

- Starting from Homework 1 you are allowed to work in of group up to 3 people (for undergrads there is no limit to the group size). Please write the names of *all* group members on the first page of your submission. *One, and only one* person from each group is responsible in submitting the solution to Canvas.
- *Each member* of the group is required to cite all the people and resources you used when trying to solve the problems; but you don't have to cite other members from your group. The standard citation rules applies (see the course webpage).

Let γ be a *generic* planar curve where all self-intersection points are transverse double-crossings. Any planar curve can be made generic by slightly perturbing the curve. In this homework we will be playing with generic planar curves.

1. **Gauss code.** A **Gauss code** is a cyclic string of $2n$ symbols where each symbol occurs exactly two times; it is **signed** if in addition each symbol x is attached with a plus/minus sign \pm , one for each occurrence of x . A Gauss code is **planar** if it encodes the sequence of crossings we see as we traverse an n -vertex planar curve γ ; the signing of the Gauss code correspond to the Gauss signs of the crossings of γ .

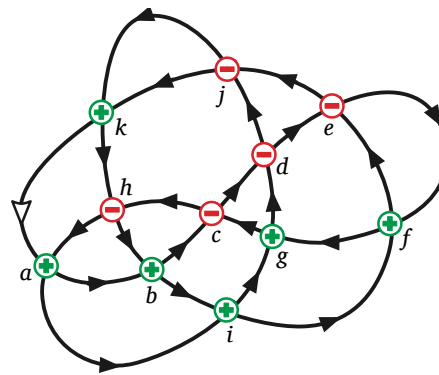


Figure 1. A planar curve with Gauss code `[abcdefgchaigdkhbfekj]` and signing `[++-+++-++-++-+-]`.

- (a) Describe and analyze an algorithm whether a given *signed* Gauss code is planar.
- (b) Construct an (unsigned) non-planar Gauss code. Identify the reason why your code cannot be realized by a planar curve (as a condition of the code).
- (c) Construct another Gauss code that is not planar, for a *different* reason than the one you stated above.
- * (d) Try to prove that any Gauss code excluding the above conditions must be planar. If the proof fails, identify more reasons to why a code can be non-planar, and try again. Generate a list of necessary conditions that becomes sufficient. [Hint: $\text{Lemma} \geq 2$ and ≤ 2 for the Jordan polygon theorem.]
- ★ (e) Can you provide such characterization for Gauss codes encoding curves on the torus?¹

¹The *really* big star here indicates that the problem, as far as I know, is open.

2. **Simplifying planar curves.** Any homotopy between two generic planar curves can be decomposed into a finite number of local operations:

- $1 \rightarrow 0$: removing an empty *loop*;
- $2 \rightarrow 0$: removing an empty *bigon*;
- $3 \rightarrow 3$: flipping a *triangle*, or equivalently, moving a strand across another crossing.

We refer to these operations, *together with their inverses*, as the **homotopy moves**.



Figure 2. Homotopy moves $1 \rightarrow 0$, $2 \rightarrow 0$, and $3 \rightarrow 3$.

During the lectures we showed that any planar curve can be reduced to some canonical curve with the same rotation number using only *regular* homotopy moves (where only $0 \rightarrow 2$, $2 \rightarrow 0$ and $3 \rightarrow 3$ are allowed), by iteratively emptying the loops. From there one can always simplify the canonical curves (that is, remove all crossings) using $1 \rightarrow 0$ moves.

The goal of this problem is to simply a planar curve *without ever creating new self-intersections*. In other words, any planar curve can be simplified using only $1 \rightarrow 0$, $2 \rightarrow 0$, and $3 \rightarrow 3$ moves. Such moves are referred as **monotonic**.

- (a) As a first step, we show that at least one of the monotonic homotopy moves can be applied at any given time. In other words, prove that there is always a face of degree at most 3 in any planar curve.

The main difficulty to adapt the loop-removal approach here is that emptying a loop in general requires $0 \rightarrow 2$ moves, which is not monotonic. However, we can search for a (possibly non-empty) *bigon* instead. A **bigon** is the region bounded by two simple interiorly-disjoint subpaths of γ that shares the same two endpoints. (See Figure 3.)

- (b) Prove that after all empty loops are removed, there must be a bigon in γ (or γ is already simple).
- (c) A bigon is (*inclusion-wise*) *minimal* if no other bigon lies inside. Prove that any minimal bigon in γ must have a triangle on the bounding curves (and therefore can be removed using a single $3 \rightarrow 3$ move).
- (d) Prove our main statement that any planar curve can be simplified monotonically. How many moves does your algorithm take?

As an alternative approach, one can prove termination by introducing a *potential*. Consider the breadth-first search tree on the dual graph of γ , starting at the external face (called the *root*). Each face is labelled with its distance to the root on the BFS tree, called the **depth**. Define the **depth potential** to be the sum of depths on all faces of γ .

- ★(e) Prove that there is always a homotopy move that decreases the depth potential.²

²The only proof I know conducts an ugly case-by-case-by-case analysis, and is not done using the curve language. Is there a way that the combinatorial Gauss-Bonnet theorem can help?

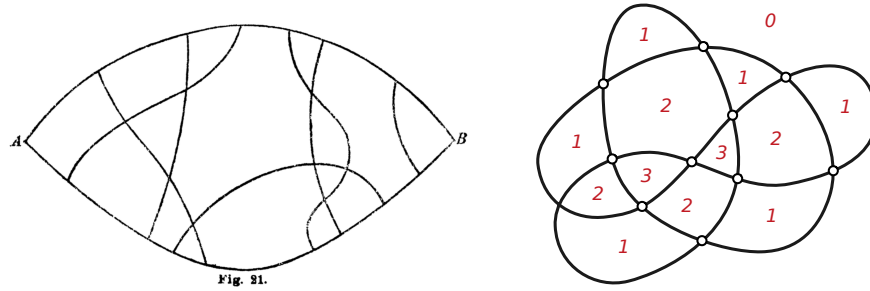


Figure 3. Left: A (non-minimal) bigon. Right: Face depths.

Since the depth potential of any n -vertex planar curve is bounded by $O(n^2)$, the algorithm will terminate after at most $O(n^2)$ moves.

★(f) Can you simplify an arbitrary n -vertex planar curve monotonically in $o(n^2)$ moves?