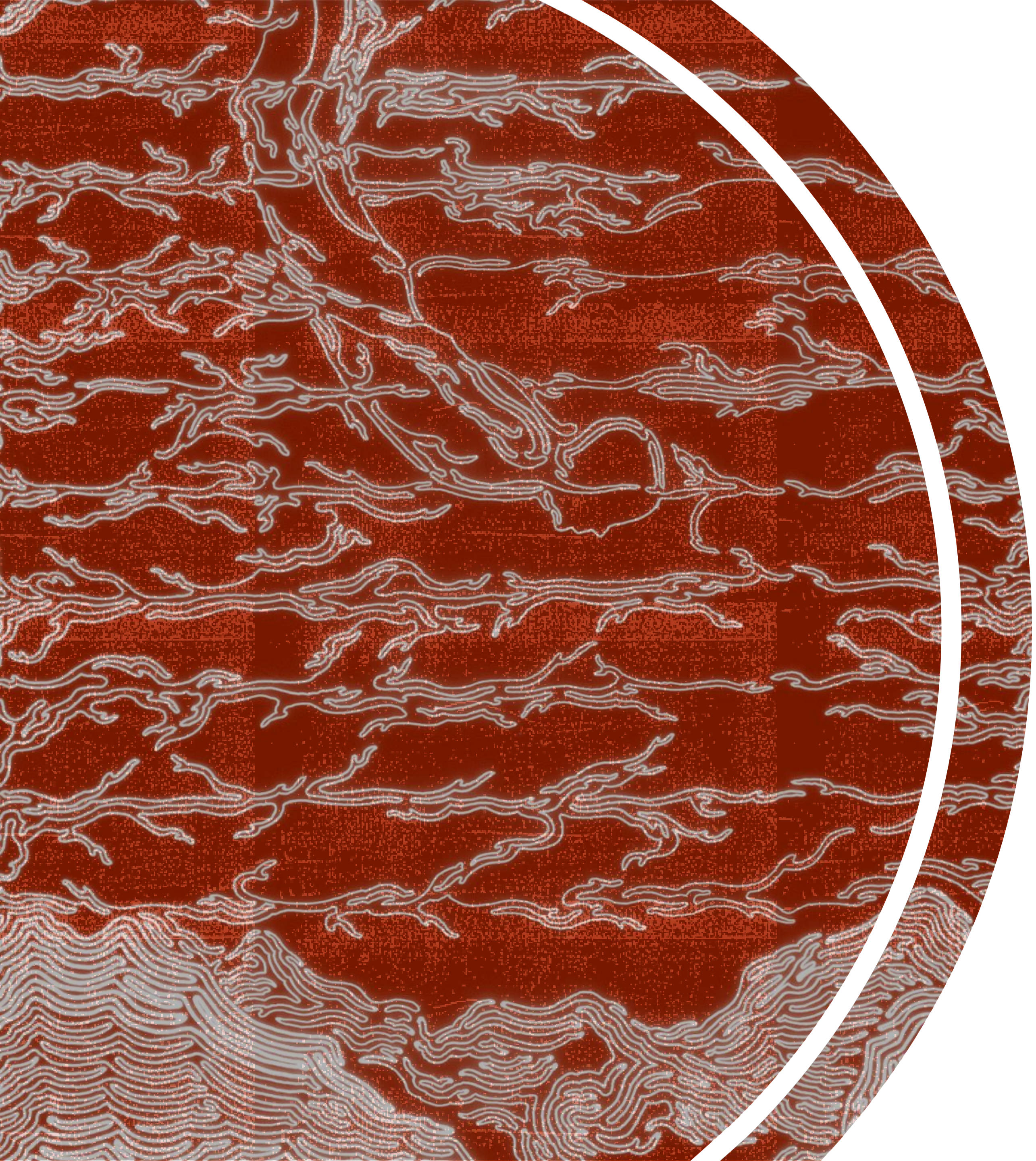


INTRODUCTION TO COMPUTATIONAL TOPOLOGY

HSIEN-CHIH CHANG

LECTURE 11, OCTOBER 19, 2021

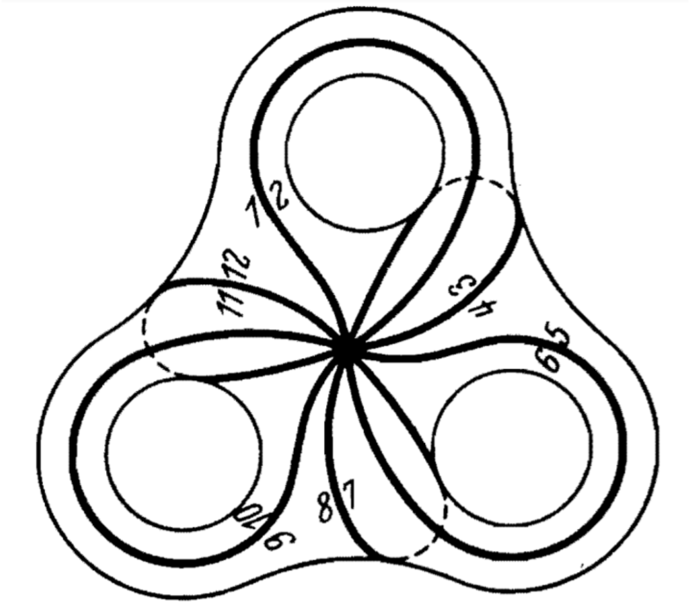
Administrivia

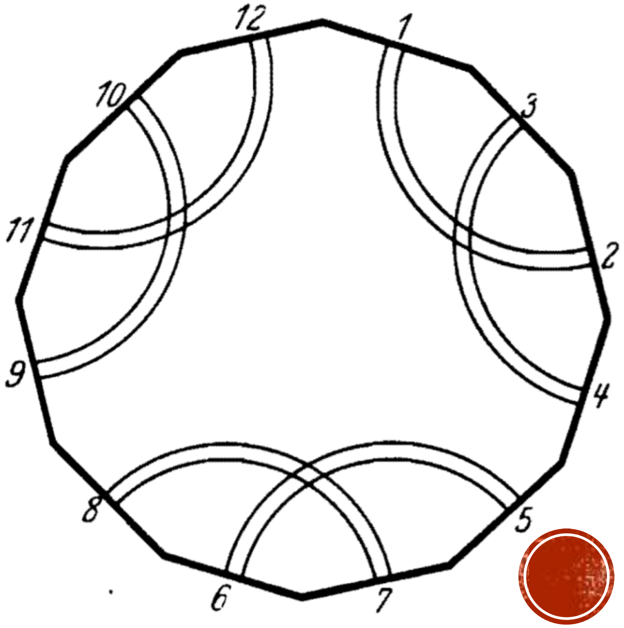
- Homework β will be out soon
- Remember to submit your project proposal!

Homology

Fundamental Groups of Surfaces

- $\pi_1(\Sigma(g,0)) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \overline{a_1 b_1} \ldots a_g b_g \overline{a_g b_g} \rangle$
- $\pi_1(\Sigma(0,r)) = \langle a_1, \ldots, a_r \mid a_1 a_1 \ldots a_r a_r \rangle$





$$
\left\langle\ a,b,c,d,e,p,q,r,t,k\ \middle|\
\begin{array}{lll}
p^{10}a = ap, & pacqr = rpcaq, & ra = ar, \\
p^{10}b = bp, & p^2adq^2r = rp^2daq^2, & rb = br, \\
p^{10}c = cp, & p^3bcq^3r = rp^3cbq^3, & rc = cr, \\
p^{10}d = dp, & p^4bdq^4r = rp^4dbq^4, & rd = dr, \\
p^{10}e = ep, & p^5ceq^5r = rp^5ecaq^5, & re = er, \\
aq^{10} = qa, & p^6deq^6r = rp^6edbq^6, & pt = tp, \\
bq^{10} = qb, & p^7cdcq^7r = rp^7cdceq^7, & qt = tq, \\
cq^{10} = qc, & p^8ca^3q^8r = rp^8a^3q^8, & \\
dq^{10} = qd, & p^9da^3q^9r = rp^9a^3q^9, & \\
eq^{10} = qe, & a^{-3}ta^3k = ka^{-3}ta^3 &
\end{array}
\right\rangle
$$

[Collins 1986]

Undecidability of π_1 [Novikov 1955] [Boone 1958]

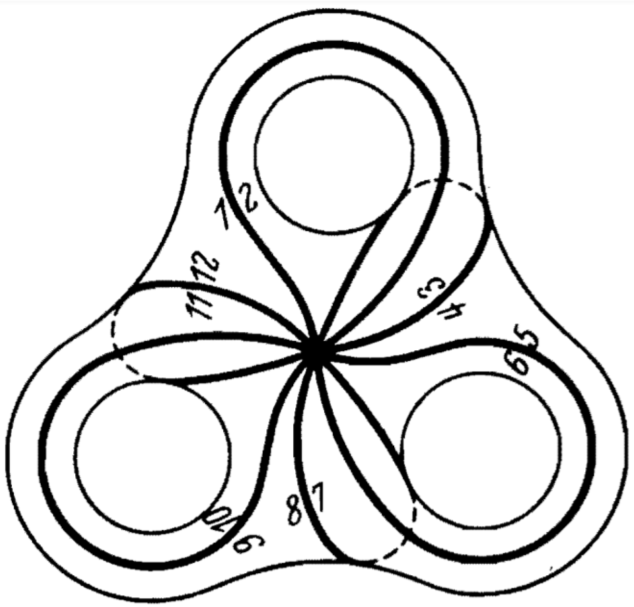
Checking if a 2-complex has trivial π_1 is undecidable

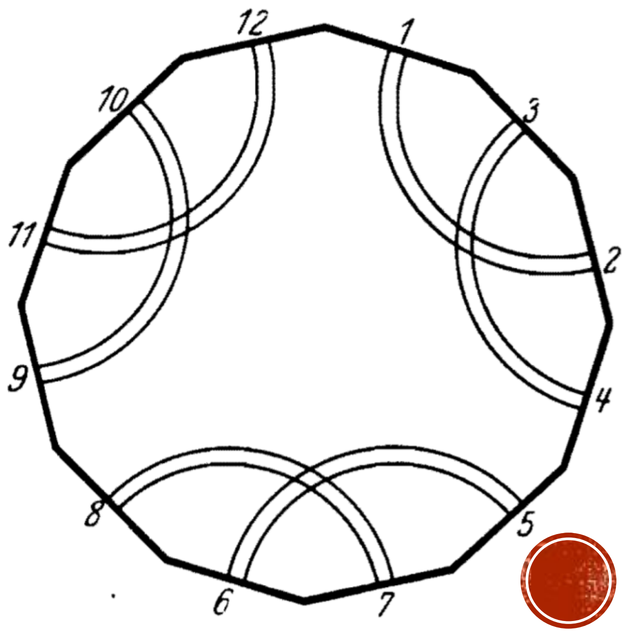
What if we use Vector Spaces?

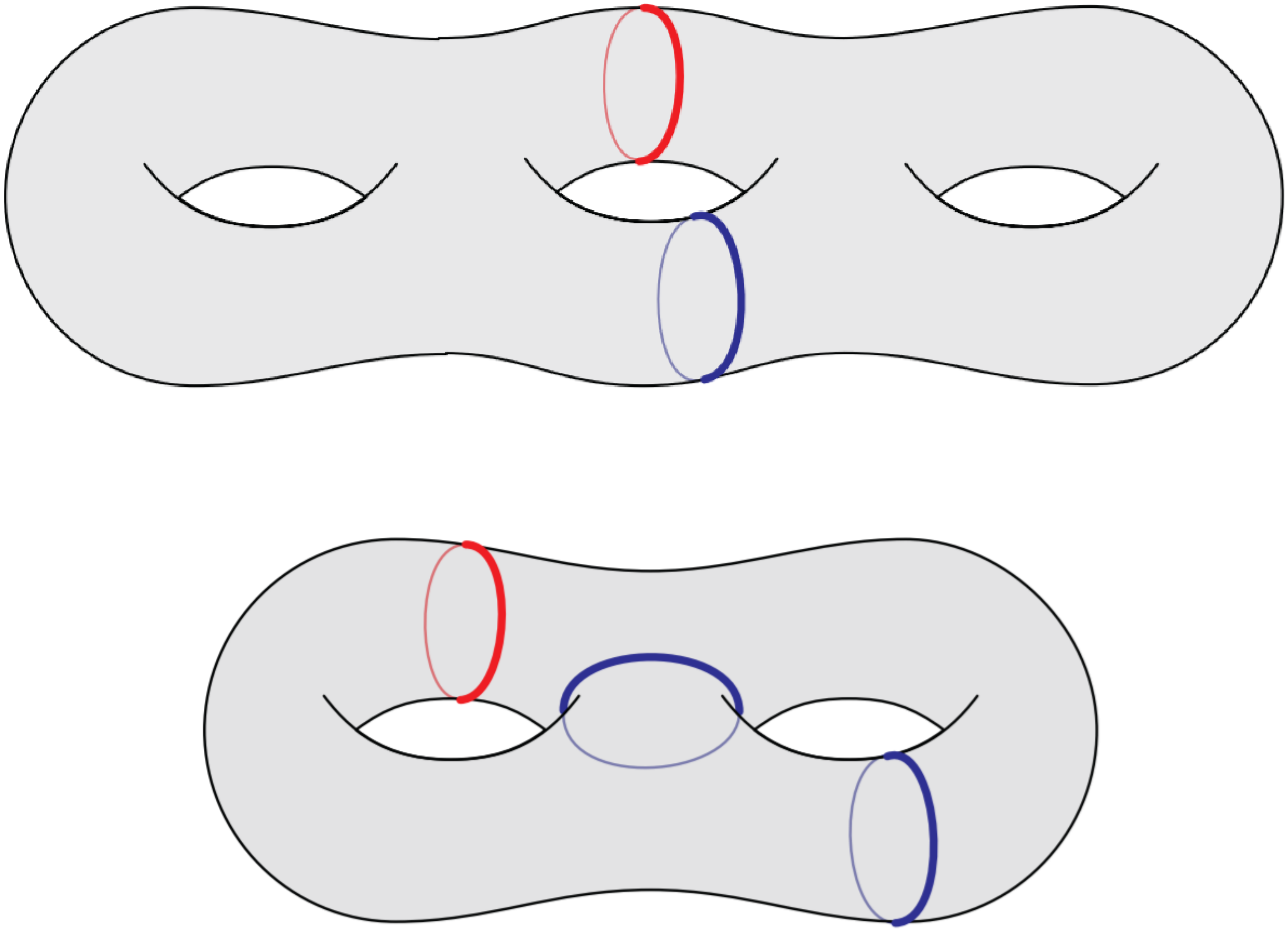
Spark of Idea

- $\langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 \overline{a_1 b_1} \ldots a_g b_g \overline{a_g b_g} \rangle$

- $\mathbb{Z}\langle a_1, b_1, \ldots, a_g, b_g \rangle \,/\, \langle a_1 + b_1 - a_1 - b_1 \ldots \rangle$



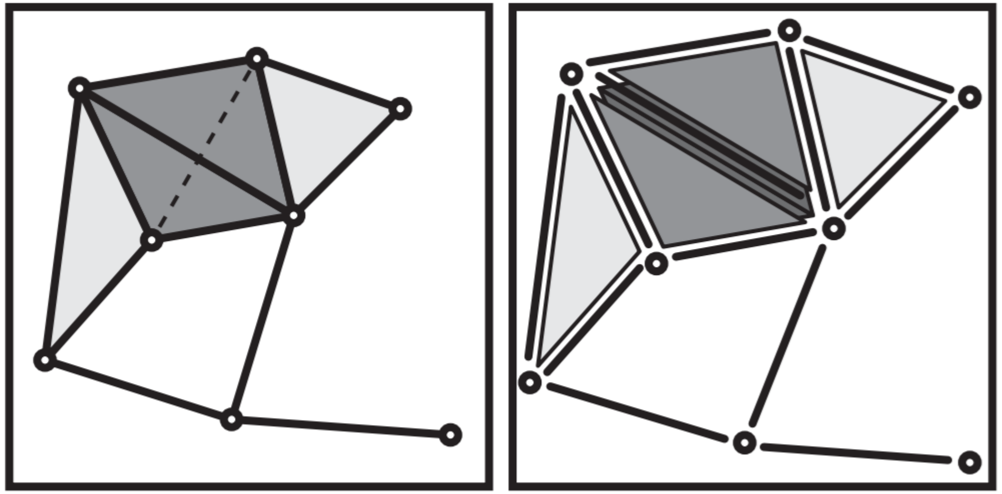




Homology Equivalence

- Two "cycles" are **homologous** if together they are the boundary of some region

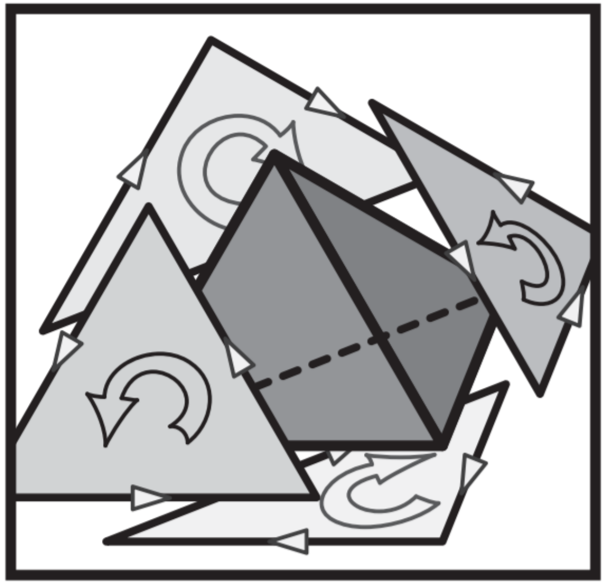
Chain Complex



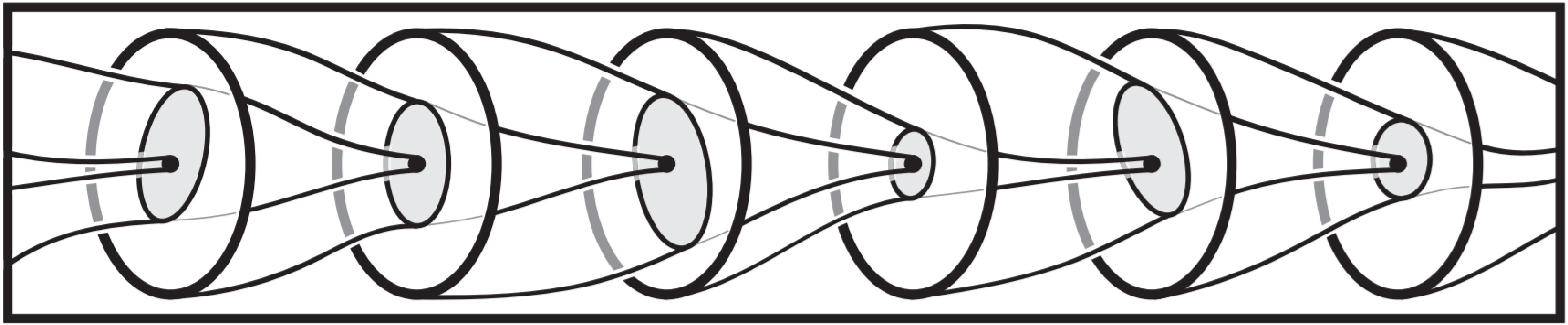
- Vector spaces over elements

- 0-complex
- 1-complex
- 2-complex
- …

Boundary Map



- $\partial_n: C_n(X) \longrightarrow C_{n-1}(X)$



Fundamental Lemma of Homology

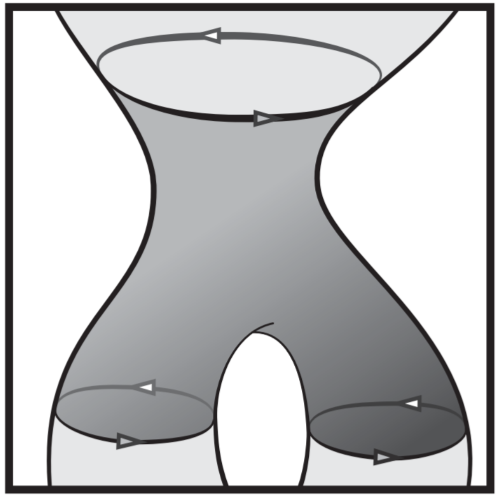
$$\partial \cdot \partial = 0$$

"Boundary of a region must be boundaryless"

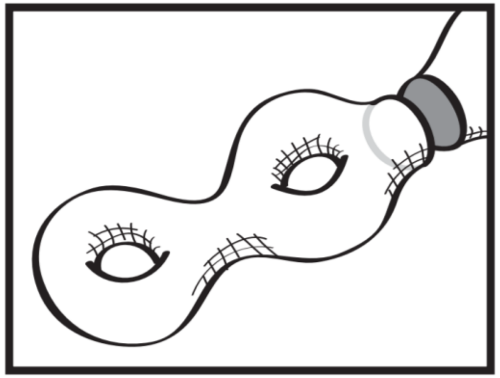
Proof of $\partial \bullet \partial = 0$.

Cycles and Boundaries

- Cycle space $Z_n(X) = \ker \partial_n$



- Boundary space $B_n(X) = \operatorname{im} \partial_{n+1}$



Homology Groups

- Simplicial homology group

$$H_n(X) = Z_n(X) \,/\, B_n(X)$$
$$= \text{Cycles/Boundaries}$$

Computing Homology

Homology of Torus

- $H_n(X) = Z_n(X) \;/\; B_n(X)$
  $= \ker \partial_n / \mathrm{im}\, \partial_{n+1}$

Intermission

Food for Thought.

What do vector spaces give us?

Homology of Klein Bottle

- $H_n(X) = Z_n(X) / B_n(X)$
  $= \ker \partial_n / \operatorname{im} \partial_{n+1}$

What is Good about Homology?

Christmas Wishlist

- Fast computation
- Functoriality:
  - $f: X \to Y$ implies $f_*: H_n(X) \to H_n(Y)$ for all n
  - $(f \bullet g)_* = f_* \bullet g_*$, $id_* = id$, $\partial \bullet f_* = f_* \bullet \partial$, …
- Invariance:
  - $f, g: X \to Y$ homotopic implies $f_* = g_*$
- …

VOL. 31, 1945 MATHEMATICS: EILENBERG AND STEENROD 117

AXIOMATIC APPROACH TO HOMOLOGY THEORY

BY SAMUEL EILENBERG AND NORMAN E. STEENROD

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN

Communicated February 21, 1945

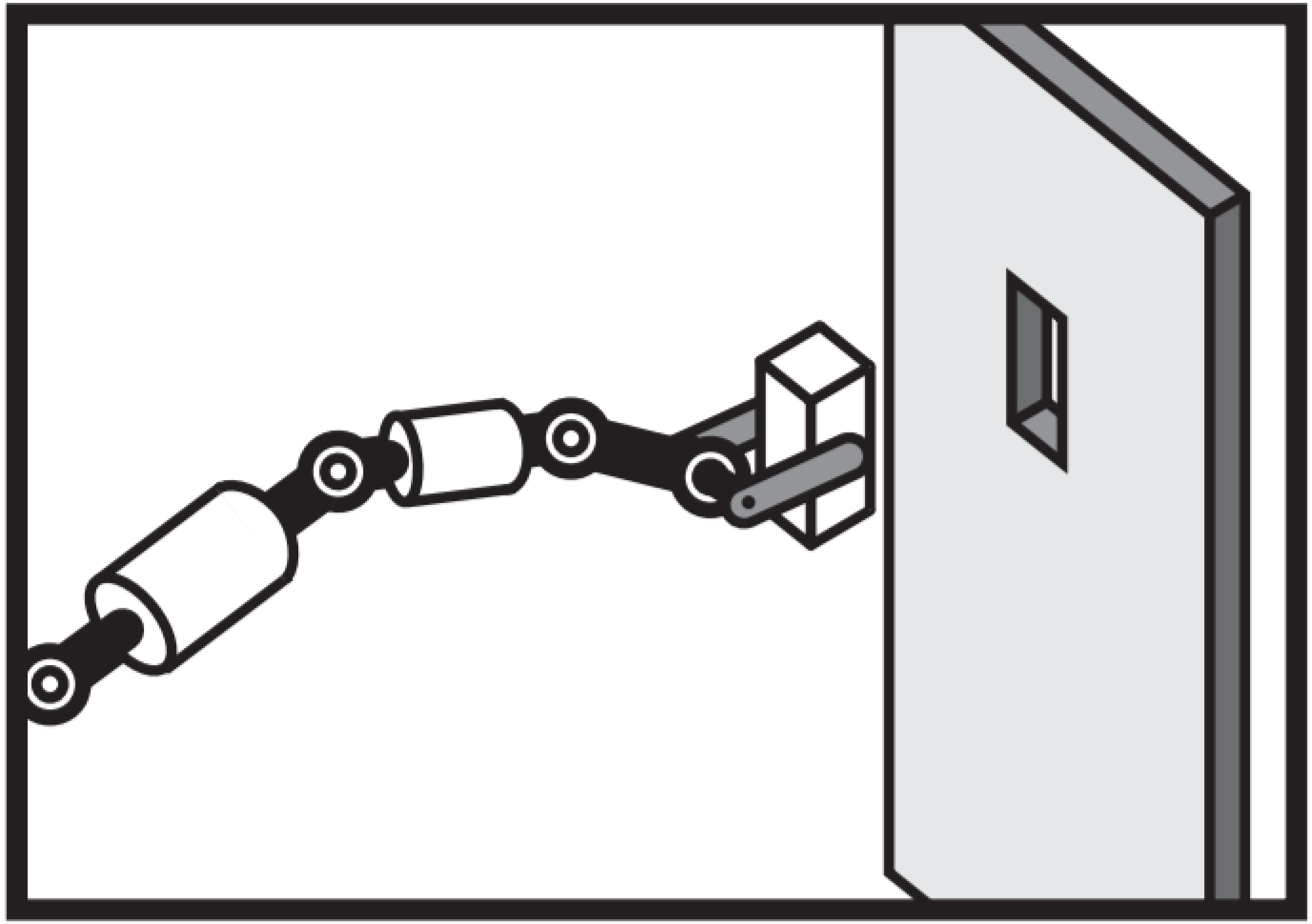
1. Introduction.—The present paper provides a brief outline of an axiomatic approach to the concept: homology group. It is intended that a full development should appear in book form.

The usual approach to homology theory is by way of the somewhat complicated idea of a complex. In order to arrive at a purely topological concept, the student of the subject is required to wade patiently through a large amount of analytic geometry. Many of the ideas used in the constructions, such as orientation, chain and algebraic boundary, seem artificial. The motivation for their use appears only in retrospect.

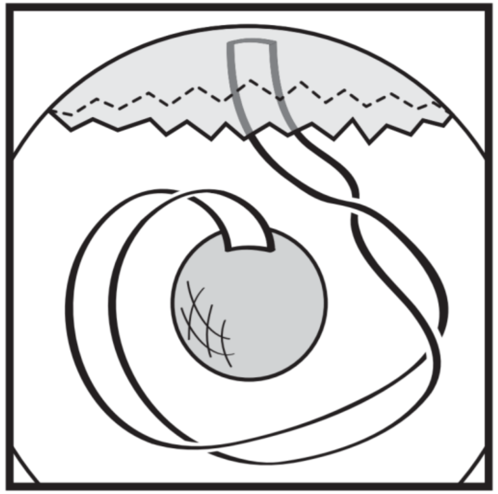
Since, in the case of homology groups, the definition by construction is so unwieldy, it is to be expected that an axiomatic approach or definition by properties should result in greater logical simplicity and in a broadened

SINGULAR HOMOLOGY THEORY [Eilenberg-Steenrod 1945]

"Homology group has everything you want from Christmas."

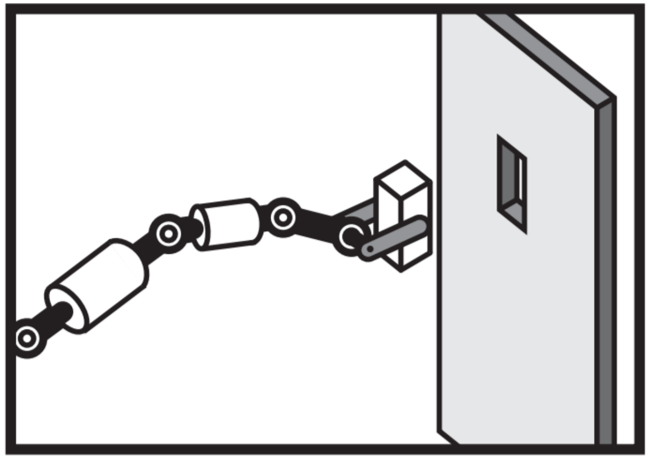


Robot Arm Puzzle



Robot Arm Puzzle

- What is $H_1(SO_3)$?



Robot Arm Puzzle

- What is $H_1(SO_3)$?

- Given $\kappa: S_1^N \longrightarrow SO_3$,
is there $s: SO_3 \longrightarrow S_1^N$,
such that $s \bullet \kappa = id$?

Euler Characteristic Redux

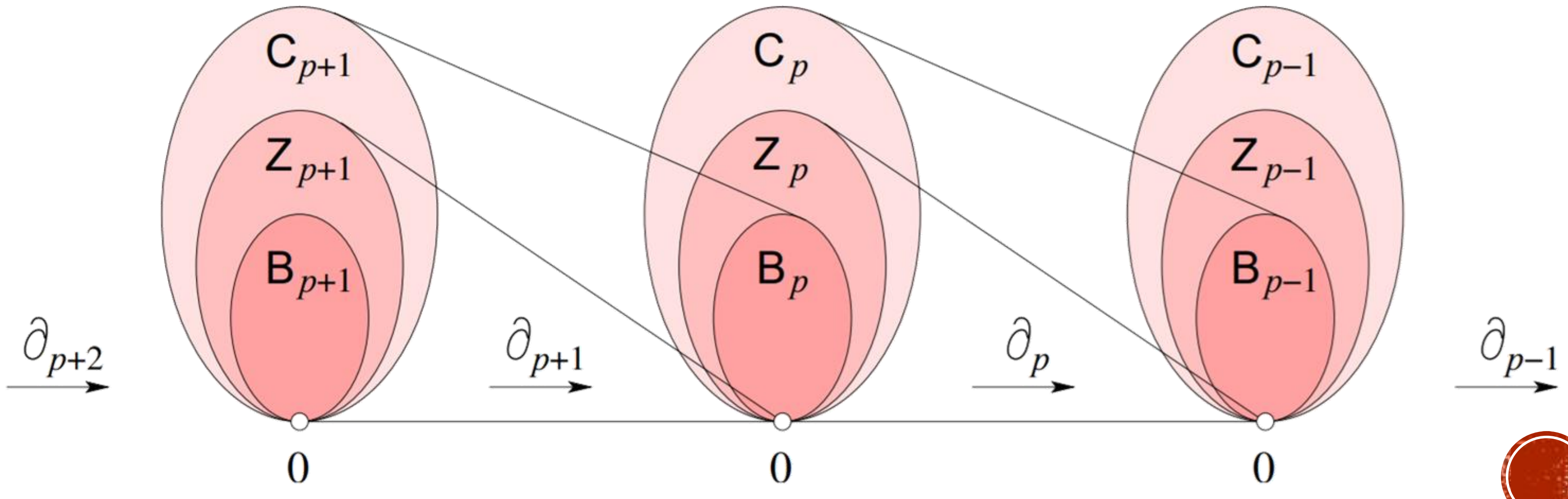
Dimension

- Betti numbers β_n: dim $H_n(X)$

Euler-Poincaré Formula

$$\chi(X) = \Sigma_n (-1)^n \cdot \dim H_n(X)$$

Proof of Euler-Poincaré Theorem.



Order Now for Christmas!

On Thursday.

Homology in action.

(Optional: homology groups has internal structures.)