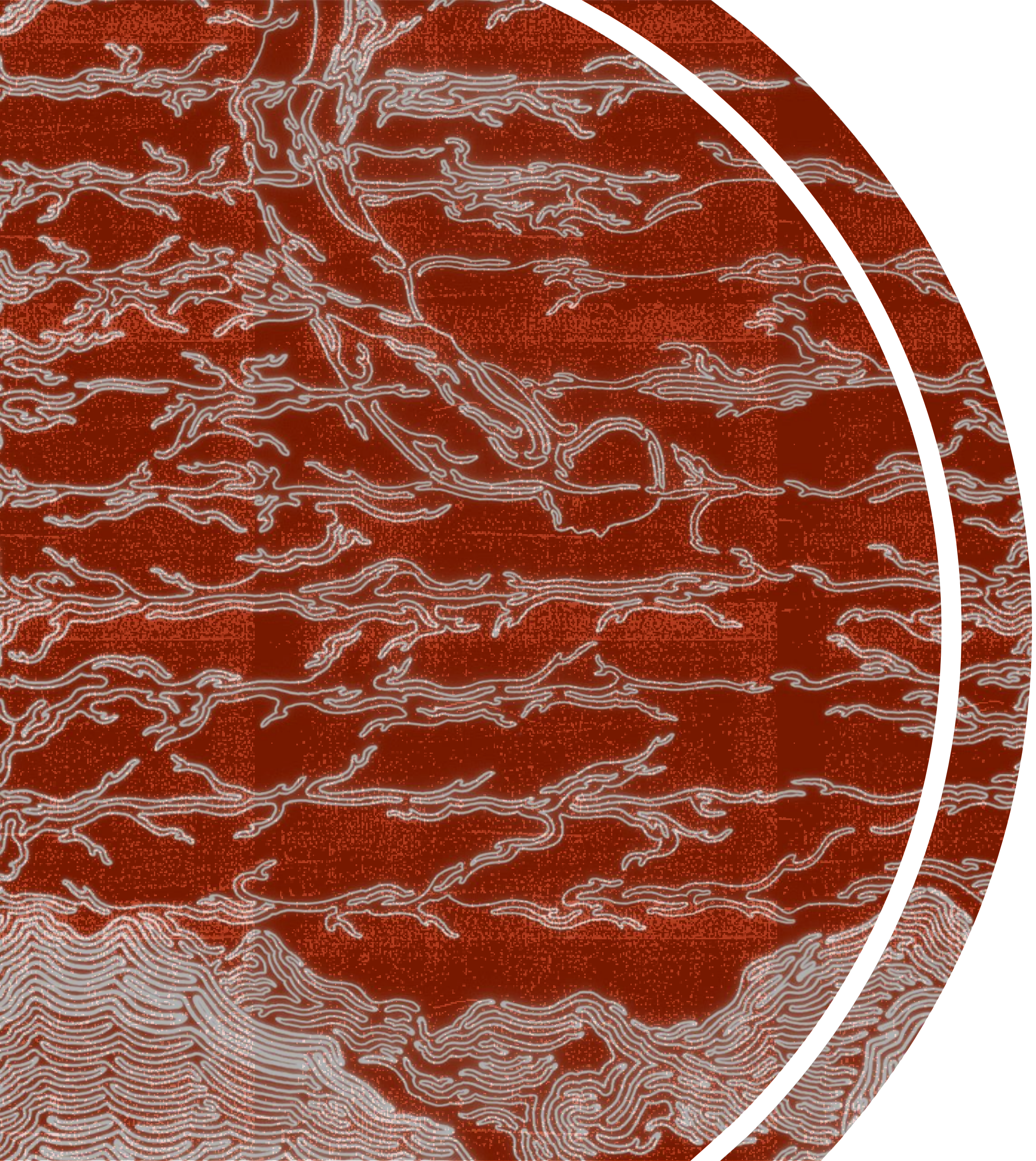


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 7, October 5, 2021

Administrivia

- Homework a will be out later today.

Homotopy

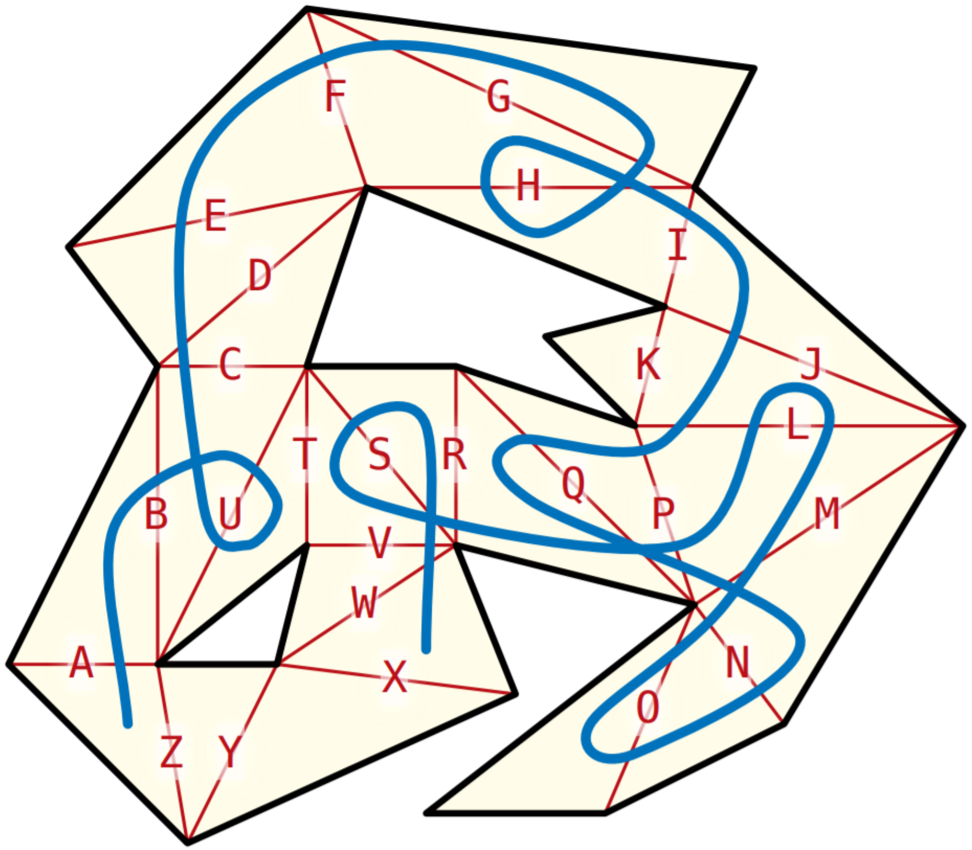
- Homotopy of curves
  - $H: S^1 \times [0,1] \rightarrow R^2$

- Homotopy of two functions f and g from X to Y
  - $H: X \times [0,1] \rightarrow Y$

Are Two Curves Homotopic?

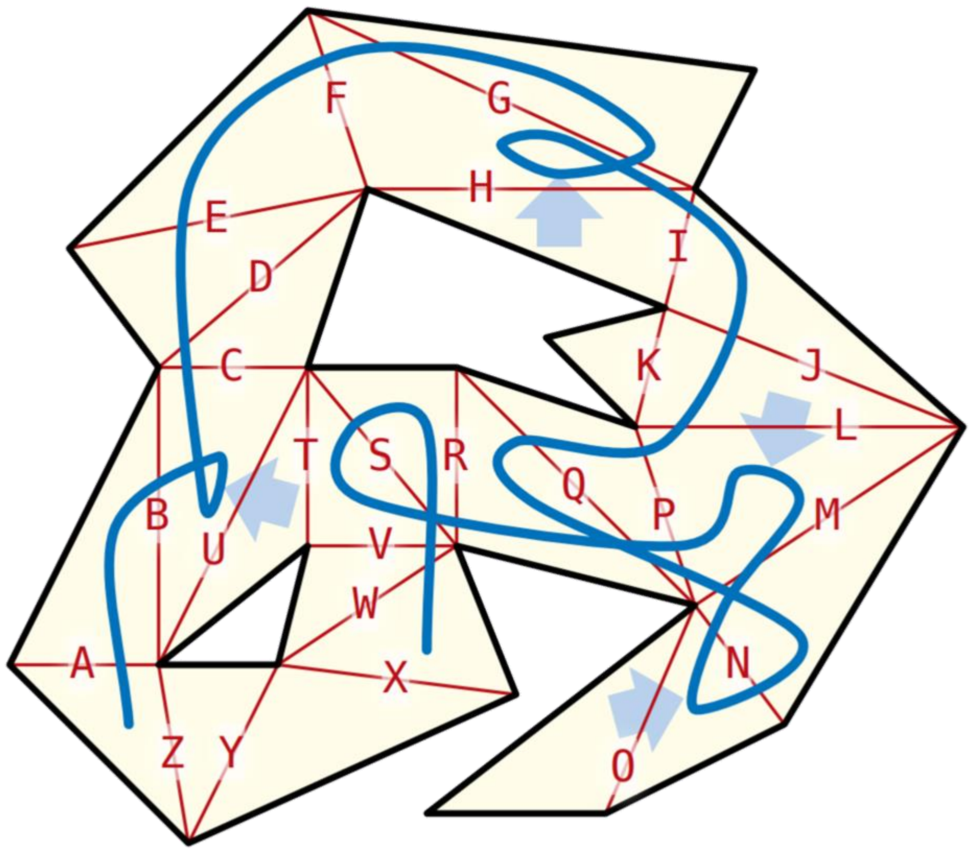
Homotopy Testing

- Cut surface into polygonal schema
- Keep track of how the curve crosses the cuts

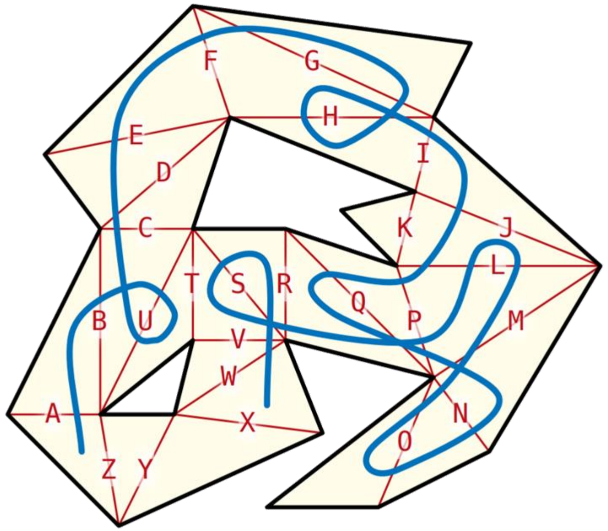


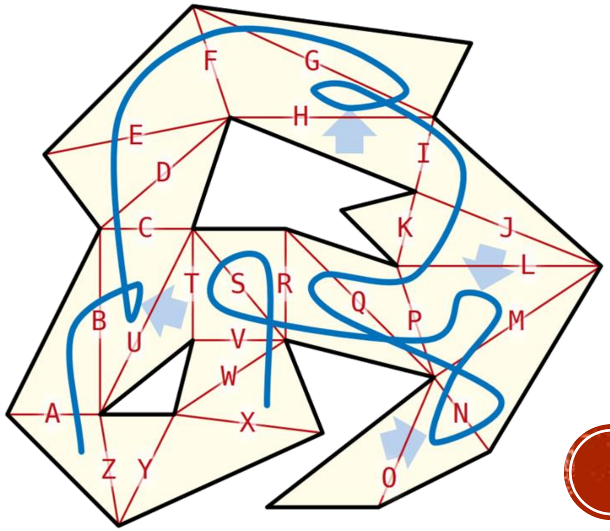
Homotopy Testing

- Cut surface into polygonal schema
- Keep track of how the curve crosses the cuts
- Reduce the crossing sequence

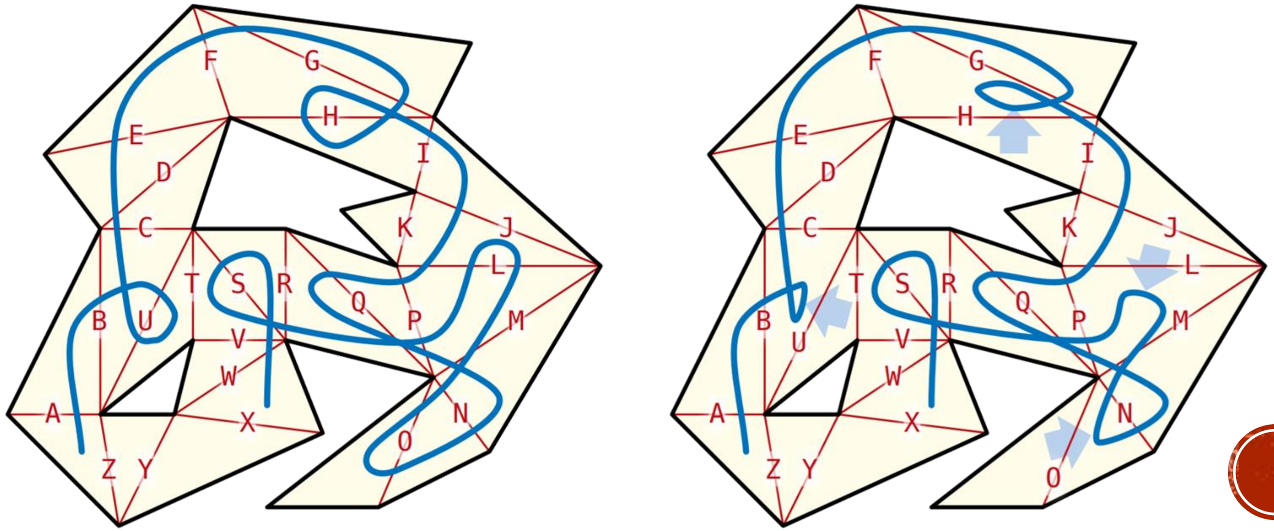


LEMMA. Every crossing sequence reduces uniquely.

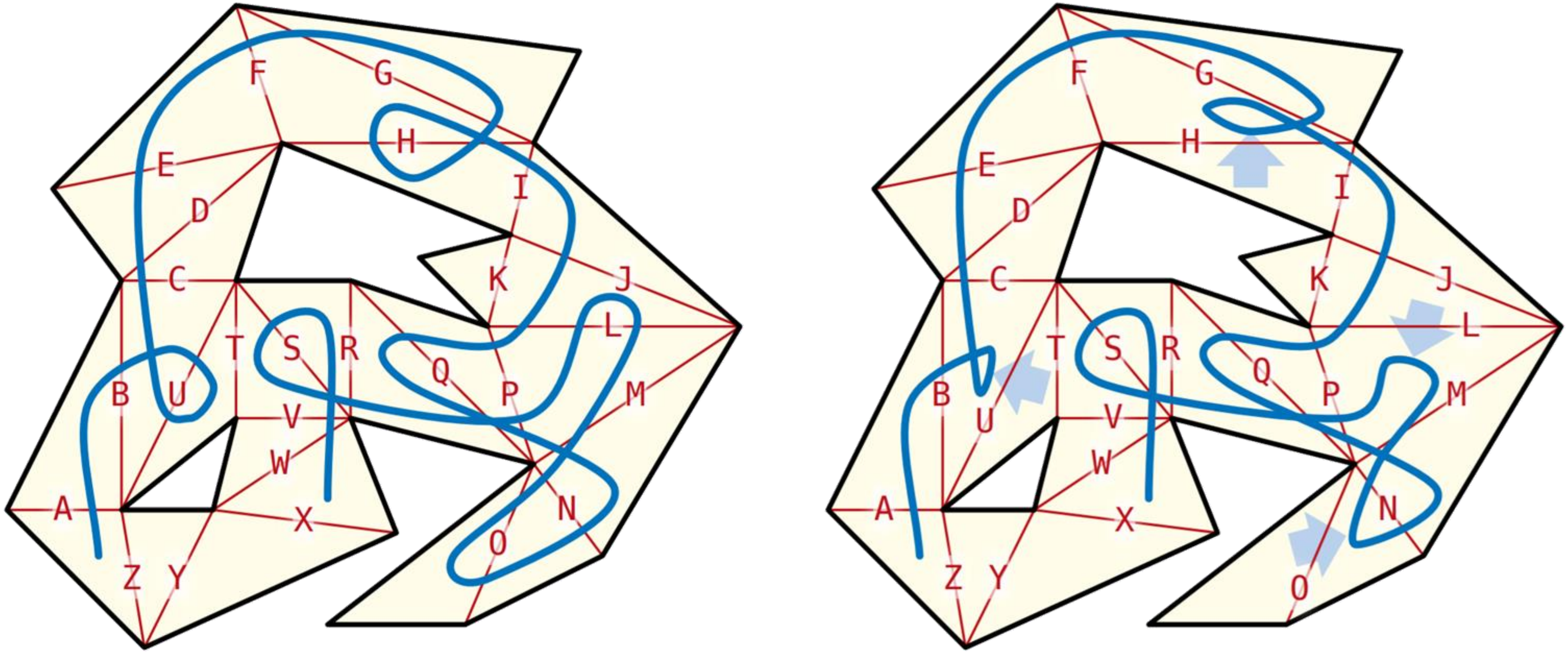




Proposition. Two curves are homotopic if and only if they share the same reduced crossing sequence.

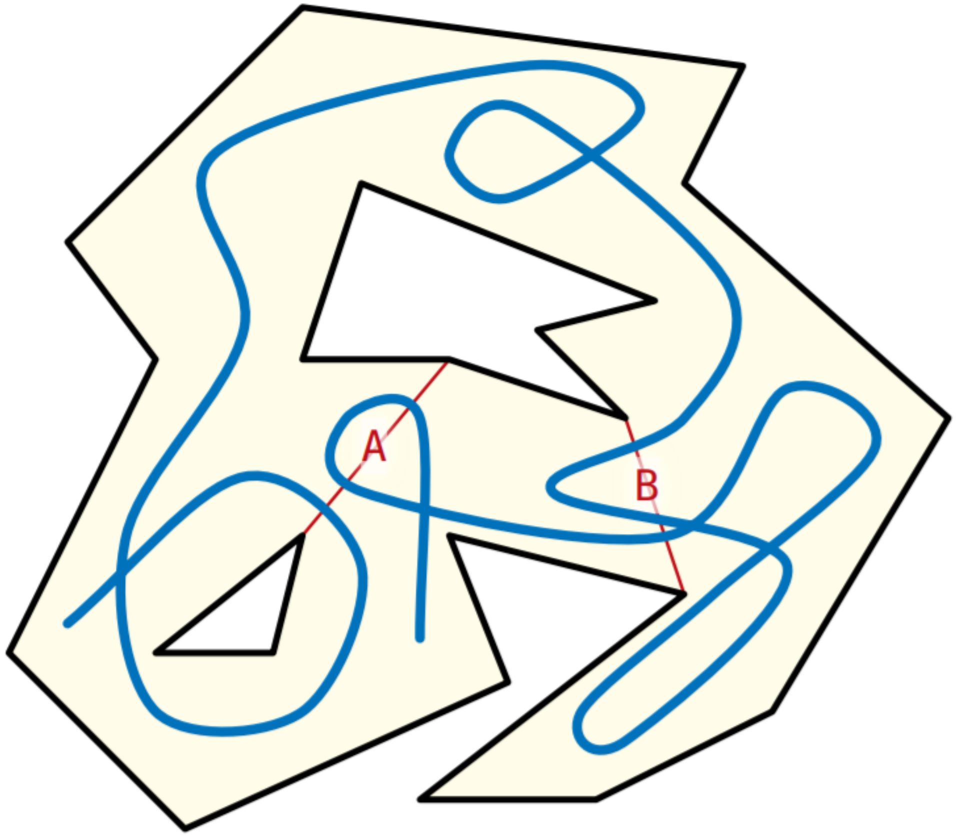


Theorem. Homotopy testing between two k-edge planar polygonal curves takes $O(n \log n + nk)$ time.



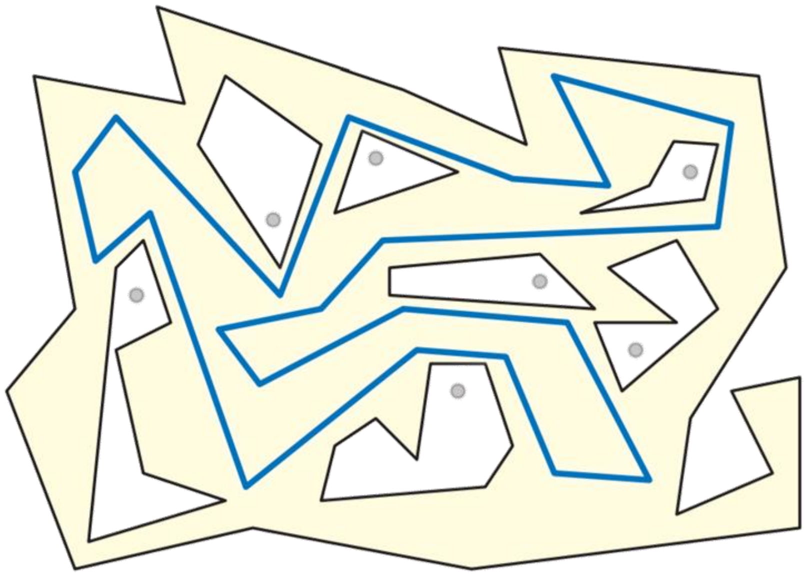
Observations

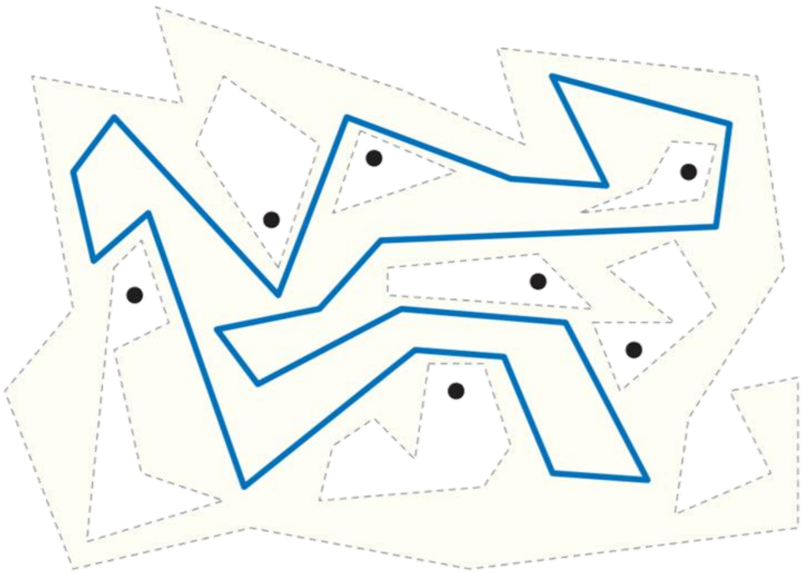
- System of loops are enough

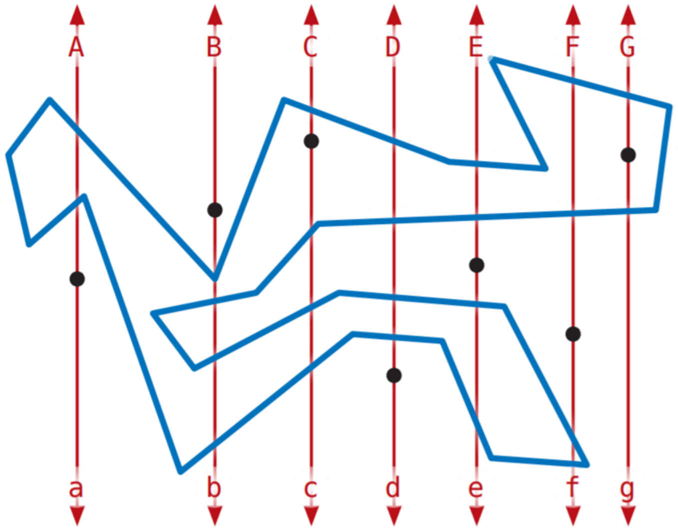


Observations

- Triangulation doesn’t matter; replace it with punctures

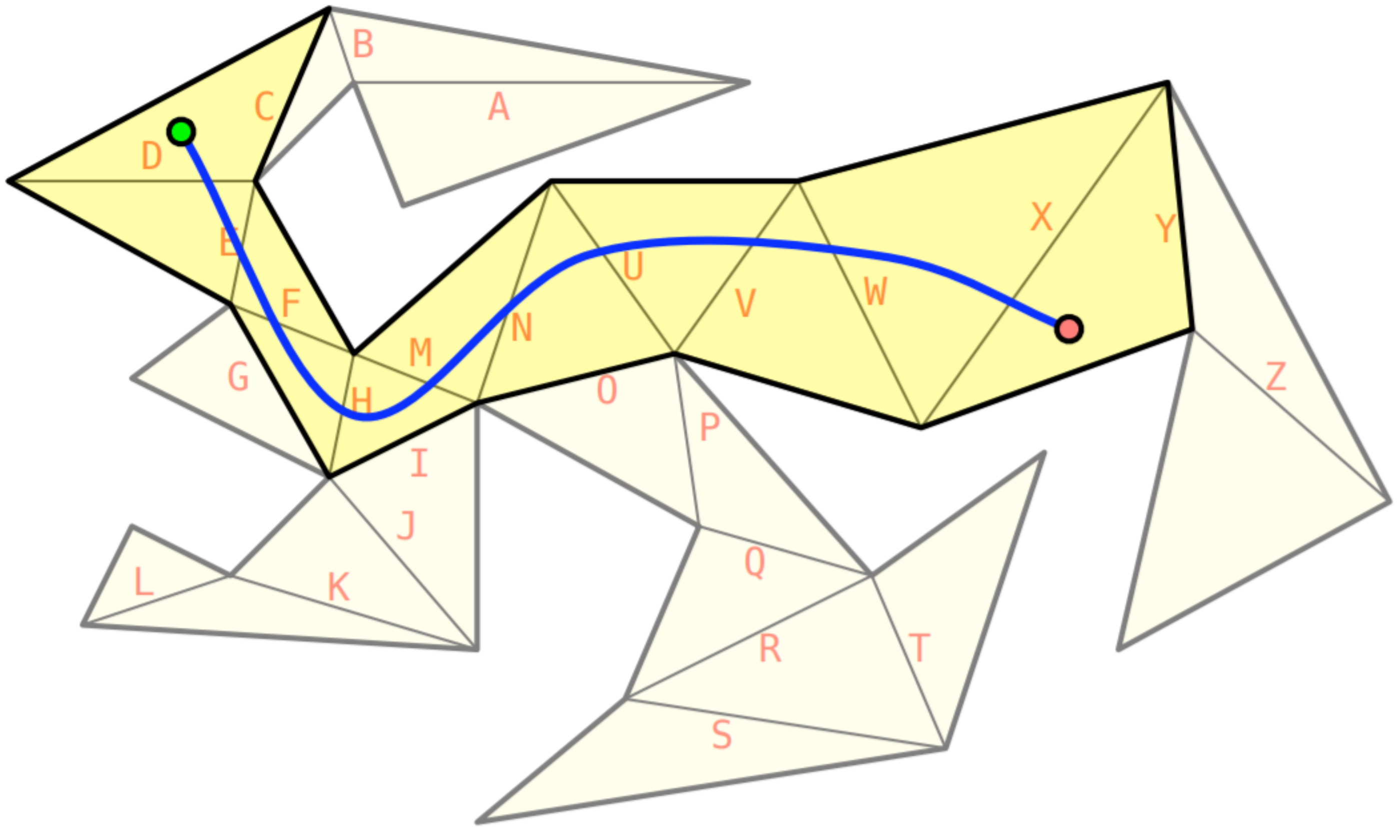






A partition of $\mathbb{R}^2 \setminus S$ into vertical slabs and a loop with crossing sequence AbcDeffeDcbbcDEFgGFEDCbA.

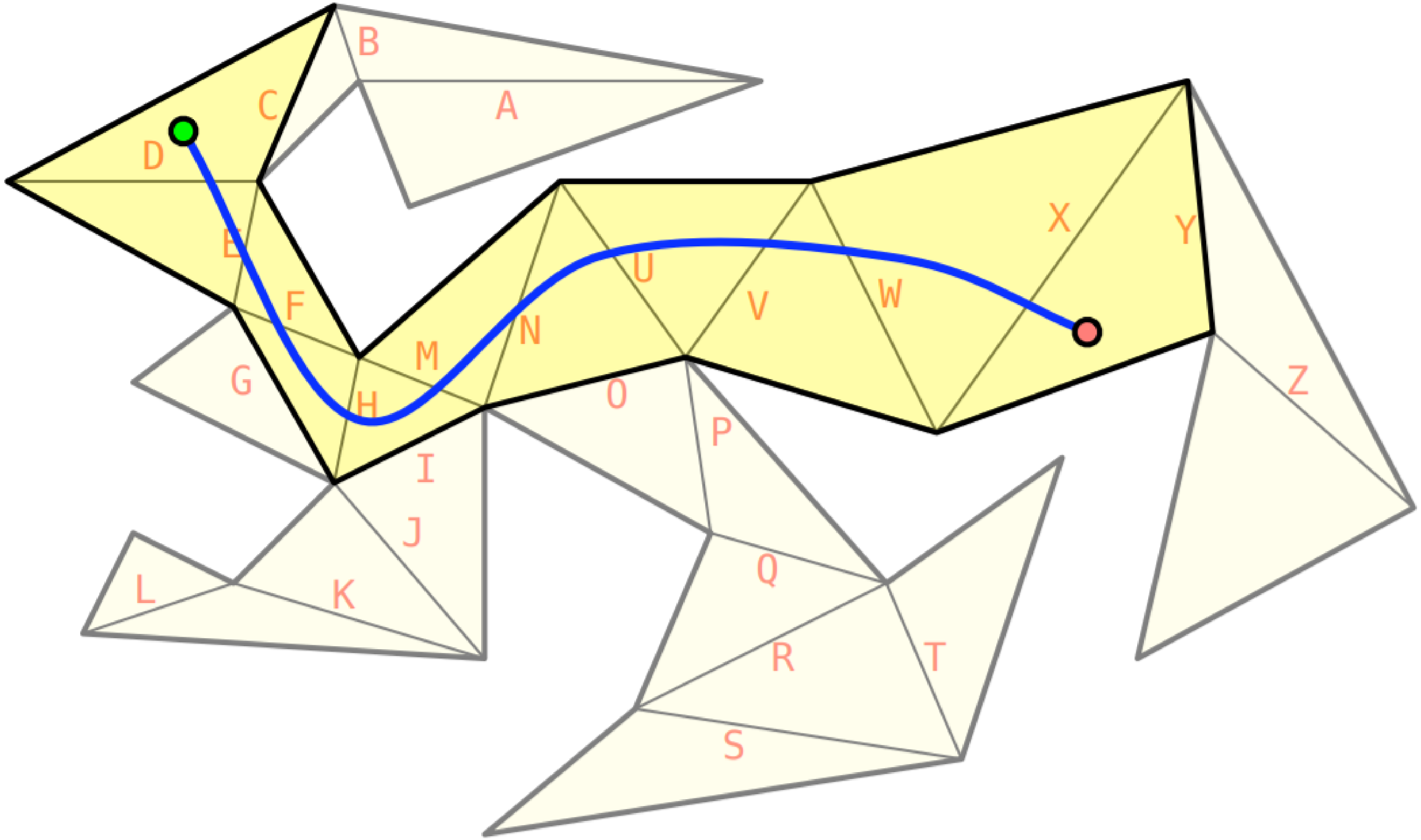
Shortest Homotopic Path?

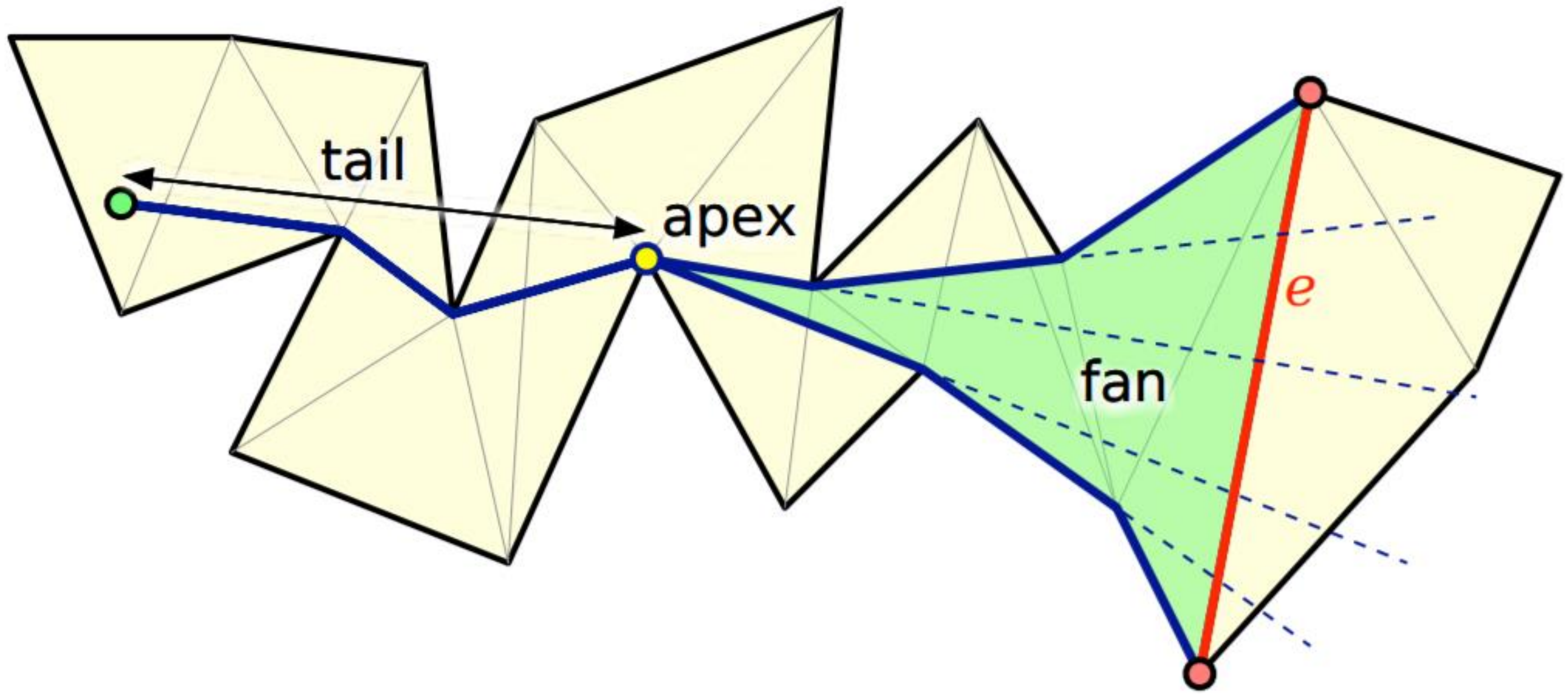


FUNNEL ALGORITHM [Tompa 1981] [Chazelle 1982] [Lee-Preparata 1984]

Given a k-edge path π in a simple polygon,
find the shortest path homotopic to π takes O(nk) time

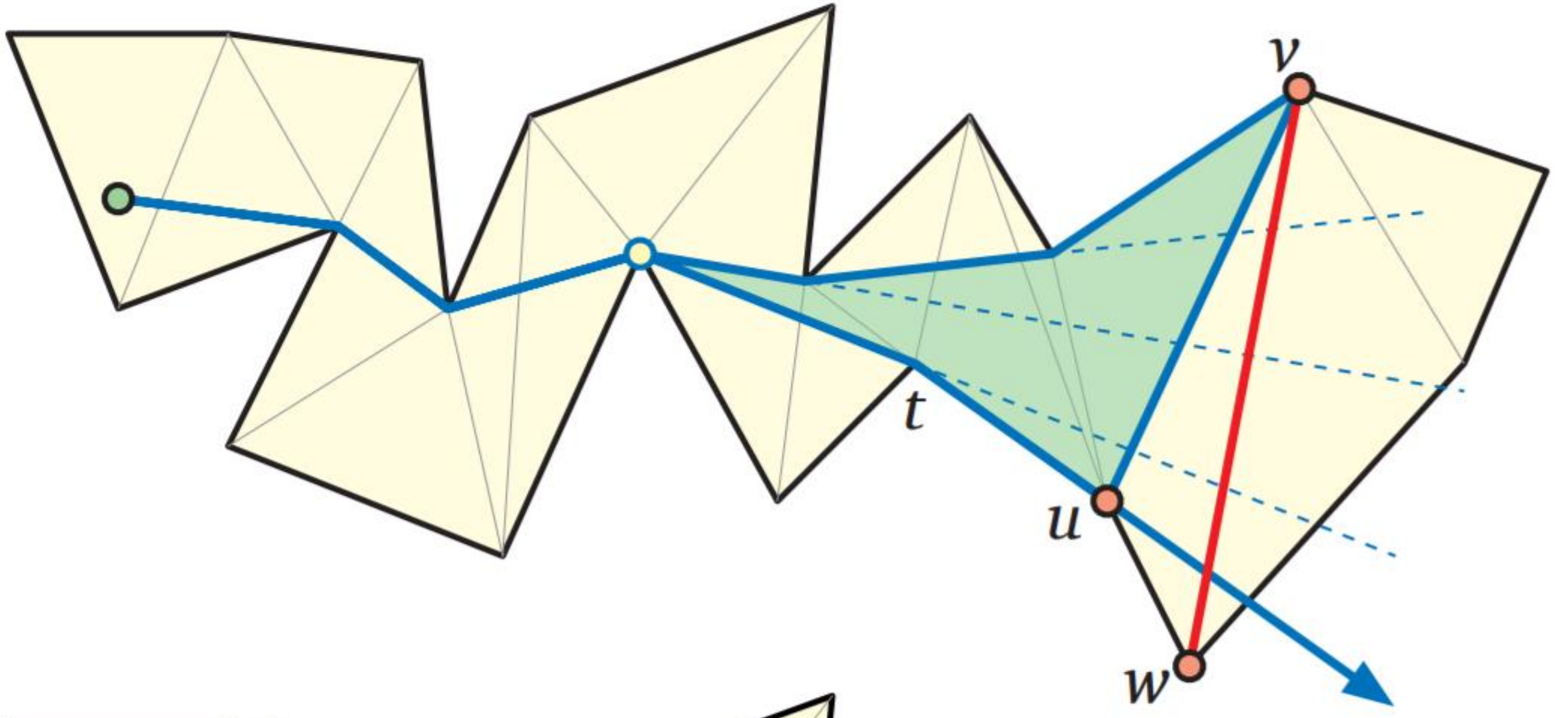
Sleeve

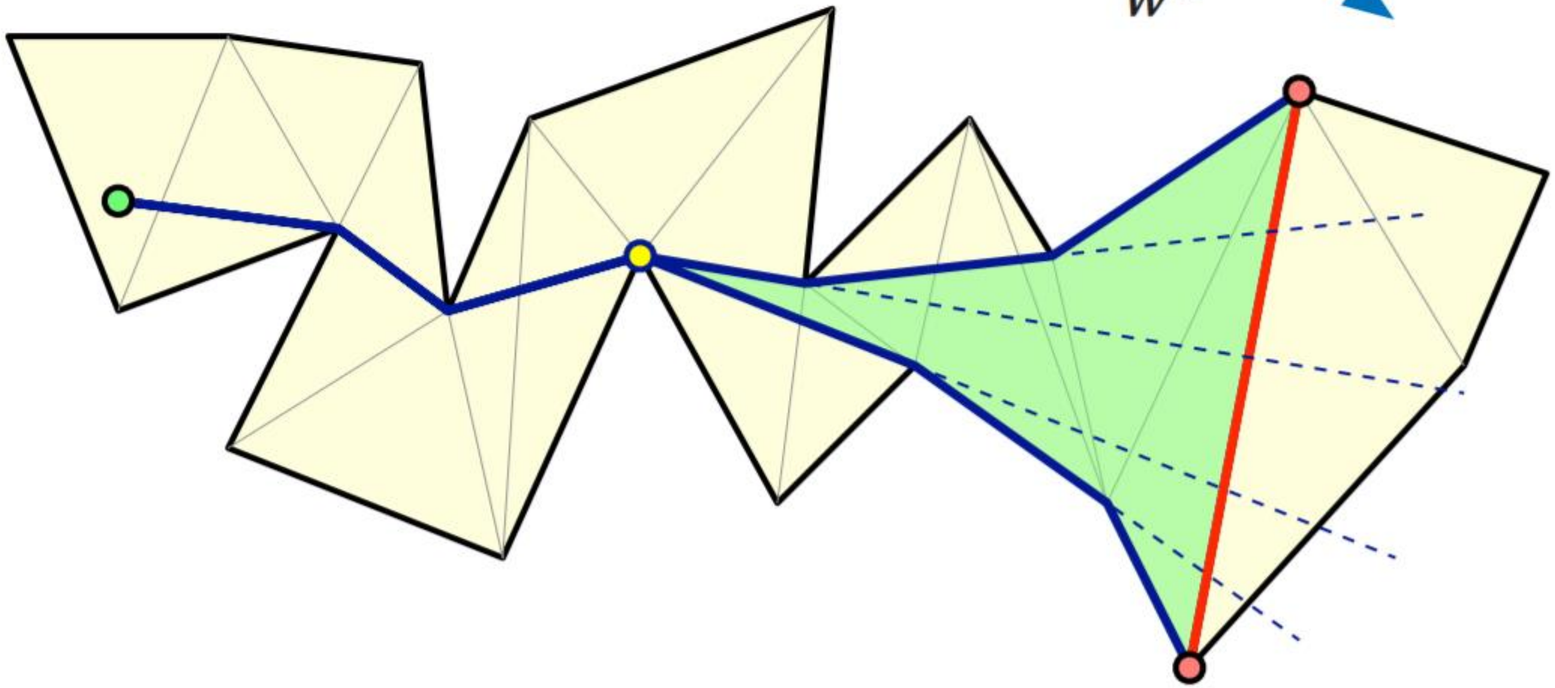


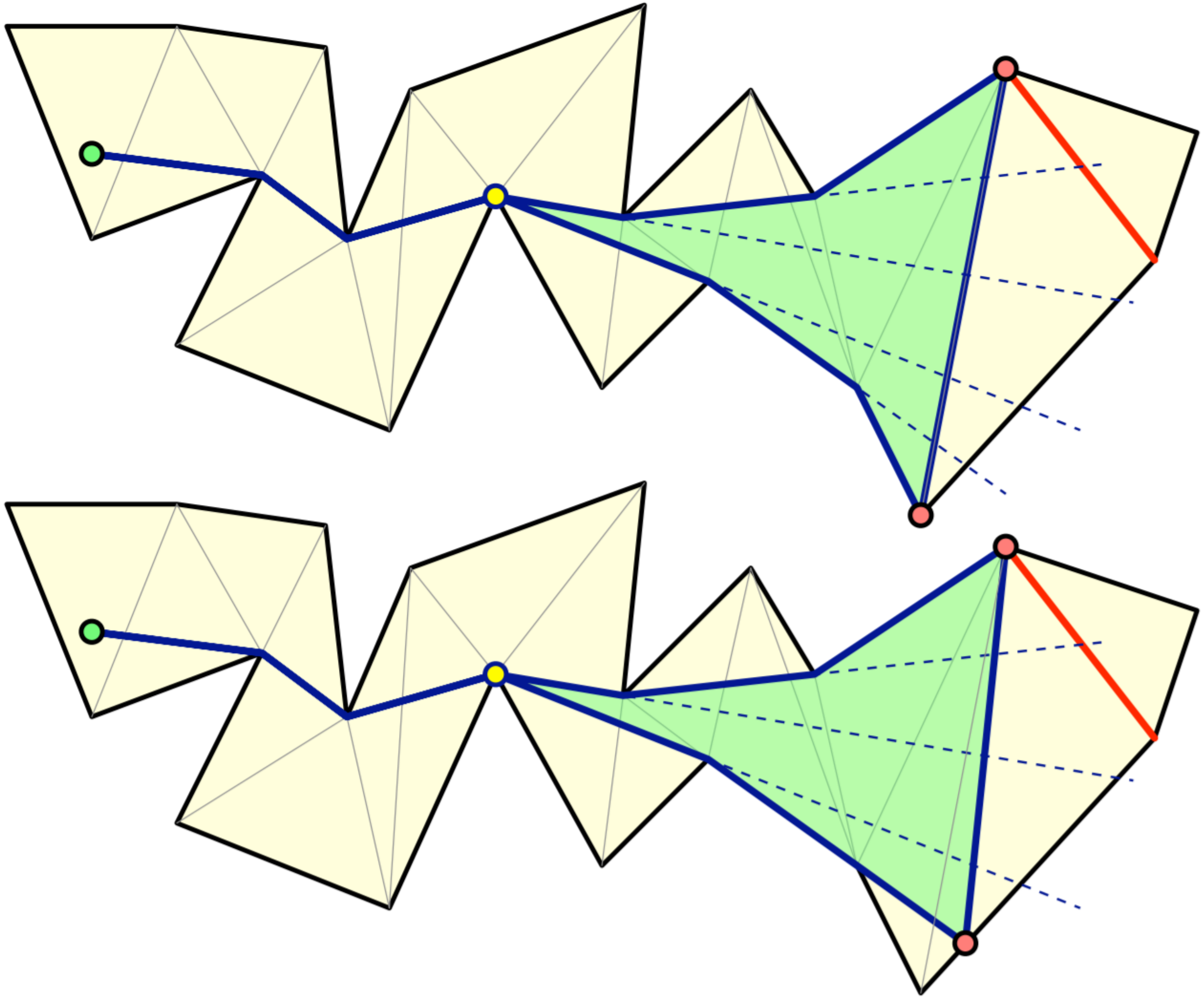


Funnel

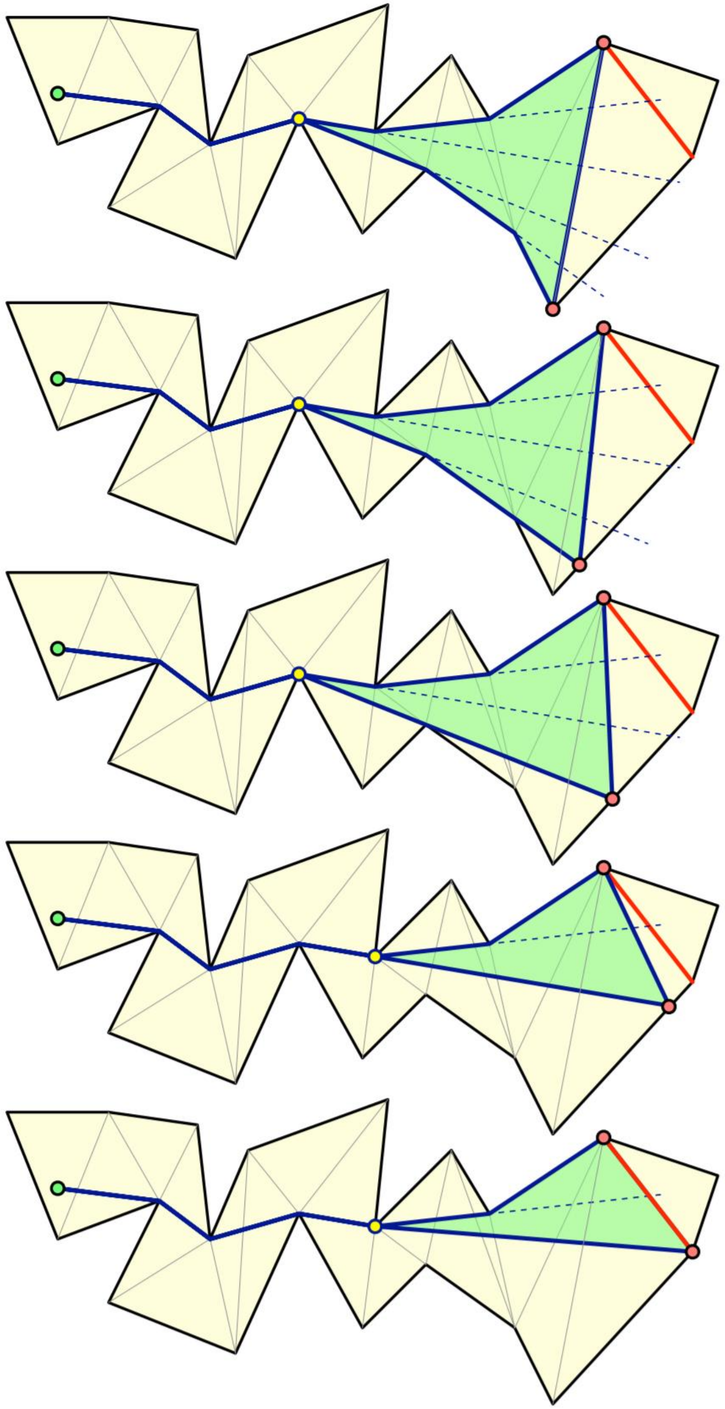
Extending Funnel



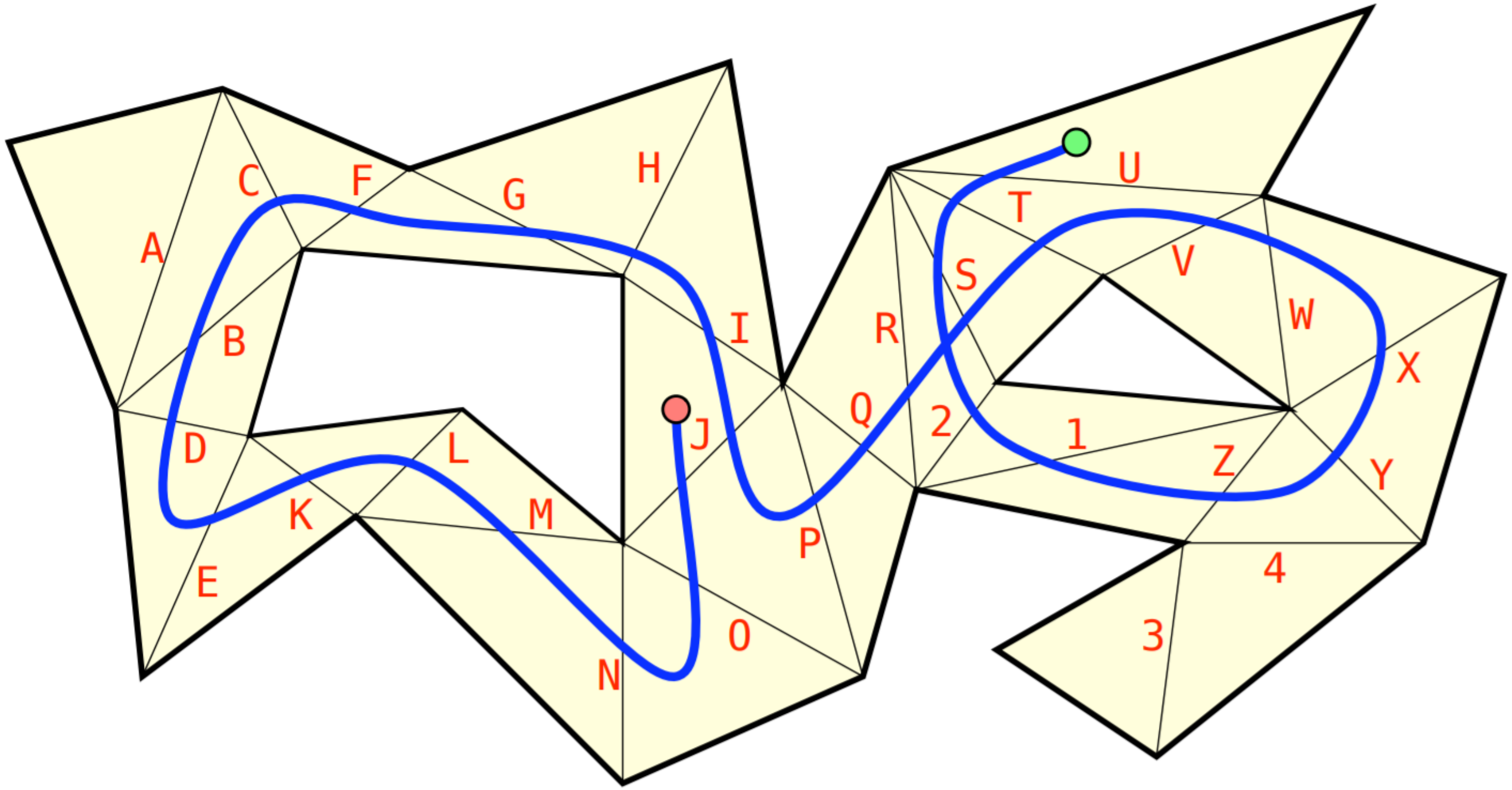




Narrowing Funnel



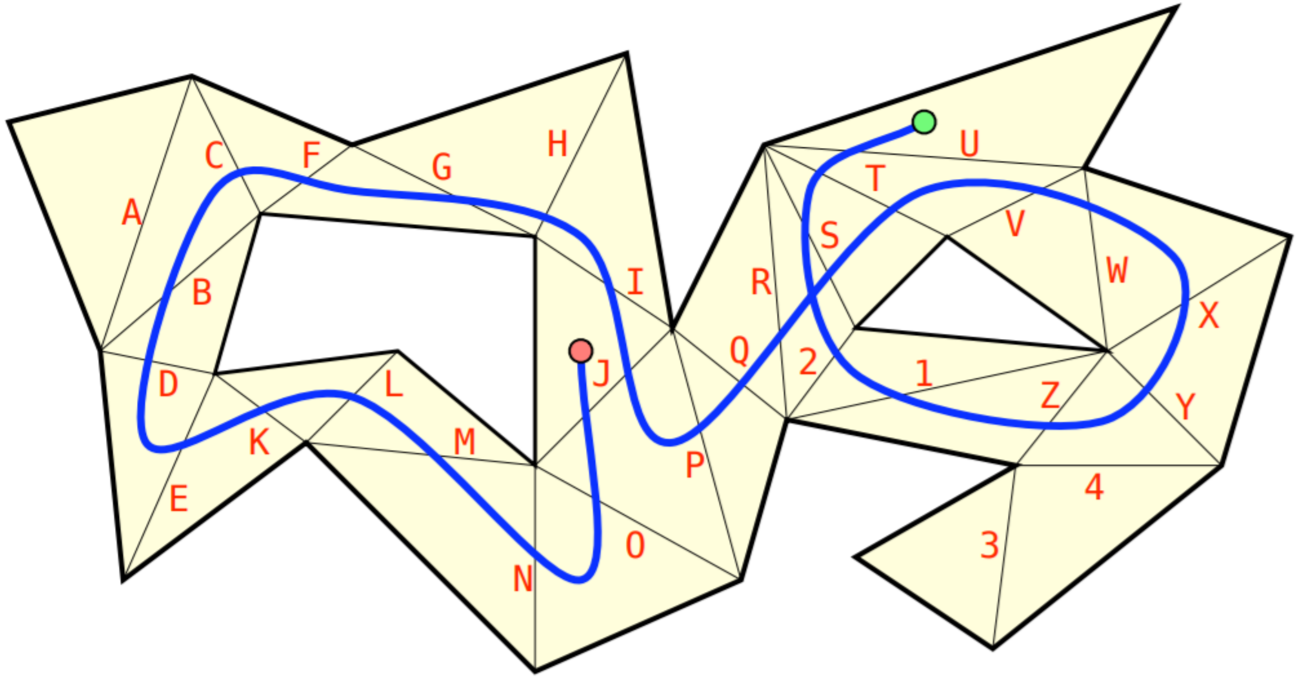
Contracting Funnel

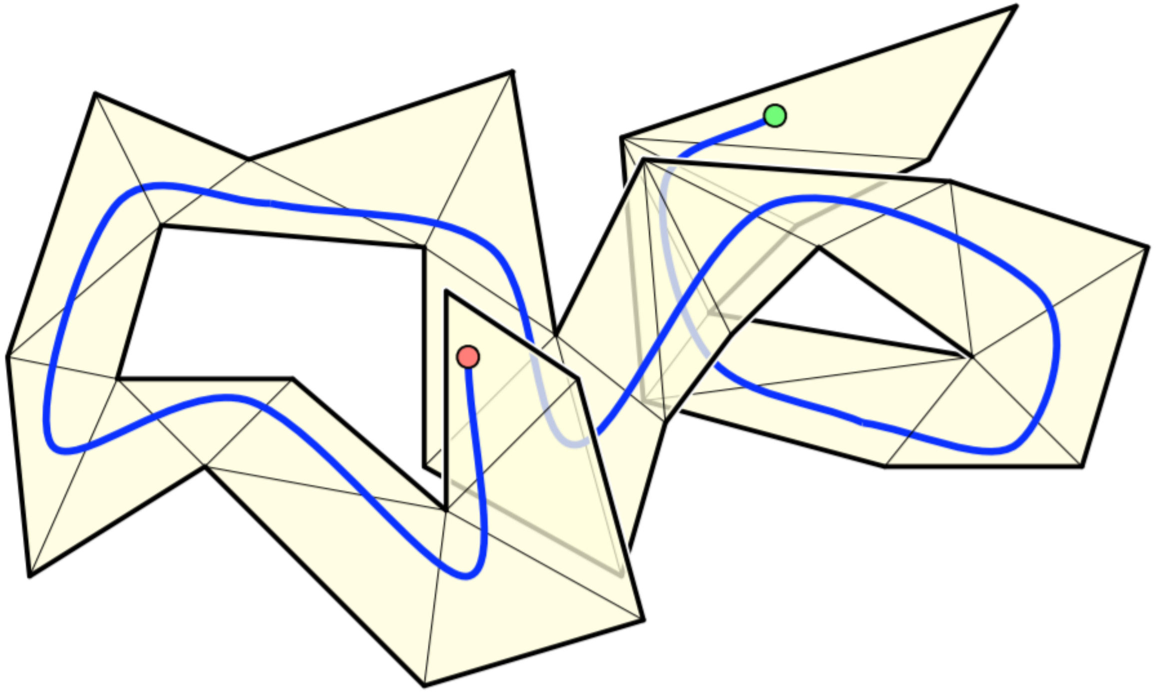


Funnel Algorithm

[Leiserson-Maley 1985] [Hershberger-Snoeyink 1994]

Given a k-edge path π in a polygon with obstacles, find the shortest path homotopic to π takes O(nk) time





Without Modification

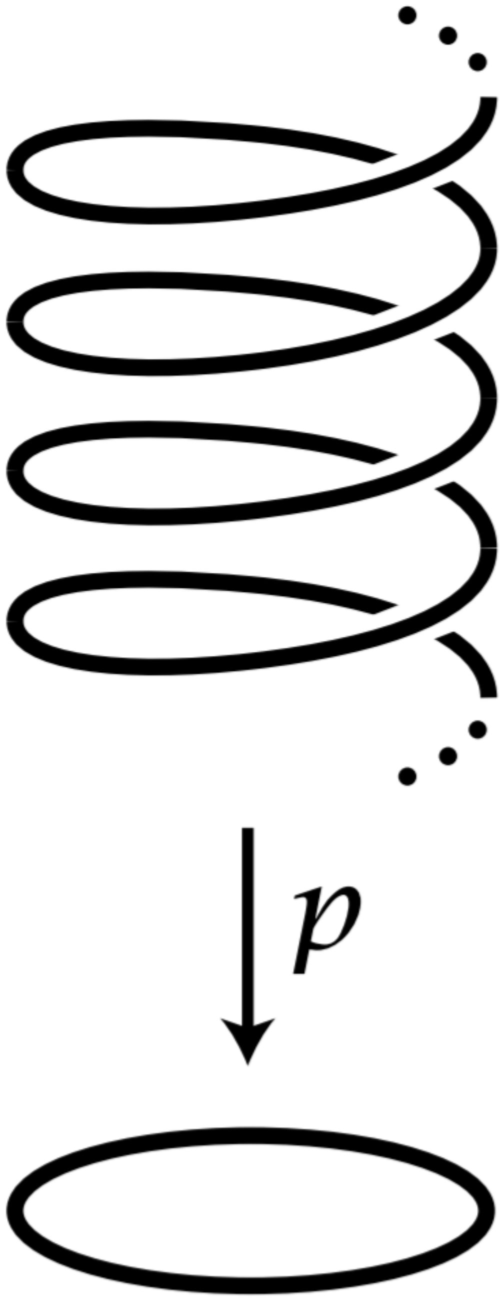
Intermission

Food for Thought.

Can the "lifted space" have non-trivial topology?

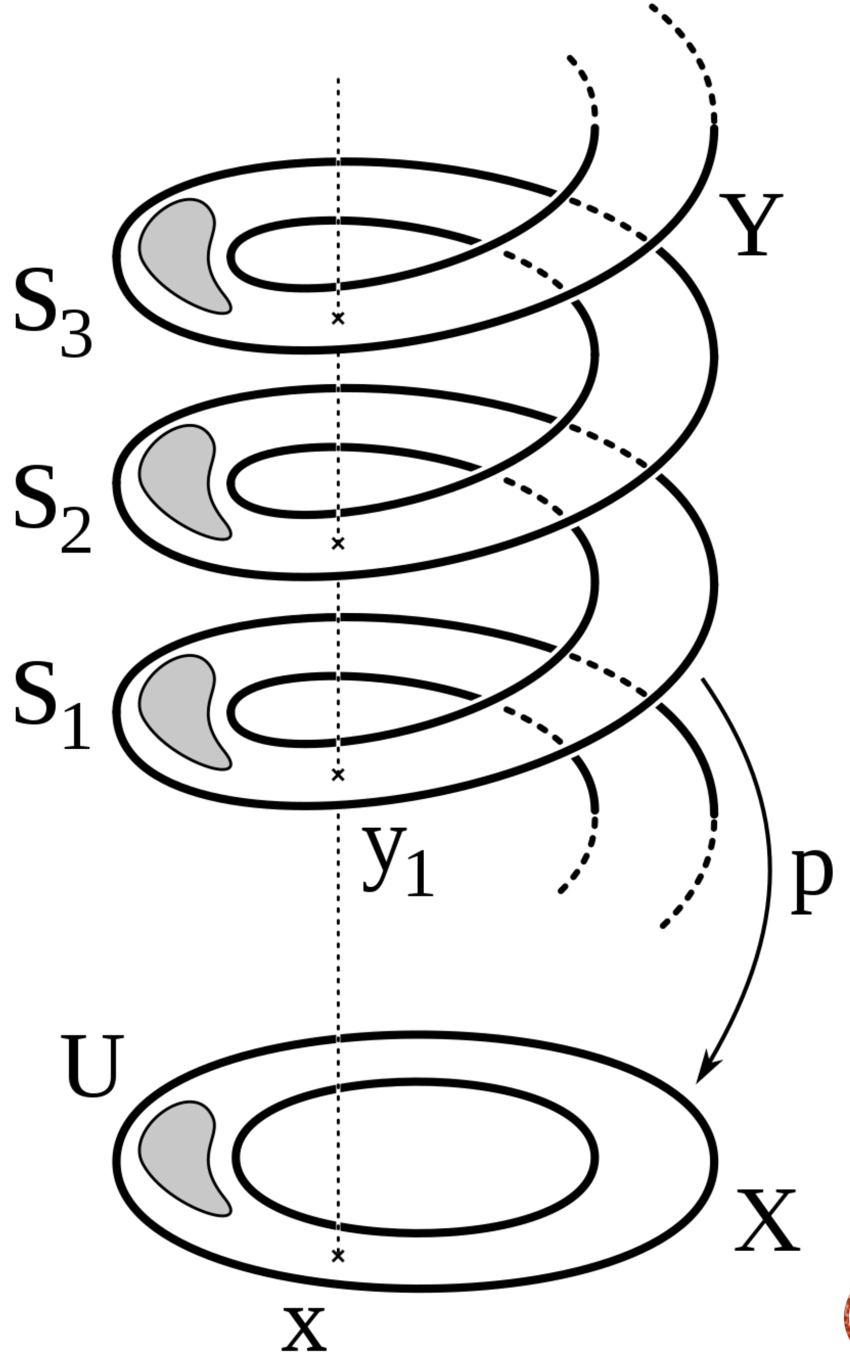
Covering Space and Fundamental Group

Covering Space of S^1



Covering Space

- Space Z' and local homeomorphism p: Z' $\rightarrow$ Z
  - For every point x in Z, there's an open disk U_x such that $p^{-1}(U_x)$ is a union of disjoint open disks, each maps homeomorphically unto U_x by p



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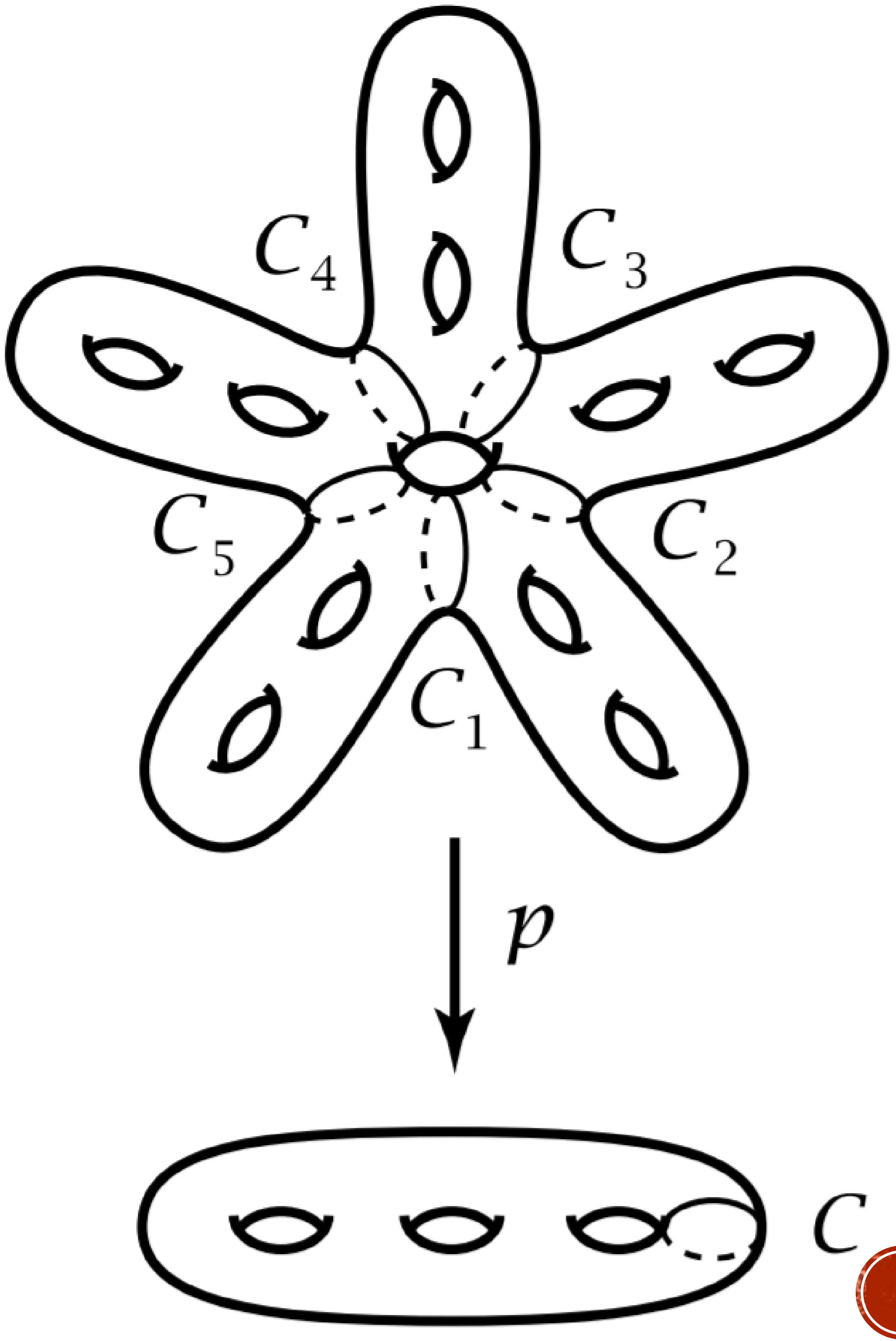
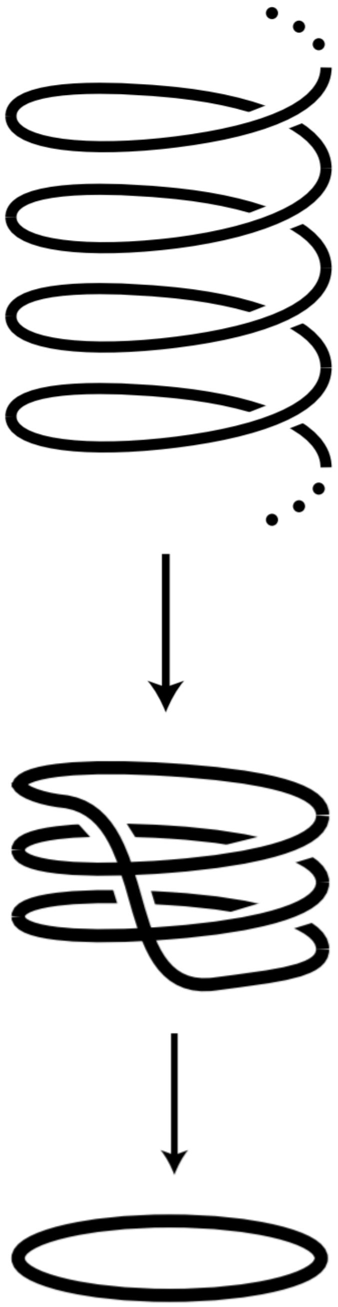
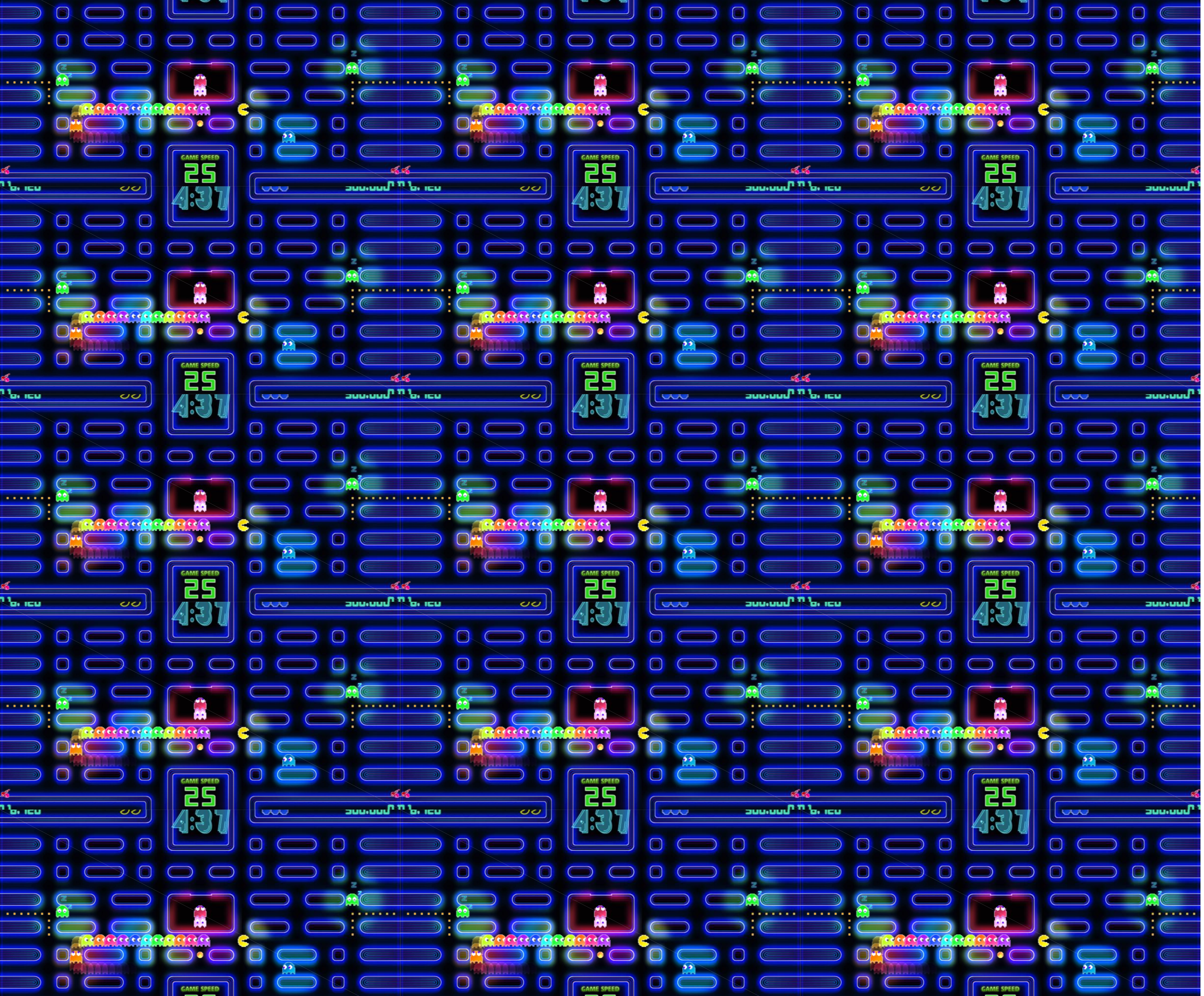


Covering Space

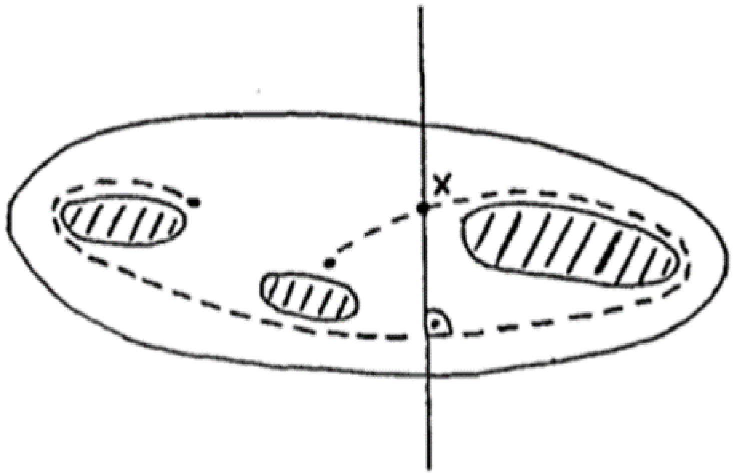
- Space Z' and local homeomorphism p: Z' $\rightarrow$ Z
  - For every point x in Z, there's an open disk U_x such that $p^{-1}(U_x)$ is a union of disjoint open disks, each maps homeomorphically unto U_x by p
- Universal cover Ž

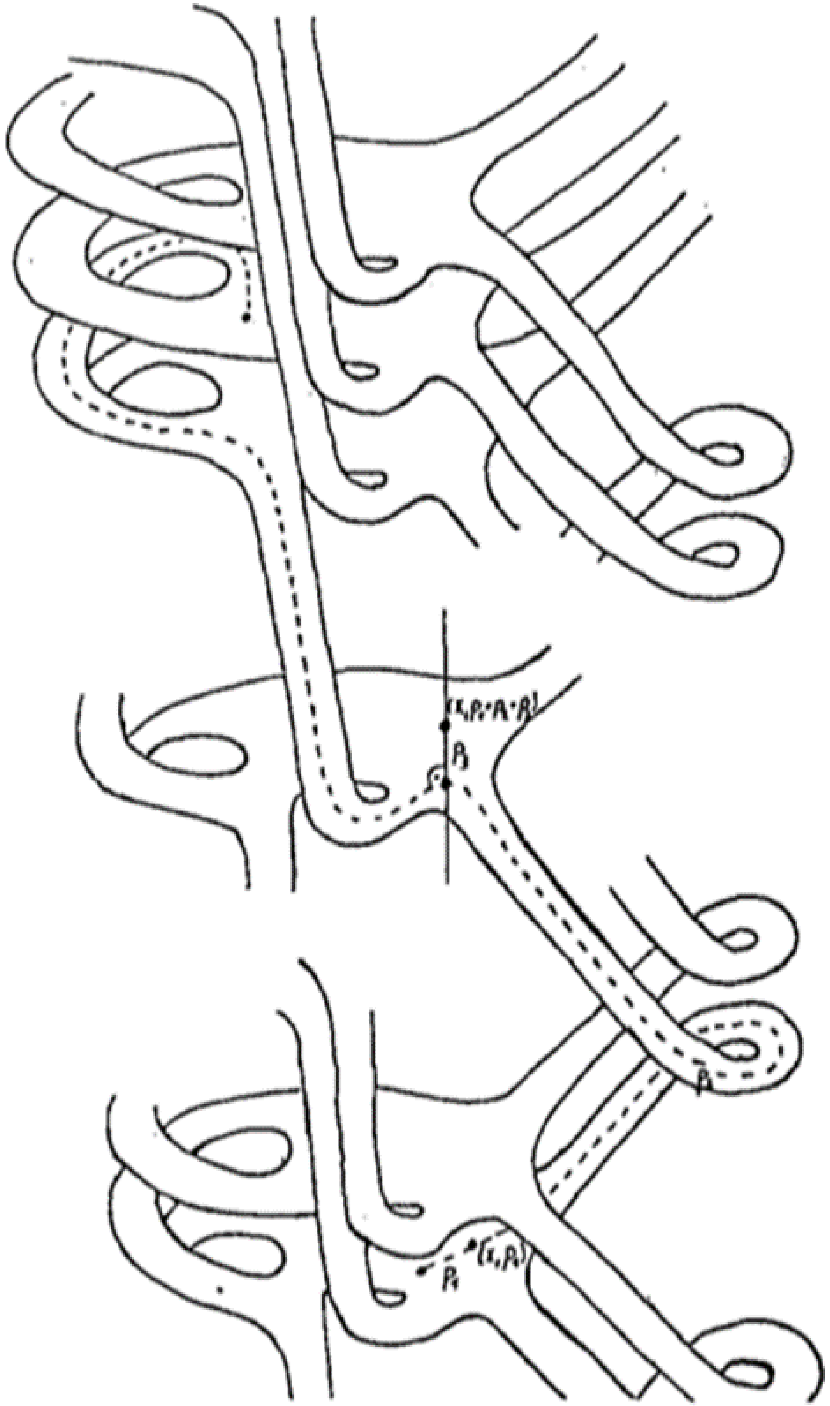


Covering of Pacman Space



Covering of Pacman Space





Lifting a Path

Proposition. Two paths are homotopic if and only if their lifts start and end at the same endpoints in Ž.

Fundamental Group

- $[\gamma]$ is the class of closed paths homotopic to γ in space Z

- $\pi_1(Z, z_0) =$

$$\{[\gamma] : \text{closed path } \gamma \text{ in } Z \text{ starting and ending at } z_0\}$$

Proposition. $\pi_1(Z, z_0)$ is a group.

Proposition. $\pi_1(Z, z_0) \cong \pi_1(Z, z_1)$ as groups.

Relation between two notions

- $\pi_1(Z, z_0) =$

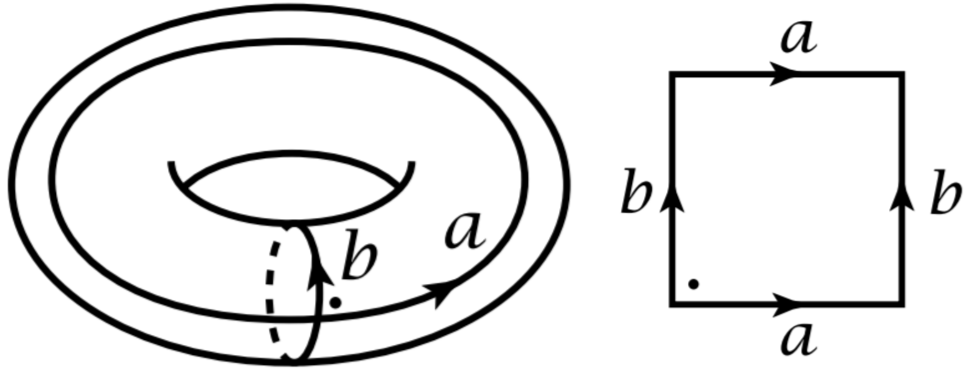
$$\{[\gamma] : \text{closed path } \gamma \text{ in } Z \text{ starting and ending at } z_0\}$$

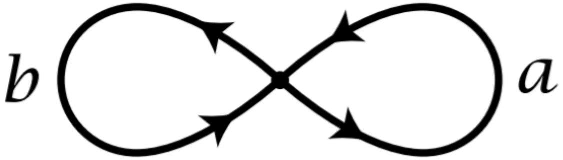
- $\check{Z} =$

$$\{[\gamma] : \text{path } \gamma \text{ in } Z \text{ starting at } z_0\}$$

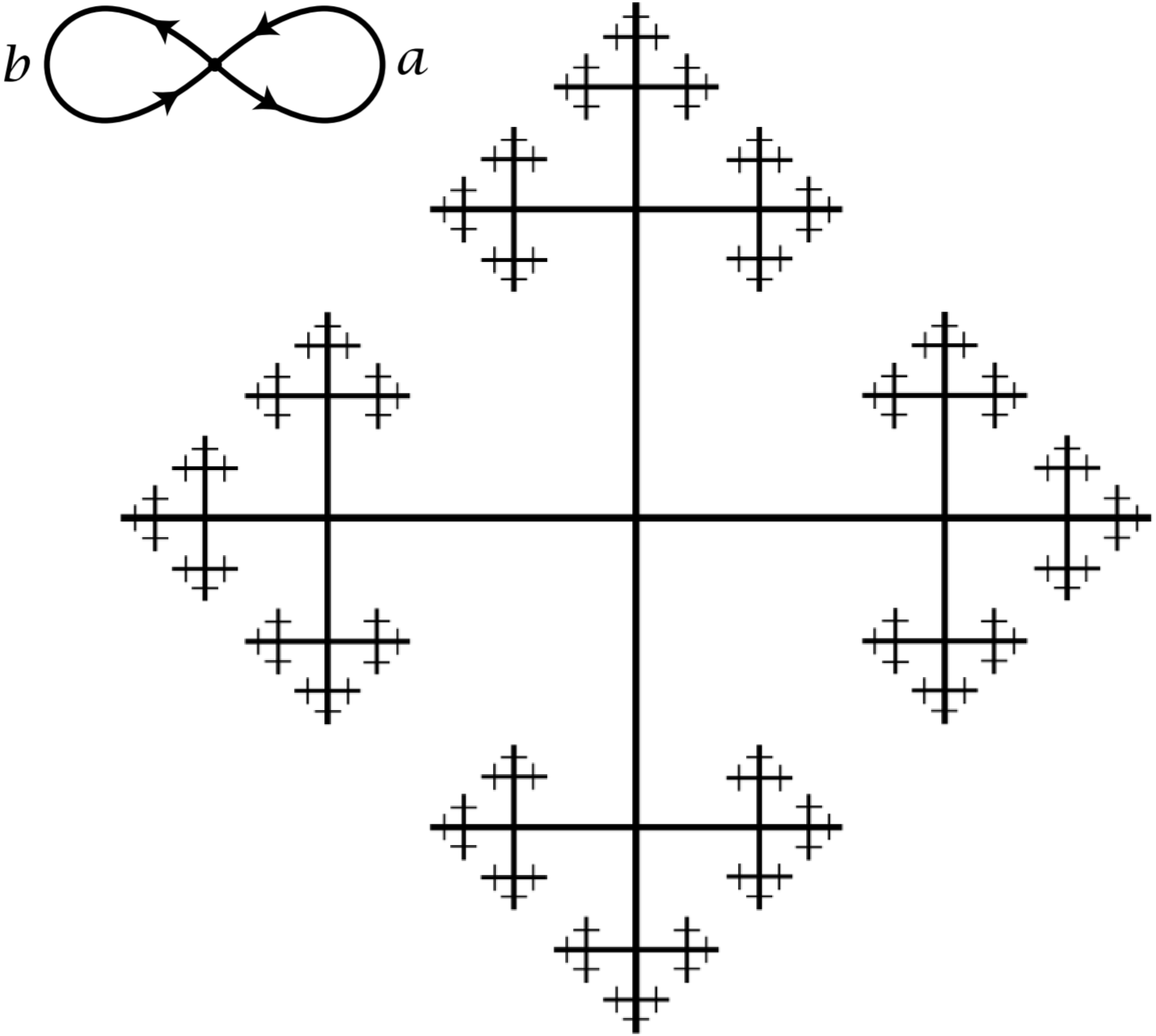
Theorem. $\pi_1(S^1) \cong \mathbb{Z}$.

π_1(Pacman)





π_1(2-Loops)



π_1(2-Loops)

Things under the rug

- Space Z has to be
  - path-connected
  - locally path-connected
  - semilocally simply-connected

Element : Fundamental Group
::
Lift : Covering Space

Next time.

Fundamental group $\pi_1(X)$ is a homotopy invariant of X.

Induced Homomorphism

- $\phi: X \longrightarrow Y$ induces $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$

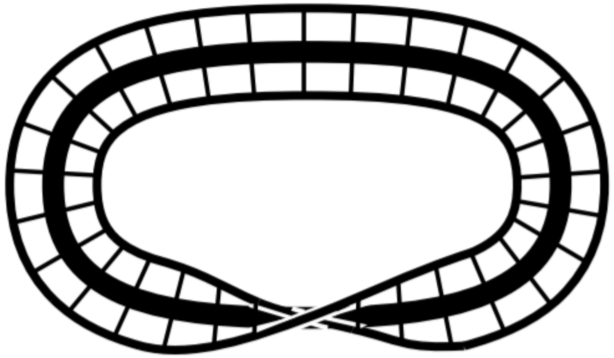
Induced Homomorphism

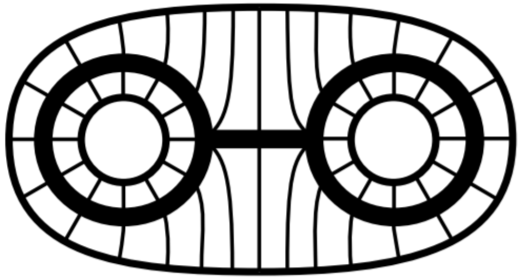
- $\phi: X \longrightarrow Y$ induces $\phi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \phi(x_0))$

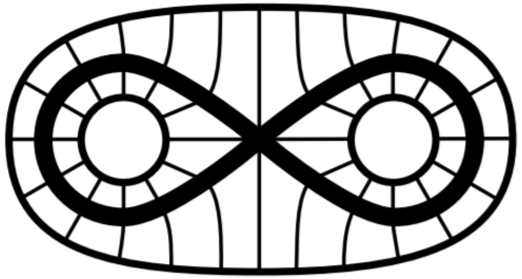
Proposition. ϕ_* is a group homomorphism.

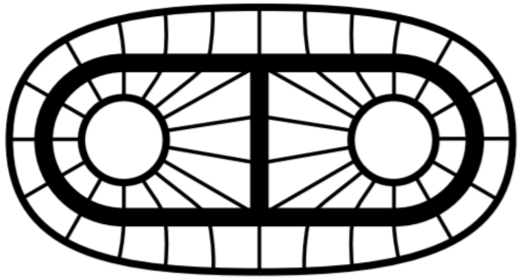
Equivalence

- Homeomorphism
  - $f: X \longrightarrow Y$ continuous bijection
  - $g: Y \longrightarrow X$ continuous bijection
  - $f \bullet g = id_X$
  - $g \bullet f = id_Y$
- Homotopy equivalence
  - $f: X \longrightarrow Y$ continuous bijection
  - $g: Y \longrightarrow X$ continuous bijection
  - $f \bullet g$ homotopic to id_X
  - $g \bullet f$ homotopic to id_Y

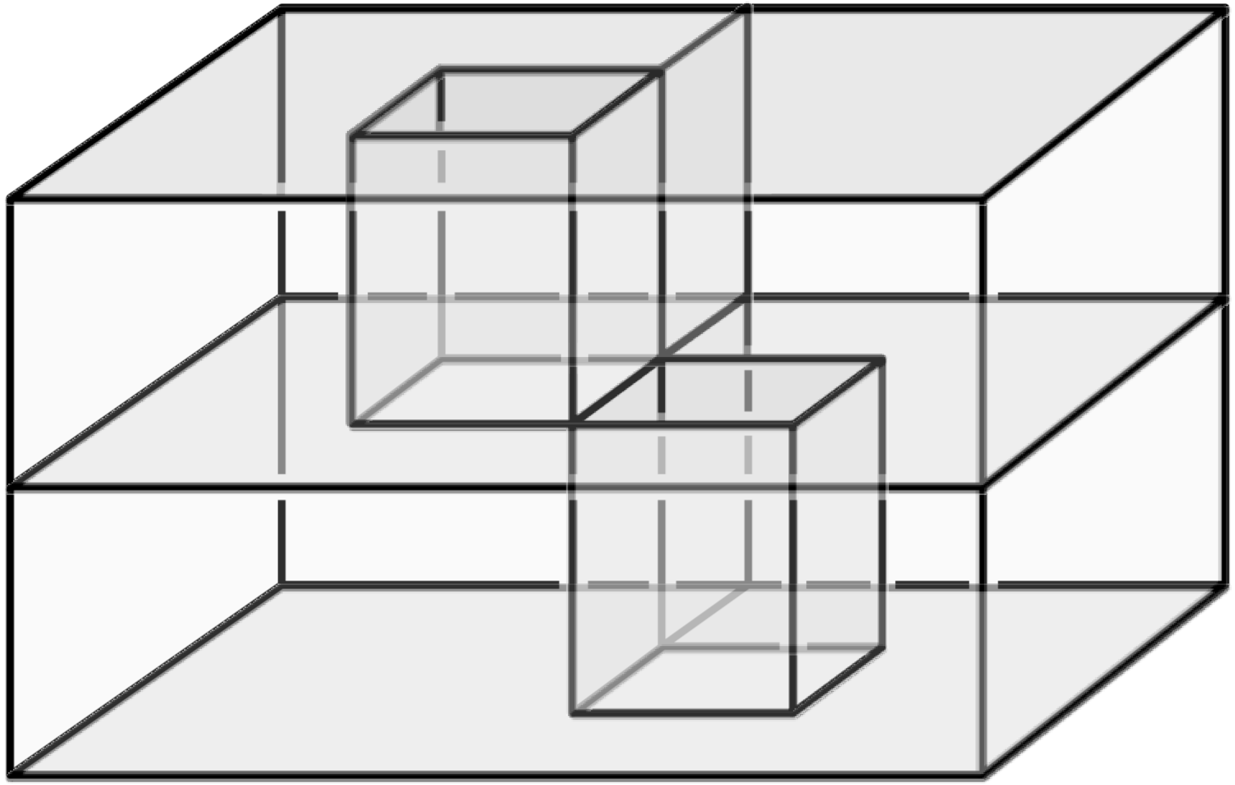








Homotopy Equivalence



Homotopy Equivalence

- House with two rooms

Theorem. Homotopy equivalence induces group isomorphism on π_1.