Computing Assignment 3

A)

The table represents the values of approximation at each of 10 iterations; Equation C could get to the close enough approximation only after 6 iterations and Equation D is also getting close to the actual value, but it requires 163 iterations to have error within 1e-3. In other words, the convergence of Equation C is faster than the convergence of Equation D. Furthermore, Equation A and B do not converge as you can observe in the table.

Speed of convergence
(C), (D)

n	Equation A	Equation B	Equation C	Equation D
0	1	1	1	1
1	343	7	2.2	1.5
2	-2.25393e25	-335.857	1.819763677	1.450520833
3	-3.38385e253	3.78844e7	1.583474830	1.498749661
4	-Inf	-5.43726e22	1.489460974	1.451903535
5	NaN	1.60746e68	1.476022436	1.497577067
6	NaN	-Inf	1.475773246	1.453192290
7	NaN	NaN	1.475773162	1.496475364
8	NaN	NaN	1.475773162	1.454396119
9	NaN	NaN	1.475773162	1.495438587
10	NaN	NaN	1.475773162	1.455522810

B)

n	Equation A	Equation B	Equation C	Equation D
0	0.475773162	0.475773162	0.475773162	0.475773162
1	341.5242268	5.524226838	0.724226838	0.024226838
2	-2.25393×10^{25}	337.3327732	0.343990515	0.025252329
3	3.38385×10^{253}	37884398.52	0.107701668	0.022976499
4	Inf	5.43726×10^{22}	0.013687812	0.023869627
5	NaN	1.60746×10^{68}	2.4927×10^{-4}	0.021803905
6	NaN	Inf	0	0.022580872
7	NaN	NaN	0	0.020702202
8	NaN	NaN	0	0.021377043
9	NaN	NaN	0	0.019665425
10	NaN	NaN	0	0.020250352

$p = 7^{(1/5)} = 1.475773162$
This table shows the absolute error $|p_n - p|$ against n.

The convergence can be determined for the 4 different equations by looking at the table.

Equation A; **diverges;** the abs.error gets larger as n increases.
Equation B; **diverges**; the abs.error gets larger as n increases.
Equation C; **converges**; the abs.error gets smaller towards 0 as n increases.
Equation D; **converges**; the abs.error gets smaller towards 0 as n increases.

C) As Equation_A and Equation_B diverge, it is not possible for them to estimate the order of convergence α and the asymptotic error constant λ because the definition 2.7 has an assumption saying that if wanting to have positive constants λ and α exist, then the sequence must converge to p. Therefore, α and λ are going to be estimated only for Equation_C and Equation_D by the definition 2.7 in the textbook, $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$. This definition can be manipulated to solve for α and λ. →

$e_{n+1} = \lambda |e_n|^\alpha$ where $|e_n| = |p_n - p|$
$e_n = \lambda |e_{n-1}|^\alpha$

therefore, $\alpha = \dfrac{\log\left(\frac{e_{n+1}}{e_n}\right)}{\log\left(\frac{e_n}{e_{n-1}}\right)}$

and $\lambda = \dfrac{|e_{n+1}|}{|e_n|^\alpha}$

For equation C,

$$p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{5p_{n-1}^4}$$

$g(p) = p - \dfrac{p^5 - 7}{5p^4}$
$g'(p) = \dfrac{4(x^5-7)}{5p^5}$; $g'(7^{1/5}) = 0$
$g''(p) = \dfrac{28}{p^6}$; $g''(7^{1/5}) \approx 2.71044$

Theorem 2.9 says that if g'(p) =0 and g''(p) ≠ 0 is then the sequence is quadratically convergent with asymptotic error constant of $\frac{|g''(p)|}{2}$. It is shown that first derivative of Equation C is 0 at p and the second derivative is continuous having non-zero at p; therefore, the sequence is quadratically convergent having α =2 and error constant of 1.355. This have been proven by using the manipulated formula to compute α and λ at each iteration. (I omit some n)

n	P_n	α	λ
1	2.2	-1.7719	0.1942
2	1.819763677	1.5598	0.5690
4	1.489460974	1.9418	1.0365
5	1.476022436	1.9955	1.3050

For equation d

$$\textbf{d.}\quad p_n = p_{n-1} - \frac{p_{n-1}^5 - 7}{12}$$

$g(p) = p - \dfrac{p^5 - 7}{12}$
$g'(p) = 1 - \dfrac{5p^4}{12}$; $|g'(7^{1/5})| = 0.9764$

Theorem 2.8 says that if g'(p)≠0, the fixed-point iteration exhibits linear convergence with asymptotic error constant |g'(p)|. It is shown that the first derivative of Equation D is continuous having non-zero at p; therefore, the sequence is linearly convergent having α =1 and error constant of 0.9764. This also have been proven by using the manipulated formula to compute α and λ at each iteration. (I omit some n)

n	P_n	α	λ
1	1.5	-0.0139	0.0240
163	1.4752763892	1.00000	0.9764

	α	λ
Equation A	N/A	N/A
Equation B	N/A	N/A
Equation C	2	1.355
Equation D	1	0.9764