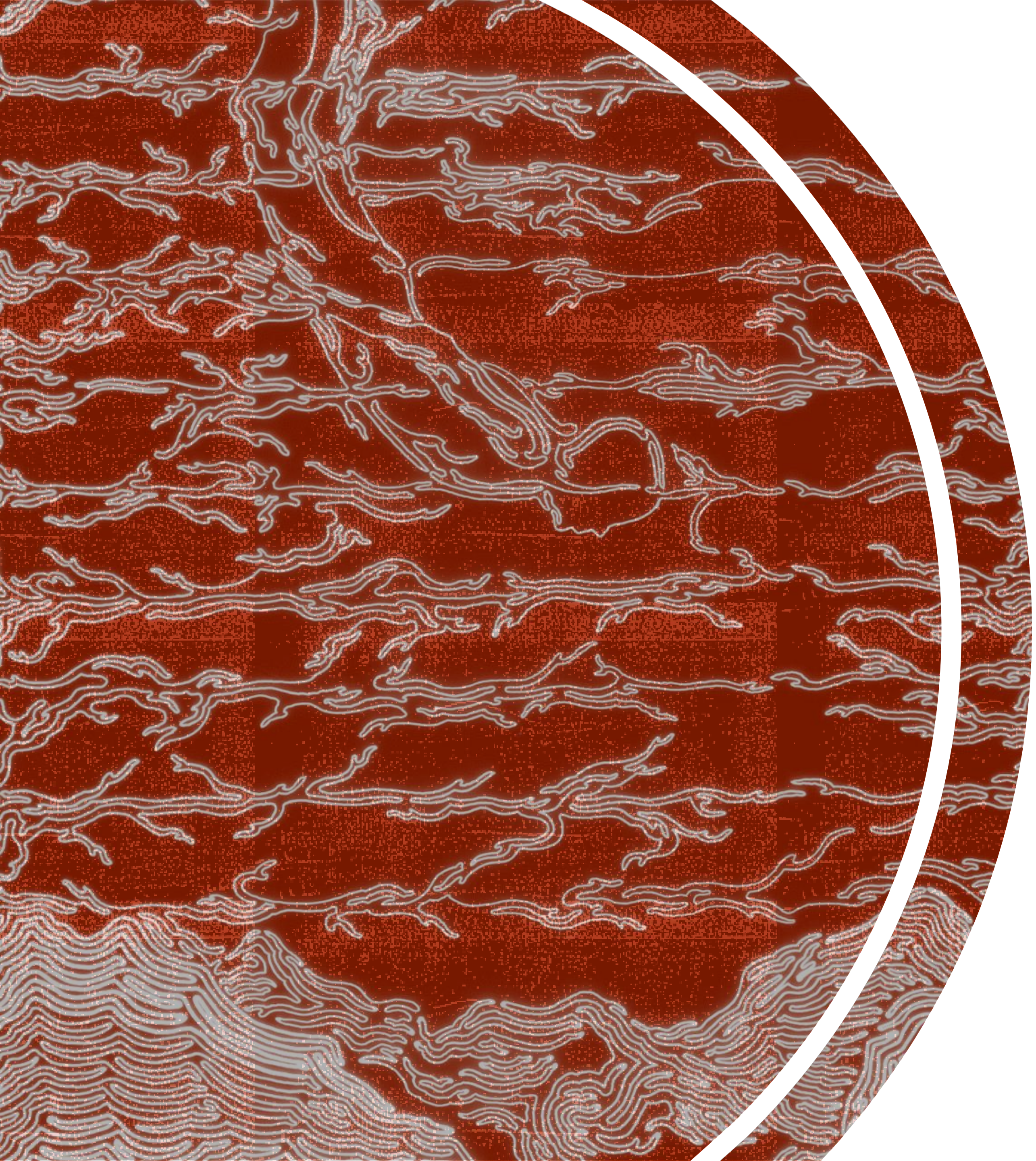


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 2, September 16, 2021

Administrivia

- Homework 0 is due 9/20 (next Monday)
  - Starting from Homework 1, collaboration up to 2 people
  - Open-everything
- Come to the office hour tomorrow!
- Again, STOP me anything you have questions

Where were we?

- $\text{Wind}_q(P)$

- Discrete homotopy (vertex moves)

- **Lemma.** $\text{Wind}_q(P)$ is invariant under safe vertex moves.

THEOREM. Two polygons P and Q are homotopic in $R^2 \setminus q$ if and only if they have the same $Wind_q$. [Hopf 1935]

Wind_q is a Complete Homotopic Invariant!

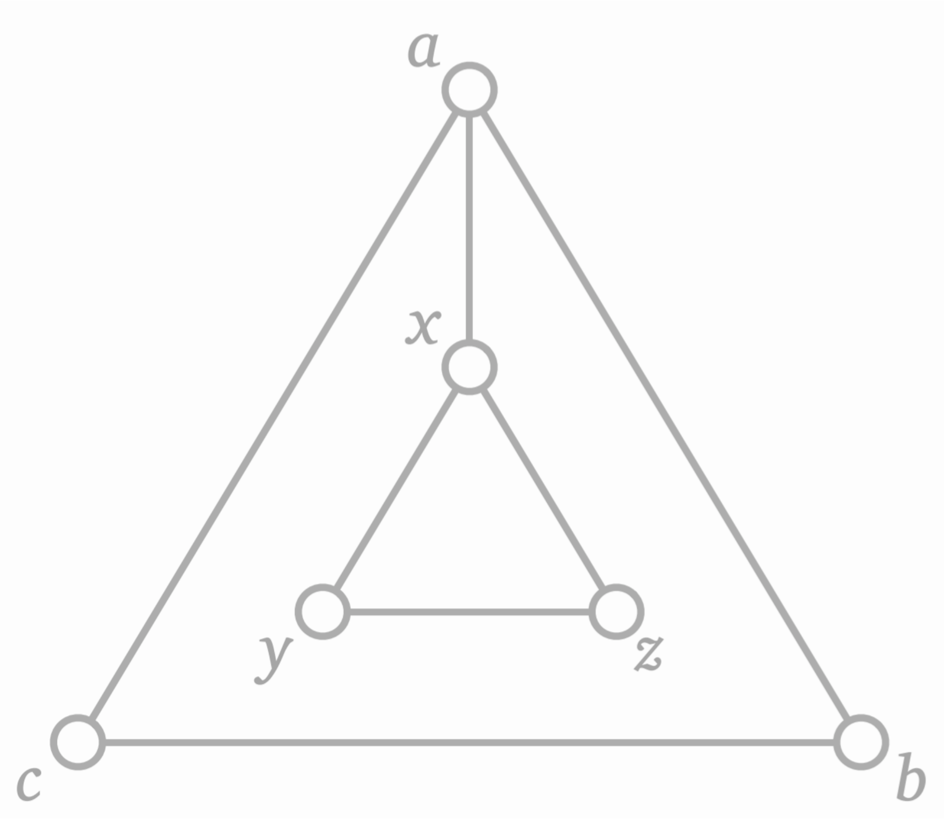
Takeaway.
Planar curve in punctured plane is described by #times it goes around the puncture.

Remark. The punctured plane $R^2\backslash q$ is different from the plane as spaces.

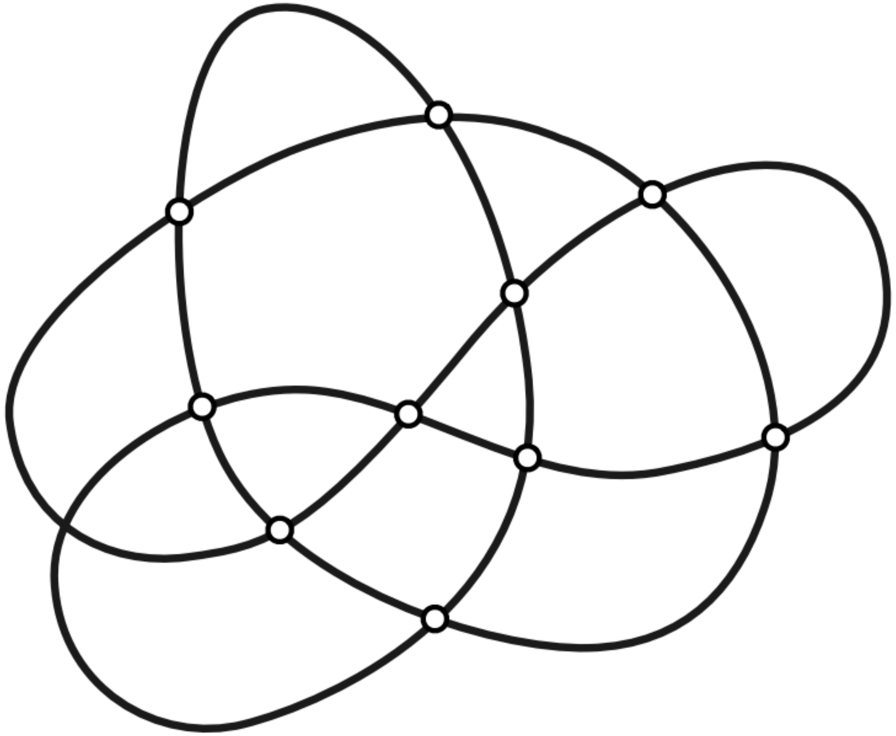
Regular Homotopy and Rotation Number

Switching Views

- Polygonal



- Generic

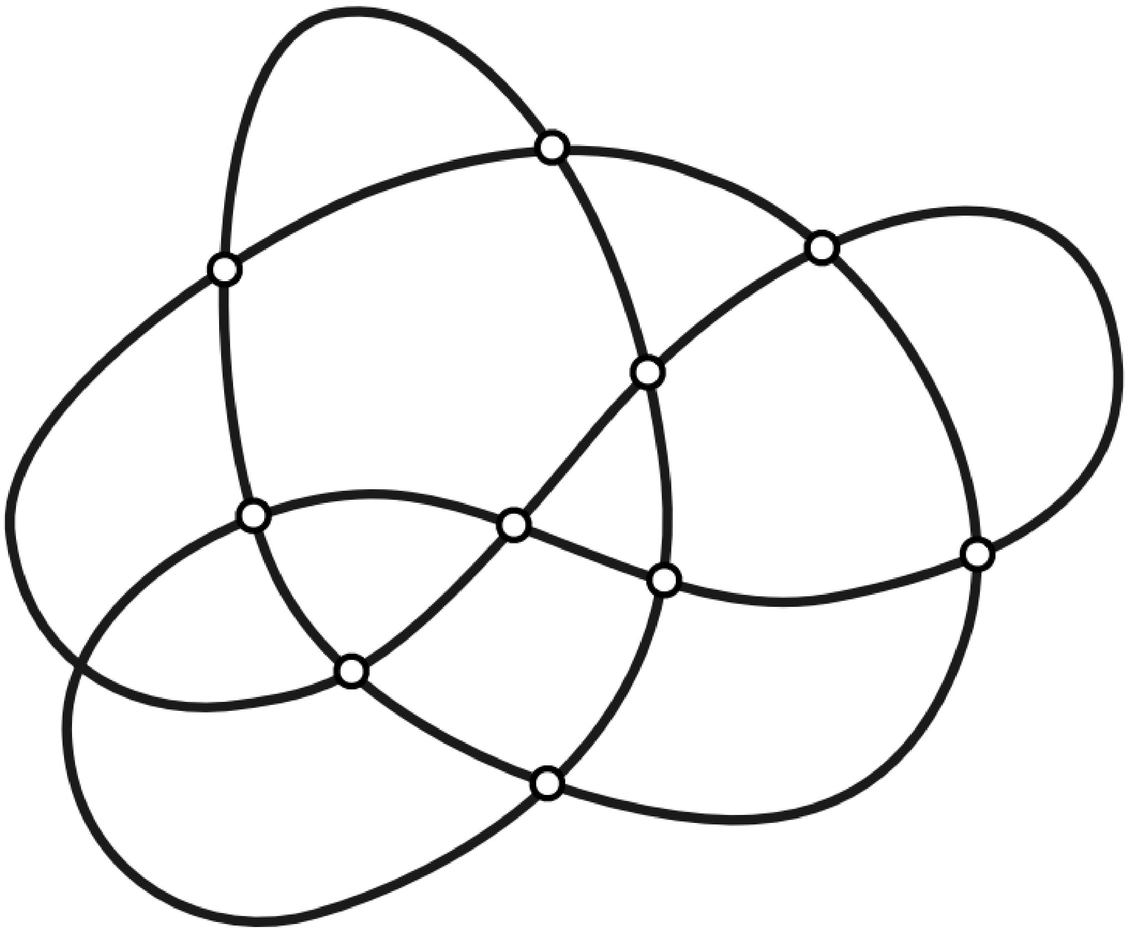


Untangling Gardening Hose

Can you untangle the hose without lifting or twisting?

(Cables magically pass through each other)

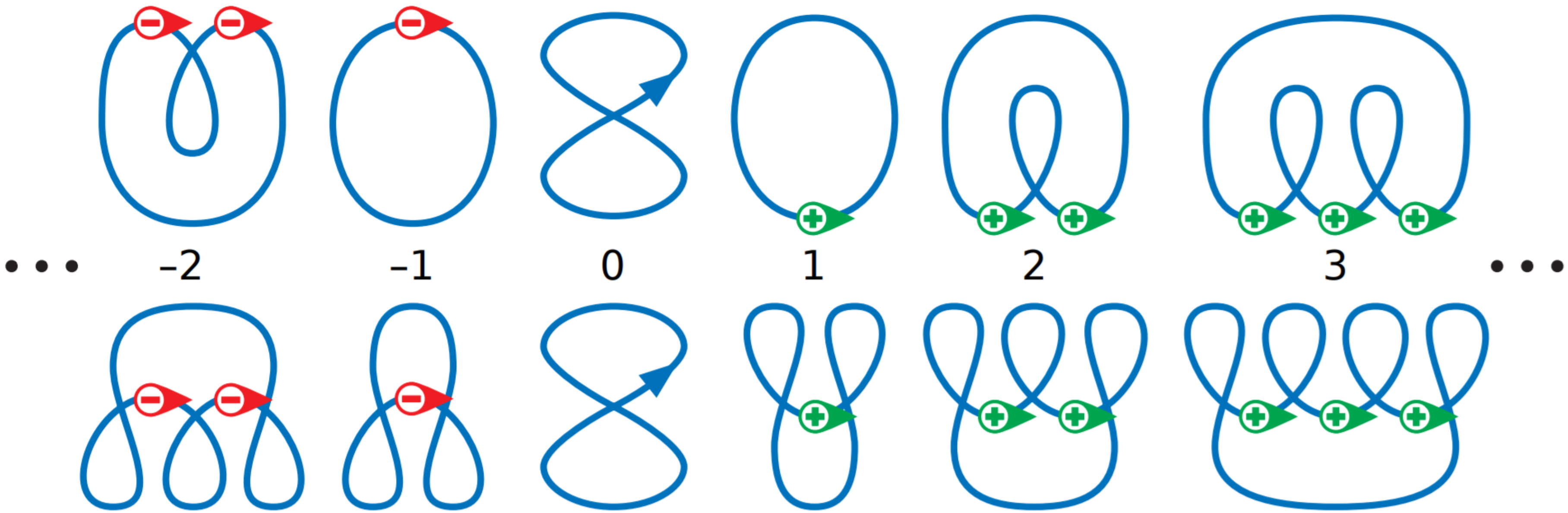
Rotation number



Definitions

- Homotopy moves

- Regular homotopy



Whitney-Graustein Theorem

[Whitney 1937] [Boy 1901] [Meister 1770]

Any two regular curves are regular homotopic if their rotation numbers are the same.

Proof of Whitney-Graustein Theorem

- Rotation number is invariant under regular homotopy:

Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 1. Shrink an arbitrary loop

Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 2. Move empty loop to outside

Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 3. ???

Proof of Whitney-Graustein Theorem

- Turn generic curve into canonical curves $0^{Rot(C)}$:
  - Step 4. PROFIT

Rot is a Complete Regular-Homotopic Invariant!

Takeaway.
Planar curve can be described by how many times its derivative winds around the origin.

Intermission

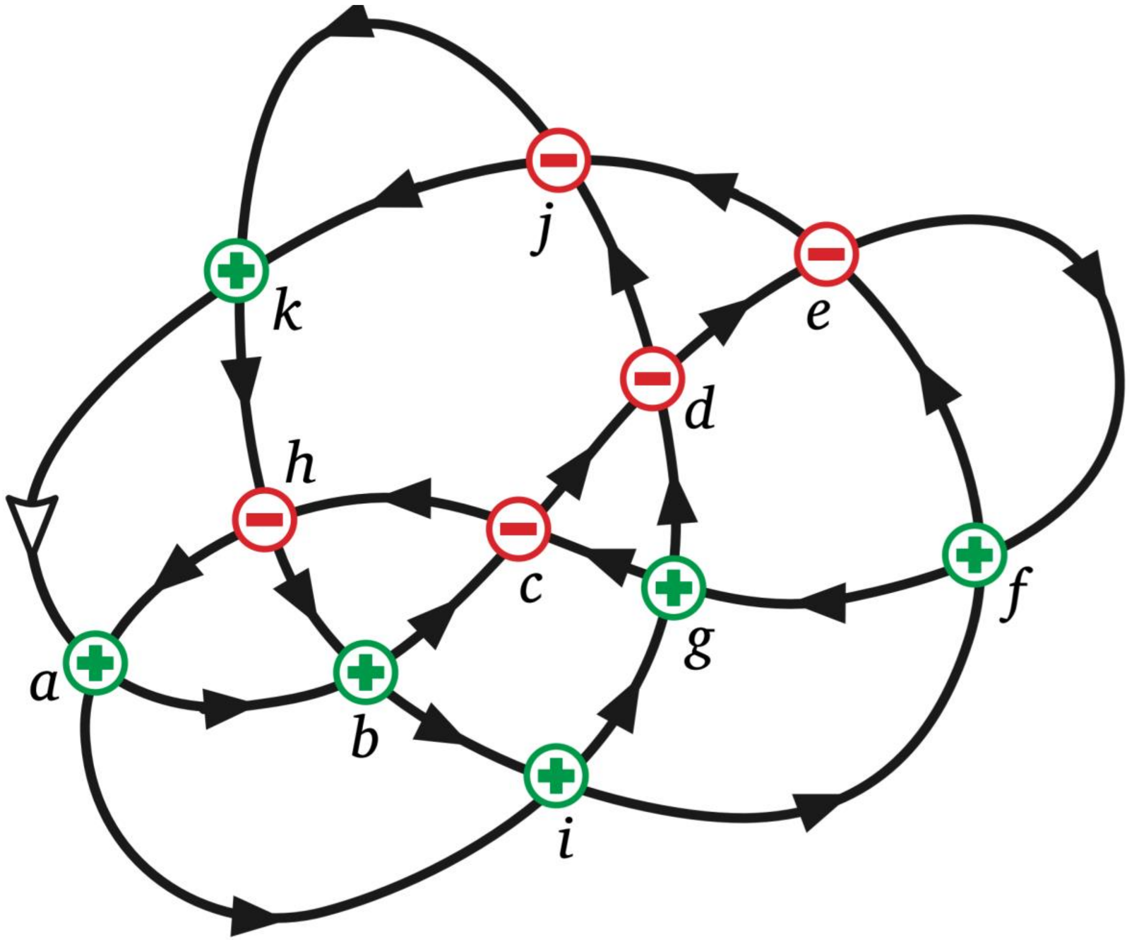
Food for thought.

WIND_q and Rot are really the same. Why?

Combinatorics of Curves

Q. Is the $O(n^2)$ Bound Tight?

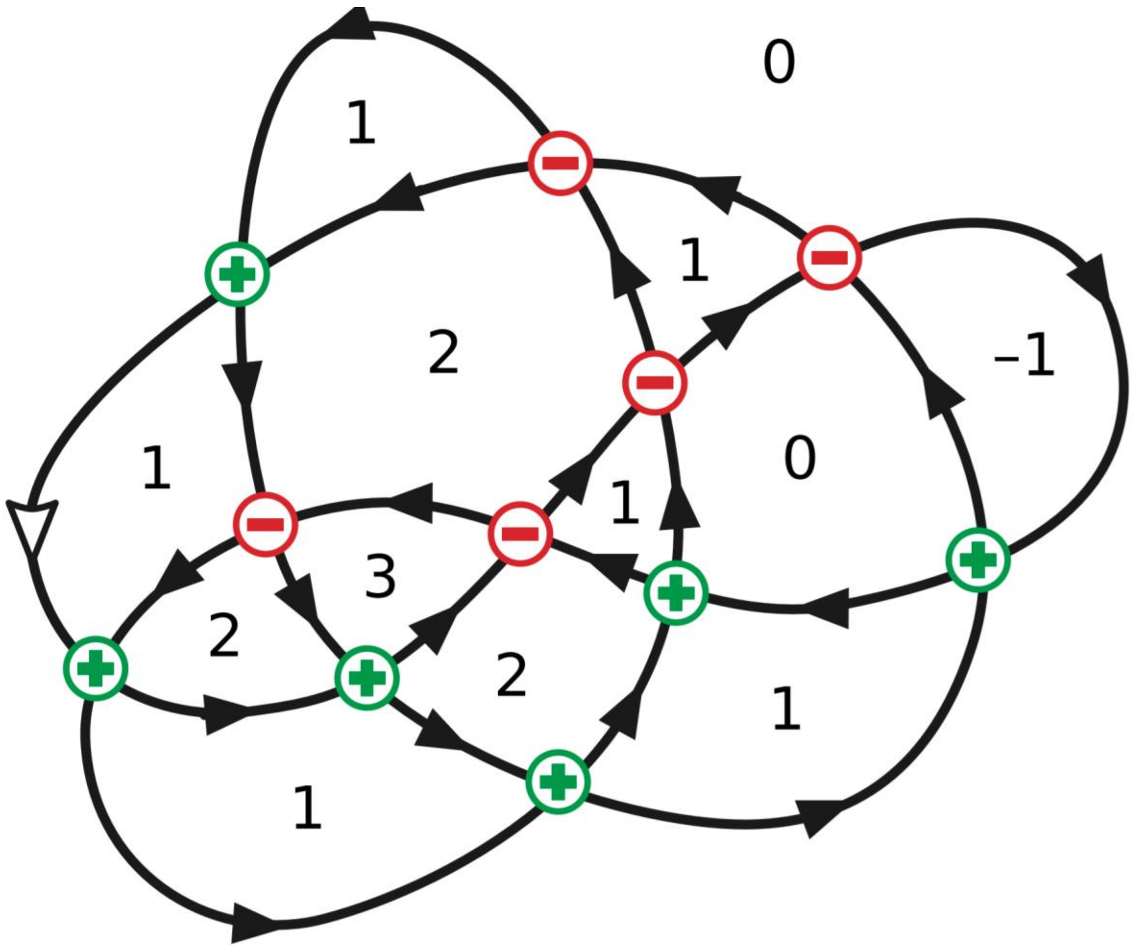
Gauss Signing and Writhe



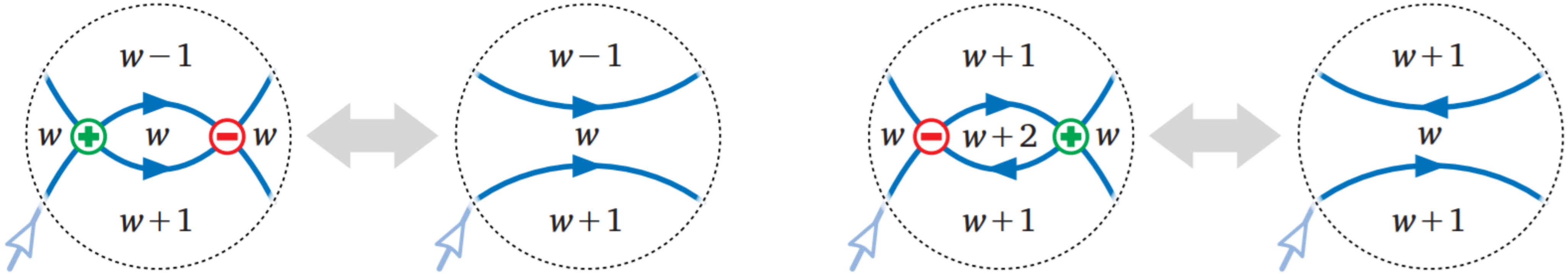
Proposition. $\mathrm{Rot}(C) = 2\mathrm{Wind}_{\nabla}(C) + \mathrm{Writhe}(C).$ [Titus 1960] [Gauss ~1823]

STRANGENESS [Arnaud 1994]

- $\mathrm{St}(C) = \sum_{\text{vertex } x} \mathrm{sgn}(x) \cdot \mathrm{Wind}_x(C)$



Theorem. Some generic curves require $\Omega(n^2)$ regular homotopy moves to untangle. [Nowik 2009]



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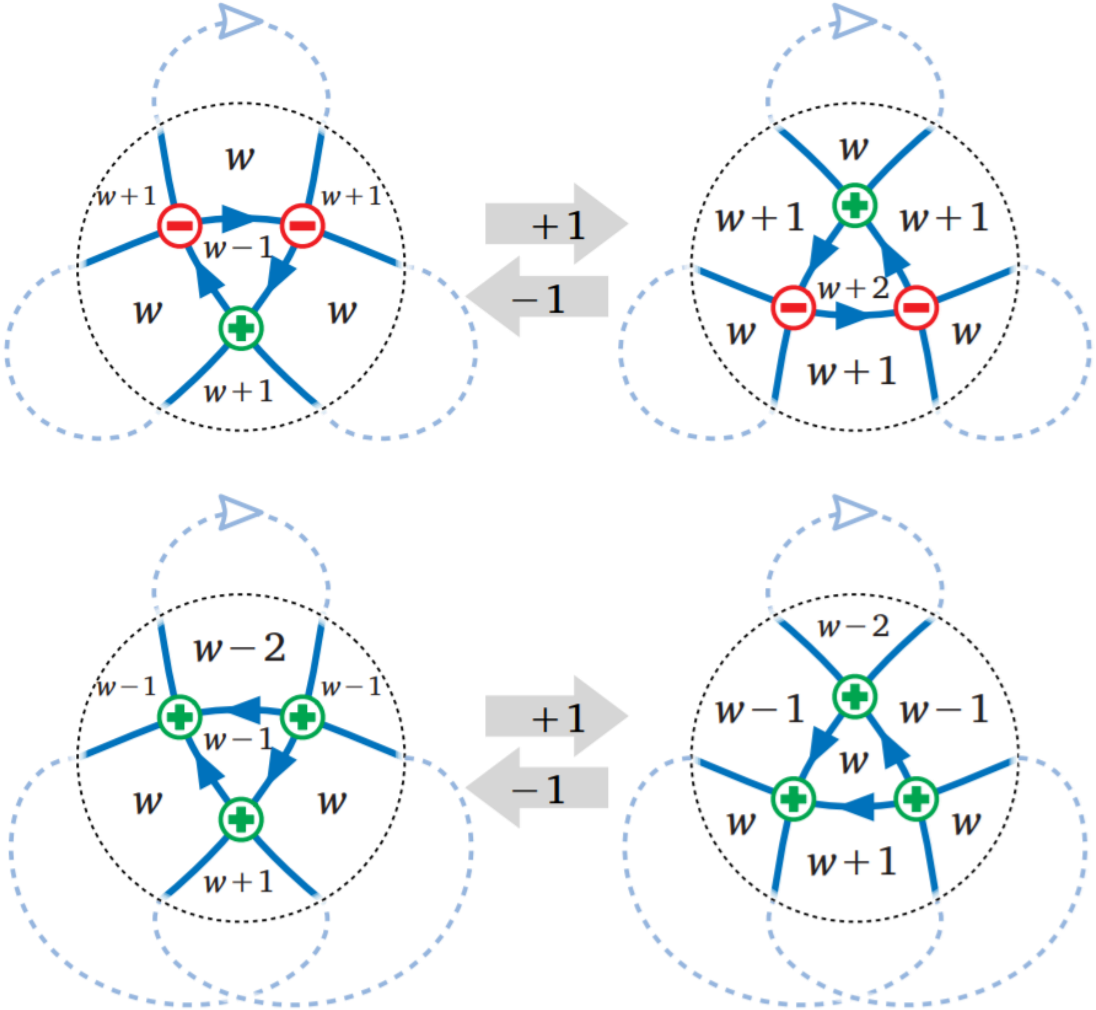
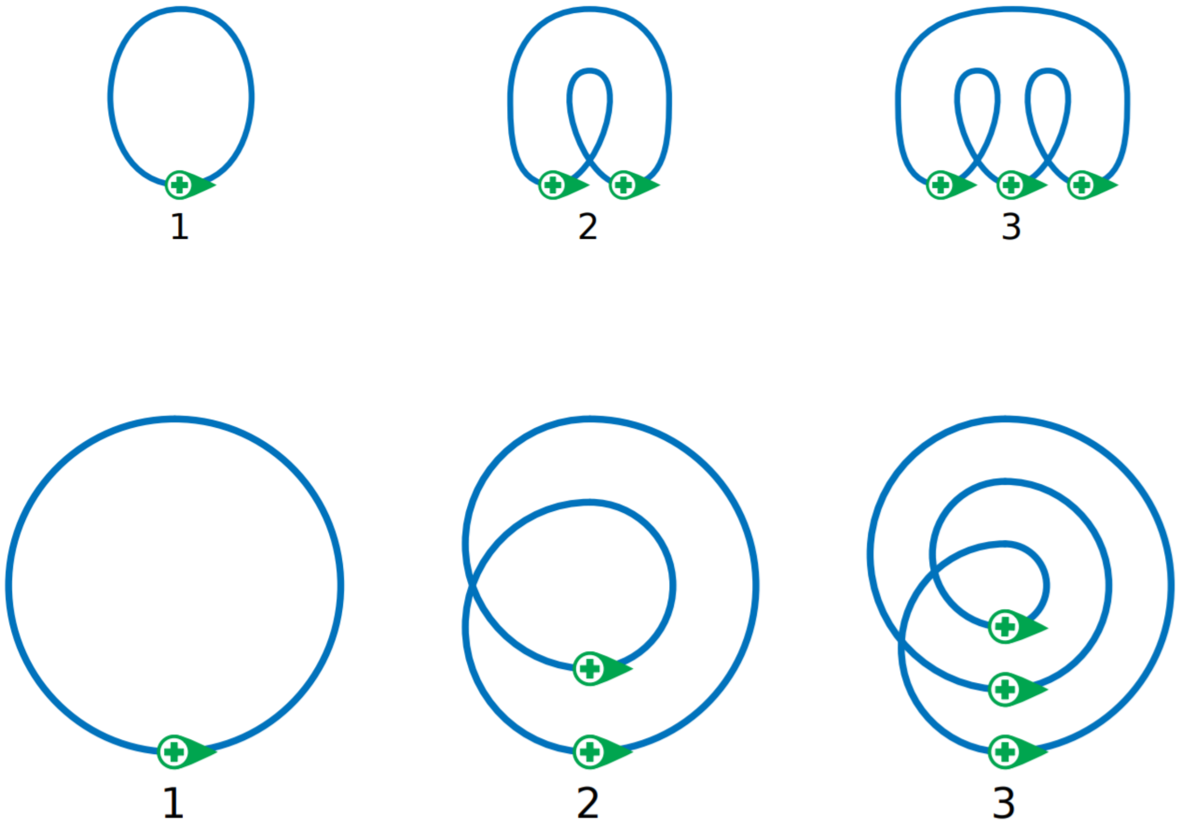


Canonical Curves

- $\mathrm{St}(C) = \sum_{\text{vertex } x} \mathrm{sgn}(x) \cdot \mathrm{Wind}_x(C)$



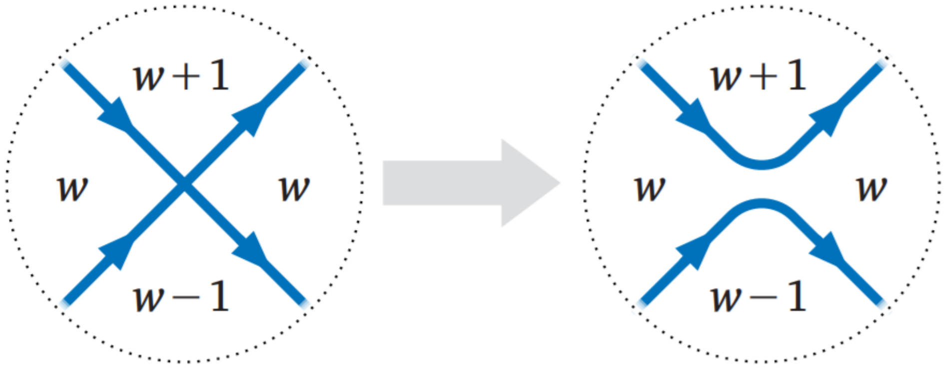
Closing Q. Canonical Curves in Pacman space?

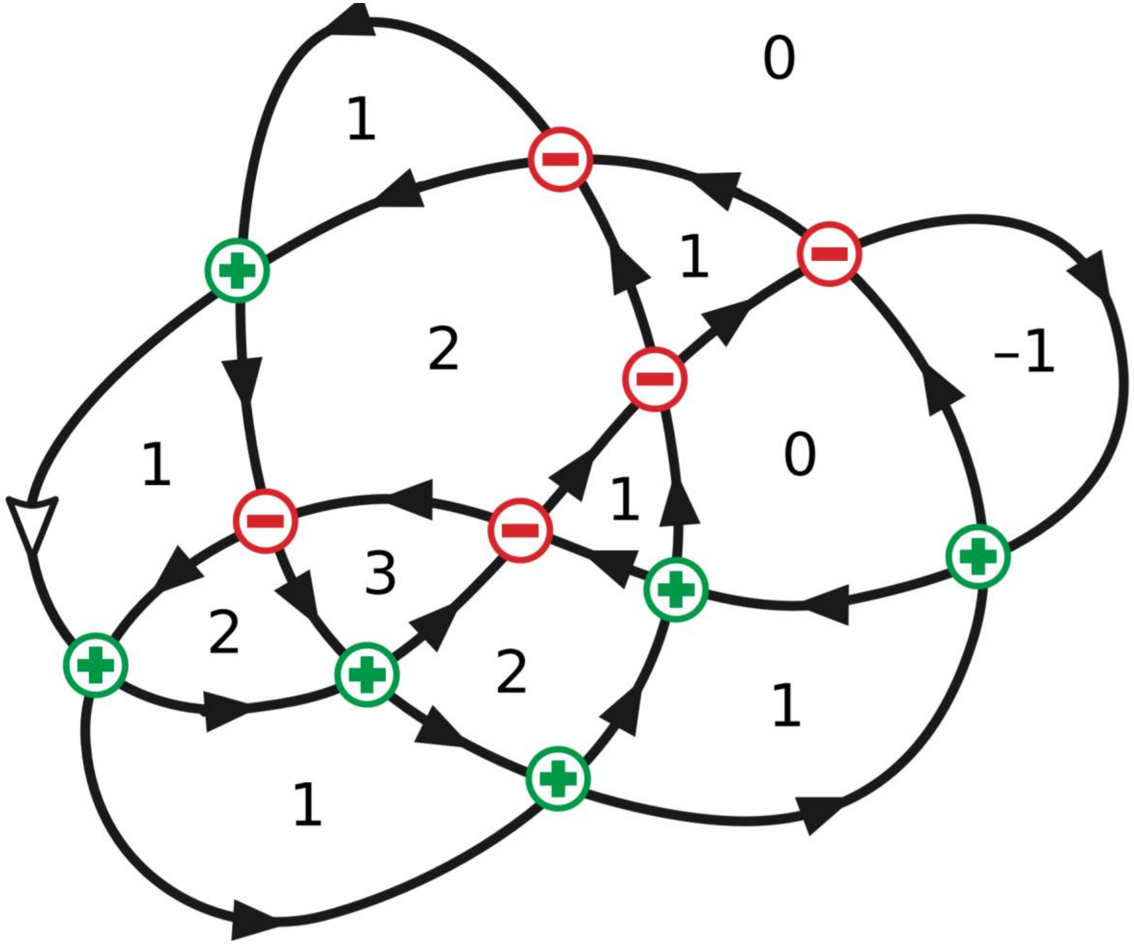
Coming up next week.

Going upwards, one-dimension Higher.

Smoothing and Seifert Decomposition

[Seifert 1931]
[Gauss ~1823]





Smoothing and Seifert Decomposition

[Seifert 1931]
[Gauss ~1823]