1. ***Regular or not?*** *Prove or disprove* that each of the languages below is regular (or not). Let $\Sigma^+$ denote the set of all *nonempty* strings over alphabet $\Sigma$; in other words, $\Sigma^+ = \Sigma \cdot \Sigma^*$. Denote $n(w)$ the integer corresponding to the binary string $w$.

   (a) $\left\{ \texttt{3}x\texttt{=}y : x, y \in \{\texttt{0}, \texttt{1}\}^*, n(y) = 3n(x) \right\}$

   **Solution:** Denote the language in the problem 1(a) as $L_a$. We prove that $L_a$ is not regular by constructing a fooling set for $L_a$ of infinite size.

   Let $F = \{\texttt{310}^i : i \geq 0\}$. For two distinct prefixes $x = \texttt{310}^i$ and $y = \texttt{310}^j$ in $F$, let $z$ be $\texttt{=110}^i$.

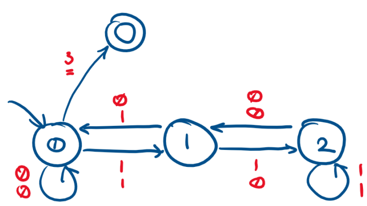
   - $xz = \texttt{310}^i\texttt{=110}^i$; because $n(\texttt{110}^i) = 3n(\texttt{10}^i)$, we have $xz$ in $F$.
   - $yz = \texttt{310}^j\texttt{=110}^i$; because $n(\texttt{110}^i) \neq 3n(\texttt{10}^j)$ if $i \neq j$, we have $yz$ not in $F$.

   This implies that $F$ is a fooling set of infinite size, and thus $L_a$ is not regular. ■

   (b) $\left\{ \genfrac{}{}{0pt}{}{\texttt{3x}}{\texttt{=y}} : \genfrac{}{}{0pt}{}{\texttt{x}}{\texttt{y}} \in \left\{ \genfrac{}{}{0pt}{}{\texttt{0}}{\texttt{0}}, \genfrac{}{}{0pt}{}{\texttt{0}}{\texttt{1}}, \genfrac{}{}{0pt}{}{\texttt{1}}{\texttt{0}}, \genfrac{}{}{0pt}{}{\texttt{1}}{\texttt{1}} \right\}^*, n(y) = 3n(x) \right\}$

   **Solution:** Denote the language in the problem 1(b) as $L_b$. We prove that $L_b$ is regular by constructing an NFA recognizing $L_b$.

   We construct NFA recognizing the reverse of the language, $L_b^R$; by the exercise problems, $L_b$ is regular if and only if $L_b^R$ is regular.



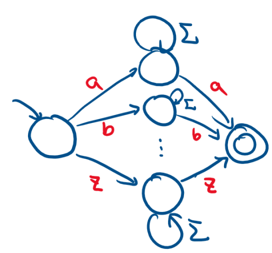

   The NFA reads the input from the least significant bits of $x$ and $y$, and records the amount of carry at any moment. The transitions are implemented so that the machine only continues if the current digit of $y$ equals to (the least significant bit of) three times the corresponding digit in the $x$ plus the carry. After reading the full strings $x$ and $y$, if there is any carry left then we reject; otherwise the NFA finishes off by reading the leading $\genfrac{}{}{0pt}{}{\texttt{3}}{\texttt{=}}$ and accepts. ■

   (c) $\left\{ wxw^R : w, x \in \Sigma^+ \right\}$

   **Solution:** Denote the language in the problem 1(c) as $L_c$. We prove that $L_c$ is regular by constructing an NFA recognizing $L_c$, which is equivalent to the following language:

$$L_c' := \left\{ \sigma x' \sigma : x' \in \Sigma^+, \sigma \in \Sigma \right\}.$$

   For $L_c' \subseteq L_c$, take $w = \sigma$ and $x = x'$; for $L_c \subseteq L_c'$, take $\sigma$ to be the first symbol in $w$ and $x'$ to be whatever is left.




   The constructed NFA reads the first and the last symbol, and accepts if they match; therefore the NFA correctly recognizes language $L_c'$. More formally, create one state $q_\sigma$ for each symbol

$\sigma \in \Sigma$; and add two extra states $s$ and $t$. Let $s$ be the only starting state and $t$ be the only accepting state. For each symbol $\sigma$, add transitions $s$ to $q_\sigma$ and $q_\sigma$ to $t$ on reading $\sigma$, and self-loop transition at $q_\sigma$ on reading all symbols.

■

(d) $\{ww^R x : w, x \in \Sigma^+\}$

**Solution:** Denote the language in the problem 1(d) as $L_d$. We prove that $L_d$ is not regular by constructing a fooling set for $L_d$ of infinite size. Without loss of generality we assume that `0` and `1` are in $\Sigma$.

Let $F = \{\mathtt{0}^i\mathtt{1} : i \geq 0\}$. For two distinct prefixes $u = \mathtt{0}^i\mathtt{1}$ and $v = \mathtt{0}^j\mathtt{1}$ in $F$ (without loss of generality assuming $i > j$), consider the suffix $z = \mathtt{10}^i\mathtt{1}$.

- $uz = \mathtt{0}^i\mathtt{110}^i\mathtt{1}$; by taking $w = \mathtt{0}^i\mathtt{1}$ and $x = \mathtt{1}$, this shows that $uz$ is in $F$.
- $vz = \mathtt{0}^j\mathtt{110}^i\mathtt{1}$. Observe that $x$ can only be taken as `1`; because $i > j \geq 0$, there are no possible $w$ that will evenly split the number of `0`s and `1`s (which is a necessary condition for separating into $ww^R$). This shows that $vz$ is not in $F$.

This implies that $F$ is a fooling set of infinite size, and thus $L_d$ is not regular. ■

**Rubric:** 5-point grading scale (plus deadly-sins and sudden-death rules) *for each pair of problems (a–b) and (c–d)*. Maximum 3 points if one tries to prove a regular language to be non-regular, or vice versa. Maximum 1 point if both guesses are wrong.

Full credit for subproblem (d) if one correctly proves the language to be regular *when $\Sigma$ is unary*. This was an oversight.

2. ***Telling DFAs apart.***

Let $M_1$ and $M_2$ be two DFAs, each with exactly $n$ states. Assume that the languages associated with the two machines are different (that is, $L(M_1) \neq L(M_2)$), there is always some string in the symmetric difference of the two languages.

Prove that there is a string $w$ of length polynomial in $n$ in the symmetric difference of $L(M_1)$ and $L(M_2)$. What is the best upper bound you can get on the length of $w$?

**Solution:** First we construct a DFA $M'$, described by $(Q', s', A', \Sigma', \delta')$, that recognizes the symmetric difference of the two languages $L(M_1)$ and $L(M_2)$, using the product construction. Denote $M_i$ by the tuple $(Q_i, s_i, A_i, \Sigma_i, \delta_i)$ for $i \in \{1, 2\}$.

- States $Q'$: $Q_1 \times Q_2$ — pairs of states, one from each $M_i$
- Starting state $s'$: $(s_1, s_2)$
- Accepting states $A'$: $\{(r_1, r_2) \in Q' : \text{either } r_1 \in A_1 \text{ and } r_2 \notin A_2, \text{ or } r_1 \notin A_1 \text{ and } r_2 \in A_2\}$
- Alphabet $\Sigma'$: $\Sigma_1 \cup \Sigma_2$
- Transition function $\delta'$: $\delta_1 \times \delta_2$, mapping $\delta'((q_1, q_2), \mathsf{a})$ to $(\delta_1(q_1, \mathsf{a}), \delta(q_2, \mathsf{a}))$ on reading any symbol $\mathsf{a} \in \Sigma'$

DFA $M'$ recognizes the symmetric difference of the two languages $L(M_1)$ and $L(M_2)$ and has $n^2$ states. Now by problem statement $M'$ accepts at least one word. Now any walk from the starting state $s'$ to an accepting state in $M'$ can be turned into a simple path between the same two endpoints, without ever visiting the same state twice. This shows that there is a word of length at most $n^2$ that is accepted by $M'$, and thus in the symmetric difference of $L(M_1)$ and $L(M_2)$. ■

**Rubric:** Any complete solution with a (justified) polynomial bound receives full credit. Any *subquadratic* bound receives extra credit.

Standard 5-point grading scale plus deadly-sins and sudden-death rules.