1. ***Gauss code.*** A ***Gauss code*** is a cyclic string of $2n$ symbols where each symbol occurs exactly two times; it is ***signed*** if in addition each symbol x is attached with a plus/minus sign +/-, one for each occurrence of x. A Gauss code is ***planar*** if it encodes the sequence of crossings we see as we traverse an $n$-vertex planar curve $\gamma$; the signing of the Gauss code correspond to the Gauss signs of the crossings of $\gamma$.

   Describe and analyze an algorithm whether a given signed Gauss code is planar.

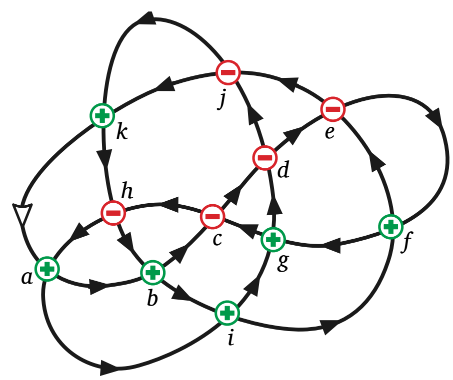


**Figure 1.** A planar curve with Gauss code `[abcdefgchaigdjkhbifejk]` and signing `[++---+++--+-+-++---++-]`.

2. ***Spanning trees as α-orientations.*** Let $G$ be a plane graph and $G^*$ be its dual, drawn in the plane in such a way that every crossing correspond to exactly one primal-dual edge pair from $(G, G^*)$. Consider the ***overlay graph*** $G^+$:

   - Add all vertices in $G$ and $G^*$, and all the crossings in the drawing as vertices of $G^+$;
   - Subdivide each edge $(u, v)$ in $G$ and $G^*$ at the crossing point $x$, and add the two edges $(u, x)$ and $(x, v)$ as edges of $G^+$.

   (Alternatively, one can construct the overlay graph by performing the radial construction twice on the primal graph $G$: $G^+ := G^{\diamond\diamond}$.[1])

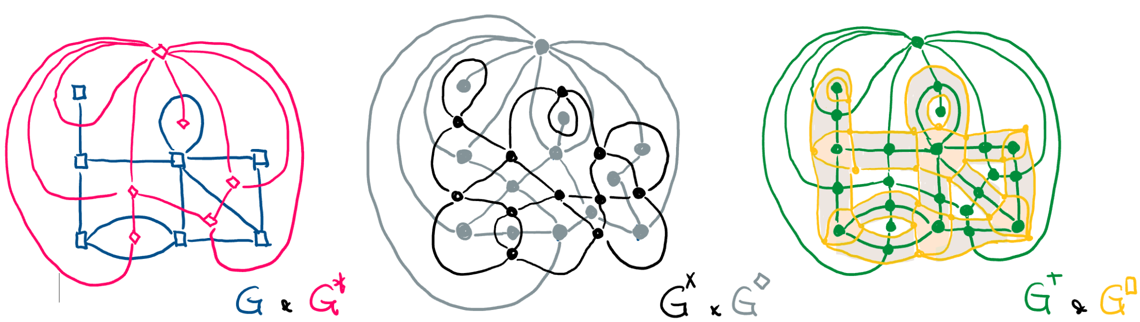


**Figure 2.** (a) Plane graph $G$ and its dual $G^*$. (b) Medial graph $G^\times$ and radial graph $G^\diamond$. (c) Overlay graph $G^+$ and its dual $G^\square$.

[1]The overlay graph $G^+$, obtained by performing the radial construction twice, is a subgraph of the barycentric subdivision of $G$. The dual graph of $G^+$, conveniented denoted as $G^\square$, can be obatined by performing the medial construction twice ($G^\square := G^{\times\times}$), and is a *minor* of the band decomposition/ribbon graph of $G$.

(a) Prove that there is a feasible function $\alpha$ defined on the overlay graph $G^+$, such that a tree-cotree pair in the primal-dual plane graph $(G, G^*)$ is in bijection with $\alpha$-orientations of $G^+$, after fixing one primal "root" and one dual "root" from the vertices of $G^+$.

(b) Prove that the essential cycles for the above collection of $\alpha$-orientations are exactly the faces of $G^+$ (which are exactly the corners of $G$) not incident to the two roots.

$^\star$3. ***Improving presentation.*** In class we showed that given any $\pm$-labeling on the edges of a planar graph with vertex set $V$, we have

$$\sum_{v \in V} alt(v) < 4|V|$$

where $alt(v)$ is the number of sign alternation around the vertex $v$.

Provide a new proof to the result using discrete Gauss-Bonnet Theorem.