Introduction to Computational Topology

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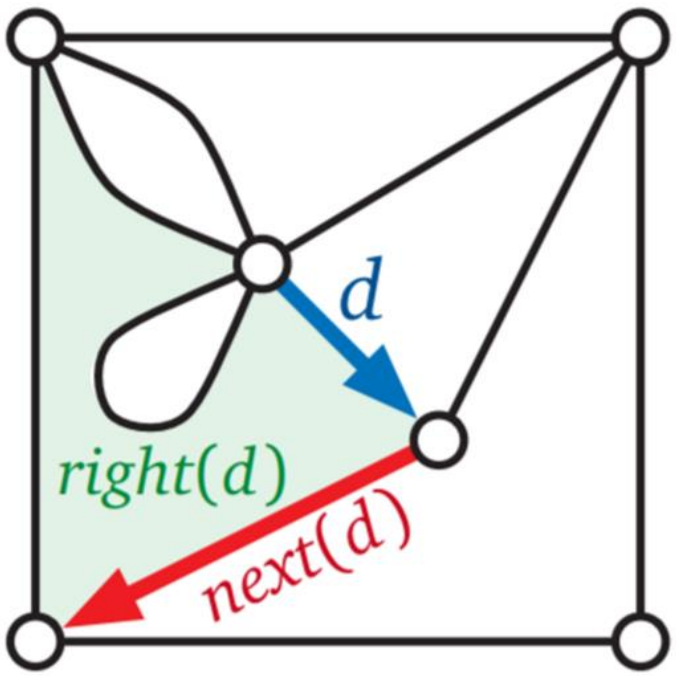
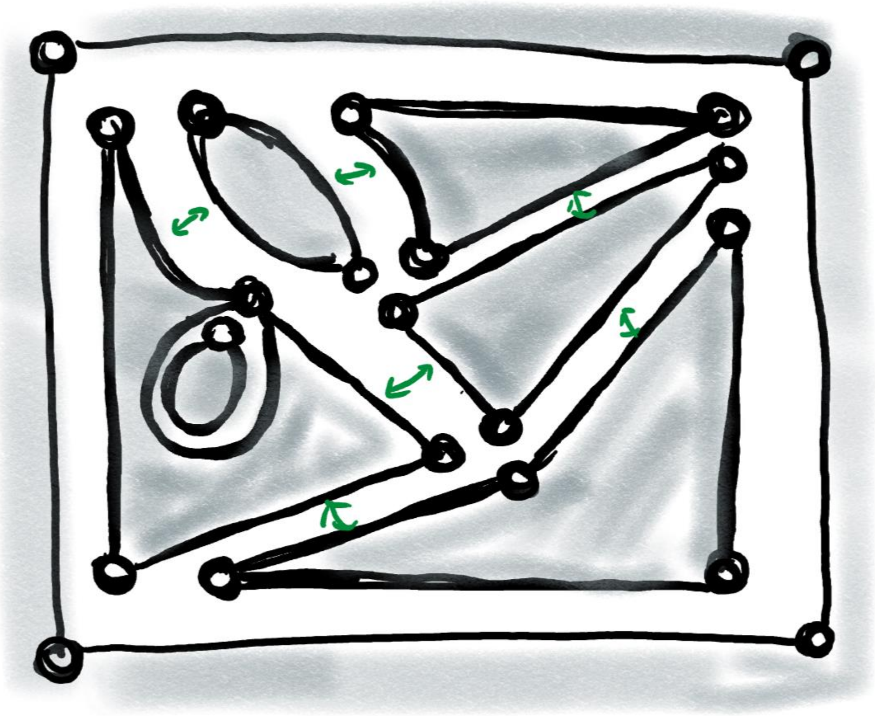
Lecture 4, September 23, 2021

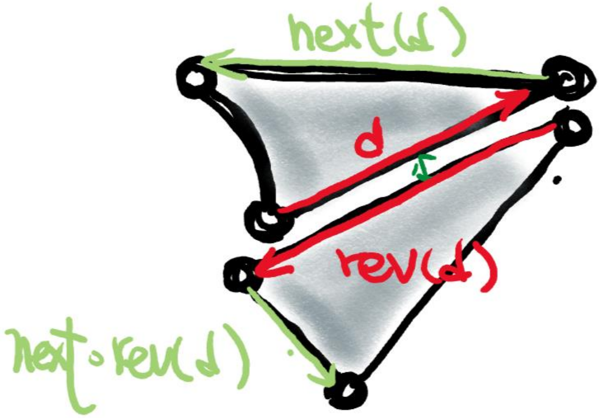
Acknowledgement

- Most of the figures today comes from
  - Jeff Erickson, *One-Dimensional Computational Topology*
  - Robert Ghrist, *Elementary Applied Topology*
  - Keenan Crane, *Discrete differential geometry: An applied introduction*

Recap: Polygonal Schema IS Rotation System







Euler's Formula

Let's Focus on Plane Graphs

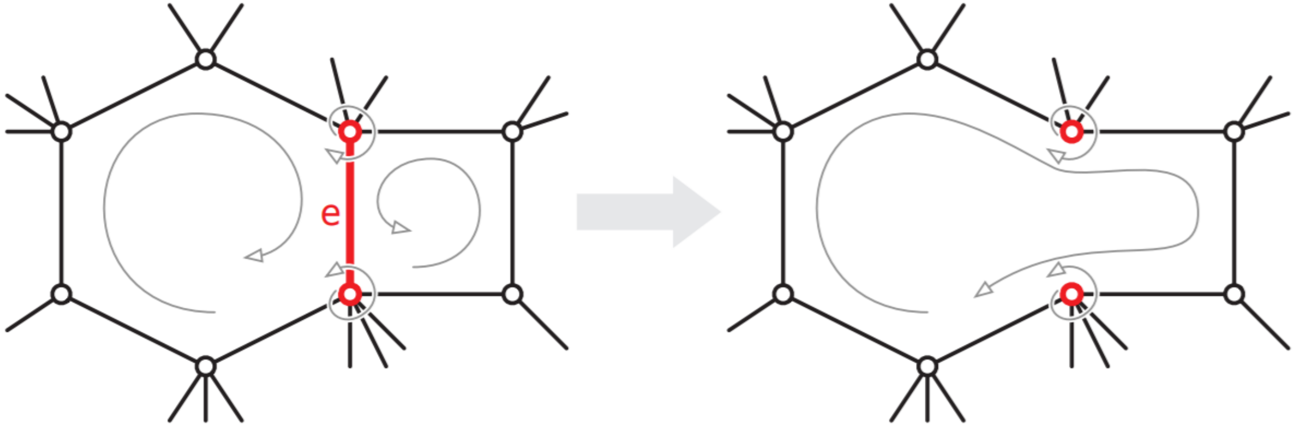
Spanning Tree Terminology

- Loop

- Bridge

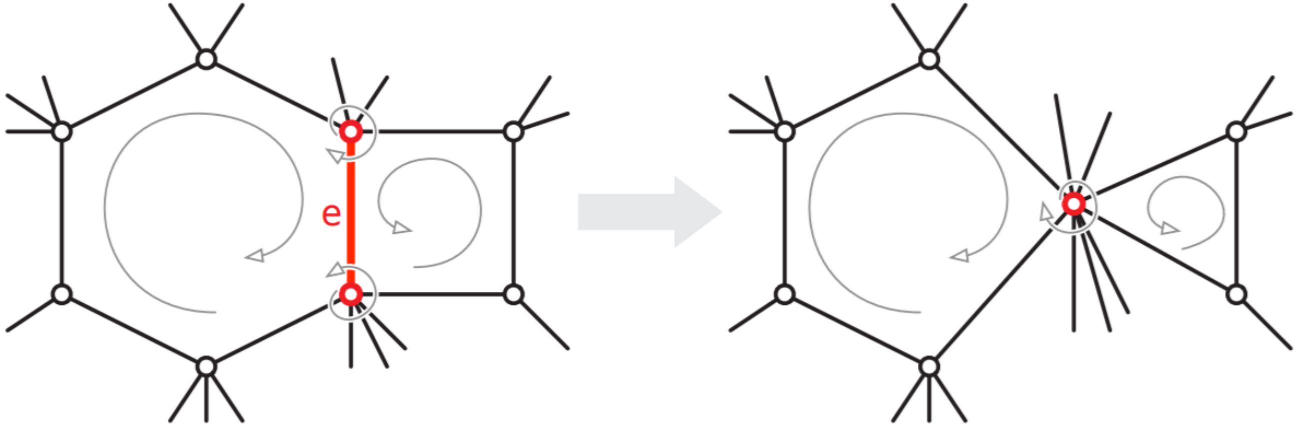
Spanning Tree Terminology

- Delete



Deleting an edge between differently oriented faces.

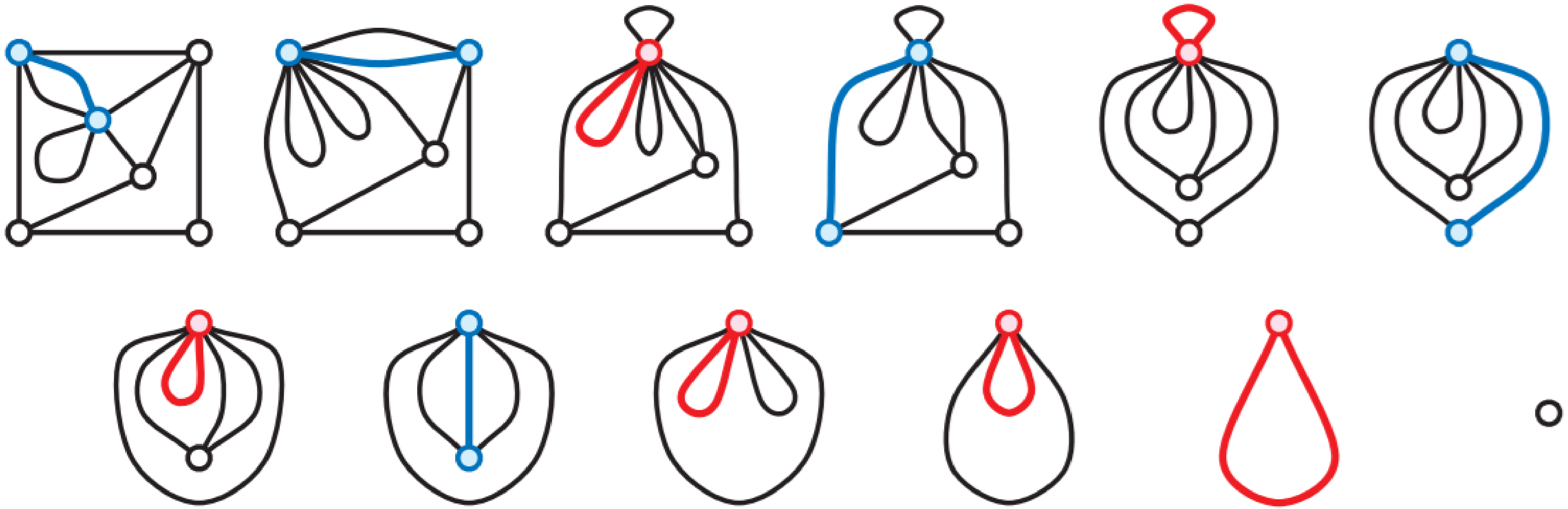
- Contract



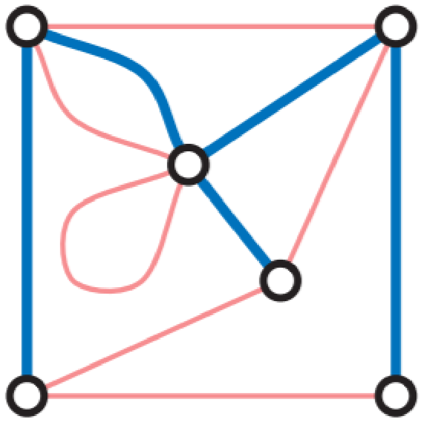
Contracting an edge between differently oriented vertices.

THE SPANNING TREE ALGORITHM

The Spanning Tree Algorithm



Computing a spanning tree of a graph.



The Spanning Tree Algorithm

```
WhateverFirstSearch(s):
  put (∅, s) in bag
  while the bag is not empty
      take (p, v) from the bag                  (⋆)
      if v is unmarked
          mark v
          parent(v) ← p
          for each edge vw                      (†)
              put (v, w) into the bag           (⋆⋆)
```

What Happened in the Dual?

```
SpanningTree(G):
  for any edge e in G:
    if e is a loop:
      delete e.
    if e is a bridge:
      contract e
    o.w.
      delete or contract e
  return all contracted edges T
```


```
SpanningTree(G):
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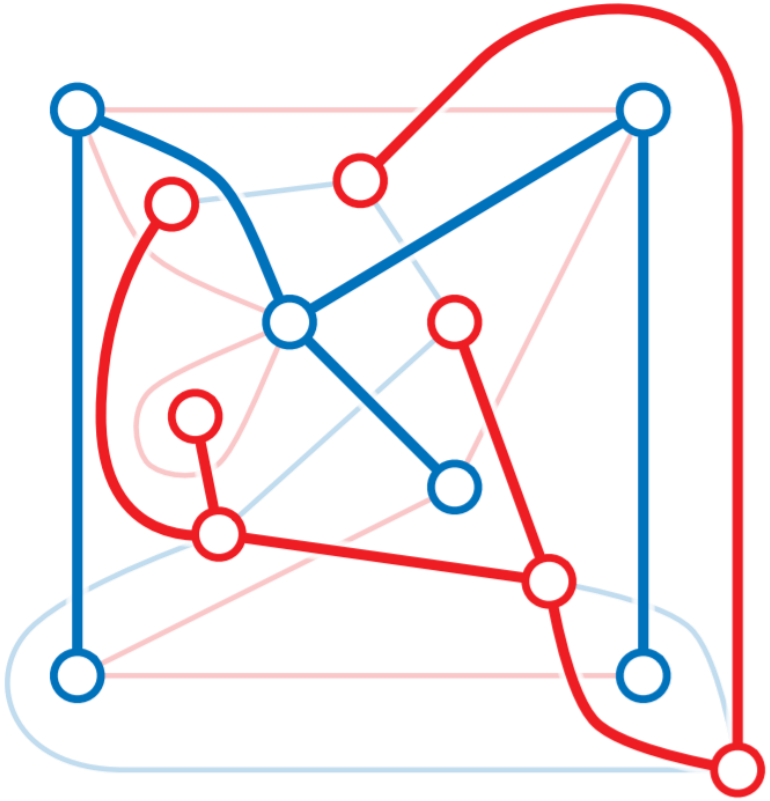
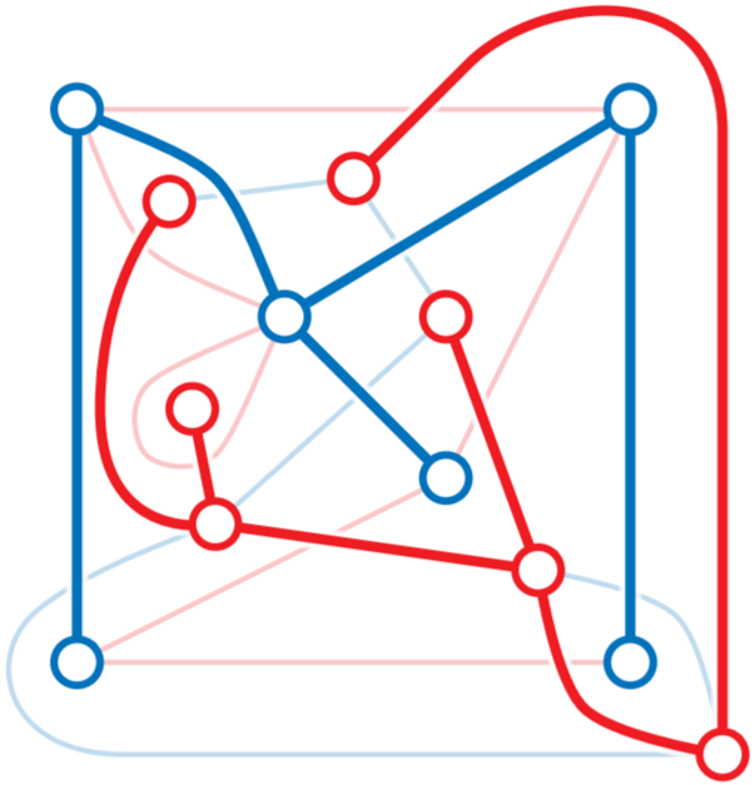
Tree-Cotree Decomposition

- Plane graph G decomposes into
  - Primal spanning tree T
  - Dual spanning cotree C





Euler's Formula

[Euler 1750] [Legendre 1794] [Cayley-Listing 1861]

For any plane graph G,

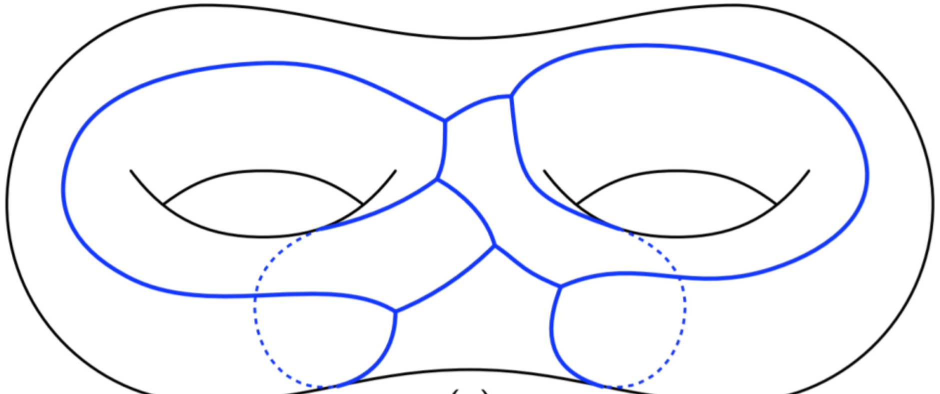
$$V_G - E_G + F_G = 2$$

What about Surface Graph?

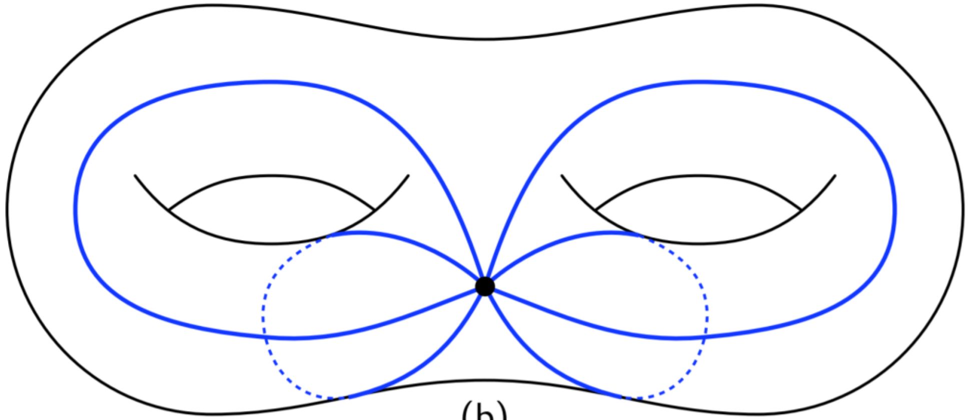
Spanning Tree

```
SpanningTree(G):
  for any edge e in G:
    if e is a loop:
      delete e.
    if e is a bridge:
      contract e
    o.w.
      delete or contract e
  return all contracted edges T
```

System of Loops

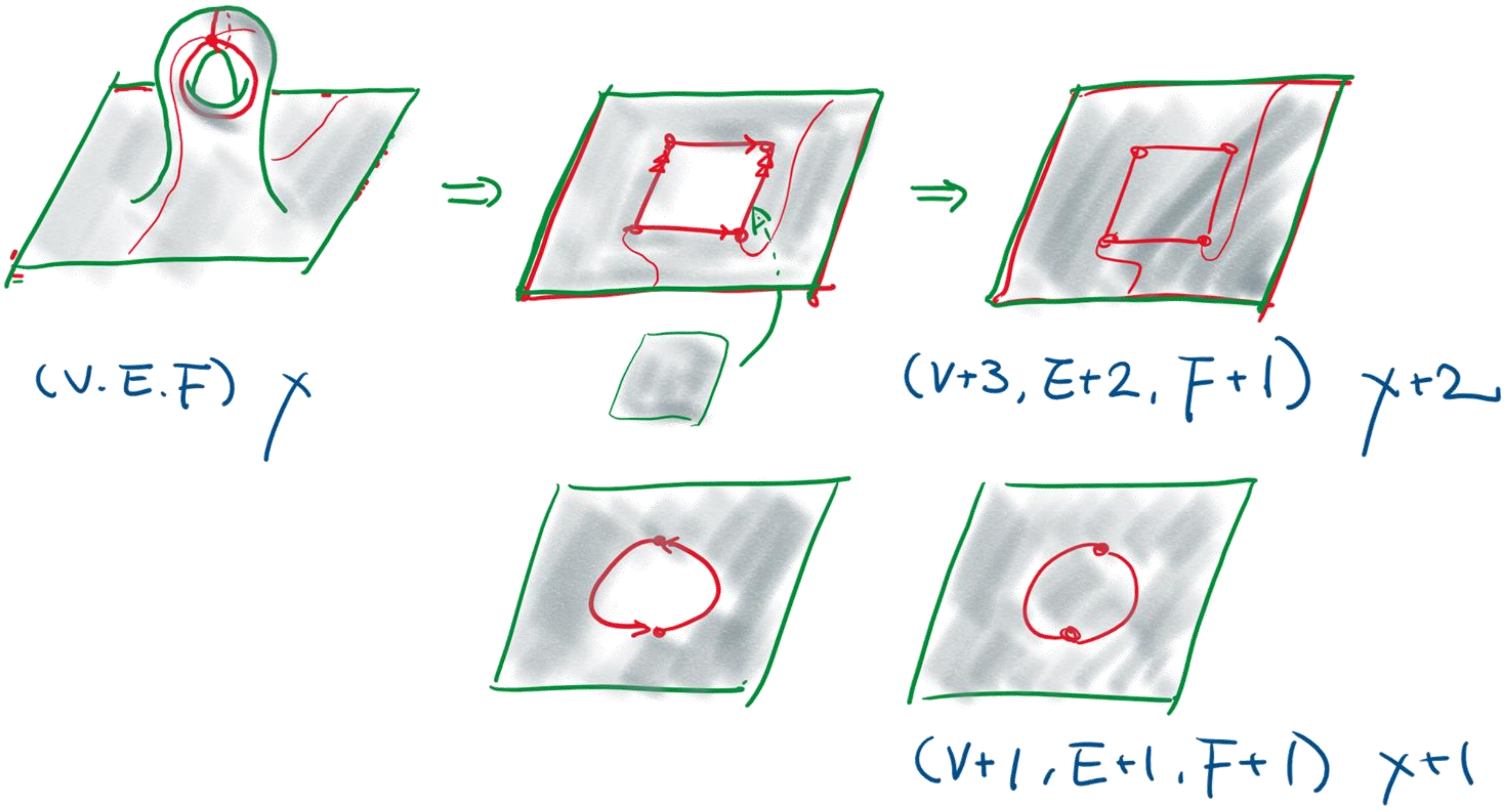


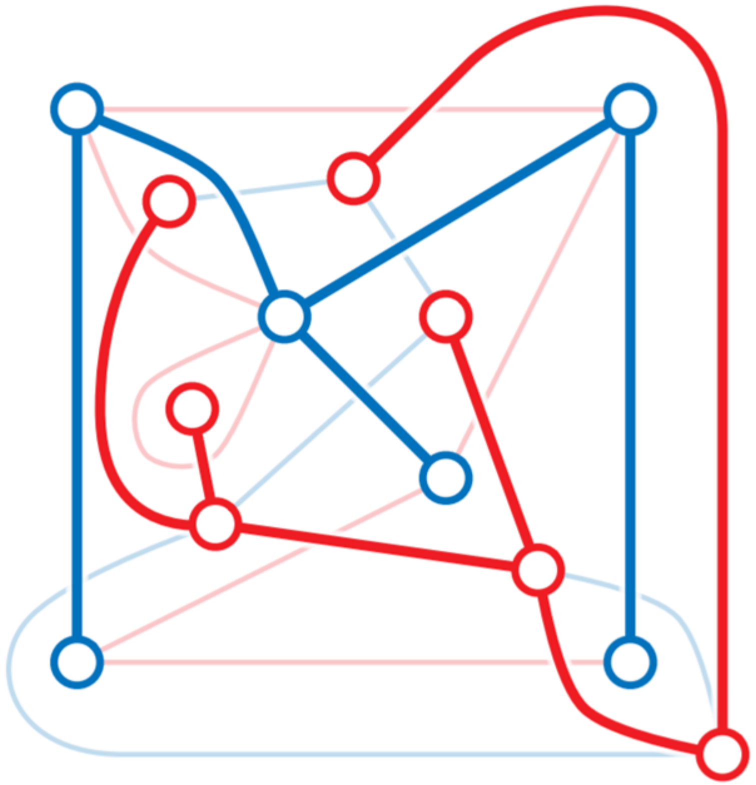
(a)



(b)

How Many Leftover Edges?





Euler's Formula

[Euler 1750] [Legendre 1794] [Cayley-Listing 1861]

For any graph G embedded on surface $\Sigma(g,r,b)$,

$$V_G - E_G + F_G = 2 - 2g - r - b$$

Euler Characteristic is a "Complete" Invar. of Surfaces

Ponder.

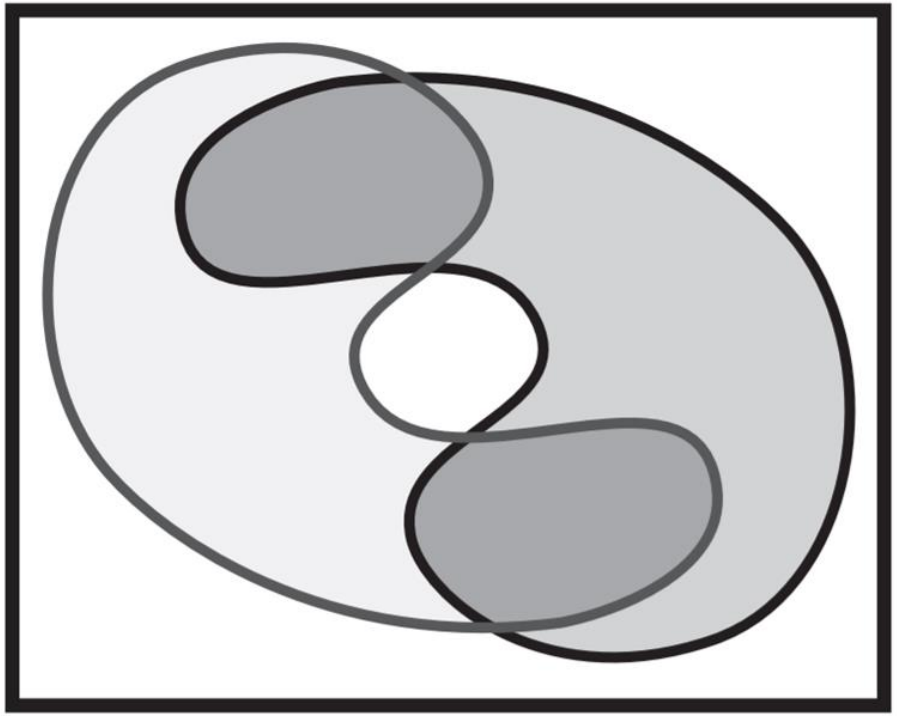
Torus and Möbius band have the same χ?

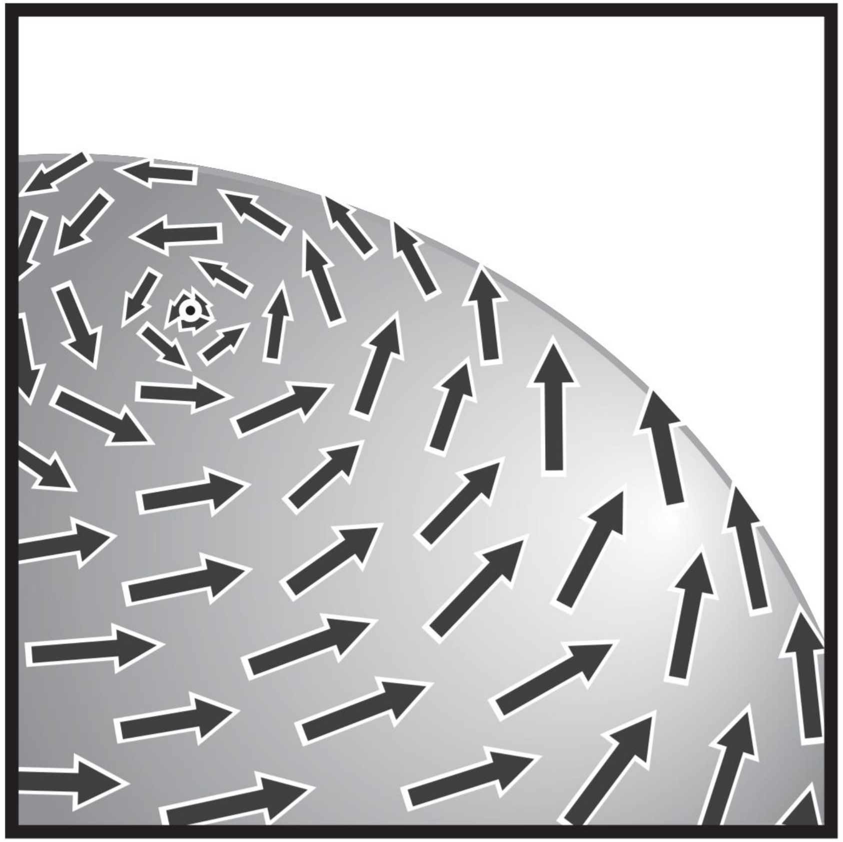
Intermission

Euler Calculus and Curvature

Euler Characteristic is Additive!

- $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$
- Watch out for open/closed



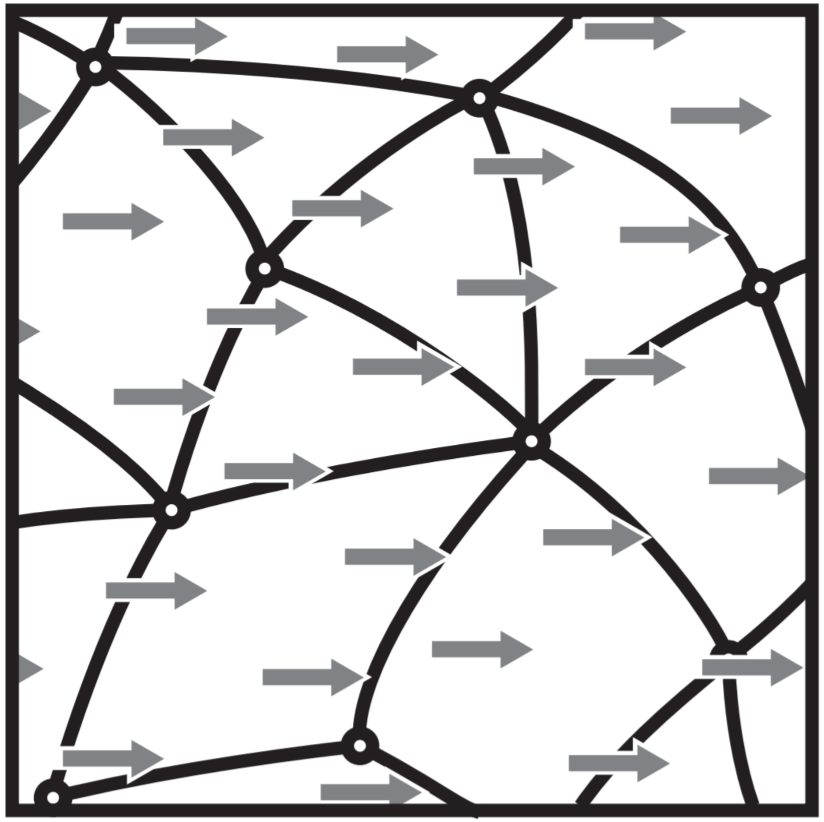


Hedgehog Theorem

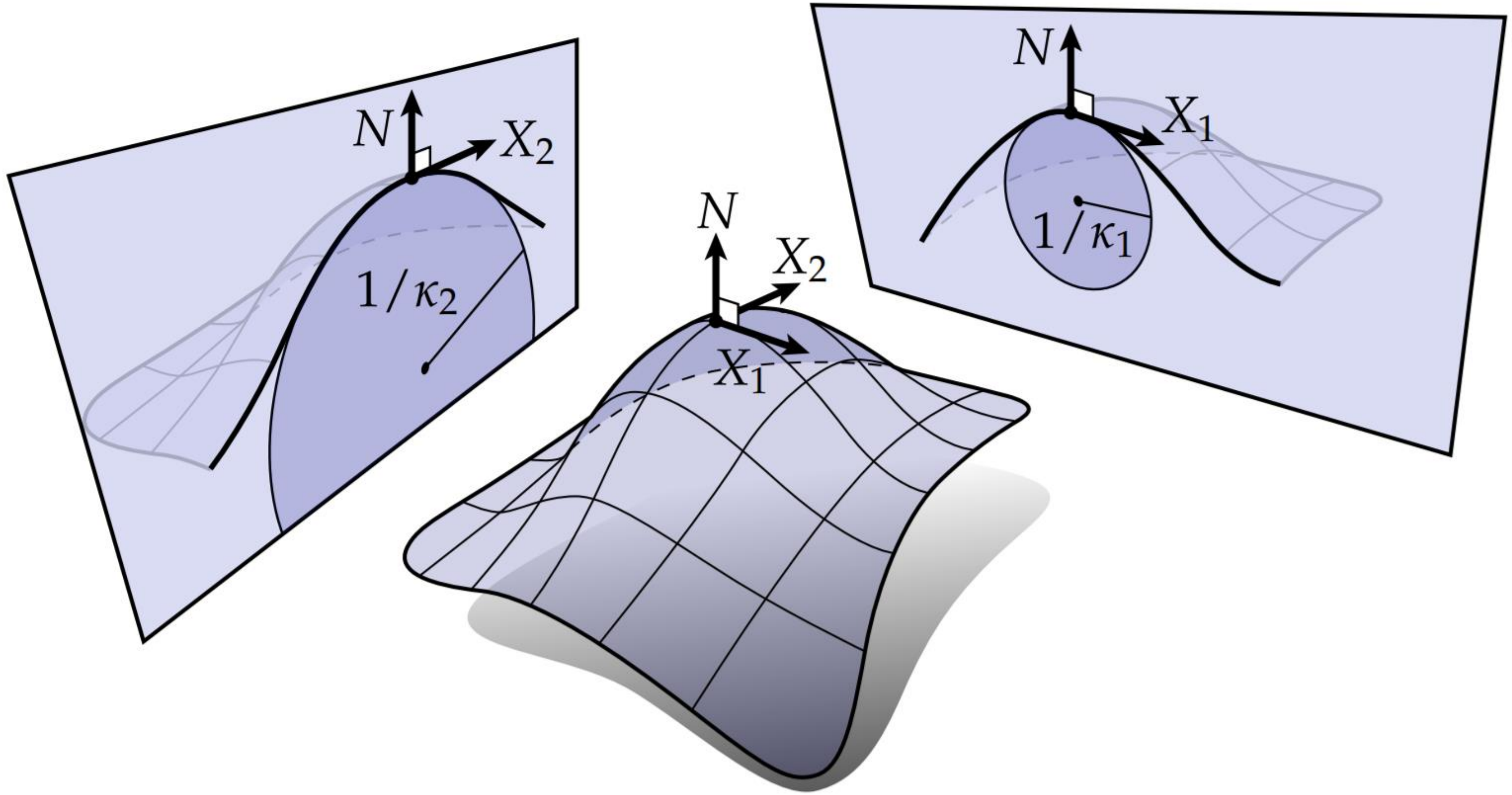
[Poincaré 1885] [Brouwer 1912]

There is no non-vanishing continuous vector field on closed surfaces of non-zero Euler characteristics.

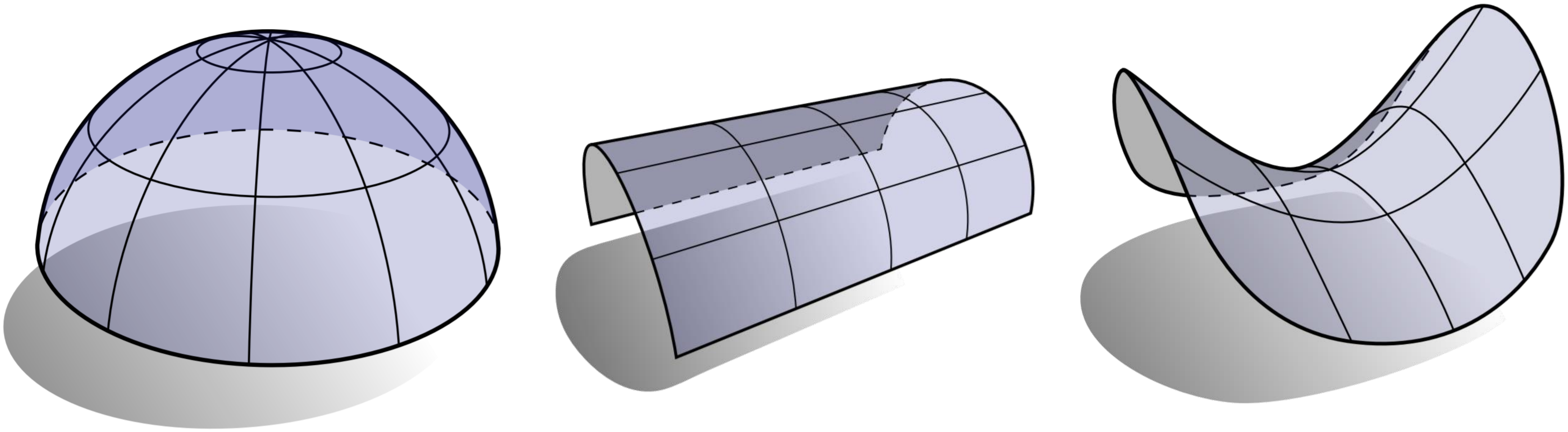
Euler Calculus Proof

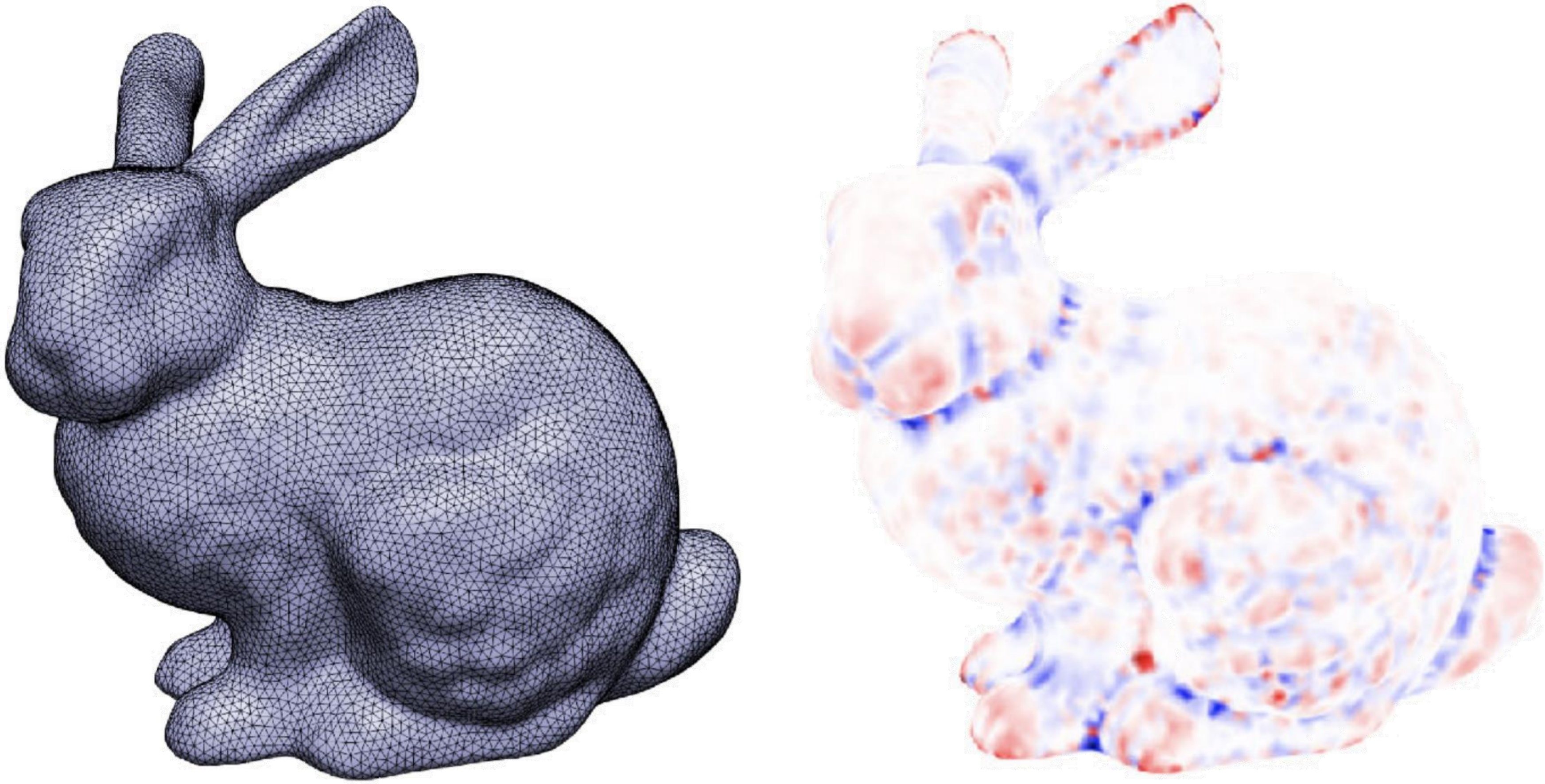


Curvature



Curvature





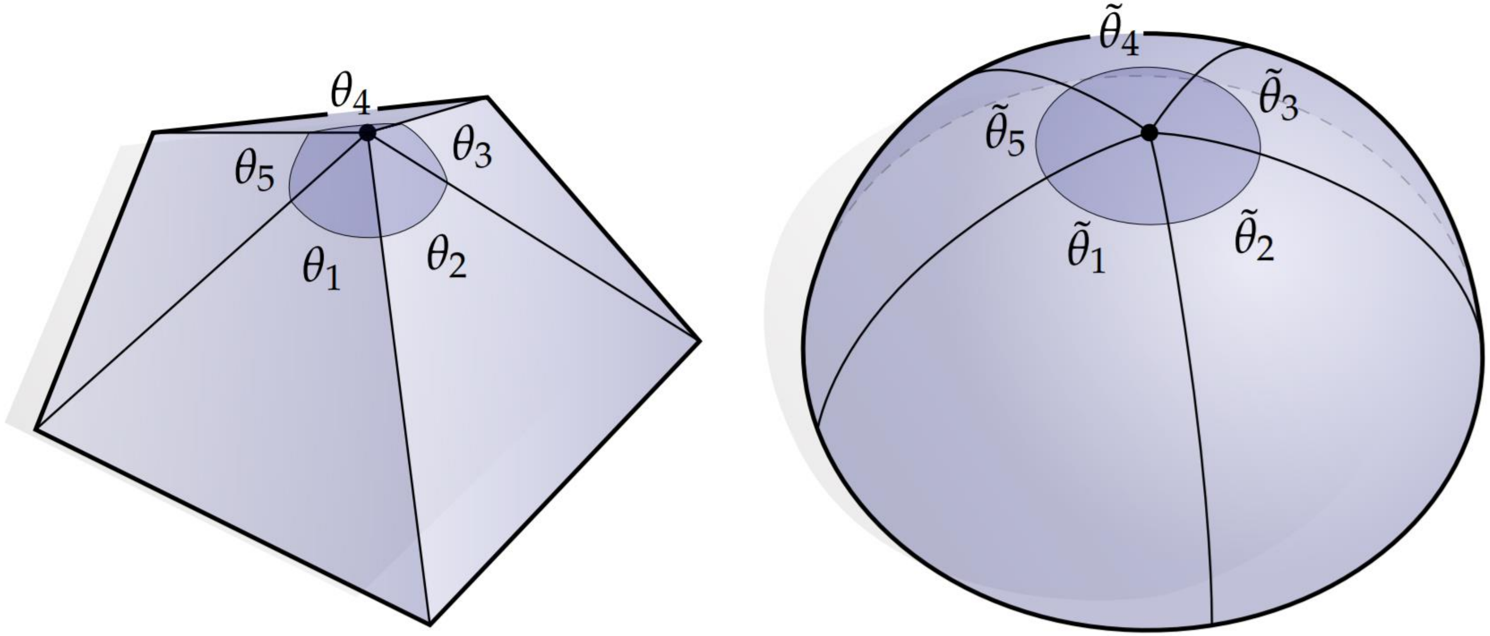
Gauss-Bonnet Theorem [Gauß 1827] [Bonnet 1848]

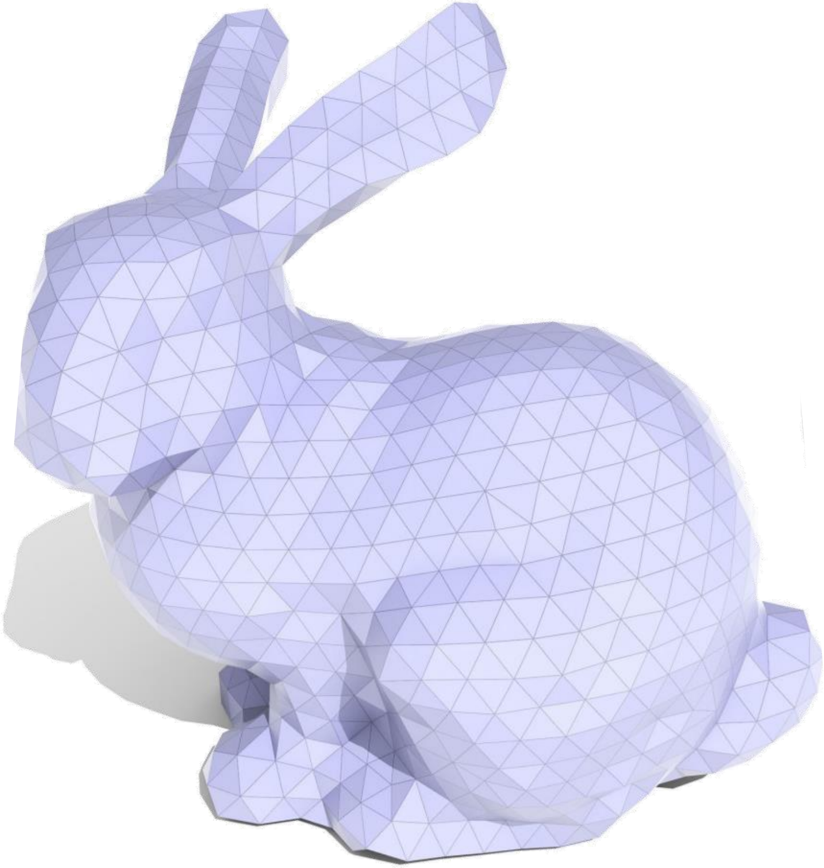
For any surface Σ with Euler characteristic χ,

$$\int_{\Sigma} d\kappa = 2\pi\chi$$

Curvature

- Curvature κ(x)





Discrete Gauss-Bonnet Theorem

For any discrete surface Σ with Euler characteristic χ,

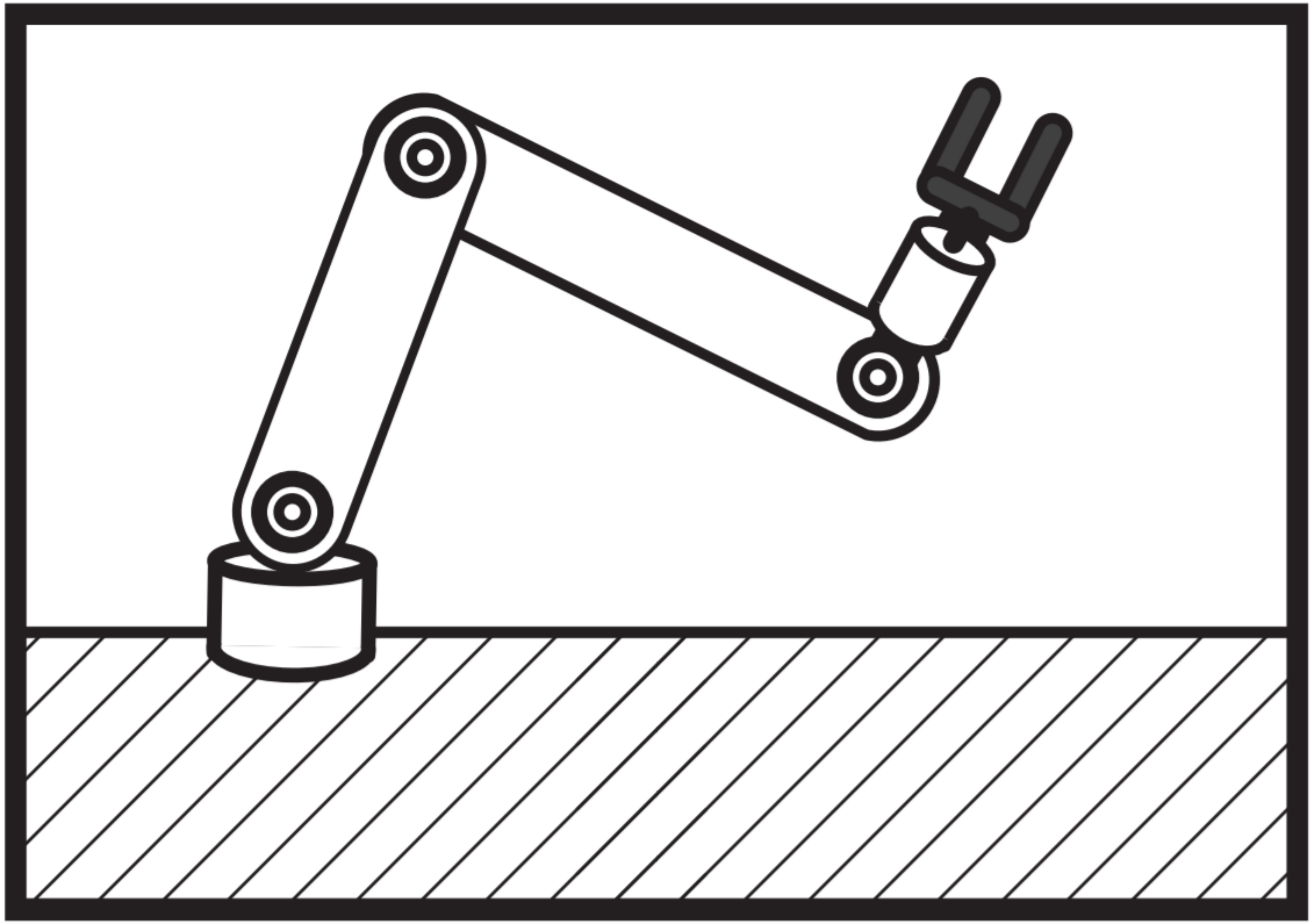
$$\sum_{x} \kappa(x) = 2\pi\chi$$

Curvatures can be Moved Around, but not Removed

To think about later.

Can you prove Hedgehog Theorem using Gauss-Bonnet?

Configuration Space and Complex



Example: Robot Arms

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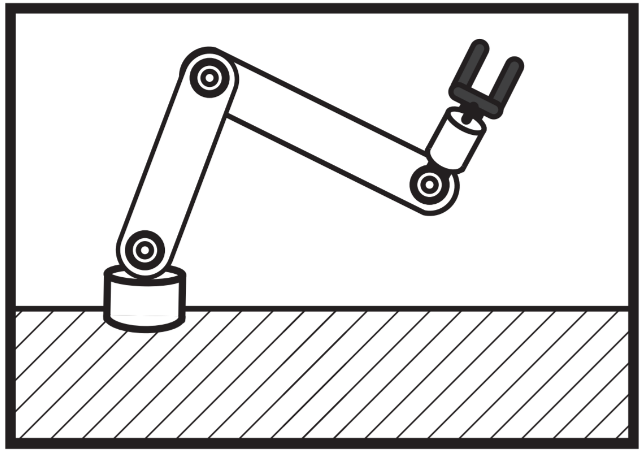
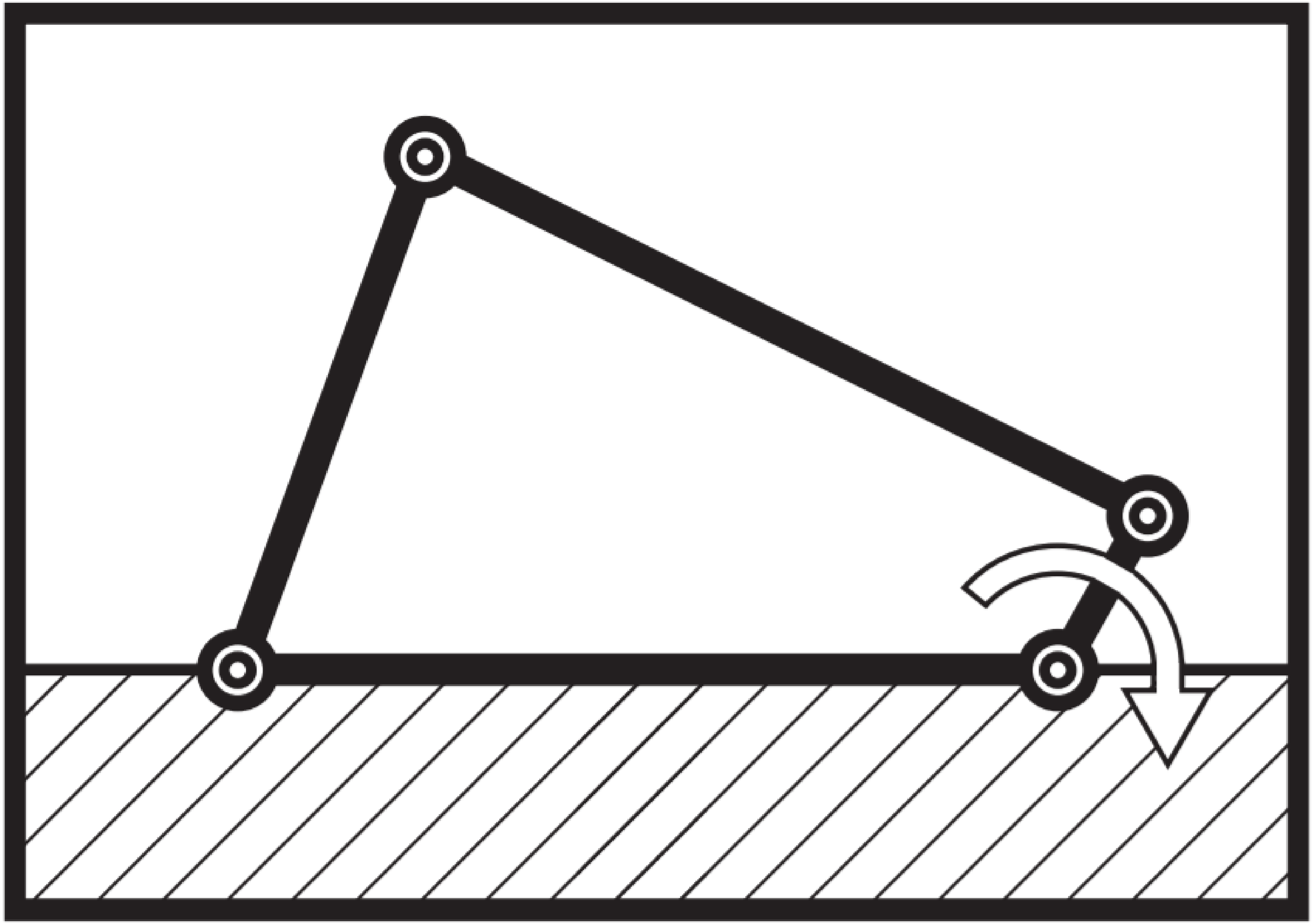
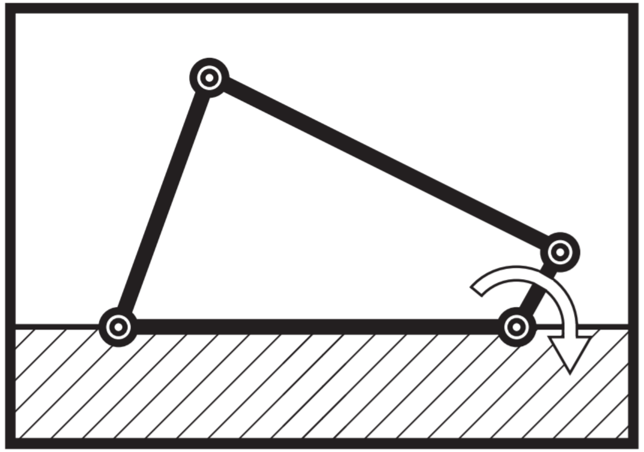


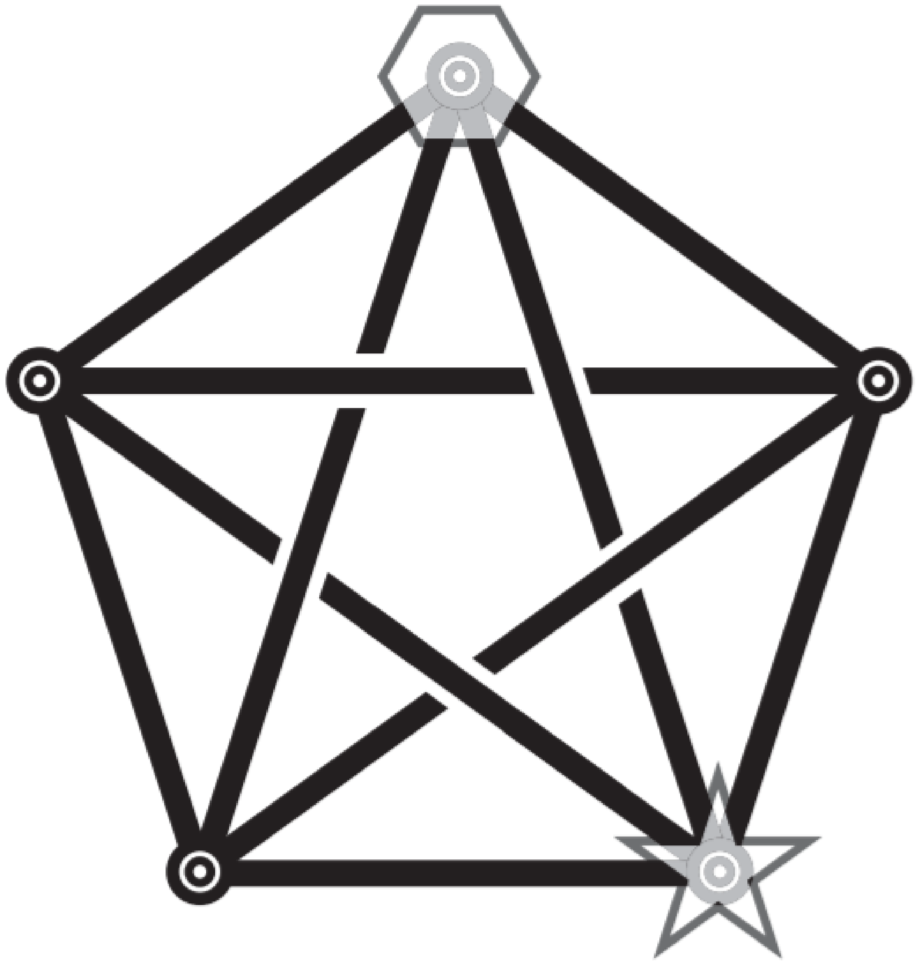


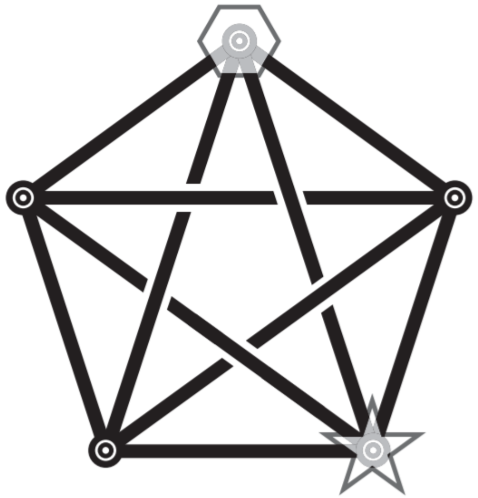
Example:
4-Bar Linkage



Example:
4-Bar Linkage

Example: Space of Vertex Pairs

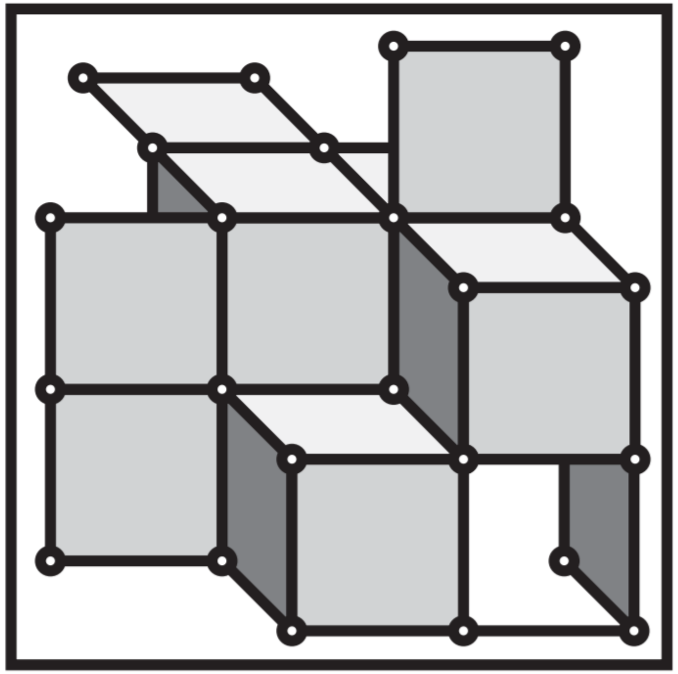


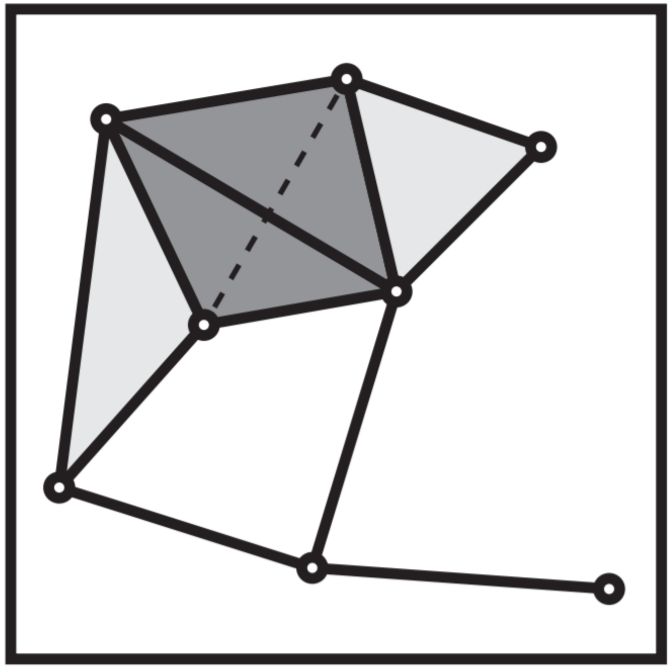


Example: Space of Vertex Pairs

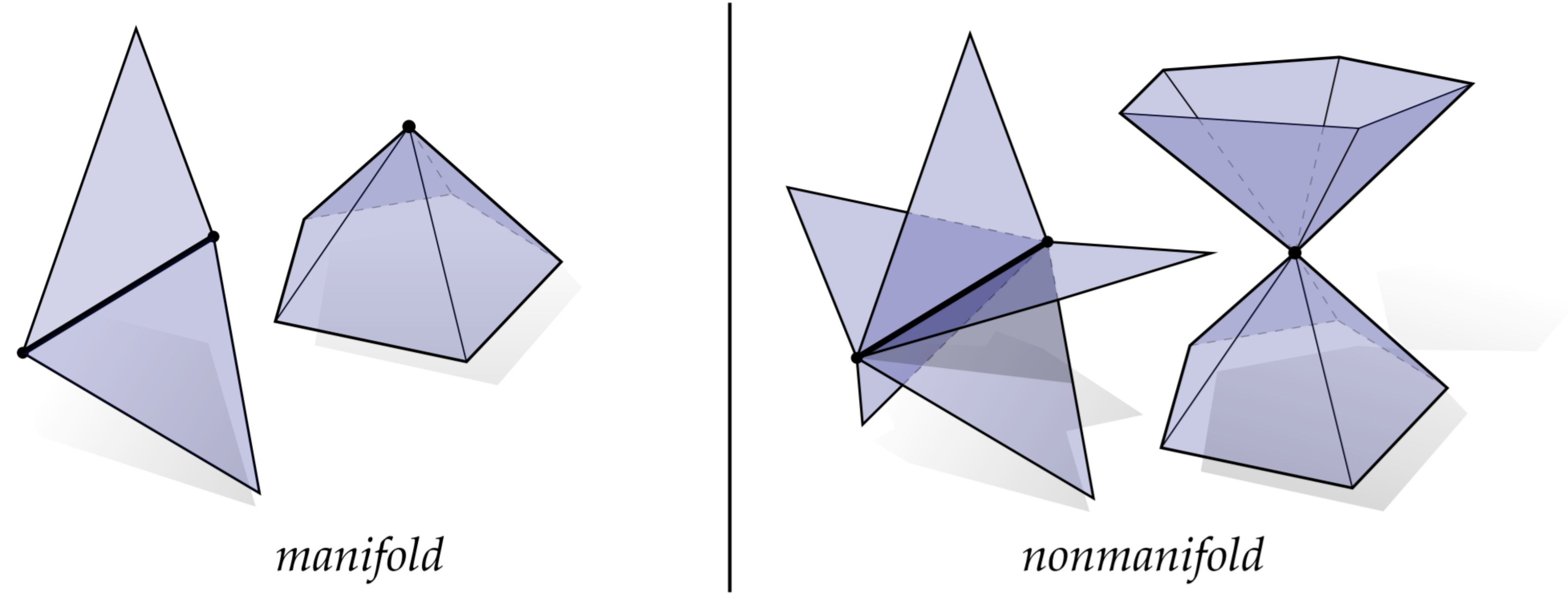
Complexes

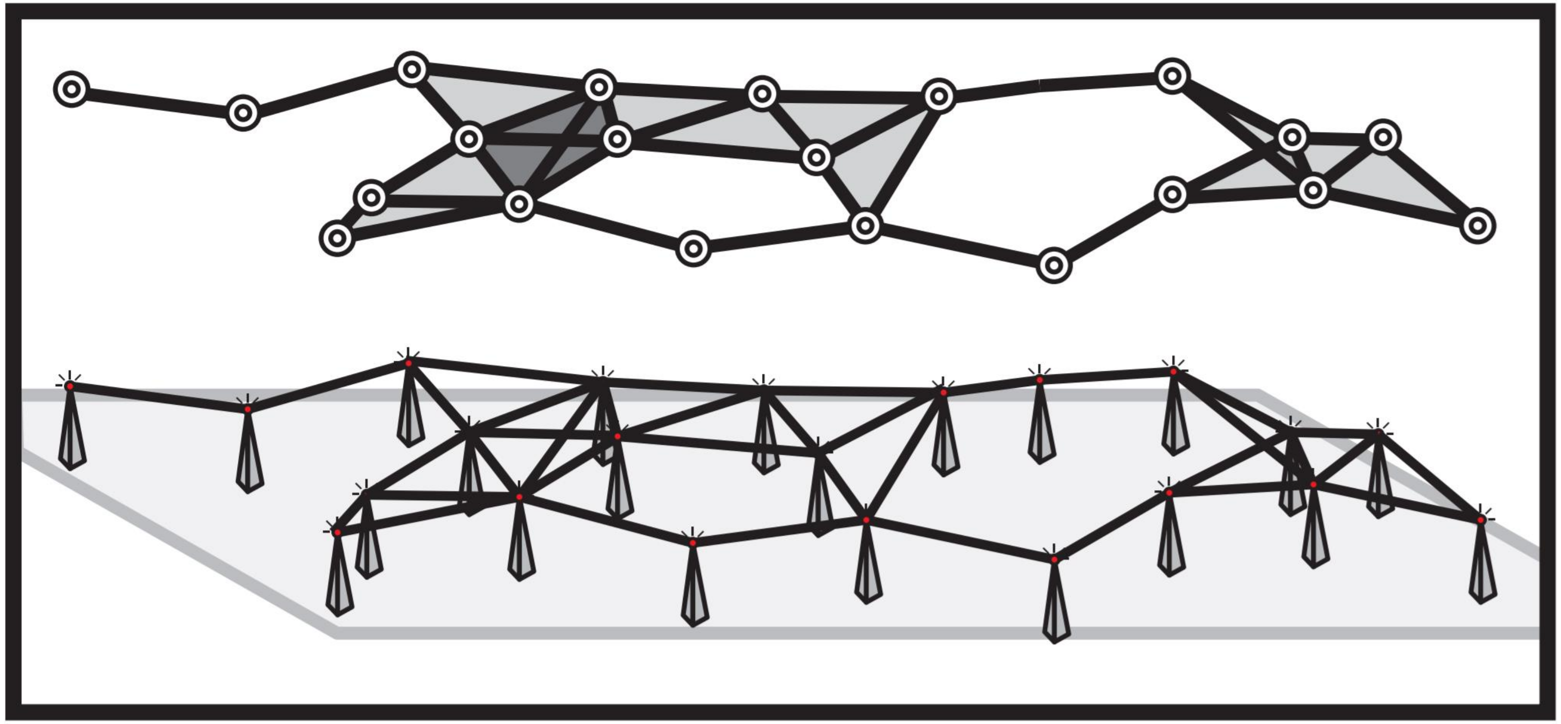
- Gluing a bunch of simplexes together





Extra Features





Vietoris-Rips Complex

- Connect any two points of distance at most r
- Add all simplexes inside a clique

Closing Q. How do we Tell Apart Complexes?

Choose your own adventure:

(A) applications of curves and surfaces: Linkage and Folding, or
(B) behavior of closed curves dictates the shape of space