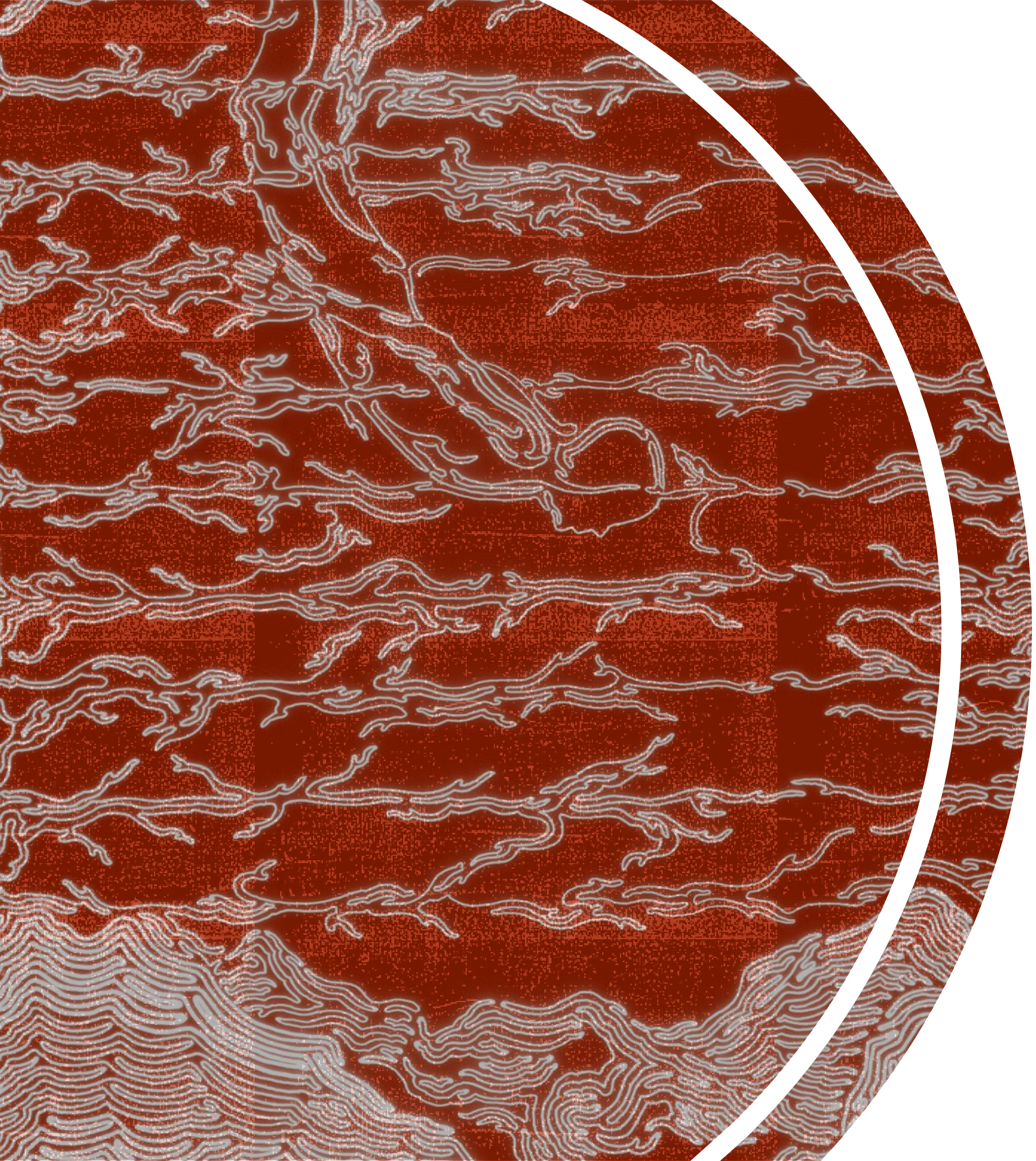

Introduction to Computational Topology

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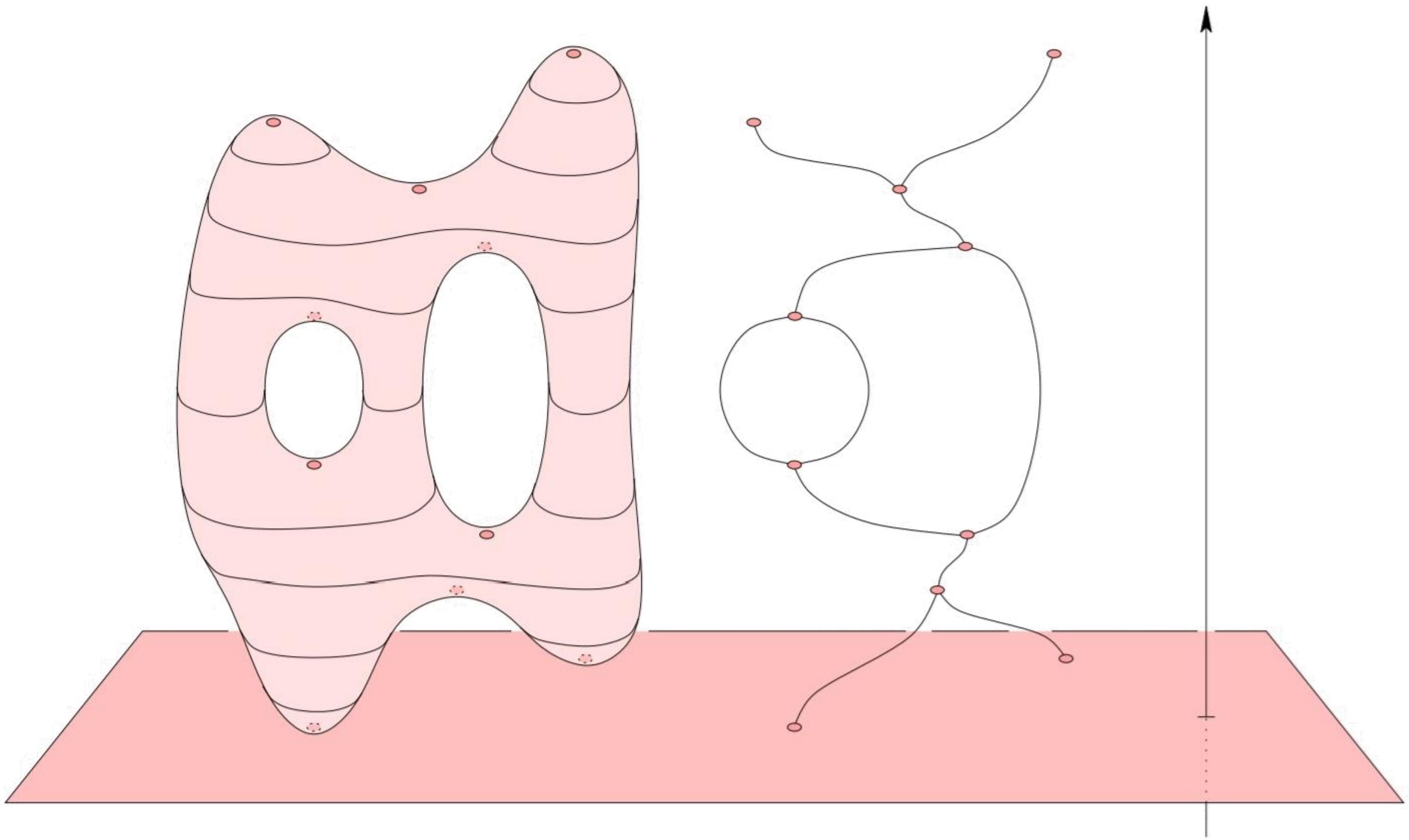
Lecture 15, November 2, 2021

Morse Theory



Reeb Graph

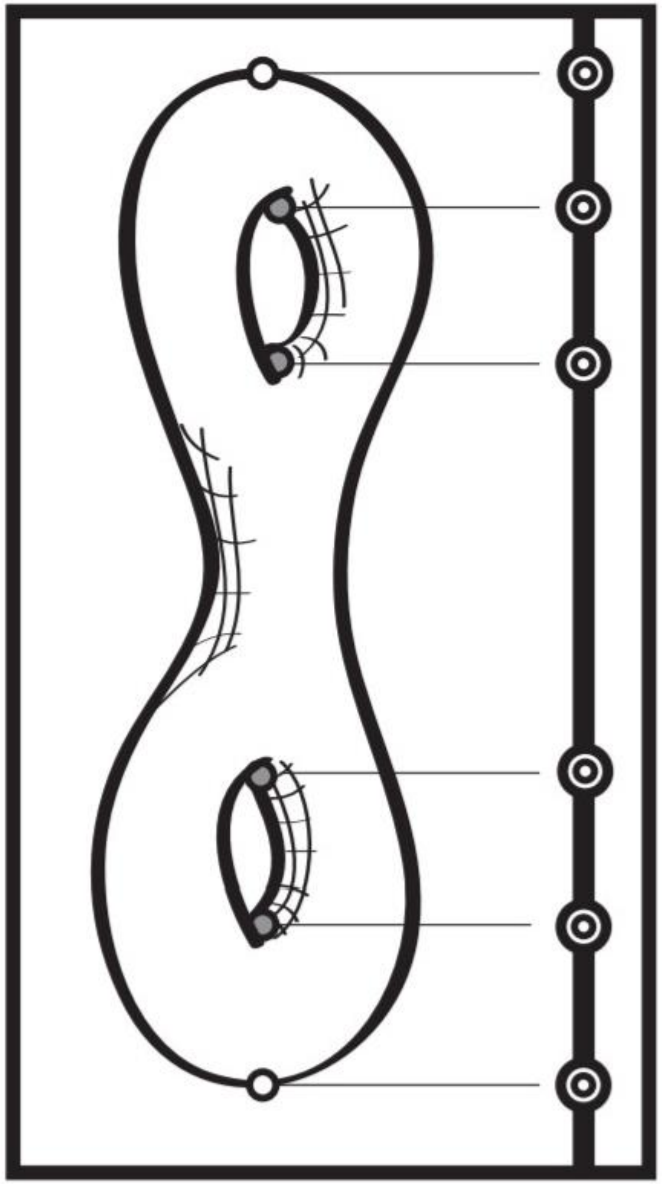
- $\beta_1(\mathrm{Reeb}(M)) \leq \beta_1(M)$

Morse Theory

- Topology is still useful when the surface is just a terrain!

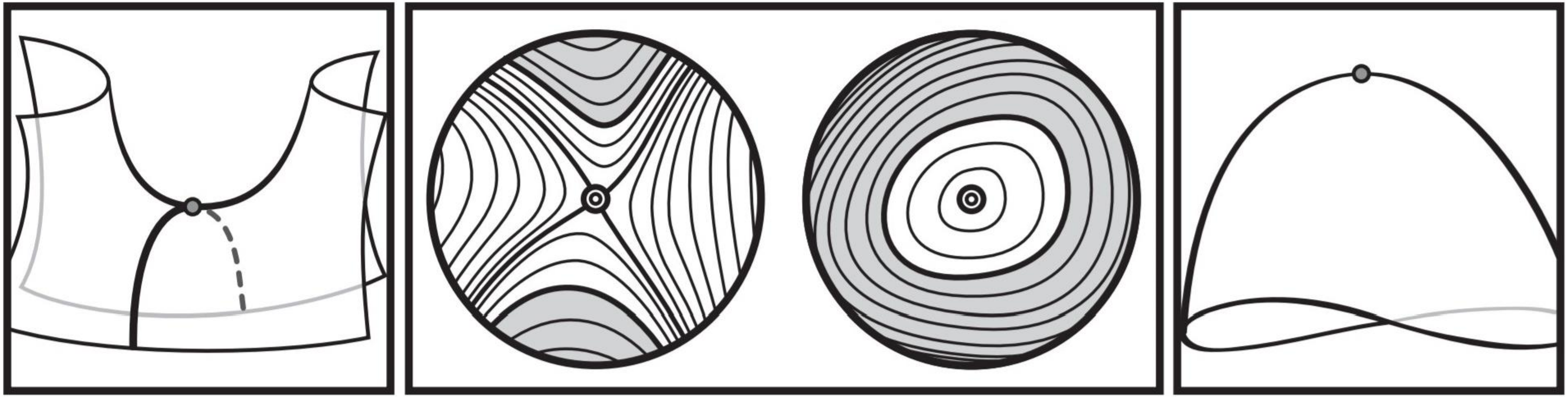
Definitions

- Height function h: M -> R
- Sub-level set $M_{\leq a}$: $h^{-1}(-\infty, a] = \{x : h(x) \leq a\}$
- Critical points: where the topology changes



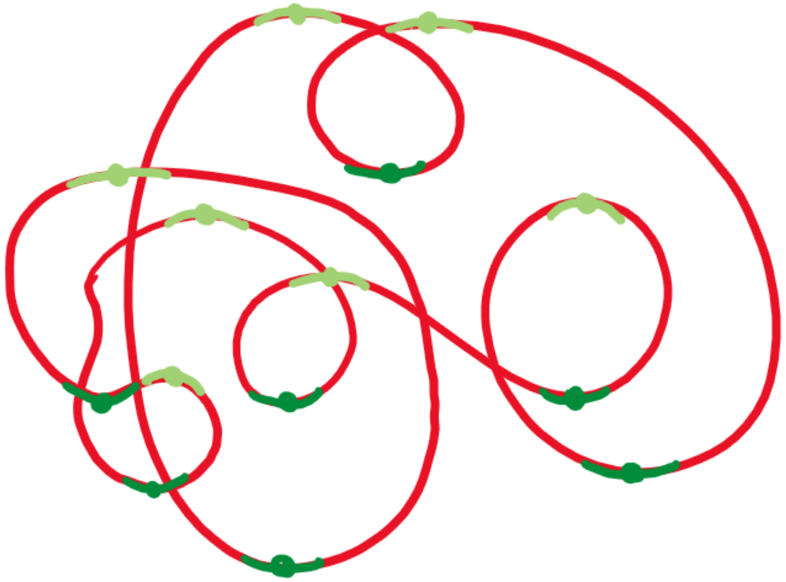
Morse Function

- All critical points are non-degenerate and have distinct function values



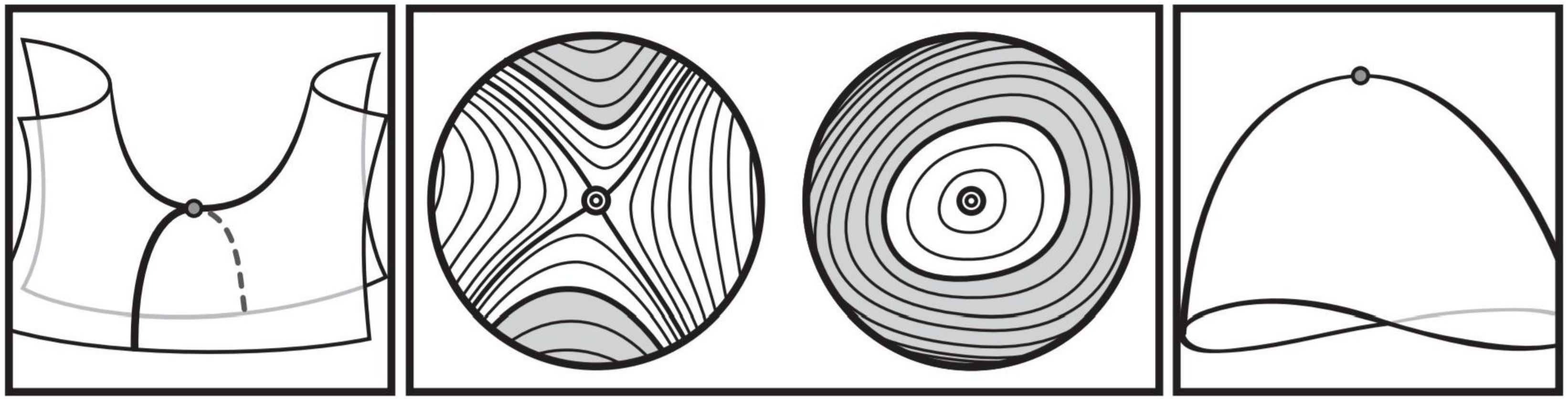
Morse Lemma [Morse 1934]

Given Morse function h and critical point p, locally U(p) looks like $f(x) = f(p) - x_1^2 \ldots - x_s^2 + x_{s+1}^2 \ldots + x_d^2$



Example

- Rotation number redux
- Morse index $\mu(p)$: number of negative quadratic terms

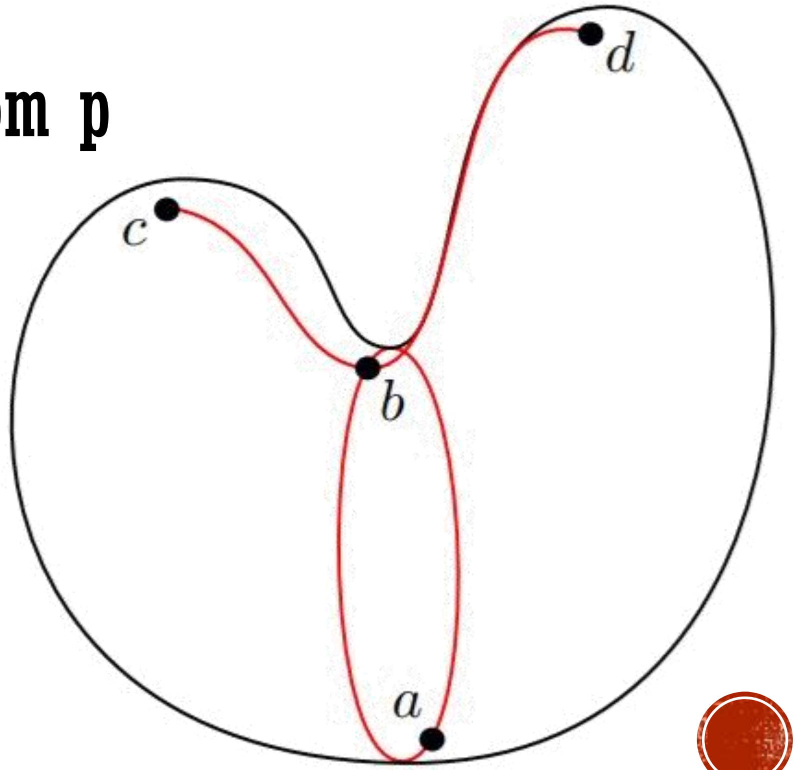


Example

- $U = D^n$
- $L = D^{n-(\mu-1)} \times S^{\mu-1}$

Flowlines

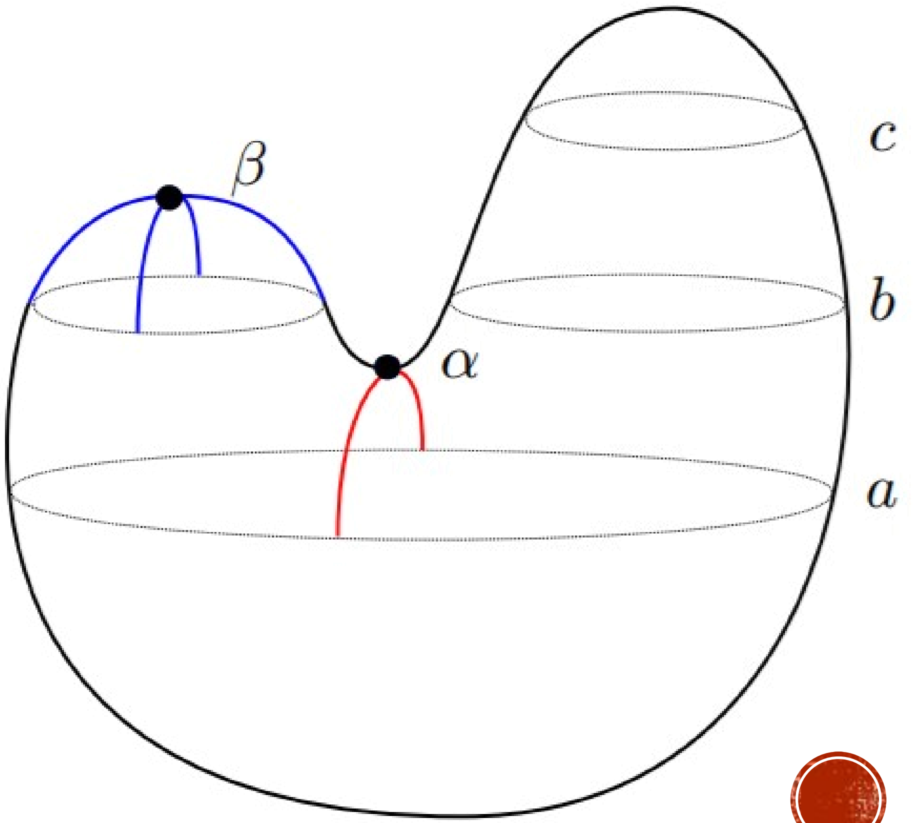
- Gradient field ∇h defines flowlines between critical points
  - M decomposes into flowlines
- Descending manifold $M{\downarrow}(p)$: flowlines originated from p



- **Proposition.** $M{\downarrow}(p)$ has dimension $\mu(p)$

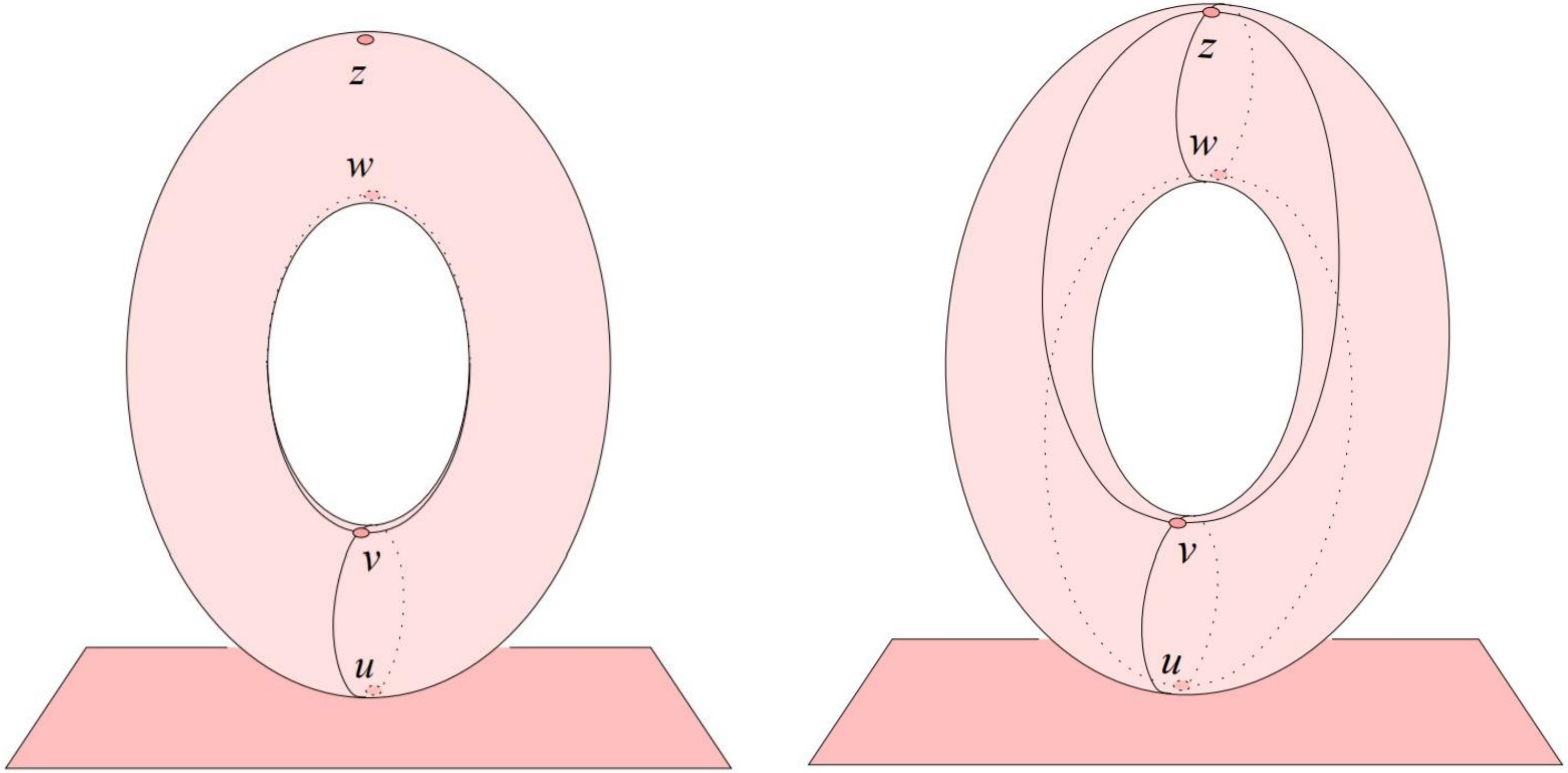
Properties

- Descending manifold $M^{\downarrow}(p)$: flowlines originated from p
- **Proposition.**
  - $M_b \simeq M_a$ if no critical points in $h^{-1}[a, b]$
  - $M_{\leq b} \simeq M_{\leq a} \cup M^{\downarrow}(p)$ if $h^{-1}[a, b]$ has critical point p



Morse-Smale Function

- All flowlines go from k-dim critical pts to (k-1)-dim critical pts



Intermission

Food for Thought.

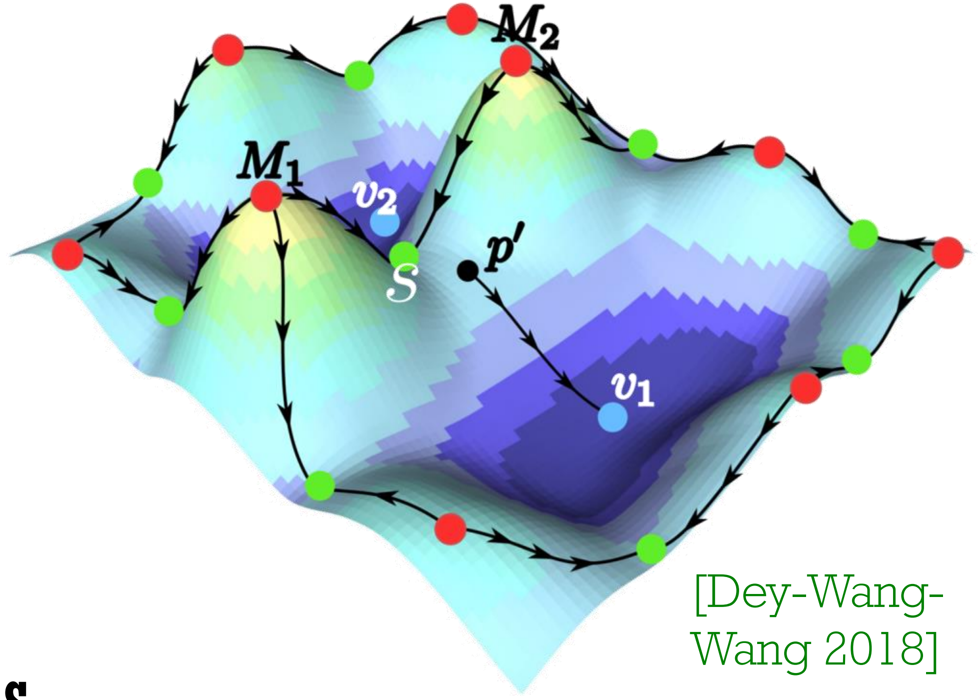
Flowlines going one-dimension lower.

What are we trying to do?

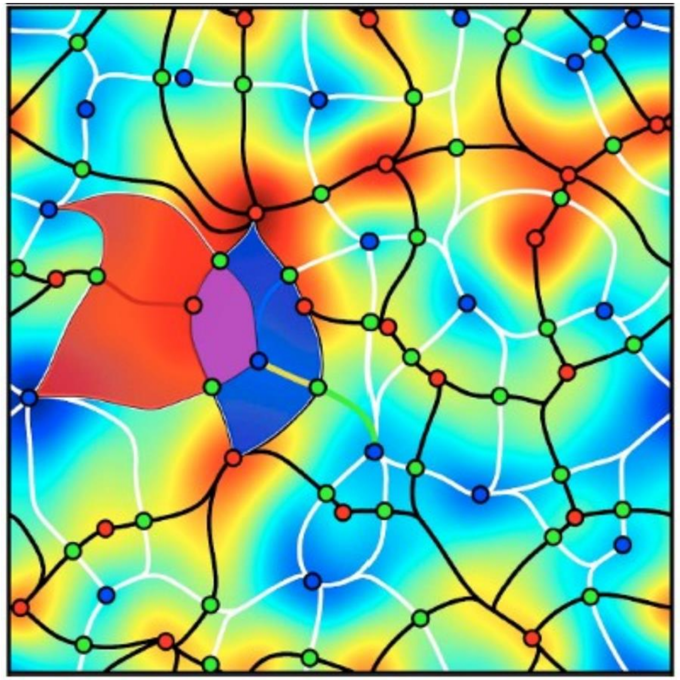
Morse Homology

Morse Complex

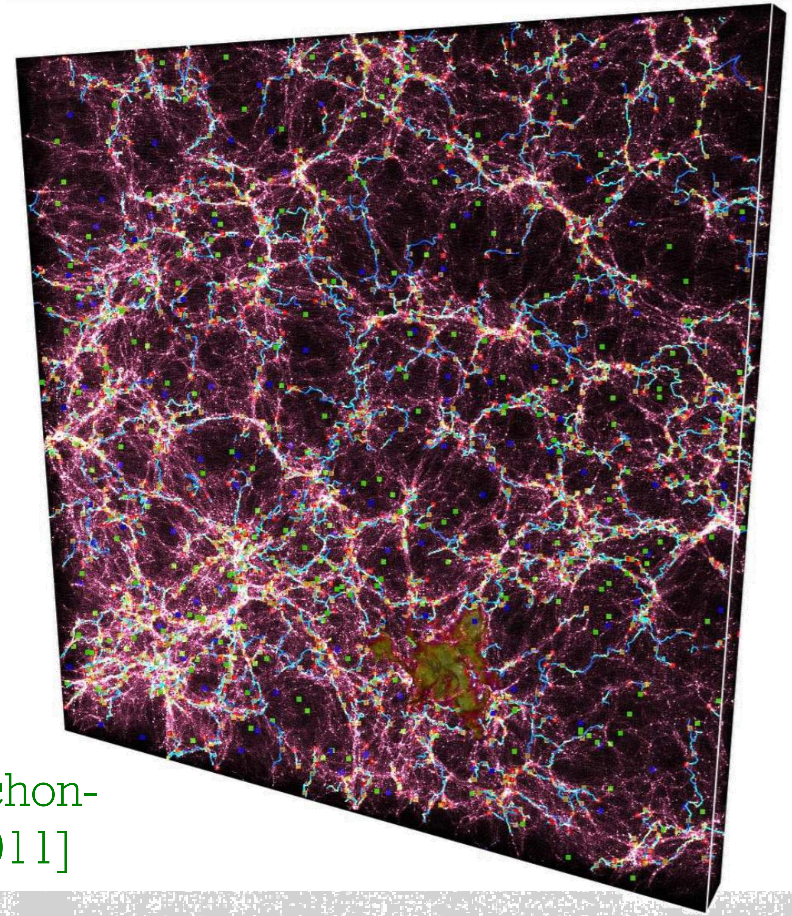
- k-chain-complex MC_k: $\langle$k-dim critical pts$\rangle$
- Boundary map ∂_k:
  all (k-1)-dim critical pts reachable by flowlines



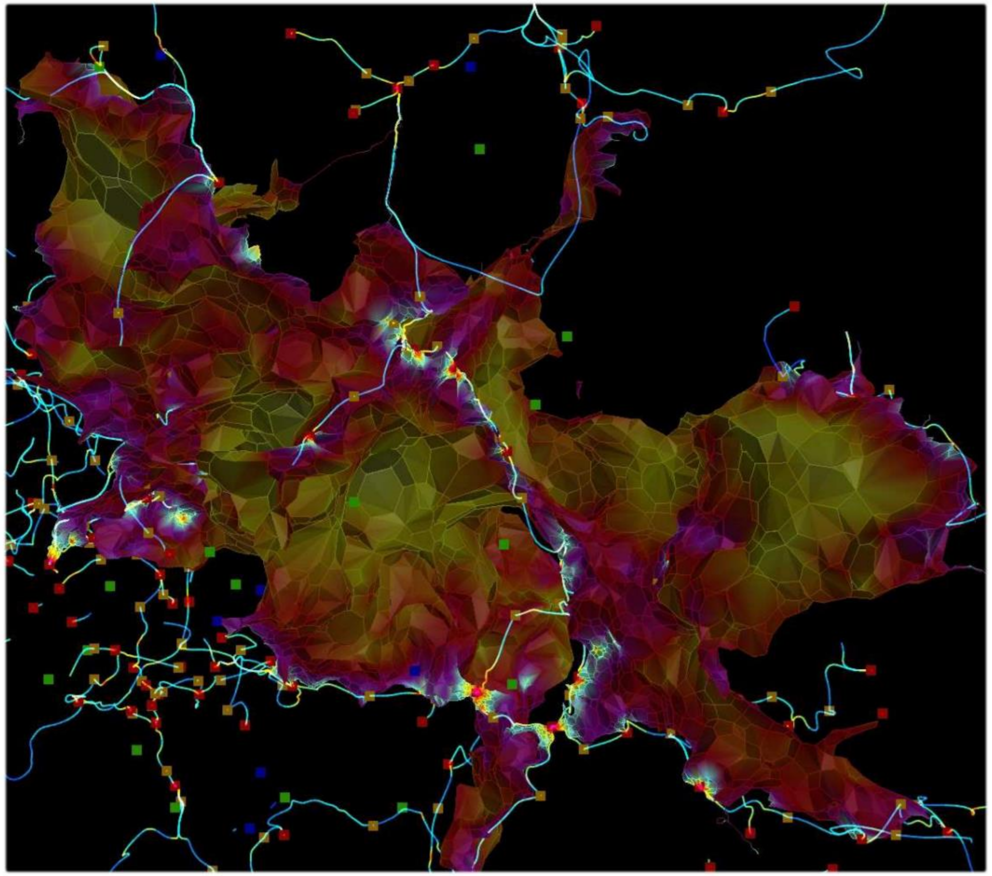
[Dey-Wang-Wang 2018]



[Sousbie 2011]



[Sousbie-Pichon-Kawahara 2011]



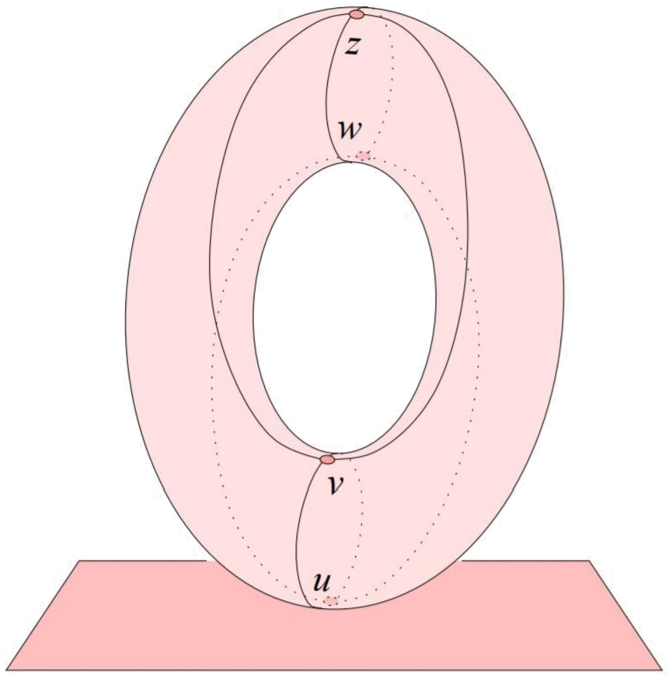
Morse Homology Theorem

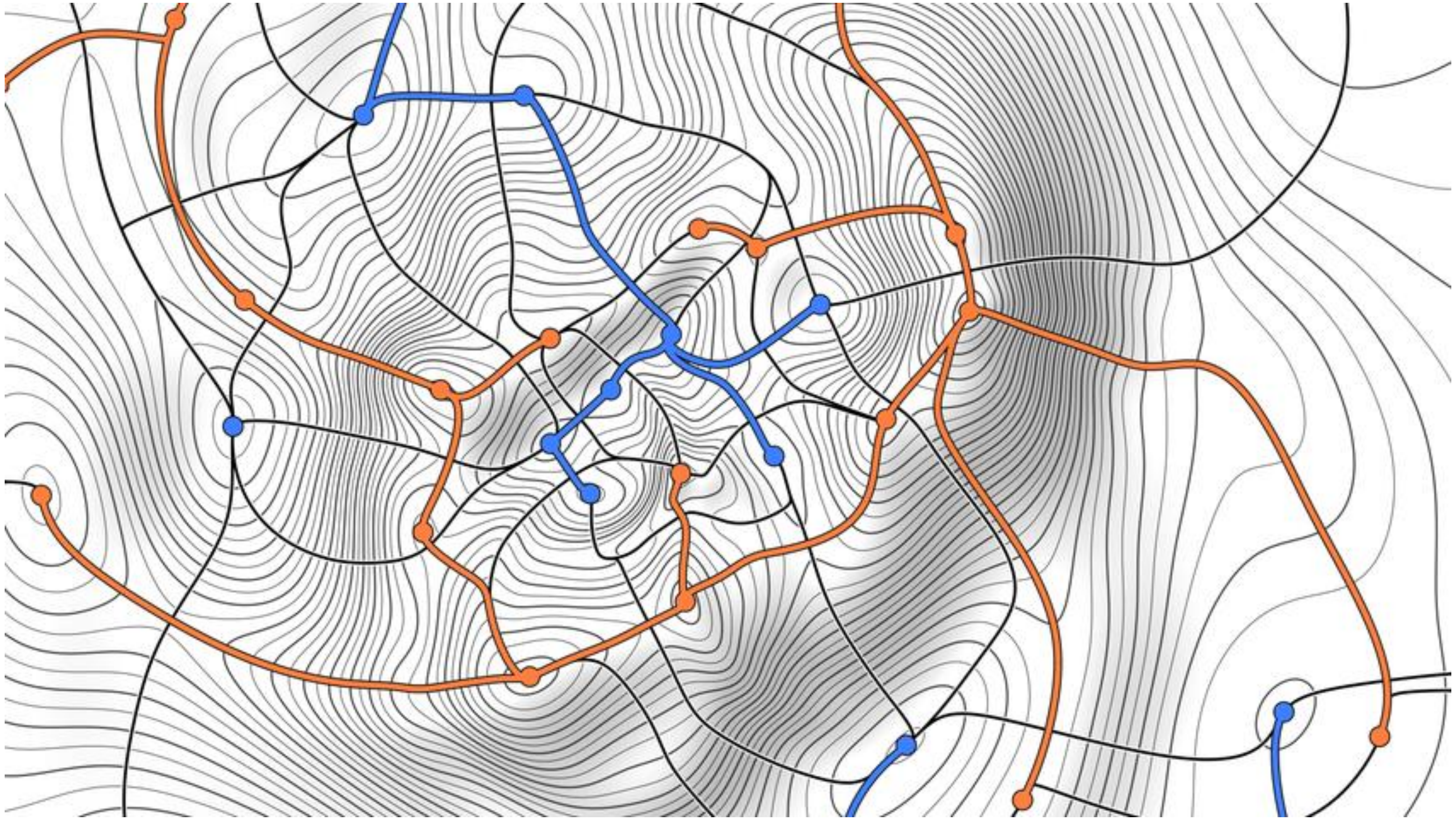
[Thom 1949] [Milnor 1963] [Smale 1967]

$$MH_n(M) \cong H_n(M)$$

(independent to the choose of height function h)

Example





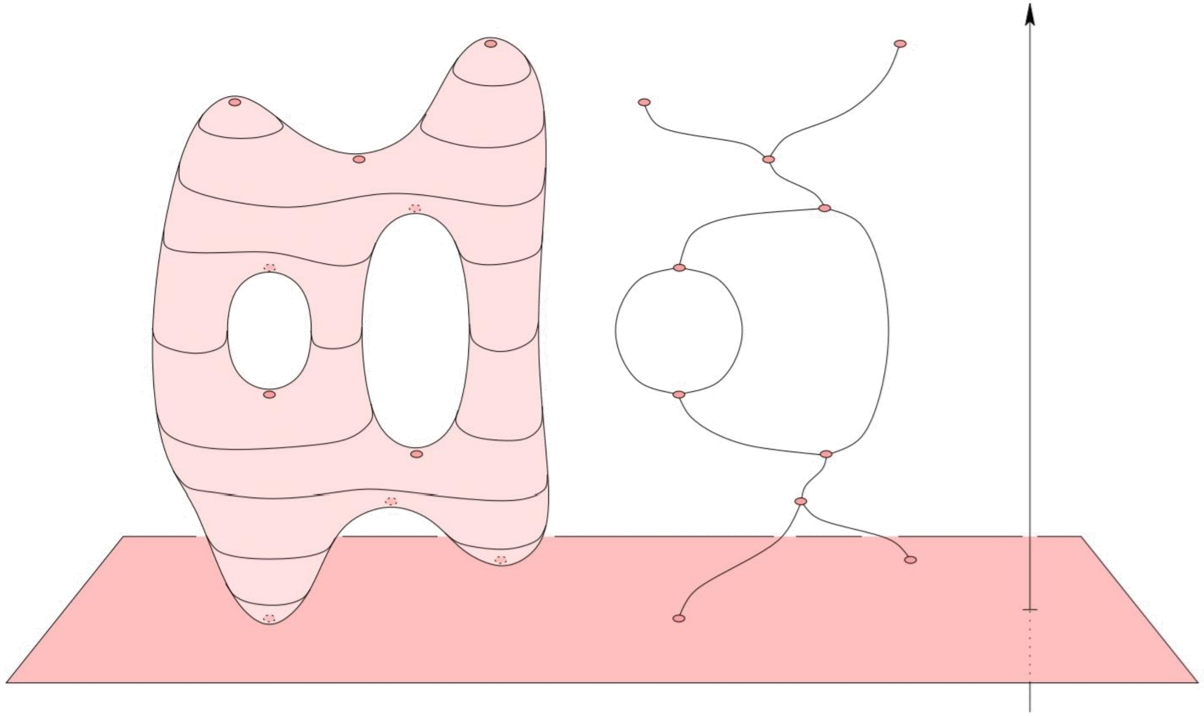
Morse Inequalities

$$\Sigma_p\, t^{\mu(p)} = \Sigma_k\, \beta_k \cdot t^k + (1+t)\cdot Q(t)$$

$$\#k\text{-dim critical pts} \geq \beta_k$$

Corollary. $\chi(X) = \Sigma_n (-1)^n \cdot \dim H_n(X)$

Corollary. $\beta_1(\mathrm{Reeb}(M)) = \beta_1(M)$ if $M=\Sigma(g,0)$



Water-rising puts Topology in Geometry

Next Time.

More applications!

What to do when the space is not a surface?