THE DEHN TWIST ON $T^4\#T^4$ IS NOT SMOOTHLY ISOTOPIC TO THE IDENTITY

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Abstract. In this project we construct an exotic diffeomorphism on a nonsimply connected manifold without the need for an exotic smooth structure. We use Pin(2)-equivariant family Bauer-Furuta invariant to show that, if $X1, X_2$ are two homology tori such that the determinants r_1, r_2 of them are odd. Then the Dehn twist along a 3-sphere in the neck of $X_1\#X_2$ is not smoothly isotopic to the identity.

Contents

1. Introduction

In this project we construct an exotic diffeomorphism on a nonsimply connected manifold without the need for an exotic smooth structure.

A homology 4-torus is a smooth 4-manifold that has the same homology groups as a 4-dimensional torus T^4. The connected sum of two manifolds X_1 and X_2 can be written as

$$X_1\#X_2 = (X_1 - D^4) \cup_{S^3} ([0,1] \times S^3) \cup_{S^3} (X_2 - D^4),$$

where $[0,1] \times S^3$ is called the neck of the connected sum. The Dehn twist along a 3-sphere in the neck is a diffeomorphism $d : X_1\#X_2 \to X_1\#X_2$ such that d is the identity outside the neck, and on the neck it has the form

$$\begin{aligned}[0,1] \times S^3 &\to [0,1] \times S^3\\ (t,s) &\mapsto (t, \alpha_t(s))\end{aligned}$$

where $\alpha \in \pi_1(SO(4), Id) = \mathbb{Z}/2$ is the nontrivial element. It looks like you rotate your head by 2π: your head and body are in the original position, and the only part that changes is your neck.

For a homology torus X, its cohomology groups are isomorphic to the ones of T^4, but the ring structure might be different. Let $\alpha_1, \cdots, \alpha_4$ be a basis of $H^1(X;\mathbb{Z})$, and define the determinant of X by

$$r := |\langle \alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4, [X]\rangle|$$

where $[X]$ is the fundamental class. The main theorem of this project is

Theorem 1.1. *If $X1, X_2$ are two homology tori such that the determinants r_1, r_2 of them are odd. Then the Dehn twist along a 3-sphere in the neck of $X_1\#X_2$ is not smoothly isotopic to the identity.*

The main tool we use is the Bauer-Furuta invariant [BF02]. Its idea is to regard the Seiberg-Witten equation as an Pin(2)-equivariant map, and consider the property of the map. By a finite dimensional approximation, it is an equivariant stable mapping class for a spin manifold X:

$$\mathrm{BF}^{\mathrm{Pin}(2)}(X,\mathfrak{s}) \in \{TF_0, S^{n\mathbb{H}+b_2^+(X)\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}.$$

where $\mathfrak{s}$ is a Spin^c-structure of X, and TF_0 is the Thom space of a rank m quarternion bundle over $Pic^{\mathfrak{s}}(X) = T^{b_1(X)}$, such that

$$m - n = \frac{\sigma(X)}{4}$$

where $\sigma(X)$ is the signature of X.

One can also forget the Pin(2)-action and define the nonequivariant Bauer-Furuta invariant by

$$\mathrm{BF}^{\{e\}}(X,\mathfrak{s}) := \mathrm{Res}^{\mathrm{Pin}(2)}_{\{e\}}\mathrm{BF}^{\mathrm{Pin}(2)}(X,\mathfrak{s}) \in \{TF_0, S^{4n+b_2^+(X)}\}.$$

Now one has a sequence of invariants that can detect exotic phenomena: the Seiberg-Witten invariant, the nonequivariant Bauer-Furuta invariant, and the Pin(2)-equivariant Bauer-Furuta invariant. They contain more and more infomation, but the computations get more and more complicated.

The main theorem comes from a sequence of results:

First, by a perturbation of the SW equation proposed by Ruberman-Strle[RS00], and a computation of the bundle TF_0 via the index theorem and the Steenrod square, we get

Theorem 1.2. *If X is a homology torus with odd determinant, and $\mathfrak{s}$ is the trivial structure, then*

$$BF^{\{e\}}(X,\mathfrak{s}) = (0,0,0,0,1) \in 4\mathbb{Z} \oplus \mathbb{Z}/2.$$

Actually $BF^{\{e\}}(X,\mathfrak{s})$ is the Hopf element η.

Second, we compute the nonequivariant family Bauer-Furuta invariant for the mapping torus of the Dehn twist $d : X_1\#X_2 \to X_1\#X_2$. It is denoted by

$$BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau})) \in \{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}.$$

We compute the bundle F by the index theorem, and prove that there exists a Hopf element ν in the stable CW structure of TF. Therefore, by Atiyah-Hirzebruch spectral sequence, $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ must be trivial. This vanishing result is similar to the fact that, a 3-sphere can not be mapped to $\mathbf{C}P^2$ nontrivially, because the 4-cell in $\mathbf{C}P^2$ is attached to the 2-cell by the Hopf element η.

Finally, we compute the equivariant family Bauer-Furuta invariant $BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$. By a cofiber sequence we can throw away the fixed points in the equivariant map, and then apply the equivariant Hopf theorem to convert $BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ to a nonequivariant stable mapping class. Now the dimension is changed and the Hopf invariant mentioned above has no effect. Hence we can apply the method of Kronheimer-Mrowka[KM20], and show that

Theorem 1.3. *$BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^{\tau}))$ is nontrivial.*

Compared with the previous results, there are some issues when computing the Bauer-Furuta invariant for T^4:

- The Bauer-Furuta invariant of T^4 is unkonwn. The map from the Bauer-Furuta invariant to the Seiberg-Witten invariant is not well-defined because T^4 doesn't satisfy the condition $b^+ - b_1 \geqslant 2$ in [BF02], and indeed, [RS00] shows that the perturbation used in the computation of the Seiberg-Witten invariant contains reducible solutions.
- Proposition 4.1 in [KM20] doen't work for T^4. The index of the twisted Dirac operator on T^4 is zero, which leads to a vanishing twist in the parameterized Bauer-Furuta invariant. Therefore, such invariant cannot distinguish the twisted spin structure and the product spin structure on a family of T^4.
- The (parameterized) Bauer-Furuta invariant of T^4 is a stable mapping class from T^4 to a sphere, which is hard to compute. The $K3$ surface considered in [KM20] and [Lin23], however, has $b_1 = 0$. Hence the (parameterized) Bauer-Furuta invariant of it is an element in the stable homotopy group of spheres, which is well known in low dimension (1, 2 and 3), and moreover, the equivariant version can be computed by algebraic topology.

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2. The Bauer-Furuta invariant of homology tori

Theorem 2.1. *Suppose X is a homology torus with*

$$\langle \alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4, X \rangle = d, \tag{2.1}$$

where $\{\alpha_i\}$ is a basis for $H^1(X;\mathbb{Z})$, and d is odd. Let $\mathfrak{s}$ be the spin^c structure on X with trivial determinent line. Then the nonequivariant Bauer-Furuta invariant $BF^{\{e\}}(X,\mathfrak{s})$ is the generator of $\mathbb{Z}/2\mathbb{Z}$ in a group $4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. First we compute the group that the invariant lives. Let F_0 be the bundle $\mathbb{H} \to F_0 \to \mathcal{T}^4$ with $c_1(T_0) = 0$ and $c_2(T_0) = d$. Let TF_0 be its Thom space. The equivariant Bauer-Furuta invariant is

$$\mathrm{BF}^{\mathrm{Pin}(2)}(X,\mathfrak{s}) \in \{TF_0, S^{\mathbb{H}+3\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}.$$

Forget the Pin(2)-action then we get the non equivariant Bauer-Furuta invariant

$$\mathrm{BF}^{\{e\}}(X,\mathfrak{s}) := \mathrm{Res}^{\mathrm{Pin}(2)}_{\{e\}} \mathrm{BF}^{\mathrm{Pin}(2)}(X,\mathfrak{s}) \in \{TF_0, S^7\}.$$

A sketch of the proof: TF_0 can be regarded as a CW complex with one 0-cell, one 4-cell (corresponds to $S^{\mathbb{H}}$), four 5-cells (obtained from the 1-cells of $\mathcal{T}^4$ by multiplying a 4-cell corresponds to $S^{\mathbb{H}}$), six 6-cells (obtained from the 2-cells of $\mathcal{T}^4$ similarly), four 7-cells (obtained from the 3-cells of $\mathcal{T}^4$), one 8-cell (obtained from the 4-cell of $\mathcal{T}^4$). By a cofiber sequence and the CW approximation theorem, there is an isomorphism

$$\{TF_0, S^7\} \cong \{TF_0/TF_0^{(5)}, S^7\}$$

where $TF_0^{(5)}$ is the 5-th skeleton of TF_0. We want to show that all attaching maps of $TF_0/TF_0^{(5)}$ are trivial.

First by Thom isomorphism and the cohomology of $\mathcal{T}^4$, we deduce that all cells of $TF_0/TF_0^{(5)}$ survive in the cohomology group, so all adjacent attaching maps are trivial.

Now the only possible nontrivial attaching map is the one from the 8-cell to 6-cells. Since the only nontrival element of π_1 is the Hopf map η, which can be detected by the Steenrod square, it suffices to show that Sq^2 is trivial. Let u be the Thom class of F_0. Note that u is an element in $H^*(F_0, F_0 - \mathcal{T}^4)$ represented by the zero section of F_0. The cup product of $H^*(F_0, F_0 - \mathcal{T}^4)$ with $H^*(\mathcal{T}^4)$ still produces closed submanifolds, so $H^*(\mathcal{T}^4)$ acts on $H^*(F_0, F_0 - \mathcal{T}^4)$. By Cartan formula we have for any $x \in H^*(\mathcal{T}^4)$

$$
\begin{aligned}
Sq^n(ux) &= \sum_{i+j=n} Sq^i(u) Sq^j(x) \\
&= \sum_{i+j=n} u w_i(F_0) Sq^j(x)
\end{aligned}
$$

where w_i is the i-th Stiefel–Whitney class. The cohomology of $\mathcal{T}^4$ is an exterior algebra generated by four 1-classes which the Steenrod algebra acts on trivially. By the Cartan formula and the Adem relations, the Steenrod algebra acts on $H^*(\mathcal{T}^4)$ trivially. Hence $Sq^j(x) \neq 0$ iff $j = 0$. So

$$Sq^2(ux) = u w_2(F_0) Sq^0(x).$$

But $w_2(F_0) \equiv c_1(F_0) \mod 2$ and from the structure of F_0 we have $c_1(F_0) = 0$. So the attaching maps from the 8-cell to 6-cells are trivial.

Now we conclude that $TF_0/TF_0^{(5)}$ is equivalent to $6\mathbb{S}^6 \vee 4\mathbb{S}^7 \vee \mathbb{S}^8$ in the stable category. Hence

$$\{TF_0, S^7\} \cong \{TF_0/TF_0^{(5)}, S^7\} \cong [6\mathbb{S}^6 \vee 4\mathbb{S}^7 \vee \mathbb{S}^8, \mathbb{S}^7] \cong 4\mathbb{Z} \oplus \mathbb{Z}/2.$$

By Ruberman-Strle[RS00], the preimage of a genric point under $BF^{\{e\}}(X, \mathfrak{s})$ is a Lie framed circle in a fiber of TF_0. So the restriction $BF^{\{e\}}(X, \mathfrak{s})|_{\mathbb{S}^8}$ is a suspension of the Hopf map

$$\Sigma^5 \eta : \mathbb{S}^8 \to \mathbb{S}^7.$$

And the restricitons of $BF^{\{e\}}(X, \mathfrak{s})$ on those 7-cells have degree 0, otherwise the preimage of a generic point would contain discrete points. Therefore,

$$BF^{\{e\}}(X, \mathfrak{s}) = (0, 0, 0, 0, 1) \in 4\mathbb{Z} \oplus \mathbb{Z}/2.$$

□

3. Family Bauer-Furuta invariant of the connected sum of two tori

Suppose X_1 and X_2 are homology tori with determinant r_1 and r_2. Let $X = X_1 \# X_2$ and $\mathcal{T}^8 \cong H^1(X, S^1)$. Let $\mathfrak{s}$ be the spin^c structure on X with trivial determinent line. Denote the family of Dirac operators by $\not{\mathfrak{D}}$. The index bundle $Ind(\not{\mathfrak{D}})$ is an $\mathbb{H}$-bundle over $\mathcal{T}^8$ since X is spin. Let $\mathbf{L} \to X \times \mathcal{T}^8$ be the universal line bundle. Since the $\hat{A}$-genus of X is zero, the Chern character of the index bundle is

$$\mathrm{ch}(Ind(\not{\mathfrak{D}})) = \int_X \mathrm{ch}(\mathbf{L})$$

by Atiyah-Singer.

Suppose $\mathbf{L}$ is equipped with a connection

$$\mathbf{A} = 2\pi i \sum_{k=1}^{8} t_k \alpha_k$$

where $\alpha_1, \ldots, \alpha_4$ is a basis for $H^1(X_1, \mathbb{Z})$ and $\alpha_5, \ldots, \alpha_8$ is a basis for $H^1(X_2, \mathbb{Z})$, and $t_k \mapsto 2\pi i t_k \alpha_k$ are coordinates on $\mathcal{T}^8 \cong H^1(X, i\mathbb{R})/H^1(X, 2\pi i\mathbb{Z})$. Then the first Chern class of $\mathbf{L}$ is

$$\Omega = \sum_k \alpha_k \wedge dt_k.$$

By the dimension reason

$$\mathrm{ch}(\mathbf{L}) = 1 + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{6}\Omega^3 + \frac{1}{24}\Omega^4.$$

A term in $\mathrm{ch}(\mathbf{L})$ contains a volumn form of X if and only if it contains $\alpha_1 \smile \alpha_2 \smile \alpha_3 \smile \alpha_4$ or $\alpha_5 \smile \alpha_6 \smile \alpha_7 \smile \alpha_8$. Therefore,

$$\mathrm{ch}(Ind(\not{\mathfrak{D}})) = \pm r_1[vol_{\mathcal{T}_1^4}] \pm r_2[vol_{\mathcal{T}_2^4}]$$

where $\mathcal{T}_i^4 \cong H^1(X_i, S^1)$ are submanifolds of $\mathcal{T}^8$. Hence $c_2(Ind(\not{\mathfrak{D}})) = \pm r_1[vol_{\mathcal{T}_1^4}] \pm r_2[vol_{\mathcal{T}_2^4}]$ and $c_i(Ind(\not{\mathfrak{D}})) = 0$ for $i \neq 2$. Hence

$$\mathrm{BF}^{\{e\}}(X, \mathfrak{s}) \in \{TF, S^{\mathbb{H}+6\mathbb{R}}\}$$

where TF is the Thom space of the $\mathbb{H}$-bundle $\mathbb{H} \to F \to \mathcal{T}^8$. Hence we have

Lemma 3.1. *$BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{S^{\mathbb{R}} \wedge TF, S^{\mathbb{H}+6\mathbb{R}}\} = \{S^{\mathbb{R}} \wedge TF, S^{10}\}$*

This also follows from the gluing theorem of Bauer-Furuta invariant, which asserts that the domain of the stable cohomotopy element for a connected sum is an extenal product of the domains of two elements.

Theorem 3.2. *Suppose X_1 and X_2 are homology tori with odd determinant. Let $\mathfrak{s}_i$ be the spin^c structure on X_i with trivial determinent line bundle. Then the nonequivariant family Bauer-Furuta invariant $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1) \# (X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{S^{\mathbb{R}} \wedge TF, S^{10}\}$ is trivial.*

Proof. First we compute the structure of TF: TF can be regarded as a CW complex with one 0-cell, one 4-cell (corresponds to $S^{\mathbb{H}}$), several n-cells ($n = 5, 6, 7, 8, 9, 10, 11$, obtained from the $(n-4)$-cells of $\mathcal{T}^8$ by multiplying a 4-cell corresponds to $S^{\mathbb{H}}$), and one 12-cell (obtained from the 8-cell of $\mathcal{T}^8$).

We claim that the attaching maps from the 12-cell to two of the 8-cells are Hopf element ν. Let u be the Thom class of F. Note that u is an element in $H^*(F, F - \mathcal{T}^8)$ represented by the zero section of F. The cup product of $H^*(F, F - \mathcal{T}^8)$ with $H^*(\mathcal{T}^8)$ still produces closed submanfolds, so $H^*(\mathcal{T}^8)$ acts on $H^*(F, F - \mathcal{T}^8)$. By Cartan formula we have for any $x \in H^*(\mathcal{T}^8)$

$$\begin{aligned} Sq^n(ux) &= \sum_{i+j=n} Sq^i(u) Sq^j(x) \\ &= \sum_{i+j=n} u w_i(F_0) Sq^j(x) \end{aligned}$$

where w_i is the i-th Stiefel–Whitney class. The cohomology of $\mathcal{T}^8$ is an exterior algebra generated by four 1-classes which the Steenrod algebra acts on trivially. By the Cartan formula and the Adem relations, the Steenrod algebra acts on $H^*(\mathcal{T}^8)$ trivially. Hence $Sq^j(x) \neq 0$ iff $j = 0$. So

$$Sq^4(ux) = u w_4(F) Sq^0(x).$$

But $w_4(F) \equiv c_2(F) \mod 2$ and from the structure of F we have $c_2(F) = \pm r_1[vol_{\mathcal{T}_1^4}] \pm [vol_{\mathcal{T}_2^4}]$. Note that $[vol_{\mathcal{T}_1^4}] \cup [vol_{\mathcal{T}_2^4}] = [vol_{\mathcal{T}^8}]$ and $[vol_{\mathcal{T}_i^4}] \cup [vol_{\mathcal{T}_i^4}] = 0$. Hence if $\{i, j\} = \{1, 2\}$ and r_i is odd,

$$Sq^4(u[vol_{\mathcal{T}_j^4}]) = u[vol_{\mathcal{T}_i^4}] \cup [vol_{\mathcal{T}_j^4}] = u[vol_{\mathcal{T}^8}]$$

is dual to the 12-cell. Since the Hopf element ν is detected by Sq^4, the attaching map from the 12-cell (dual to $u[vol_{\mathfrak{T}^8}]$) to a 8-cell (dual to $u[vol_{\mathfrak{T}^4_j}]$) is ν.

Therefore, in $S^{\mathbb{R}} \wedge TF$, the attaching map from the 13-cell to a 9-cell is ν. Hence the generator of

$$\pi^{10}(\Sigma TF^{(13)}/\Sigma TF^{(12)}) = H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-3}(*))$$

in the E_2 page of the Atiyah-Hirzebruch spectral sequence:

$$\begin{tikzcd}
H^9(\Sigma TF^{(9)}/\Sigma TF^{(8)}; \pi^0(*)) \arrow[dddrr, dashed, bend right, "d_4"'] & \cdots & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^0(*)) \\
 & & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-1}(*)) \\
 & & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-2}(*)) \\
 & & H^{13}(\Sigma TF^{(13)}/\Sigma TF^{(12)}; \pi^{-3}(*))
\end{tikzcd}$$

doesn't survive to the E_∞ page. So the 13-cell can only be mapped to S^{10} trivially.

From the observation of Kronheimer-Mrowka[KM20], the preimage of a genric point under $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau))$ is η^2 in a fiber of TF smash a Lie framed circle in $\mathbb{S}^{\mathbb{R}}$. Hence for $n \neq 13$, any n-cell in $S^{\mathbb{R}} \wedge TF$ is mapped to S^{10} trivially by $BF^{\{e\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau))$. □

4. Equivariant family Bauer-Furuta invariant of the connected sum of two tori

To address the dimension issue in the previous section, we can consider the S^1-equivariant Bauer-Furuta invariant. By the equivariant Hopf theorem, we can convert

$$BF^{\{S^1\}}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1}$$

to a nonequivariant stable mapping class, if $\{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1}$ has no fixed points. However, the base of the bundle F is the Picard torus $\mathfrak{T}^8$ fixed by the S^1-action.

To address this issue we have to use a refinement of the Bauer-Furuta invariant:

Definition 4.1. For a spin manifold X, define its free Bauer-Furuta invariant of the Spin^c-structure $\mathfrak{s}$ to be:

$$\mathrm{BF}^{\mathrm{Pin}(2)}_{free}(X, \mathfrak{s}) \in \{TF_0/Pic^{\mathfrak{s}}(X), S^{n\mathbb{H}+b_2^+(X)\tilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)},$$

where TF_0 is the Thom space of a rank m quarternion bundle over $Pic^{\mathfrak{s}}(X) = T^{b_1(X)}$, such that

$$m - n = \frac{\sigma(X)}{4}.$$

For family invariant we can similarly define an invariant with domain acted freely by a subgroup of Pin(2). For example:

Definition 4.2. Define the free S^1-equivariant Bauer-Furuta invariant Bauer-Furuta invariant of the Dehn twist on a sum of two homology tori to be:

$$BF^{\{S^1\}}_{free}((X_1 \times S^1, \tilde{\mathfrak{s}}_1)\#(X_2 \times S^1, \tilde{\mathfrak{s}}_2^\tau)) \in \{(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathfrak{T}^8), S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1}.$$

These invariants work as well as the ordinary BF invariant, because in the Seiberg-Witten equation, the Picard torus is always mapped to zero (while the kernel of the spinor bundle might be mapped to nonzero self-dual 2-form).

By a cofiber sequence we

$$\{(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8), S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1} \cong \{S^{\mathbb{R}} \wedge TF, S^{2\mathbb{H}+6\mathbb{R}}\}^{S^1}.$$

Since the S^1-action on $(S^{\mathbb{R}} \wedge TF)/(S^{\mathbb{R}} \wedge \mathcal{T}^8)$ is free away from the base point, we can mod out the S^1-action on both side.

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