



# **INTRODUCTION TO COMPUTATIONAL TOPOLOGY**

**HSIEN-CHIH CHANG**  
**LECTURE 6, SEPTEMBER 30, 2021**

# ADMINISTRIVIA

- Thanks for all the feedback!



# GEOMETRIC FOLDING ALGORITHMS

## 6.849: Geometric Folding Algorithms: Linkages, Origami, Polyhedra (Fall 2020)

Prof. [Erik Demaine](#); [Martin Demaine](#); TAs Yevhenii Diomidov & Klara Mundilova

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[Home] [\[Lectures\]](#) [\[Problem Sets\]](#) [\[Project\]](#) [\[Coauthor\]](#) [\[Github\]](#) [\[Accessibility\]](#)

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### Overview

*the algorithms behind building TRANSFORMERS and designing ORIGAMI*

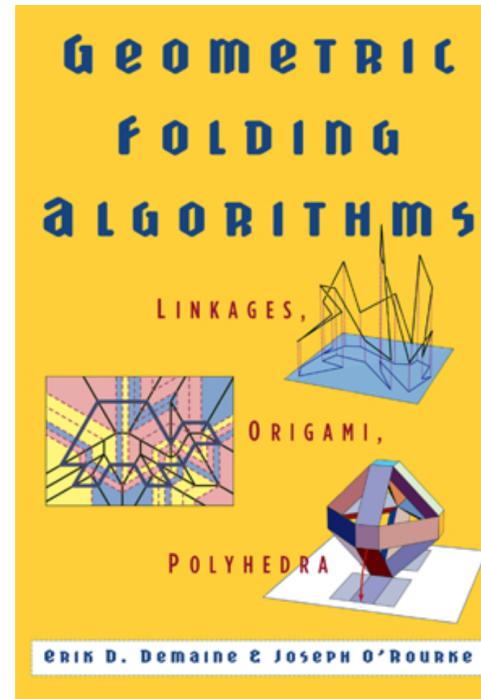
Whenever you have a physical object to be reconfigured, **geometric folding** often comes into play. This class is about algorithms for analyzing and designing such folds. Motivating applications include

- automated design of [new](#) and [complex](#) origami
- using 2D fabrication technology to manufacture [complex 3D objects](#)
- transforming robots by [self-folding sheets](#) or chains
- how to [fold robotic arms without collision](#)
- how to [bend sheet metal](#) into [desired 3D shapes](#)
- understanding how [proteins fold](#)

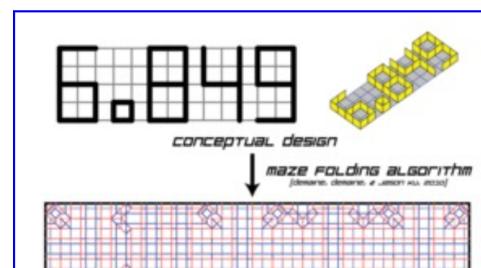
Major progress have been made in recent years in many of these directions, thanks to a growing understanding of the mathematics and algorithms underlying folding. Nonetheless, many fundamental questions remain tantalizingly unsolved. This class covers the state-of-the-art in folding research, including a variety of open problems, enabling the student to do research and advance the field.

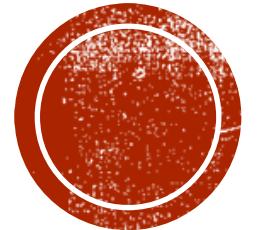
### Fully Online Format

Most course material is covered in [video lectures](#) recorded in 2010 (already watched by over 19,000 people), which you can conveniently play at faster speed than real time. There may also be some **new material** presented by the professor and/or guest lecturers, which will be recorded for asynchronous viewing.



[textbook](#)

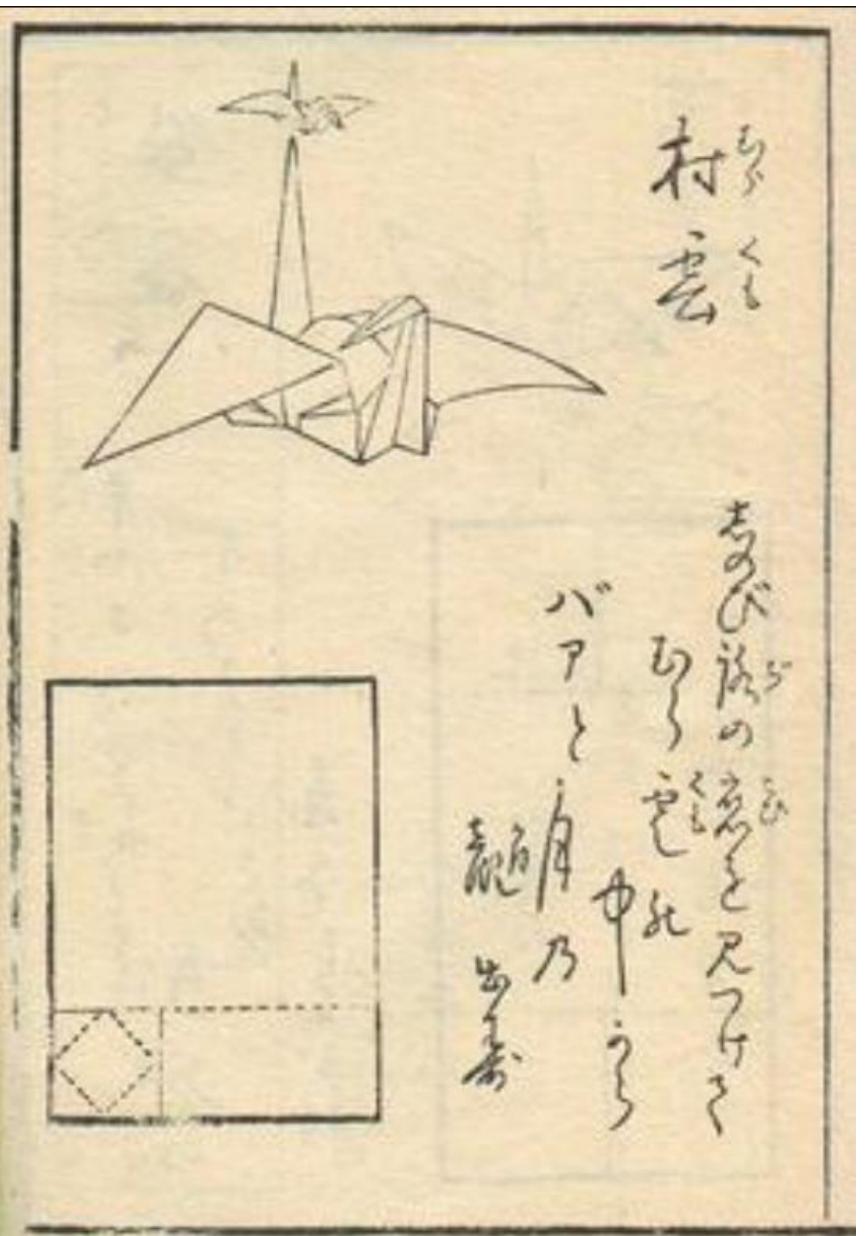




# PAPER FOLDING

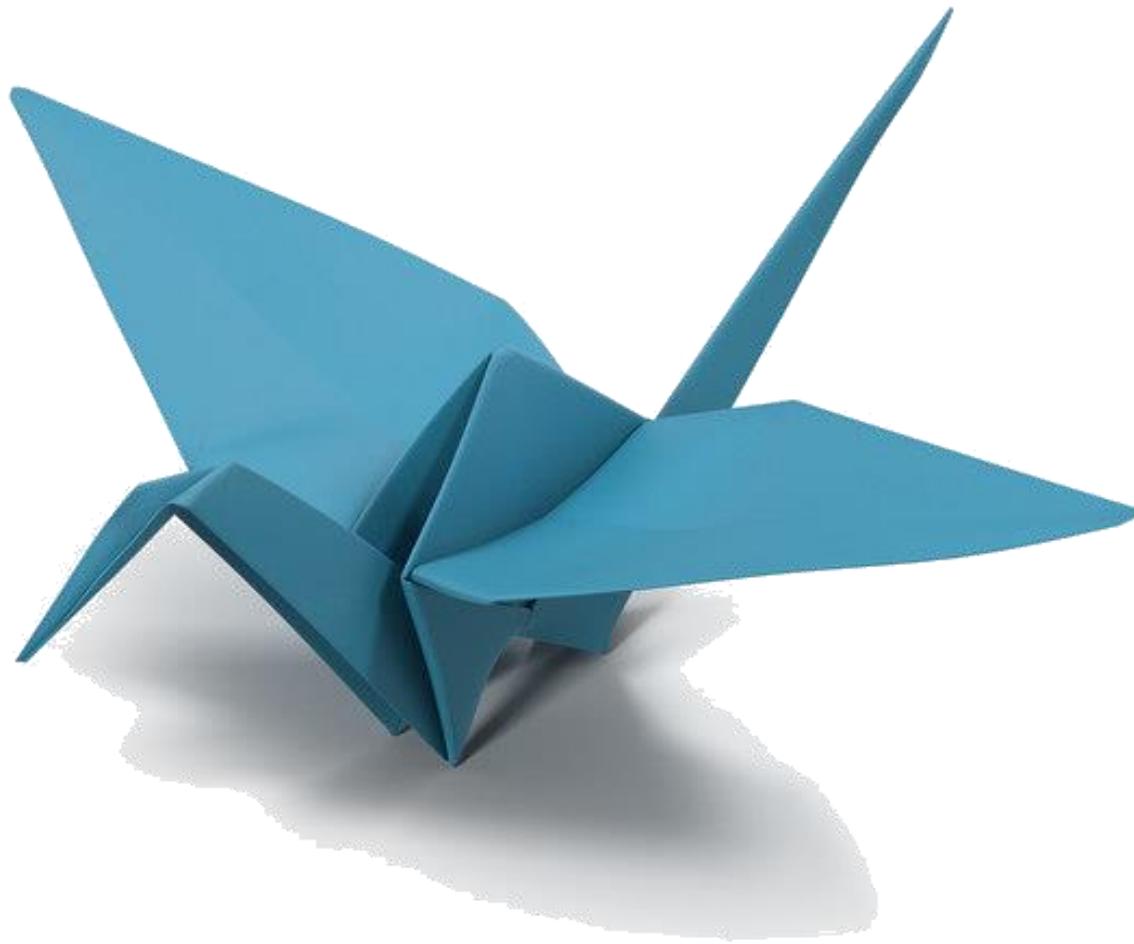


# 折り紙 ORIGAMI

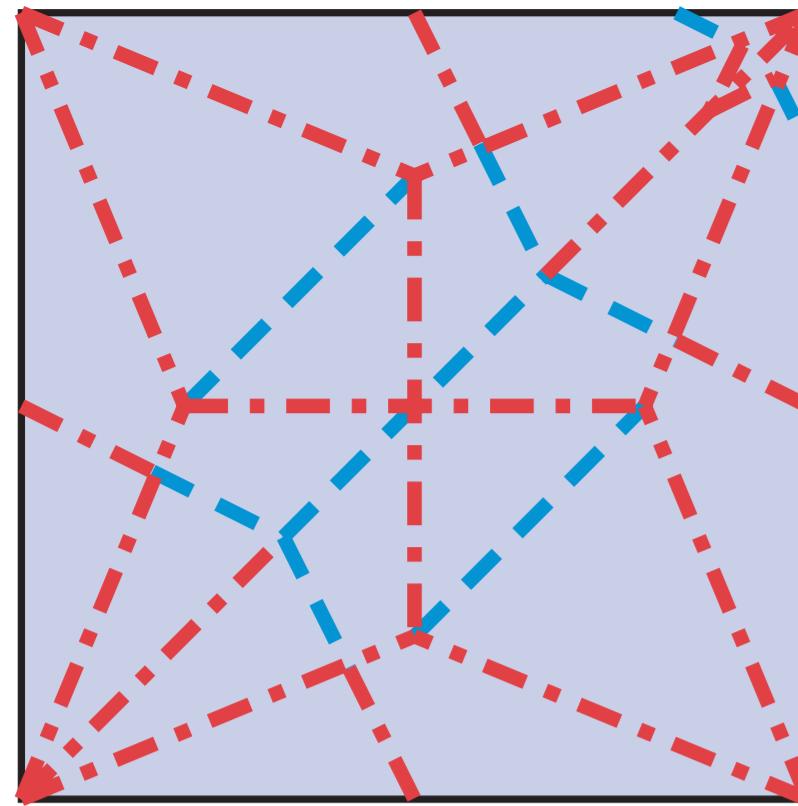
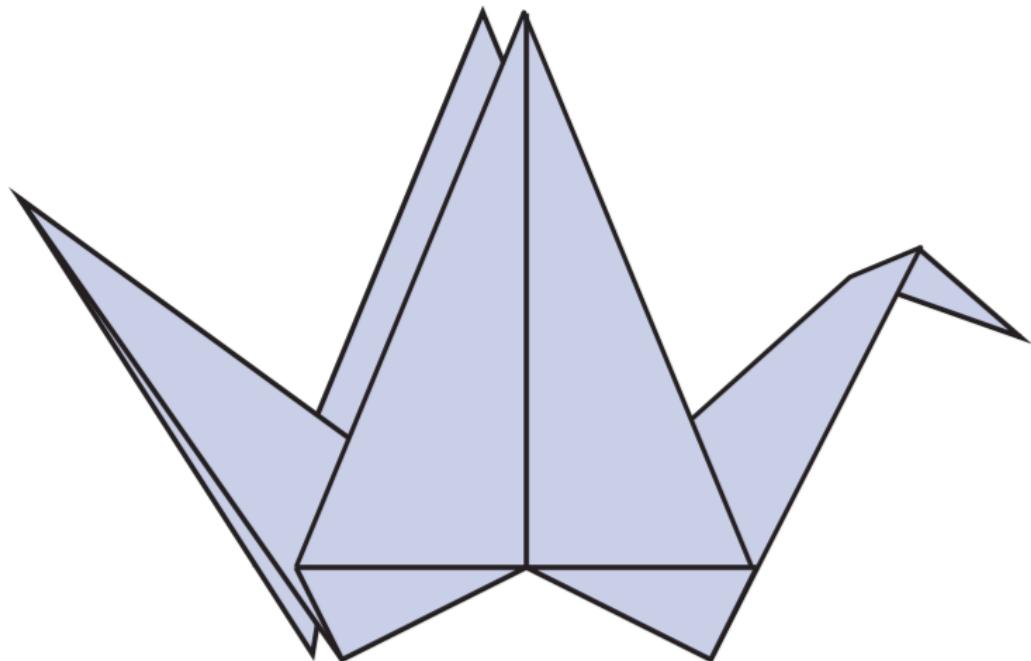


[秋里籬島 Akisato Ritō, 1798]

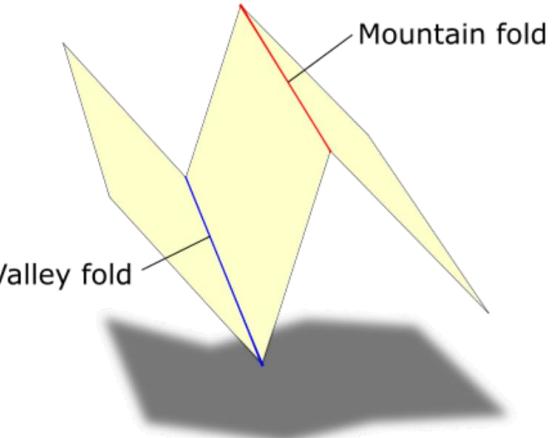
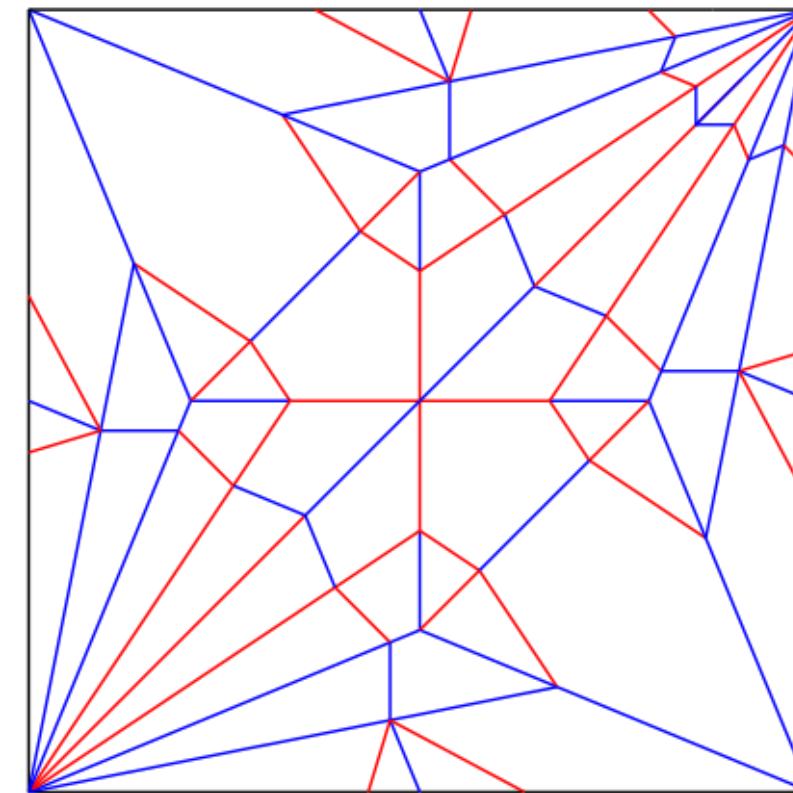
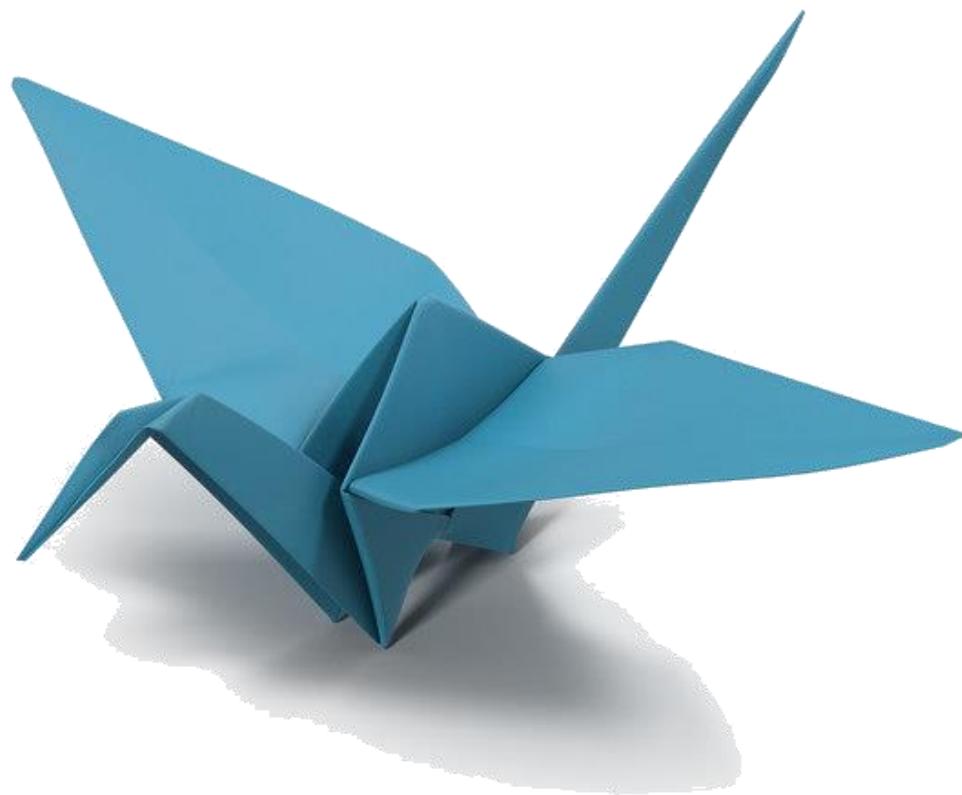
# FOLDED STATE



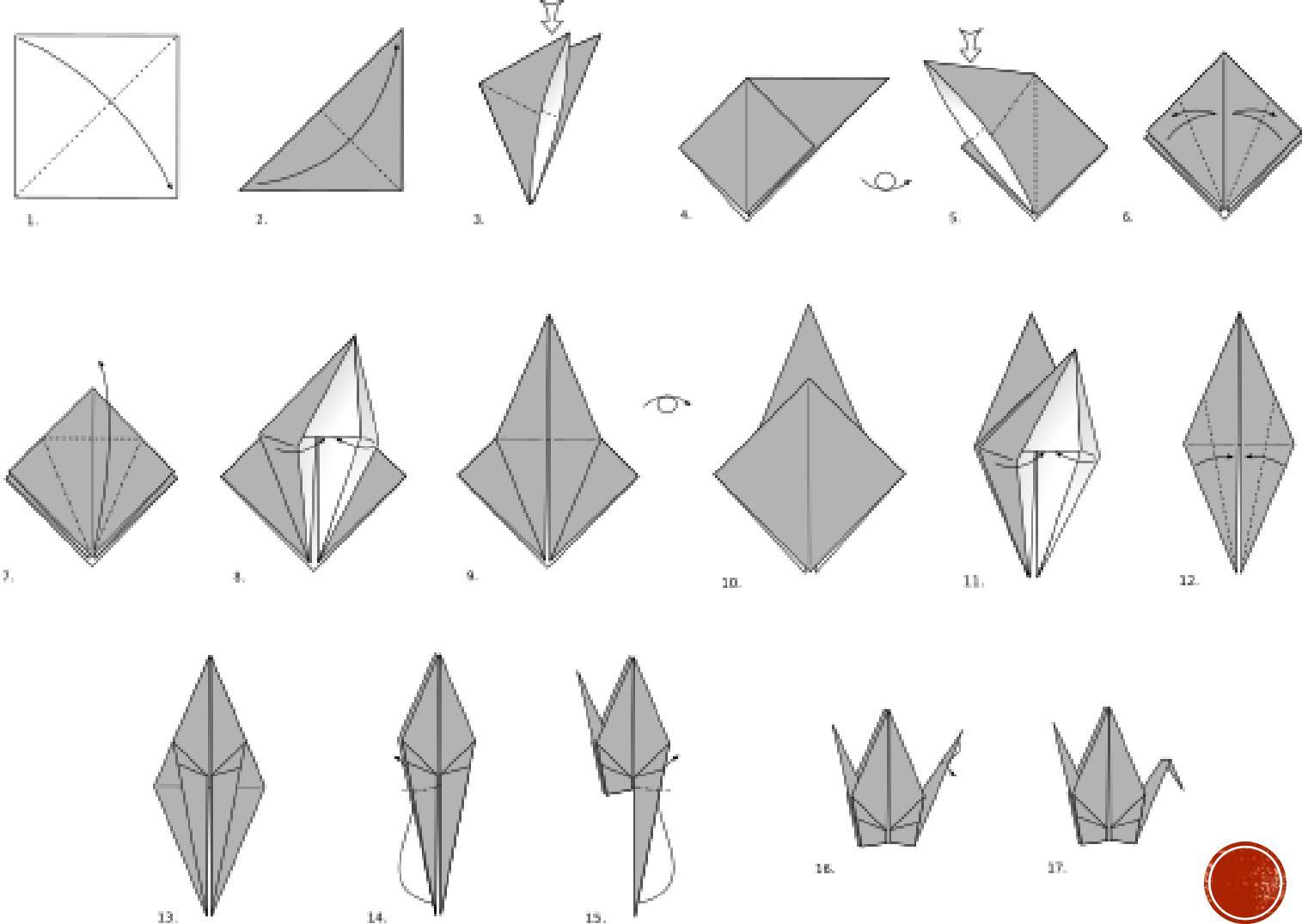
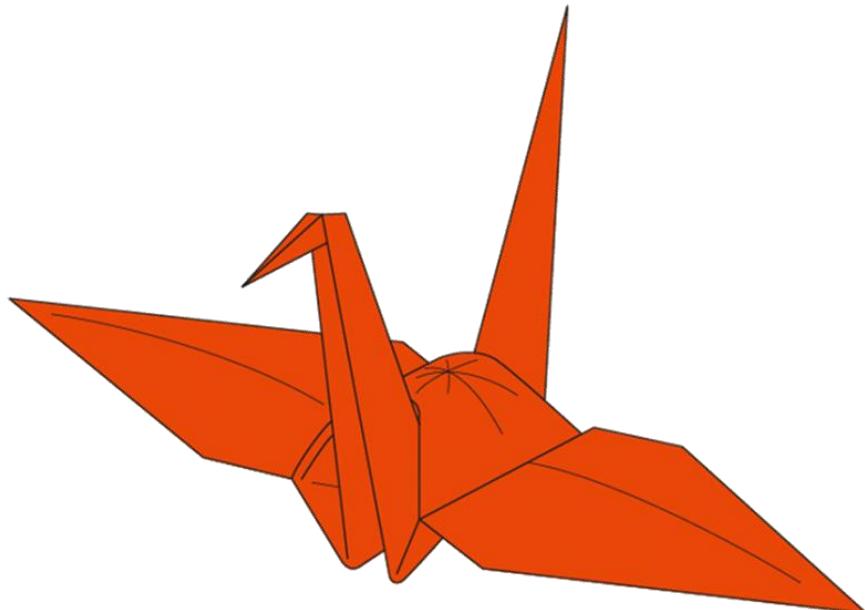
# CREASE PATTERN



# CREASE PATTERN



# FOLDED MOTION



# FOLDING PROBLEMS

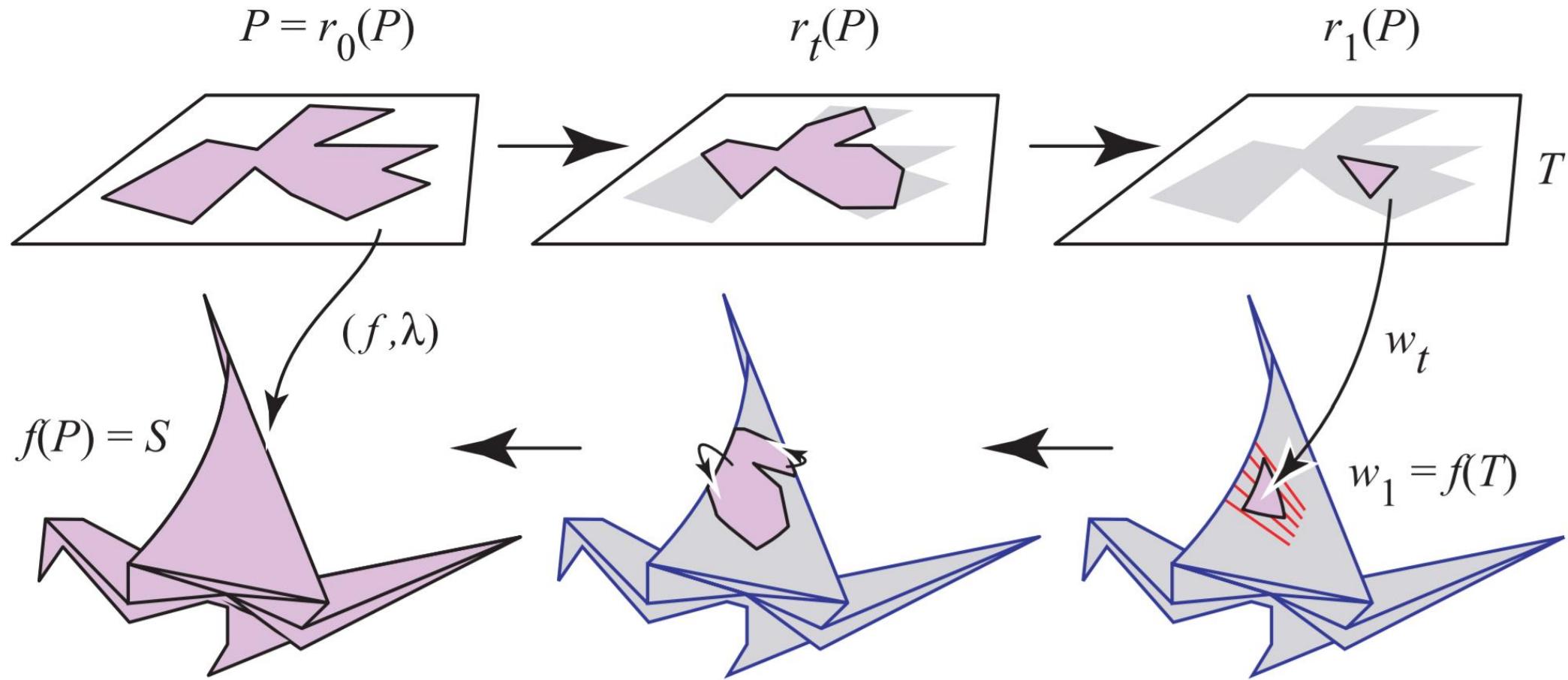
- Given a crease pattern, can it be flat-folded?
- Given a flat-foldable mountain-valley pattern, what is the final state?
- Folding into arbitrary 2D/3D shape?



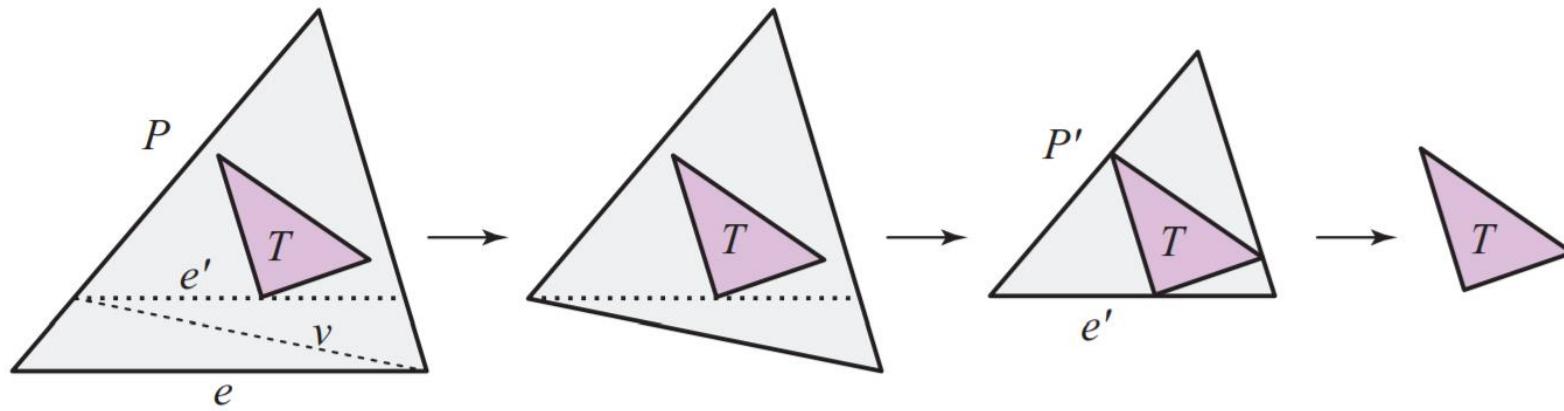
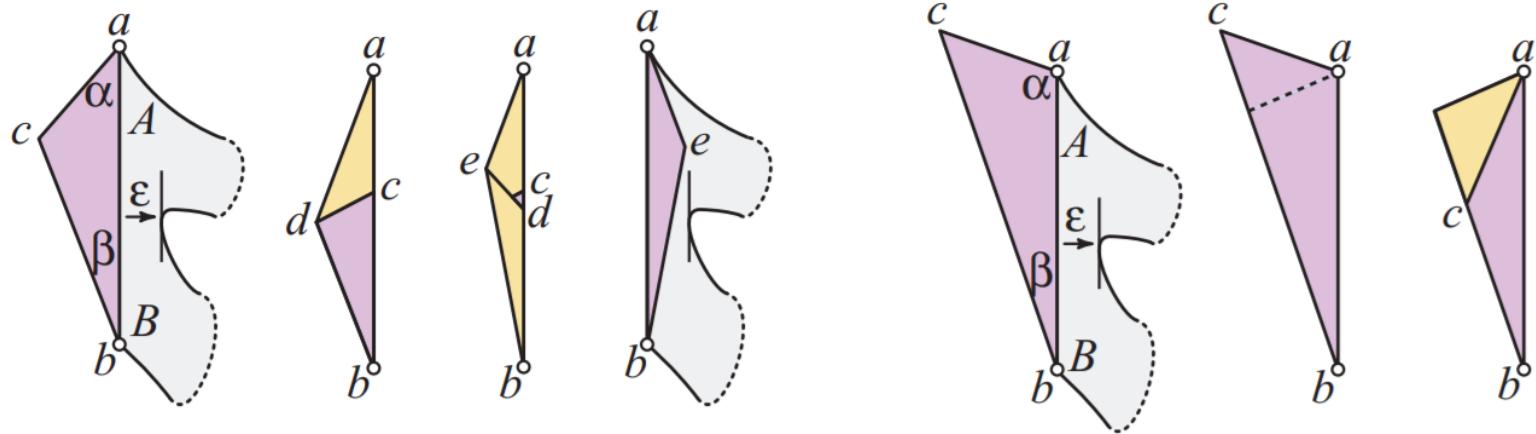
**PROPOSITION.** There is always a folding motion from the unfolded state to any folded state.



**PROPOSITION.** There is always a folding motion from the unfolded state to any folded state for flat papers.

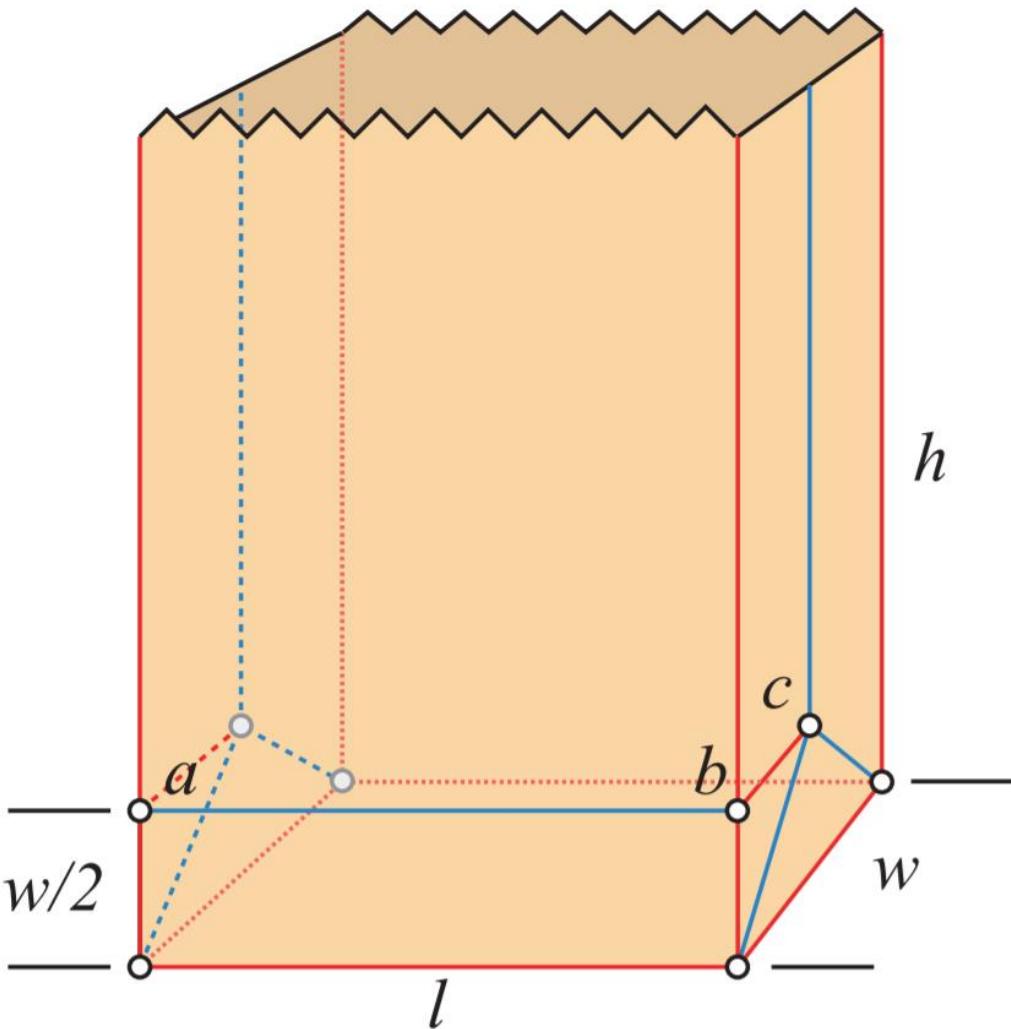


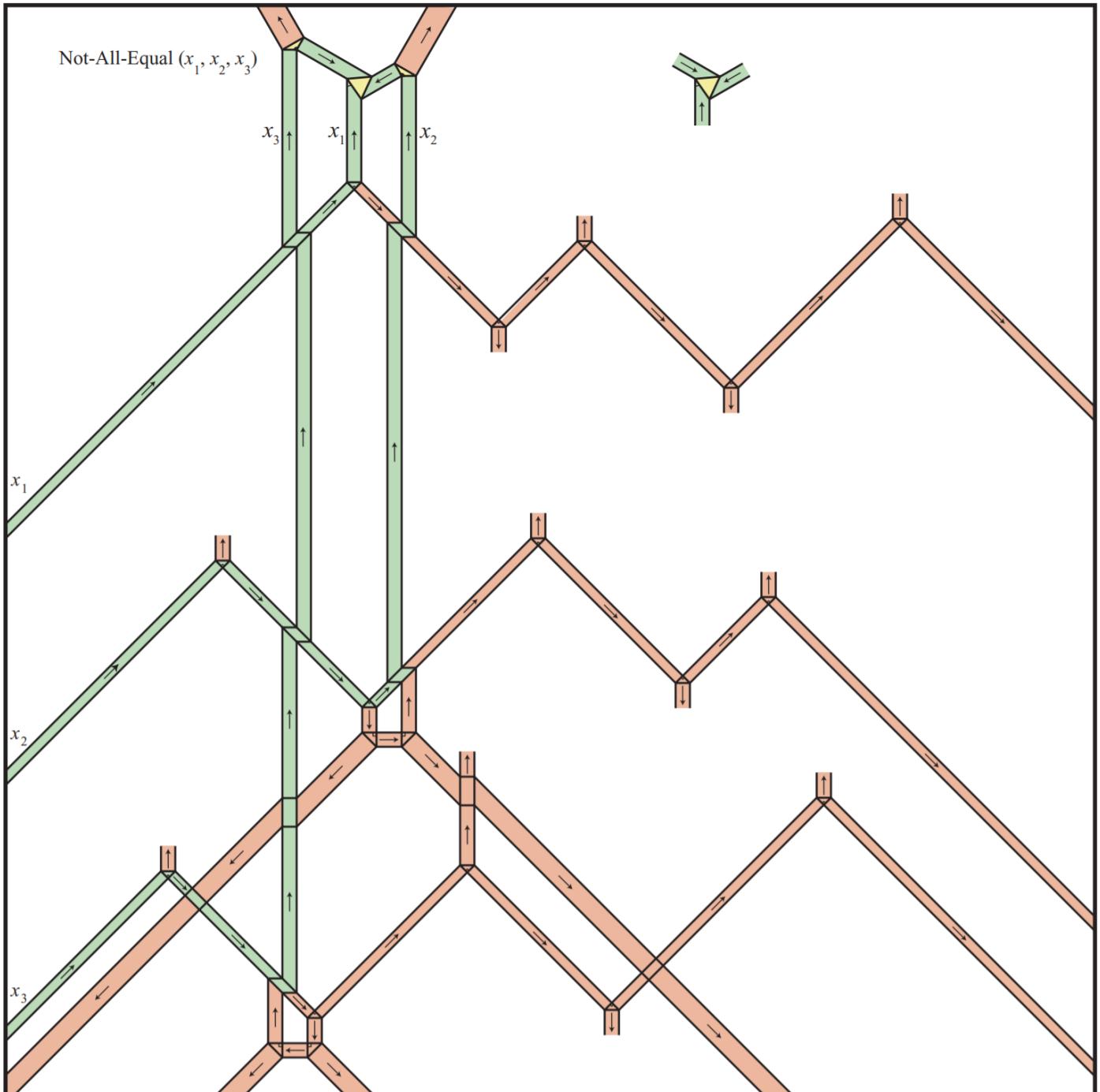
**PROPOSITION.** There is always a folding motion from the unfolded state to any folded state.



# BENDING IS NECESSARY!

- OPEN:  
Does folding motion exist  
when the original paper  
is not flat?



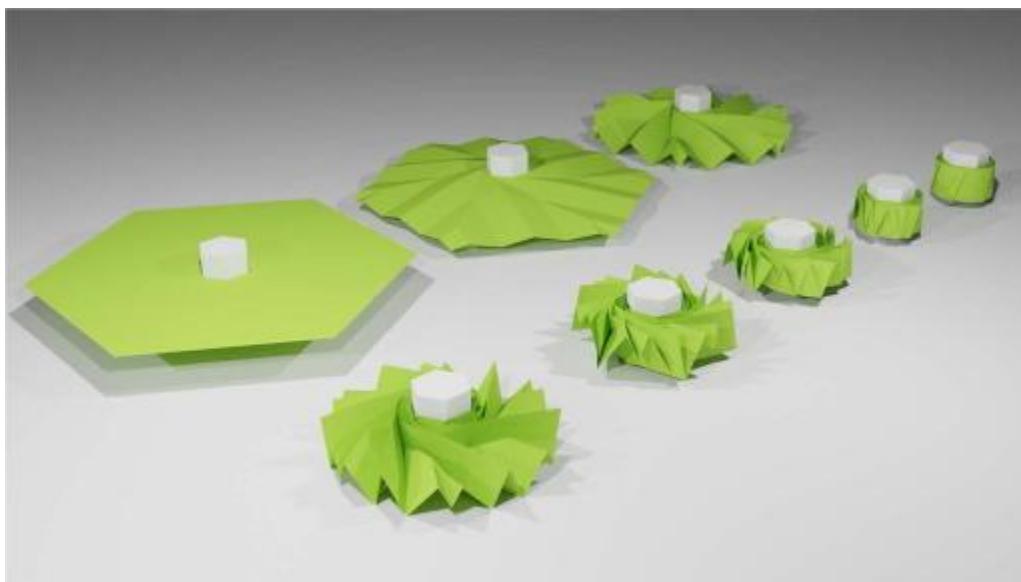
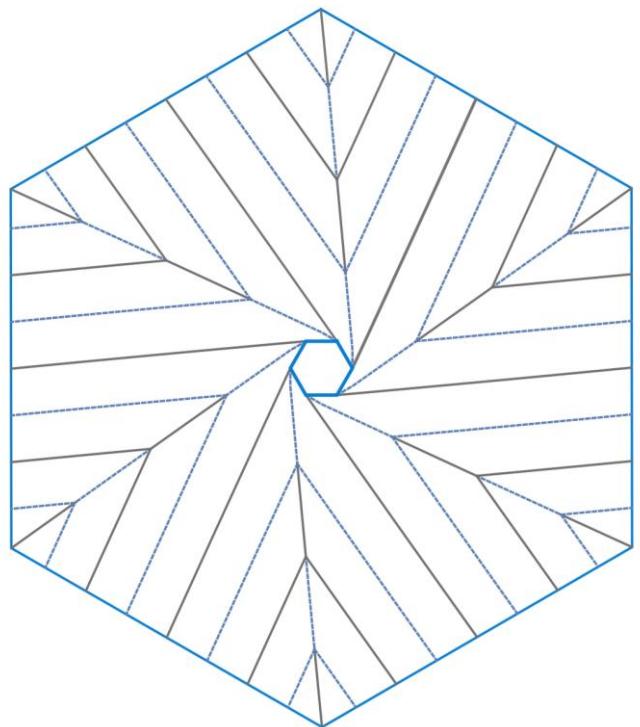
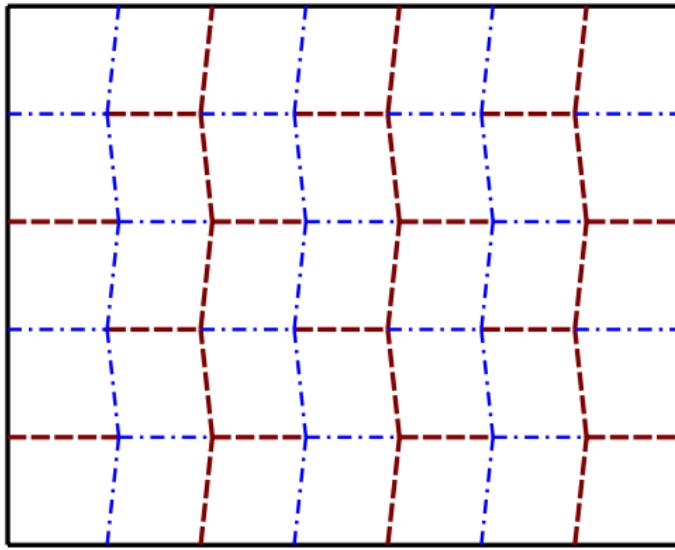


# FOLDING IS HARD

- Given a crease pattern, it is NP-hard to decide if can be flat-folded.
- Given a flat-foldable mountain-valley pattern, it is NP-hard to compute the flat-folded state.

[Bern-Hayes 1996]





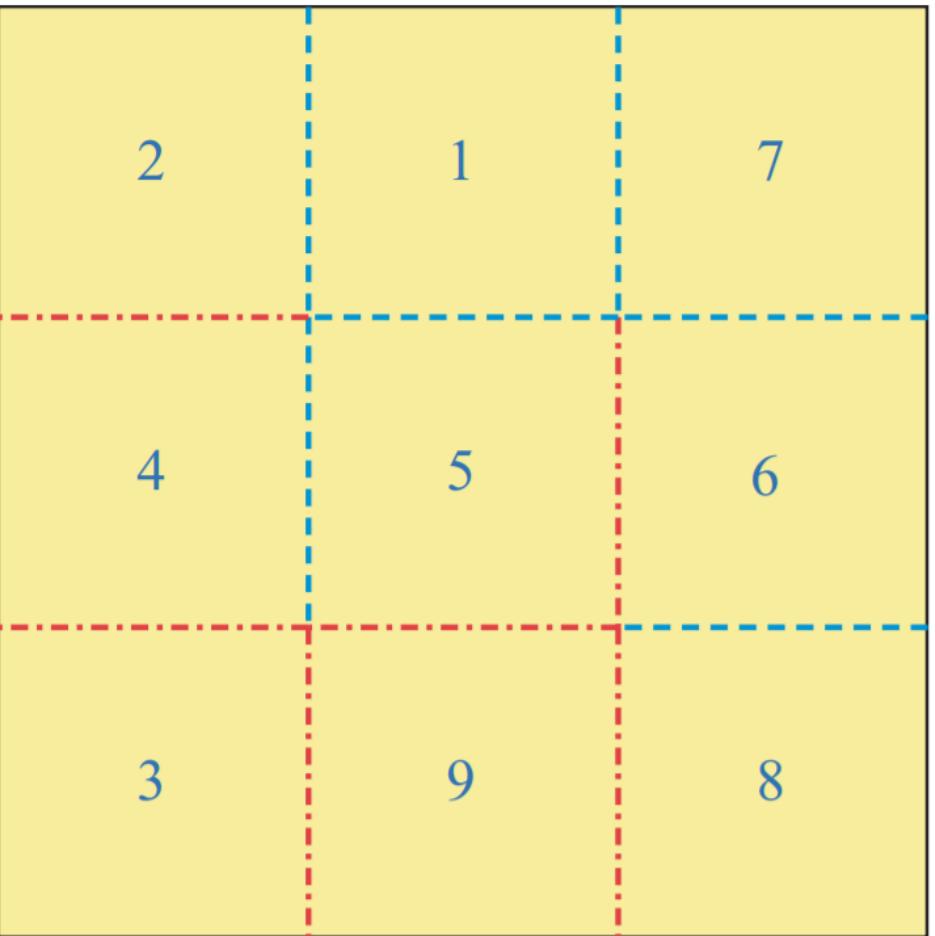
# FOLDING IS HARD

- Not all folding motions are simple.

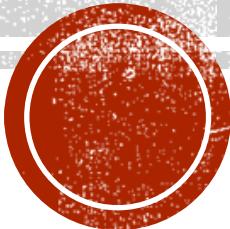


# FOLDING IS HARD

- **OPEN:**  
Given a 2-by-n grid with mountain-valley pattern,  
can it fold flat?

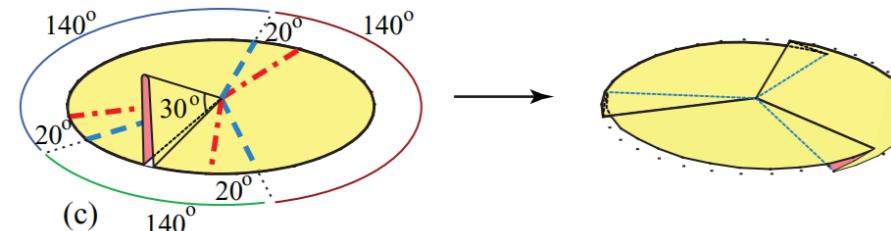
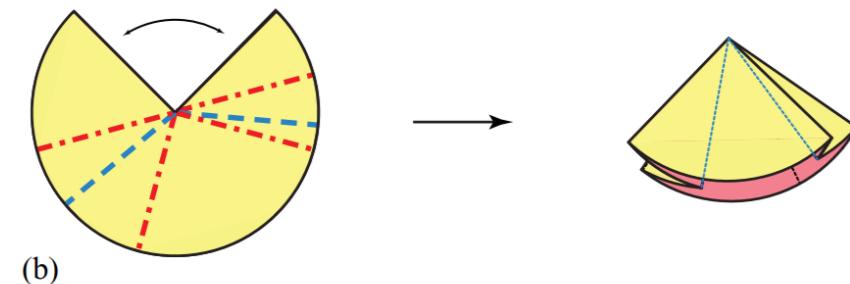
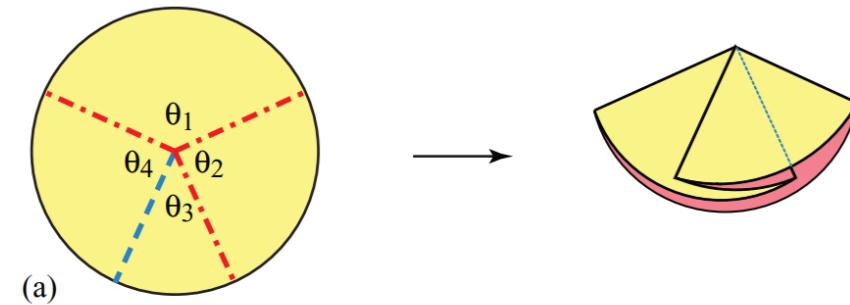


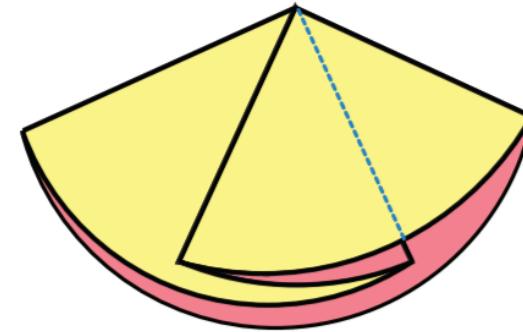
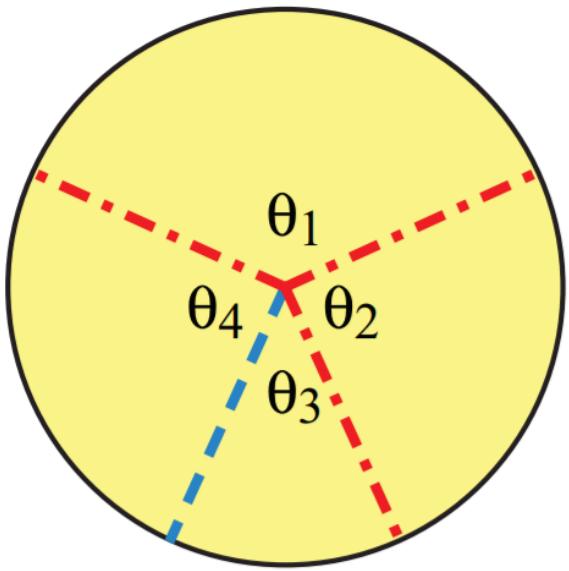
**LET'S LOOK AT SIMPLE FOLDING  
AT A SINGLE VERTEX**



# SINGLE-VERTEX CREASE PATTERN

- Disk of paper
- Exactly  $n$  creases
- Angles  $\theta_1, \theta_2, \dots, \theta_n$

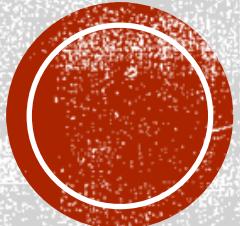




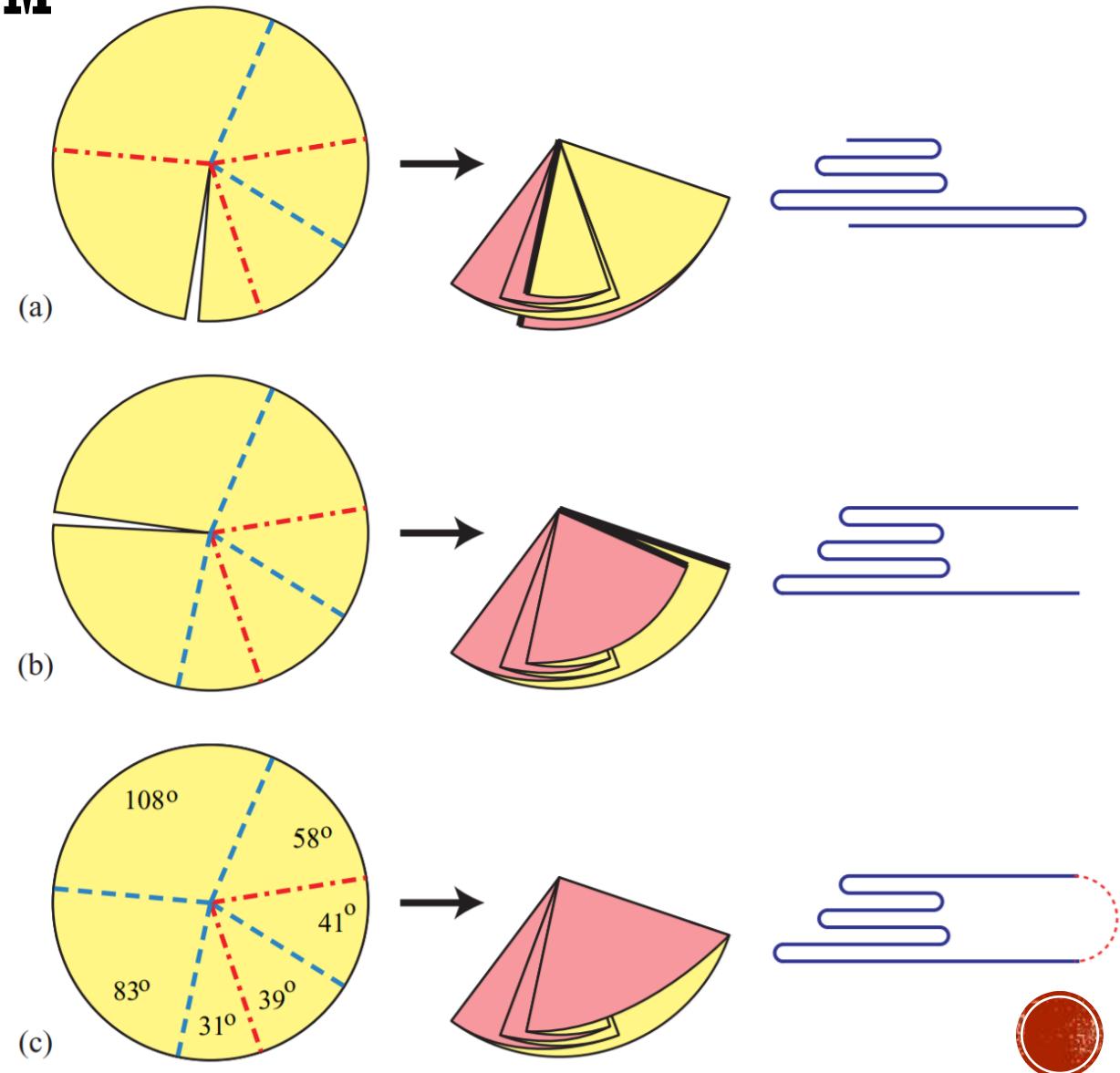
## KAWASAKI'S THEOREM

[Kasahara 1989, Justin 1989]

A single-vertex crease pattern is flat-foldable if  
 $(\theta_1 + \theta_3 + \dots + \theta_{n-1}) - (\theta_2 + \theta_4 + \dots + \theta_n) \in \{0, \pm 2\pi\}$

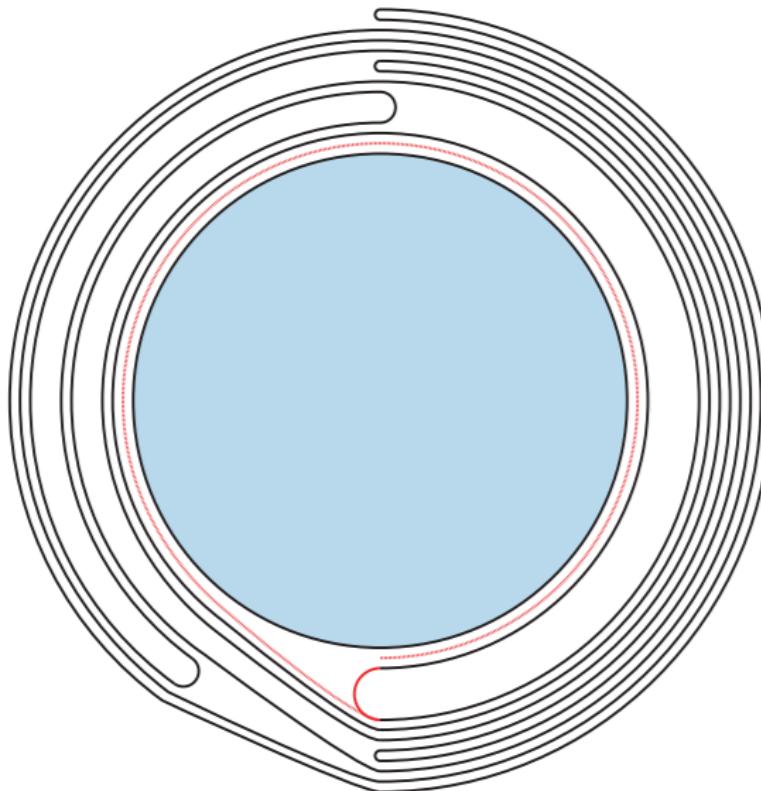


# PROOF OF Kawasaki's THEOREM



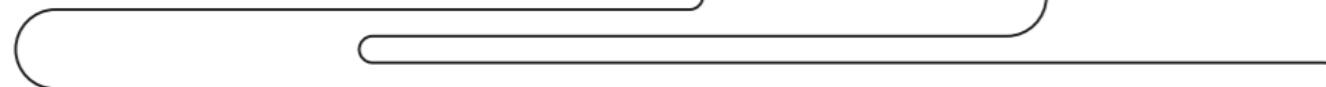
# PROOF OF Kawasaki's THEOREM

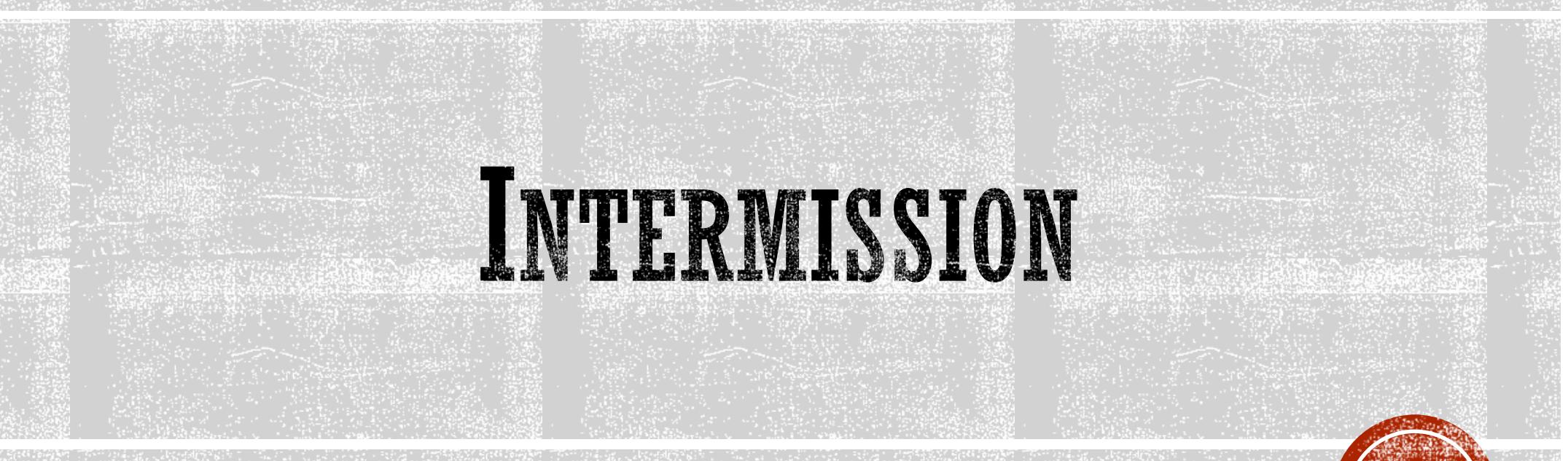
(a)



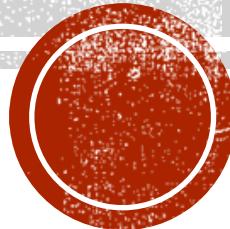
$0\pi$        $1\pi$        $2\pi$        $3\pi$        $4\pi$        $5\pi$

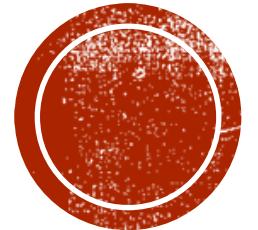
(b)





**INTERMISSION**





# PAPER UNFOLDING



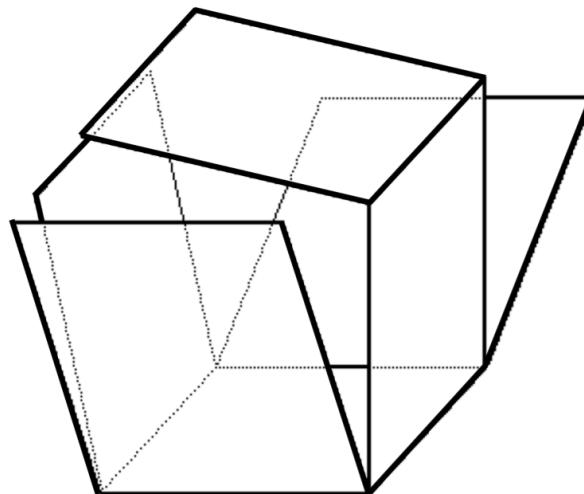
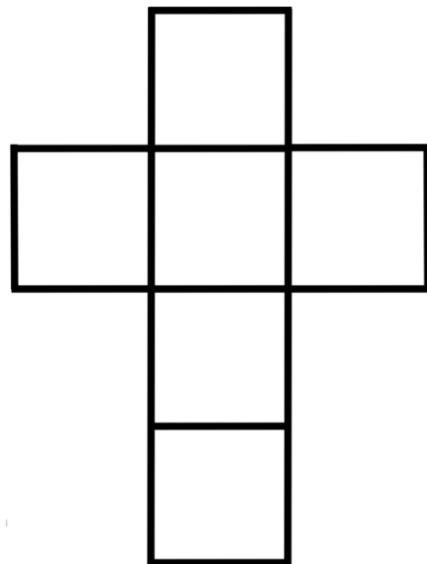
# UNFOLDING PROBLEMS

- Given a 3D object, can it be unfolded into something flat?

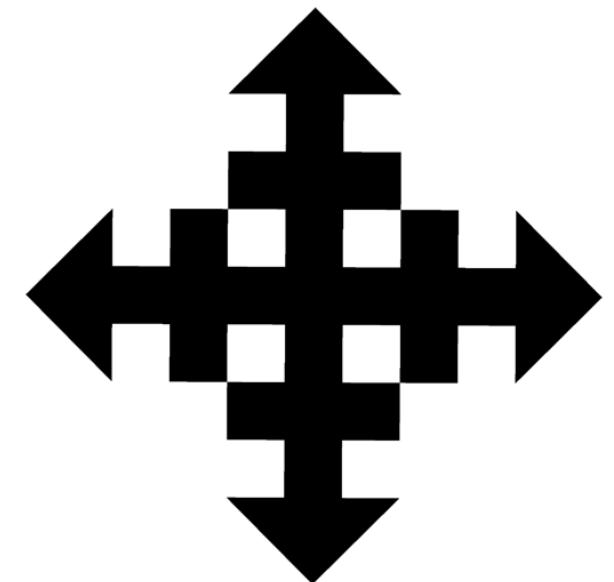
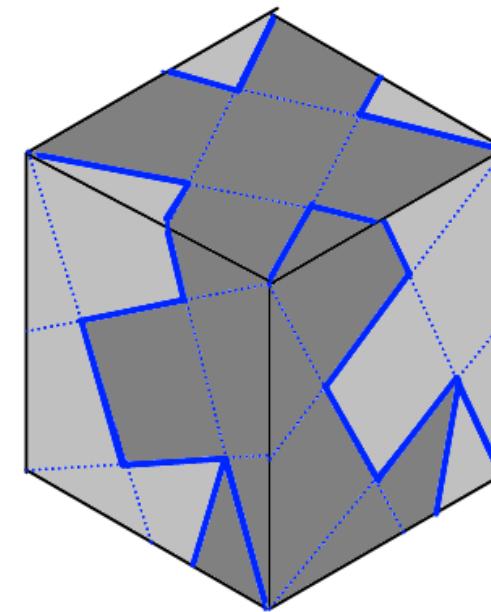


# Two DIFFERENT UNFOLDING RULES

- Edge-unfolding

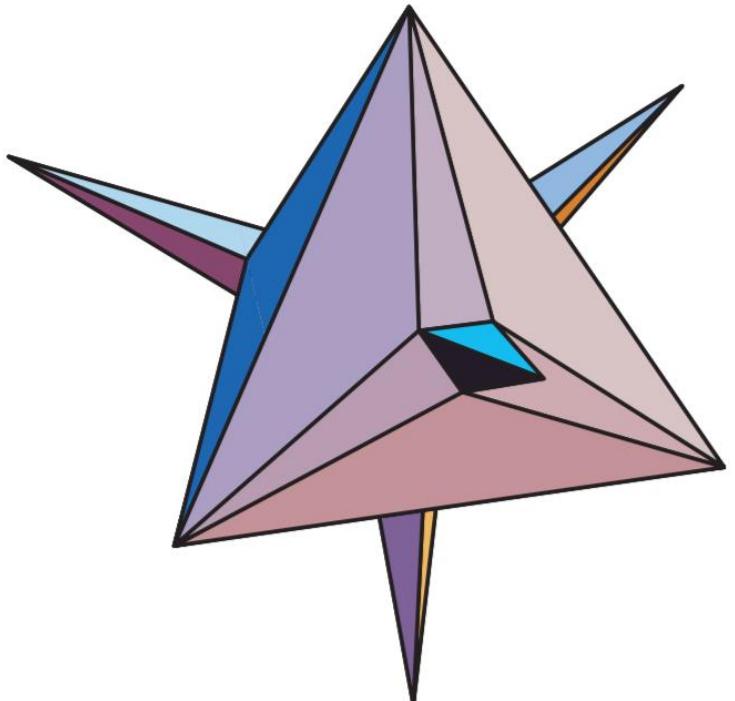


- General-unfolding

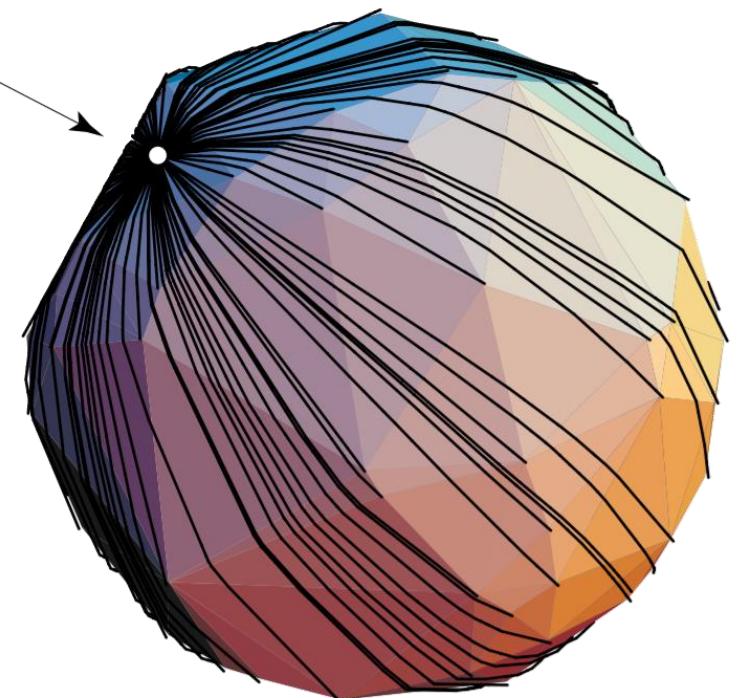


**Table 22.1: Status of main questions concerning nonoverlapping unfoldings**

Shapes	Edge unfolding?	General unfolding?
Convex polyhedra	Open	YES
Nonconvex polyhedra	NO	Open



[Bern-Demaine-Eppstein-Kuo-Mantler-Snoeyink 2003]



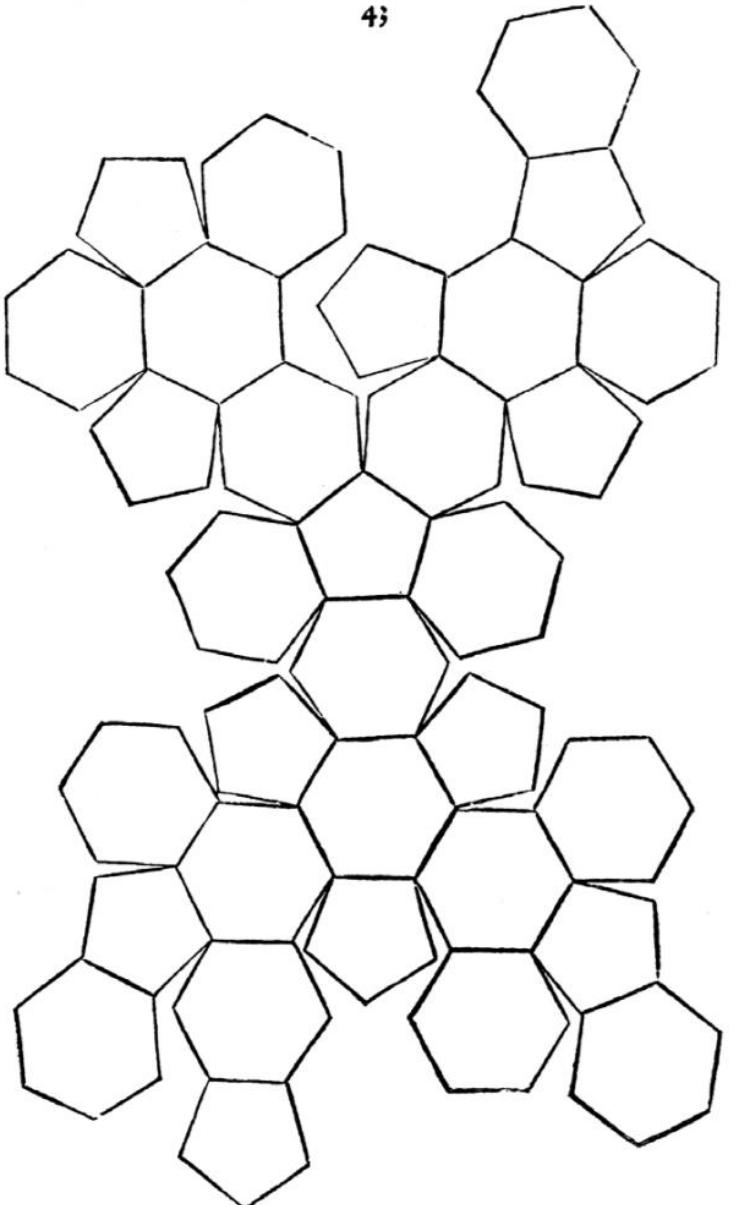
[Xu 1996]

# KNOWN RESULTS



**S**In anders das mach aus zweinsig sechsecketer flachen vnd windlich/  
so man darzu thut zwölß fünfecketer flacher felder/ so die gleichsietig gegen den sechsecken  
seiden sind/ vnd in jnen selbs auch gleich windlich vnd ordentlich an cynander gesetzt wer-  
den/ wie ich das offen im plano hernach hab aufgerissen / So man dann das alles zusammen  
schleust/ so würt ein corpus daraus/ das gewinnet zwey vnd sechzig ecke/ vnd neunzig scharfster  
stüten/ dis Corpus rüret in einer holen kugeln mit allen seinen ecken an.

43



# UNFOLDING POLYHEDRON

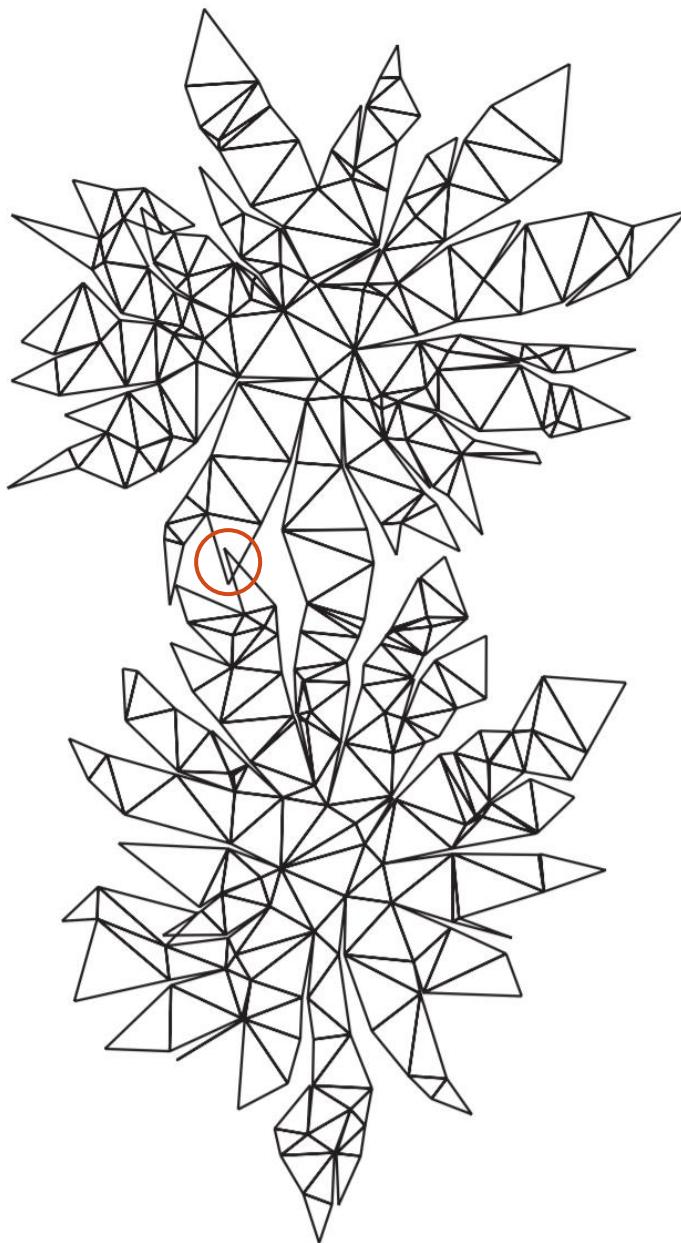
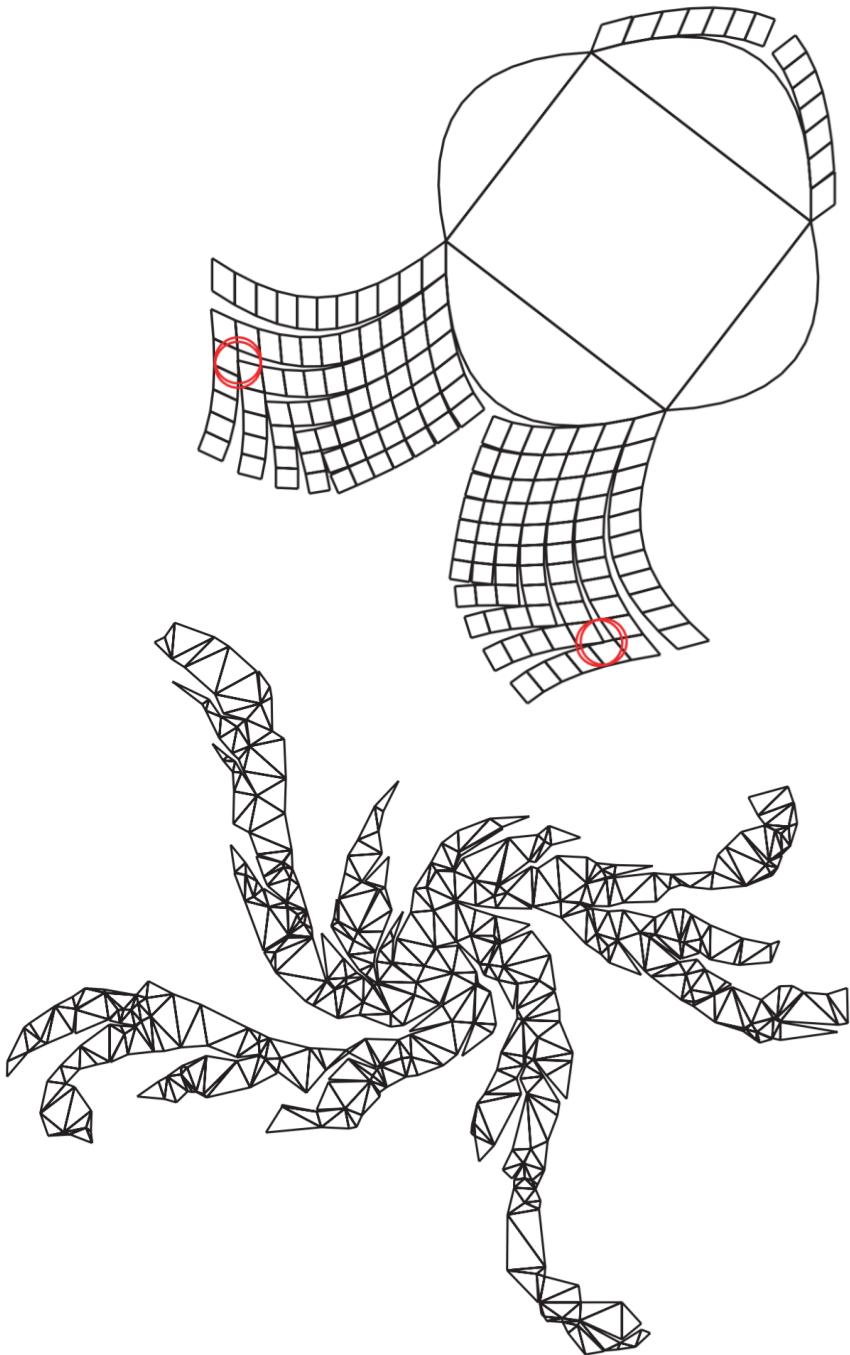
- **OPEN:**  
Does every convex polyhedron have an edge-unfolding?

[Albrecht Dürer 1525]

[Shephard 1975]

[Schlickenrieder 1997]





# UNFOLDING POLYHEDRON

- **OPEN:**  
Does every convex polyhedron have an edge-unfolding?

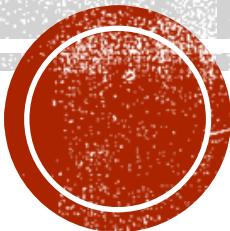
[Albrecht Dürer 1525]

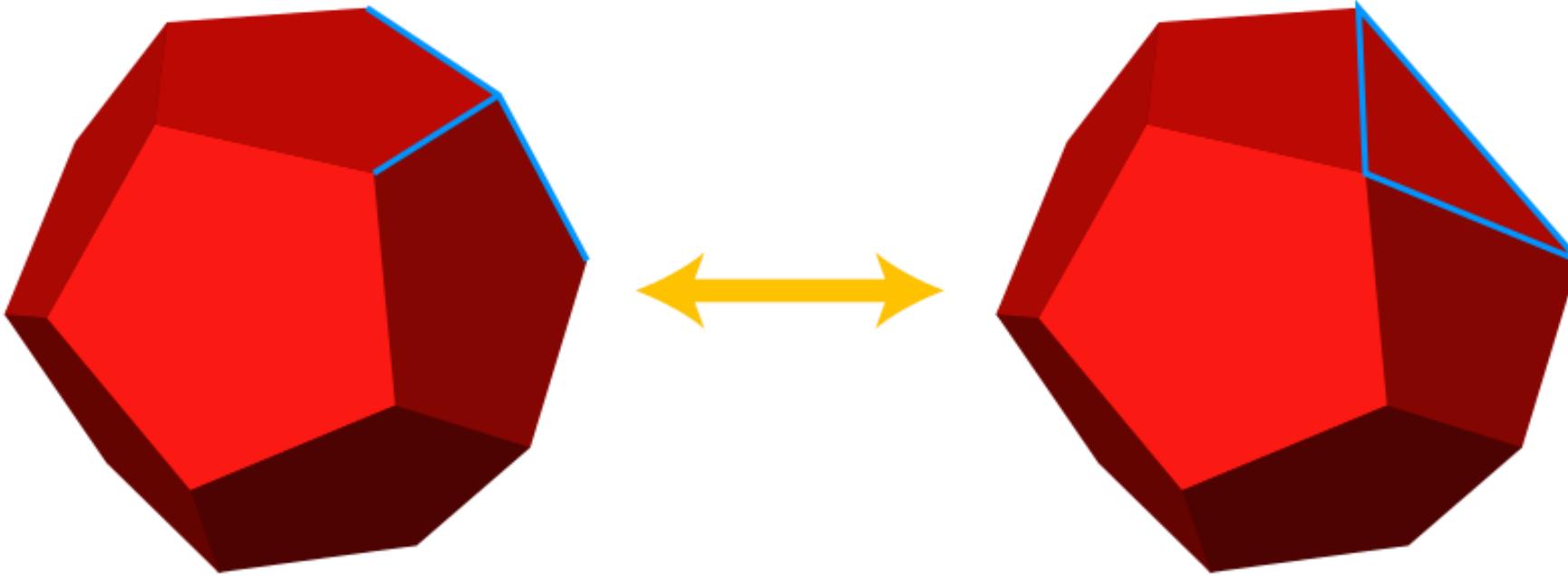
[Shephard 1975]

[Schlickenrieder 1997]



**WHEN DOES AN UNFOLDING  
COMING FROM A POLYHEDRON?**

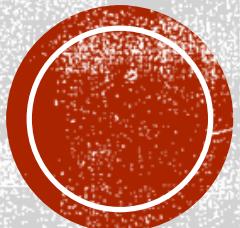


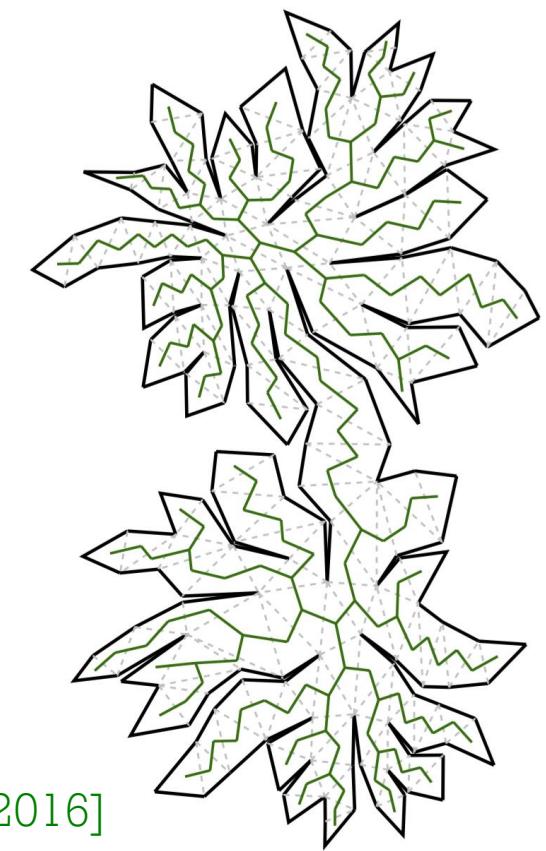
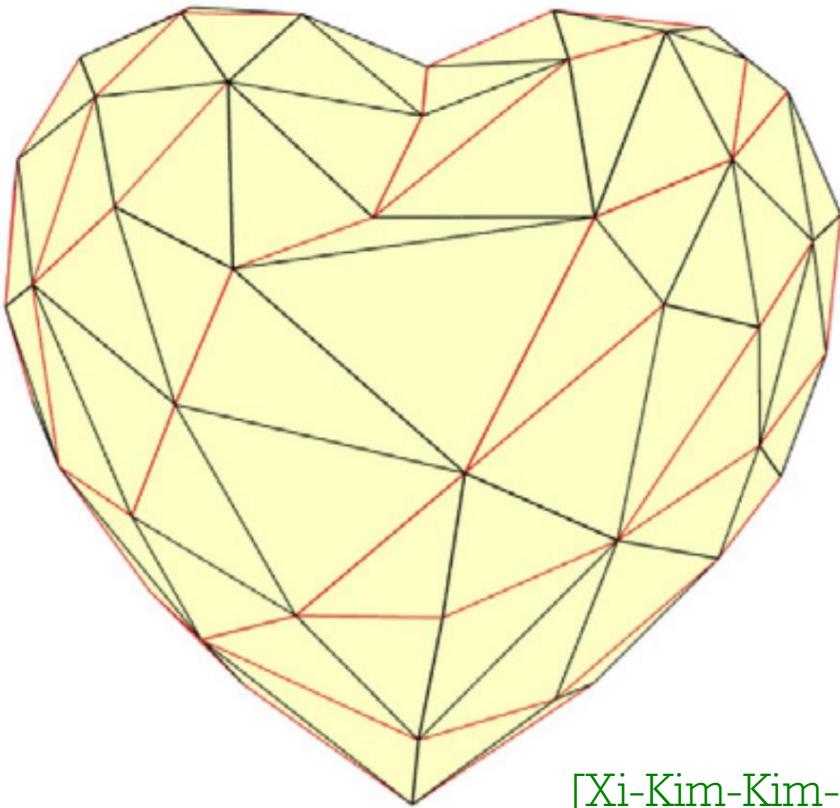
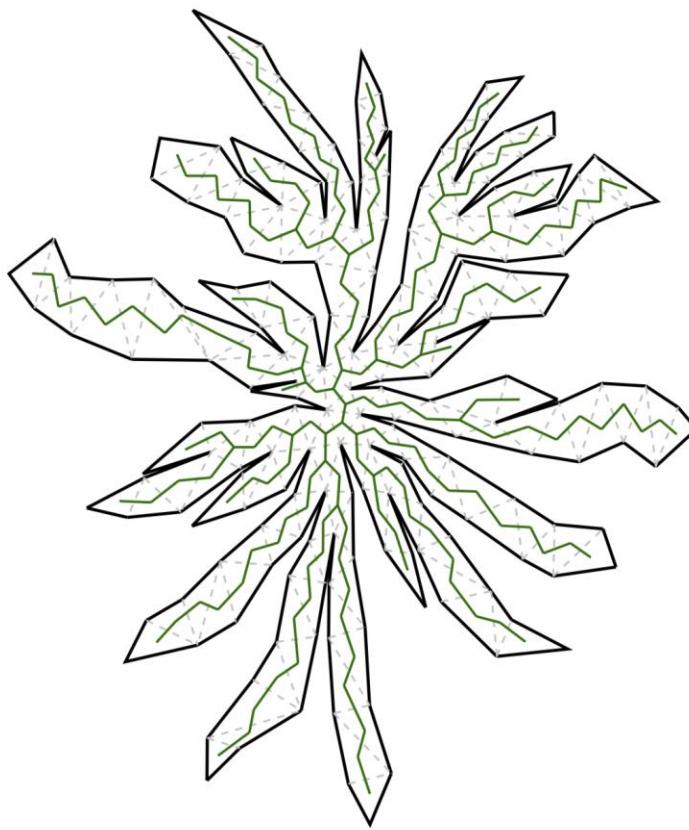


## STEINITZ'S THEOREM

[Steinitz 1916]

A graph  $G$  forms a convex polyhedron if and only if  
 $G$  is simple, planar, and 3-connected



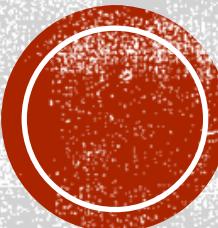


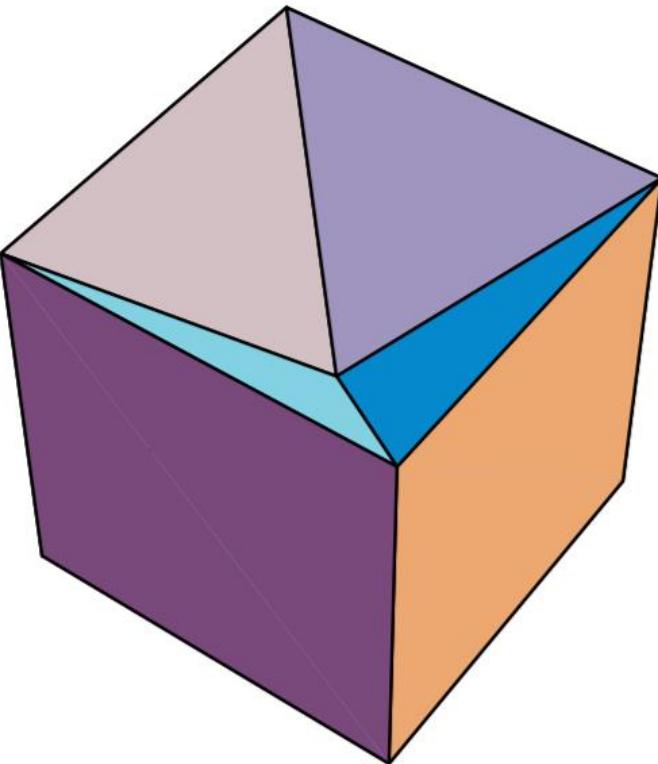
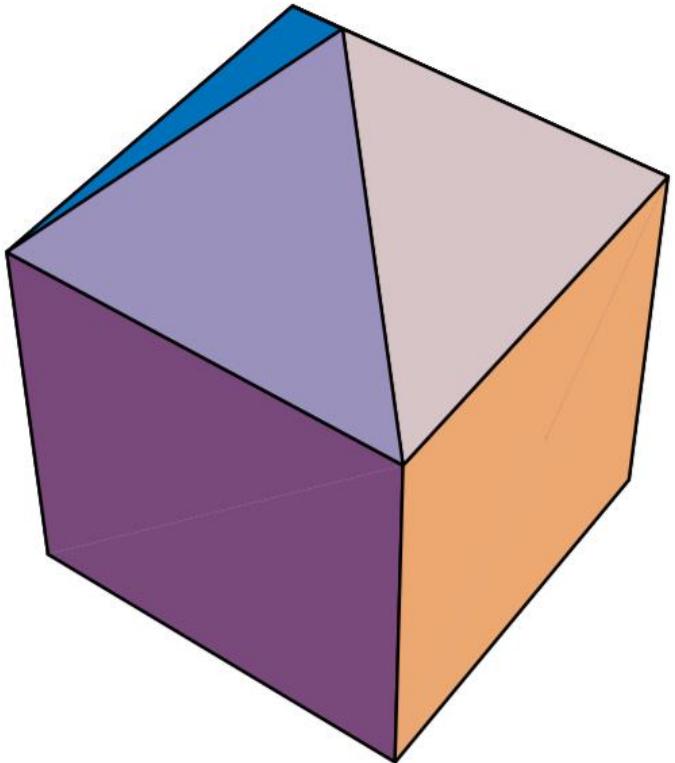
[Xi-Kim-Kim-Lien 2016]

# CAUCHY RIGIDITY THEOREM

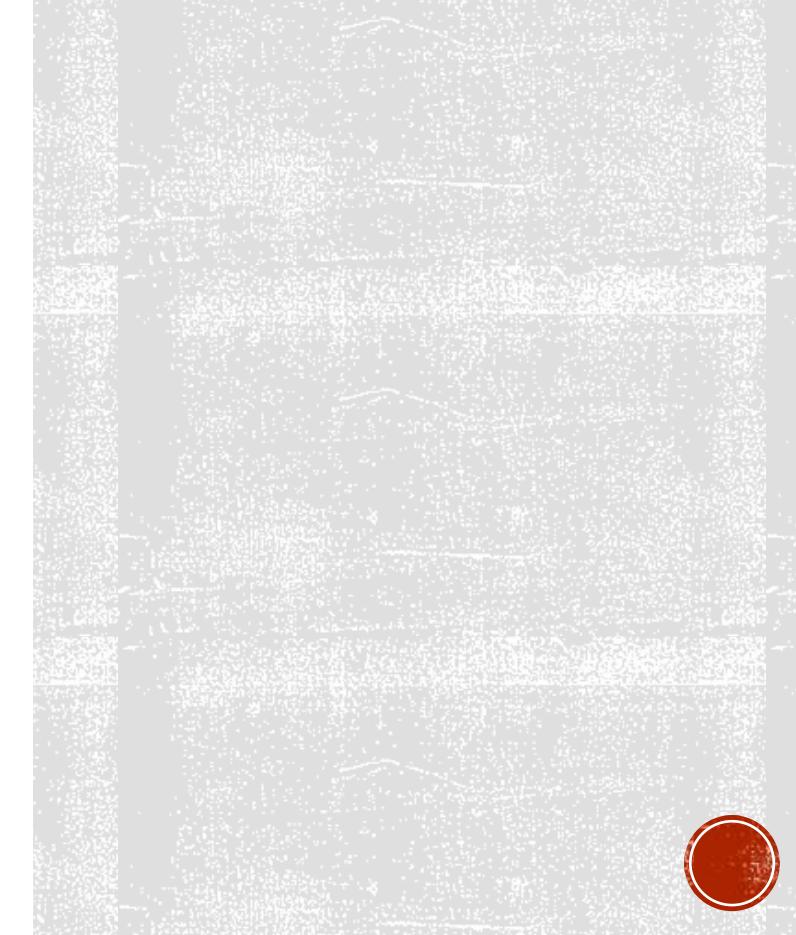
Two combinatorially equivalent, convex polyhedral  
must be congruent

[Cauchy 1813] [Steinitz-Rademacher 1934]

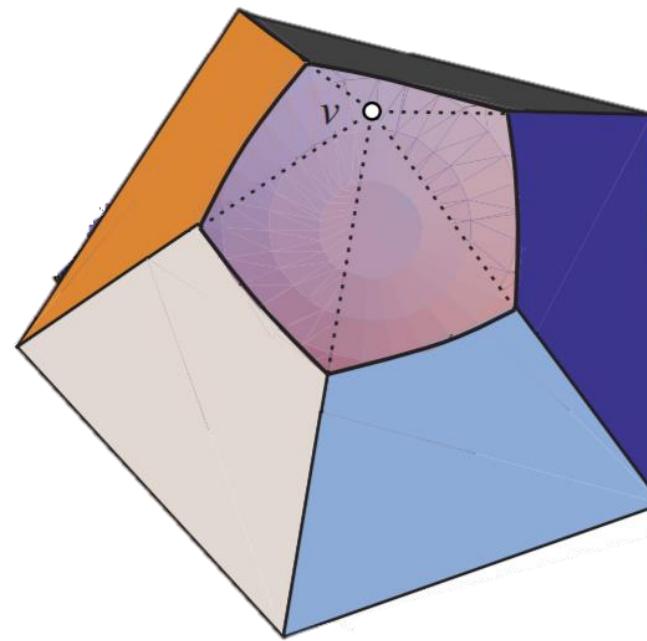
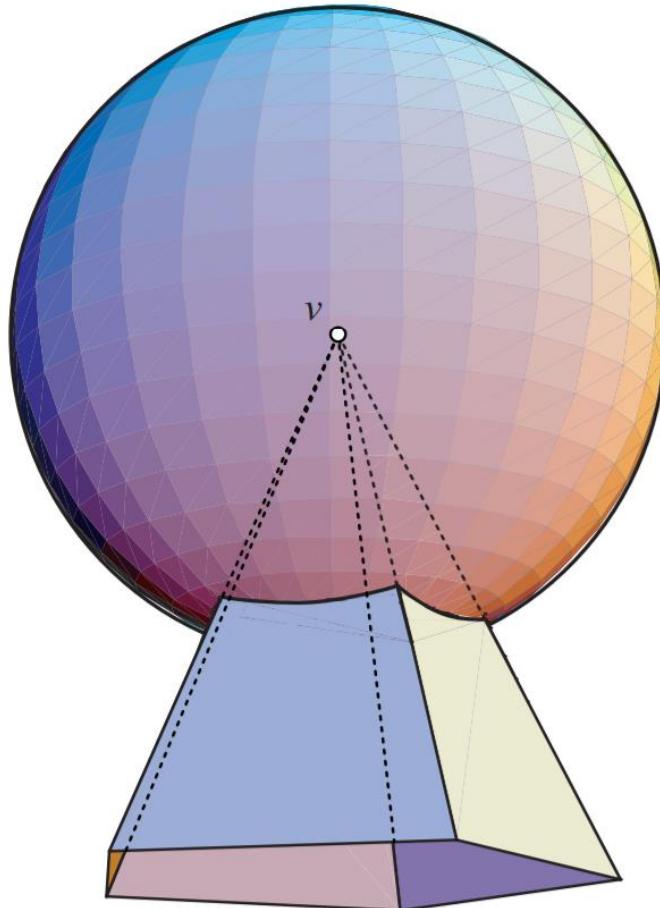




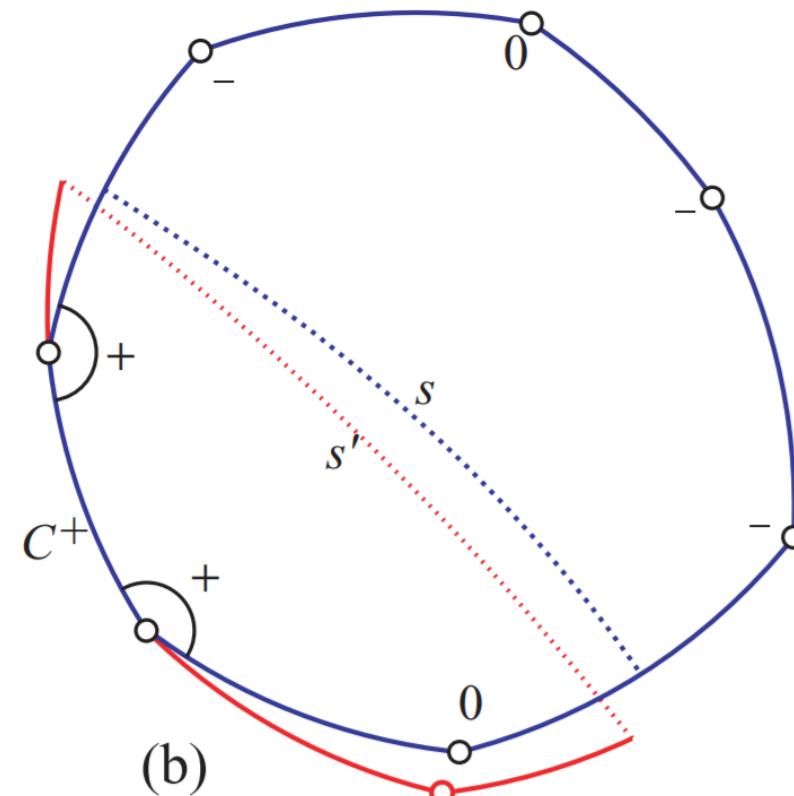
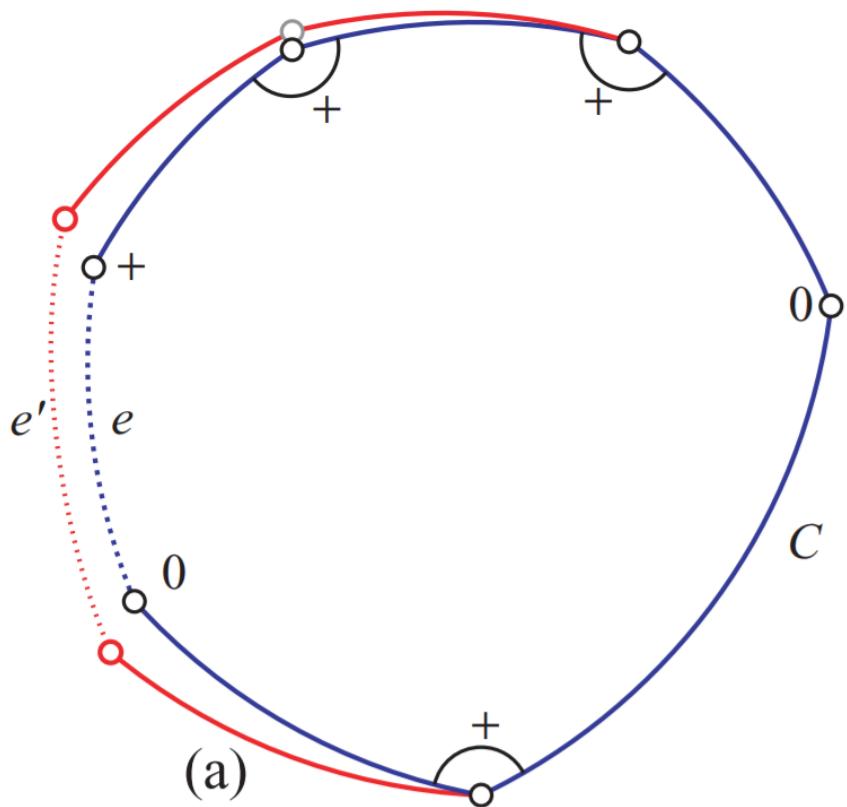
**Not TRUE IF  
Non-Convex**



# PROOF OF RIGIDITY THEOREM

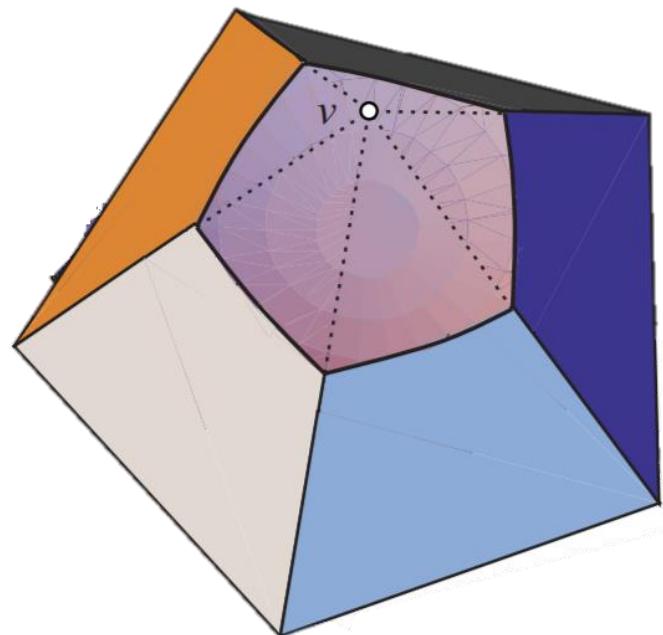


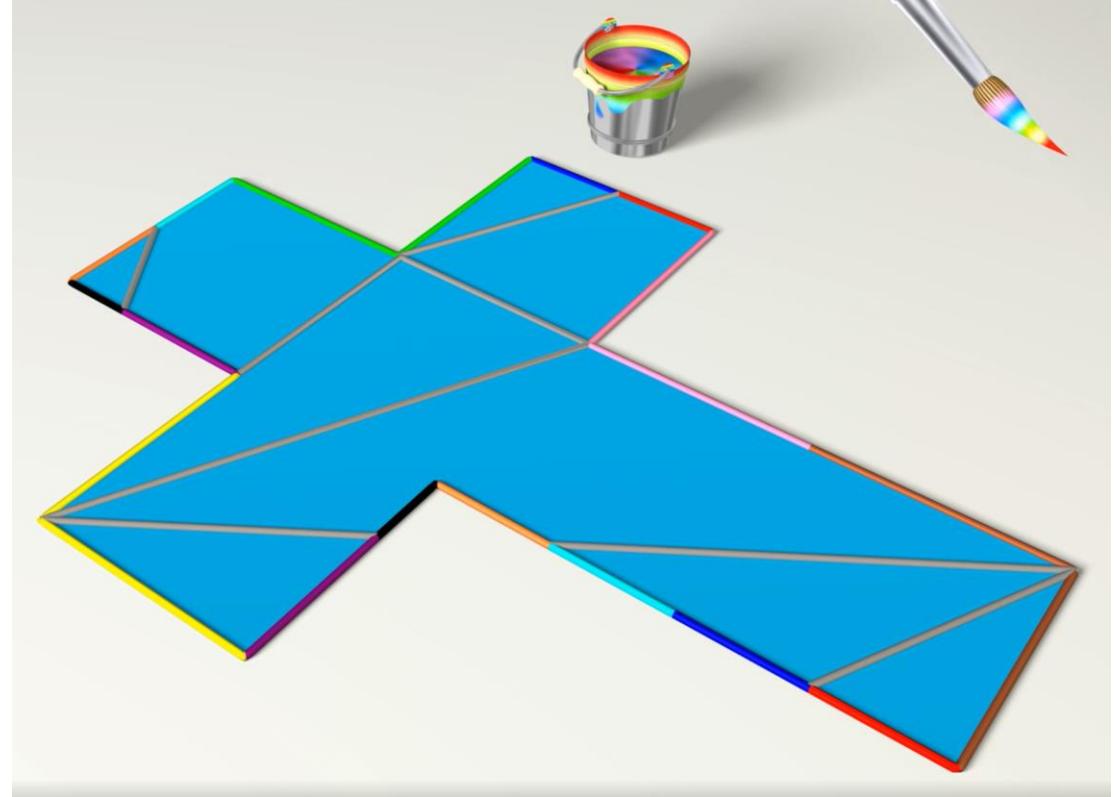
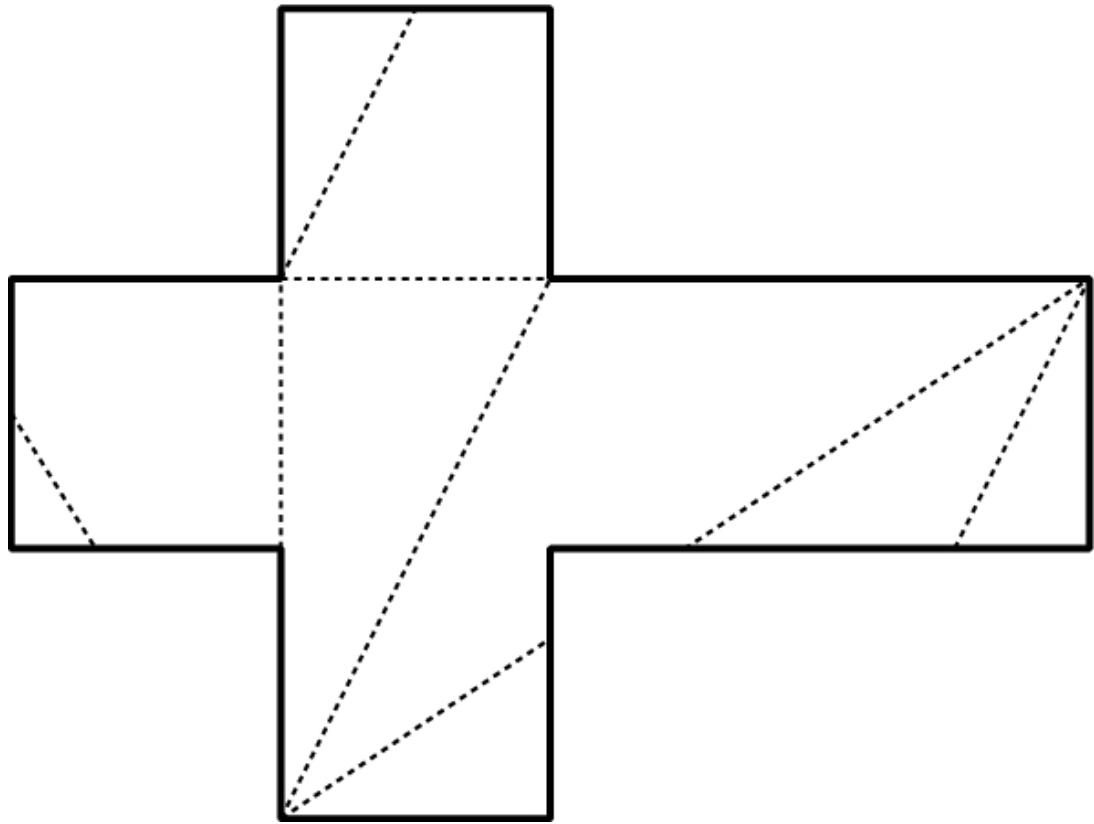
**CAUCHY-STEINITZ LEMMA.** Any  $\pm$ -labeling on the corners of convex polygon must have 4 alternations.



# PROOF OF RIGIDITY THEOREM

C-S LEMMA.  $\geq 4$  alternations.

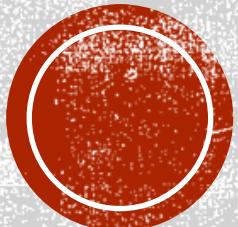


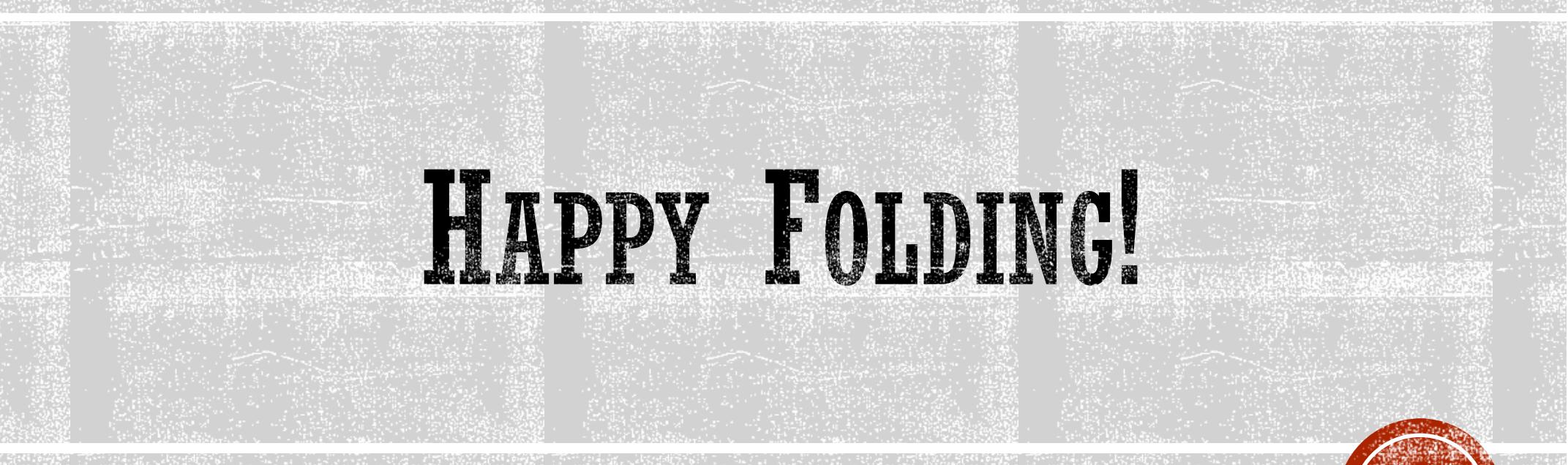


## ALEXANDROV'S THEOREM

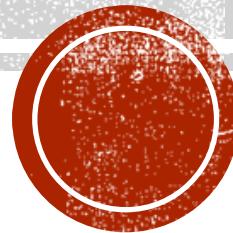
[Alexandrov 1941]

Any gluing of polygons that (1) uses  $\leq 2\pi$  angle per point  
and (2) results in a topological sphere  
must uniquely determine a convex polygon





# HAPPY FOLDING!



**NEXT TIME.**  
Behavior of closed curves  
dictates the shape of space.