1. ***Grading homework problems.*** You are a teaching assistant of the upcoming exciting course on theory of computation. One of the problems in Homework 0 asks

   > A ***complete*** binary tree is a rooted binary tree where every node is either a *leaf* (with no children), or an *internal node* where both its children are presented.
   >
   > Prove that any complete binary tree has more leaves than the internal nodes.

   One of the students Amelia submit the following answer:

   > Let's prove the statement by induction on the number of nodes in the tree. Given a complete binary tree T with n nodes, consider all the possible way to add nodes to the tree. If we add two leaves the same leaf node x in T, x becomes an internal node. By induction hypothesis T has more leaves than the internal nodes, and while we remove one leaf x and turn it into an internal node, we also added two leaves and thus the difference between number of leaves and internal nodes remains unchanged.

   Another student Bo submitted this answer:

   > T has n nodes. If r is a leaf then we are done. If r is an internal node, remove r from T and now we have T_1 and T_2. Let ℓ_i be the number of leaves and m_i be the number of internal nodes in T_i. By induction, we have $\ell_i > m_i$ and $\ell_i + m_i = n_i$ for each i, and $n_1 + n_2 = n - 1$. Now adding r back, the leaves in T_i are still leaves and the internal nodes in T_i are still internal nodes, thus $\ell = \ell_1 + \ell_2 \geq (m_1 + 1) + (m_2 + 1)$ by the inequalities above. Because r itself is an internal node, one has $m = m_1 + m_2 + 1$, and thus $\ell > m$.

   A bright student Charlie wrote this:

   > Charge every leaf to its parent; if an internal node has more than one charge, charge it to its own parent. Because the tree is binary and complete, every node will receive exactly two charges and then transfer one excess charge to its parent. In the end the whole graph is charged and the root has one extra charge, which show that there are one more leaves than the internal nodes.

   Yet another student Delta wrote this:

   Yet another student Erin wrote this:

   > Look at the picture. There is always an internal node near the bottom of the tree that has two children, both being leaves. Removing the two children will decrease the number of leaves by two, but at the same time turning their parent into a leaf, thus keeping

   Criticize all the answers from the students by pointing out any false statements, logic flaw (where a true statement does not follow immediately from the previous true statement), and sentences that are not parsable.[1]

[1]Bonus points: Criticize your own answers to the rest of the homework problems. As a rule of thumb, your answers should at least pass your *own* criticism.

2. ***Neighborhoods in graphs.*** Let G be an undirected graph with exactly one edge between any pair of vertices, but possibly with self-loops (an edge whose head and tail is the same vertex). Define V and E to be the vertex set and edge set of G, respectively.

   The ***neighborhood*** function $N : V \to 2^V$ takes a vertex v and maps it to the subset of vertices of G that each of them is adjacent to v. Fix a starting vertex s in V. The ***k-th neighborhood*** of s is defined to be $N(N(\cdots N(s)\ldots))$, where $N(\cdot)$ is applied k times.

   (a) Describe an algorithm to decide if for any vertex t in V, there is an integer k such that the k-th neighborhood of s contains t.

   (b) Describe an algorithm to decide if there is an integer k, such that for any vertex t in V the k-th neighborhood of s contains t.

3. ***Balanced parentheses.*** A ***balanced parenthesis*** is a string over the two symbols [and], defined *recursively* as one of the following:

   - an empty string ε;
   - string [w] for some balanced parenthesis w;
   - string xy for some *nonempty* balanced parentheses x and y.

   For example, [[[][]]][][[[]]][] is a balanced parenthesis of length 18.

   (a) Prove that removing a pair of consecutive symbols [] (if exists) from any balanced parenthesis results in another balanced parenthesis.

   (b) Prove that removing a pair of consecutive symbols][(if exists) from any balanced parenthesis results in another balanced parenthesis.