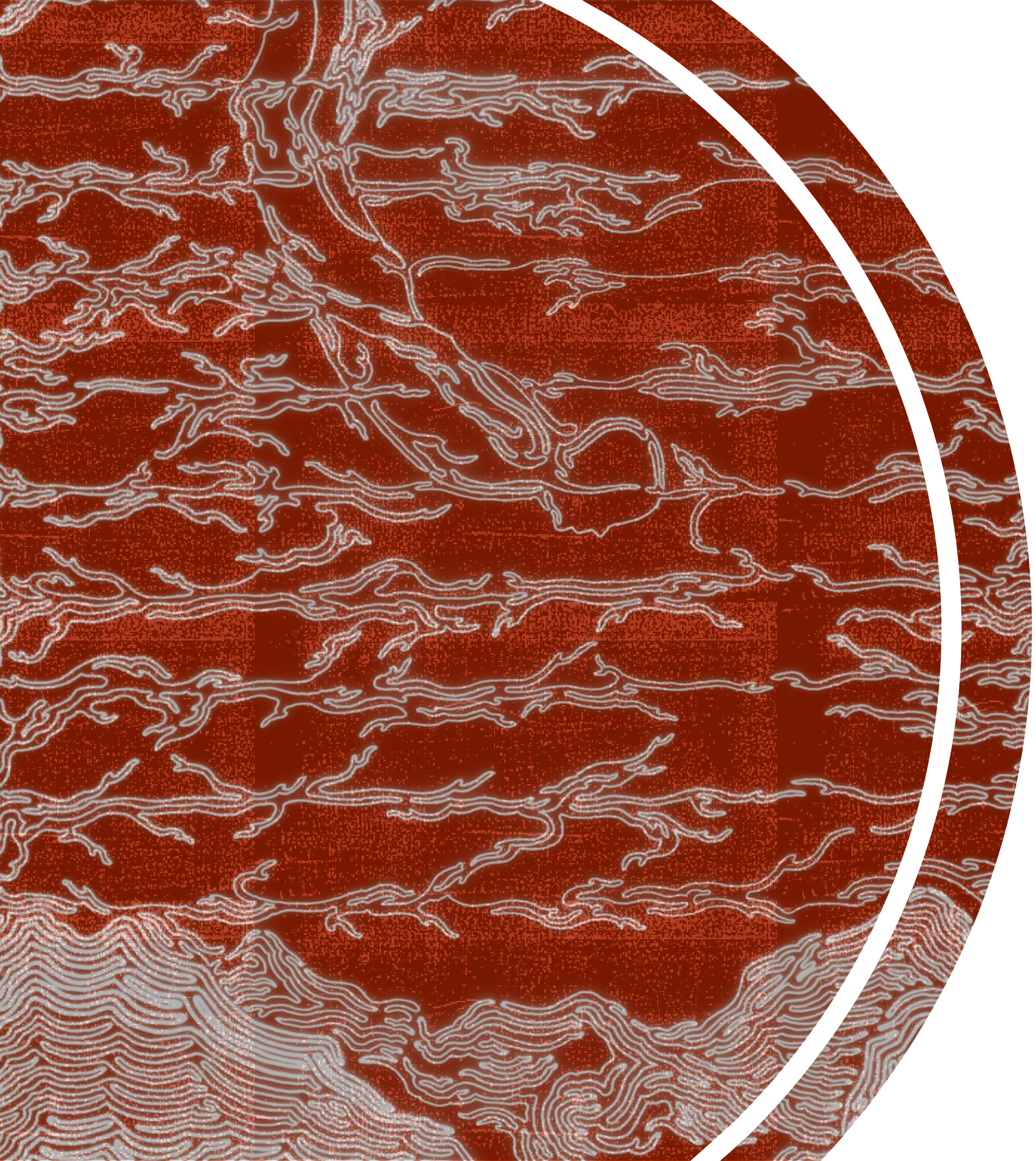


Introduction to Computational Topology

Hsien-Chih Chang

Lecture 3, September 21, 2021

Administrivia

- Homework 1 is due 9/27 (next Monday)
  - Starting from Homework 1, group submission up to 2 people

Surfaces (2d Manifolds)

What is a Surface?

- Formally, a surface (without boundary) is

A Haussdorff 2nd-countable topological space,
that is locally homeomorphic to the plane.

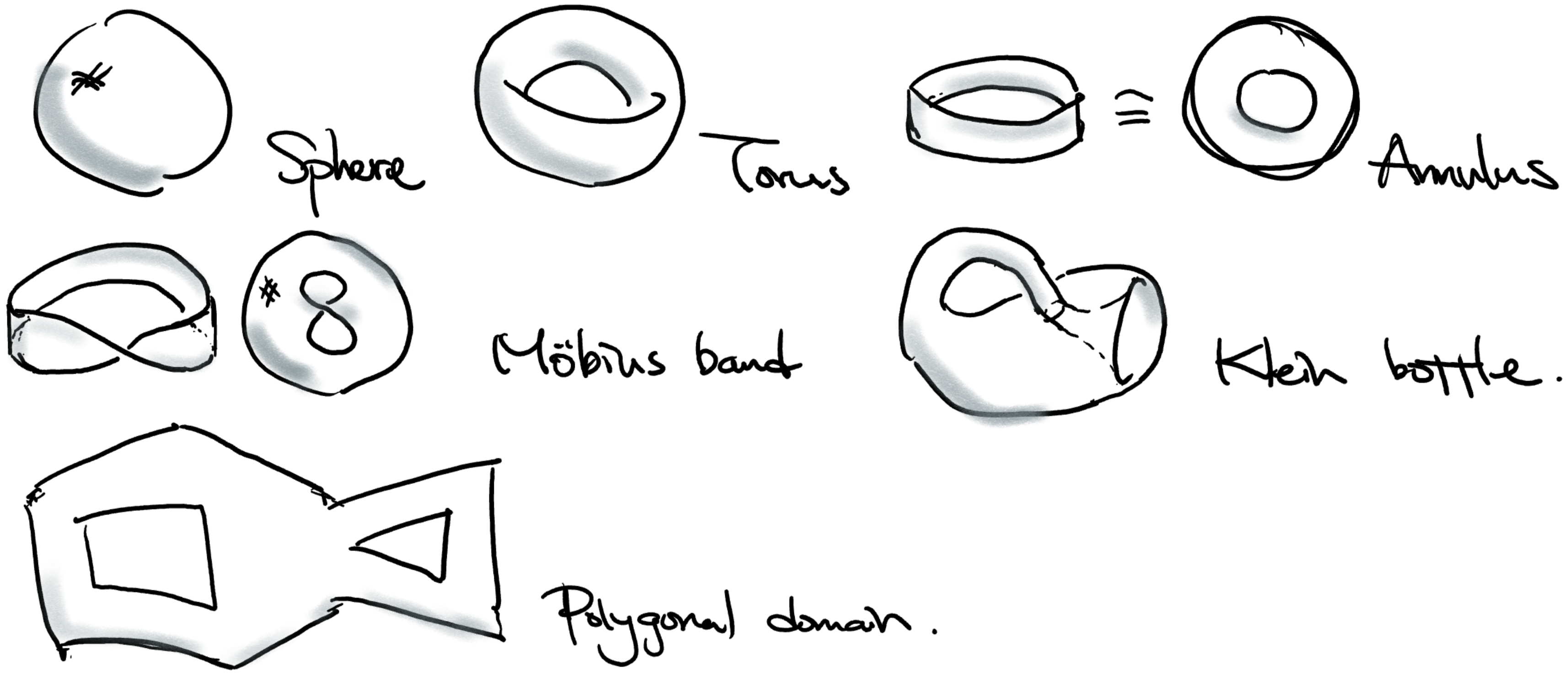
What is a Surface?

- Formally, a surface (with boundary) is

A Haussdorff 2nd-countable topological space,
that is locally homeomorphic to the plane or the half-plane.

Mythic Creature Exhibition

Mythic Creature Exhibition

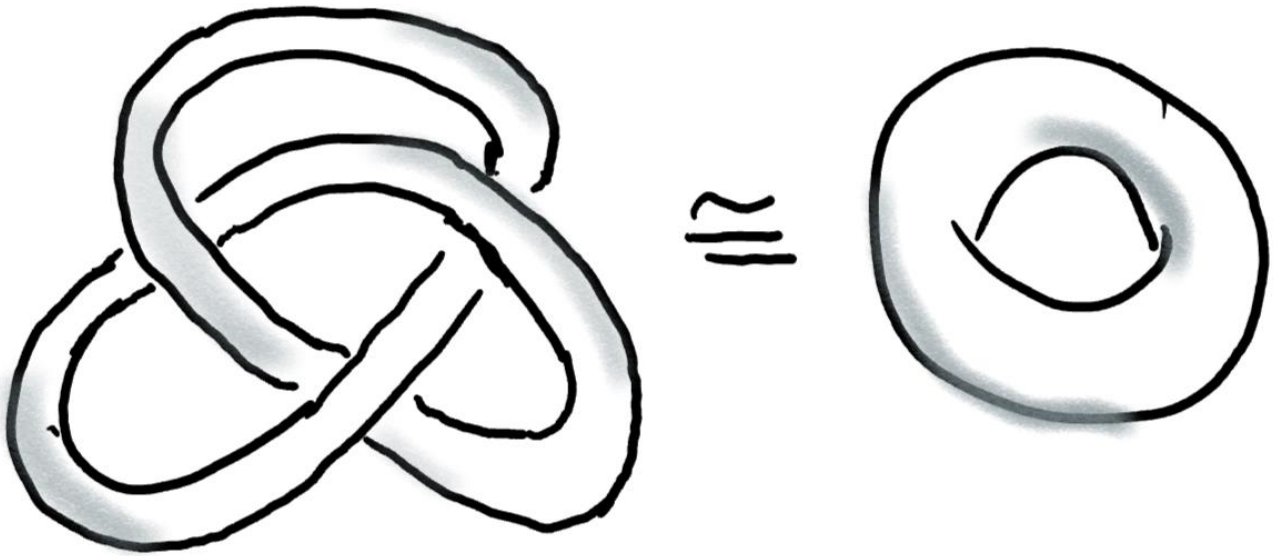


Tech-spec of the Fabric

- Bendable and stretchable

Tech-spec of the Fabric

- Bendable and stretchable
- Can phase-through itself

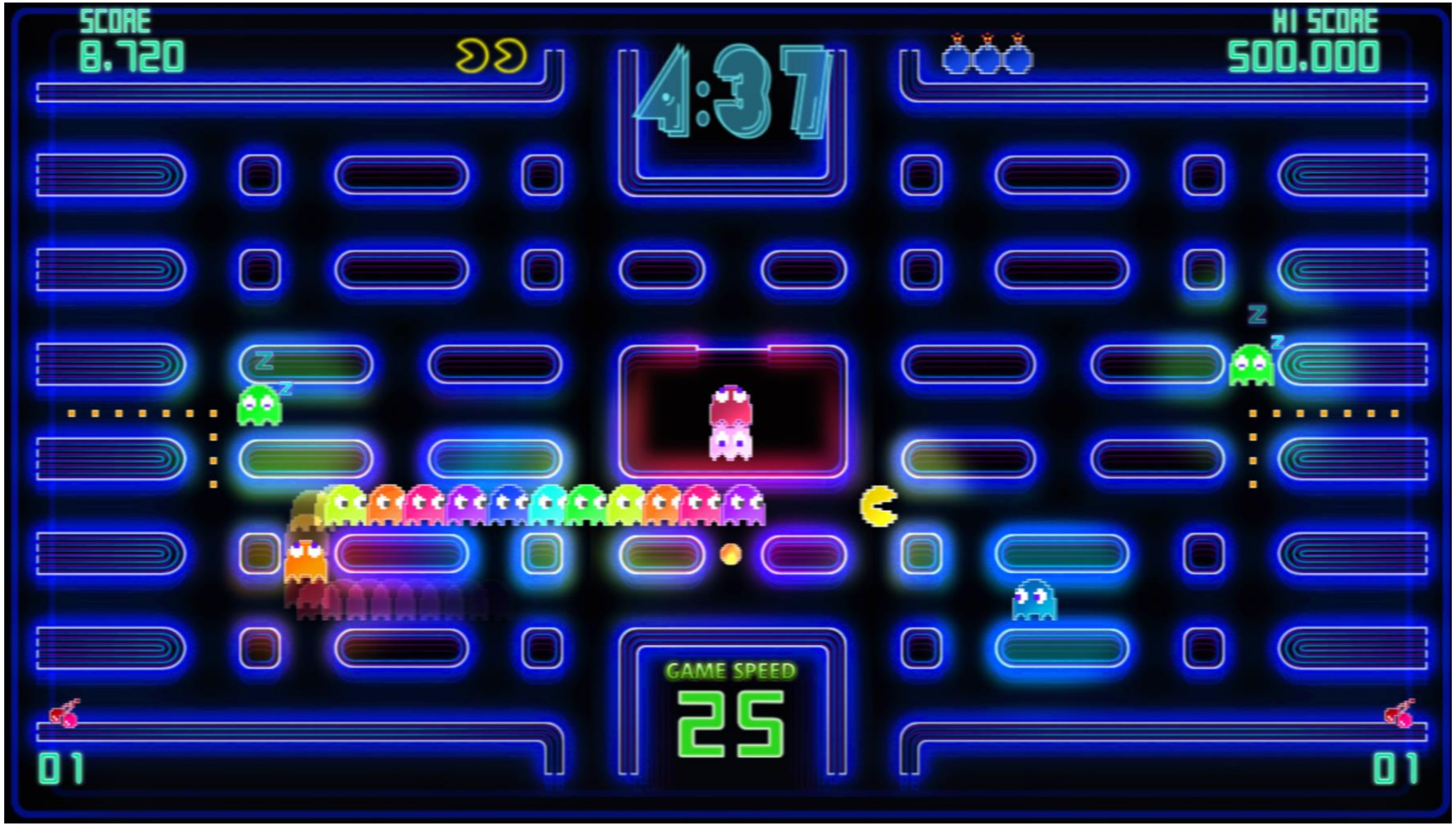


Tech-spec of the Fabric

- Bendable and stretchable
- Can phase-through itself
- NOT cuttable…

Tech-spec of the Fabric

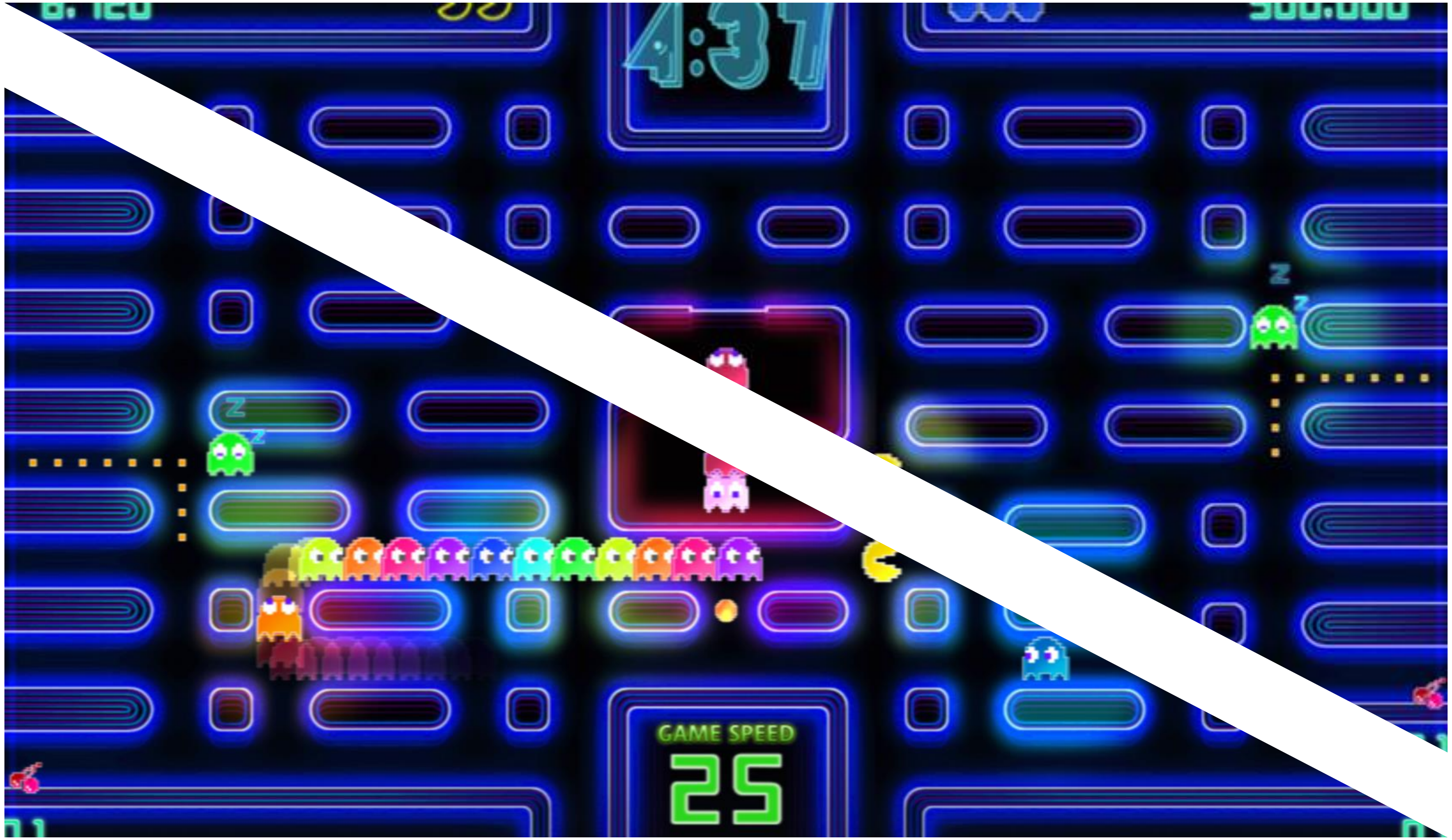
- Bendable and stretchable
- Can phase-through itself
- NOT cuttable…
  …unless you glue it back

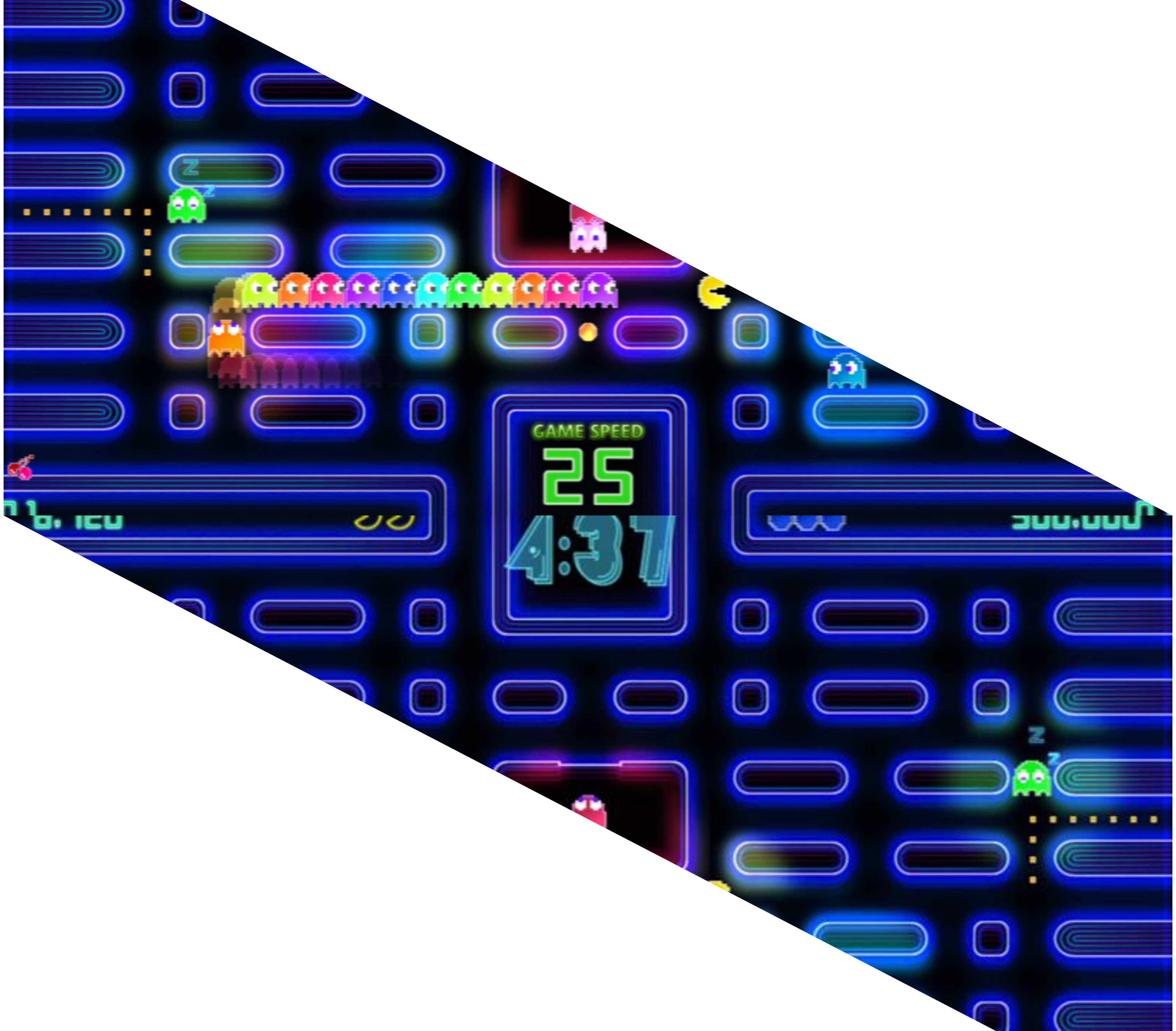


Example: Pacman Space

Example: Pacman Space

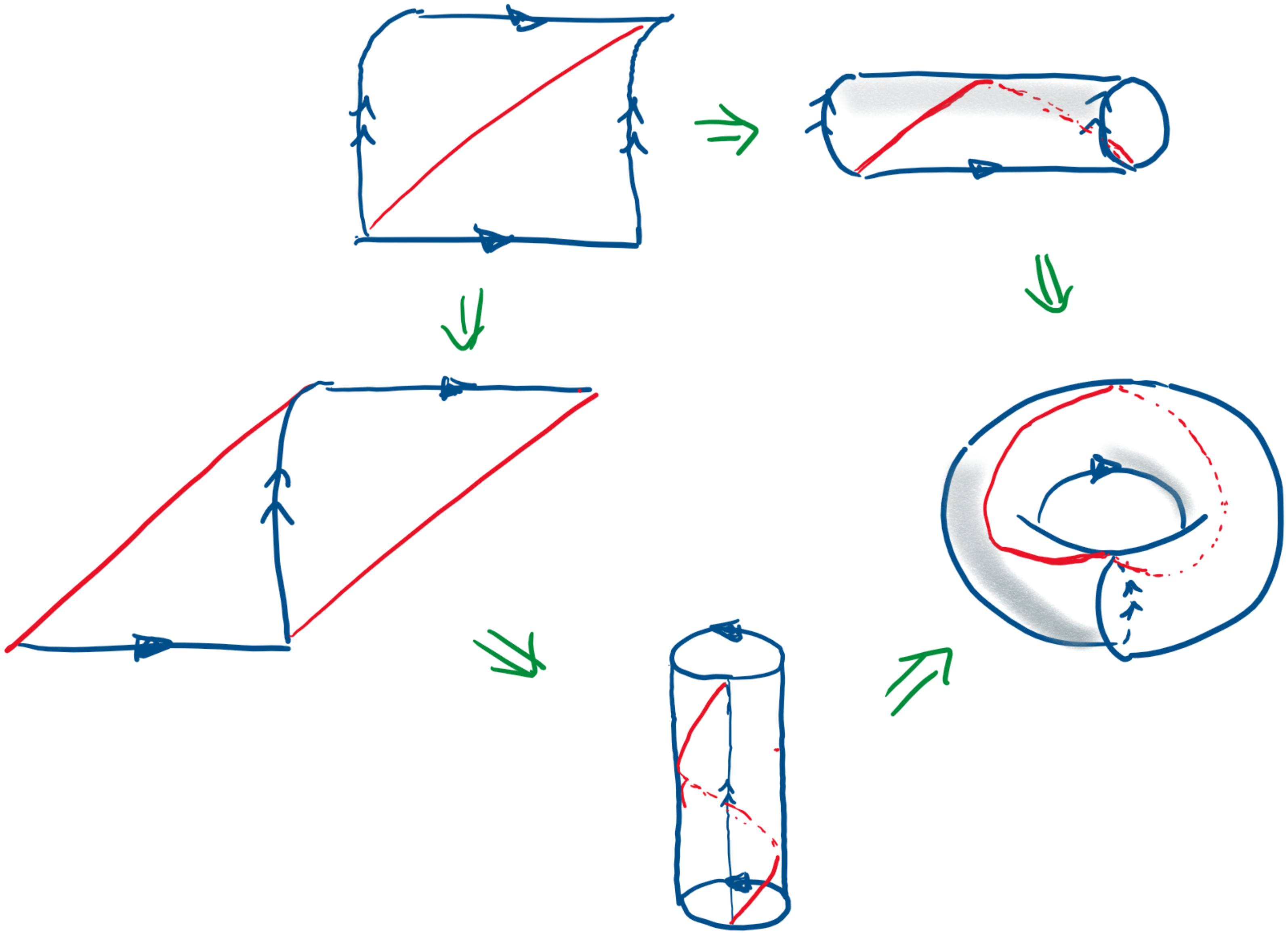
Example: Pacman Space

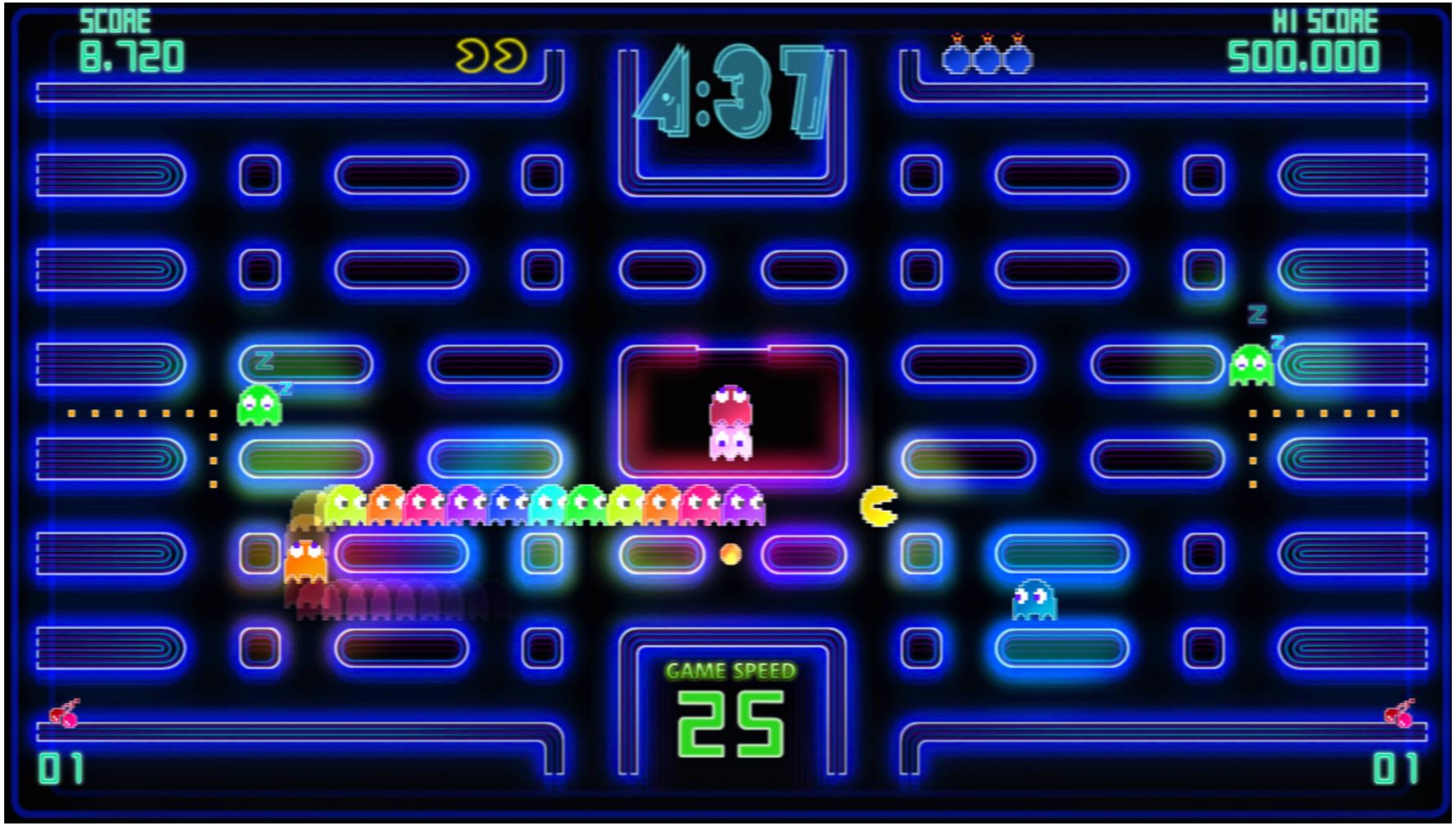




Example: Pacman Space

Example: Pacman Space





Upside-down Pacman?

Upside-down Pacman?

Exercise: What is this Surface?

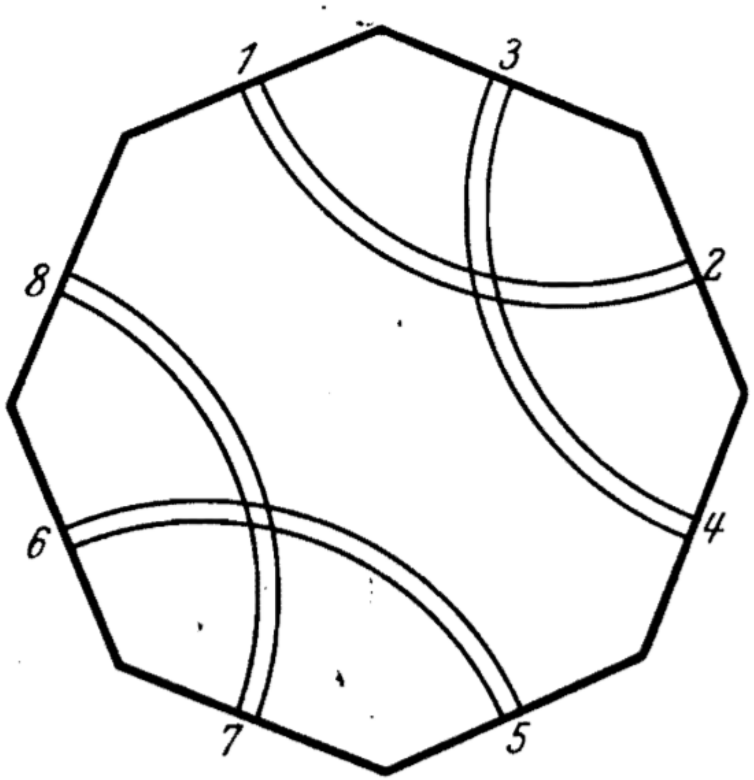


Fig. 286a

Exercise: What is this Surface?

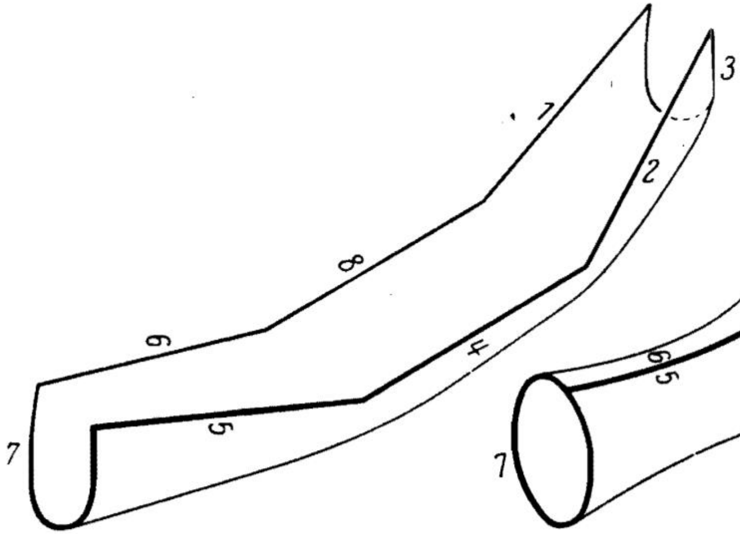


Fig. 286b

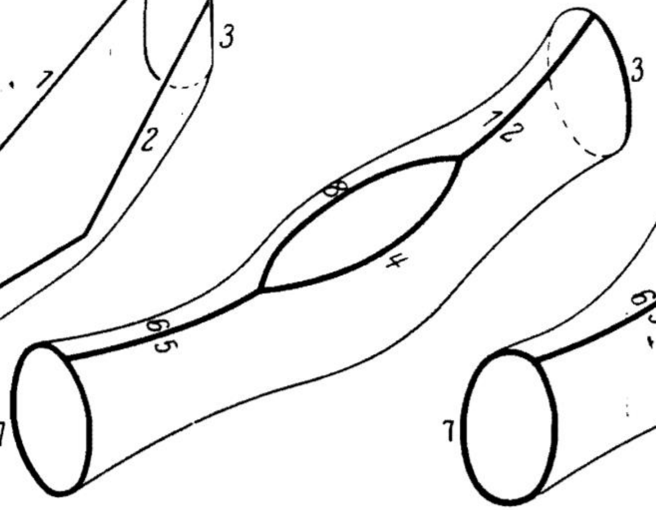


Fig. 286c

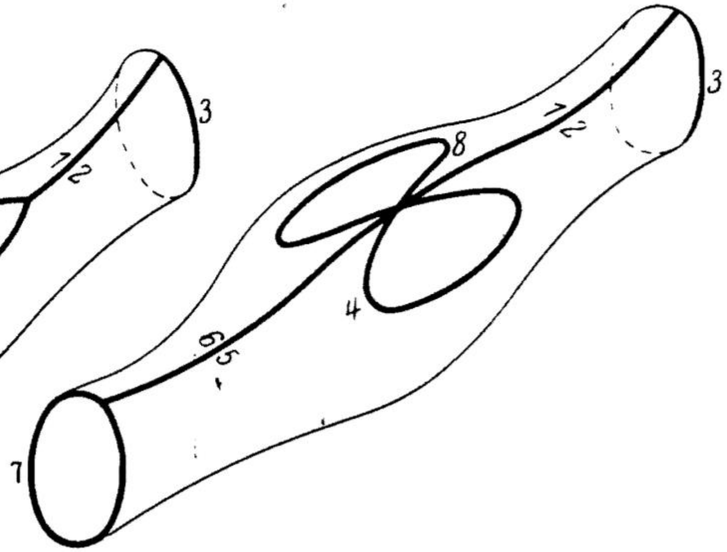


Fig. 286d

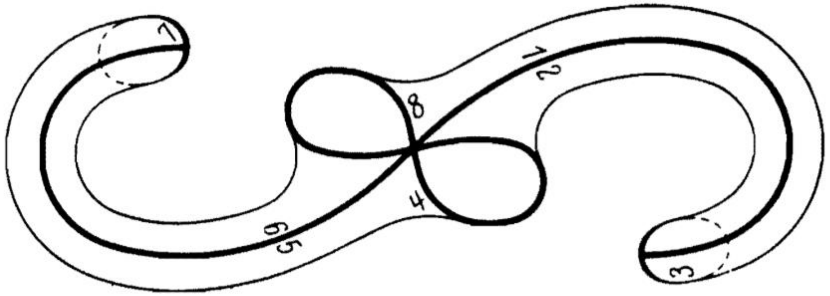


Fig. 286e

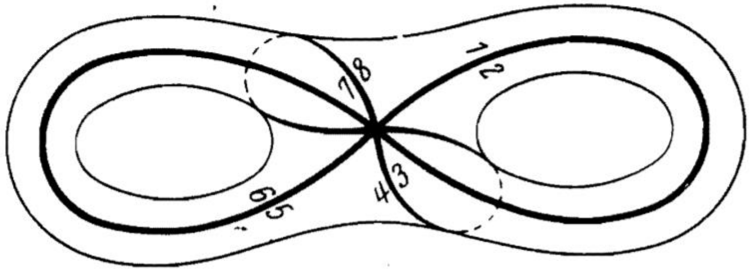
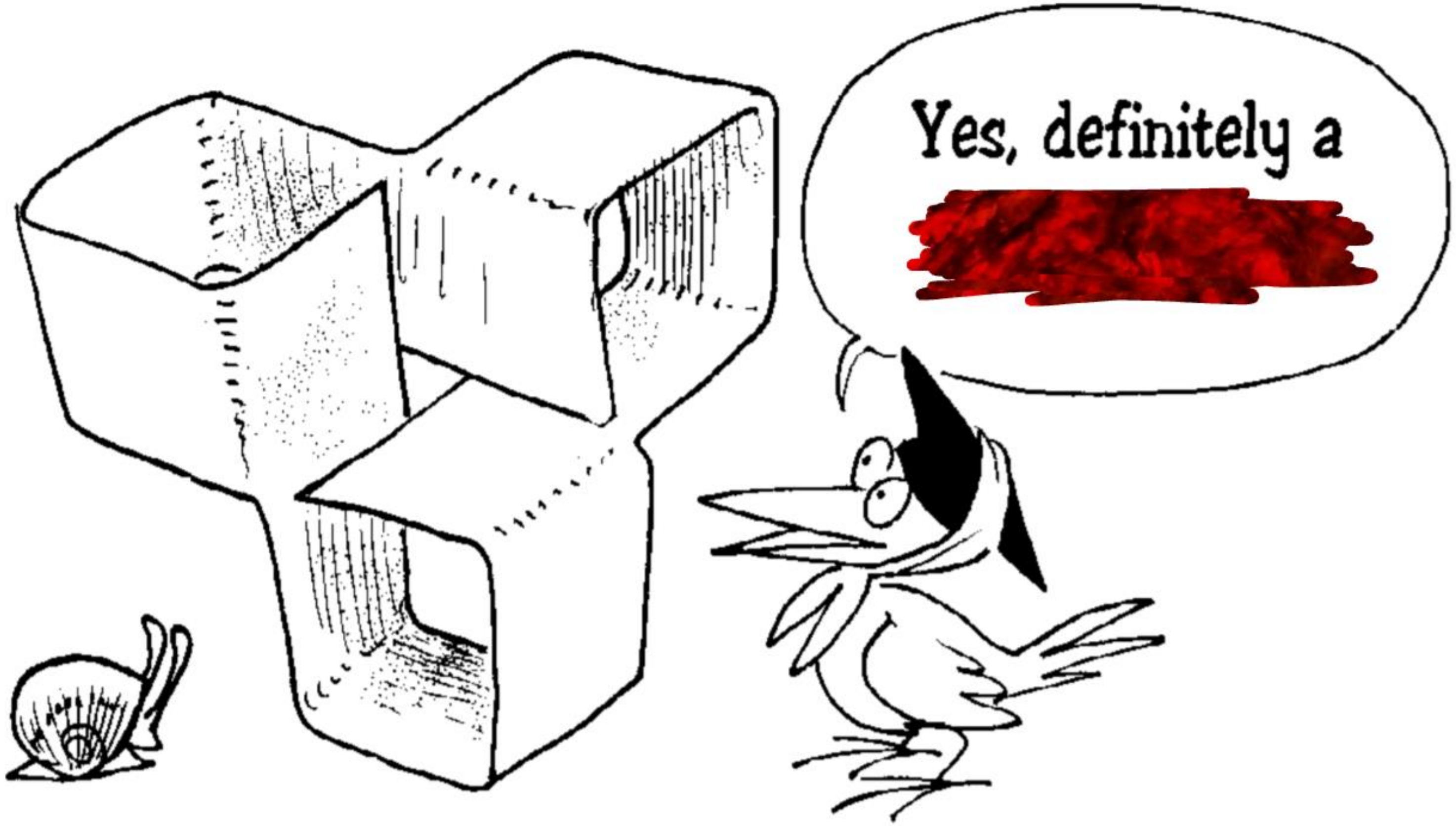
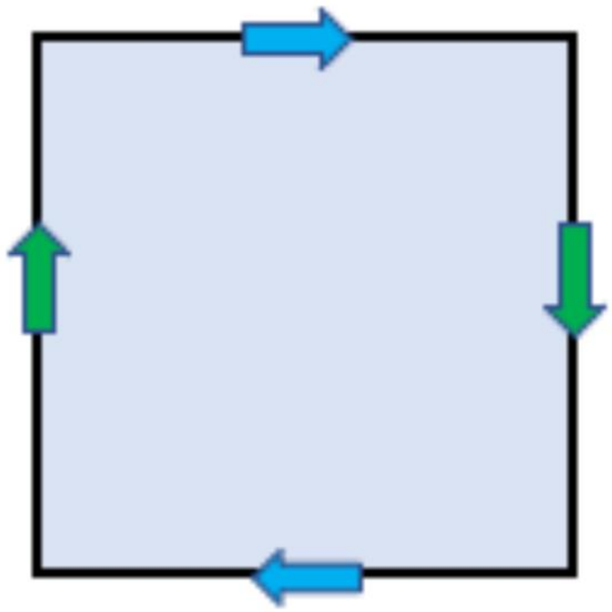


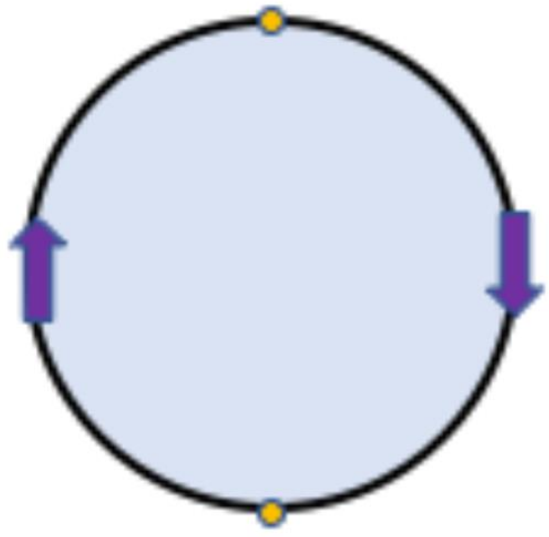
Fig. 286f

Exercise: What is this Surface?

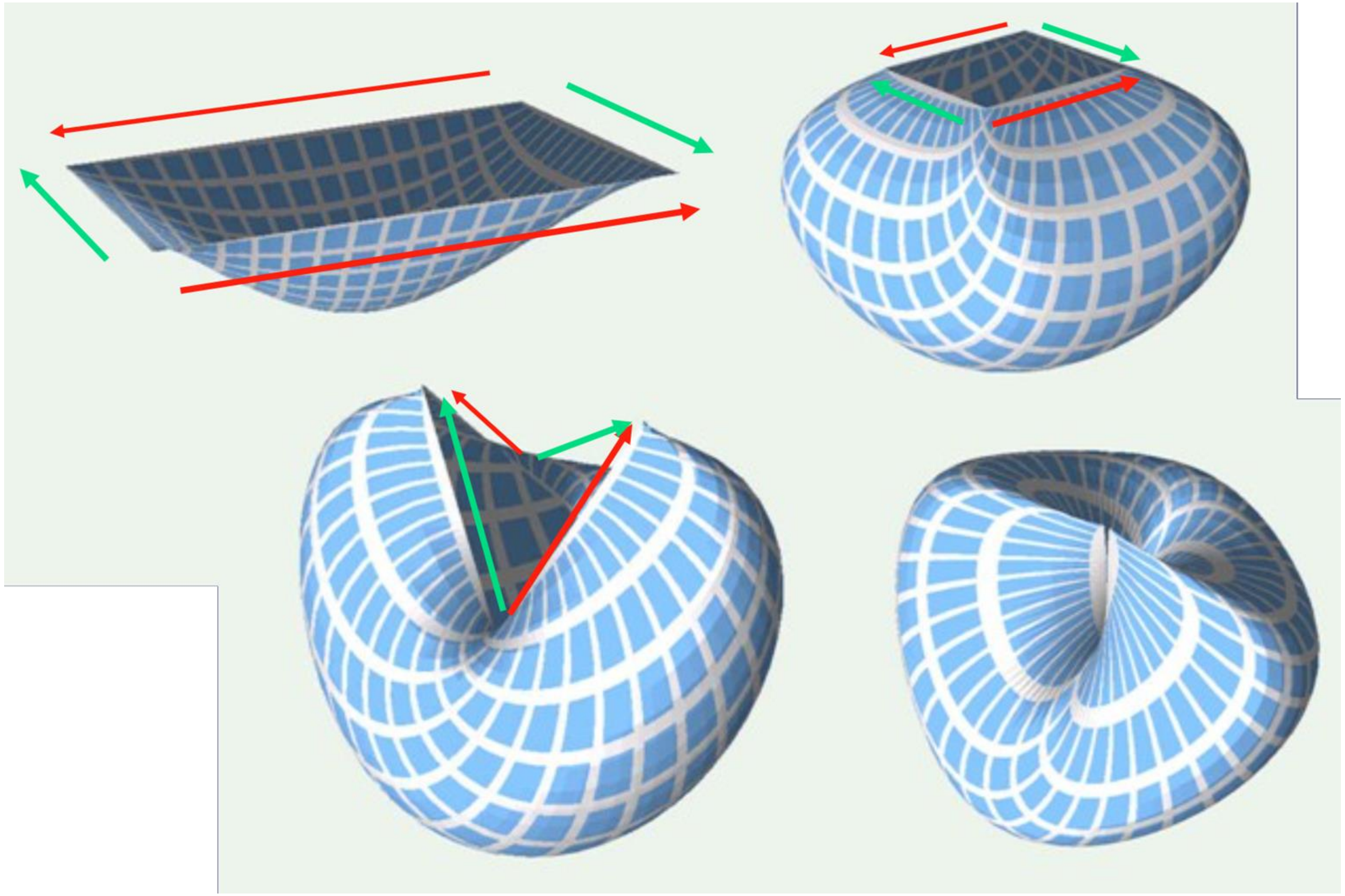


Exercise: What is this Surface?

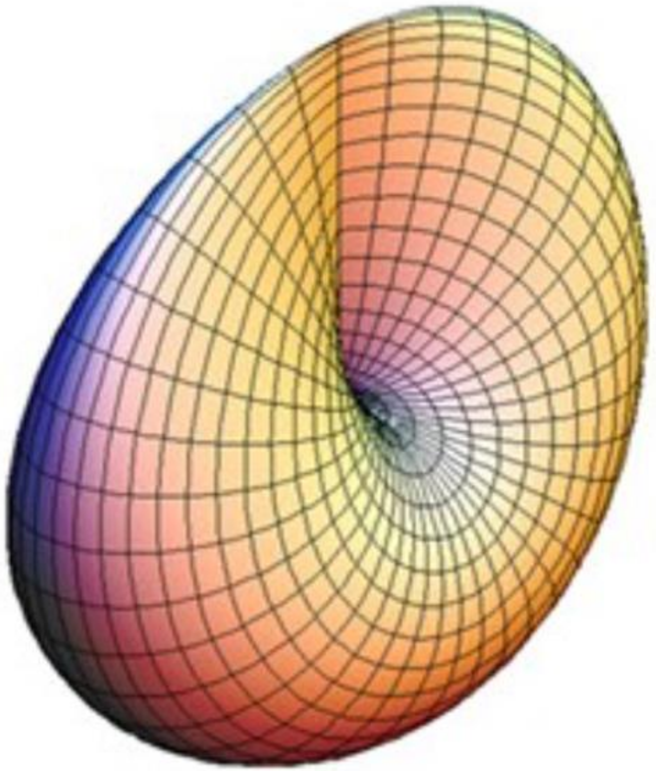


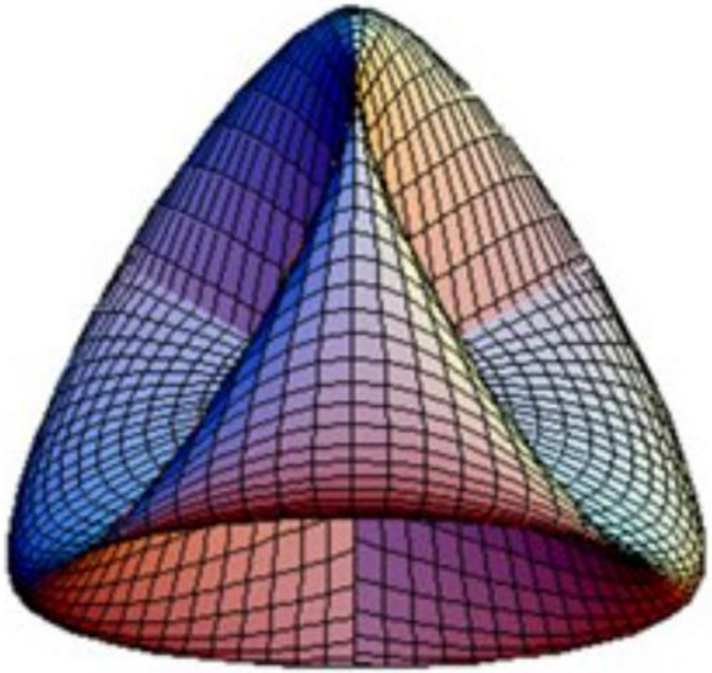


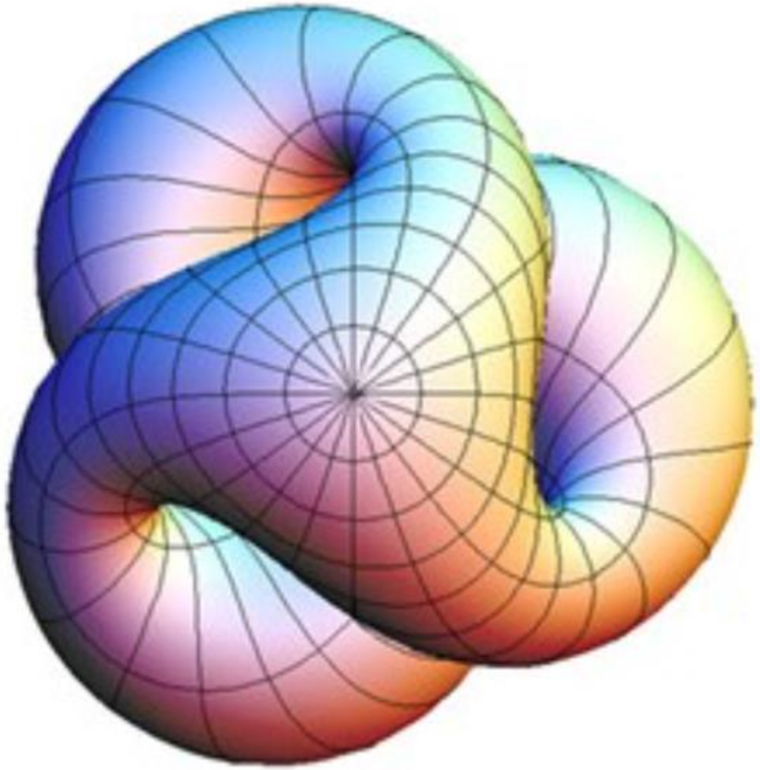
Exercise: What is this Surface?



Exercise: What is this Surface?





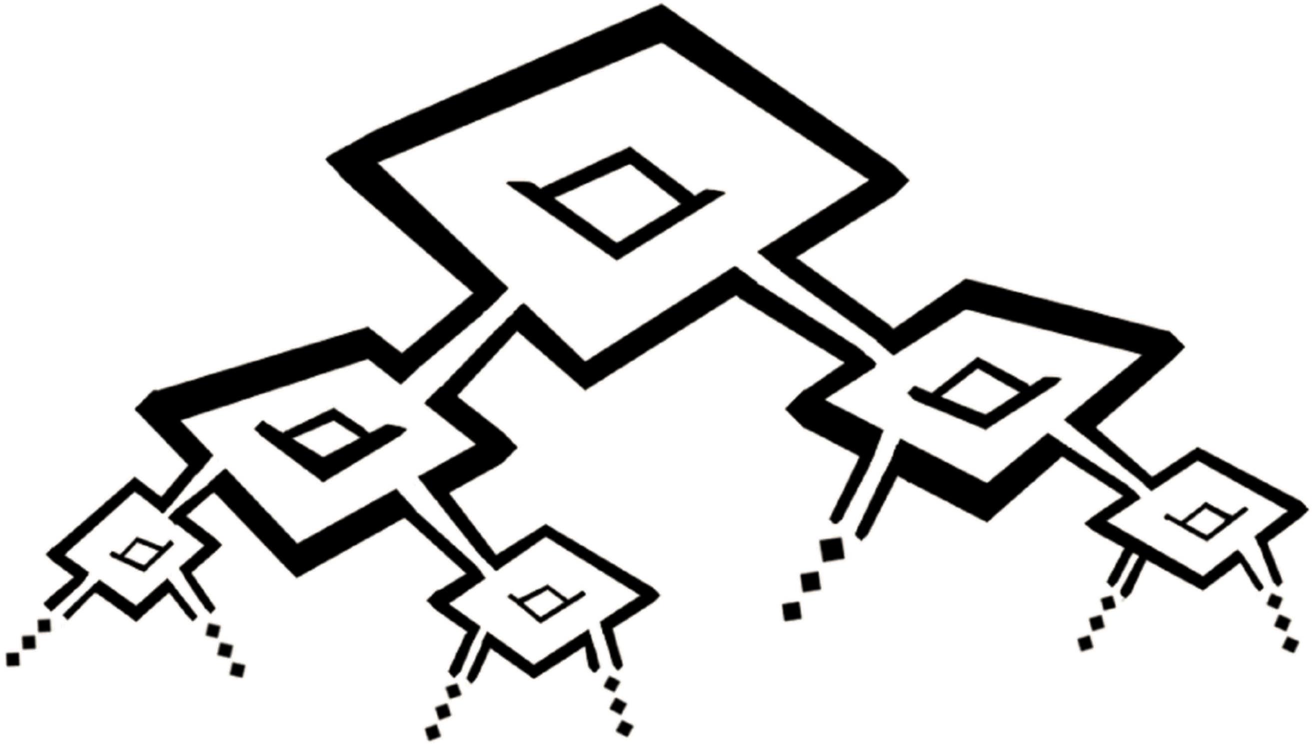


Can we get all surfaces through cut-and-paste?

Equivalence

- Homeomorphism

- Homotopy equivalence



Surface Classification

[Möbius 1861] [Dehn-Heegaard 1907] [Radó 1925]

Every connected surface is homeomorphic to the following:

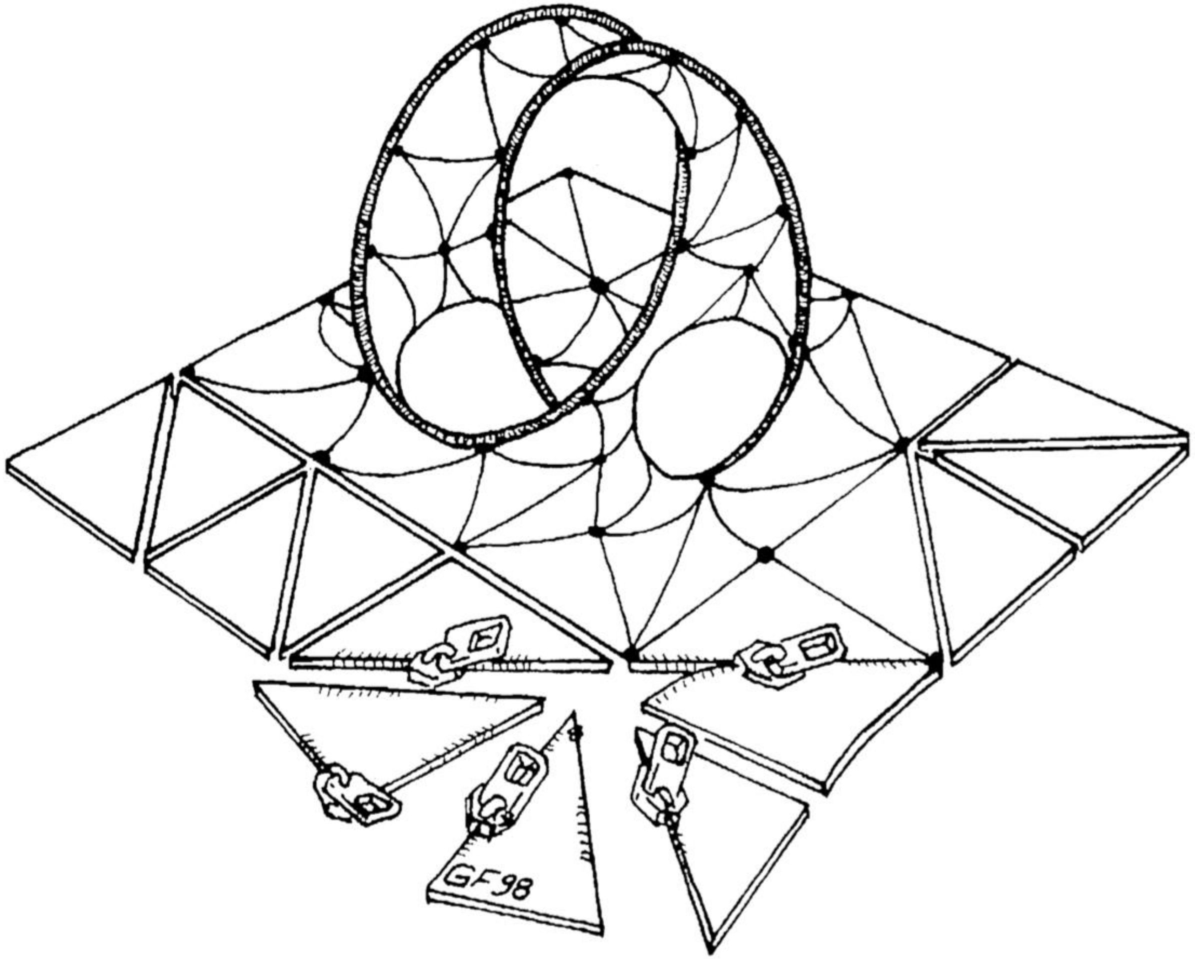
(plus boundaries)

- Sphere with g handles $\Sigma(g, 0)$
- Sphere with r cross-caps $\Sigma(0, r)$

Theorems we secretly assumed

- Triangulation Theorem [Kerékjártó-Radó 1925]
  - Any surface can be cut into triangles
- Refinement Theorem [Moise 1977]
  - Any two triangulations have a common refinement

Conway's ZIP Proof



Conway's ZIP Proof

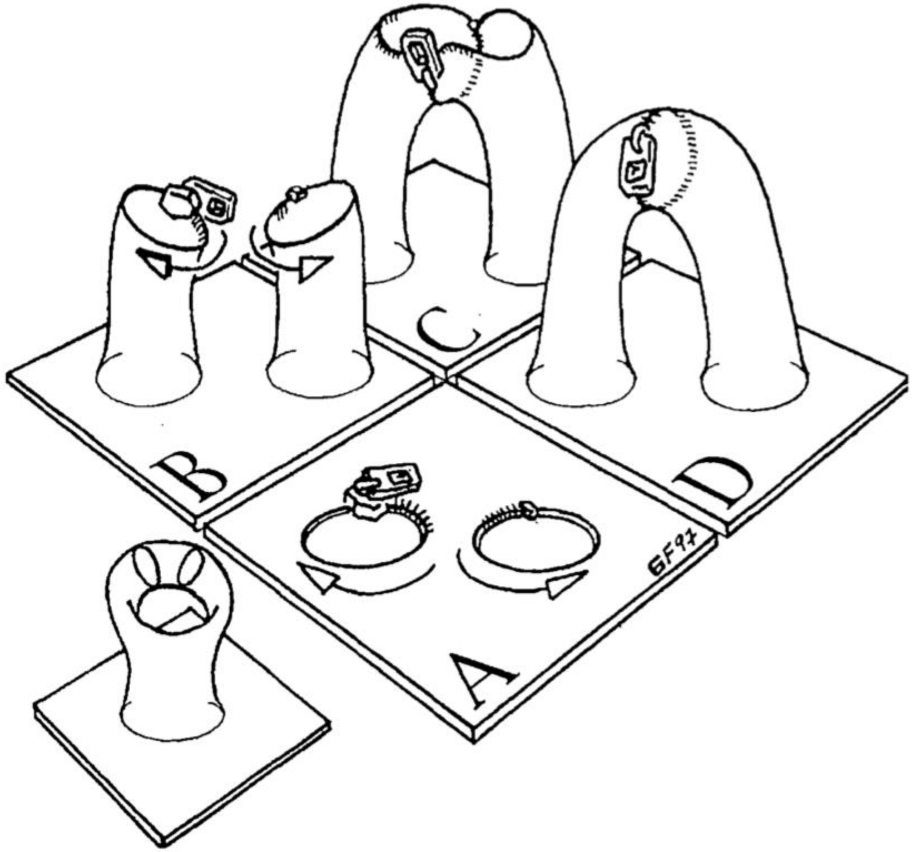


Figure 1. Handle

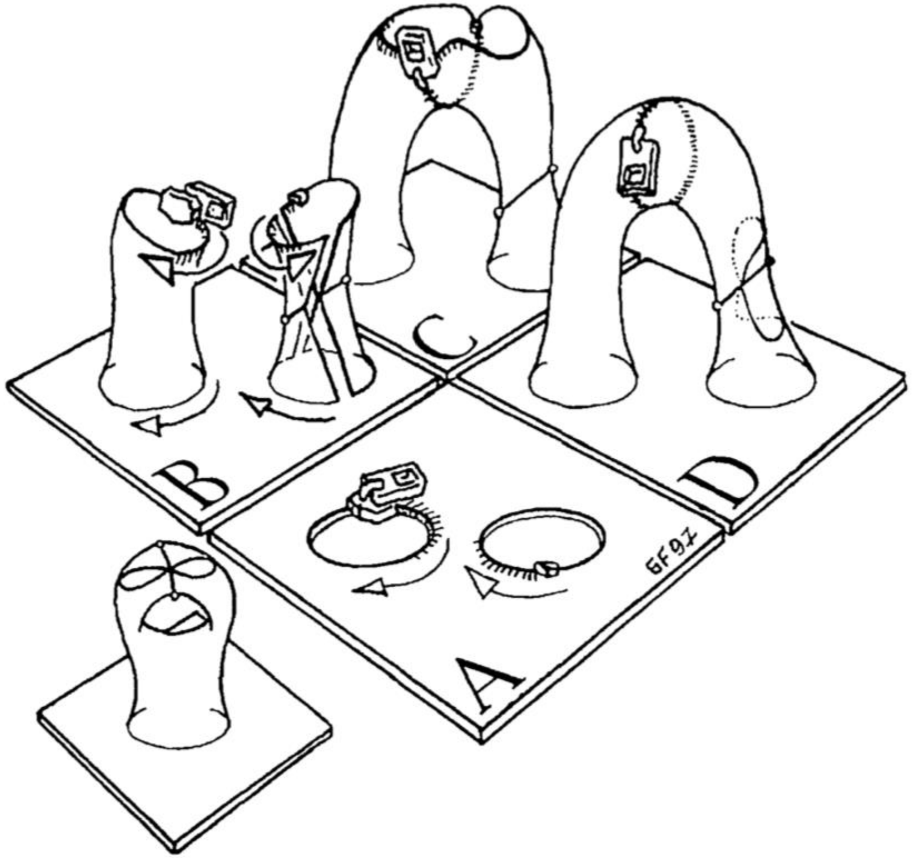


Figure 2. Crosshandle

Conway's ZIP Proof

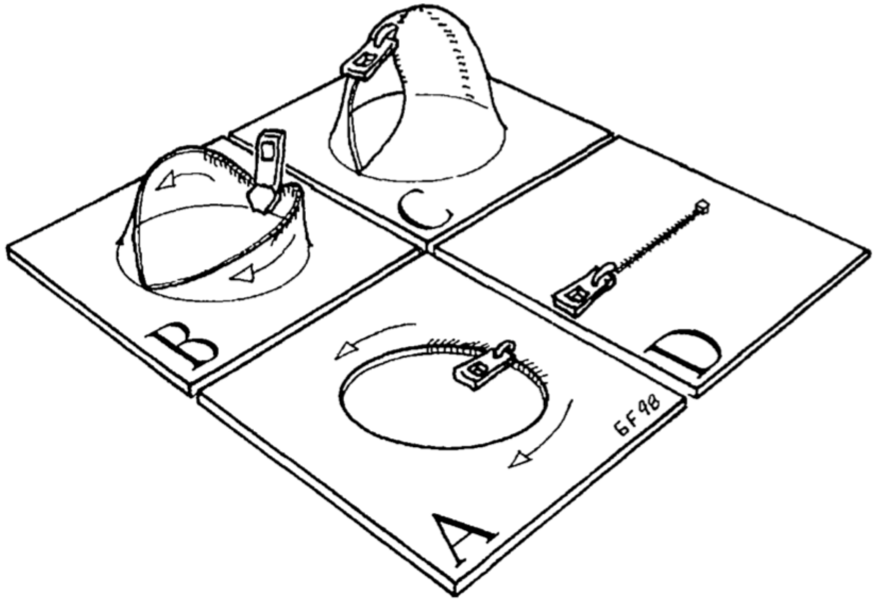


Figure 3. Cap

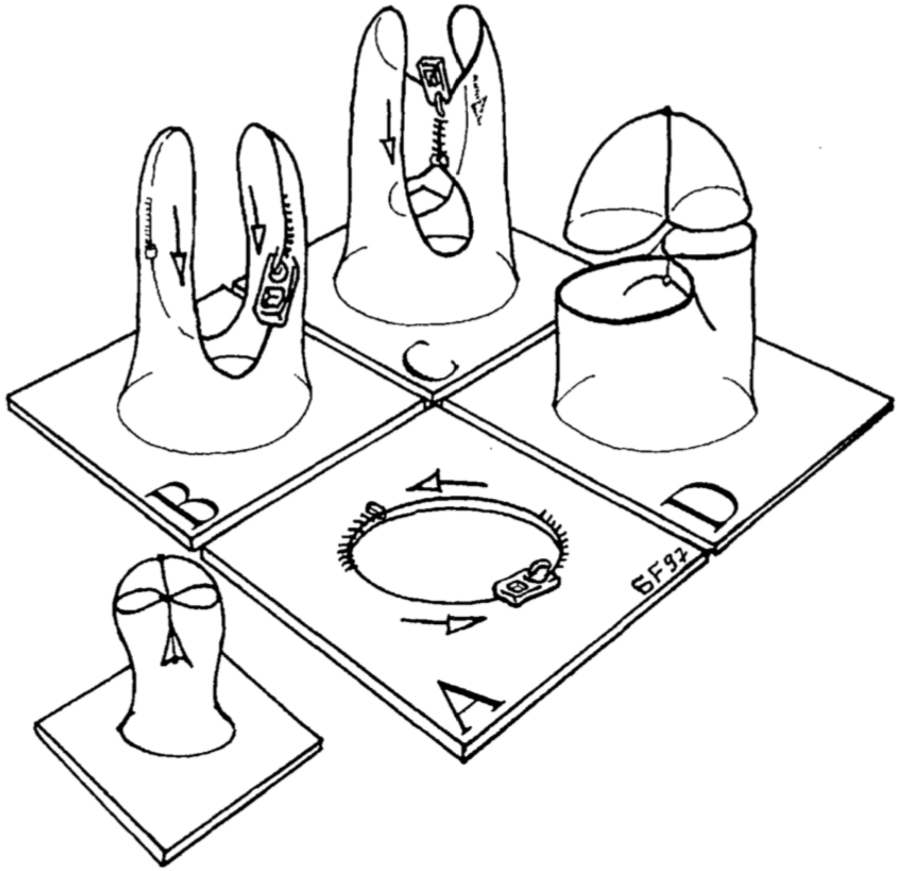
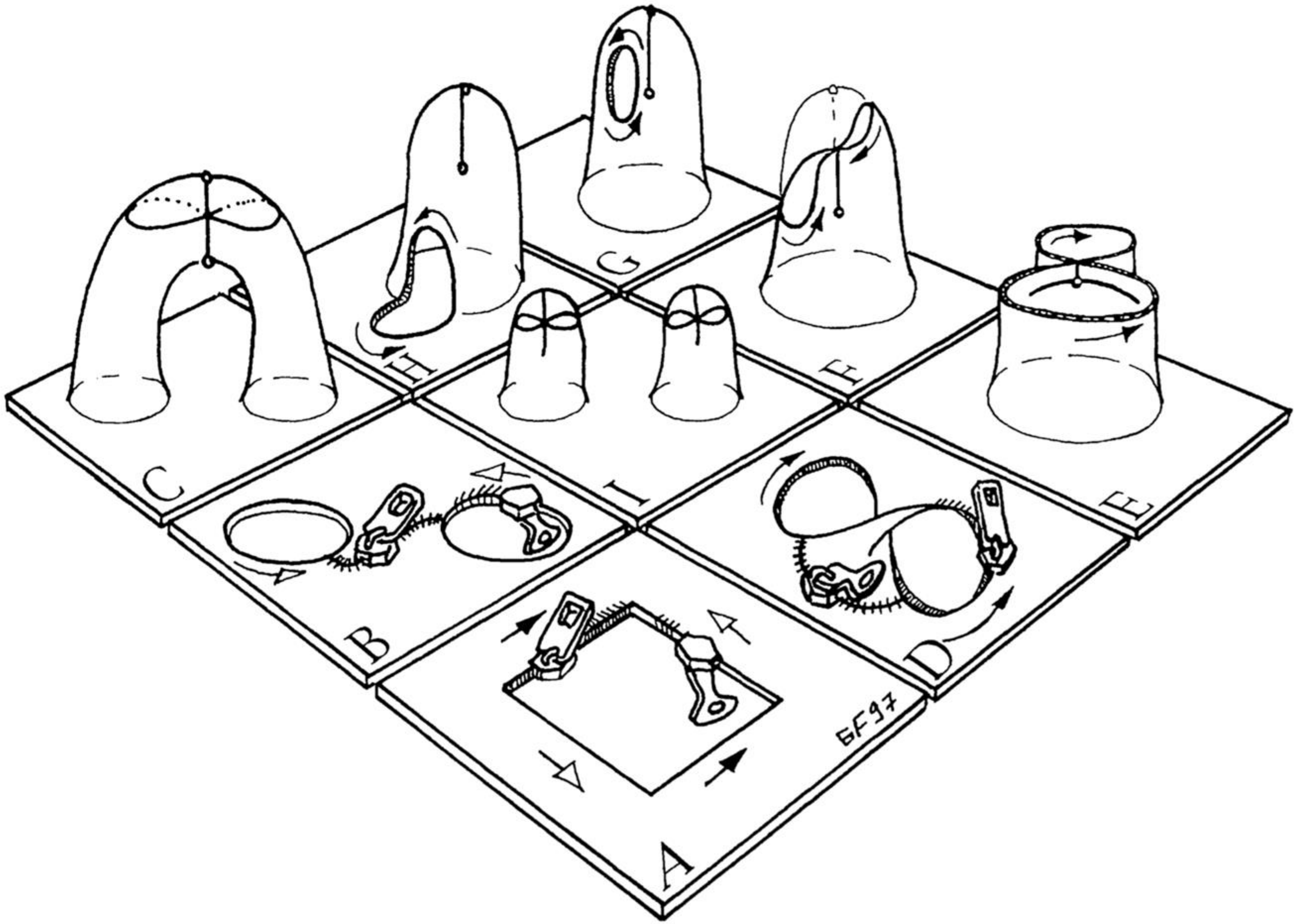
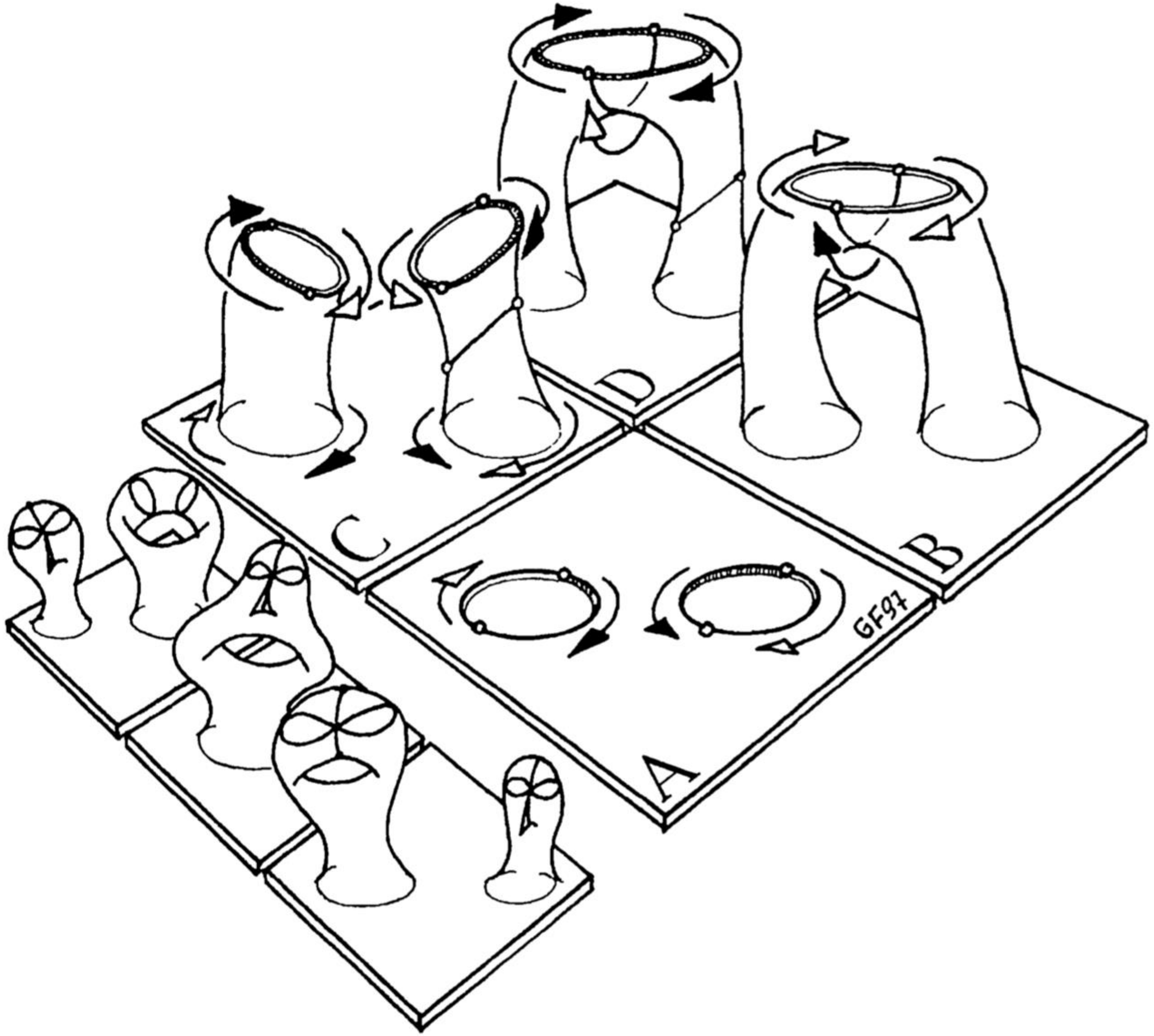


Figure 4. Crosscap



K=P#P

Exchanging two cross-caps for a cross-handle



T#P=K#P

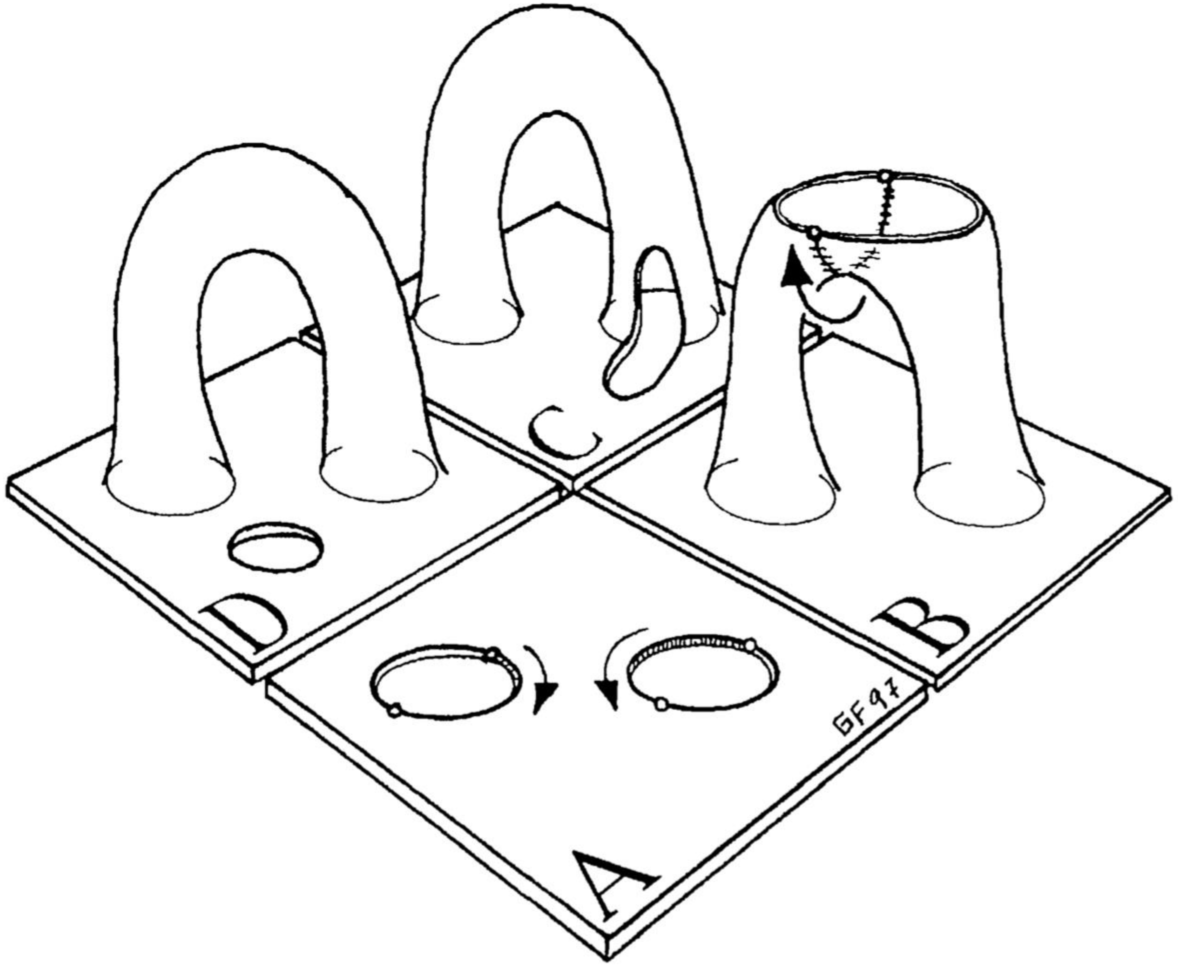
Handles and cross-handles are the same when cross-caps are presented

Trading

- When cross-handles or cross-caps are presented
  - Turn all handles and cross-handles into cross-caps

- Otherwise, only handles exist

Dealing with Boundaries



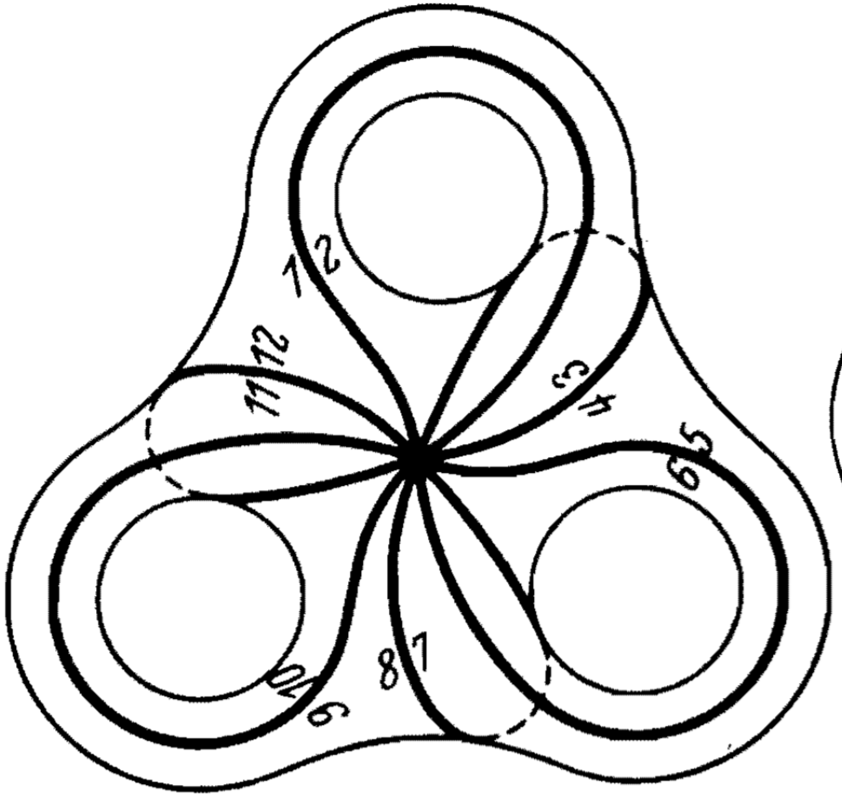
INTERMISSION

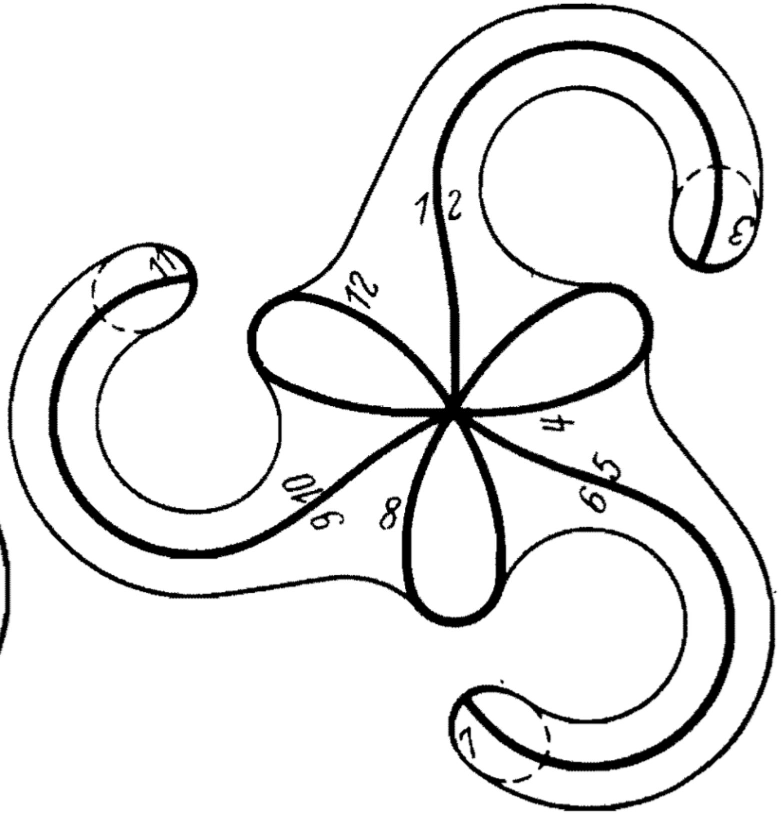
EXERCISE:

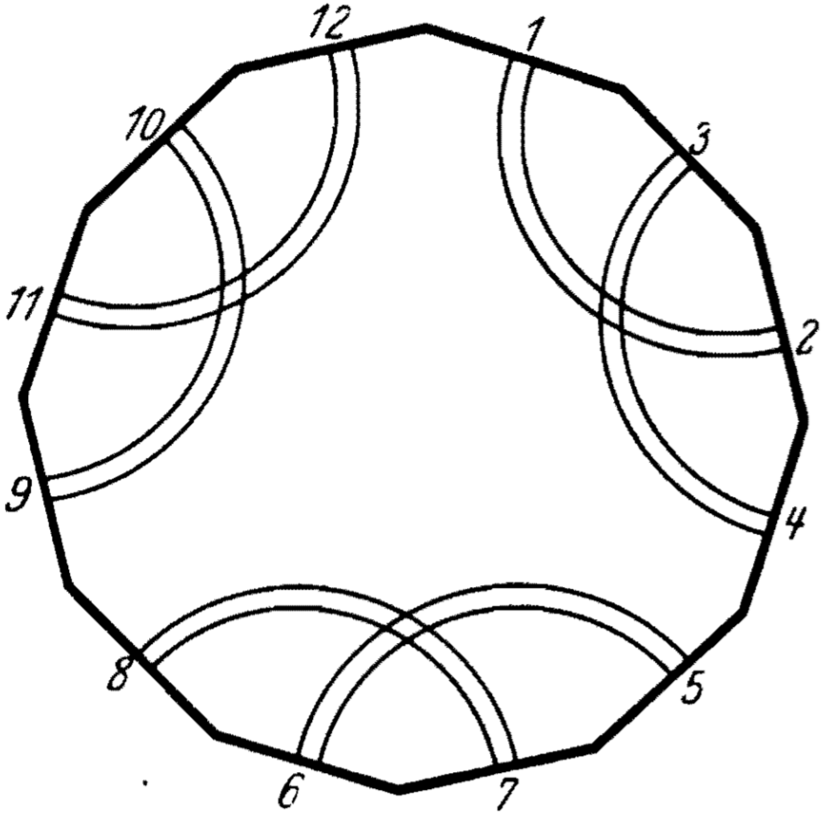
What is this surface?

Surface Graph and Rotation System

Treat Cutting-lines as Graphs

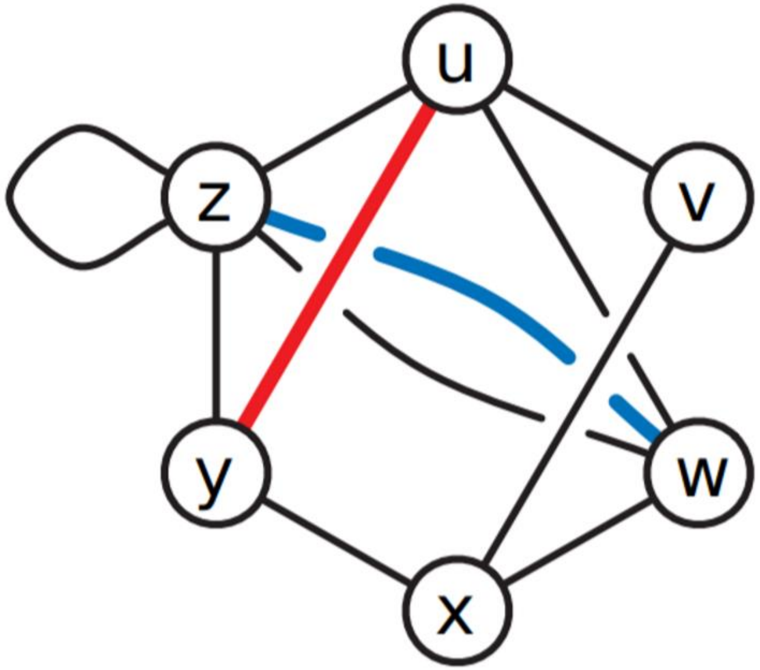




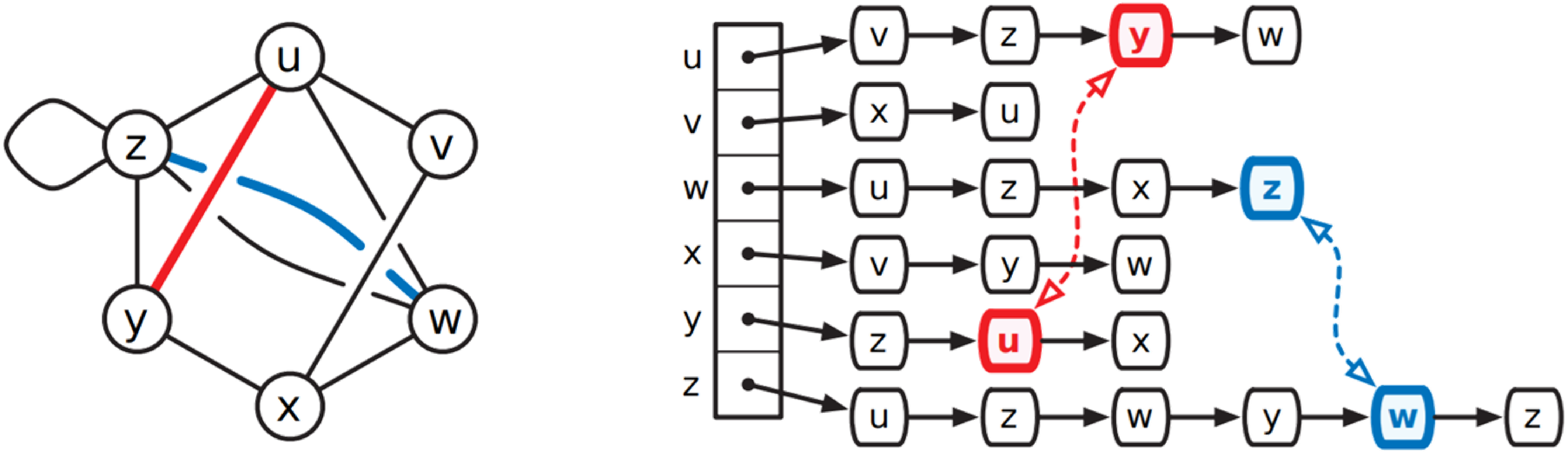


Treat Cutting-lines as Graphs

Abstract Graph

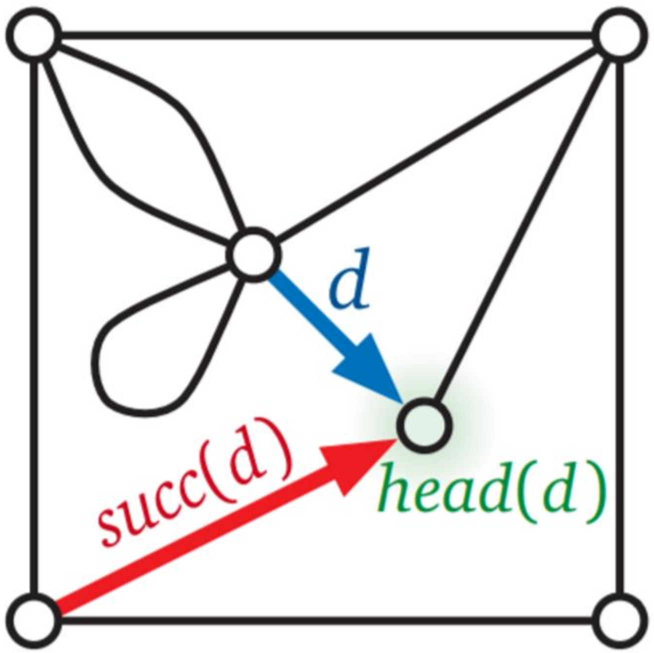


Graph Data Structure

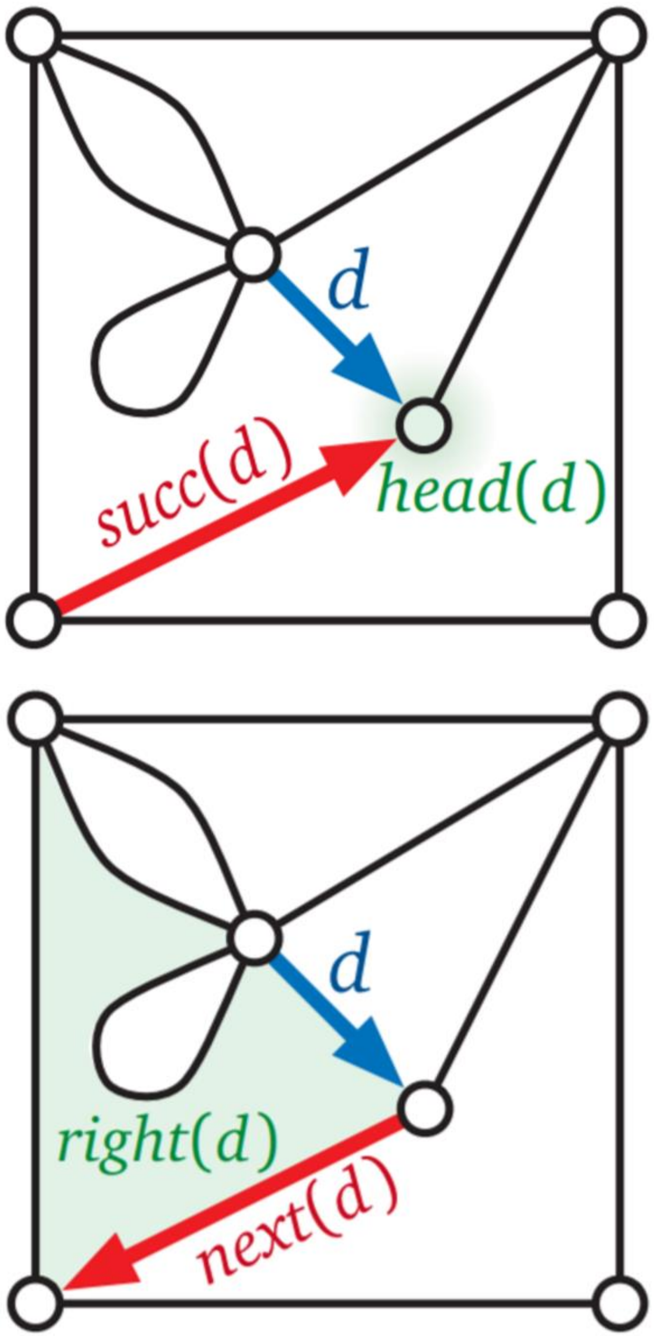


An incidence list representation of a graph, with the dart records for two edges emphasized. For clarity, most reversal pointers are omitted.

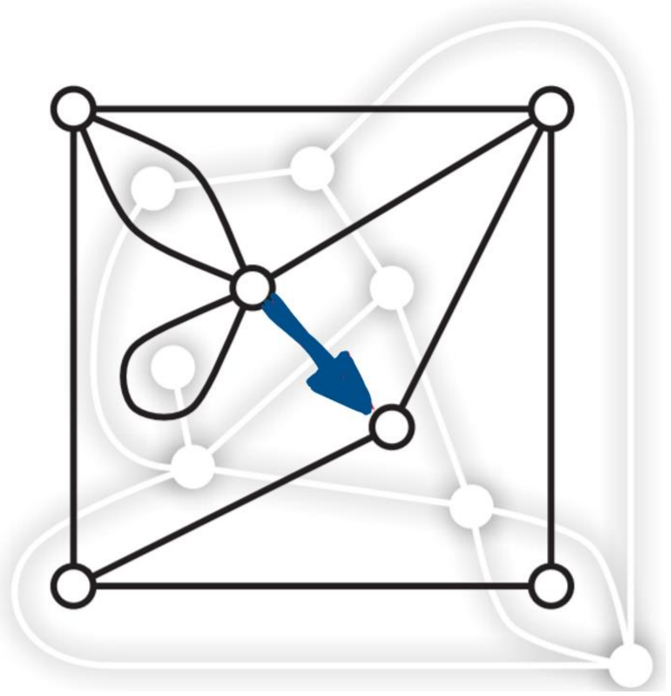
Surface Graph

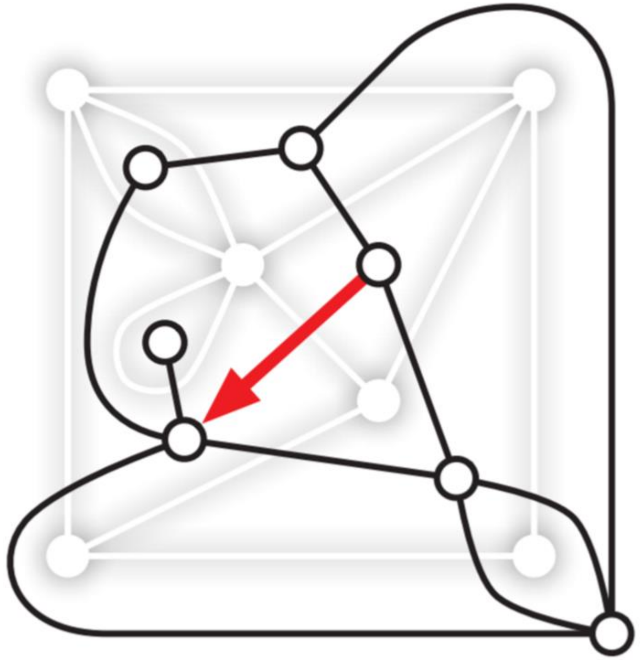


Surface Graph

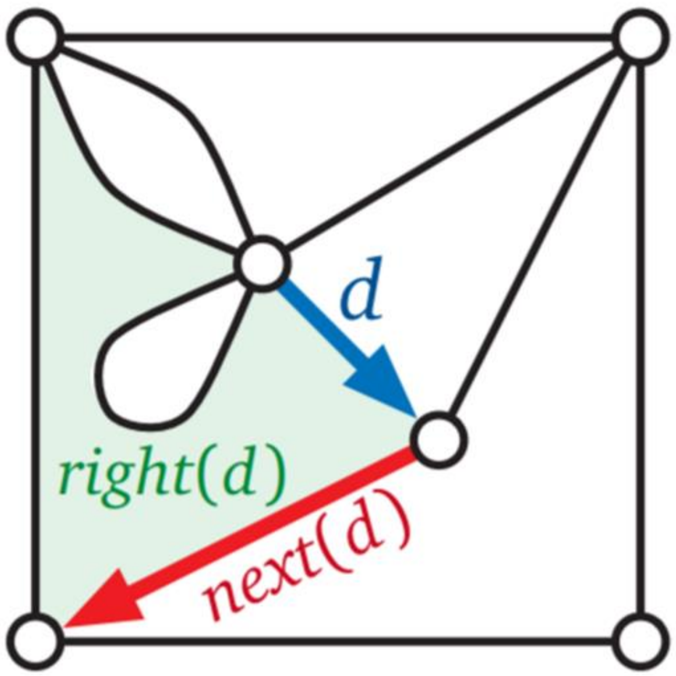


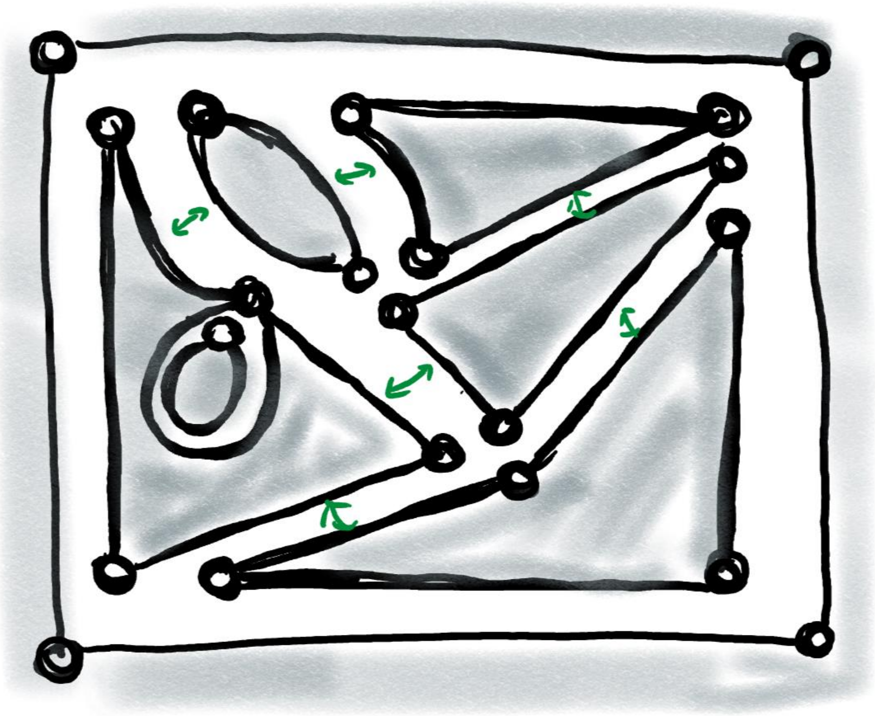
Dual Surface Graph

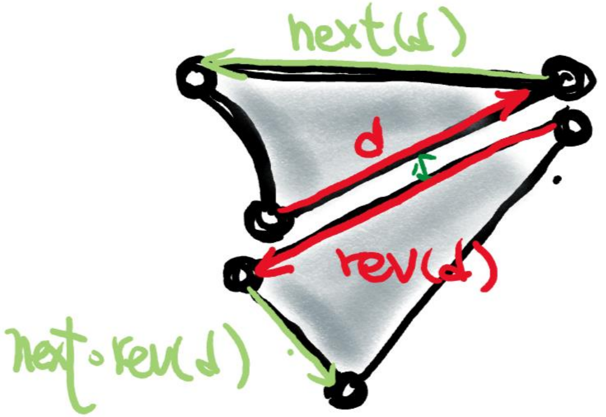




Polygonal Schema IS Rotation System







Theorems we secretly assumed

- Triangulation Theorem [Kerékjártó-Radó 1925]
  - Any surface can be cut into triangles
- Refinement Theorem [Moise 1977]
  - Any two triangulations have a common refinement

- Existence of rotation system
  - Every surface-embedded graph has a rotation system

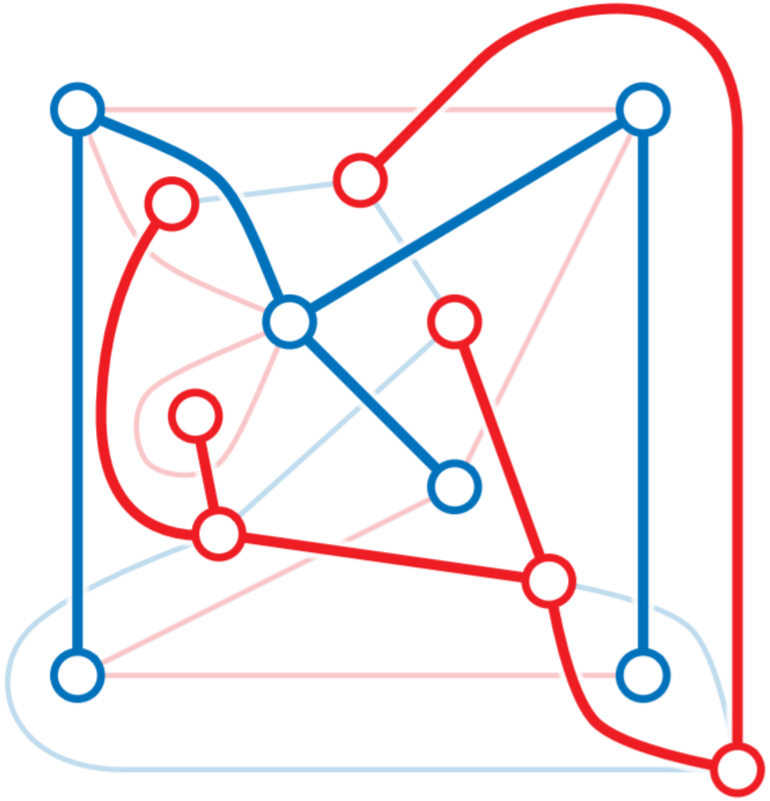
Let's Focus on Plane Graphs:

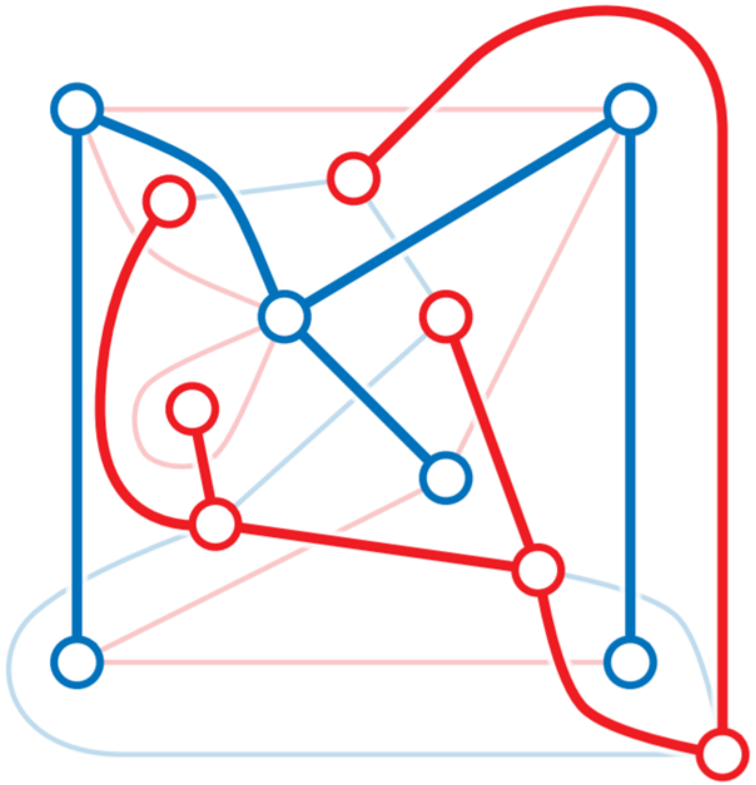
Tree-Cotree Decomposition

primal G	**dual G^***
vertex v	face v^*
dart d	dart d^*
edge e	edge e^*
face f	vertex f^*
$\mathit{tail}(d)$	$\mathit{left}(d^*)$
$\mathit{head}(d)$	$\mathit{right}(d^*)$
succ	$\mathit{rev} \circ \mathit{succ}$
clockwise	counterclockwise

primal G	**dual G^***
empty loop	spur
loop	bridge
cycle	bond
even subgraph	edge cut
spanning tree	complement of spanning tree
$G \setminus e$	G^* / e^*
G / e	$G^* \setminus e^*$
minor $G \setminus X / Y$	minor $G^* \setminus Y^* / X^*$

Correspondences between features of primal and dual planar maps





Euler's Formula

[Euler 1750] [Legendre 1794] [Cayley-Listing 1861]

For any plane graph G,

$$V_G - E_G + F_G = 2$$

Q. Does Euler's Formula Hold for Surface Graphs?

Next time.

Surface hard to visualize?

The space is weird?