



INTRODUCTION TO COMPUTATIONAL TOPOLOGY

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LECTURE 6, SEPTEMBER 30, 2021

ADMINISTRIVIA

- Thanks for all the feedback!



GEOMETRIC FOLDING ALGORITHMS

6.849: Geometric Folding Algorithms: Linkages, Origami, Polyhedra (Fall 2020)

Prof. [Erik Demaine](#); [Martin Demaine](#); TAs Yevhenii Diomidov & Klara Mundilova

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Overview

the algorithms behind building TRANSFORMERS and designing ORIGAMI

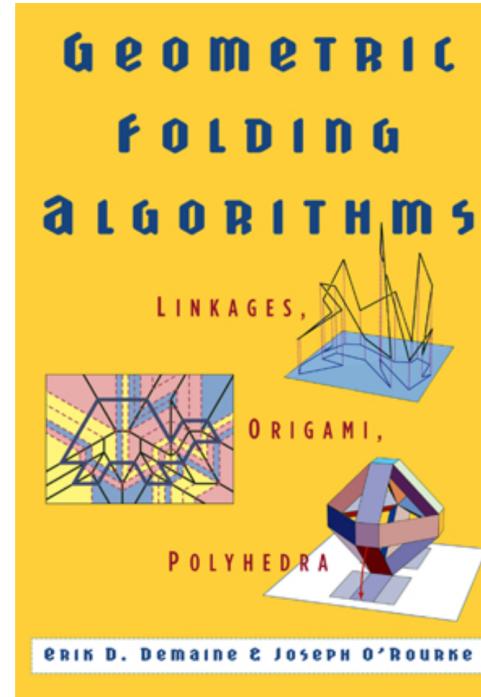
Whenever you have a physical object to be reconfigured, **geometric folding** often comes into play. This class is about algorithms for analyzing and designing such folds. Motivating applications include

- automated design of [new](#) and [complex](#) origami
- using 2D fabrication technology to manufacture [complex 3D objects](#)
- transforming robots by [self-folding sheets](#) or chains
- how to [fold robotic arms without collision](#)
- how to [bend sheet metal](#) into [desired 3D shapes](#)
- understanding how [proteins fold](#)

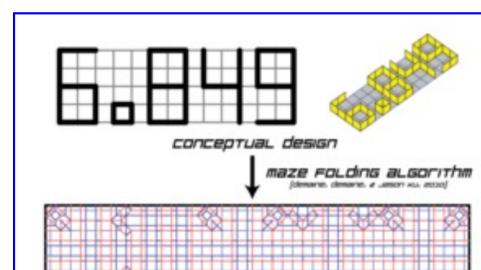
Major progress have been made in recent years in many of these directions, thanks to a growing understanding of the mathematics and algorithms underlying folding. Nonetheless, many fundamental questions remain tantalizingly unsolved. This class covers the state-of-the-art in folding research, including a variety of open problems, enabling the student to do research and advance the field.

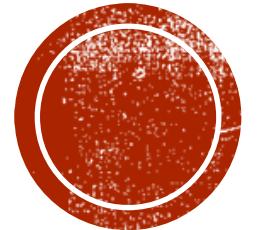
Fully Online Format

Most course material is covered in [video lectures](#) recorded in 2010 (already watched by over 19,000 people), which you can conveniently play at faster speed than real time. There may also be some **new material** presented by the professor and/or guest lecturers, which will be recorded for asynchronous viewing.



[textbook](#)

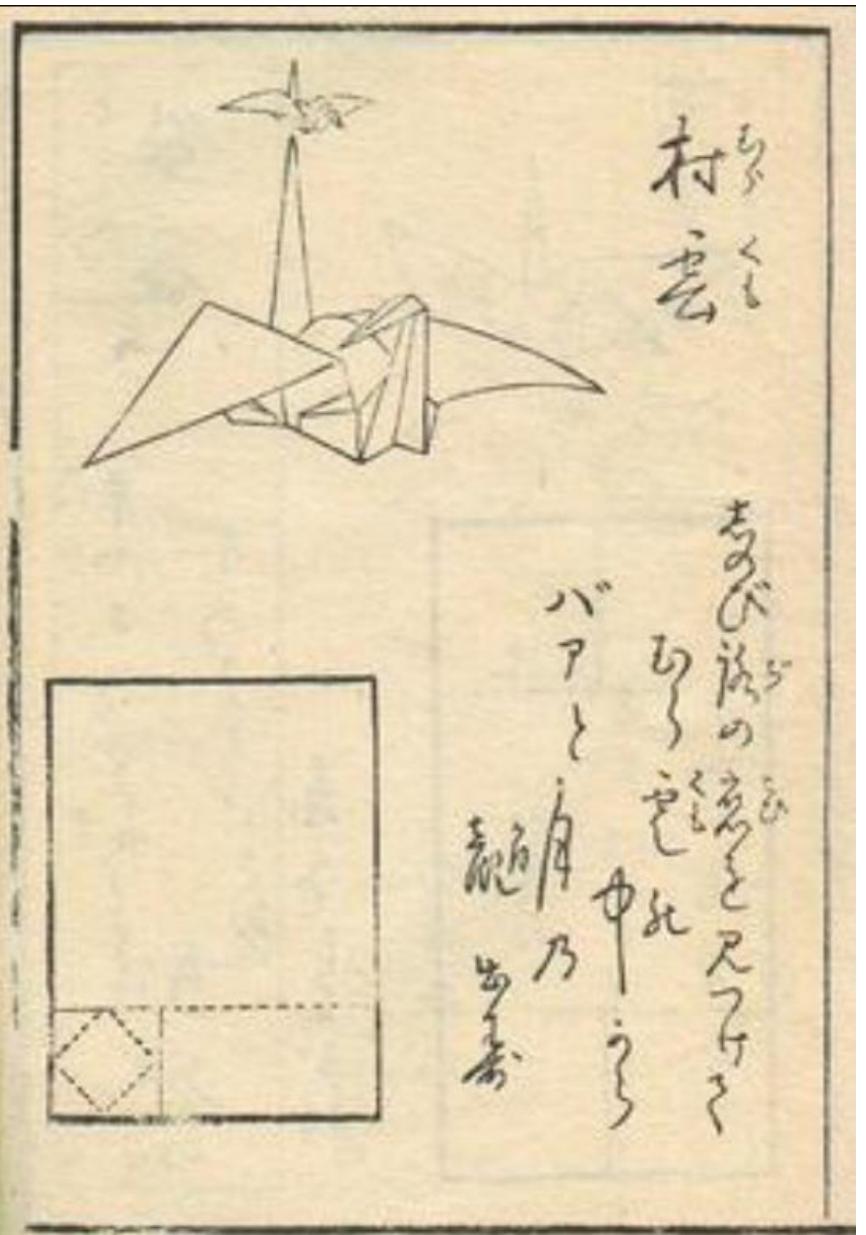




PAPER FOLDING

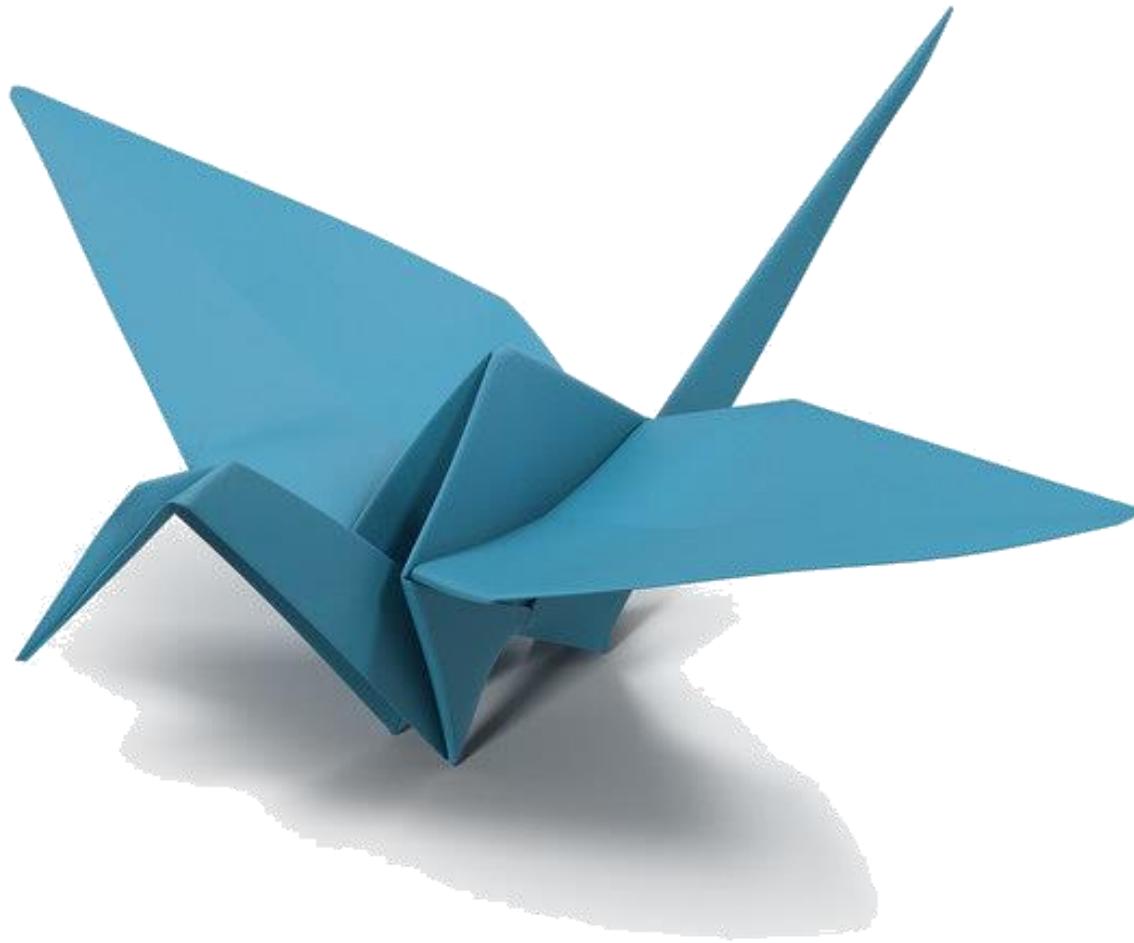


折り紙 ORIGAMI

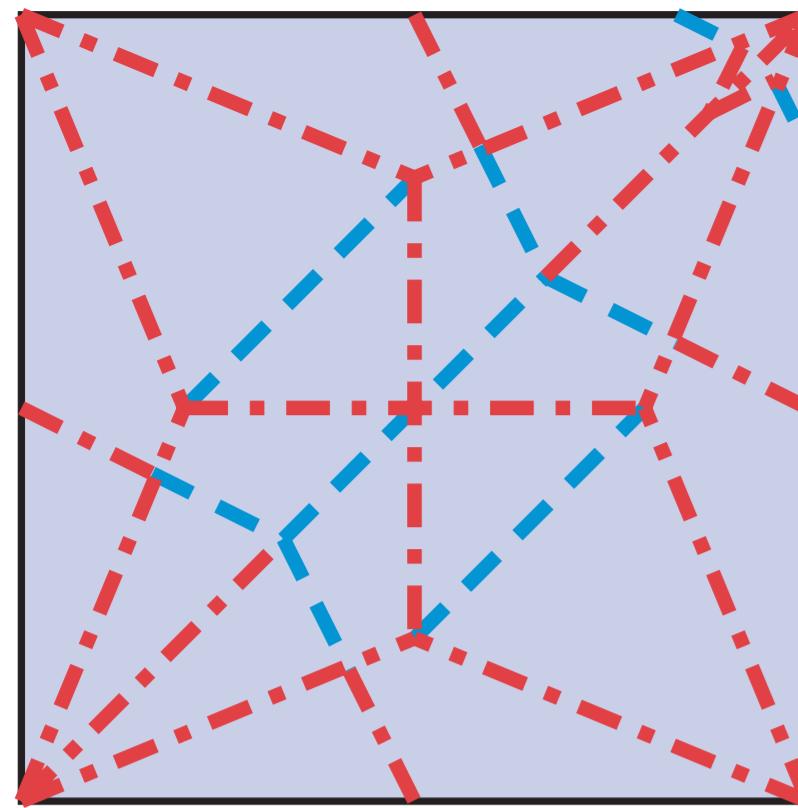
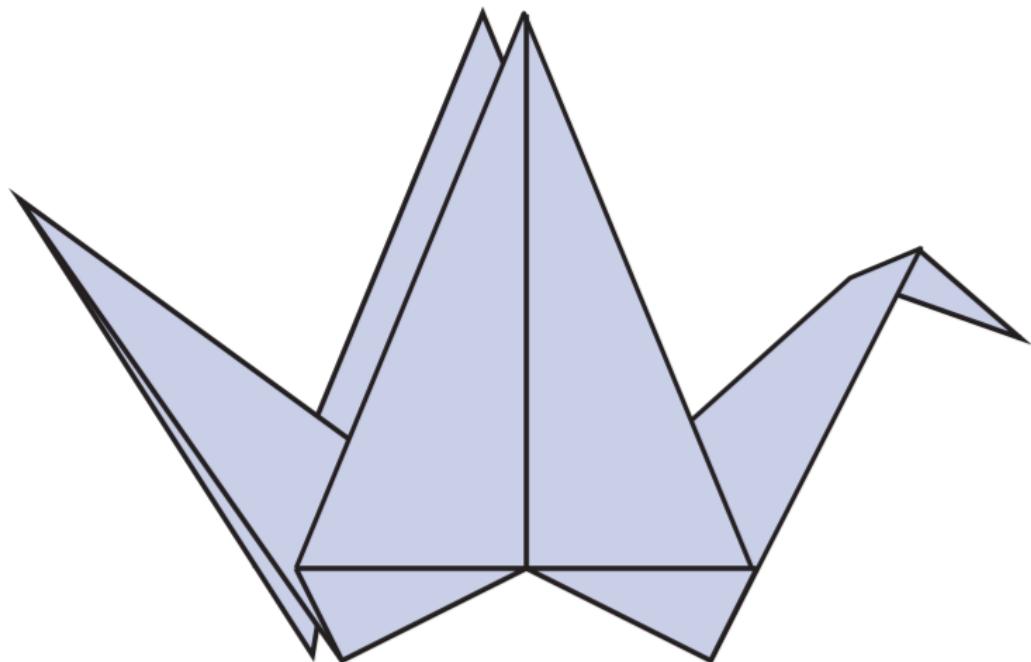


[秋里籬島 Akisato Ritō, 1798]

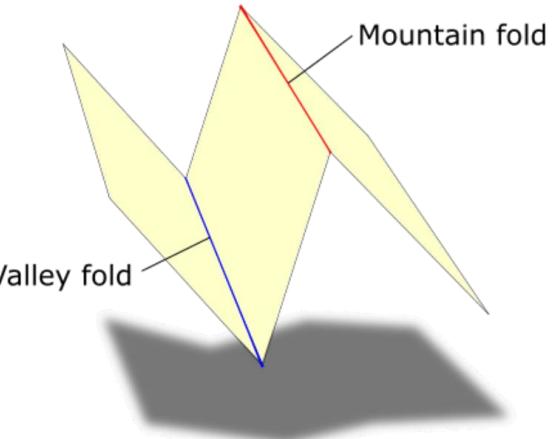
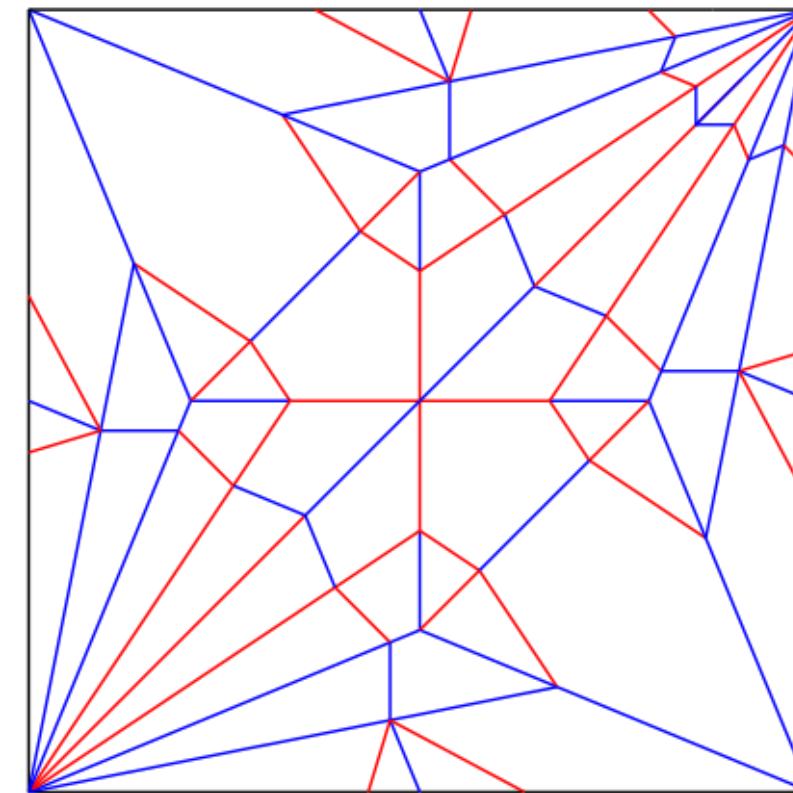
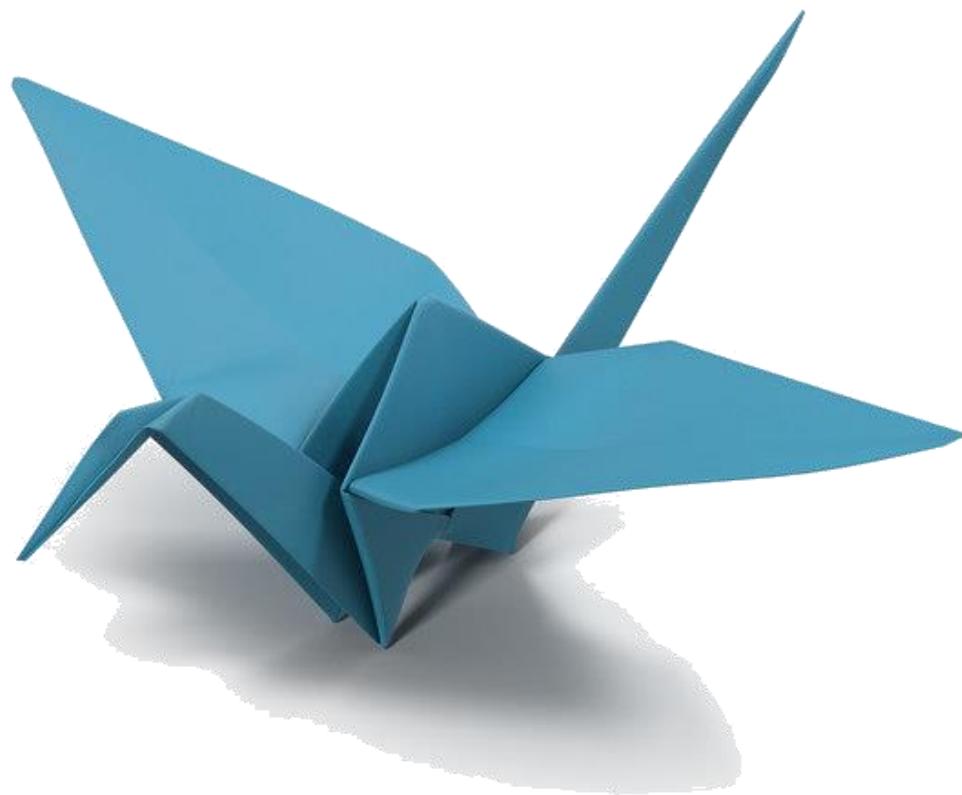
FOLDED STATE



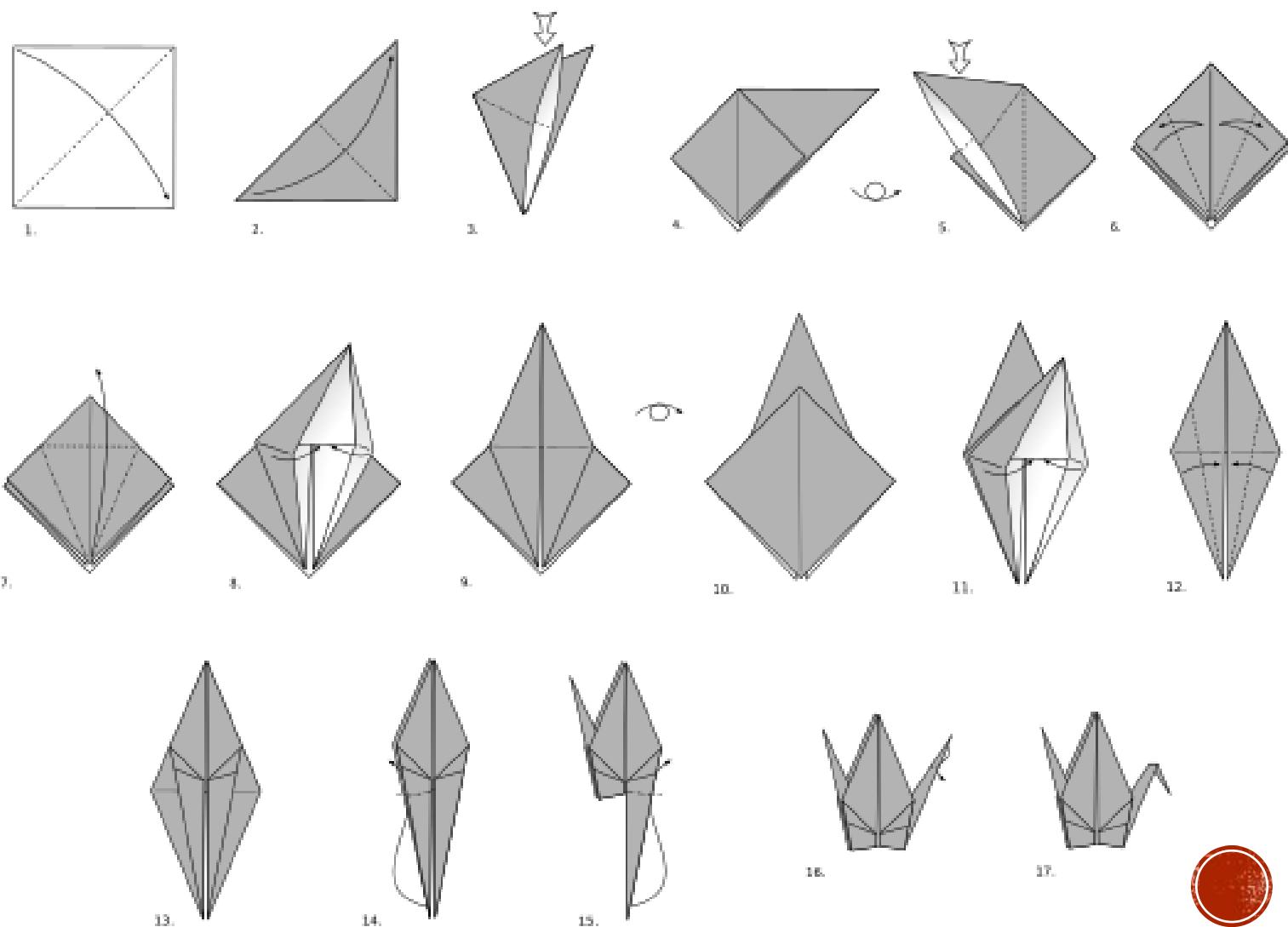
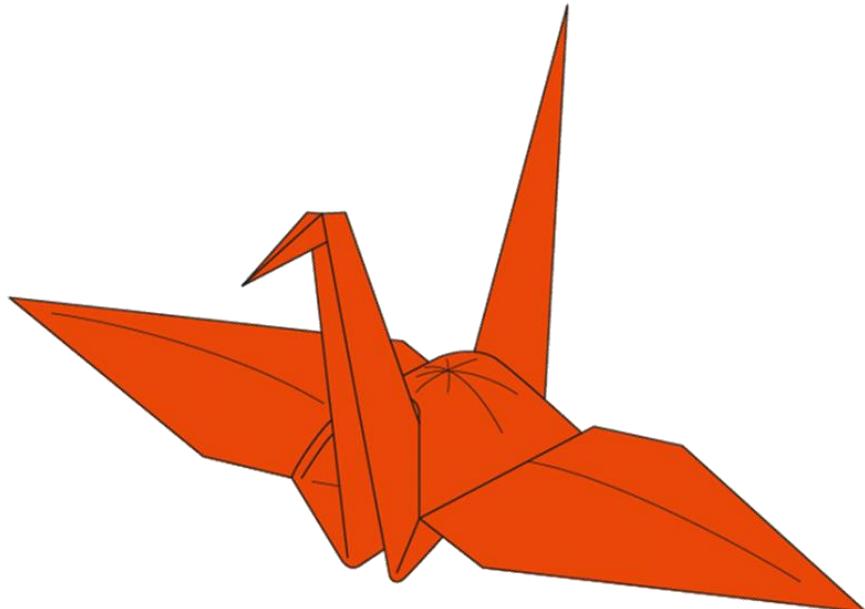
CREASE PATTERN



CREASE PATTERN



FOLDED MOTION

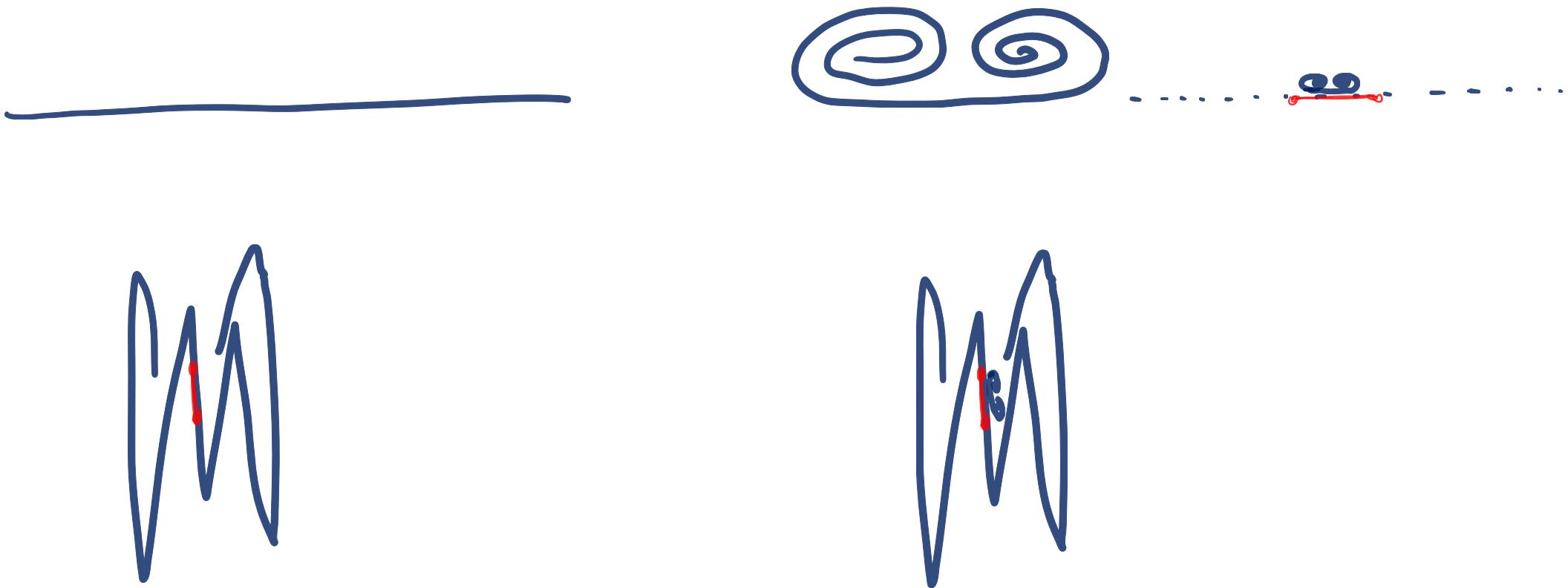


FOLDING PROBLEMS

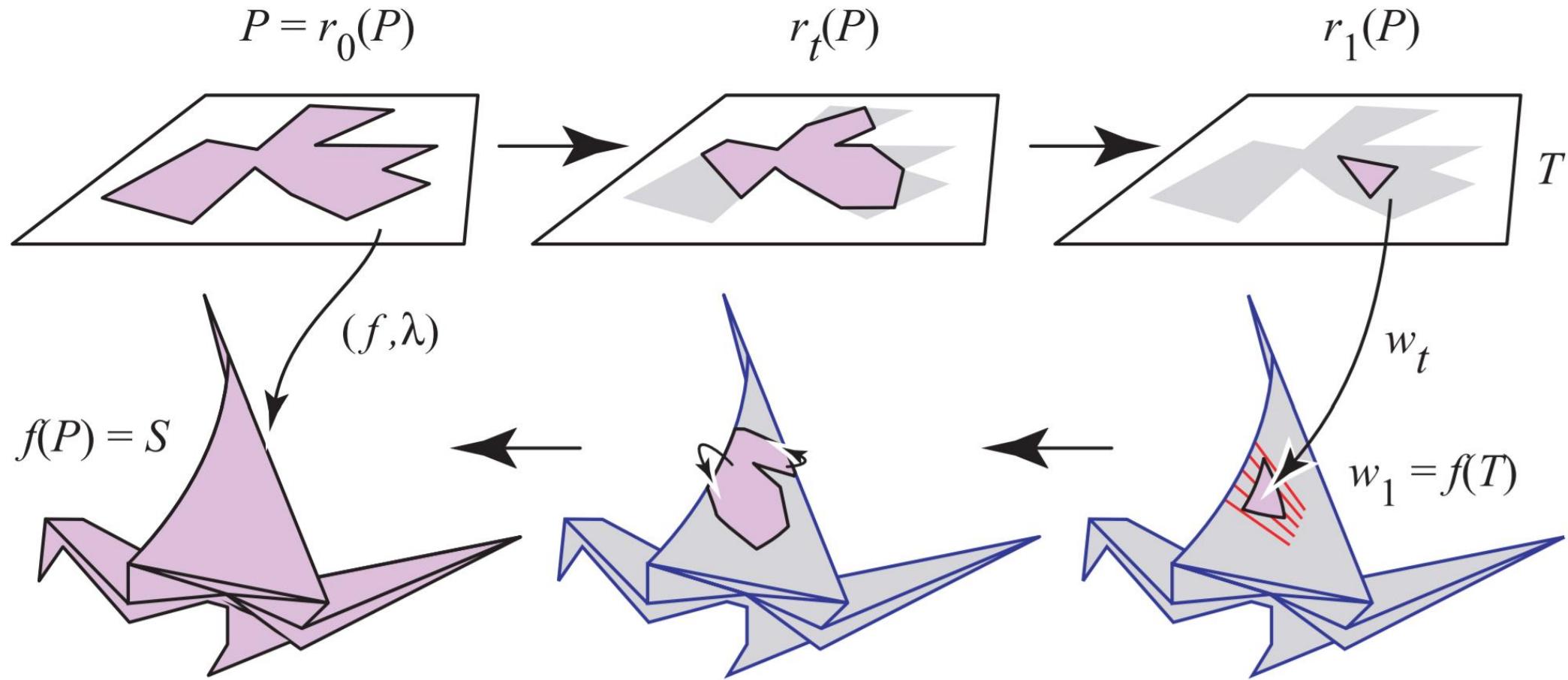
- Given a crease pattern, can it be flat-folded?
- Given a flat-foldable mountain-valley pattern, what is the final state?
- Folding into arbitrary 2D/3D shape?



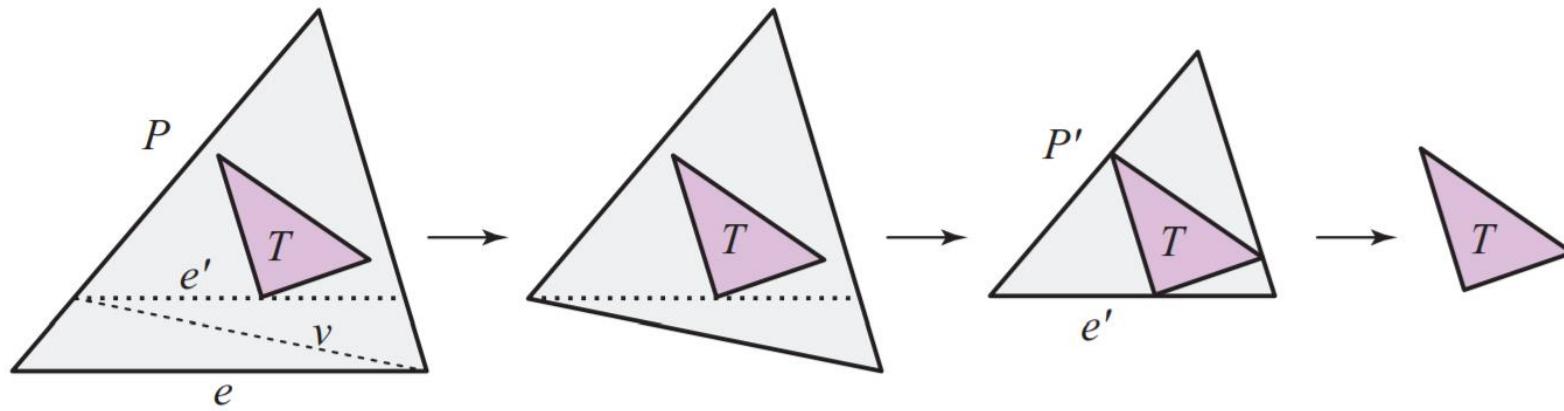
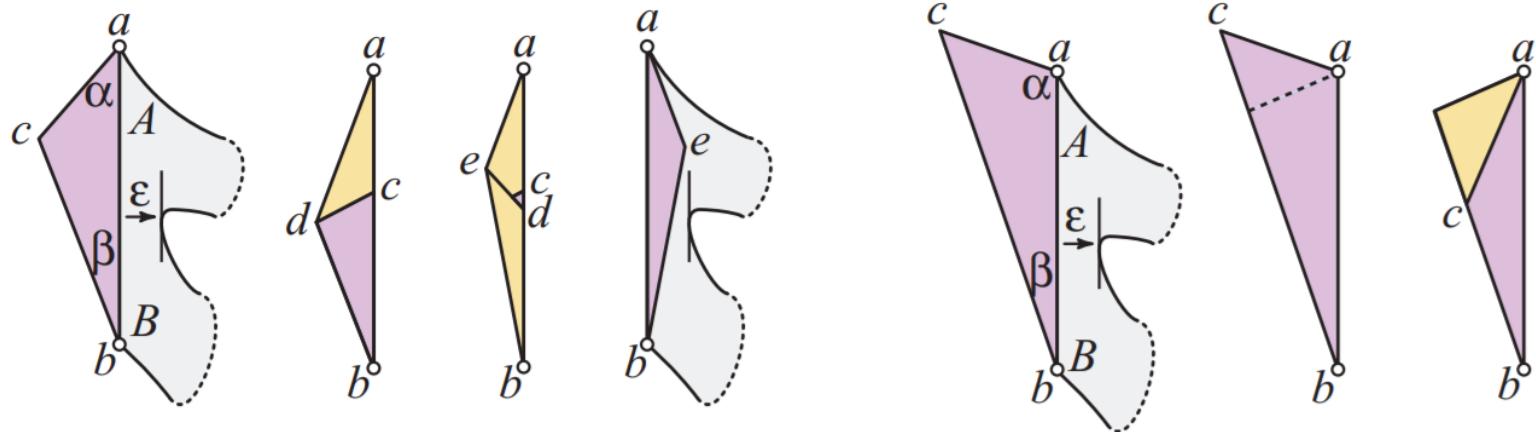
PROPOSITION. There is always a folding motion from the unfolded state to any folded state.



PROPOSITION. There is always a folding motion from the unfolded state to any folded state for flat papers.

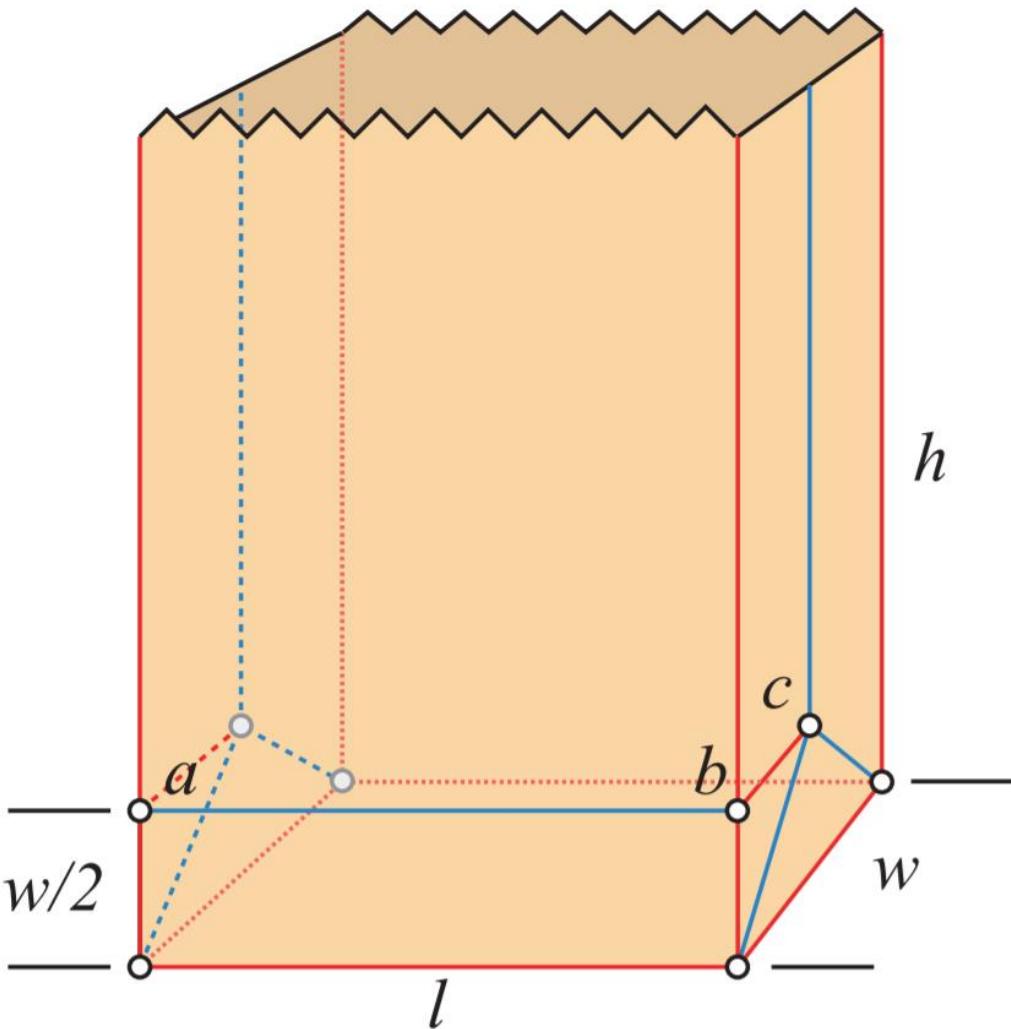


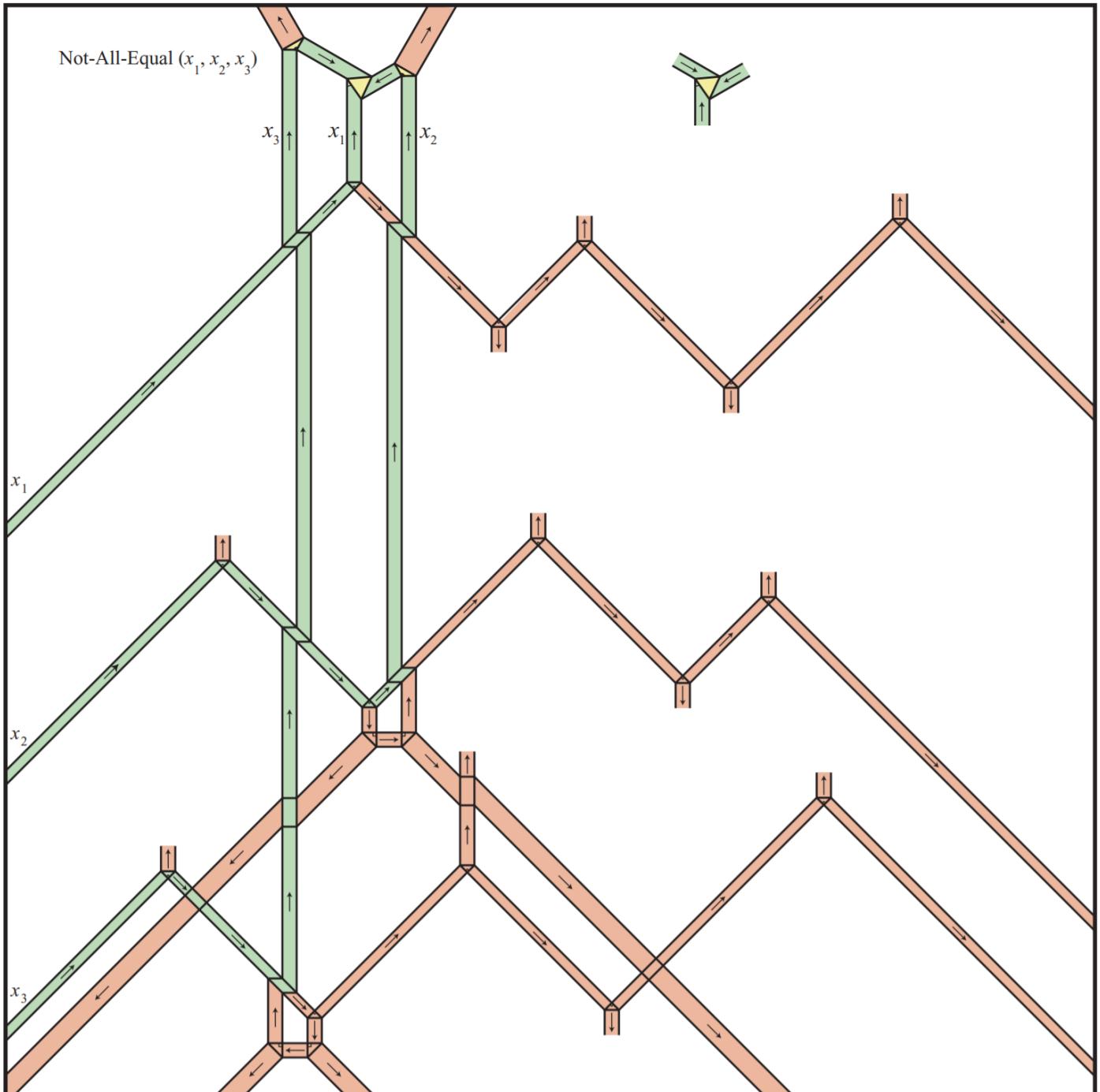
PROPOSITION. There is always a folding motion from the unfolded state to any folded state.



BENDING IS NECESSARY!

- OPEN:
Does folding motion exist
when the original paper
is not flat?



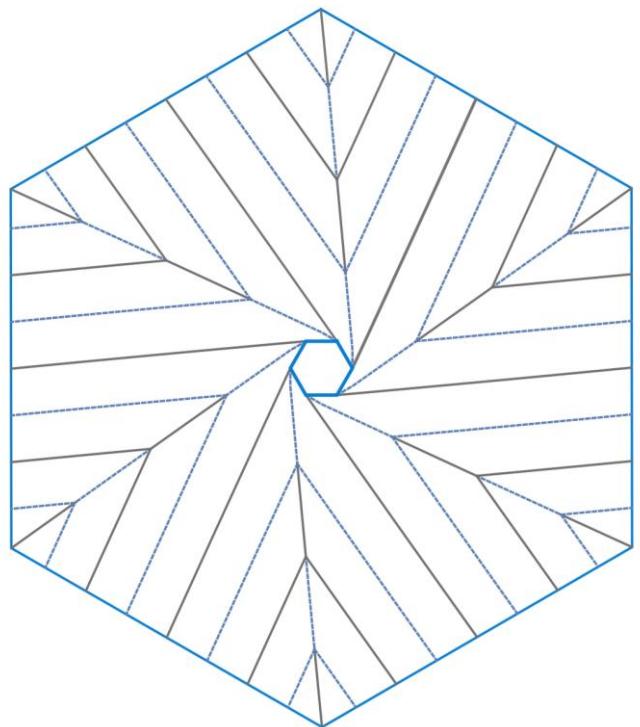
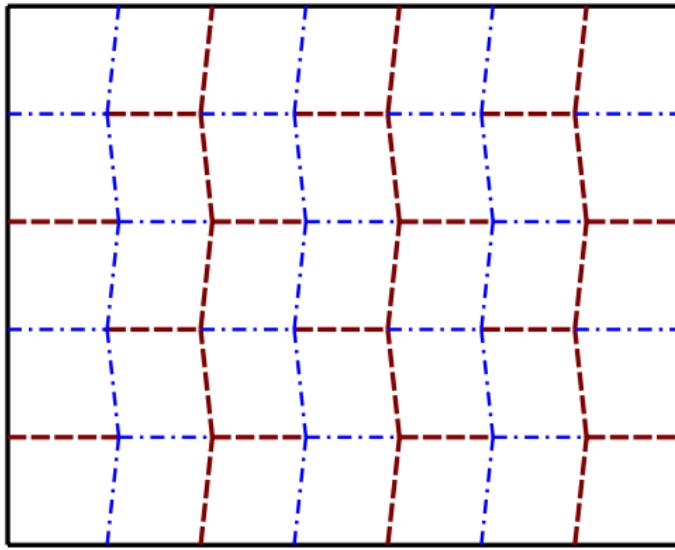


FOLDING IS HARD

- Given a crease pattern, it is NP-hard to decide if can be flat-folded.
- Given a flat-foldable mountain-valley pattern, it is NP-hard to compute the flat-folded state.

[Bern-Hayes 1996]





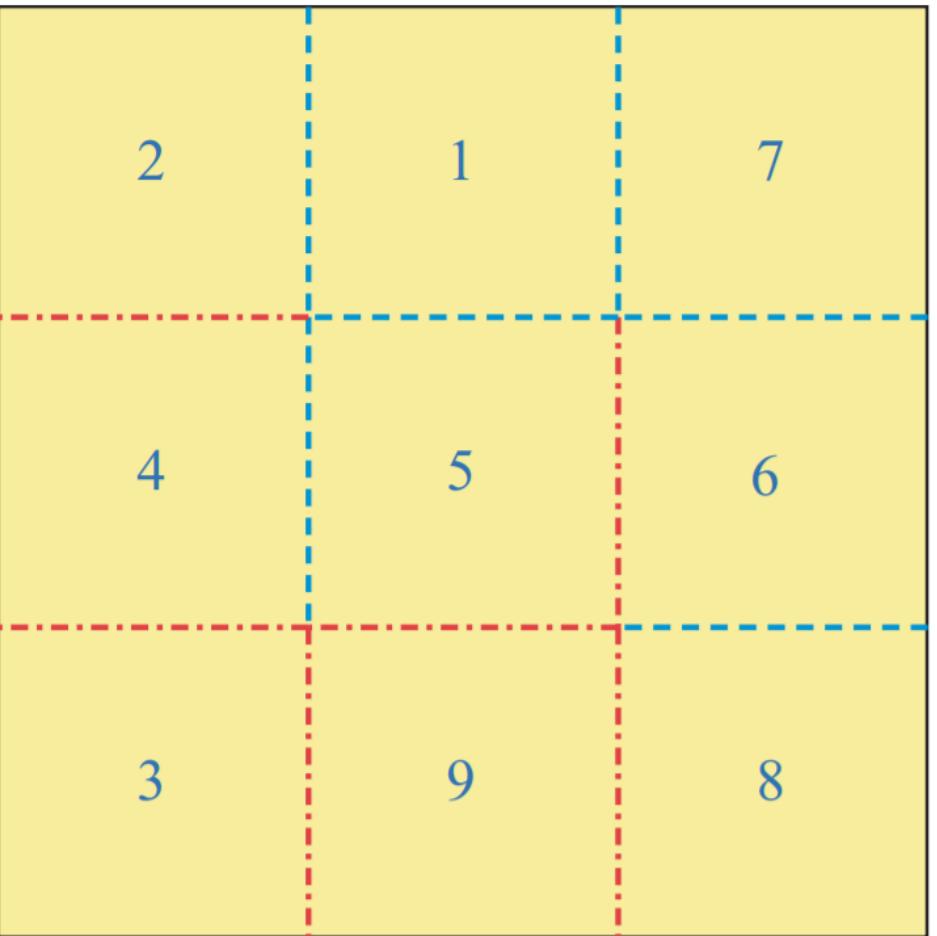
FOLDING IS HARD

- Not all folding motions are simple.

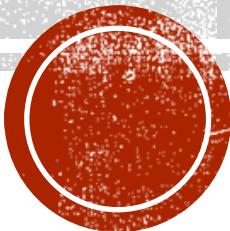


FOLDING IS HARD

- **OPEN:**
Given a 2-by-n grid with mountain-valley pattern,
can it fold flat?

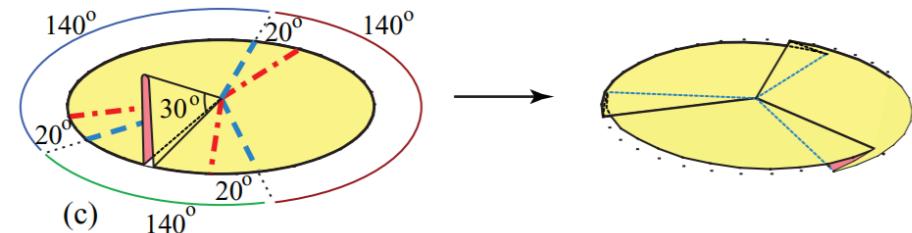
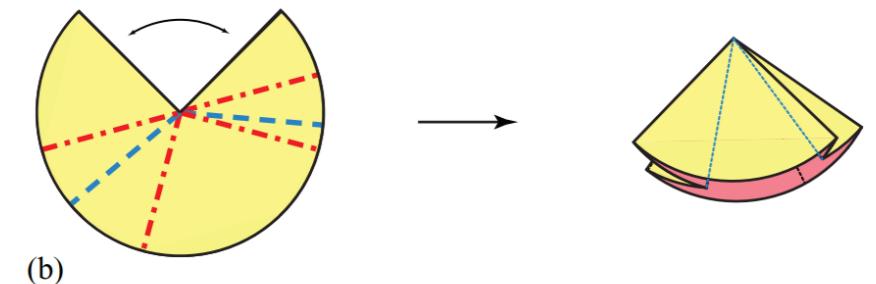
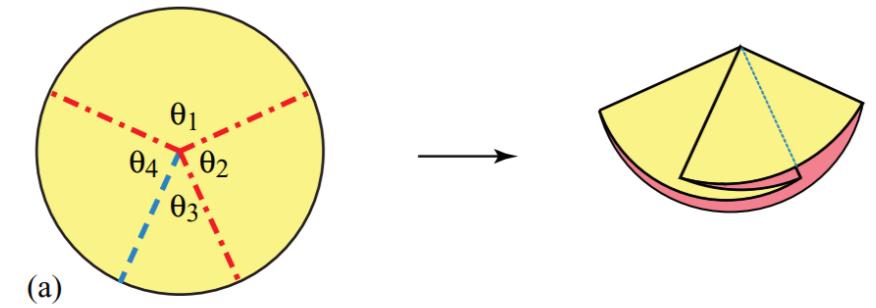


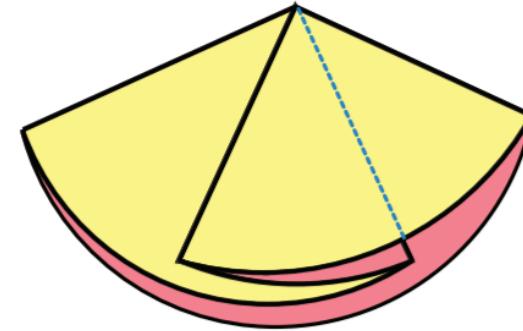
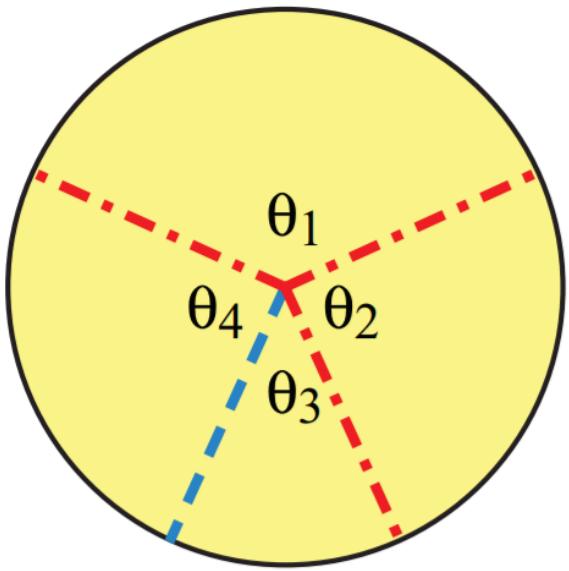
**LET'S LOOK AT SIMPLE FOLDING
AT A SINGLE VERTEX**



SINGLE-VERTEX CREASE PATTERN

- Disk of paper
- Exactly n creases
- Angles $\theta_1, \theta_2, \dots, \theta_n$

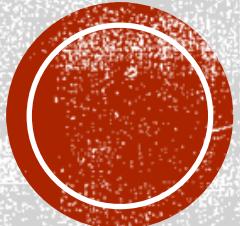




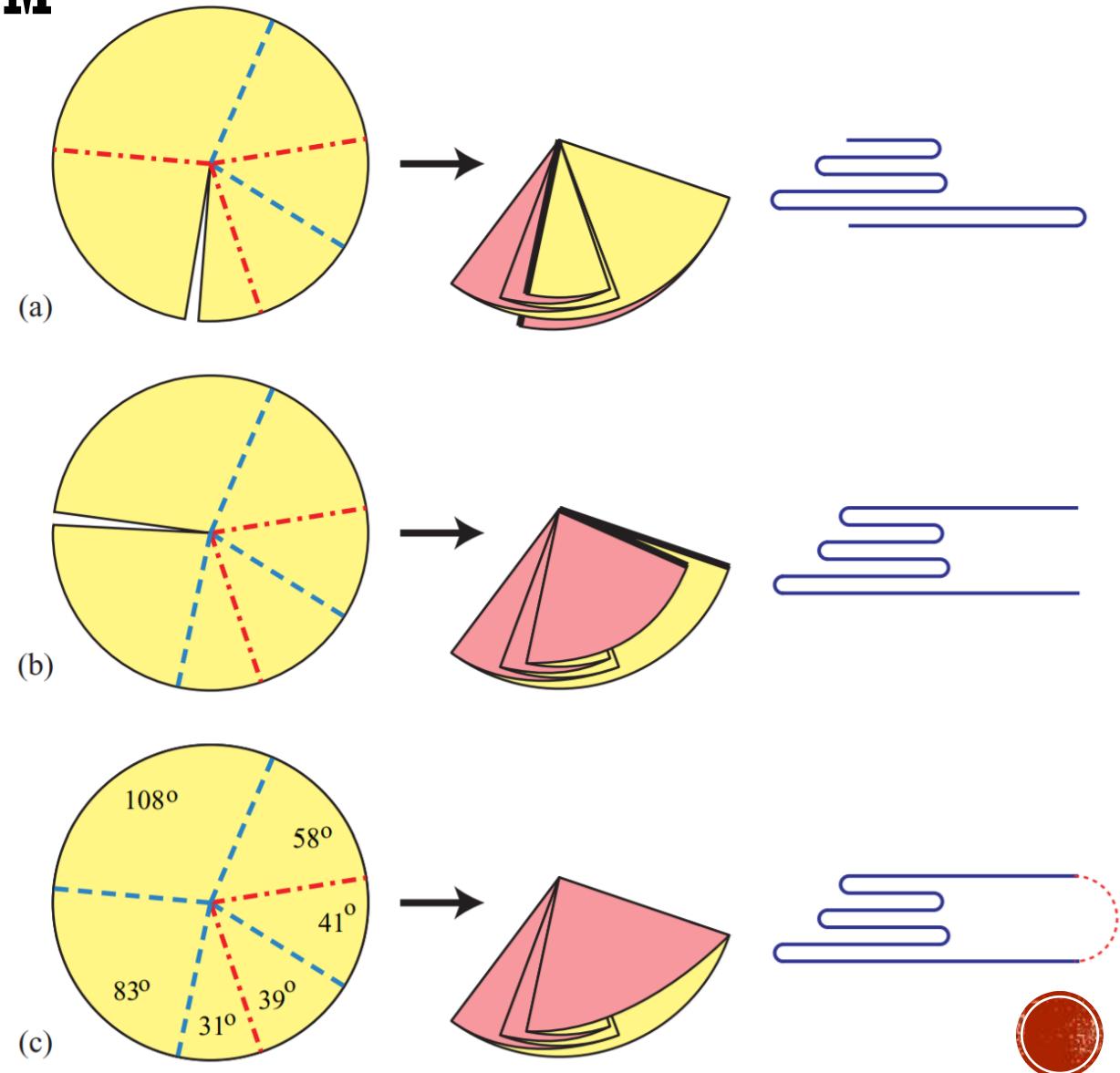
KAWASAKI'S THEOREM

[Kasahara 1989, Justin 1989]

A single-vertex crease pattern is flat-foldable if
 $(\theta_1 + \theta_3 + \dots + \theta_{n-1}) - (\theta_2 + \theta_4 + \dots + \theta_n) \in \{0, \pm 2\pi\}$

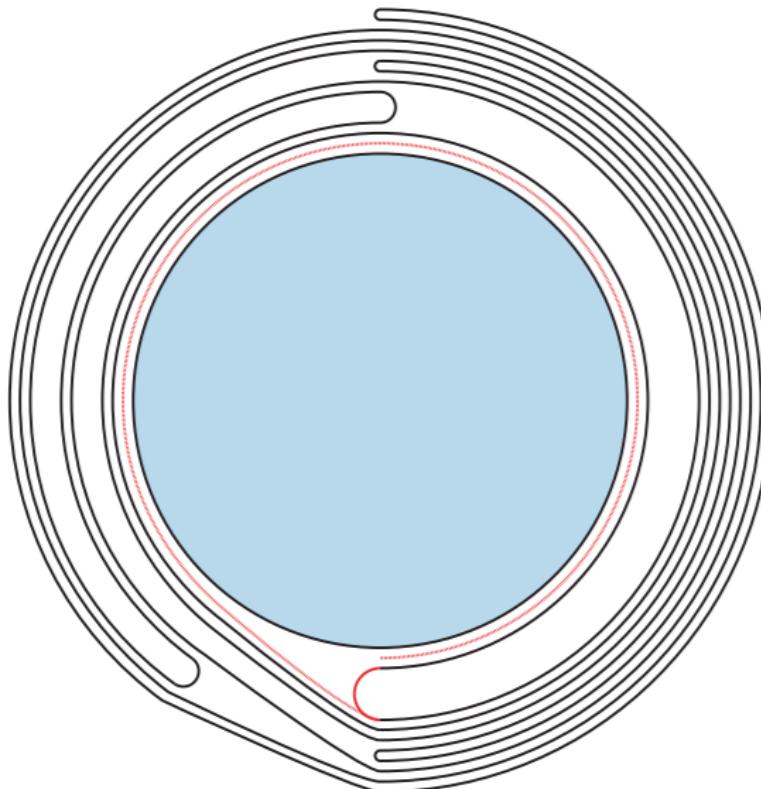


PROOF OF Kawasaki's THEOREM



PROOF OF Kawasaki's THEOREM

(a)



0π

1π

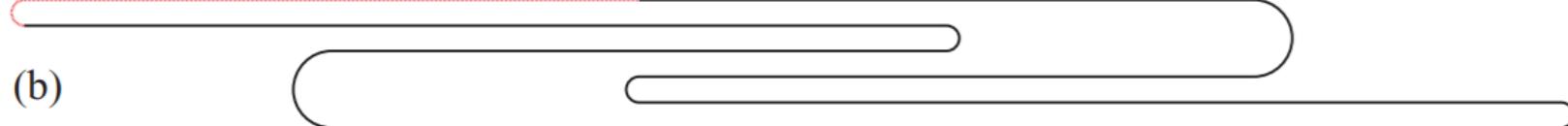
2π

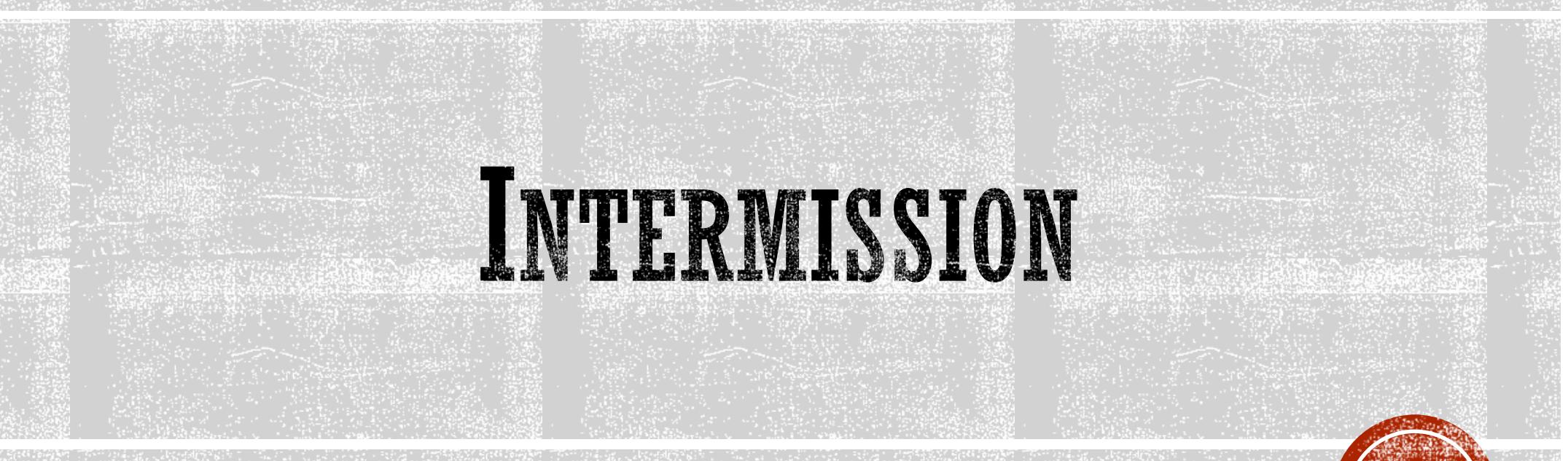
3π

4π

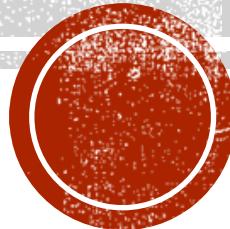
5π

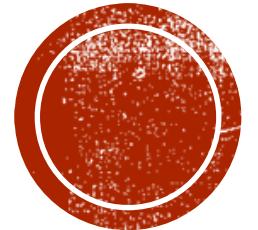
(b)





INTERMISSION





PAPER UNFOLDING



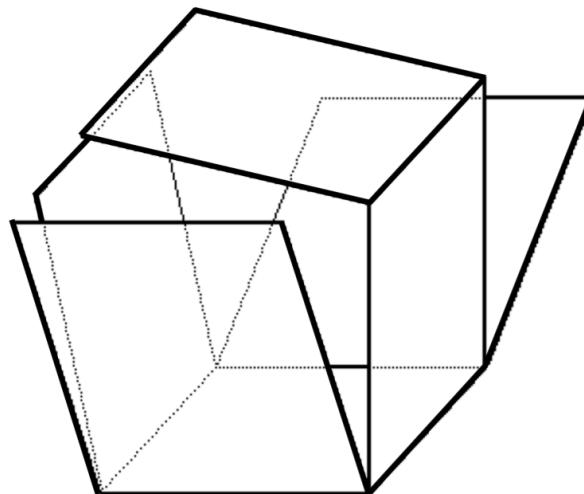
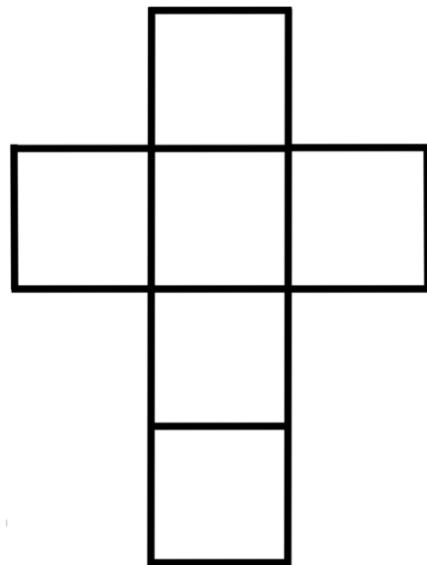
UNFOLDING PROBLEMS

- Given a 3D object, can it be unfolded into something flat?



Two Different Unfolding Rules

- Edge-unfolding



- General-unfolding

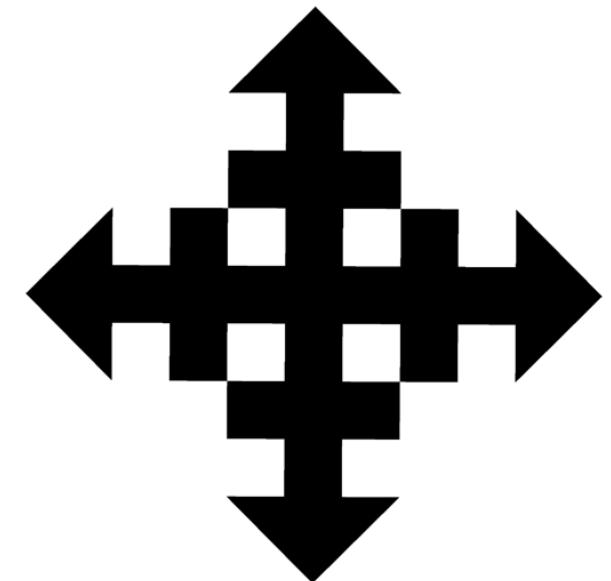
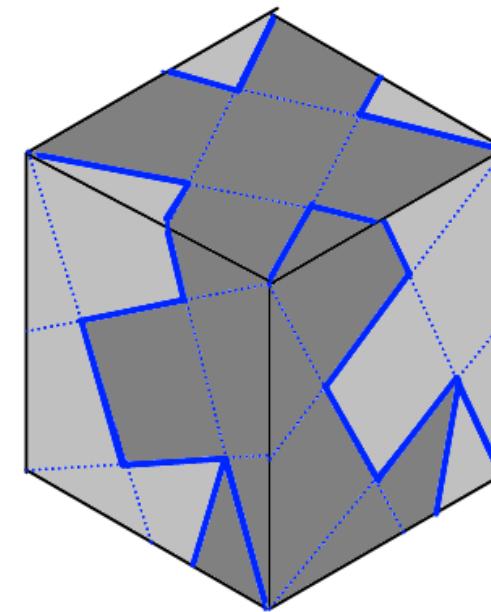
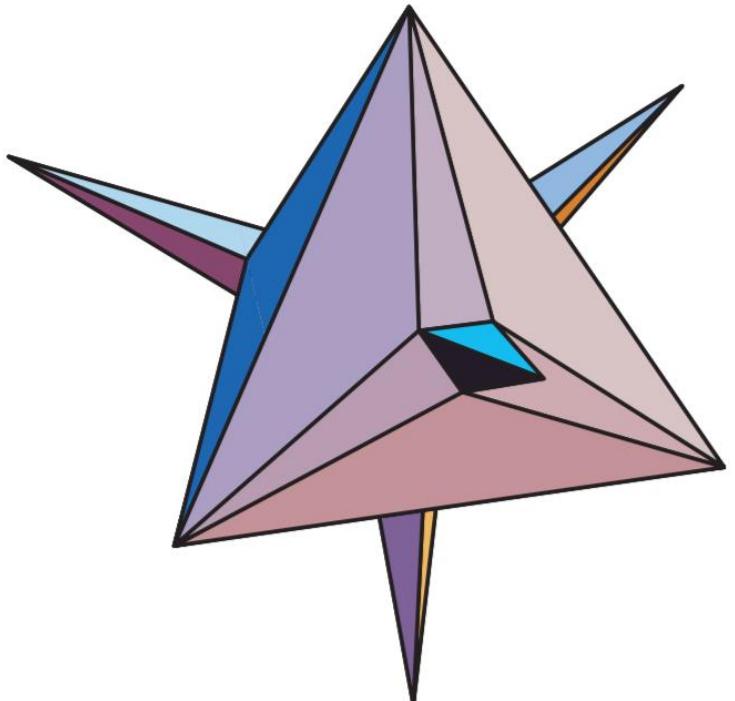
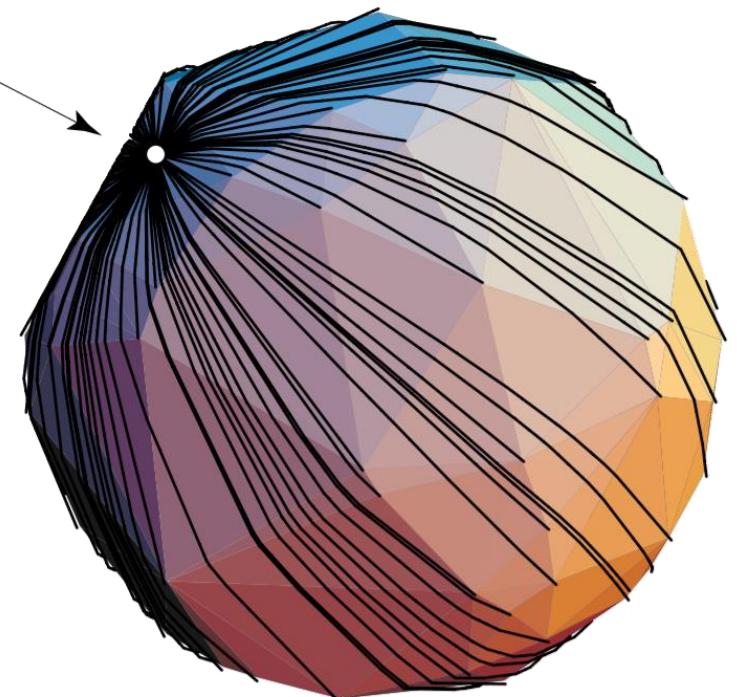


Table 22.1: Status of main questions concerning nonoverlapping unfoldings

Shapes	Edge unfolding?	General unfolding?
Convex polyhedra	Open	YES
Nonconvex polyhedra	NO	Open



[Bern-Demaine-Eppstein-Kuo-Mantler-Snoeyink 2003]



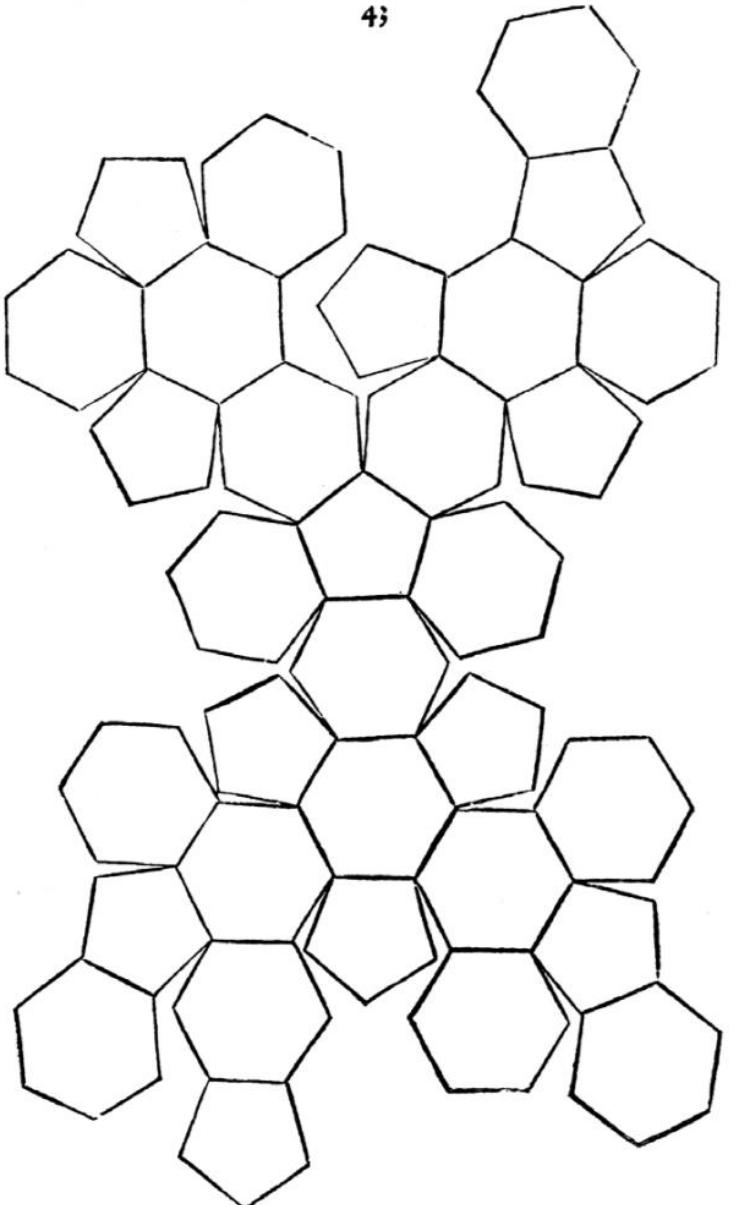
[Xu 1996]

KNOWN RESULTS



SIn anders das mach aus zweinsig sechsecketer flachen vnd windlich/
so man darzu thut zwölß fünfecketer flacher felder/ so die gleichsietig gegen den sechsecken
seiden sind/ vnd in jnen selbs auch gleich windlich vnd ordentlich an cynander gesetzt wer-
den/ wie ich das offen im plano hernach hab aufgerissen / So man dann das alles zusammen
schleust/ so würt ein corpus daraus/ das gewinnet zwey vnd sechzig ecke/ vnd neunzig scharfster
stüten/ dis Corpus rüret in einer holen kugeln mit allen seinen ecken an.

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UNFOLDING POLYHEDRON

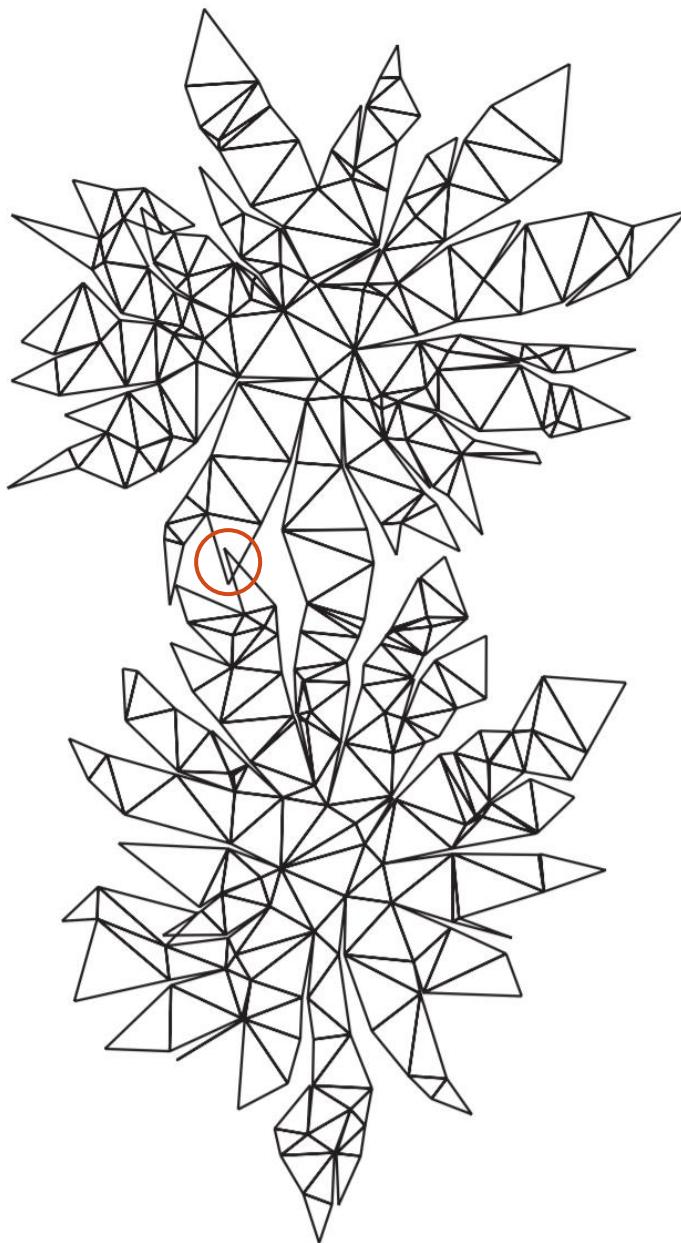
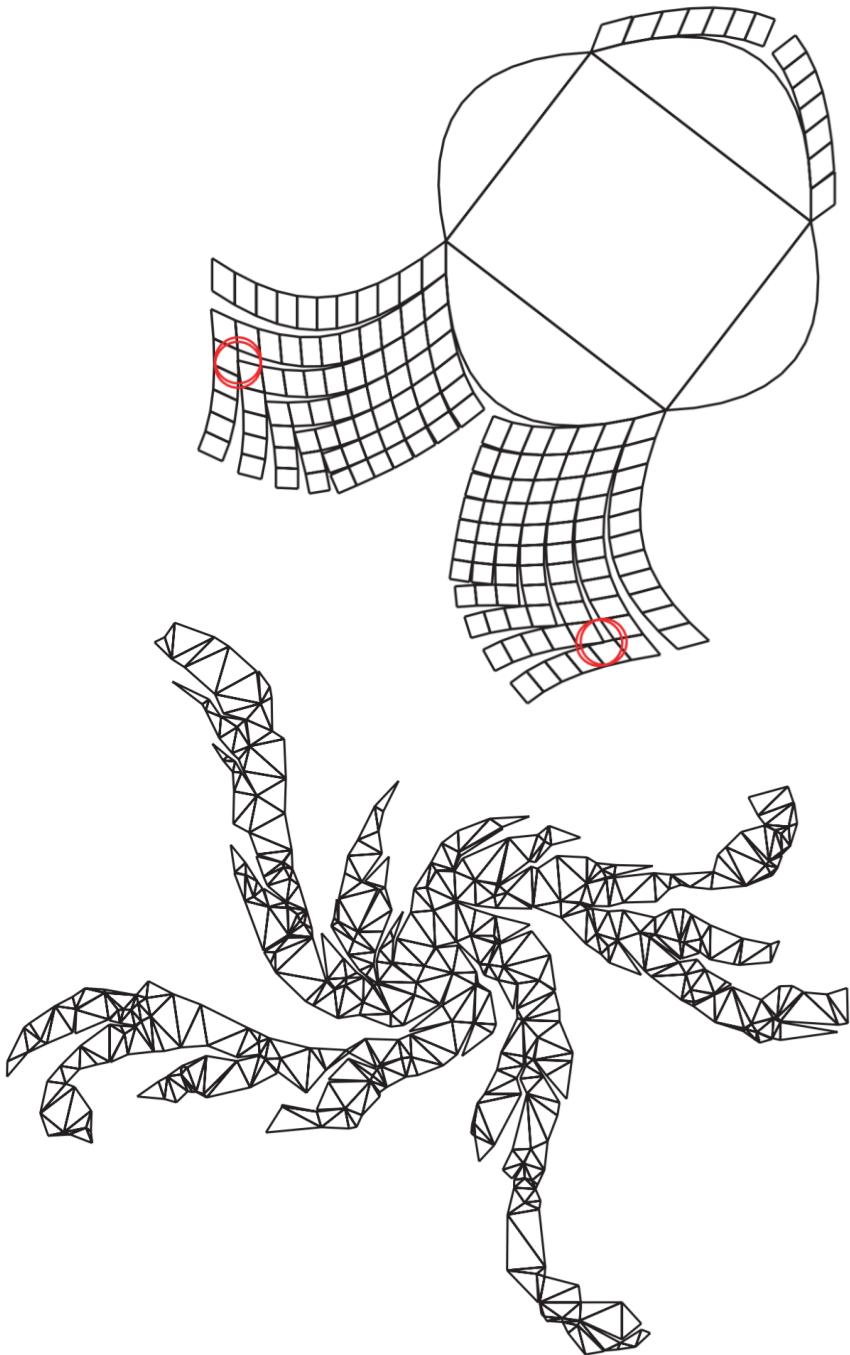
- **OPEN:**
Does every convex polyhedron have an edge-unfolding?

[Albrecht Dürer 1525]

[Shephard 1975]

[Schlickenrieder 1997]





UNFOLDING POLYHEDRON

- **OPEN:**
Does every convex polyhedron have an edge-unfolding?

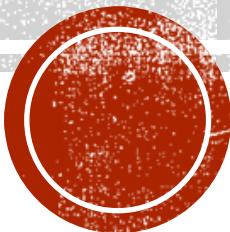
[Albrecht Dürer 1525]

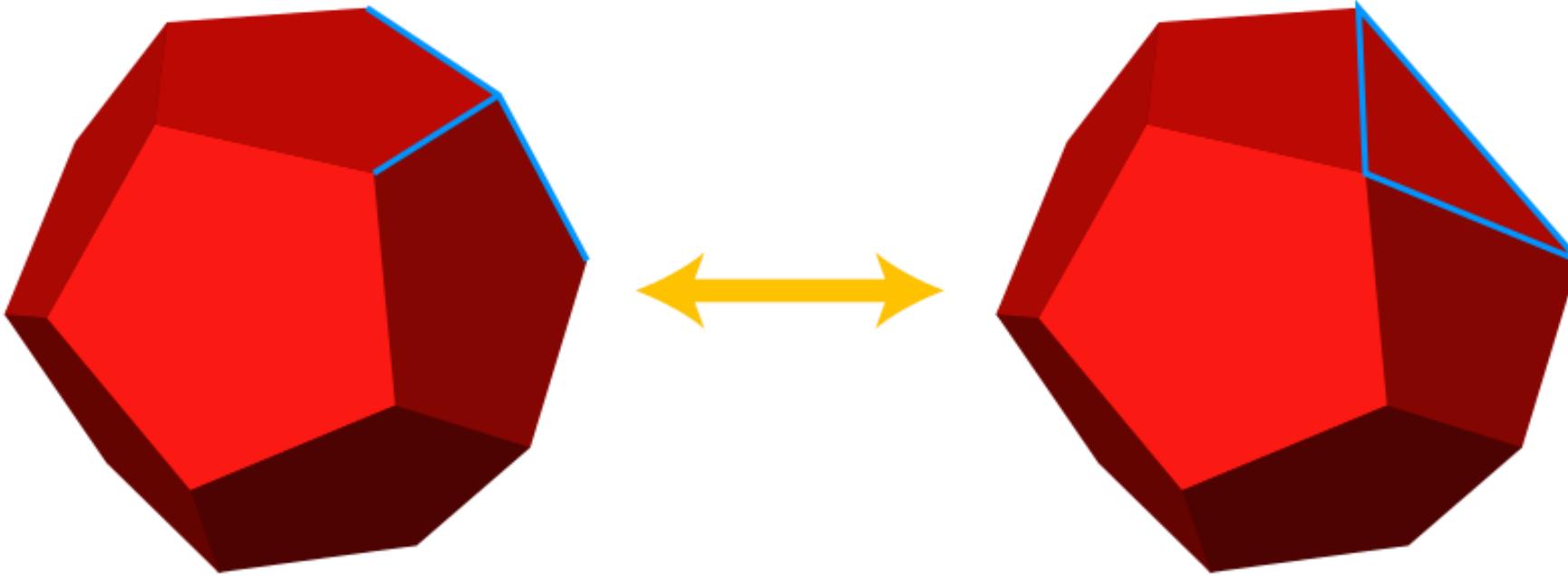
[Shephard 1975]

[Schlickenrieder 1997]



**WHEN DOES AN UNFOLDING
COMING FROM A POLYHEDRON?**

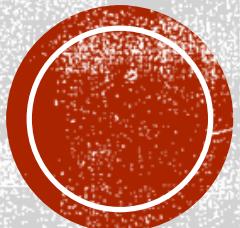


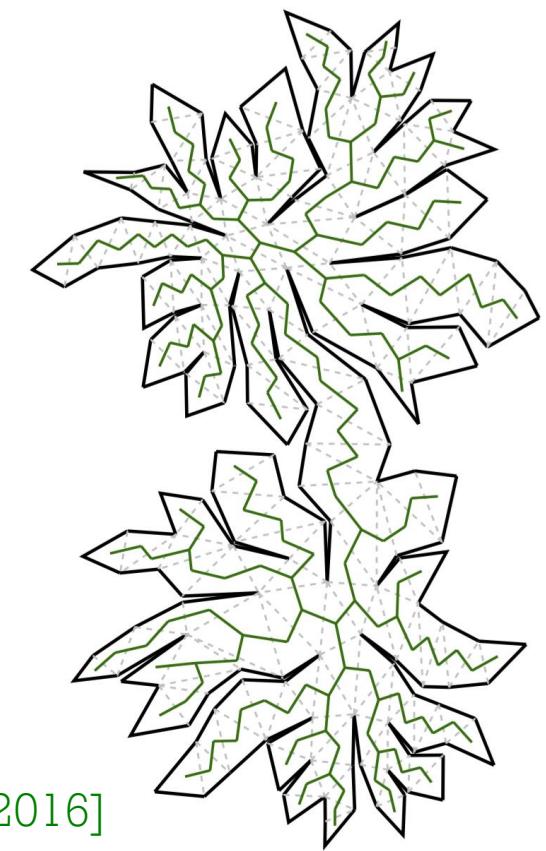
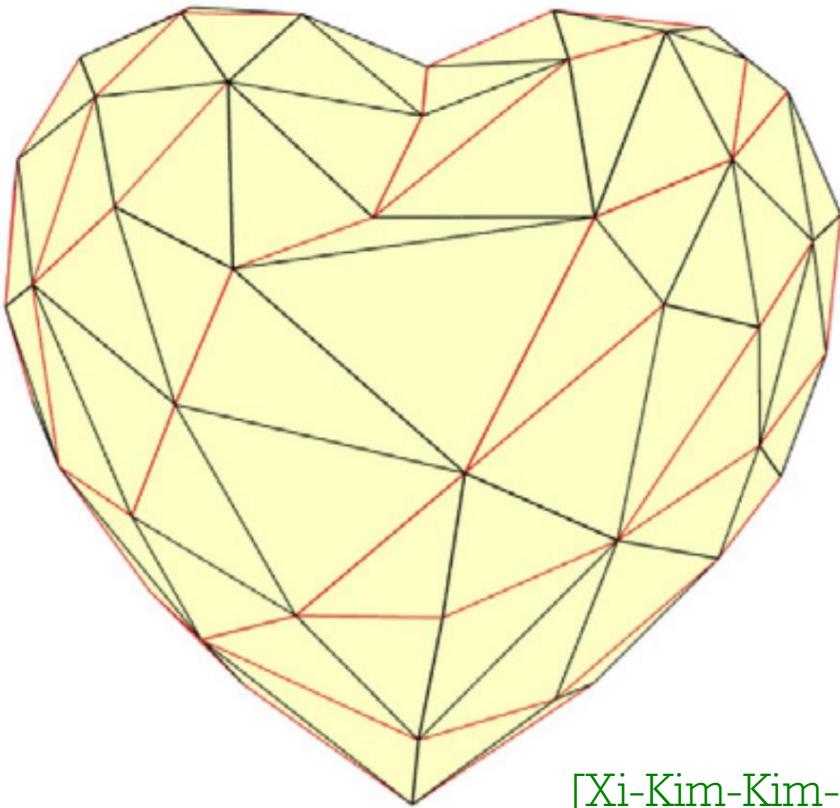
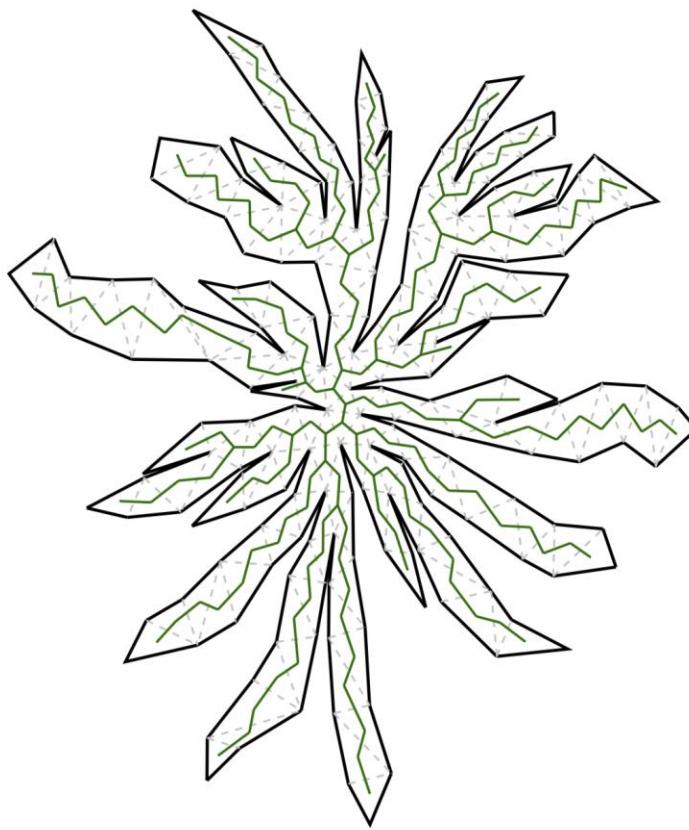


STEINITZ'S THEOREM

[Steinitz 1916]

A graph G forms a convex polyhedron if and only if
 G is simple, planar, and 3-connected



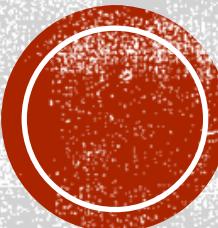


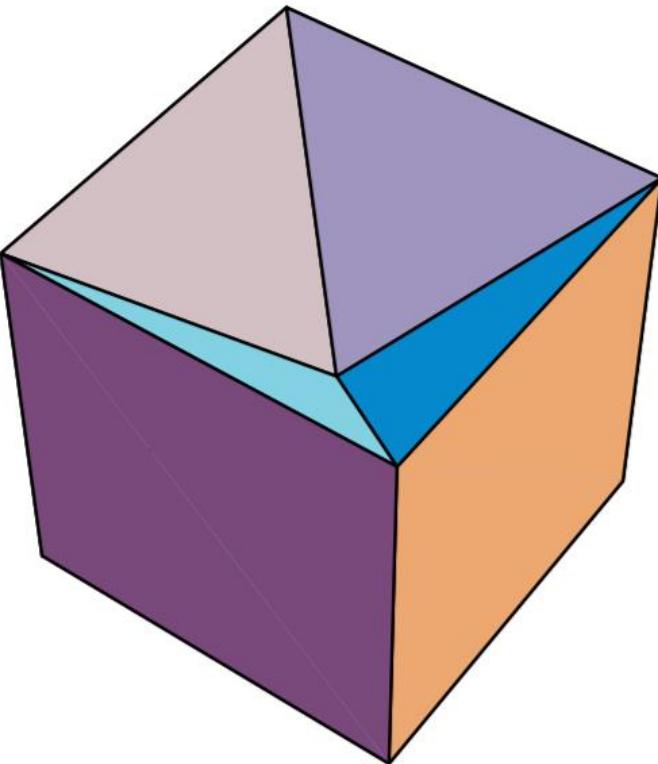
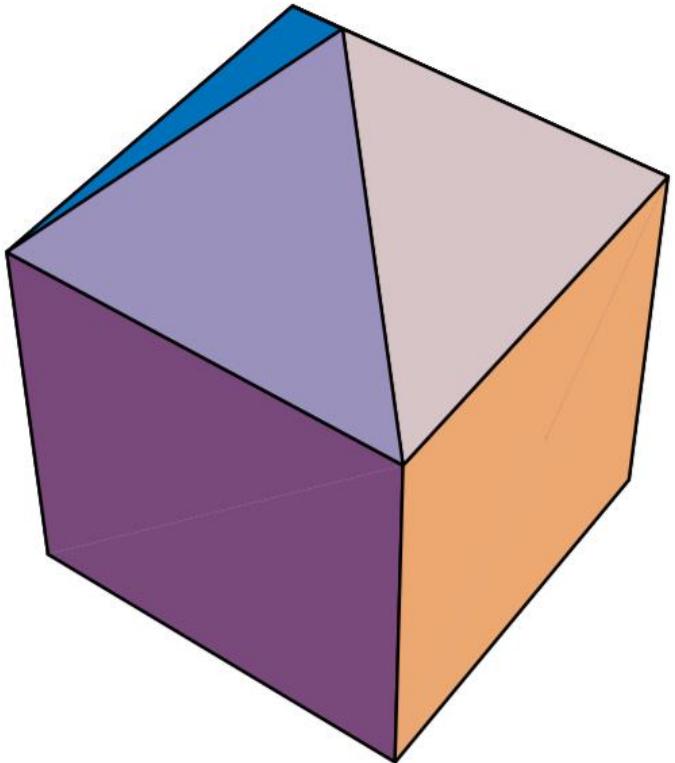
[Xi-Kim-Kim-Lien 2016]

CAUCHY RIGIDITY THEOREM

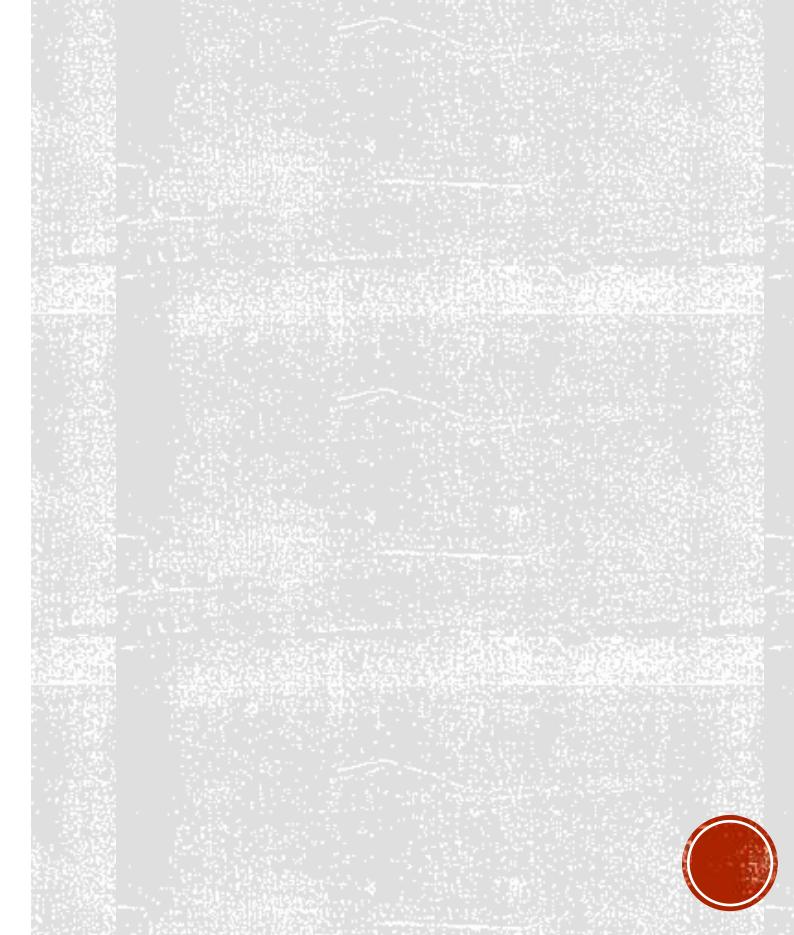
[Cauchy 1813] [Steinitz-Rademacher 1934]

Two combinatorially equivalent, convex polyhedral
must be congruent

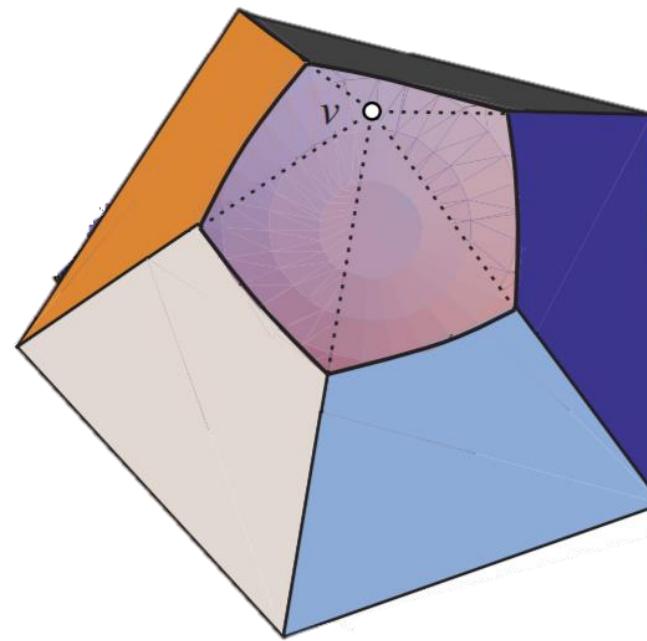
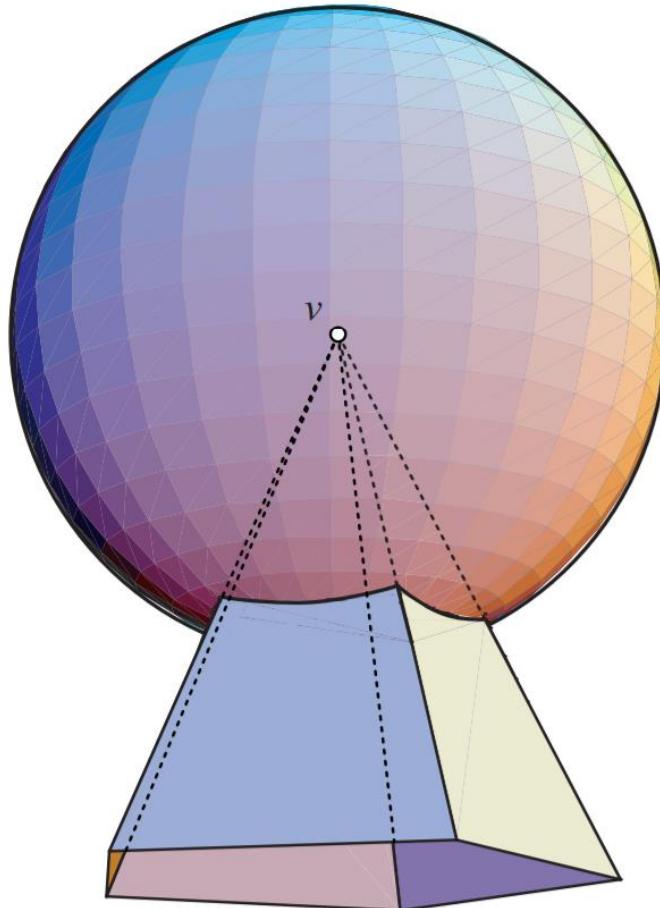




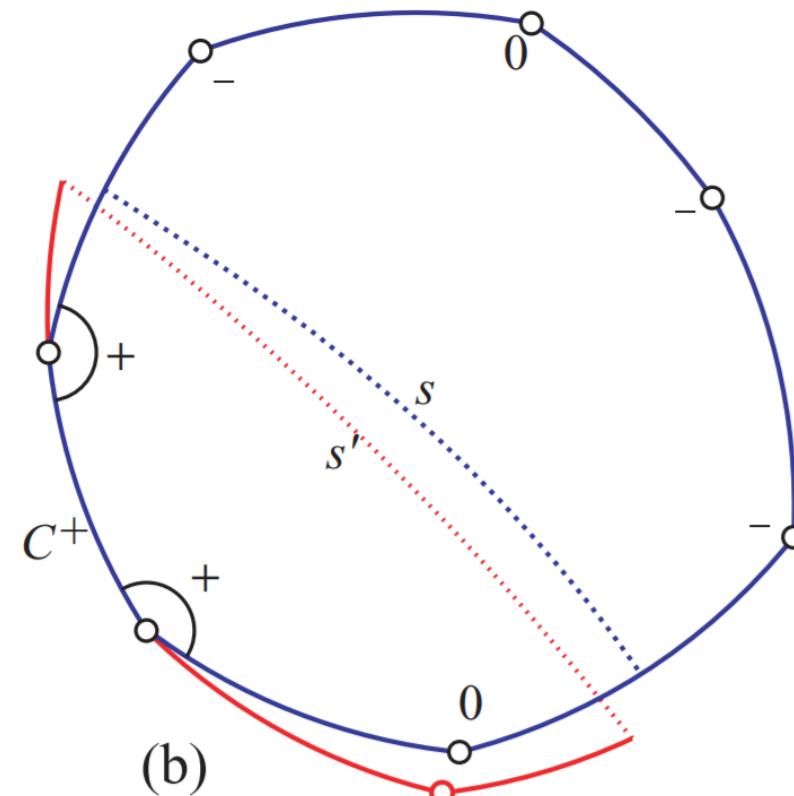
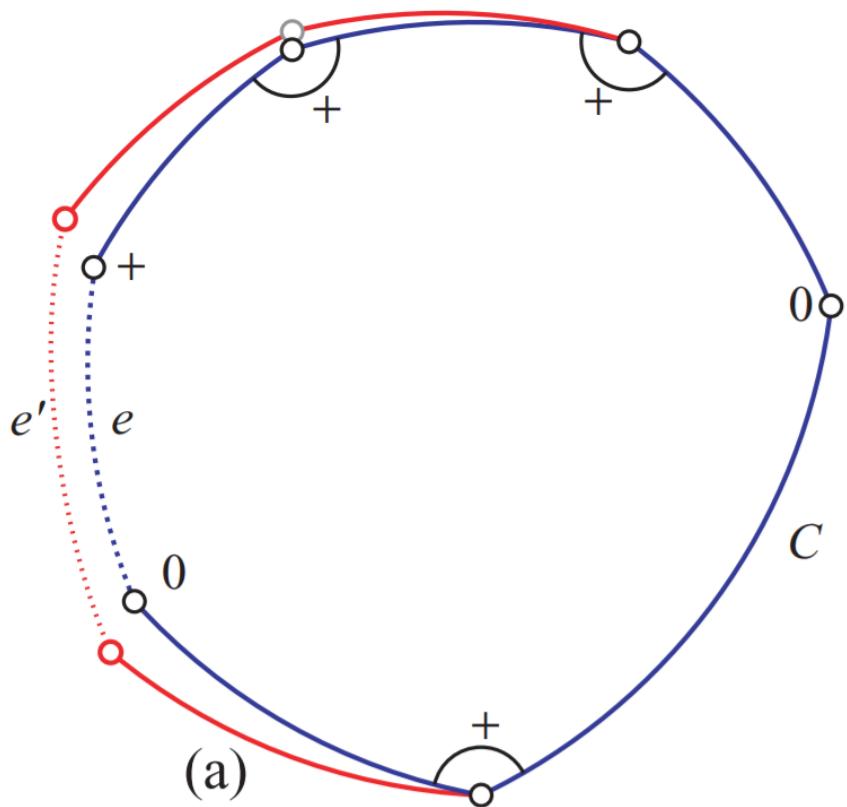
**Not TRUE IF
Non-Convex**



PROOF OF RIGIDITY THEOREM

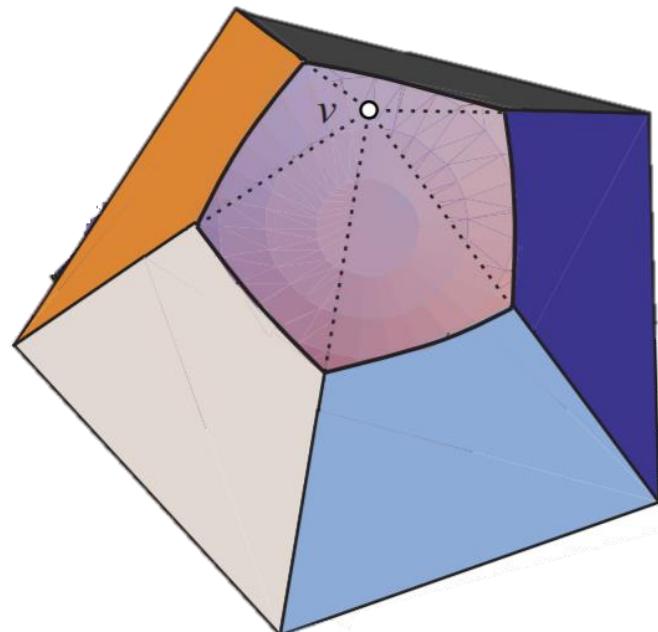


CAUCHY-STEINITZ LEMMA. Any \pm -labeling on the corners of convex polygon must have 4 alternations.



PROOF OF RIGIDITY THEOREM

C-S LEMMA. ≥ 4 alternations.



$$V - E + F = 2 \quad 4V = (2 + E - F) \cdot 4$$

$$\sum := \text{# alternations} = \sum_{v \in P} \text{alt}(v)$$

$$\sum \geq 4 \cdot V \quad (\text{by C-S Lemma})$$

$$\boxed{4V > \sum^+ \geq \sum \geq 4 \cdot V}$$

X - $\angle 4V$

$$\sum^+ = 2 \cdot f_3 + 4 \cdot f_4 + 4 \cdot f_5 + 6 \cdot f_6 + \dots$$

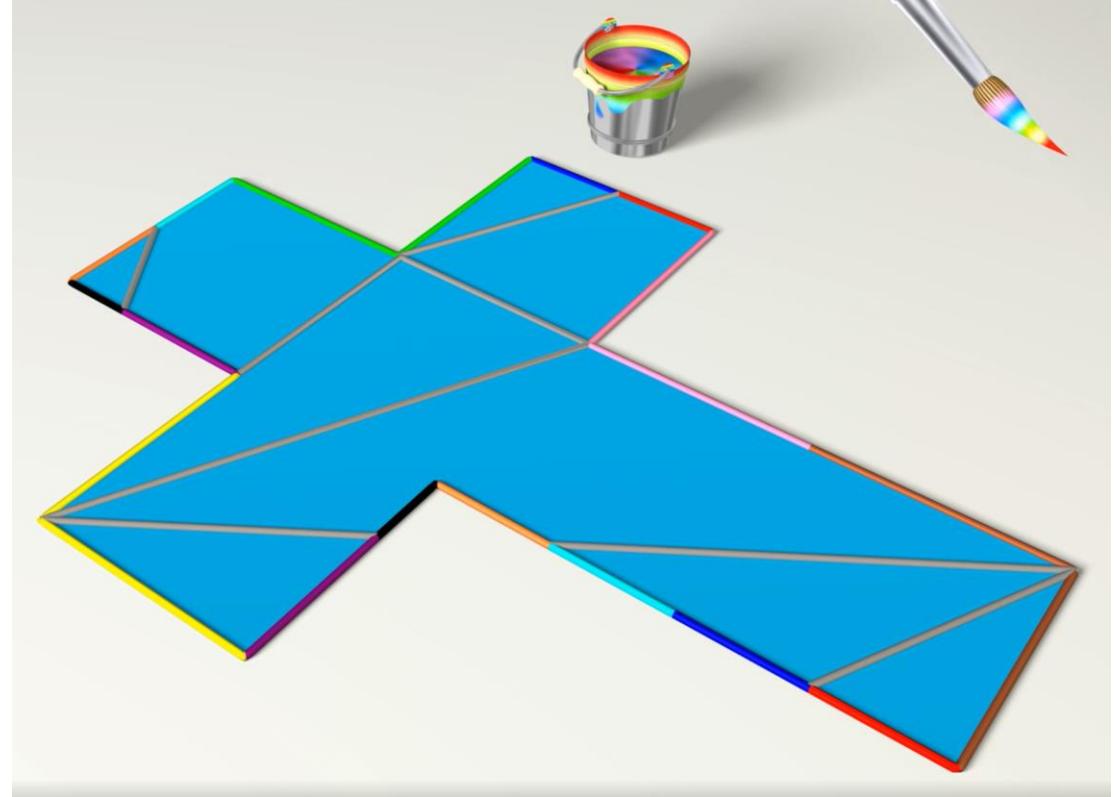
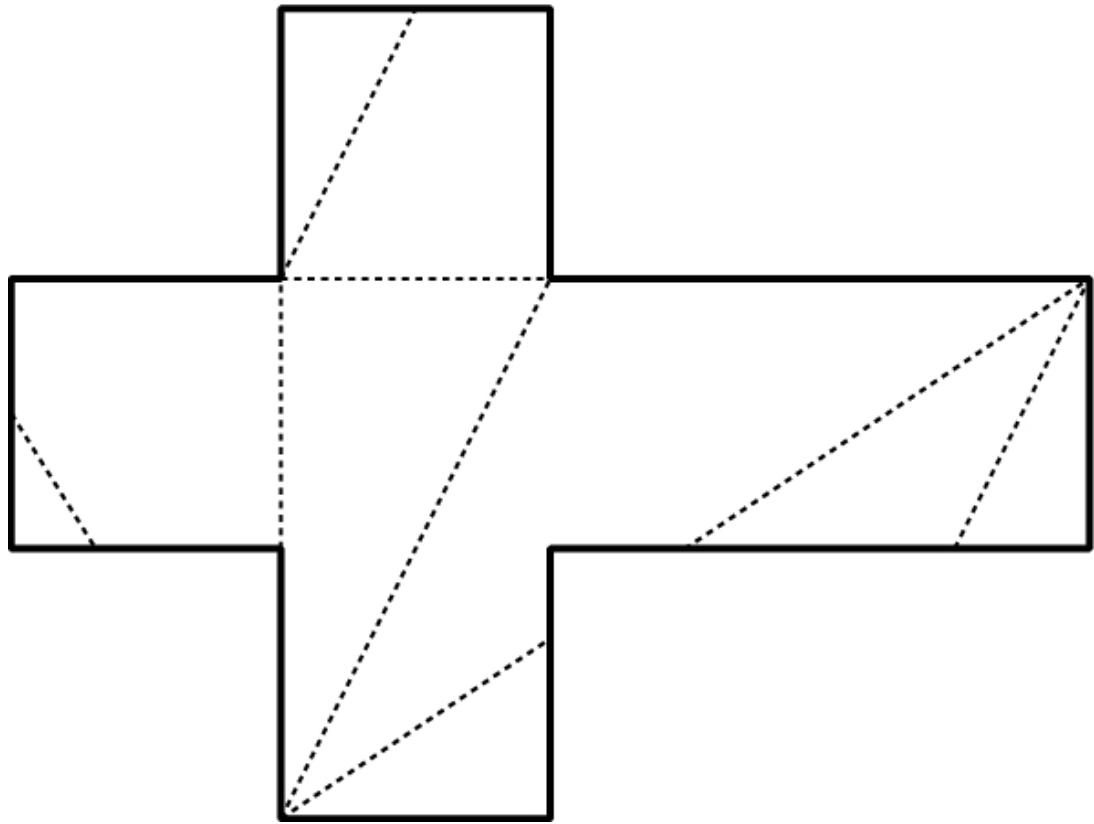
$$2E = 3 \cdot f_3 + 4 \cdot f_4 + 5 \cdot f_5 + 6 \cdot f_6 + \dots$$

$$F = f_3 + f_4 + f_5 + f_6 + \dots$$

$$4V = 8 + 6f_3 + 8f_4 + 10f_5 + \dots$$

$$- 4f_3 - 4f_4 - 4f_5 + \dots$$

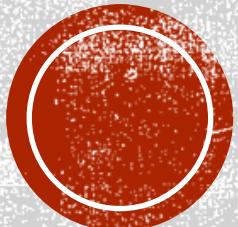
$$= 8 + 2f_3 + 4f_4 + 6f_5 + 8f_6 + \dots$$

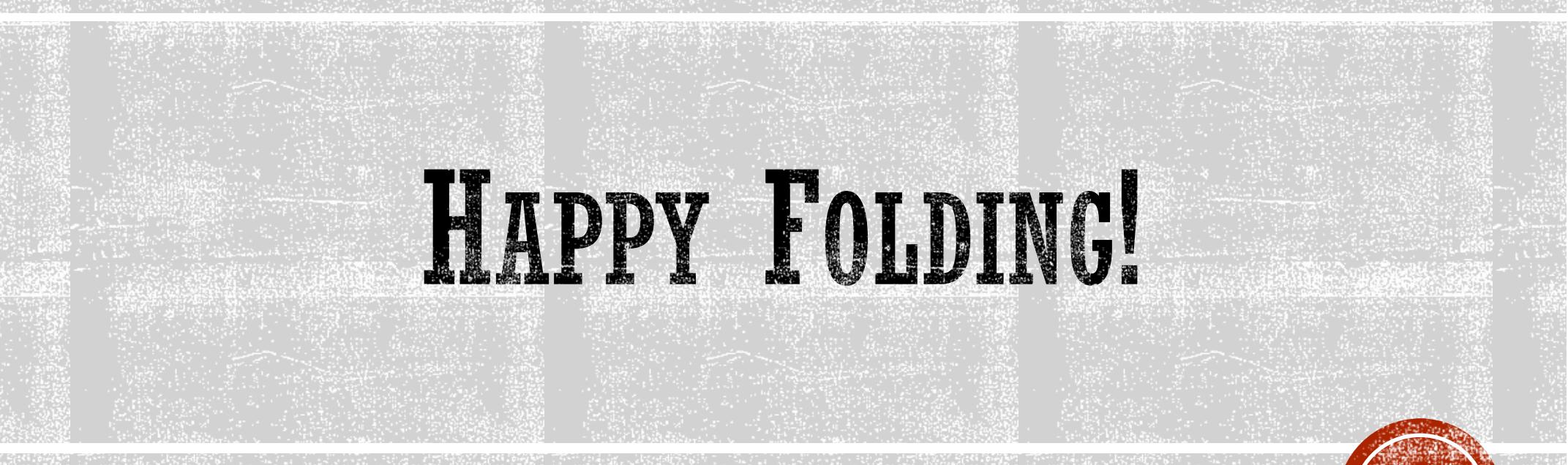


ALEXANDROV'S THEOREM

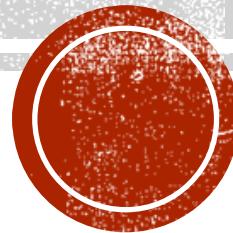
[Alexandrov 1941]

Any gluing of polygons that (1) uses $<2\pi$ angle per point
and (2) results in a topological sphere
must uniquely determine a convex polygon





HAPPY FOLDING!



NEXT TIME.
Behavior of closed curves
dictates the shape of space.