Exterior Calculus and Isolation Lemma

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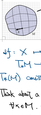
- Case study on what to do when the project of you heard?
- Intuition feedback from: Help me out please!

Crash Course on Differential Geometry [Calculus, linear algebra]

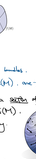
Differential.



$df : X \to \mathbb{R}$ smooth
$df(x) \to T_x(M)$ tangent bundles.

$T_x(M)$ consists of vector space $T_x(M)$, one for each $x \in M$.

Think about a vector field on M as a section of $T_x(M)$:
$\forall x \in M$, pick a vector from $T_x(M)$.
so that the vector flows smoothly.



One can define inner product on $T_x(M)$: $\langle df(x), df(x) \rangle$
which induces a metric $\langle x, x \rangle_g$ on M.

We can also define normal vectors on $T_x(M)$: $\langle df(x), v \rangle = 0$
$df : M \to \mathbb{R}$ Gen map!!

(Ask the for advisors which "technical" thing is useful ver!)

Geometric Representation of $df(x)$:

$$df(x) = \lim_{\Delta x \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix}$$

function vector

In Tensor, $df(x)$ as vector-vector product of X:
In other words, df function vector is a change of basis.

Exterior Calculus.

calculus studies the change of scalars.
vect. calculus " (string) vectors.
exterior calculus " (string) signed volumes.

Wedge product.

example. signed area.

$u \times v = -v \times u$
$(u+v) \times u = (u \times v)$
$(u+v) \times u = u \times v$

example. signed volume.

$u \times v \times w = v \times u \times w = u \times v \times w$
$= -u \times w \times v = -v \times u \times w$

Forms. Given vector space V.

$\Lambda(V)$:= forms on V. forming an algebra of V.
algebra is always product.

Let $\{e_i\}$ be basis of V. the generators of $\Lambda(V)$ are.
$de^i \in V^*$ (dual space of V) s.t. $de^i(e_j) = \delta_{ij}$
dual basis

Naturally, $\Lambda(V) = \bigoplus \Lambda^k(V)$. where.

$\Lambda^k(V)$:= $\{de^i \wedge \cdots \wedge de^k \mid 1 \le i_1 < \cdots < i_k \le \dim(V)\}$
only k vectors and k k-vectors = number.

$\Lambda(T_x M)$ (a collection of $\Lambda^k(T_x M)$, $\forall x \in M$)
differential forms on M.

$\dim(M)$
p-form: a section on M = $\Lambda^p(T_x M)$ $\forall x$ do you usually.

example. $\int_M \omega$ ω a 2-form on $\mathbb{R}$.

$\int_M \omega = \int_M \underbrace{(dx \wedge dy)}_{\text{dx, dy, a 2-form on } \mathbb{R}^2}$.

$\int_M \omega = \int_M \underbrace{\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)}_{\text{2 form on } \mathbb{R}^2} \wedge \cdots$

$\dim(M)$
Test your intuition: $\Lambda(T_x M) = \bigoplus_{k=0}^{\dim(M)} \Lambda^k(T_x M)$ $\forall x$ under product
$dx \wedge dy$ $\wedge$ dz $\Rightarrow$ $dx \wedge dy \wedge dz$ Λ^3.

What are the rules to do?

- Linear algebra and so forth.

$de^i \wedge de^j$ de^i de^j $\in V^*$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

How generally
3-vector X $X = \sum_{i,j,k} V_{ijk} = V_{ijk}^1 + \cdots + V_{ijk}^n$
3-form α $\alpha = \sum_{i,j,k} \alpha(de^i) \cdots \alpha_{ijk}$

$de^i \cdot V_{ijk}^1 = 1 \cdot 1 \cdot 1$

Then $\alpha(X) = \alpha(X) = \sum_{i,j,k} \alpha(de^i \left(\sum_{i,j,k} V_{ijk}^1 \right) = \sum_{i,j,k} \alpha(X)$

- Wedge product with linear algebra.

Consider vectors u, v. 3-forms α, β.
$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ $\alpha \wedge \beta$ vs $\alpha \cdot \beta$ α β.

Then. $u \wedge v = \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ $\alpha \wedge \beta = \det \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$

satisfying all properties of wedge product (Check!)

Exterior Derivative.

$d : \Omega^k \to \Omega^{k+1}$

(1) linear: $d(\alpha + \beta) = d(\alpha) + d(\beta)$
(2) Leibniz: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + \alpha \wedge d\beta$
(3) $d(d\alpha) = d\alpha \wedge \beta + \alpha \wedge d\beta = d\alpha \wedge \beta - \alpha \wedge d\beta$
(4) $d\beta$ is the unique differential on 3-form/p-form β.
(5) $d(d\beta) = 0$ $\forall \alpha = \beta$

Test your intuition yourself. What are things to do?

example.

- Take $f : \mathbb{R}^2 \to \mathbb{R}^2$ 0-form.
$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$ $df = (df)^*$ gradient.

- Take $\omega : \mathbb{R}^2 \to \Omega^1(\mathbb{R}^2) \cong \mathbb{R}^2$ 1-form.
$d\omega = d(\omega_1 dx + \omega_2 dy + \omega_3 dz)$
$d(\omega_1 dx) + d(\omega_2 dy + \omega_3 dz) = \sum_{i=1}^{3} \frac{\partial \omega_i}{\partial x} dx \wedge dx$
apply chain rule, using $\omega_1 \beta = -\beta \wedge \omega$, one has
$d\omega = \left(\frac{\partial \omega_3}{\partial y} - \frac{\partial \omega_2}{\partial z} \right) dy \wedge dz$ Help you $\omega = (f_1 + f_2)^*$.
$+ \left(\frac{\partial \omega_1}{\partial z} - \frac{\partial \omega_3}{\partial x} \right) dz \wedge dx$ $\omega(dx) = (f_1 + f_2)\, dx = \cdots + dx$
$+ \left(\frac{\partial \omega_2}{\partial x} - \frac{\partial \omega_1}{\partial y} \right) dx \wedge dy$ $\nabla \times \omega = (\omega_3 \times \nabla)^*$ curl.
$\nabla \times \omega$ (curl)

- Take $\beta : \mathbb{R}^3 \to \Omega^1(\mathbb{R}) \cong \mathbb{R}^3$ 2-form.
$d\beta = d(\beta_1 dx \wedge dy + \beta_2 dz \wedge dx + \beta_3 dx \wedge dy)$
$= \left(\frac{\partial \beta_1}{\partial x} + \frac{\partial \beta_2}{\partial y} + \frac{\partial \beta_3}{\partial z} \right) dx \wedge dy \wedge dz$
$\nabla \cdot X = \alpha \circ X^*$ div.

— ⊙ —

the Poincare Lemma.

Consider contract complex $\mathbb{R}\mathrm{Cm}(\mathbb{R}, \Omega^k, d)$.
define the Poincare cohomology $H_{dR}^k(M)$.

complex/closed form $\{\omega\}$: $d\omega = 0$.
cohomology/exact form $\{\omega\}$: $\exists \eta \in \text{form } \beta : d\beta = \alpha$.

example. $d\theta = \frac{1}{r^2}$

is a closed form (check!)
but not exact : θ not a 0-form.
(0-form: map manifold)

Stokes' Thm. ω compact-supported p-form.
Σ a oriented $(p+1)$-manifold.

Then $\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega$

example. $\int_{[0,1]} \frac{1}{x^2} dx = \int_{\{0,1\}} f(x) = f(1) - f(0)$
$\Sigma = \mathbb{R}^2$.

Formal: $\int_{\Sigma} (\nabla \times \omega) dx = \int_{\Sigma} \left(\frac{\partial \omega_3}{\partial y} - \frac{\partial \omega_2}{\partial z} \right) dx \wedge dz$
$\Sigma = \mathbb{R}^3$.
$= \int_{\Sigma} d(\omega_1 dx + \omega_2 dz)$
$= \int_{\partial\Sigma} (\omega_1 dx + \omega_2 dz) = \int_{\partial\Sigma} \omega \cdot d\Sigma$

(Stokes), (Green's).

Wedge product forms $H_{dR}^k(M)$ $\cong$ ring. (check!)

The Künneth Decomposition. For M manifold, $H_{dR}^k(M) \cong H_{dR}^k(M)$

Let the given manifold be obvious (α complex).
Integrate α p-form on α over p-simplex σ : $\int_\sigma \omega$
Now $\partial\sigma = \sum_{i=1}^{p+1} \sigma_i$

example. Say Δ = triangle (2-simplex) w/ edges e_1, e_2, e_3.
ω denote 1-form (on edges).
$\int_{\partial\Delta} \omega = \int_{\partial\Delta} \omega = \sum_{i=1}^{3} \int_{e_i} \omega = \sum_{i=1}^{3} \omega_i$
Now we can define $\int_{\partial\Delta} \omega = \Delta \cdot d\Omega$.

Intuition to cohomology on our familiar cartesian complex!

Thm (Poincare Duality). M compact orientable manifold.

$\int : H_k(M) \to H_{dR}^k(M)$ $\Rightarrow$ $H_{dR}^k(M)$
$\int : H_k(M) \to H_{dR}^k(M)$ $\Rightarrow$ $H_{dR}^k(M)$

(integration works at the homology/cohomology level because.
$\int_{\partial\Sigma=0} (\omega + d\eta) = \int_{\partial\Sigma} \omega + \int_{\partial\Sigma} d\eta = \int_{\partial\Sigma} \omega + \int_{\Sigma} d^2\eta = \int_{\partial\Sigma} \omega$
$\int_{\partial\Sigma=0} \omega$ $\int_{\partial\Sigma} \omega$ $\int_{\partial\Sigma} \omega$ $\int_{\partial\Sigma} \omega$
Thus $\int_{\partial\Sigma} \omega$ is well-defined.)

Laplacian and Hodge Theorem.

Define Laplacian $\Delta = (d + d^*)^2 = d d^* + d^* d = (d + d^*)(d + d^*)$ $d = d$
$\forall \omega \in \Omega^k = \Omega^{k+1} \to \Omega^{k+2}$ $d^* = d^*$
$\to \Omega^{k-1} \to \Omega^{k-2}$ $d^* = d^*$
Harmonic form $H_{dR}^k(M) = \ker \Delta$.

Hodge Thm. M compact oriented manifold, $\Omega^k(M)$ has decomposition
$\Omega^k = d\Omega^{k-1} \oplus H_{dR}^k \oplus \Omega^{k+1}$
in other words, every form has exact, harmonic, & closed components.

Conclusion. gradient/co-gradient. ∇ curl/curlfree.

Cor. $H_k^*(M) \cong H_{dR}^k(M)$.
furthermore, Hodge star $\star : H_{dR}^k(M) \to H_{dR}^{n-k}(M)$
Proof: Harmonic decomposition

pf. $L(\omega\star) = \omega\star^{n-k}$ $(d\omega\star = \omega\star d\omega)$ $\omega\star = d\omega\star^{-1}\omega\star\omega = 0$
$\star\omega$ is harmonic.

Hard. Every cohomology class has a harmonic representative.

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Application. Isolation lemma in planar graphs.
(2011)
Isolation Lemma. Let G be a collection of subsets of $[n]$.
G each element in $[n]$ is assigned ω / weight, weight ω in $[2n]$.
Then w/ pr $\geq 1/2$, min-weight subset in G is unique.
Proof: Thirdwork counting

example. Think about graph w/ n edges, and pick between two vertices.

Randomizing isolation lemma is a useful open problem.

Question. Can we get deterministic isolation on planar graphs?
(2011) (2012)
Thm. Given planar graph G, $O(\log^2 n)$-bit logspace
Isolation weighting

pf. Idea: Use Gomory Thm!

$\omega(e) = \begin{cases} 2^i & e \text{ is vertical} \\ 2^j & e \text{ is vertical} \end{cases}$

do 2 is a 2-form, positive coordinates.

$\int_\omega = \int_\Omega \omega = \int_\Omega d\omega > 0$.

(2011)(2012)

Cor. Planar Reachability $\in$ UL.
Bipartite Planar Matching $\in$ SPL.
$\in$ UL. (2011) (2012)

Open Question. Do we have deterministic logspace isolation lemma for general planar graph?
pf. $\Omega(n)$ $\Omega(n)$. (2012)

Near-planar