EXOTIC DIFFEOMORPHISM ON 4-MANIFOLDS WITH $b_2^+ = 2$

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ABSTRACT. While the exotic diffeomorphisms turned out to be very rich, we know much less about the $b_2^+ = 2$ case, because in this case all parameterized gauge-theoretic invariants are not well defined. In this paper we present a method (that is, comparing the winding number of parameter families) to find exotic diffeomorphisms on simply-connected smooth closed 4-manifolds with $b_2^+ = 2$, and as a result we obtain that $2\mathbf{C}P^2\#10\overline{\mathbf{C}P^2}$ admits exotic diffeomorphisms.

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1. INTRODUCTION

The first examples of exotic diffeomorphisms on simply-connected smooth closed 4-manifolds were found by Ruberman[Rub98] using parameterized Donaldson invariant, and his examples have $b_2^+ \geqslant 4$. While the exotic diffeomorphisms turned out to be very rich, we know much less about the $b_2^+ = 2$ case, because parameterized gauge-theoretic invariants are not well defined. In this paper we present a method to find exotic diffeomorphisms on simply-connected smooth closed 4-manifolds with $b_2^+ = 2$, and as a result we obtain

Theorem 1.1. $2\mathbf{C}P^2\#10\overline{\mathbf{C}P^2}$ *admits exotic diffeomorphisms.*

To motivate the method, we first discuss the complications due to small b_2^+ in the ordinary case.

The ordinary Seiberg-Witten invariant depends on the choice of the metric and the perturbing self-dual 2-form. All of such parameters can be separated to some "chambers". The Seiberg-Witten invariant is constant for parameters in the same chamber, and is the same for a parameter and its pushforward by a diffeomorphism. The space of parameters is equivalent to $S^{b_2^+-1}$, hence when $b_2^+ > 1$, there is only one chamber and the ordinary Seiberg-Witten invariant is a well-defined smooth invariant. When $b_2^+ = 1$, there are two chambers. Szabó's result[Sza96] says there are two homeomorphic smooth 4-manifolds with $b_2^+ = 1$, such that the Seiberg-Witten invariant for one of them is some m for one chamber, and the invariant for another one can not be m for any chamber. This proves that they are not diffeomorphic. Such 4-manifolds are the smallest ones (in the sense of b_2^+) that admit exotic smooth structures detected by the gauge theory.

To detect exotic diffeomorphisms, we compute the family Seiberg-Witten invariants (FSW) for the mapping tori of two diffeomorphisms, and if they are different, these tori are not diffeomorphic, hence these diffeomorphisms are not smoothly isotopic. By this machinery Ruberman and Baraglia-Konno prove that for X with an exotic smooth structure detected by the Seiberg-Witten invariant and $b_2^+(X) > 1$, $X \# \mathbf{C}P^2 \# 2\overline{\mathbf{C}P^2}$ and $X \# S^2 \times S^2$ admit exotic diffeomorphisms. Note that these manifolds have $b_2^+ > 2$.

Our work generalize such results to $b_2^+ = 2$. The main issue in this case is that, the family invariant FSW on a family of manifolds, depends on the family of parameters. The mapping torus is an S^1-family of manifolds, so the space of parameter families is an S^1-family of $S^{b_2^+-1} = S^1$. The set of chambers corresponds to the set of fiberwise homotopy classes of these parameter families, which has more than one elements. If there exists a bundle isomorphism between two mapping tori, it would bring a chamber on one mapping torus to a chamber on another one. To disprove this hypothesis, we need to compare the FSW for these chambers. But the situation is a bit more complicated than in the ordinary case treated by Szabó:

- FSW may run over all possible values ($\mathbb{Z}$ if it is an integer invariant, or $\mathbb{Z}/2$ for the mod 2 invariant) as the chambers change.
- The set of chambers corresponds to $\mathbb{Z}$ or $\mathbb{Z}/2$ only noncanonically, which means we can only measure the difference between two parameter families on the same mapping torus. But we cannot compare two chambers on different mapping tori.
- The family of metrics will also determine the chamber, but we don't know how the diffeomorphism in the hypothesis acts on the families of metrics.

To solve all these problems, we construct a homotopy invariant of the parameter families, which is called the winding number. We prove that this is an invariant under the diffeomorphism of mapping tori. This viewpoint symplifies the chamber structure and decouples the families of metrics and the family of perturbing 2-forms. By additional assumption on b_2^- we can throw out the influences of the metric family and the Spin^c-structure, such that we can apply the traditional wall-crossing and gluing arguments.

As a remark, Konno-Mallick-Taniguchi[KMT24] noted that $2\mathbf{C}P^2 \# n\overline{\mathbf{C}P^2}$ admits exotic diffeomorphisms for $n > 10$ by a fairly different method.

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1.1. **Prerequisite.** Let $H^2(X;\mathbb{R})$ be the vector space of harmonic 2-forms. The cup product gives a nondegenerate quadratic form on $H^2(X;\mathbb{R})$ with positive index b_2^+. All the positive definite rank b_2^+ subspaces in $H^2(X;\mathbb{R})$ form an open subset of the rank b_2^+ Grassmanian in $H^2(X;\mathbb{R})$. It will be denoted by $G^+_{b_2^+}(H^2(X;\mathbb{R}))$. Every metric g on X gives a unique Hodge star operator $\star_g$ by the formula

$$\int \alpha \wedge \star_g \beta = \langle \alpha, \beta \rangle_g.$$

We say a 2-form α is self-dual if $\star_g \alpha = \alpha$. From the above formula we see that all self-dual harmonic 2-forms under $\star_g$ form a positive definite rank b_2^+ subspace H_g^+.

For later usage we state the following proposition from the homotopy theory.

Proposition 1.2. *Let B be a circle and $\mathcal{P}$ be an $\mathbb{S}^1$-bundle over B. The fiberwise homotopy class of sections on $\mathcal{P}$, denoted by $[B,\mathcal{P}]_f$, can be identified with $\mathbb{Z}$ or $\mathbb{Z}/2$ noncanonically. It becomes canonical after choosing a trivialization of $\mathcal{P}$.*

Proof. For two sections of $\mathcal{P}$, the obstruction of the fiberwise homotopy is measured by a sequence of elements which live in $H^r(B;\pi_r^{loc}(\mathcal{P}))$, where $\pi_r^{loc}(\mathcal{P})$ is the local system of r-th homotopy groups of the fiber ([Ste51]).

Since the fiber of $\mathcal{P}$ is just $\mathbb{S}^1$, we only need to consider $r=1$. If $\mathcal{P}$ is orientable, $\pi_1^{loc}(\mathcal{P})$ is the constant sheaf $\underline{\mathbb{Z}}$ and $H^1(B;\pi_1^{loc}(\mathcal{P}))=\mathbb{Z}$. If $\mathcal{P}$ is a Klein bottle, $\pi_1^{loc}(\mathcal{P})$ is the twisted coefficient and $H^1(B;\pi_1^{loc}(\mathcal{P}))=\mathbb{Z}/2$ (see [BT82] Exercise 10.7). □

2. Winding number of parameter families

Definition 2.1. Let $B=\mathbb{S}^1$ be the parameter space and X be a smooth closed oriented 4-manifold with $b_2^+=2$. For any metric family g indexed by B, the family of nonzero self-dual harmonic 2-forms is homotopic to an $\mathbb{S}^1$-bundle

$$\mathcal{S}_g := \bigsqcup_{b\in B}(H^+_{g(b)}-\{0\})$$

over B.

Since in the construction that two diffeomorphisms have different family invariants, we cannot compare two chambers on the same manifold (see [BK20] Theorem 9.7 for the analogue in large b_2^+ case), we do need a canonical trivialization of the bundle $\mathcal{S}_g$ to identify "absolute positions" for all chambers:

Choose a base point P in $G^+_{b_2^+}(H^2(X;\mathbb{R}))$, then project every positive planes to P. This projection is nondegenerate. This gives a trivialization of $\mathcal{S}_g$.

Definition 2.2. Adopt the settings in Definition 2.1. If the family E_X over B has structure group that preserves the orientation of the positive cone of $H^2(X,\mathbb{R})$, then the **winding number** *wind* of a parameter family (g,μ) is defined by the degree of the projection of $\mathcal{H}(\mu)$ to the fiber of $\mathcal{S}_g$ (note that the trivialization of $\mathcal{S}_g$ is canonical). If $\mathcal{S}_g$ is nonorientable, the winding number of (g,μ) can be defined similarly, but with values in $\mathbb{Z}/2$.

Example 2.3. If $X=Y\#(\mathbb{S}^2\times\mathbb{S}^2)$ with $b_2^+(Y)=1$, and E_X is the mapping torus of $id_Y\# r$ where r sends $(a,b)\in H^2(\mathbb{S}^2\times\mathbb{S}^2)$ to $(-a,-b)$. Then there are two homotopy classes of purterbation families. They are represented by two paths in Figure 1.

2.1. Properties of the winding number.

To disprove the hypothesis that there exists a diffeomorphism of mapping tori, we have to first figure out how such diffeomorphisms can change the parameter family. It turns out that the winding numbeer is an invariant under the diffeomorphism of any $\mathbb{S}^1$-parameter family of positve index 2 manifolds.

Lemma 2.4. *Let $B=\mathbb{S}^1$ be the parameter space and X be a smooth closed oriented 4-manifold with $b_2^+=2$. Let E_X be a smooth family of X over B. Let (g,μ) be a parameter family that doesn't cross any wall. Then for any diffeomorphism f of E_X,*

$$wind(g,\mu)=wind(f^*g,f^*\mu).$$

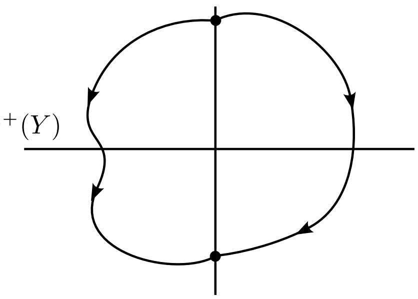

FIGURE 1

It turns out that when b_2^- is not so large, we don't really need to know how a diffeomorphism acts on the set of parameter families. Hence we omit the proof of the above lemma.

For each g, the harmonic projection gives an identification between the period bundle in [LL01] Section 3.1 and the bundle $\mathcal{S}_g$, so the winding number actually determines the chamber:

Lemma 2.5. *Let X be a smooth closed oriented 4-manifold with $H_1(X,\mathbb{Z}) = 0$ and $b_2^+(X) = 2$. Fix a homology orientation of $H^{2,+}(X,\mathbb{R})$. Let E_X be a family of X over the circle B. For each Riemannian metric family g let the bundle $\mathcal{S}_g$ be the one defined in Definition 2.1 with the orientation compatible with the homology orientation. Then for each characteristic element $K \in H^2(X;\mathbb{Z})$ with $(K^2 - 2e(X) - 3sign(X))/4 \geqslant 0$ we have*

- *If the winding number of $(g_1, 2\pi K^+ + \mu_1)$ is equal to the winding number of $(g_2, 2\pi K^+ + \mu_2)$, then*

$$FSW(E_X, K, g_1, \mu_1) = FSW(E_X, K, g_2, \mu_2)$$

Proof. It suffices to find a homotopy from (g_1, μ_1) to (g_2, μ_2) without crossing any wall. Since the space of all metrics is contractible, there exsists a homotopy $g(b,t)$ between the loops g_1 and g_2. To avoid $2\pi K^+ + \mathcal{H}(\mu_1)$ being zero during this process, we may change the perturbation family by the following procedure:

For each $g(b,t)$, the projection of the base plane P to $H^+_{g(b,t)}$ is nondegenerate. Hence a frame in P will induce a smooth family of frames on H^+_g. For each $g(b,t)$, the pairing between the attached frame on $H^+_{g(b,t)}$ and an element in $H^+_{g(b,t)}$ gives an element in $\mathbb{R}^2$ (this identification is compatible with the previous trivialization of $\mathcal{S}_g$). Denote this identification by

$$i_{b,t} : H^+_{g(b,t)} \to \mathbb{R}^2.$$

Let

$$\mu(b,t) := \mu_1(b) + 2\pi(i_{b,t}^{-1} i_{b,0} K^{+_{g(b,0)}} - K^{+_{g(b,t)}})$$

where $K^{+_{g(b,t)}}$ is the self dual part of K with respect to the netric $g(b,t)$. Then $(g(b,t), \mu(b,t))$ gives a homotopy from (g_1, μ_1) to (g_2, μ_1'), and $2\pi K^{+_g} + \mathcal{H}(\mu)$ is always nonzero. Moreover this homotopy preserves the winding number since $i_{b,t}\mu(b,t)$ is actually unchanged. This means that

$$wind(g_2, 2\pi K^+ + \mu_1') = wind(g_1, 2\pi K^+ + \mu_1) = wind(g_2, 2\pi K^+ + \mu_2).$$

Then by Proposition 1.2, (g_2, μ_1') is fiberwise homotopy equivalent to (g_2, μ_2). The concatenation of these two homotopies is a homotopy from (g_1, μ_1) to (g_2, μ_2) without crossing any wall. This proves that

$$FSW(E_X, K, g_1, \mu_1) = FSW(E_X, K, g_2, \mu_2).$$

□

3. Proof of the main theorem

If the characteristic element $K \in H^2(X, \mathbb{Z})$ is not in the positive cone of $H^2(X, \mathbb{Z})$, there is a risk that the metric family g indexed over a circle would make K^+ wind around the origin of H_g^+. This means that the chamber of the parameter family may depend on g, which is something we cannot control in the diffeomorphism. We have to assume some conditions on $b_2^-(X)$ to avoid this situation:

Lemma 3.1. *Let X be a smooth closed oriented 4-manifold with $H_1(X, \mathbb{Z}) = 0$, $b_2^+(X) = 2$ and $b_2^-(X) \leqslant 10$. Let E_X be a family of X over a circle B. Then for every characteristic element $K \in H^2(X, \mathbb{Z})$, family of metrics g_1, g_2, and sufficiently small (with respect to K^+) perturbation family μ_1, μ_2, we have*

$$FSW(E_X, K, g_1, \mu_1) = FSW(E_X, K, g_2, \mu_2).$$

Proof. Let $K \in H^2(X; \mathbb{Z})$ be a characteristic element for which the formal dimension of the parameterized moduli space is at least 0. This means that the formal dimension of the ordinary moduli space is -1. Then

$$2e(X) + 3\text{sign}(X) = 4 + 5b_2^+(X) - b_2^-(X) \geqslant 4$$

implies $K^2 \geqslant 0$. Hence the distance between $2\pi K$ and the subspace perpendicular to $H_g^+ \subset H^2(X; \mathbb{R})$ has a lower bound. As a corollary we have that for small enough μ_1 and μ_2,

$$wind(g_1, 2\pi K^+ - \mu_1) = wind(g_2, 2\pi K^+ - \mu_2).$$

Now Lemma 3.1 follows from Lemma 2.5. □

Theorem 3.2. *Let M and M' be two closed oriented simply connected smooth 4-manifolds with indefinite intersection forms. Suppose that $\phi : M \to M'$ is a homeomorphism. Suppose that $b_2^+(M) = 1, b_2^-(M) \leqslant 9$ and that $\mathfrak{s}_M$ is a Spin^c -structure on M with $d(M, \mathfrak{s}_M) = 0$. We make the following assumptions:*

1) *Suppose that for any matric g_M and sufficiently small (with respect to $c_1(\mathfrak{s}_M)^+$) perturbation μ_M, all solutions on M are irreducible and $SW(M, \mathfrak{s}_M, g_M, \mu_M)$ has the same value. Similarly, suppose that for any matric $g_{M'}$ and sufficiently small perturbation $\mu_{M'}$, all solutions on M' are irreducible and $SW(M', \phi(\mathfrak{s}_M), g_{M'}, \mu_{M'})$ is well defined.;*
2) *Suppose that*

$$SW(M, \mathfrak{s}_M, g_M, \mu_M) \neq SW(M', \phi(\mathfrak{s}_M), g_{M'}, \mu_{M'}) \mod 2; \tag{3.1}$$

3) *Lastly, suppose that $M\#(\mathbb{S}^2 \times \mathbb{S}^2)$ is diffeomorphic to $M'\#(\mathbb{S}^2 \times \mathbb{S}^2)$.*

Then $M\#(\mathbb{S}^2 \times \mathbb{S}^2)$ admits an exotic diffeomorphism.

Proof. Let $\mathfrak{s}_0$ be the Spin^c -structure on $\mathbb{S}^2 \times \mathbb{S}^2$ which has trivial determinant line. Let r be the reflection of $\mathbb{S}^2 \times \mathbb{S}^2$ in Example 2.3.

Let $\phi' : M\#(\mathbb{S}^2 \times \mathbb{S}^2) \to M'\#(\mathbb{S}^2 \times \mathbb{S}^2)$ be a homeomorphism induced by ϕ and $id_{\mathbb{S}^2\times\mathbb{S}^2}$. Since M is indeinite, there exsists a diffeomorphism $\psi : M\#(\mathbb{S}^2 \times \mathbb{S}^2) \to M'\#(\mathbb{S}^2 \times \mathbb{S}^2)$ such that ψ and ϕ' induce the same map on cohomology groups (see [Wal64]). Then we have

$$\psi(\mathfrak{s}_M\#\mathfrak{s}_0) = \phi(\mathfrak{s}_M)\#\mathfrak{s}_0.$$

Define

$$f_1 := id_M\#r : M\#(\mathbb{S}^2 \times \mathbb{S}^2) \to M\#(\mathbb{S}^2 \times \mathbb{S}^2)$$

and

$$f_2 := \psi^{-1} \circ (id_{M'}\#r) \circ \psi : M\#(\mathbb{S}^2 \times \mathbb{S}^2) \to M\#(\mathbb{S}^2 \times \mathbb{S}^2).$$

We will show that they are continuously isotopic but not smoothly isotopic. Then $f_1 \circ f_2^{-1}$ will be an exotic diffeomorphism.

Since r is a reflection, it induces the identity on $H^2(\mathbb{S}^2 \times \mathbb{S}^2; \mathbb{Z}/2)$, and therefore so does $id_{M'}\#r$ on the $H^2(\mathbb{S}^2 \times \mathbb{S}^2; \mathbb{Z}/2)$-summand of $H^2(M'\#(\mathbb{S}^2 \times \mathbb{S}^2); \mathbb{Z}/2)$. On the other hand $\psi^* = (\phi')^*$ acts on the $H^2(M'; \mathbb{Z}/2)$ summand of $H^2(M'\#(\mathbb{S}^2 \times \mathbb{S}^2); \mathbb{Z}/2)$. Hence f_1 and f_2 induce the same map on $H^2(M\#(\mathbb{S}^2 \times \mathbb{S}^2); \mathbb{Z}/2)$. Therefore by [Qui86], f_1 and f_2 are continuously isotopic.

Define

$$f_1' := id_{M'}\#r : M'\#(\mathbb{S}^2 \times \mathbb{S}^2) \to M'\#(\mathbb{S}^2 \times \mathbb{S}^2).$$

Let E_1, E_1' and E_2 be the mapping tori of f_1, f_1' and f_2. Choose g_2, μ_2 to be any generic family of parameters on E_2. Note that ψ gives a bundle isomorphism between E_1' and E_2. Hence

$$\begin{aligned} \text{(3.2)} \qquad \mathrm{FSW}^{\mathbb{Z}/2}(E_2, \mathfrak{s}_M\#\mathfrak{s}_0, g_2, \mu_2) &= \mathrm{FSW}^{\mathbb{Z}/2}(E_1', \psi(\mathfrak{s}_M\#\mathfrak{s}_0), \psi(g_2), \psi(\mu_2)) \\ &= \mathrm{FSW}^{\mathbb{Z}/2}(E_1', \phi(\mathfrak{s}_M)\#\mathfrak{s}_0, \psi(g_2), \psi(\mu_2)). \end{aligned}$$

Choose g_1, μ_1 to be any generic family of parameters on E_1 with μ_1 small enough (with respect to $c_1(\mathfrak{s}_M)^+$). By the gluing construction [BK20] or the surgery construction [Qiu24], the parameterized moduli space of E_1 comes from gluing an irreducible solution on M and a reducible solution on $\mathbb{S}^2 \times \mathbb{S}^2$, which locates on a parameter where the self-dual 2-form on $\mathbb{S}^2 \times \mathbb{S}^2$ is 0 (see Figure 2). Similarly, choose g_2, μ_2 to be any generic family of parameters on E_2 with μ_2 small enough (with respect to $c_1(\mathfrak{s}_M)^+$).

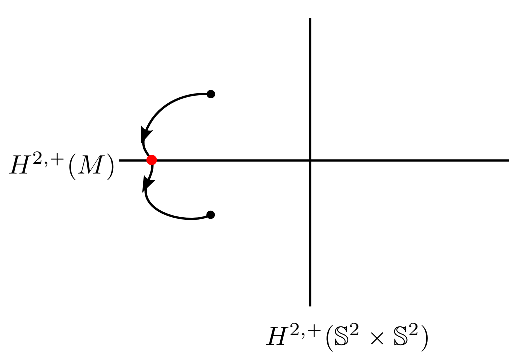

Figure 2. The only solutions on $\mathbb{S}^2 \times \mathbb{S}^2$ are reducible by the dimension reason and therefore locate on the point with 0 perturbation on $H^{2,+}(\mathbb{S}^2 \times \mathbb{S}^2)$, and after the gluing, the parameterized moduli space of E_1 locates on nearby parameters.

Hence we have

$$\mathrm{FSW}^{\mathbb{Z}/2}(E_1, \mathfrak{s}_M \# \mathfrak{s}_0, g_1, \mu_1) = SW(M, \mathfrak{s}_M, g_1(b)|_M, \mu_1(b)|_M) \mod 2$$
$$\mathrm{FSW}^{\mathbb{Z}/2}(E_1', \phi(\mathfrak{s}_M) \# \mathfrak{s}_0, \psi(g_2), \psi(\mu_2)) = SW(M', \phi(\mathfrak{s}_M), \psi(g_2)(b')|_{M'}, \psi(\mu_2)(b')|_{M'}) \mod 2$$

for some b in the base of E_1, and some b' on the base of E_1' by the gluing results. Since μ_1, μ_2 are much smaller than $c_1(\mathfrak{s}_M)^+$, we have $\mu_1(b)|_M << c_1(\mathfrak{s}_M)^+$ and $\psi(\mu_2)(b')|_{M'} << \phi(c_1(\mathfrak{s}_M))^+ = c_1(\phi(\mathfrak{s}_M))^+$. Therefore, by Assumption (1) and (2) on the ordinary Seiberg-Witten invariant, we have

$$\mathrm{FSW}^{\mathbb{Z}/2}(E_1, \mathfrak{s}_M \# \mathfrak{s}_0, g_1, \mu_1) \neq \mathrm{FSW}^{\mathbb{Z}/2}(E_1', \phi(\mathfrak{s}_M) \# \mathfrak{s}_0, \psi(g_2), \psi(\mu_2)).$$

Combine this and (3.2) we have

$$\mathrm{FSW}^{\mathbb{Z}/2}(E_1, \mathfrak{s}_M \# \mathfrak{s}_0, g_1, \mu_1) \neq \mathrm{FSW}^{\mathbb{Z}/2}(E_2, \mathfrak{s}_M \# \mathfrak{s}_0, g_2, \mu_2).$$

Suppose that f_1 is smoothly isotopic to f_2, then the isotopy forms a bundle isomorphism H from E_1 to E_2. Since f_1 and f_2 act by the identity on $H^2(M)$ and $\mathfrak{s}_0$ is a Spin^c-structure with trivial determinant line on $\mathbb{S}^2 \times \mathbb{S}^2$, f_1 and f_2 and therefore H preserve $\mathfrak{s}_M \# \mathfrak{s}_0$. Hence

$$\mathrm{FSW}^{\mathbb{Z}/2}(E_2, \mathfrak{s}_M \# \mathfrak{s}_0, H(g_1), H(\mu_1)) \neq \mathrm{FSW}^{\mathbb{Z}/2}(E_2, \mathfrak{s}_M \# \mathfrak{s}_0, g_2, \mu_2).$$

Also, H would bring (g_1, μ_1) to a generic family of parameters on E_2 with μ_2 small enough (with respect to $c_1(\mathfrak{s}_M)^+$). Hence the above equation contradicts Lemma 3.1. □

Corollary 3.3. $2\mathbf{C}P^2 \# 10\overline{\mathbf{C}P^2}$ *admits exotic diffeomorphisms.*

Proof. [Sza96] proves that there exists a family of manifolds $E(1)_{K_k}$ (where K_k is the k-twist knot, as mentioned in the last part of [FS97]) that are homeomorphic to $E(1)$. Choose $M = E(1)_{K_m}$ and $M' = E(1)_{K_n}$ for $m, n \geqslant 0$, m odd and n even. We have that $b_2^+(M) = 1, b_2^-(M) \leqslant 9$. By [Sza96] Lemma 3.2, Assumption (1) in Theorem 3.2 is satisfied. Choose $\mathfrak{s}_M$ such that $c_1(\mathfrak{s}_M)$ is the Poincaré dual of a regular elliptic fiber. By [Sza96] Theorem 3.3, $SW(M, \mathfrak{s}_M, g_M, \mu_M) = m$ and $SW(M', \phi(\mathfrak{s}_M), g_{M'}, \mu_{M'})$ can be only n or 0. Hence Assumption (2) in Theorem 3.2 is satisfied. By [Akb02], [Auc03] and [Bay18], both $M \# (\mathbb{S}^2 \times \mathbb{S}^2)$ and $M' \# (\mathbb{S}^2 \times \mathbb{S}^2)$ dissolve. Hence Assumption (3) in Theorem 3.2 is satisfied. □

References

[Akb02] Selman Akbulut. Variations on Fintushel-Stern knot surgery on 4-manifolds. *Turkish J. Math.*, 26(1):81–92, 2002. (Cited on page 7.)

[Auc03] David Auckly. Families of four-dimensional manifolds that become mutually diffeomorphic after one stabilization. In *Proceedings of the Pacific Institute for the Mathematical Sciences Workshop "Invariants of Three-Manifolds" (Calgary, AB, 1999)*, volume 127, pages 277–298, 2003. (Cited on page 7.)

[Bay18] R. Inanc Baykur. Dissolving knot surgered 4-manifolds by classical cobordism arguments, 2018. (Cited on page 7.)

[BK20] David Baraglia and Hokuto Konno. A gluing formula for families Seiberg–Witten invariants. *Geometry Topology*, 24:1381–1456, 2020. (Cited on pages 3 and 6.)

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. (Cited on page 3.)

[FS97] Ronald Fintushel and Ronald J. Stern. Knots, links, and 4-manifolds, 1997. (Cited on page 7.)

[KMT24] Hokuto Konno, Abhishek Mallick, and Masaki Taniguchi. Exotic dehn twists on 4-manifolds, 2024. (Cited on page 2.)

[LL01] Tian-Jun Li and Ai-Ko Liu. Family Seiberg-Witten invariants and wall crossing formulas. *Communications in Analysis and Geometry*, 9, 08 2001. (Cited on page 4.)

[Qui86] F Quinn. Isotopy of 4-manifolds. *J. Differential Geom.*, 24:343–372, 1986. (Cited on page 6.)

[Rub98] Daniel Ruberman. An obstruction to smooth isotopy in dimension 4. *Math. Res. Lett.*, 5:743–758, 1998. (Cited on page 1.)

[Ste51] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951. (Cited on page 3.)

[Sza96] Zoltán Szabó. Exotic 4-manifolds with $b_2^+ = 1$. *Math. Res. Lett.*, 3(6):731–741, 1996. (Cited on pages 1 and 7.)

[Wal64] C. T. C. Wall. Diffeomorphisms of 4-manifolds. *J. London Math. Soc.*, 39:131–140, 1964. (Cited on page 6.)