STAT 668
Homework 3
Assigned: March 11, 2019
Due: April 2, 2019

The α-permanent of an $n \times n$ real-valued matrix $A = (A_{ij})_{1 \le i,j \le n}$ is defined as

$$\mathrm{per}_\alpha(A) = \sum_{\sigma \in \mathcal{S}_n} \alpha^{\#\sigma} \prod_{j=1}^{n} A_{j,\sigma(j)},$$

where $\alpha \in \mathbb{R}$, $\mathcal{S}_n$ is the set of all permutations of $[n] = \{1, \ldots, n\}$, and $\#\sigma$ is the number of cycles of a permutation σ. For example, for $n = 3$, there are exactly 6 permutations of $\{1, 2, 3\}$, expressed in cycle notation as

$$(1)(2)(3), \quad (12)(3), \quad (13)(2), \quad (1)(23), \quad (123), \quad (132).$$

Each expression in parentheses represents a cycle. You can imagine as placing the elements $(i_1 \ldots i_k)$ around the perimeter of a circle in the order they are written. This notation appearing in the expression for σ implies that $\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_k) = i_1$, so that the last element in sequence "cycles back" to the first.

For a 3×3 matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

the α-permanent is computed by

$$\mathrm{per}_\alpha(A) = \underbrace{\alpha^3 A_{11} A_{22} A_{33}}_{\sigma=(1)(2)(3)} + \underbrace{\alpha^2 A_{12} A_{21} A_{33}}_{\sigma=(12)(3)} + \underbrace{\alpha^2 A_{13} A_{31} A_{22}}_{\sigma=(13)(2)} + \underbrace{\alpha^2 A_{11} A_{23} A_{32}}_{\sigma=(1)(23)} + \underbrace{\alpha^1 A_{12} A_{23} A_{31}}_{\sigma=(123)} + \underbrace{\alpha^1 A_{13} A_{21} A_{32}}_{\sigma=(132)}.$$

Permanents are of interest to a number of applied and theoretical researchers. Note that the case $\alpha = -1$ is related to the determinant of a matrix,

$$\det(A) = (-1)^n \mathrm{per}_{-1}(A),$$

so the α-permanent is a natural object of interest to mathematicians. Computer scientists are interested in permanent because of its daunting computational properties: although the determinant of a matrix is easy to compute quickly, the 1-permanent is known to be practically impossible (in the #P-complexity class), and similar difficulty is believed for $\alpha \neq 0, -1$. Statisticians' interest in the α-permanent stems from its appearance in a class of models called permanental point processes. For statistical applications, it is necessary to compute the value of $\mathrm{per}_\alpha(A)$ for different α values of interest, but as mentioned above this calculation cannot be done efficiently for sufficiently large matrices.

The task in this problem is to implement an importance sampling algorithm for approximating the α-permanent.

1. The article

S.C. Kou and P. McCullagh. (2009). Appoximating the α-permanent. *Biometrika*, **96**(2), 635–644.

can be accessed at

http://www.stat.uchicago.edu/ pmcc/pubs/perm.pdf

Read the article for additional context and information on the algorithm to be implemented below.

2. Write code that computes the α-permanent of a matrix exactly. (Note that the cases $\alpha = \pm 1$ can be done more efficiently than general α. For $\alpha = -1$, there is the connection to determinant given above. For $\alpha = 1$, Ryser's method gives the fastest known way to compute. For general α you'll have to compute by brute force using the above definition.)

3. Verify that your algorithm for exact computation returns the correct answer for some small matrices (2×2 and 3×3) for which the value can be computed by hand.

4. For $\alpha = -1, +1, +2$, display in a table the running times of the algorithm for computing the α-permanent of $n \times n$ matrices for $n = 1, 2, \ldots, 10$.

5. Implement the importance sampling algorithm given on p. 4 of the Kou–McCullagh article.

6. Assess the accuracy of the Kou–McCullagh algorithm by obtaining 100 estimates of the α-permanent for $\alpha = 1$ and $\alpha = 2$ for the same matrix A obtained by putting $A_{ij} \sim \text{Uniform}[0, 1]$ i.i.d. for each $1 \le i, j \le 20$. Compute the mean and standard error of these estimates.

7. Discuss any limitations of the proposed algorithm.