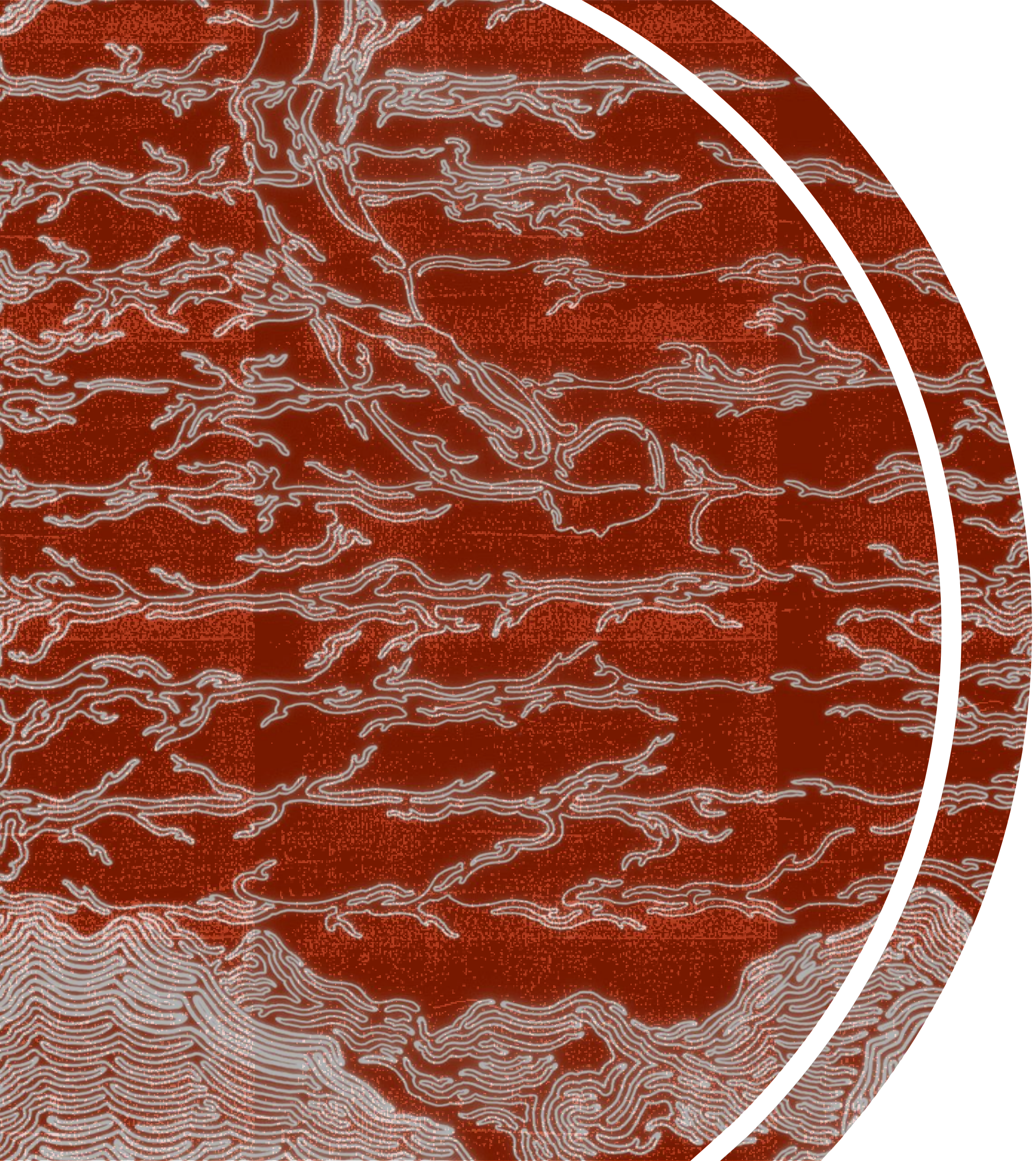


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 8, October 7, 2021

Administrivia

- Homework a is out, due 11/15 (end of term)

Homotopy Equivalence and Induced Homomorphism

Last Time on Algebraic Topology

- $[\gamma]$ is the class of closed paths homotopic to γ in space X

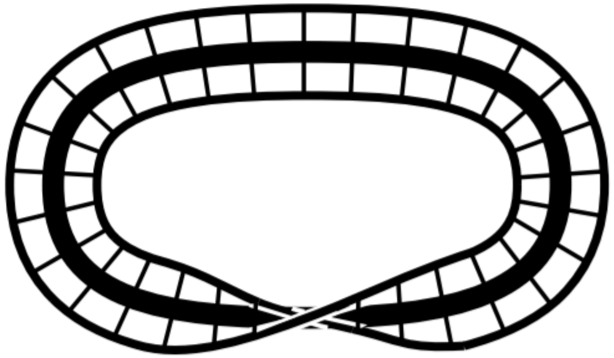
$$\pi_1(X, x_0) = \{[\gamma] : \text{closed path } \gamma \text{ in } X \text{ starting and ending at } x_0\}$$

Does $\pi_1(X)$ Classify Spaces?

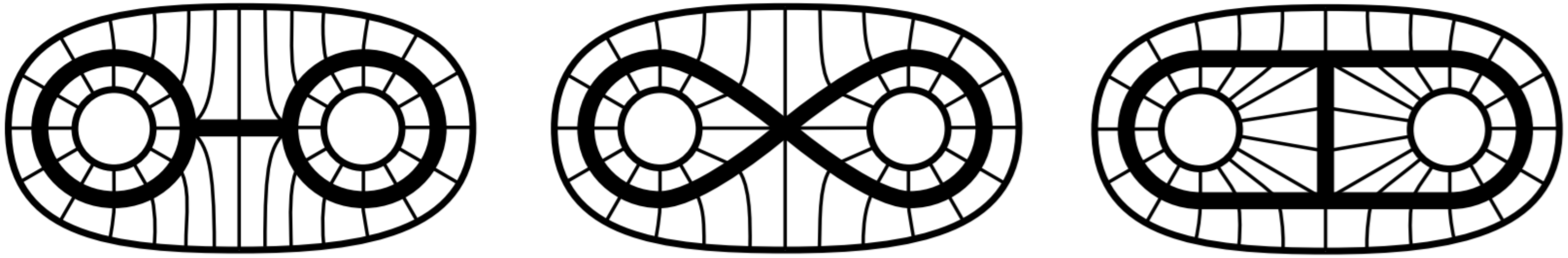
Equivalence

- Homeomorphism
  - f: $X \longrightarrow Y$ continuous bijection
  - g: $Y \longrightarrow X$ continuous bijection
  - $f \bullet g = id_X$
  - $g \bullet f = id_Y$
- Homotopy equivalence
  - f: $X \longrightarrow Y$ continuous bijection
  - g: $Y \longrightarrow X$ continuous bijection
  - $f \bullet g$ homotopic to id_X
  - $g \bullet f$ homotopic to id_Y

Homotopy Equivalence



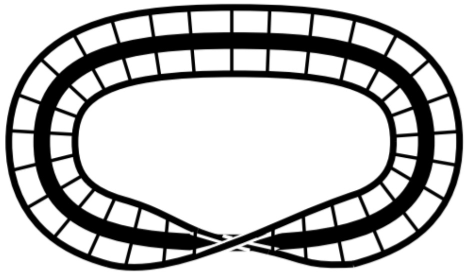
Homotopy Equivalence



Homotopy ≠ Homotopy Equivalence

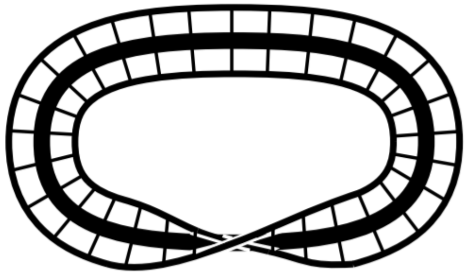
- Homotopy:
  Morph within the same space

- Homotopy Equivalence:
  Morph between identity and maps between two spaces



Where is the Homotopy?

- Retraction
  - $r: X \longrightarrow A$
  - $r|_A = id_A$
- Inclusion
  - $i: A \longrightarrow X$
  - $i|_A = id_A$

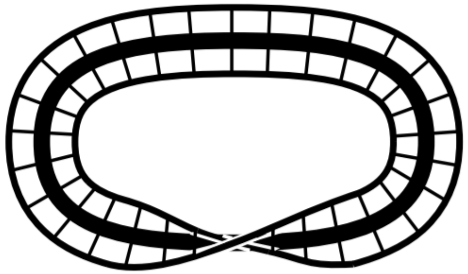


- Deformation retract
  - $f_t: X \longrightarrow X$
  - $f_1(X) = A$
  - $f_t|_A = id_A$
  - $f_0 = id_X$

$= $ Homotopy from id_X to $r \cdot i$

Where is the Homotopy Equivalence?

- Retraction
  - $r: X \to A$
  - $r|_A = id_A$
- Inclusion
  - $i: A \to X$
  - $i|_A = id_A$

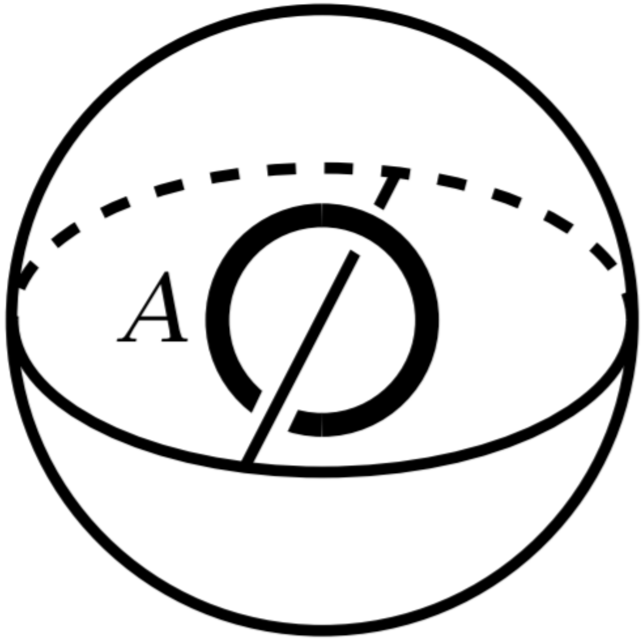


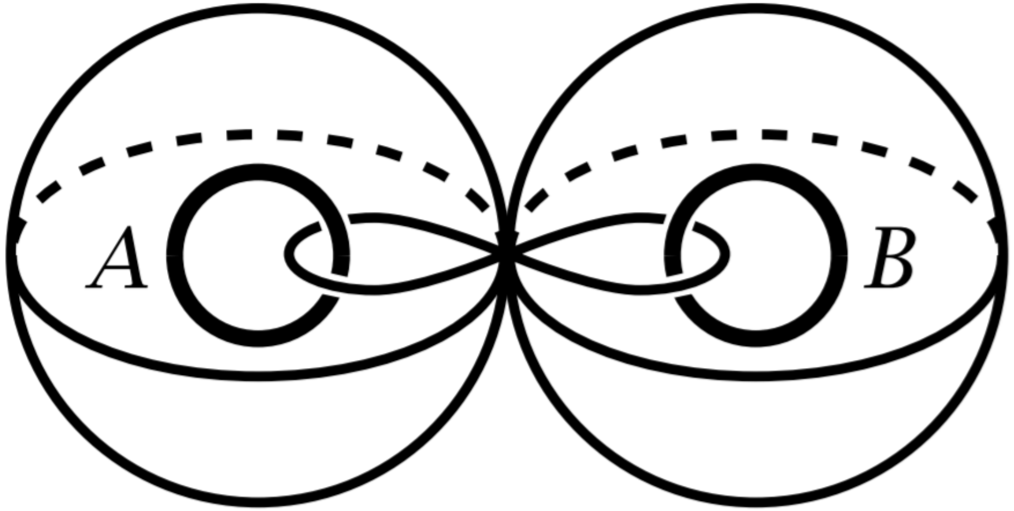
- $i \circ r = id_A$
- $r \circ i$ homotopic to id_X
  - Through deformation retract from X to A
    = homotopy from id_X to $r \circ i$

PROPOSITION. Deformation retract provides homotopy equivalence between space X and subspace A.

Visualization Exercise

- Complement of circles



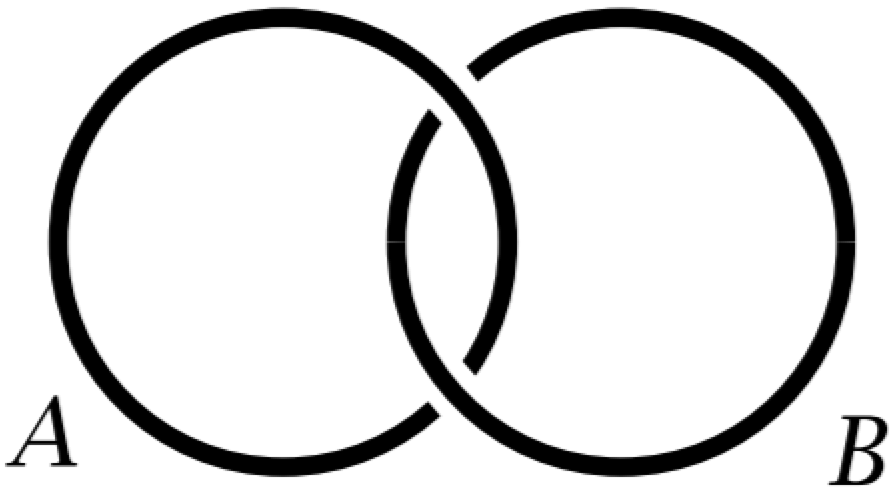


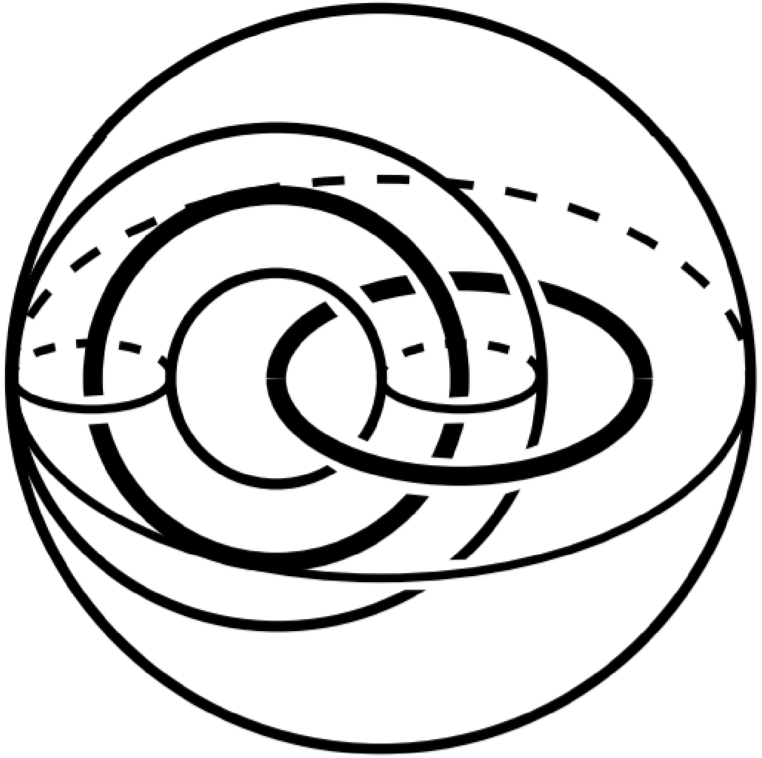
Visualization Exercise

- Complement of circles

Visualization Exercise

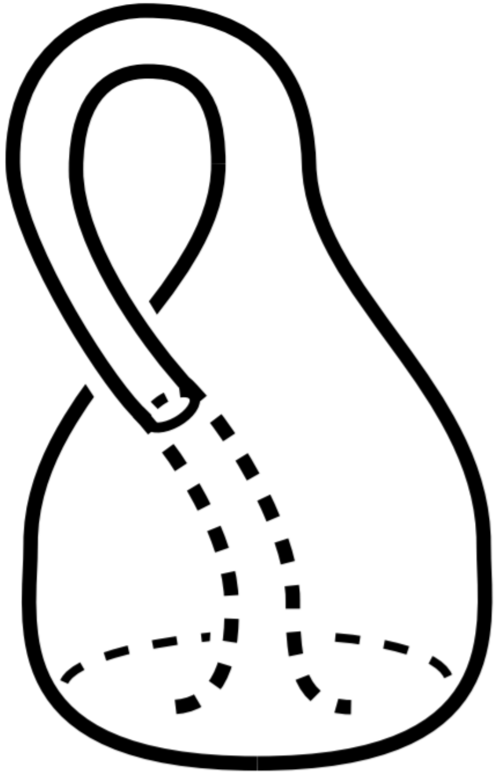
- Complement of linked circles





Visualization Exercise

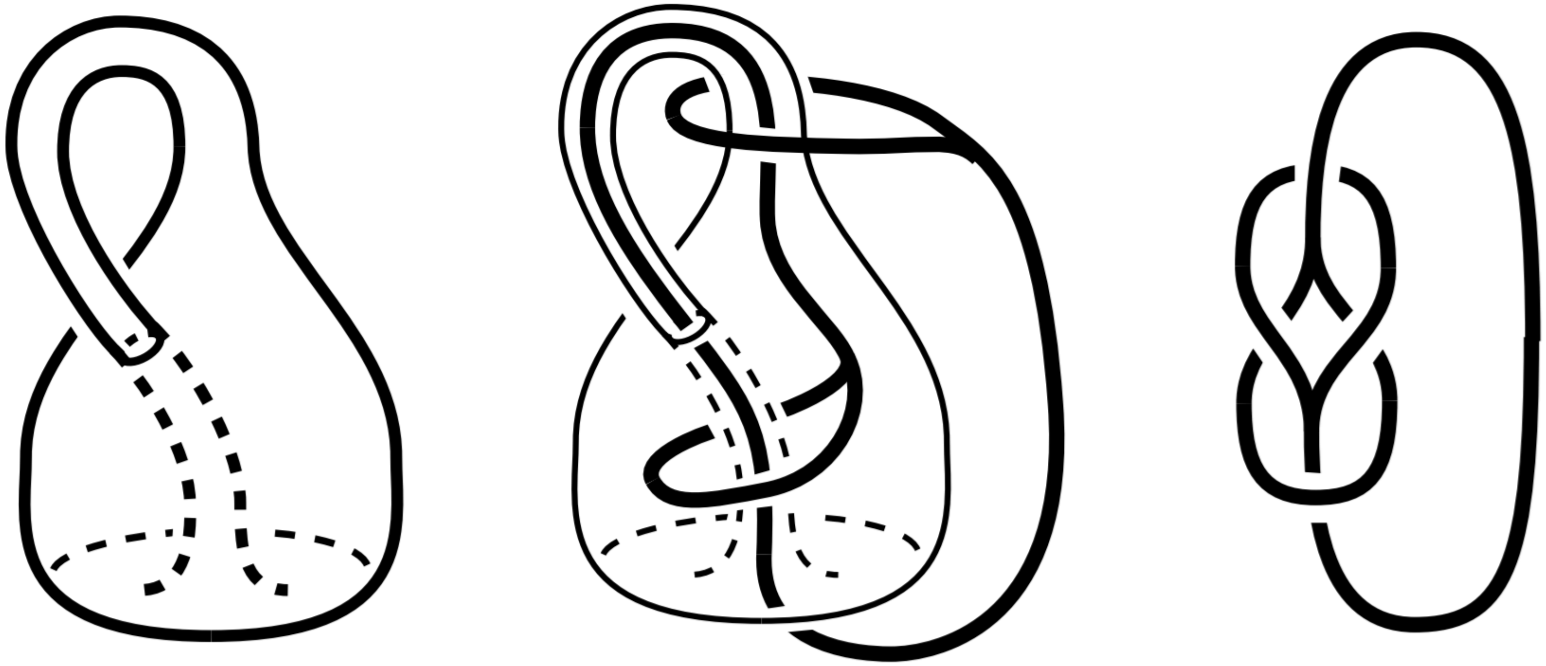
- Complement of linked circles



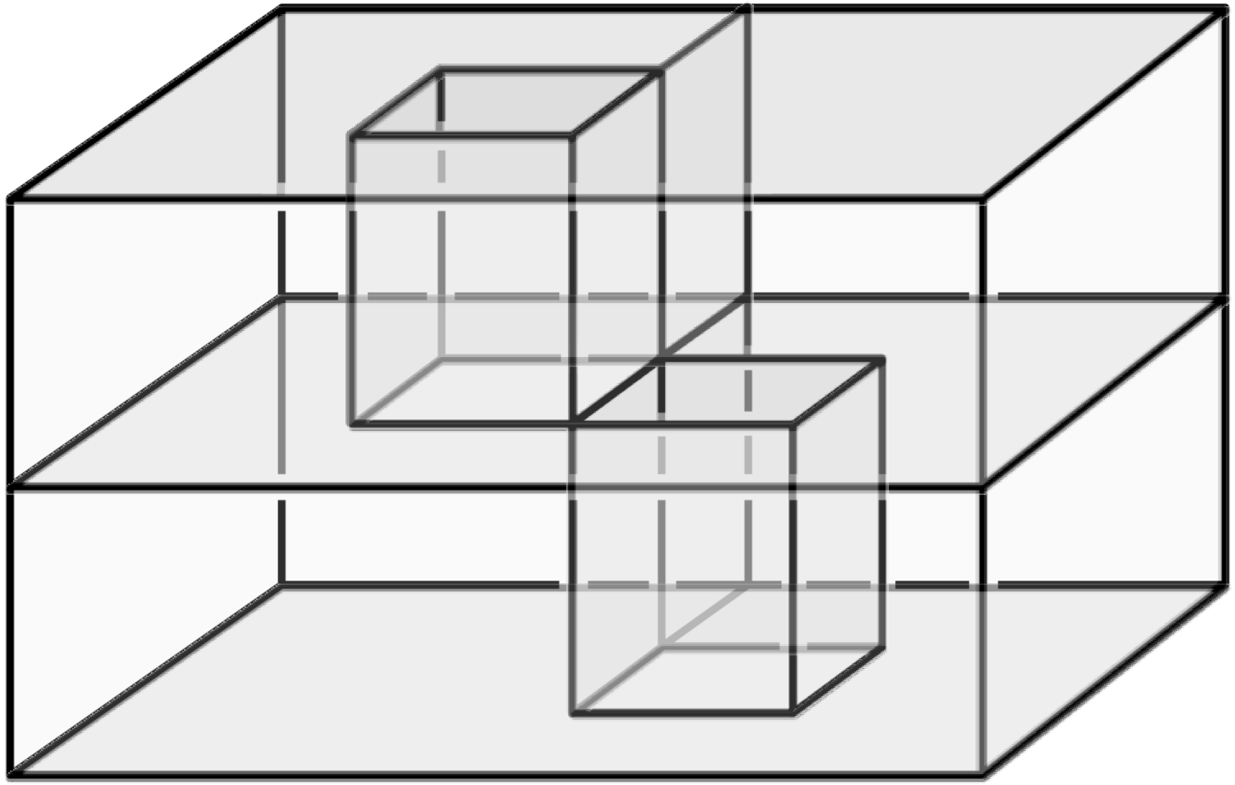
Visualization Exercise

- Complement of Klein bottle

Visualization Exercise



- Complement of Klein bottle



Visualization Exercise

- House with two rooms

Induced Homomorphism

- $\phi: X \longrightarrow Y$ induces $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$

PROPOSITION. ϕ_* is a group homomorphism.

Lemma. Retraction from X to A induces an injective inclusion map $i_*: \pi_1(A) \longrightarrow \pi_1(X)$.

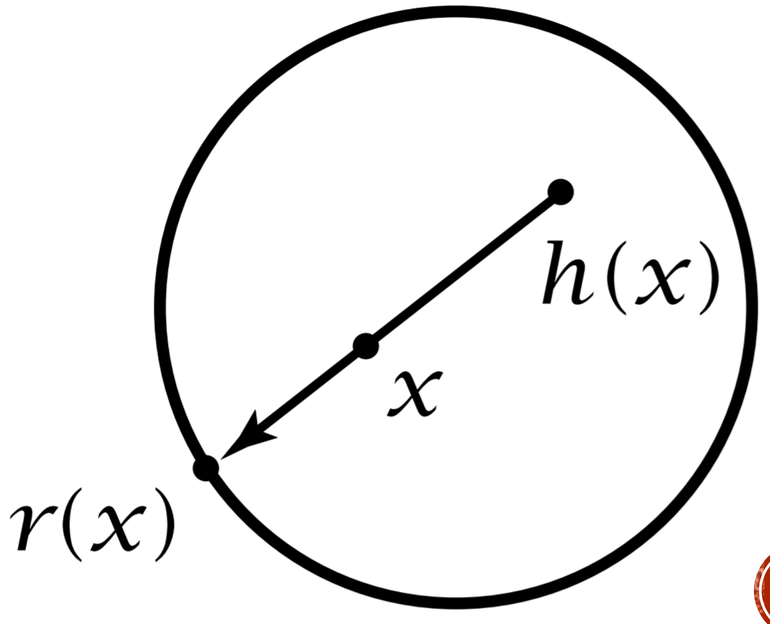
Lemma. Deformation retract from X to A induces an isomorphism $i_*: \pi_1(A) \longrightarrow \pi_1(X)$.

Theorem. Homotopy equivalence induces group isomorphism on π_1.

2D Brouwer Fixed-point Theorem [Bohl 1904] [Brouwer 1909]

Every map from a disk to itself has a fixed point

Proof of 2d Brouwer Fixed-point Theorem.



Intermission

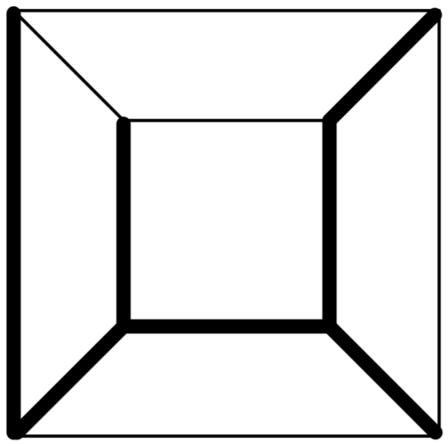
Food for Thought.

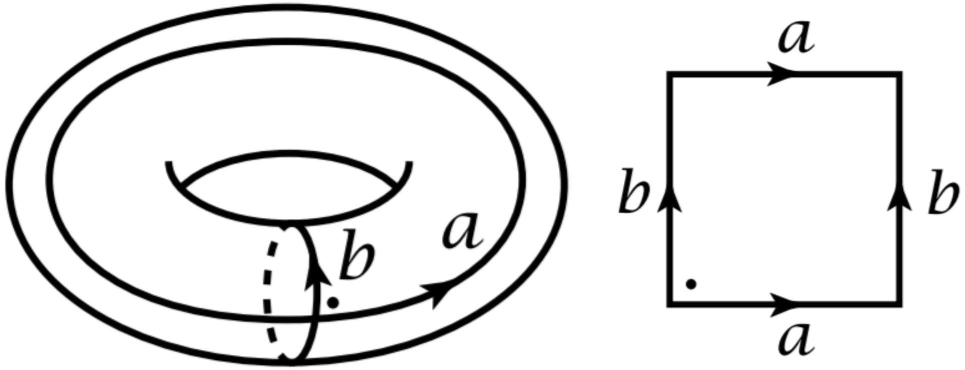
Does trivial π_1 imply contractibility?

Computing Fundamental Groups

Can we compute $\pi_1(\Sigma(g,r))$?

π_1(Graph)

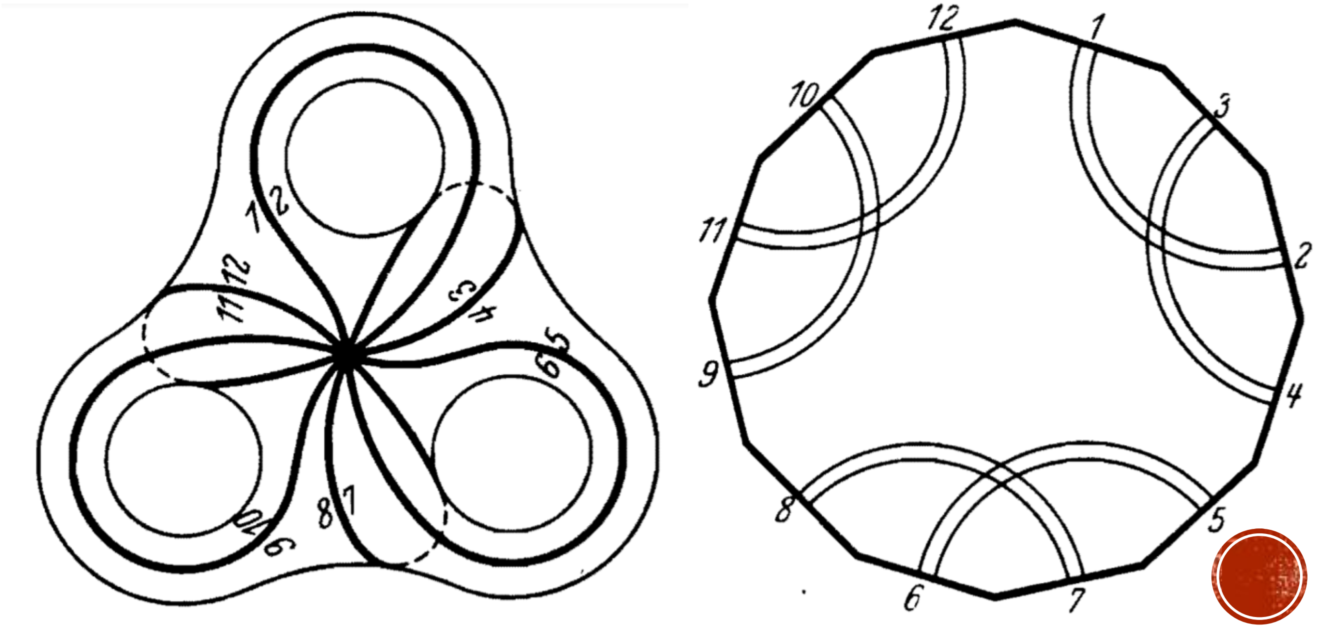




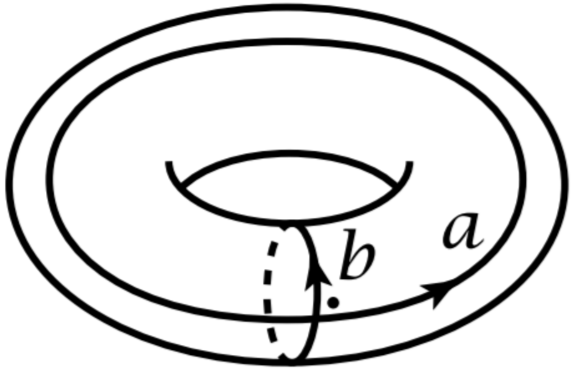
π_1(Torus)

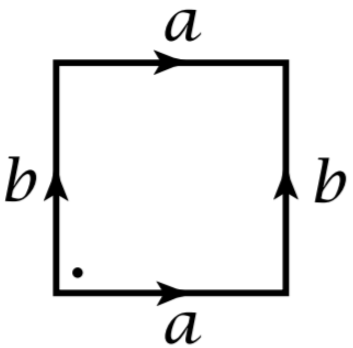
Fundamental Groups of Surfaces

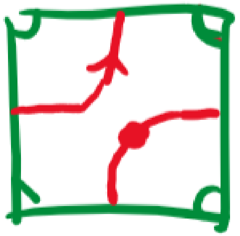
- $\pi_1(\Sigma(g,0)) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \overline{a_1 b_1} \ldots a_g b_g \overline{a_g b_g} \rangle$
- $\pi_1(\Sigma(0,r)) = \langle a_1, \ldots, a_r \mid a_1 a_1 \ldots a_r a_r \rangle$



What about Punctures?







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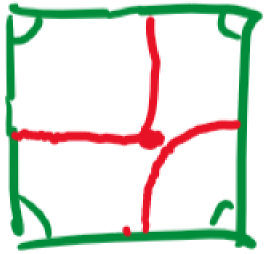

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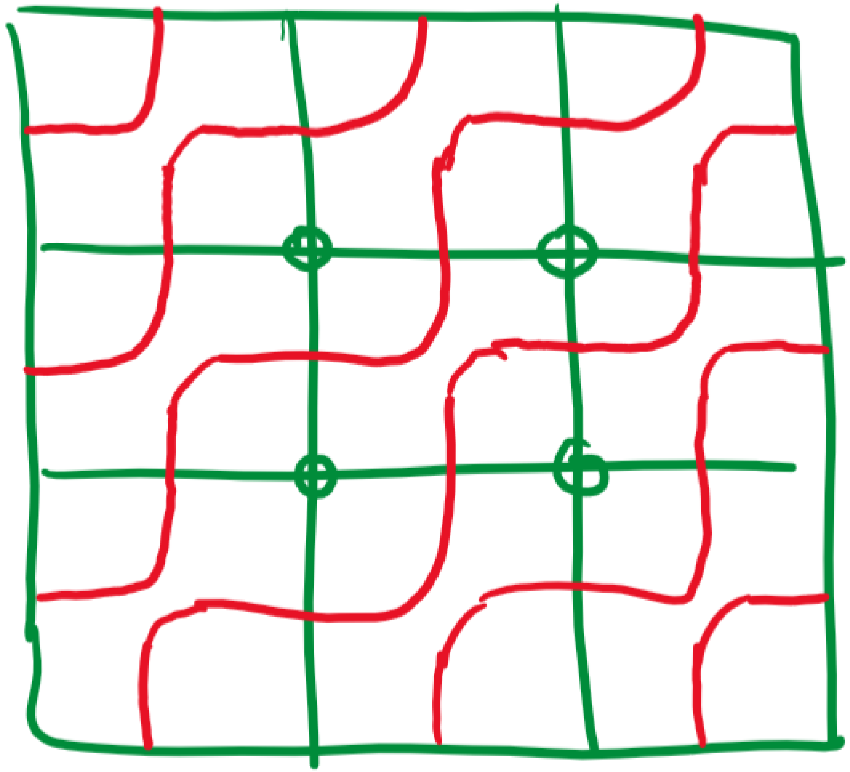
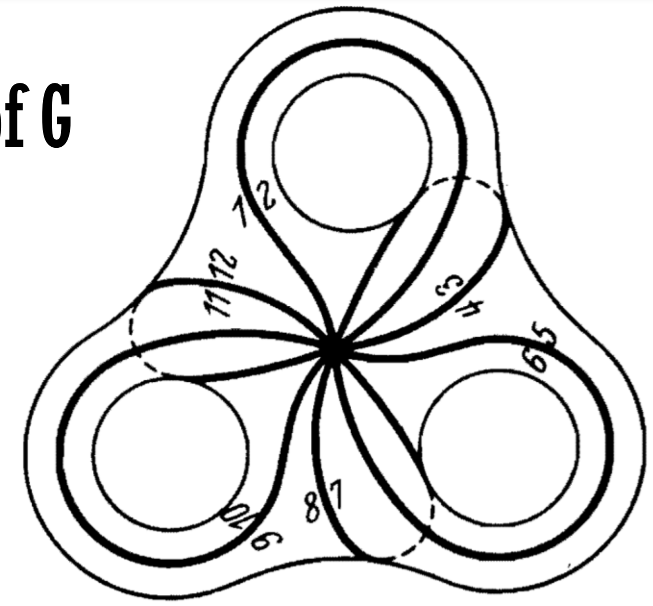
$\Sigma(1,0,1)$

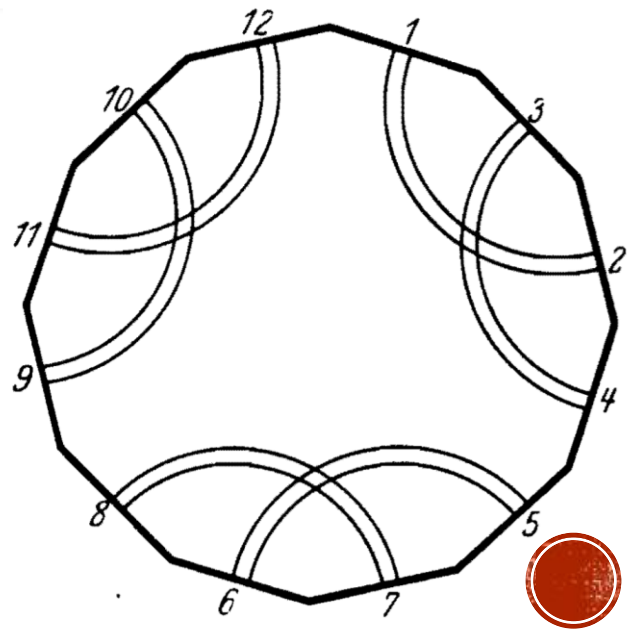


Fundamental Groups of 2-Complex

- $\pi_1(\Sigma) = \langle C \mid F \rangle$
  - C: cotree edges
  - F: faces

- $\pi_1(\Sigma)$ is independent to the choice of G





$$
\left\langle \; a,b,c,d,e,p,q,r,t,k \;\middle|\;
\begin{array}{lll}
p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\
p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\
p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\
p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\
p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\
aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\
bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\
cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\
dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\
eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 &
\end{array}
\right\rangle
$$

[Collins 1986]

Undecidability of π_1 [Novikov 1955] [Boone 1958]

Checking if a 2-complex has trivial π_1 is undecidable

$\pi_1(X)$ is Homotopic Invariant but Useless for Computation

Choose your own adventure:

more (A)lgorithms on curve homotopy, or
something (B)etter than fundamental groups