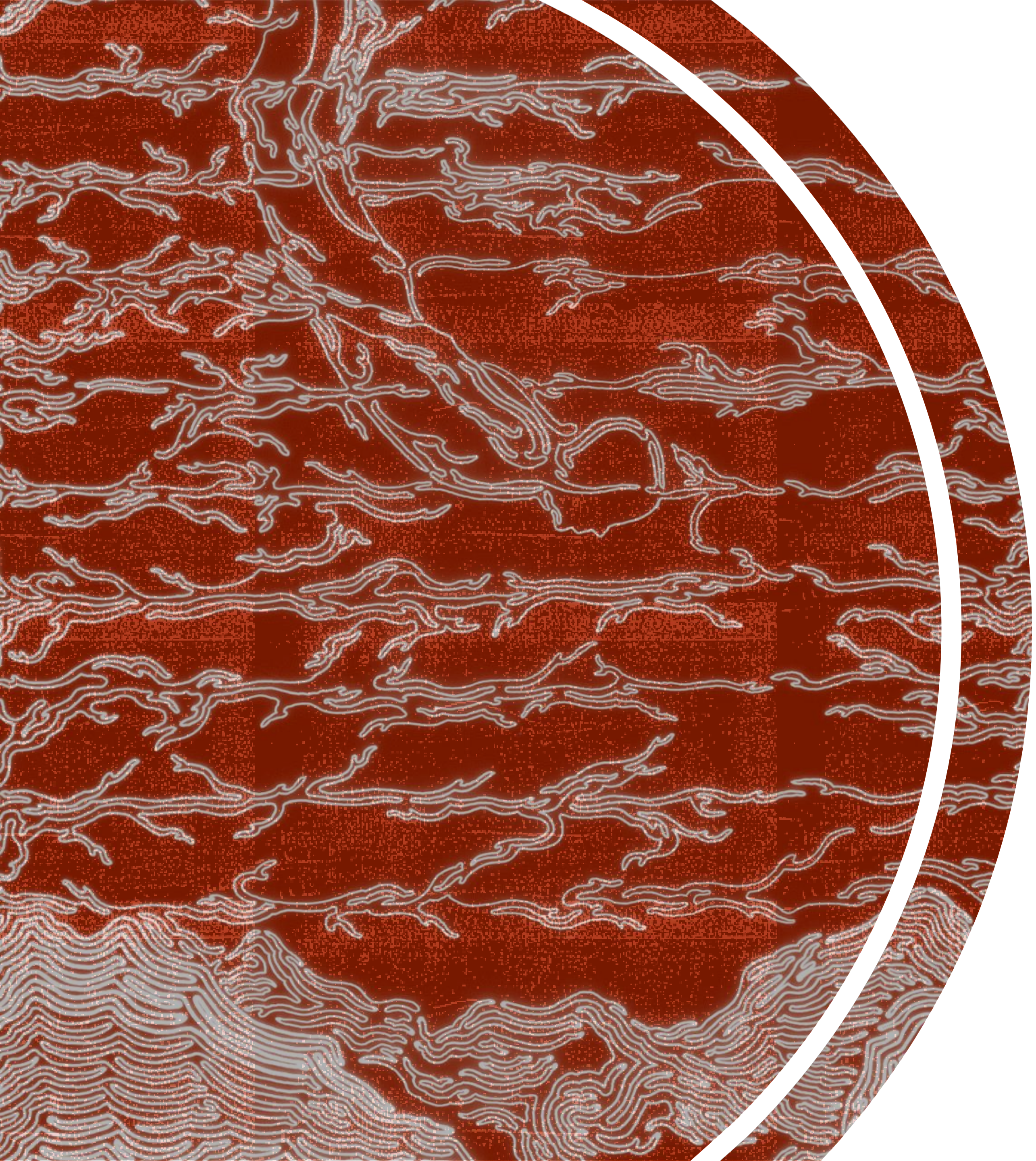


# Introduction to Computational Topology

Hsien-Chih Chang

Lecture 9, October 12, 2021

# Administrivia

- Homework 3 is out, due 10/25 (Mon)

- Optional Final Project:
  - Project proposal is due 10/18 (Mon)
  - Presentation during finals week (likely to be 11/23 (Tue))
  - Project report due 11/29 (Mon)

# Minimum Cut in Planar Graphs

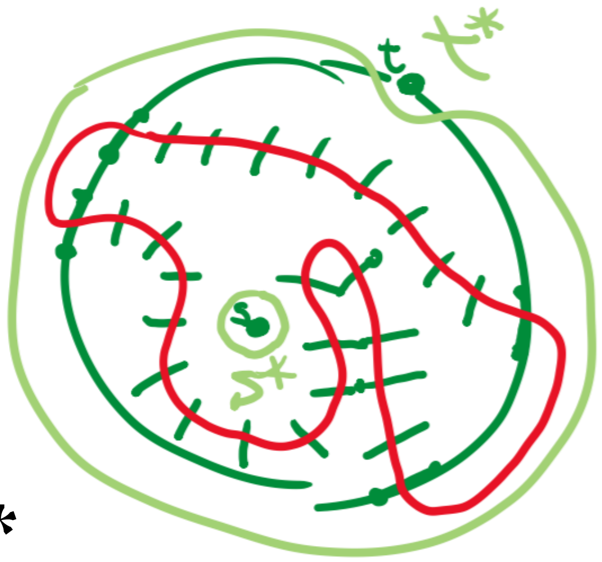
# Minimum Cut in a Graph

- Given undirected graph G with positive edge-weights and two vertices s and t, find a minimum-weight edge cut separating s and t

# Minimum Cut in Planar Graph

- Given undirected planar graph G with positive edge-weights and two vertices s and t, find a minimum-weight edge cut separating s and t

$$\{\text{edge cuts}\} \Leftrightarrow \{\text{circuit} = \text{union of cycles}\}$$

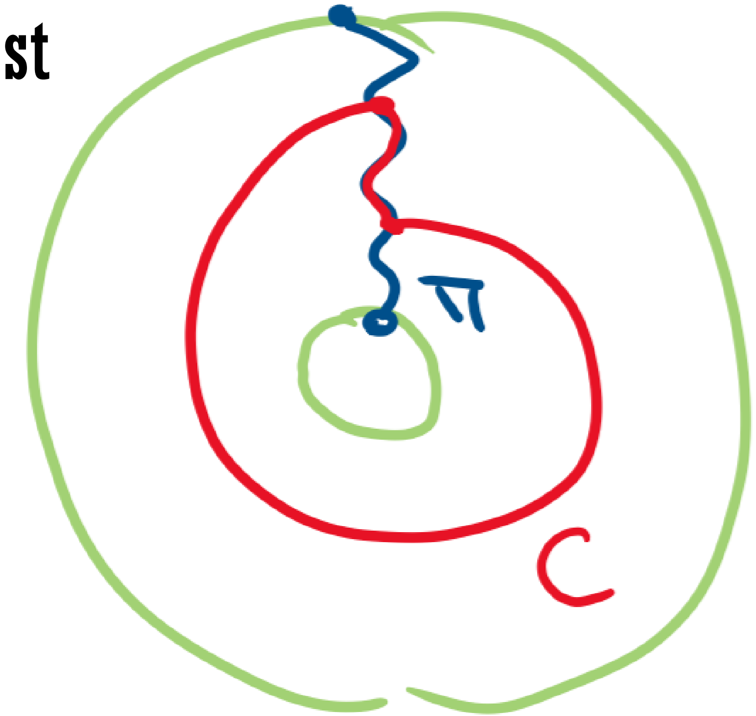



min (s,t)-cut $\Leftrightarrow$ minimum cycle separating s* and t*

# Find a Min Homotopic Cycle!

# Observations

- Shortest cycle C must pass through any path $\pi$ from $s^*$ to $t^*$
- Cycle C intersects $\pi$ at one segment if $\pi$ is shortest

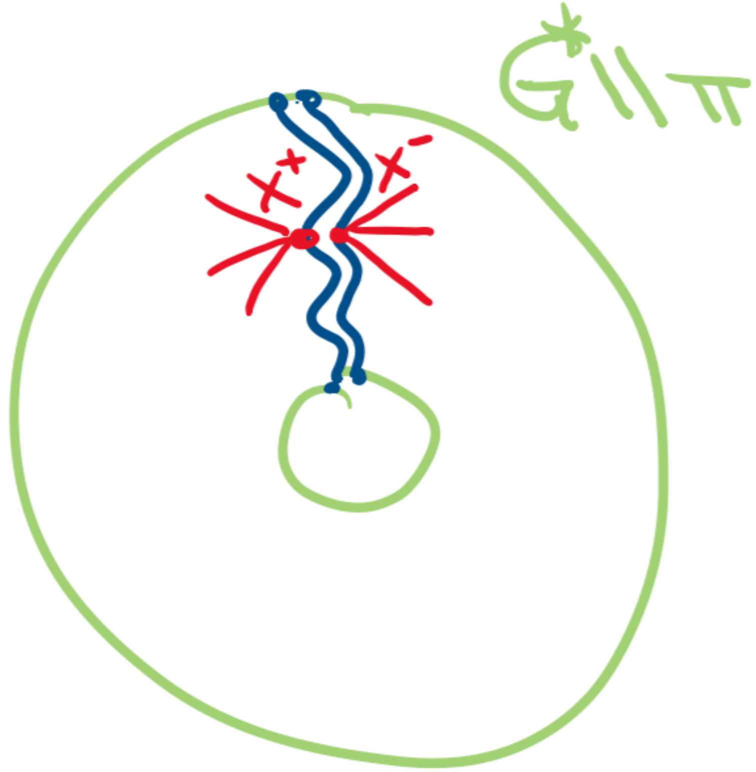


# Naïve Algorithm

**MinCut$(G, s, t)$:**

Find shortest path $\pi$ from $s^* \rightsquigarrow t^*$
Cut open $G^*$ along $\pi$.
for each vertex $x$ on $\pi$:
    find shortest path $x^+ \rightsquigarrow x^-$

Return length of $\min\{x^+ \rightsquigarrow x^-\}$



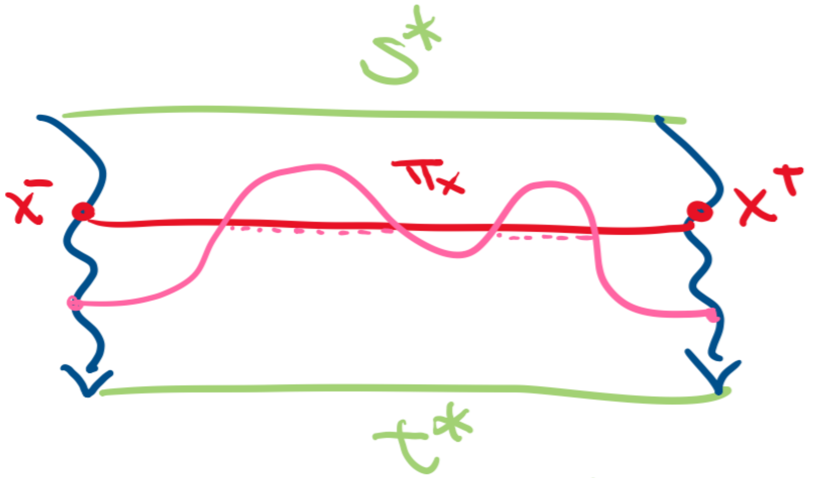
# Reif's Algorithm [Reif 1983]

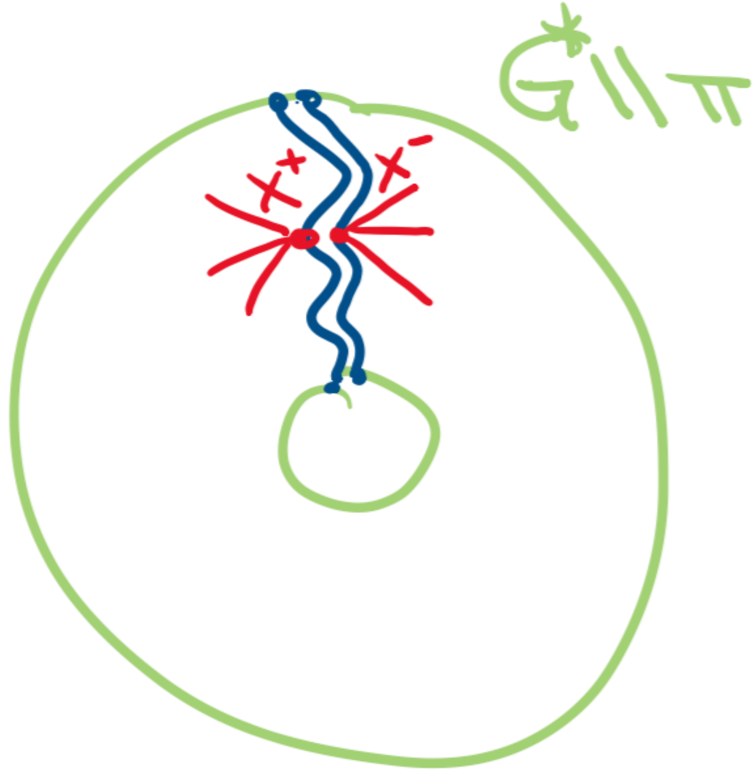
[Reif 1983]

MinCut(G, s, t):

Find shortest path $\pi$ from $s^* \rightsquigarrow t^*$
Cut open $G^*$ along $\pi$.
for ~~each~~ middle vertex $x$ on $\pi$:
    find shortest path $\pi_x : x^+ \rightsquigarrow x^-$
    MinCut($G^* \backslash\backslash \pi_x$, $s^*$, $\pi_x^*$)
    MinCut($G^* \backslash\backslash \pi_x$, $\pi_x^*$, $t^*$)

Return length of min $\{x^+ \rightsquigarrow x^-\}$



# Improved Algorithms for Min Cut and Max Flow in Undirected Planar Graphs

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## ABSTRACT

We study the min $st$-cut and max $st$-flow problems in planar graphs, both in static and in dynamic settings. First, we present an algorithm that given an undirected planar graph and two vertices $s$ and $t$ computes a min $st$-cut in $O(n \log \log n)$ time. Second, we show how to achieve the same bound for the problem of computing a max $st$-flow

## Categories and Subject Descriptors

G.2.2 [**Graph Theory**]: Graph algorithms

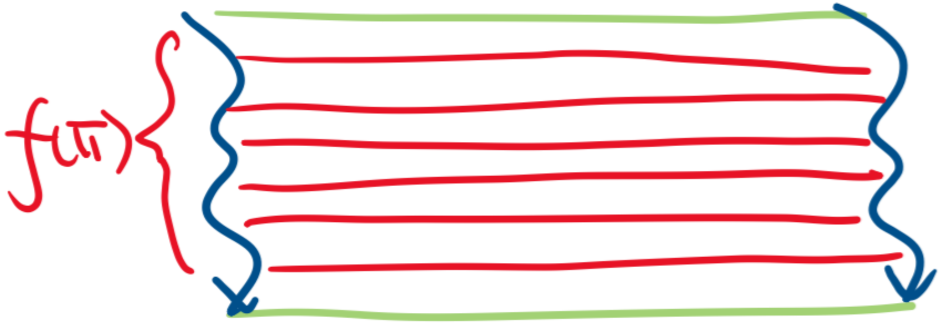
## General Terms

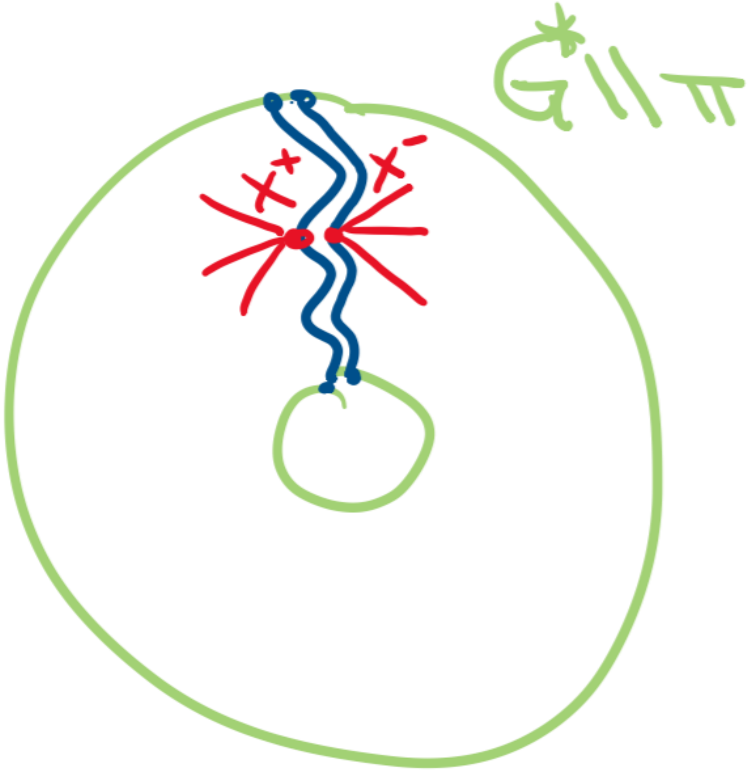
Algorithms,Theory

# Faster Planar Min-cut [Italiano-Nussbaum-Sankowski-Wulff-Nilsen 2011]

Planar min-cut can be computed in O(n loglog n) time

# High-Level Ideas



# Toolbox to be Built

- Multiple-source shortest paths [Klein 2005] [Cabello-Chambers-Erickson 2013]
- Cycle separator decomposition/r-division [Frederickson 1989] [Klein-Mozes-Sommer 2012]
- Monge heap/dense distance graph [Aggarwal-Klawe-Moran-Shor-Wilber 1987]
- FR-Dijkstra [Fakcharoenphol-Rao 2001]

- Monge emulator [Chang-Ophelders 2020] [Chang-Krauthgamer-Tan 2022]

# Intermission

## Food for Thought.

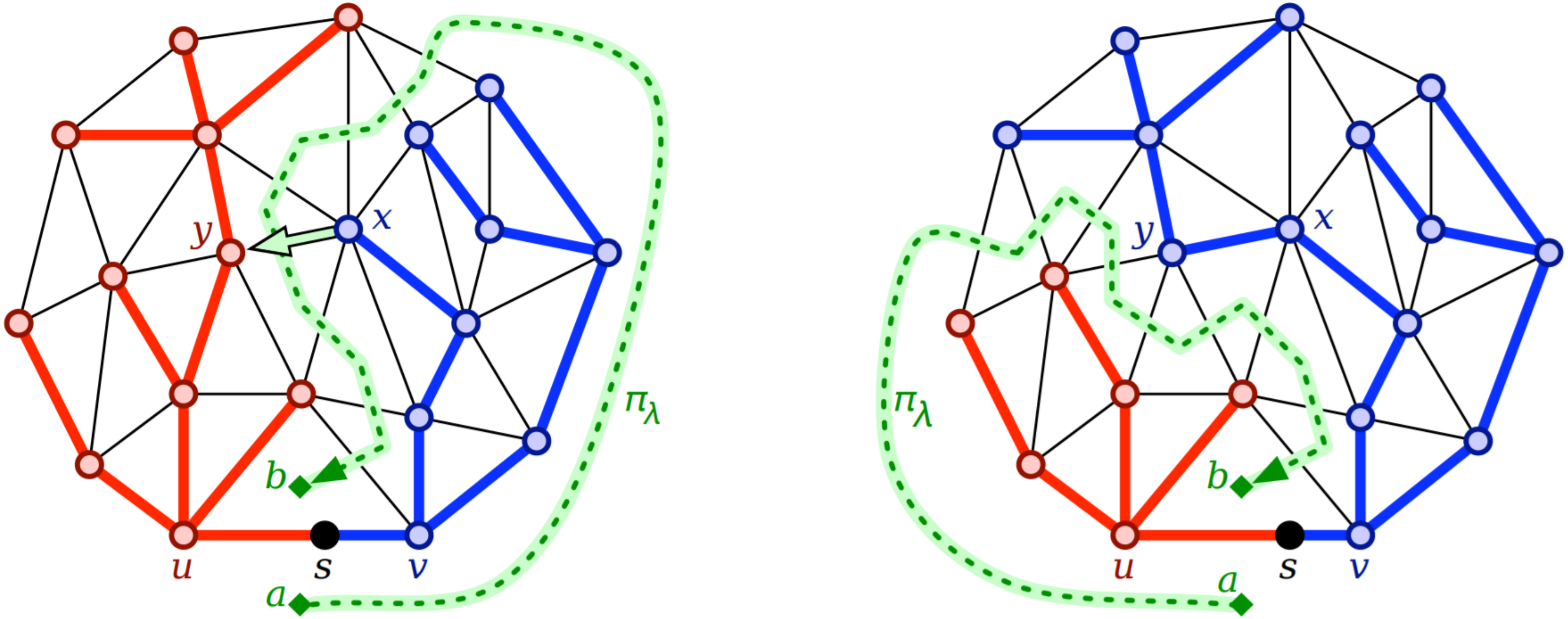
Does trivial $\pi_1$ imply contractibility?

# Multiple-Source Shortest Paths

# MSSP Problem Definition

- Given a planar directed graph G with and sources all on the outer-face, and edges weights $w: E(G) \longrightarrow R_+$

- Compute shortest paths between every source s and every vertex x
  - Represented implicitly

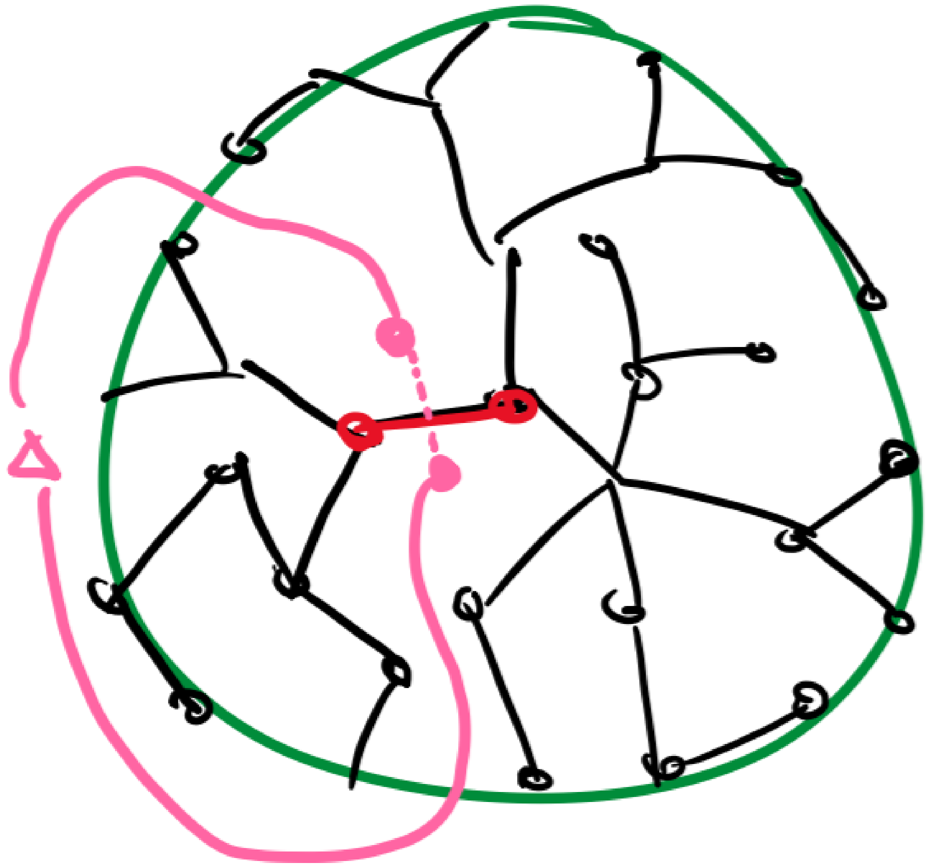



# Multiple-Source Shortest Paths

[Klein 2005]
[Cabello-Chambers-Erickson 2013]

MSSP problem can be solved in O(n log n) time,
such that each distance can be queried in O(log n) time

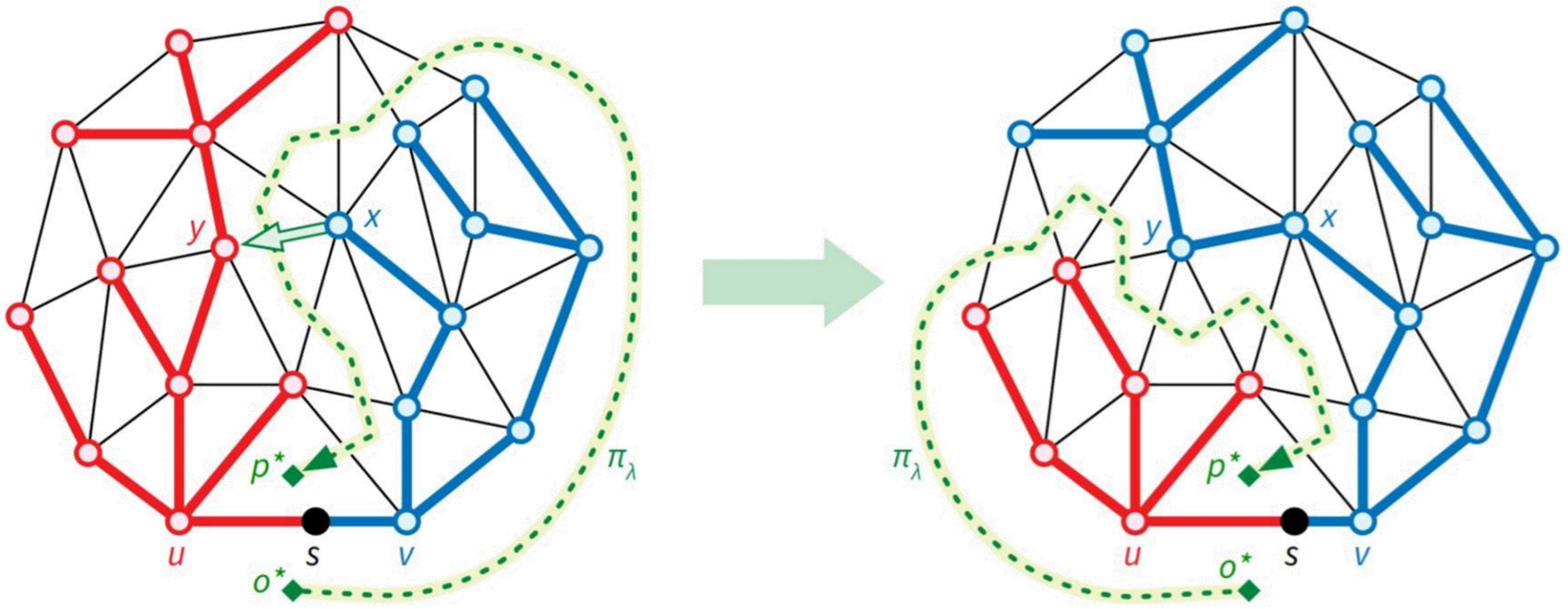
**Disk-Tree Lemma.** For any spanning tree T and tree-edge e, the boundary vertices in components of T-e are consecutive.



**Corollary.** Let $T_0, \ldots, T_{k-1}$ be shortest path trees in sequence. Then any edge $x \rightarrow y$ belongs to a consecutive interval of shortest-path trees: $T_i, \ldots, T_{i+j \bmod k}$.

# Parametric Shortest Paths

- Shortest path tree pivots as one moves the source



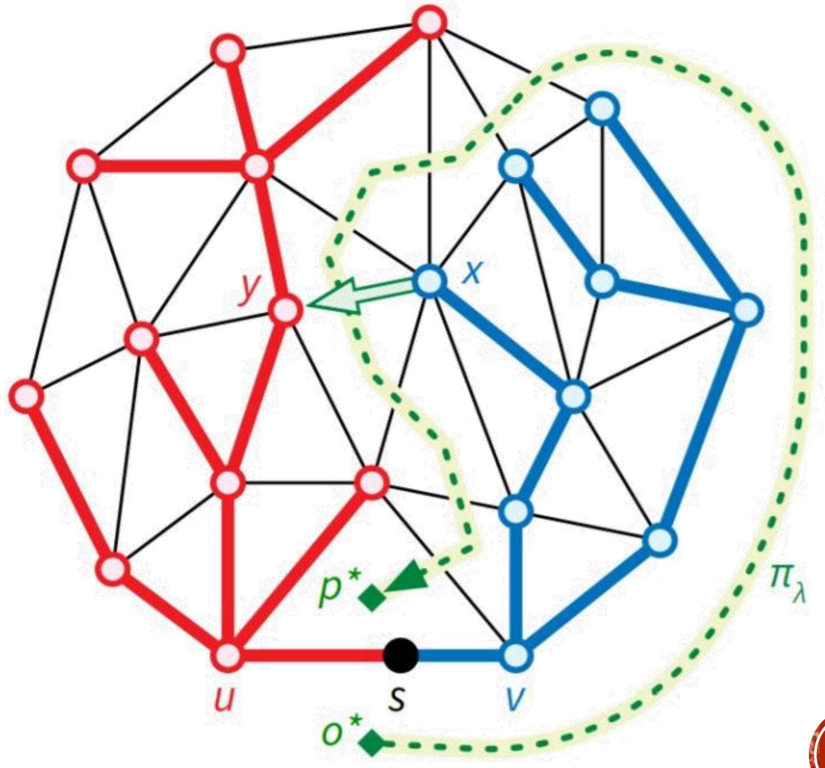
- $d_\lambda(x)$: distance from s to x under $w_\lambda$
- $\text{slack}_\lambda(x{\to}y) = d_\lambda(x) + w(x{\to}y) - d_\lambda(y)$

**Observation.** Any shortest-path tree has

- non-negative slack on all darts
- zero slack on tree darts
- positive slack on non-tree darts

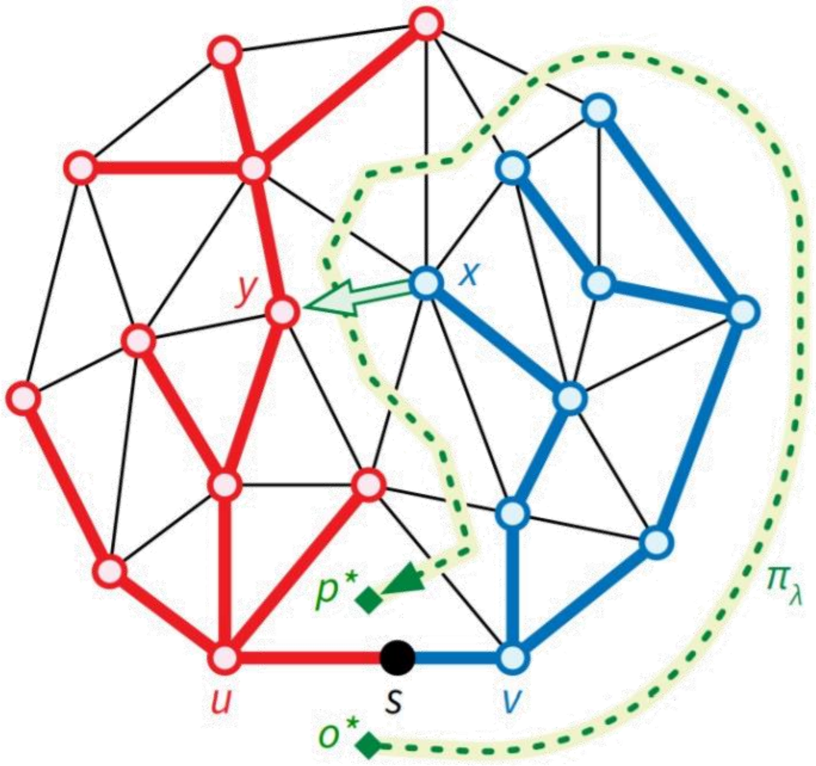
# Parametric Shortest Paths

- A vertex x is
  - red if $d_\lambda(x)$ goes up as $\lambda$ goes up
  - blue if $d_\lambda(x)$ goes down as $\lambda$ goes up
- Dart $x \to y$ is active if
  - $\text{slack}_\lambda(x \to y)$ goes up as $\lambda$ goes up



## RED-BLUE LEMMA. For any $\lambda$:

- All vertices behind u are red
- All vertices in front of v are blue
- $x \rightarrow y$ active if x blue and y red



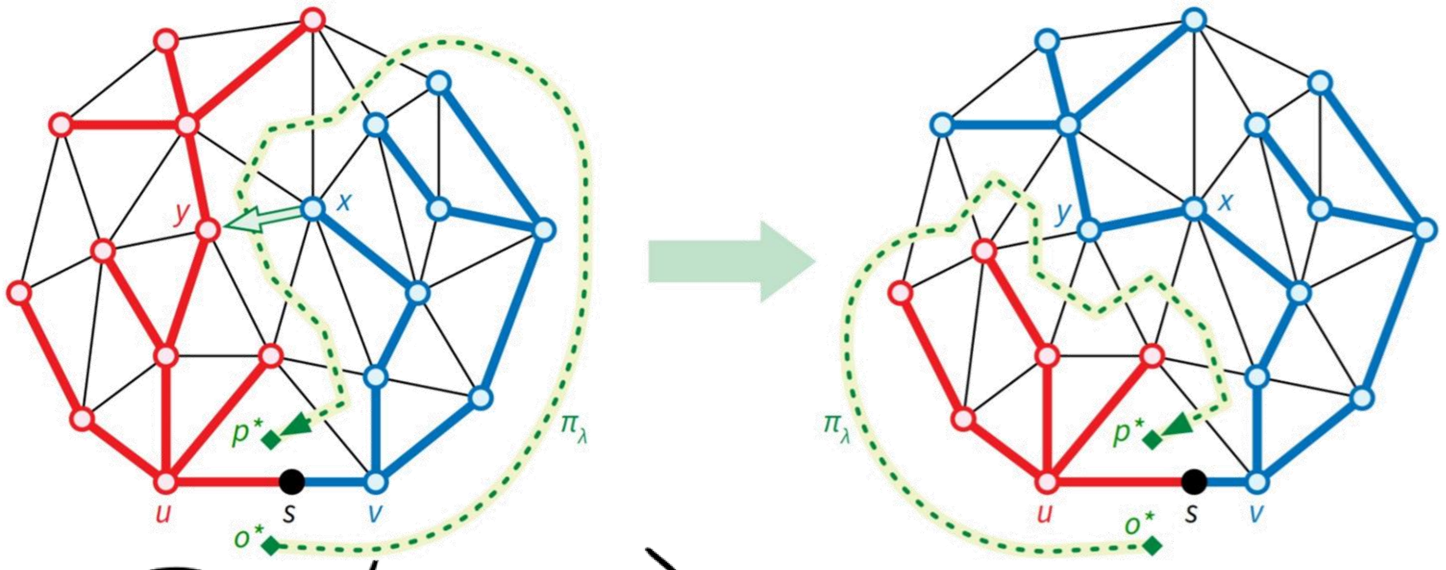

## COROLLARY.

Active darts form dual path $\pi_\lambda$ between $o^*$ and $p^*$.

## COROLLARY.

The min-slack active dart on $\pi_\lambda$ is the next pivot.

# MSSP Algorithm




**NextPivot(G, $T_\lambda$):**

$x \to y \leftarrow$ MinPathSlack($o^*$, $p^*$)
$\Delta \leftarrow slack_\lambda(x \to y)/2$

if $\lambda + \Delta / w(u \to v) < 1$ :
  Pivot($x \to y$, $\Delta$)
  return $\lambda + \Delta / w(u \to v)$
else
  return 1.

**Pivot($x \to y$, $\Delta$):**

AddSubtreeDist($\Delta$, u)
AddSubtreeDist($-\Delta$, v)

AddPathSlack($-2\Delta$, $o^*$, $p^*$)

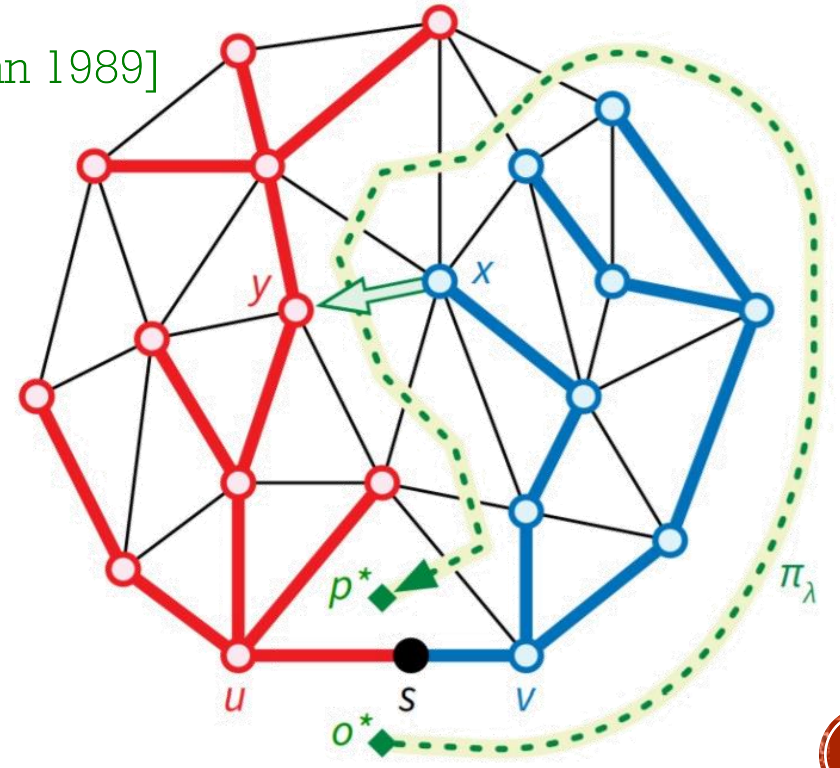
$z \leftarrow pred(y)$, $pred(y) \leftarrow x$

Cut(yz), Link(xy)
Cut($(xy)^*$), Link($(z \to y)^*$).

# Implementation and Analysis

- Implement tree-cotree using dynamic tree data structure
  - Splay tree into link-cut tree [Sleator-Tarjan 1982-1985]
  - Persistent data structure [Driscoll-Sarnak-Sleator-Tarjan 1989]

- Summary:
  - O(n) pivots (by disk-tree lemma)
  - Correctly identify next pivot (by red-blue lemma)
  - O(log n) amortized update time (by data structure magic)
- Thus O(n log n) time in total



# Topology+Data Structures= Fast Algorithms

## Next Time:

Two more tools from the toolbox

assemble our faster min-cut algorithm