - S^1 \times D^3$. Then gluing X_0 with $S^1 \times D^3$ produces X, while gluing X_0 with $D^2 \times S^2$ produces X'. The classical gluing result says, if a certain "obstruction space" is trivial on X_0, then $\mathfrak{M}(X)$ is the fiber product $\mathfrak{M}(X_0) \times_{\mathfrak{M}(S^1\times S^2)} \mathfrak{M}(S^1 \times D^3)$ while $\mathfrak{M}(X')$ is the fiber product $\mathfrak{M}(X_0) \times_{\mathfrak{M}(S^1\times S^2)} \mathfrak{M}(D^2 \times S^2)$. We prove that since γ is homologically nontrivial, for generic parameters such obstruction space is trivial. Furthermore, we can choose suitable metrics such that $\mathfrak{M}(S^1 \times D^3) \to \mathfrak{M}(S^1 \times S^2)$ is the identity map of a circle, and $\mathfrak{M}(D^2 \times S^2) \to \mathfrak{M}(S^1 \times S^2)$ is the inclusion of one point into a circle. Hence if we cut $\mathfrak{M}(X')$, we get $\mathfrak{M}(X)$, and the theorem follows.

As lots of exotic smooth structures are detected by SW, we can now generalize those results to nonsimply connected manifolds, for example:

Corollary 2.2. *$E(n)\#S^1 \times S^3$ admits infinitely many exotic smooth structures.*

The method developed in this project also works for the homologically trivial case. Let $\gamma \subset X$ be a loop that represents $0 \in H_1(X;\mathbb{Z})$. Suppose a surgery along γ produces X'. We show that for any extension $\mathfrak{s}'$ of any Spin^c-structure $\mathfrak{s}$ on X with $\dim \mathfrak{M}(X,\mathfrak{s}) = 0$, we have $\dim \mathfrak{M}(X',\mathfrak{s}') = 0$. Since γ is homologically trivial, we will have

Theorem 2.3. *$SW(X',\mathfrak{s}') = 0$.*

This generalizes the vanishing result of the connected sum with $S^2\times S^2$. Theorem 2.3 can also be obtained by the generalized adjunction formula ([KM94]), but the method in this project fits in the proof of family surgery formula below, where a homologically trivial loop has nontrivial higher exotic phenomena.

2.2. **Family surgery formula for homologically nontrivial loop.** In this project we consider a smooth family E_X of X indexed by the parameter space B. Let E_{S^1} be a subbundle such that each fiber of E_{S^1} is a loop that represents a generator of $H_1(X;\mathbb{Z})=\mathbb{Z}$. Suppose a family of surgeries along E_{S^1} produces $E_{X'}$. Suppose $\mathfrak{s}$ is a Spin^c-structure such that $\dim\mathfrak{M}(X,\mathfrak{s})=\dim B+1$. As before any Spin^c-structure $\mathfrak{s}$ on X can be extended to a unique Spin^c-structure $\mathfrak{s}'$ on X', and we are able to define Θ similarly. Since the surgery kills the first cohomology group, $H^1(X';\mathbb{Z})=0$ and therefore the parameterized moduli space on X' has dimension $\dim\mathfrak{FM}(X',\mathfrak{s}')=0$. Hence $FSW(X',\mathfrak{s}')$ is defined by counting points in $\mathfrak{FM}(X',\mathfrak{s})$. The main theorem of this project is

Theorem 2.4. $FSW^{\Theta}(E_X,\mathfrak{s})=FSW(E_{X'},\mathfrak{s}')$.

The main issue here is that the parameterized moduli space on X is 1-dimensional. Then locally there would be two cases:

1) For an isolated parameter the solution is 1-dimensional, and there is no other nearby parameter such that the equation has solutions;
2) There exists a 1-dimensional family of parameters such that the solutions are 0-dimensional for each of them.

By analysing Hodge star operator and an exact sequence, it turns out that these cases depend purely on topological properties of X_0. When γ is homologically nontrivial, we prove that for a generic parameter, the parameterized moduli space on X_0 is of case 1, and the dimension of the obstruction space on X_0 is equal to $\dim B$, and therefore we can apply a method developed by Baraglia-Konno[BK20].

This cut-down family invariant generalizes exotic diffeomorphisms found by Ruberman[Rub98] and Baraglia-Konno[BK20]. For example:

Corollary 2.5. *Let X be one of the following manifolds:*

- $\mathbb{CP}^2\#(\#^2\overline{\mathbb{CP}}^2)\#Y$ *for* $n\geqslant 2$ *and* $b_2^+(Y)>2$.
- $\#^n(\mathbb{S}^2\times\mathbb{S}^2)\#(\#^nK3)$ *for* $n\geqslant 2$.
- $\#^{2n}\mathbb{CP}^2\#(\#^m\overline{\mathbb{CP}}^2)$ *for* $n\geqslant 2$ *and* $m\geqslant 10n+1$.

Then $X\#(\mathbb{S}^1\times\mathbb{S}^3)$ admits an exotic diffeomorphism.

Ruberman[Rub02] gives examples of simply connected manifolds for which the space of positive scalar curvature (psc) metrics is disconnected. This is demonstrated using family Seiberg-Witten invariant. We can generalize these results by the family surgery formula:

Corollary 2.6. *Let X be one of the following manifolds:*

- $\mathbb{CP}^2\#(\#^2\overline{\mathbb{CP}}^2)\#Y$ *for* $n\geqslant 2$ *and* $b_2^+(Y)\geqslant 3$.
- $\#^{2n}\mathbb{CP}^2\#(\#^m\overline{\mathbb{CP}}^2)$ *for* $n\geqslant 2$ *and* $m\geqslant 10n+1$.

Then the space of psc metrics on $X\#(\mathbb{S}^1\times\mathbb{S}^3)$ has infinite many path components.

Konno proves that $\pi_0(\mathrm{Diff}(X))$ is not finitely generated for some simply connected 4-manifold. We can generalize his result to nonsimply connected 4-manifolds:

Corollary 2.7. *There exists a simply connected* 4*-manifold* X *that is not a sphere, such that*

$$\pi_0(\mathit{Diff}(X\#(\mathbb{S}^1\times\mathbb{S}^3)))$$

is not finitely generated.

2.3. **Family surgery formula for homologically trivial loops.** In this project, we suppose each fiber of E_{S^1} is a homologically trivial loop. Then we have

Theorem 2.8. *Use the notation as before and assume the following:*

- $\dim B>0$*;*
- $E_{\mathbb{S}^1}$ *is an orientable* $\mathbb{S}^1$*-subbundle of* E_X*.*

Then

$$FSW(E_{X'},s')=0.$$

As we remark above, a surgery along a homologically trivial loop can preoduce nontrivial exotic phenomena:

Theorem 2.9. *Use the notation as before and assume the following:*

- B *is a circle;*
- $E_{\mathbb{S}^1}$ *is an* $\mathbb{S}^1$*-subbundle of* E_X*, and it is a Klein bottle;*

Then

$$FSW^{\mathbb{Z}/2}(E_{X'},s')\equiv SW(X,s)\mod 2.$$

(Here the family invariant is defined by counting the points mod 2*.)*

When γ is homologically trivial, we prove that for a generic parameter, the parameterized moduli space on X_0 is of case 2: there exists a 1-dimesional family of parameters such that the solutions are 0-dimensional for each of them. The dimension of the obstruction space on X_0 is one higher than $\dim B$, and therefore we have to generalize the method developed by Baraglia-Konno and estimate the errors by some inequalities.

A special example of these theorems is that each fiber of E_{S^1} is a homotopically trivial loop. In this case X' is $X\#(S^2\times S^2)$ or $X\#\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$, and the results for $X\#(S^2\times S^2)$ were previously obtained by Baraglia-Konno[BK20]. But Theorem 2.9 works also for a homotopically nontrivial loop, so it has the potential to produce exotic diffeomorphisms on a irreducible manifold.

3. Exotic diffeomorphisms on manifolds with $b_2^+=2$

The first examples of exotic diffeomorphisms on simply-connected smooth closed 4-manifolds were found by Ruberman[Rub98] using parameterized Donaldson invariant, and his examples have $b_2^+\geqslant 4$. While the exotic diffeomorphisms turned out to be very rich, we know much less about the $b_2^+=2$ case, because parameterized gauge-theoretic invariants are not well defined. In this project (draft available at https://hcqiu.github.io/critical.pdf) we present a method to find exotic diffeomorphisms on simply-connected smooth closed 4-manifolds with $b_2^+=2$, and as a result we obtain

Theorem 3.1. $2\mathbf{C}P^2\#10\overline{\mathbf{C}P^2}$ *admits exotic diffeomorphisms.*

To motivate the method, we first discuss the complications due to small b_2^+ in the ordinary case.

The ordinary SW invariant depends on the choice of the metric and the perturbing 2-form. All of such parameters can be separated to some "chambers". The SW invariant is constant for parameters in the same chamber, and is the same for a parameter and its pushforward by a diffeomorphism. The space of parameters is equivalent to $S^{b_2^+-1}$, hence when $b_2^+ > 1$, there is only one chamber and the ordinary SW invariant is a well-defined smooth invariant. When $b_2^+ = 1$, there are two chambers. Szabó's result[Sza96] says there are two homeomorphic smooth 4-manifolds with $b_2^+ = 1$, such that the SW invariant for one of them is some m for one chamber, and the invariant for another one can not be m for any chamber. This proves that they are not diffeomorphic. Such 4-manifolds are the smallest ones (in the sense of b_2^+) that admit exotic smooth structures detected by the gauge theory.

To detect exotic diffeomorphisms, we compute the family SW invariants (FSW) for the mapping tori of two diffeomorphisms, and if they are different, these tori are not diffeomorphic, hence these diffeomorphisms are not smoothly isotopic. By this machinery Ruberman and Baraglia-Konno prove that for X with an exotic smooth structure detected by the SW invariant and $b_2^+(X) > 1$, $X\#\mathbf{C}P^2\#2\overline{\mathbf{C}P^2}$ and $X\#S^2 \times S^2$ admit exotic diffeomorphisms. Note that these manifolds have $b_2^+ > 2$.

Our work generalize such results to $b_2^+ = 2$. The main issue in this case is that, the family invariant FSW on a family of manifolds, depends on the family of parameters. The mapping torus is an S^1-family of manifolds, so the space of parameter families is an S^1-family of $S^{b_2^+-1} = S^1$. The set of chambers corresponds to the set of fiberwise homotopy classes of these parameter families, which has more than one elements. If there exists a bundle isomorphism between two mapping tori, it would bring a chamber on one mapping torus to a chamber on another one. To disprove this hypothesis, we need to compare the FSW for these chambers. But the situation is a bit more complicated than in the ordinary case treated by Szabó:

- FSW may run over all possible values ($\mathbb{Z}$ if it is an integer invariant, or $\mathbb{Z}/2$ for the mod 2 invariant) as the chambers change.
- The set of chambers corresponds to $\mathbb{Z}$ or $\mathbb{Z}/2$ only noncanonically, which means we can only measure the difference between two parameter families on the same mapping torus. But we cannot compare two chambers on different mapping tori.
- The family of metrics will also determine the chamber, but we don't know how the diffeomorphism in the hypothesis acts on the families of metrics.

To solve all these problems, we construct a homotopy invariant of the parameter families, which is called the winding number. We prove that this is an invariant under the diffeomorphism of mapping tori. This viewpoint symplifies the chamber structure and decouples the families of metrics and the family of perturbing 2-forms. By additional assumption on b_2^- we can throw out the influences of the metric family and the Spin^c-structure, such that we can apply the traditional wall-crossing and gluing arguments.

4. Dehn twist on a sum of two homology 4-tori

Up to now all exotic diffeomorphisms we saw come from the exotic smooth structures of a 4-manifold. In this project we construct an exotic diffeomorphism on a nonsimply connected manifold without the need for an exotic smooth structure.

A homology 4-torus is a smooth 4-manifold that has the same homology groups as a 4-dimensional torus T^4. The connected sum of two manifolds X_1 and X_2 can be written as

$$X_1 \# X_2 = (X_1 - D^4) \cup_{S^3} ([0,1] \times S^3) \cup_{S^3} (X_2 - D^4),$$

where $[0,1] \times S^3$ is called the neck of the connected sum. The Dehn twist along a 3-sphere in the neck is a diffeomorphism $d : X_1 \# X_2 \to X_1 \# X_2$ such that d is the identity outside the neck, and on the neck it has the form

$$\begin{aligned} [0,1] \times S^3 &\to [0,1] \times S^3 \\ (t,s) &\mapsto (t, \alpha_t(s)) \end{aligned}$$

where $\alpha \in \pi_1(SO(4), Id) = \mathbb{Z}/2$ is the nontrivial element. It looks like you rotate your head by 2π: your head and body are in the original position, and the only part that changes is your neck.

For a homology torus X, its cohomology groups are isomorphic to the ones of T^4, but the ring structure might be different. Let $\alpha_1, \cdots, \alpha_4$ be a basis of $H^1(X;\mathbb{Z})$, and define the determinant of X by

$$r := |\langle \alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4, [X] \rangle|$$

where $[X]$ is the fundamental class. The main theorem of this project is

Theorem 4.1. *If $X1, X_2$ are two homology tori such that the determinants r_1, r_2 of them are odd. Then the Dehn twist along a 3-sphere in the neck of $X_1 \# X_2$ is not smoothly isotopic to the identity.*

The main tool we use is the Bauer-Furuta invariant [BF04]. Its idea is to regard the Seiberg-Witten equation as an Pin(2)-equivariant map, and consider the property of the map. By a finite dimensional approximation, it is an equivariant stable mapping class for a spin manifold X:

$$\mathrm{BF}^{\mathrm{Pin}(2)}(X, \mathfrak{s}) \in \{TF_0, S^{n\mathbb{H} + b_2^+(X)\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}.$$

where $\mathfrak{s}$ is a Spin^c-structure of X, and TF_0 is the Thom space of a rank m quarternion bundle over $T^{b_1(X)}$, such that

$$m - n = \frac{\sigma(X)}{4}$$

where $\sigma(X)$ is the signature of X.

One can also forget the the Pin(2)-action and define the nonequivariant Bauer-Furuta invariant by

$$\mathrm{BF}^{\{e\}}(X, \mathfrak{s}) := \mathrm{Res}^{\mathrm{Pin}(2)}_{\{e\}} \mathrm{BF}^{\mathrm{Pin}(2)}(X, \mathfrak{s}) \in \{TF_0, S^{4n + b_2^+(X)}\}.$$

Now one has a sequence of invariants that can detect exotic phenomena: the Seiberg-Witten invariant, the nonequivariant Bauer-Furuta invariant, and the Pin(2)-equivariant Bauer-Furuta invariant. They contain more and more infomation, but the computations get more and more complicated.

The main theorem comes from a sequence of results:

First, by a perturbation of the SW equation proposed by Ruberman-Strle[RS00], and a computation of the bundle TF_0 via the index theorem and the Steenrod square, we get

Theorem 4.2. *If X is a homology torus with odd determinant, and $\mathfrak{s}$ is the trivial structure, then*

$$BF^{\{e\}}(X, \mathfrak{s}) = (0,0,0,0,1) \in 4\mathbb{Z} \oplus \mathbb{Z}/2.$$

Actually $BF^{\{e\}}(X, \mathfrak{s})$ is the Hopf element η.

Second, we compute the nonequivariant family Bauer-Furuta invariant for the mapping torus of the Dehn twist $d: X_1 \# X_2 \to X_1 \# X_2$. It is denoted by

$$BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau})) \in \{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}.$$

We compute the bundle F by the index theorem, and prove that there exists a Hopf element ν in the stable CW structure of TF_0. Therefore, by Atiyah-Hirzebruch spectral sequence, $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ must be trivial. This vanishing result is similar to the fact that, a 3-sphere can not be mapped to $\mathbf{C}P^2$ nontrivially, because the 4-cell is attached to the 2-cell in $\mathbf{C}P^2$ by the Hopf element η.

Finally, we compute the equivariant family Bauer-Furuta invariant $BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$. By a cofiber sequence we can throw away the fixed points in the equivariant map, and then apply the equivariant Hopf theorem to convert $BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ to a nonequivariant stable mapping class. Now the dimension is changed and the Hopf invariant mentioned above has no effect. Hence we can apply the method of Kronheimer-Mrowka[KM20], and show that

Theorem 4.3. *$BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ is nontrivial.*

5. Current Projects and Future Work

In the next few years, I plan to think about some topics that are related to methods or problems above:

- Joint with Jianfeng Lin, we are trying to figure out if the Dehn twist on a sum of two odd homology 4-tori is still exotic after a stabilization (connected sum with $S^2 \times S^2$).
- Theorem 2.9 suggests that it's possible to produce an irreducible manifold with exotic diffeomorphism by a surgery along a homotopically nontrivial loop. A candidate is to find an analogue of log transform of elliptic surfaces, and an analogue of resolving elliptic surfaces by a stabilization.
- A forthcoming work of Ruberman-Auckly suggests that $\pi_k(\mathrm{Diff}(X))$ is related to

  $$\pi_{k+1}(\mathrm{Diff}(X \# (S^2 \times S^2))).$$

  It's possible to consider an analogue of the stable homotopy category, by replacing the suspension with the stablilization.
- Budney-Gabai prove Corollary 2.7 for $X = \mathbb{S}^4$ by considering the homotopy of a configuration space. We want to consider how to generalize their invariants to higher dimension.
- The skein-lasagna module is a generalization of Khovanov homology, and it's able to detect some exotic phenomena. We want to generalize some ideas we have in Seiberg-Witten theory to this invariant.

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